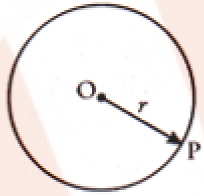
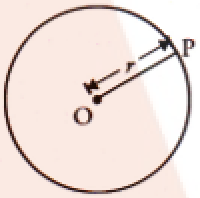
Circles – Summary Notes

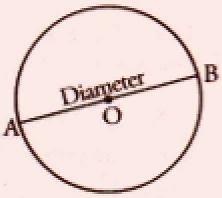
- **Circle:** The path of all points that are equidistant from a fixed point is called a circle. In other words, a circle is a set of those points in a plane that are at a given constant distance from a given fixed point in the plane.
  - The fixed point is called the centre of the circle.



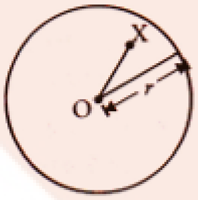
  - The constant distance of each and every point on the circle from its centre is called the radius of the circle.



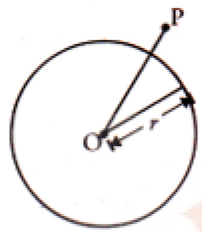
- **Diameter of a Circle:** A line segment joining two points on the circle and passing through the centre of the circle is called a diameter of the circle.
  In the given figure, AB is a diameter of the circle with centre O.
  Obviously, the length of diameter $= 2 \times \text{radius}$.



- **Interior of a Circle:** A point X lies inside a circle if and only it its distance from the centre of the circle is less than the radius of the circle.

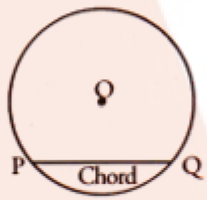


  In the given figure, $OX < r$, therefore, X lies inside the circle.
  The set of all points of the plane such that $OX < r$ forms the interior of the circle.
- **Circular Region:** The interior region of a circle along with the circumference is called the circular region.
- **Exterior of a Circle:** A point P lies outside a circle if and only if its distance from the centre of the circle is greater than the radius of the circle.

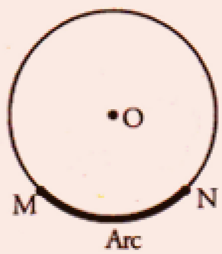


In the adjoining figure, $OP > r$, therefore, P lies outside the circle.
The set of all points of the plane such that $OP > r$ forms the exterior of the circle.

- **Chord of a Circle:** A line segment which joins any two points of a circle is called a chord of the circle.
  A circle has infinite number of chords.
  In the adjoining figure, PQ is a chord of a circle with centre O. The distance PQ is called the length of the chord.

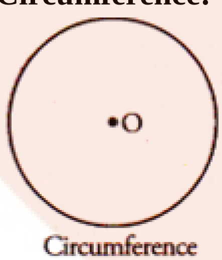


- **Arc of a Circle:** A part of a circle included between two points on the circle is called an arc of the circle. The arc of a circle is denoted by the symbol $\frown$.



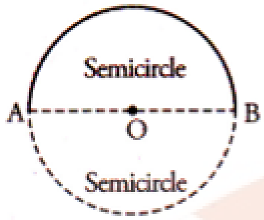
In the adjoining figure, MN denotes the arc $\widehat{MN}$ of the circle with centre O.

- **Circumference:** The whole arc of a circle is called the circumference of the circle.

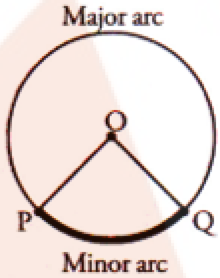


The length of the circumference of a circle is the length of its whole arc. However, in general, the term ‘circumference’ of a circle refers to its length.

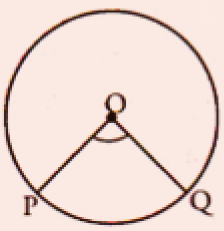
- **Semicircle:** A diameter of a circle divides the circle into two equal parts. Each part of the whole is called a semicircle of the circle.



- Minor and Major Arcs: An arc less than the length of a semicircle is called a minor arc of the circle, and an arc greater than the length of a semicircle is called a major arc of the circle.

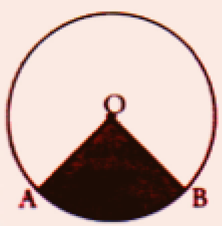


- **Angle Subtended by an Arc:** The angle formed by the two bounding radii of an arc of a circle at the centre of the circle is called the angle substended by the arc.



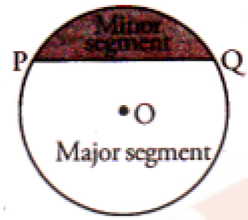
In the adjoining figure, $\angle POQ$ is the angle subtended by $\widehat{PQ}$ of a circle with centre O. The measurement of $\angle POQ$ in degrees is called degree measurement of $\widehat{PQ}$.

- **Sector of a Circle:** The region enclosed by an arc of a circle and its two bounding radii is called a sector of a circle.



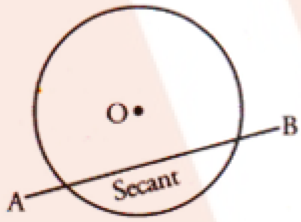
In the given figure, the region enclosed by $\widehat{AB}$ and its two boundary radii OA and OB is a sector of the circle with centre O. Generally, the term 'sector of a circle' refers to the area of this region.

- **Segment of a Circle:** A chord of a circle divides it into two parts. Each part is called a segment. The part containing the minor arc is called the minor segment, and the part containing the major arc is called the major segment.



However, in general, the terms minor segment and major segment refer to the areas of the regions enclosed by these. The minor and major segments of a circle arc called alternate segments of each other.

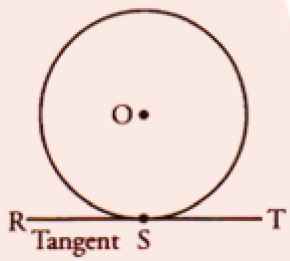
- **Secant of a Circle:** A line which intersects a circle in two distinct points is called a secant of the circle.



In the adjoining figure, line AB is a secant of the circle with centre O.

Note: *A line can meet a circle at most in two distinct points.*

- **Tangent to a Circle:** A line which meets a circle exactly at one point is called a tangent to the circle.

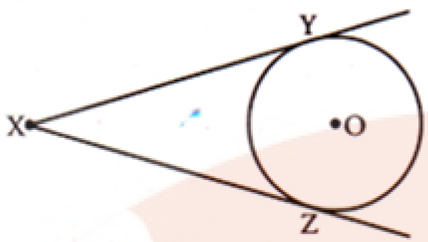


In the adjoining figure, the line RST is a tangent to the circle with centre O.

The point where the line touches the circle is called its point of contact. In the given figure, S is the point of contact.

Note:

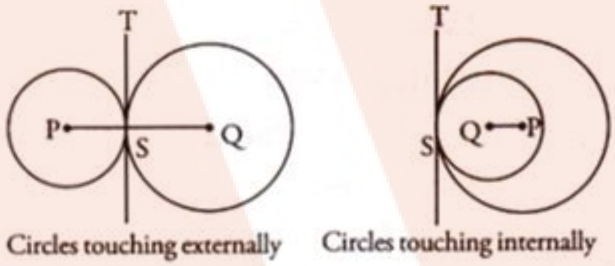
- All points other than the point of contact of a tangent to a circle lie out the circle.
- No tangent can be drawn to a circle through a point inside the circle.
- Not more than one tangent can be drawn to a circle at a point on the circumference of the circle.
- Two tangents can be drawn to a circle from a point outside the circle.



In the above figure, X is a point outside the circle with centre O. XY and XZ are two tangents drawn from the point X to the circle with centre O.

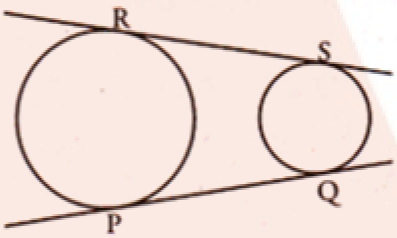
The distance XY (or XZ) is called the length of tangent.

- **Touching Circles:** Two circles are said to touch each other if and only if they have only one point in common. The common point is called the point of contact, and the line joining their centres is called the line of centres. A line touching two circles is called a common tangent. Two circles may touch internally or externally.



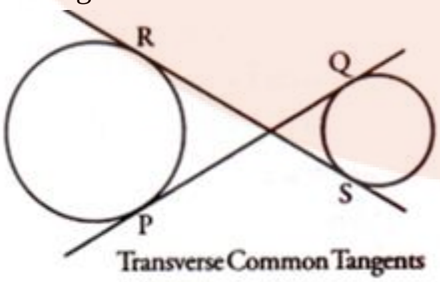
In the above figure, S is the point of contact, PQ is the line of centres and ST is a common tangent.

- **Direct Common Tangent:** A common tangent to two circles is called a direct common tangent if both the circles lie on the same side of it.



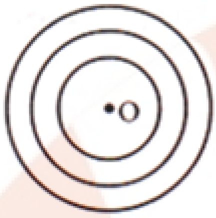
In the above figure, PQ and RS are two direct common tangents. Only two direct common tangents can be drawn to two circles and both are equal in length, i.e., PQ = RS.

- **Transverse Common Tangent:** If the circles lie on opposite sides of the common tangent, the tangent is called a transverse common tangent.



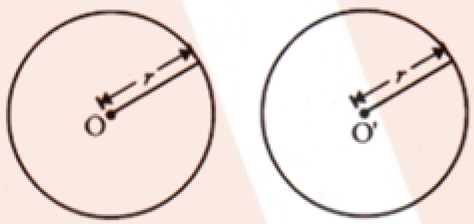
In the above figure, PQ and RS are two transverse common tangents. Only two transverse common tangents can be drawn to two circles and both are equal in length, i.e., PQ = RS.

- **Concentric Circles:** Two or more circles are said to be concentric if an only if they have the same centre and different radii.



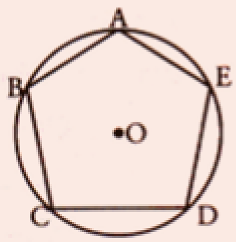
In fact, we can draw a number of circles with the same centre and different radii and such circles are called a family of concentric circles.

- **Equal (or Congruent) Circles:** Two or more circles are said to be equal (or congruent) if and only if they have the same radius.



In the above figures, circles with centres O and O' have equal radius (r), therefore, these are equal circles.

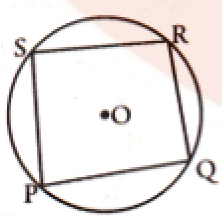
- **Circumscribed Circle:** A circle passing through all the vertices of a polygon is called a circumscribed circle of the polygon, and its centre is called circumcenter.



In the adjoining figure, the polygon ABCDE is called inscribed polygon.

The points which lie on the circumference of the same circle, are called concyclic points.

- **Cyclic Quadrilateral:** A quadrilateral is said to be cyclic when all of its vertices lie on the circumference of a circle.

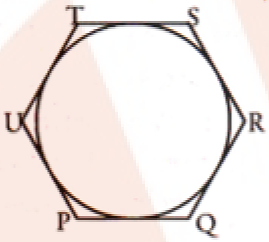


In the adjoining figure, PQRS is a cyclic quadrilateral. The points P, Q, R and S are concyclic points. The cyclic quadrilateral has two pairs of:

- Opposite vertices P, R; Q, S.
- Opposite angles $\angle P, \angle R$; $\angle Q, \angle S$.

Opposite angles of a cyclic quadrilateral are supplementary.

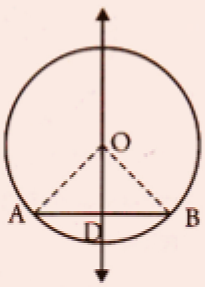
- **Inscribed Circle:** A circle touching all the sides of a polygon is called an inscribed circle of the polygon, and its centre is called incentre. The polygon is called circumscribed polygon.



Theorems

Theorem 1

- Statement: A straight line drawn from the centre of a circle to bisect a chord, which is not a diameter, is perpendicular to the chord.



- Given: A chord AB of a circle with centre O, and OD bisects the chord AB.
- To Prove: $OD \perp AB$.
- Construction: Join OA and OB.
- Proof: In right triangles ODA and ODB,

OD = OD	[Common]
OA = OB	[Radii of the same circle]
AD = DB	[$\because$ D is the midpoint of AB]
$\therefore \Delta ODA \cong \Delta ODB$	[By SSS axiom]

$\Rightarrow \angle ODA = \angle ODB$

But these are adjacent supplementary angles.

$\therefore \angle ODA + \angle ODB = 180°$

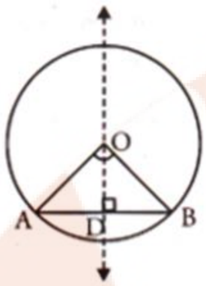
$\Rightarrow \angle ODA = \angle ODB = 90°$

Hence, $OD \perp AB$. Proved.

Theorem 2

(Converse of Theorem 1)

- **Statement:** The perpendicular to a chord from the centre of a circle bisects the chord.

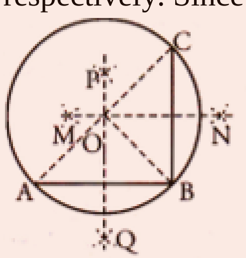


  - Given: A chord AB of a circle with centre O, and OD is perpendicular to the chord AB.
  - To Prove: AD = DB.
  - Construction: Draw $OD \perp AB$.
  - Proof: In right triangles ODA and ODB,

OD = OD	[Common]
OA = OB	[Radii of the same circle]
$\angle ADO = \angle BDO$	[Each of $90°$]
$\therefore \Delta OAD \cong \Delta ODB$	[By SAS congruency]
Hence, AD = DB.	Proved.

Theorem 3

- There is one and only one circle passing through three non-collinear points.
  - **Given:** Three non-collinear points A, B and C.
  - **To Prove**: There is one and only one circle passing through A, B and C.
  - **Construction:** Join AB and BC. Draw perpendicular bisectors PQ and MN of AB and BC respectively. Since A, B and C are not collinear,



$\therefore PQ / MN$.

Bisectors PQ and MN intersect each other at O. Join OA, OB and OC.

  - **Proof:** Since O lies on the perpendicular bisector of AB,

$\therefore$ OA = OB

Again O lies on the perpendicular bisector of BC,

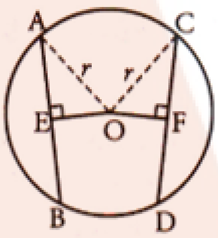
$\therefore$ OB = OC

Thus, OA = OB = OC = r (say).

Taking O as centre and radius r, we draw a circle C (O, r) which clearly passes through points A, B and C.
This proves that there is a circle. Now we prove that this is the only such circle.
If possible, let there be another circle $C(O', s)$ with same centre O' and radius s passing through the points, A, B and C, the point O' will les on the perpendicular bisectors PQ of AB and MN of BC.
Since the two lines cannot intersect at more than one point, so O' must coincide with O.
Since $OA = r$, $O'A = s, O$ and O' coincide,
$\therefore r = s$
$\Rightarrow C(O, r) \cong C(O', s)$.
Hence, there is one and only one circle passing through three non-collinear points. Proved.

Theorem 4

- **Statement:** Equal chords of a circle are equidistant from the centre.



- **Given:** A circle with centre O, two equal chords AB and CD. $OE \perp AB$ and $OF \perp CD$.
- **To Prove:** OE = OF.
- **Construction:** Join OA and OC.
- **Proof:** Since the perpendicular from the centre of a circle to a chord bisects the chord,

$\therefore AE = \frac{1}{2} AB$

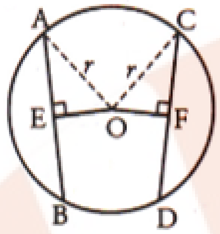
and $CF = \frac{1}{2} CD$

But AB = CD [Given]
$\therefore AE = CF$ ………… (i)
Now in right triangles OEA and OFC,
OA = OC [Radii of the same circle]
AE = CF [Proved in (i)]
$\angle AEO = \angle CFO$ [Each of $90°$]
$\therefore \Delta OEA \cong \Delta OFC$ [By SAS congruency]
$\therefore$ OE = OF [By CPCT]
This shows that AB and CD are equidistant from O.
Hence, equal chords of a circle are equidistant from the centre. Proved.

Theorem 5

(Converse of Theorem 4)

- Statement: Chords of a circle that are equidistant from the centre are equal.



  - **Given:** A circle with centre O, two chords AB and CD. $OE \perp AB$ and $OF \perp CD$ such that OE = OF.
  - **To Prove:** AB = CD.
  - **Construction:** Join OA and OC.
  - **Proof:** Since the perpendicular from the centre of a circle to a chord bisects the chord,

    $\therefore AE = \frac{1}{2}AB$

    and $CF = \frac{1}{2}CD$

    Now in right triangles OEA and OFC,

    OA = OC = r [Radii of the same circle]

    OE = OF [Given]

    $\angle OFC = \angle OEA$ [Each of $90°$]

    $\therefore \Delta OEA \cong \Delta OFC$

    $\Rightarrow AE = CF$ [By CPCT]

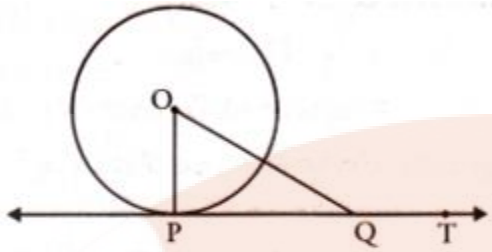
    $\Rightarrow \frac{1}{2}AB = \frac{1}{2}CD$

    $\Rightarrow AB = CD$.

    Hence, the chords of a circle which are equidistant from the centre are equal. Proved.

Theorem 6

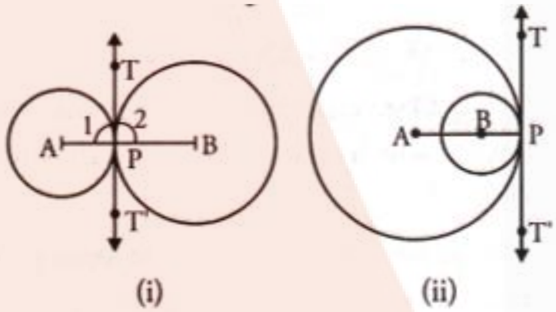
- **Statement:** The tangent at any point on a circle and the radius through that point are perpendicular to each other.
  - **Given:** A circle with the centre O, and a tangent PT at point P.
  - **To Prove:** $OP \perp PT$.
  - **Construction:** Take any point Q on the tangent PT. Join OQ.



- **Proof:** Since Q is any point other tha P on PT,
  $\therefore$ Q lies in the exterior of circle.
  $\Rightarrow OQ > OP$.
  This shows that of all segments that can be drawn from the centre O to any point on the line PT, OP is shortest.
  We know that the shortest segment that can be drawn from a given point to a given line is the perpendicular from the given point on the given line.
  Hence, $OP \perp PT$. Proved.

Theorem 7

- **Statement:** If two circles touch each other (internally or externally), the point of contact lies on the line through their centres.
  - **Given:** Two circles with their respective centres A and B touching each other at a point P.
  - **To Prove:** The point of contact lies on the line joining their centres, i.e., P lies on the line AB.
  - **Construction:** Join AP and BP. Through P, draw a common tangent PT to the circles.

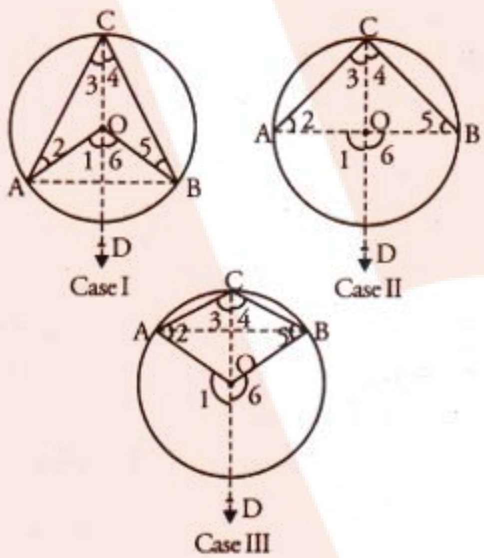


  - **Proof:** Since PT is a tangent to the circle with centre A and AP is the radius of the circle,
    $\therefore PA \perp PT$
    $\Rightarrow \angle 1 = 90°$
    Similarly, $PB \perp PT$
    $\Rightarrow \angle 2 = 90°$
    From figure (i),
    $\angle 1 = 90°$
    and $\angle 2 = 90°$ [Already proved]
    $\Rightarrow \angle 1 + \angle 2 = 90° + 90° = 180°$

$\Rightarrow$ AP and PB are on the same straight line.
Hence, A, P and B are in a straight line.
From figure (ii),
$\angle TPA = \angle TPB$ [Angles with a common arm]
$\Rightarrow$ A, P and B are on the same straight line
Hence, P lies on the line AB. Proved.

Theorem 8

- **Statement:** The angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.
  - **Given:** A circle with centre O in which an arc AB subtends $\angle AOB$ at the centre and $\angle ACB$ at any point C on the remaining part of the circle (i.e., not on arc AB).
  - **To prove:** $\angle AOB = 2\angle ACB$.
  - **Construction:** Join DOC.



  - **Proof: Case I:** When $\angle AOB$ is an acute angle
    In ΔAOC,
    OA = OC [Radii of the same circle]
    $\therefore\ \angle 2 = \angle 3$
    But $\angle 1 = \angle 2 + \angle 3$ [External angle is equal to the sum of interior opposite angles]
    $\therefore \angle 1 = 2\angle 3$ ……. (i)
    Similarly, in ΔBOC,
    $OB = OC$ [Radii of the same circle]
    $\therefore \angle 4 = \angle 5$
    $\angle 6 = \angle 4 + \angle 5$
    $\therefore\ \angle 6 = 2\angle 4$ ……. (ii)

Adding equations (i) and (ii), we get

$$\angle 1+\angle 6=2\angle 3+2\angle 4$$

$\Rightarrow \angle 1+\angle 6=2(\angle 3+\angle 4)$

Hence, $\angle AOB=2\angle ACB$. Proved.

Case II: When $\angle AOB$ is straight angle

In ΔABC

OA = OC [Radii of the same circle]

$\angle 2=\angle 3$ [Angles opposite to equal sides are equal]

$\Rightarrow \angle 1=\angle 2+\angle 3$ [External angle is equal to the sum of interior opposite angles]

$\therefore\ \angle 1=2\angle 3$

Similarly, $\angle 6=2\angle 4$

Now $\angle 1+\angle 6=2(\angle 3+\angle 4)$

$\therefore\ \angle AOB=2\angle ACB$

But $\angle AOB=180°$

$\therefore\ 2\angle ACB=180°$

$\Rightarrow \angle ACB=\frac{1}{2}\times 180°=90°$, which is true,

i.e., an angle in a semicircle is equal to $90°$.

Proved.

Case III: When $\angle AOB$ is obtuse reflex angle

In ΔOAC,

OA = OC [Radii of the same circle]

$\therefore \angle 2=\angle 3$

But $\angle 1=\angle 2+\angle 3$ [External angle is equal to the sum of interior opposite angles]

$\therefore\ \angle 1=2\angle 3$ …….. (i)

Similarly, in ΔBOC,

$OB=OC$ [Radii of the same circle]

$\therefore\ \angle 4=\angle 5$

$\angle 6=\angle 4+\angle 5$

$\therefore\ \angle 6=2\angle 4$ …….. (ii)

Adding equation (i) and (ii), we get

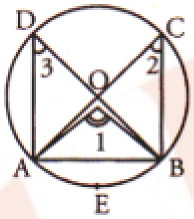
$$\angle 1+\angle 6=2\angle 3+2\angle 4$$

$\Rightarrow \angle 1+\angle 6=2(\angle 3+\angle 4)$

Hence, $\angle AOB=2\angle ACB$. Proved.

Theorem 9

- **Statement:** Angles in the same segment of a circle are equal.
  - Given: A circle with centre O, $\angle 2$ and $\angle 3$ are on the same segment.



  - **To Prove:** $\angle 2 = \angle 3$.
  - **Construction:** Join OA and OB.
  - **Proof:** AEB subtends $\angle 1$ at the centre and $\angle 3$ on the remaining part of it.

    $\therefore \angle 1 = 2\angle 3$ ………. (i)

    Similarly, $\angle 1 = 2\angle 2$ ……… (ii)

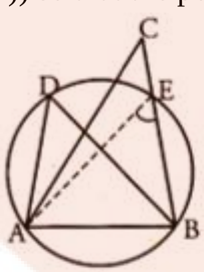
    From equations (i) and (ii), we get

    $2\angle 2 = 2\angle 3$

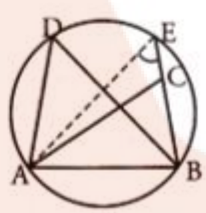
    $\Rightarrow \angle 2 = \angle 3$. Proved.

Theorem 10
(Converse of Theorem 9)

- **Statement:** If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, then the four points are concyclic.
  - **Given:** A line segment AB, two points C and D lying on the same side of the line containing the segment AB and $\angle ACB = \angle ADB$.
  - **To Prove:** Points A, B, C and D are concyclic.
  - **Construction:** If the points A, B, C and D are not concyclic, let a circle passing through three non-collinear points A, B and D meet BC at E (figure (i)) or BC produced at E (figure (ii)) so that the points A, B, E and D are concyclic.



(i)



(ii)

  - **Proof:** We have:

    $\angle ACB = \angle ADB$ …… (i) [Given]

    $\angle AEB = \angle ADB$ ……. (ii) [Angles in the same segment]

    $\therefore \angle ACB = \angle AEB$ [From (i) and (ii)]

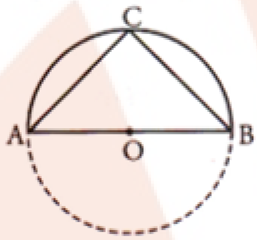
But $\angle ACB \neq \angle AEB$ [Exterior angle of a triangle cannot be equal to opposite interior angle]

$\therefore$ Our supposition is wrong.

Hence, points A, B, C and D are concyclic. Proved.

Theorem 11

- **Statement:** The angle in a semicircle is a right angle.



  - **Given:** AB is a diameter of a circle with centre O and $\angle ACB$ is an angle in a semicircle.
  - **To Prove:** $\angle ACB = 90°$
  - **Proof:** Since the angle subtended at the centre by arc AB is twice the angle formed by this arc at C,

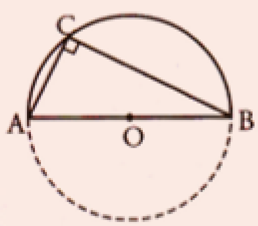
    $\therefore\ 2\angle ACB = \angle AOB = 180°$ [$\because$ AOB is a straight line]

    $\Rightarrow \angle ACB = \frac{1}{2}\times 180° = 90°$. Proved.

Theorem 12

(Converse of Theorem 11)

- **Statement:** If an arc of a circle subtends a right angle at any point on the remaining part of the circle, then the arc is a semicircle.
  - **Given:** A circle with centre O, and arc AB (shown by dotted) which subtends $\angle ACB$ on the remaining part of the circle such that $\angle ACB = 90°$.



  - **To Prove:** Arc AB is a semicircle.
  - **Construction:** Join OA and OB.
  - **Proof:** From the above figure,

    $\angle ACB = 90°$ ………… (i) [Given]

    $\angle AOB = 2\angle ACB$ ……….. (ii)

    [Angle at the centre is double the angle at the remaining part of the circle]

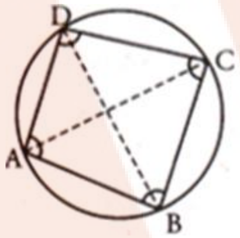
    $\Rightarrow \angle AOB = 2\times 90° = 180°$ [From (i)]

$\Rightarrow \angle AOB$ is a straight line.

Hence, arc AB is a semicircle. Proved.

Theorem 13

- **Statement:** The opposite angles of a cyclic quadrilateral are supplementary.
  - **Given:** A cyclic quadrilateral ABCD.
  - **To Prove:** $\angle A + \angle C = 180°$ and $\angle B + \angle S = 180°$.
  - **Construction:** Join AC and BD.



  - **Proof:** We have:

$\angle ACB = \angle ADB$ ………….. (i) [Angles in the same segment]

$\angle BAC = \angle BDC$ ………….. (ii)[Angles in the same segment]

Adding equations (i) and (ii), we get

$$\angle ACB + \angle BAC = \angle ADB + \angle BDC$$

$\Rightarrow \angle ACB + \angle BAC = \angle ADC$

$\Rightarrow \angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$ [Adding $\angle ABC$ to both sides]

$\Rightarrow 180° = \angle ADC + \angle ABC$

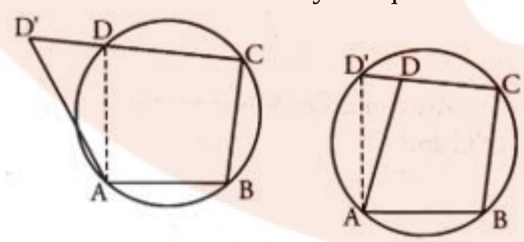
$\therefore \angle D + \angle B = 180°$

Again $\angle A + \angle C = 180°$, and $\angle B + \angle D = 180°$. Proved.

Theorem 14

(Converse of Theorem 13)

- **Statement:** If the sum of any pair of opposite angles of a quadrilateral is $180°$, the quadrilateral is cyclic.
  - **Given:** A quadrilateral ABCD in which $\angle B + \angle D = 180°$.
  - **To Prove:** ABCD is a cyclic quadrilateral.



  - **Proof:** If possible, let ABCD be not a cyclic quadrilateral.

Draw a circle passing through three non-collinear points A, B and C.

Suppose the circle meets AD or AD produced at D'. Join $D'C$.

Now $\angle ABC + \angle ADC = 180°$ .... (i) [Given]

and $\angle ABC + \angle AD'C = 180°$ .... (ii) [Opposite angles of a cyclic quadrilateral]

Comparing equations (i) and (ii), we get

$$\angle ABC + \angle ADC = \angle ABC + \angle AD'C$$

$\Rightarrow \angle ADC = \angle AD'C$

But this is not possible, since an exterior angle of a triangle can never be equal to its interior opposite angle.

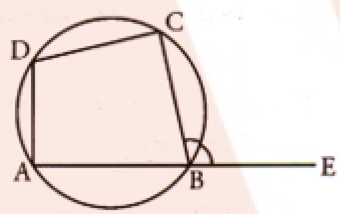
Clearly, $\angle ADC = \angle AD'C$ is possible only when D' coincides with D.

So, the circle passing through A, B and C must also pass through D.

Hence, ABCD is a cyclic quadrilateral. Proved.

Theorem 15

- **Statement:** The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
  - **Given:** A cyclic quadrilateral ABCD whose side AB is produced to a point E.
  - **To Prove:** Ext. $\angle CBE$ = int. opposite $\angle ADC$.



  - **Proof:** $\angle ABC + \angle CBE = 180°$ [AE is a straight line]

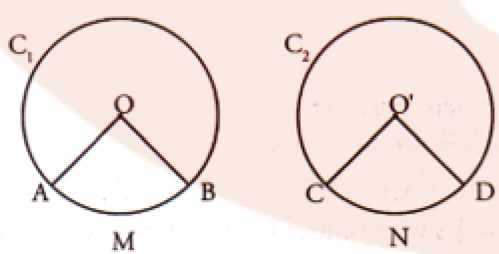
    $\angle ABC + \angle ADC = 180°$ [Opposite angles of a cyclic quadrilateral]

    $\Rightarrow \angle ABC + \angle CBE = \angle ABC + \angle ADC$

    $\therefore \angle CBE = \angle ADC$. Proved.

Theorem 16

- **Statement:** In equal circles (or in the same circle), if two arcs subtend equal angles at the centres (or centre), they are equal.
  - **Given:** Two circles C_1 and C_2 with centres O and O' respectively. Arc AMB subtends $\angle AOB$ at the centre and arc CND subtends $\angle CO'D$ such that $\angle AOB = \angle CO'D$.



  - **Proof:** Place th first circle C_1 over C_2 such that centre O falls on centre O' and radius OA falls on radius $O'C$.

$\therefore$ Point A falls on point C and OB falls on $O'D$. $[OA = O'C$ and $\angle AOB = \angle CO'D]$

Again, point B falls on point D. $[\because OB = O'D]$

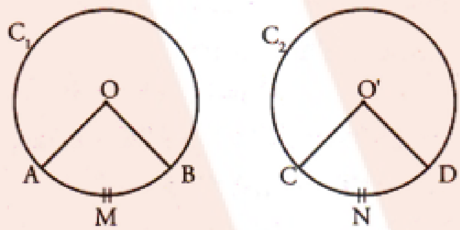
$\therefore$ $\widehat{AMB}$ coincides with $\widehat{CND}$.

$\therefore$ $\widehat{AMB} = \widehat{CND}$.

Similarly, in the same circle if $\widehat{AMB}$ and $\widehat{CND}$ subtend equal angles at the centre, arcs ar also equal. Proved.

Theorem 17
(Converse of Theorem 16)

- **Statement:** In equal circles (or in the same circle), if two arcs are equal, they subtend equal angles at the centre.
  - **Given:** Two circles C_1 and C_2 with their centres O and O'. $\widehat{AMB} = \widehat{CND}$.
  - **To Prove:** $\angle AOB = \angle CO'D$.



  - **Proof:** Please circle C_1 over C_2 such that A falls on C, AO falls on CO' and falls on.

    $\therefore$ O falls on O' $[OA = O'C = \text{radii}]$

    and B falls on $O'D$ $[\widehat{AMB} = \widehat{CND}]$

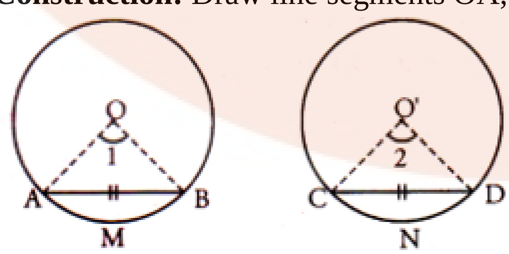
    $\Rightarrow$ OB falls on $O'D$

    Since sector AOBM completely coincides with sector $CO'DN$,

    $\therefore \angle AOB = \angle CO'D$. Proved.

Theorem 18

- **Statement:** In equal circles (or in the same circle), equal chords cut off equal arcs.
  - **Given:** AB and CD are chords of two equal circles with centres O and O' respectively, and AB = CD.
  - **To Prove:** Arc AB = Arc CD.
  - **Construction:** Draw line segments OA, OB $O'C$ and $O'D$.



- **Proof:** In triangles OAB and $O'CD$,

  AB = CD [Given]

  $OA = O'C$ [Radii of equal circles]

  $OB = O'D$ [Radii of equal circles]

  $\therefore \Delta OAB \cong \Delta O'CD$ [By SSS axiom]

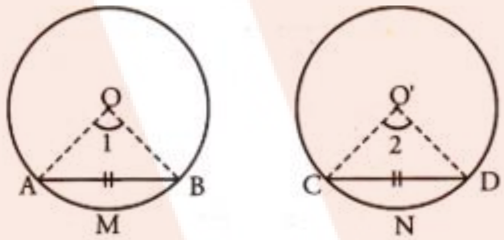
  $\therefore \angle 1 = \angle 2$ [By CPCT]

  Hence, arc AB = arc CD. Proved.

Theorem 19

(Converse of Theorem 18)

- **Statement:** In equal circles (or n the same circle), if two arcs are equal, the chords are equal.
  - **Given:** Two equal circles with their centres O and O' and arc AMB = arc CND.
  - **To prove:** AB = CD.
  - **Construction:** Draw line segments OA, OB, $O'C$ and $O'D$.



  - **Proof:** Let AB and CD be minor arcs. Then:

  $\widehat{AB} = \widehat{CD}$

  $\Rightarrow m(AB) = m(CD) \Rightarrow \angle 1 = \angle 2$

  In triangles OAB and $O'CD$,

  $OA = O'C$ [Radii of congruent circles]

  $\angle 1 = \angle 2$ [Already proved]

  $OA = O'D$ [Radii of congruent circles]

  $\therefore \Delta OAB \cong \Delta O'CD$ [By SAS axiom]

  $\therefore AB = CD$ [By CPCT]

  If AB and CD are major arcs, AMB and CND will be minor arcs.

  $\therefore AB \cong CD$

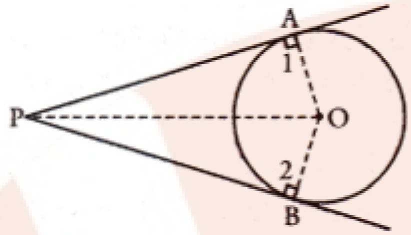
  Hence, AB = CD. Proved.

Theorem 20

- **Statement:** If two tangents are drawn from an external point to a circle, then the:
  (i) tangents are equal in length,
  (ii) tangents subtend equal angles at the centre of the circle,
  (iii) tangents are equally inclined to the line joining the point and the centre of the circle.
  - **Given:** A circle with centre O and a point P outside the circle.

PA and PB are two tangents drawn from the point P.

- **To Prove:** PA = PB.
- **Construction:** Join OP, OA and OB.



- **Proof:** Since OA is radius and PA is a tangent to the circle at A.

$\therefore\ OA \perp PA$

$\Rightarrow \angle 1 = 90°$

Similarly, $\angle 2 = 90°$

Now in triangles OAP and OBP,

$\angle 1 = \angle 2$ [Each of $90°$]

OP = OP [Common]

OA = OB [Radii of the same circle]

$\therefore \Delta OAP \cong \Delta OBP$ [By SAS axiom]

(i) PA = PB [By CPCT]

(ii) $\angle AOP = \angle BOP$

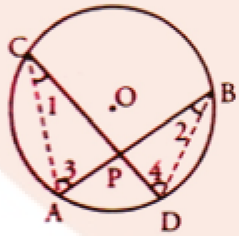
(iii) $\angle APO = \angle OPB$. Proved.

Theorem 21

- **Statement:** If two chords of a circle intersect internally or externally, the product of the lengths of the segments are equal.

Case I: When chords intersect internally

- **Given:** Two chords AB and CD intersecting inside a circle at point P.
- **To Prove:** $PA \times PB = PC \times PD$.
- **Construction:** Join A to C and B to D.



- **Proof:** In ΔCAP and ΔBDP,

$\angle 1 = \angle 2$; $\angle 3 = \angle 4$ angles in the same segment

$\therefore\ \Delta CAP \sim\ \Delta BDP$ [AA corollary]

$\therefore\ \dfrac{CA}{BD} = \dfrac{AP}{DP} = \dfrac{CP}{BP}$ [Corresponding sides are proportional]

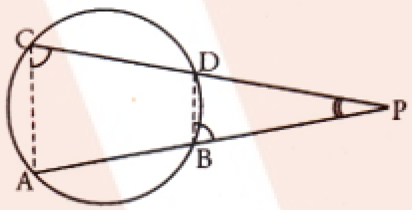
$\Rightarrow \frac{AP}{DP} = \frac{CP}{BP}$

$\Rightarrow AP \times BP = CP \times DP$

Hence, $PA \times PB = PC \times PD$. Proved.

Case II: When chords intersect externally

- **Given:** Chords AB and CD when produced, intersect each other at a point P outside the circle.
- **To Prove:** $PA \times PB = PC \times PD$
- **Construction:** Jon AC and BD.



- **Proof:** In ΔAPC and ΔBPD,

$\angle APC = \angle BPD$ [Common]

We know that an exterior angles of a cyclic quadrilateral is equal to its interior opposite angle. i.e., $\angle ACP = \angle DBP$

$\therefore \ \Delta APC \sim \Delta BPD$ [AA corollary]

Therefore, their corresponding sides must be proportional.

i.e., $\therefore \ \frac{PA}{DP} = \frac{PC}{PB} = \frac{AC}{DB}$

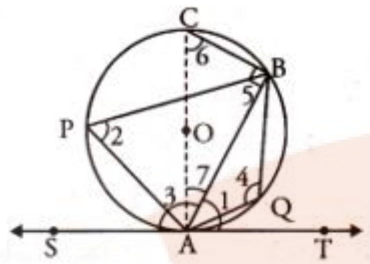
$\Rightarrow \frac{PA}{DP} = \frac{PC}{PB}$

$\Rightarrow PA \times PB = PC \times DP$

Hence, proved.

Theorem 22

- **Statement:** The angle between a tangent and a chord through the point of contact is equal to an angle in the alternate segment.
  - **Given:** A circle with centre O. SAT is a tangent to the circle at A. AB is a chord through A. P and Q are points on the opposite sides of the chord AB.
  - **To Prove:** (i) $\angle 1 = \angle 2$
    (ii) $\angle 3 = \angle 4$
  - **Construction:** Through A, draw the diameter AC of the circle. Join BC.



- **Proof:** Here, AC is the diameter of the circle and $\angle 5$ is an angle in a semicircle.
- $\therefore \angle 5 = 90°$ [Angle in a semicircle $= 90°$]

In ΔABC,

$\angle 5 + \angle 6 + \angle 7 = 180°$ [Sum of the angles of a triangles is $180°$]

$\Rightarrow 90° + \angle 6 + \angle 7 = 180°$

$\Rightarrow \angle 6 + \angle 7 = 180° - 90°$

$\Rightarrow \angle 6 + \angle 7 = 90°$ ……….. (i)

Since SAT is a tangent to the circle at A and OA is the radius through the point of contact,

$\therefore OA \perp SAT$

$\Rightarrow \angle 1 + \angle 7 = 90°$ ………. (ii)

From equations (i) and (ii), we get

$$\angle 6 + \angle 7 = \angle 1 + \angle 7$$

$\Rightarrow \angle 6 = \angle 1$

But $\angle 2 = \angle 6$ [Angles in the same segment]

$\therefore \angle 1 = \angle 2$

Since AQBP is a cyclic quadrilateral,

$\therefore \angle 4 + \angle 2 = 180°$ ……. (iii)

[Sum of the opposite angle of a cyclic quadrilateral is $180°$]

$\Rightarrow \angle 1 + \angle 3 = 180°$ …….. (iv) [Axiom of linear pair]

From equations (iii) and (iv),

$$\angle 4 + \angle 2 = \angle 1 + \angle 3$$

$\Rightarrow \angle 4 + \angle 1 = \angle 1 + \angle 3$ $[\because \angle 1 = \angle 2]$

$\Rightarrow \angle 4 = \angle 3$

Hence,

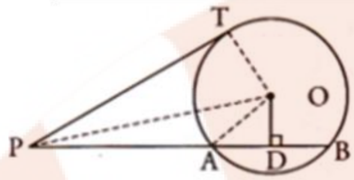
(i) $\angle 1 = \angle 2$

(ii) $\angle 3 = \angle 4$. Proved.

Theorem 23

- **Statement:** If a chord and a tangent intersect externally, the product of the lengths of the segments of the chord is equal to the square of the length of the tangent from the point of contact to the point of intersection.

- **Given:** A circle with centre O. PAB is a secant intersecting the circle at A and B. PT is a tangent to the circle.
- **To Prove:** $PA \times PB = PT^2$.
- **Construction:** Draw $OD \perp AB$. Join OP, OT and OA.



- **Proof:** Since $OD \perp \text{chord } AB$

$\therefore\ AD = DB$ [Perpendicular from the centre to the chord bisects the chord]

$\therefore\ PA \times PB = (PD - AD)(PD + DB)$

$= (PD - AD)(PD + AD)$ $[\because\ AD = BD]$

$= PD^2 - AD^2$

$= (OP^2 - OD^2) - AD^2$ $[\because\ OP^2 = PD^2 + OD^2]$

$= OP^2 - (OD^2 + AD^2)$

$= OP^2 - OA^2$

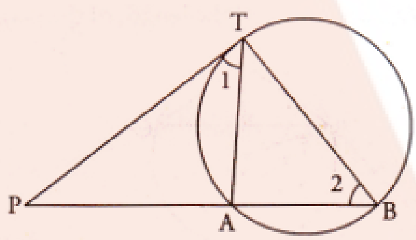
$= OP^2 - OT^2$ $[\because\ OA = OT = radii]$

$= PT^2$ $[\because\ OP^2 = OT^2 + PT^2]$

Hence, $PA \times PB = PT^2$. Proved.

OR

- **To Prove:** $PT^2 = PA \times PB$



- **Proof:** Consider ΔPAT and ΔPTB

$\angle P = \angle P$ [Common]

$\angle 1 = \angle 2$ [Angle in a alternate segment]

$\Delta PTA \sim \Delta PBT$ [By A.A. axiom]

$$\frac{PT}{PB} = \frac{PA}{PT}$$

$PT^2 = PA \times PB$

Hence proved.