MATRIX

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Definition:

A matrix is a rectangular array of elements. The horizontal arrangements are called rows and vertical arrangements are called columns.

Order of a Matrix:

If a matrix A has m number of rows and n number of columns, then the order of the matrix A is (Number of rows)*(Number of columns) that is,

- A=m*n

MATRIX

TYPES OF MATRIX:

- Row Matrix
- Column Matrix
- Square Matrix
- Diagonal Matrix
- Scalar Matrix

- Unit (or) Identity Matrix
- Zero (or) unit Matrix
- Transpose of a Matrix
- Triangular matrix

1. Lower Triangular Matrix
2. Upper Triangular Matrix

1. Row Matrix:

A matrix is said to be a row matrix if it has only one row and any number of columns. A row matrix is also called as a row vector.

For example, $A = (8 \quad 9 \quad 4 \quad 3)$, $B = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 1 & \sqrt{3} \end{pmatrix}$ are row matrices of order 1×4 and 1×3 respectively.

In general $A = (a_{11} \quad a_{12} \quad a_{13} \quad \ldots \quad a_{1n})$ is a row matrix of order $1 \times n$.

2. Column Matrix:

A matrix is said to be a column matrix if it has only one column and any number of rows. It is also called as a column vector.

For example, $A = \begin{pmatrix} \sin x \\ \cos x \\ 1 \end{pmatrix}$, $B = \begin{pmatrix} \sqrt{5} \\ 7 \end{pmatrix}$ and $C = \begin{pmatrix} 8 \\ -3 \\ 23 \\ 17 \end{pmatrix}$ are column matrices of order 3×1, 2×1 and 4×1 respectively.

In general, $A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix}$ is a column matrix of order $m \times 1$.

3. Square Matrix:

A matrix in which the number of rows is equal to the number of columns is called a square matrix.

For example, $\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}_{2\times2}$, $\begin{pmatrix} -1 & 0 & 2 \\ 3 & 6 & 8 \\ 2 & 3 & 5 \end{pmatrix}_{3\times3}$ are square matrices.

In general, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times2}$, $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ are square matrices of orders 2×2 and 3×3 respectively.

$A = (a_{ij})_{m\times m}$ is a square matrix of order m.

4.Diagonal Matrix:

A square matrix, all of whose elements, except those in the leading diagonal are zero is called a diagonal matrix.

(ie) A square matrix $A = (a_{ij})$ is said to be diagonal matrix if $a_{ij} = 0$ for $i \neq j$. Note that some elements of the leading diagonal may be zero but not all.

For example, $\begin{pmatrix} 8 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 11 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are diagonal matrices.

5.Scalar Matrix:

A diagonal matrix in which all the leading diagonal elements are equal is called a scalar matrix.

For example, $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

In general, $A = (a_{ij})_{m \times m}$ is said to be a scalar matrix if

$$a_{ij} = \begin{cases} 0 & when \quad i \neq j \\ k & when \quad i = j \end{cases} \quad \text{where } k \text{ is constant.}$$

6.Identity (or) Unit Matrix:

A square matrix in which elements in the leading diagonal are all “1” and rest are all zero is called an identity matrix or unit matrix.

Thus, the square matrix $A = (a_{ij})$ is an identity matrix if $a_{ij} = \begin{cases} 1 & if \ i = j \\ 0 & if \ i \neq j \end{cases}$

A unit matrix of order n is written as I_n.

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are identity matrices of order 2 and 3 respectively.

7. Zero (or) Null Matrix:

A matrix is said to be a zero matrix or null matrix if all its elements are zero

For example, (0), $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are all zero matrices of order 1×1, 2×2 and 3×3 but of different orders. We denote zero matrix of order $n\times n$ by O_n.

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a zero matrix of the order 2×3.

8. Transpose of a Matrix:

The matrix which is obtained by interchanging the elements in rows and columns of the given matrix A is called transpose of A and is denoted by A^T .

For example,

(a) If $A = \begin{pmatrix} 5 & 3 & -1 \\ 2 & 8 & 9 \\ -4 & 7 & 5 \end{pmatrix}_{3\times 3}$ then $A^T = \begin{pmatrix} 5 & 2 & -4 \\ 3 & 8 & 7 \\ -1 & 9 & 5 \end{pmatrix}_{3\times 3}$

(b) If $B = \begin{pmatrix} 1 & 5 \\ 8 & 9 \\ 4 & 3 \end{pmatrix}_{3\times 2}$ then $B^T = \begin{pmatrix} 1 & 8 & 4 \\ 5 & 9 & 3 \end{pmatrix}_{2\times 3}$

If order of A is $m\times n$ then order of A^T is $n\times m$.

We note that $(A^T)^T = A$.

9. Triangular Matrix:

I. Lower Triangular Matrix.
II. Upper Triangular Matrix.

Lower Triangular Matrix:

A square matrix in which all the entries above the leading diagonal are zero is called a lower triangular matrix.

Upper Triangular Matrix:

A square matrix in which all the entries below the leading diagonal are zero, then it is called an upper triangular matrix.

For example, $A = \begin{pmatrix} 1 & 7 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{pmatrix}$ is an upper triangular matrix and $B = \begin{pmatrix} 8 & 0 & 0 \\ 4 & 5 & 0 \\ -11 & 3 & 1 \end{pmatrix}$ is a lower triangular matrix.

Operations on Matrix:

In this section, we shall discuss the addition and subtraction of matrices, multiplication of a matrix by a scalar and multiplication of matrices.

Addition and subtraction of matrices

Two matrices can be added or subtracted if they have the same order. To add or subtract two matrices, simply add or subtract the corresponding elements.

Matrix Addition & Subtraction is only possible in same rows and columns,

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix}$, find $A+B$.

$$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+7 & 3+0 \\ 4+1 & 5+3 & 6+1 \\ 7+2 & 8+4 & 9+0 \end{pmatrix} = \begin{pmatrix} 2 & 9 & 3 \\ 5 & 8 & 7 \\ 9 & 12 & 9 \end{pmatrix}$$

Matrix Addition & Subtraction is not possible in different rows and columns,

If $A = \begin{pmatrix} 1 & 3 & -2 \\ 5 & -4 & 6 \\ -3 & 2 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 8 \\ 3 & 4 \\ 9 & 6 \end{pmatrix}$, find $A+B$.

It is not possible to add A and B because they have different orders.

Multiplication of Matrix by a Scalar:

We can multiply the elements of the given matrix A by a non-zero number k to obtain a new matrix kA whose elements are multiplied by k. The matrix kA is called scalar multiplication of A.

If $A = \begin{pmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{pmatrix}$ then Find $2A+B$.

Since A and B have same order 3×3, $2A+B$ is defined.

$$\text{We have } 2A+B = 2\begin{pmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{pmatrix} + \begin{pmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 16 & 12 \\ 2 & 6 & 18 \\ -8 & 6 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 27 & 9 \\ 1 & 8 & 22 \\ -1 & 11 & -2 \end{pmatrix}$$

Properties of Matrix Addition and Scalar Multiplication:

Let A, B, C be m n´ matrices and p and q be two non-zero scalars (numbers). Then we have the following properties.

(i) $A + B= B + A$ [Commutative property of matrix addition]

(ii) $A + (B + C) = (A + B) + C$ [Associative property of matrix addition]

(iii) $(pq) A = p (qA)$ [Associative property of scalar multiplication]

(iv) $IA=A$ [Scalar Identity property where I is the unit matrix]

(v) $P (A + B) = pA + pB$ [Distributive property of scalar and two matrices]

(vi) $(P + q) A = pA + qA$ [Distributive property of two scalars with a matrix]

Additive Identity:

The null matrix or zero matrix is the identity for matrix addition.

Let A be any matrix.

Then, $A + 0 = 0 + A = A$. where 0 is the null matrix or zero matrix of same order as that of A.

Additive Inverse:

If A be any given matrix then –A is the additive inverse of A.

In fact we have $A + (-A) = (-A) + A = 0$.

Multiplication of Matrices:

To multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix. Consider the multiplications of 3×3 and 3×2 matrices.

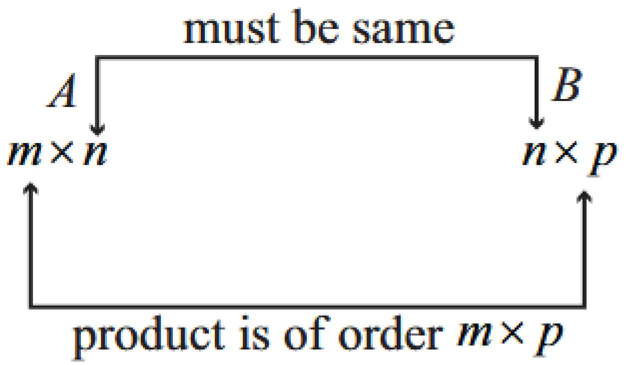
(Order of left hand matrix) * (order of right hand matrix) = (order of product matrix).

(3 × 3) * (3 × 2) = (3 × 2)

Matrices are multiplied by multiplying the elements in a row of the first matrix by the elements in a column of the second matrix, and adding the results.

For example, product of matrices $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \times \begin{pmatrix} g & h & i \\ k & l & m \end{pmatrix} = \begin{pmatrix} ag+bk & ah+bl & ai+bm \\ cg+dk & ch+dl & ci+dm \\ eg+fk & eh+fl & ei+fm \end{pmatrix}$

The product AB can be found if the number of columns of matrix A is equal to the numb matrix B. If the order of matrix A is m*n and B is n*p then the order of AB is m*p.



The order of AB is $m \times p = (\text{number of rows of } A) \times (\text{number of columns of } B)$.

Properties of Multiplication of Matrix:

(a) Matrix multiplication is not commutative in general:

If A is of order m*n and B of the order n*p then AB is defined but BA is not defined. Even if AB and BA are both defined, it is not necessary that they are equal.
In general AB≠BA .

Example for Matrix multiplication is not commutative in general

If $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ find AB and BA. Verify $AB = BA$?

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4+1 & 0+3 \\ 2+3 & 0+9 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 5 & 9 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4+0 & 2+0 \\ 2+3 & 1+9 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 10 \end{pmatrix}$$

Therefore, $AB \neq BA$.

Example for Matrix multiplication is not commutative in general

Special cases:

If $A = \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$

$$\text{LHS} = AB = \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+4 & 4\sqrt{2}-4\sqrt{2} \\ 2\sqrt{2}-2\sqrt{2} & 4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

$$\text{RHS} = BA = \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+4 & -4\sqrt{2}+4\sqrt{2} \\ -2\sqrt{2}+2\sqrt{2} & 4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

Hence LHS = RHS (ie) $AB = BA$

(b) Matrix multiplication is distributive over matrix addition:

(i) If A, B, C are m*n , n*p and n*p matrices respectively then

A (B + C) = AB + AC (Right Distributive Property)

(ii) If A, B, C are m*n , m*n and n*p matrices respectively then

(A + B) C = AC + BC (Left Distributive Property)

(c) Multiplication of a matrix by a unit matrix:

If A is a square matrix of order n*n and I is the unit matrix of same order then AI = IA = A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A \cdot I_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$I_2 \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

(D) Matrix multiplication is always associative:

If A, B, C are m*n , n*p and p*q matrices respectively then (AB) C = A (BC).

Examples for Matrix multiplication for Associative:

show that $(AB)C = A(BC)$.

If $A = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

LHS $= (AB)C$

$$AB = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}_{1\times 3} \times \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 1-2+2 & -1-1+6 \end{pmatrix} = \begin{pmatrix} 1 & 4 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 1 & 4 \end{pmatrix}_{1\times 2} \times \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 1+8 & 2-4 \end{pmatrix} = \begin{pmatrix} 9 & -2 \end{pmatrix} \quad \ldots(1)$$

RHS $= A(BC)$

$$BC = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}_{3\times 2} \times \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 1-2 & 2+1 \\ 2+2 & 4-1 \\ 1+6 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}_{1\times 3} \times \begin{pmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{pmatrix}_{3\times 2}$$

$$A(BC) = \begin{pmatrix} -1-4+14 & 3-3-2 \end{pmatrix} = \begin{pmatrix} 9 & -2 \end{pmatrix} \quad \ldots(2)$$

From (1) and (2), $(AB)C = A(BC)$.

Symmetric and Skew-symmetric Matrices:

Symmetric Matrices:

A square matrix A is said to be symmetric if $A^T = A$.

$$A = \begin{bmatrix} 3 & -6 & 9 \\ -6 & 8 & 5 \\ 9 & 5 & 2 \end{bmatrix} \text{ is a symmetric matrix since } A^T = A.$$

Skew-symmetric Matrices:

A square matrix A is said to be skew-symmetric if $A^T = -A$.

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix since } A^T = -A.$$

Inverse of a Matrix:

The inverse of a matrix A, denoted as A^{-1}, is a matrix that, when multiplied by A, results in the identity matrix (I). Mathematically, this is represented as $AA^{-1} = A^{-1}A = I$. For a matrix to have an inverse, it must be a non-singular square matrix, meaning its determinant is not zero

Not all matrices have an inverse. A matrix must be **square** (same number of rows and columns) and must be non-singular (its determinant is not zero) to have an inverse.

Inverse of a Matrix

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

Determinants:

To every square matrix A = [aij] of order n, we can associate a number called determinant of the matrix A

i) Determinants can be defined only for square matrices.

ii) For a square matrix A, |A| is read as determinant of A.

iii) Matrix is only a representation whereas determinant is a value of a matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Determinant of a matrix of order 2:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{21}\ a_{12}$.

Determinant of a Matrix of order 3:

consider the 3×3 matrix defined by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then the minors and cofactors of the elements a_{11}, a_{12}, a_{13} are given as

(i) Minor of a_{11} is M_{11} $= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Cofactor of a_{11} is A_{11} $= (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

(ii) Minor of a_{12} is M_{12} $= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

Cofactor a_{12} is A_{12} $= (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$

(iii) Minor of a_{13} is M_{13} $= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Cofactor of a_{13} is A_{13} $= (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$.

$|A| = a_{22}a_{33} - a_{32}a_{23} - (a_{21}a_{33} - a_{31}a_{23}) + a_{21}a_{32} - a_{31}a_{22}$

Singular and non-singular Matrices:

A square matrix A is said to be singular if | A | = 0. A square matrix A is said to be non-singular if | A | ≠ 0.

For instance, the matrix $A = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix}$ is a singular matrix, since

$$|A| = 3(1-1) - 8(-4+4) + 1(-4+4) = 0.$$

If $B = \begin{bmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{bmatrix}$ then $|B| = 2(0-20) - (-3)(-42-4) + 5(30-0) = -28 \neq 0.$

Thus B is a non-singular matrix.

Adjoint of a Matrix:

The adjoint of a matrix A, also called the adjugate matrix, is defined as the transpose of the cofactor matrix of A. It's denoted as adj(a). The cofactor matrix is formed by taking the determinant of the minor matrices of A, with alternating signs, and the transpose is then taken of this matrix. The adjoint is crucial for finding the inverse of a matrix

Consider a 3 x 3 matrix as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{22} & a_{33} \end{bmatrix}$$

The adjugate of this matrix is given by:

$$adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{22} & A_{33} \end{bmatrix}^T$$

The above formula can be expanded as:

$$\text{adj A} = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$