Geometry through Linear Algebra

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Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.**

1 Planes and Lines

1.1 Distance from a plane to a point

1.1.1. Solve the following

a) Find the foot of perpendicular from the point $\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ on the plane $\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\mathbf{x} = 2$.

Solution: Consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.1.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \tag{1.1.1.2}$$

$$\implies 3a + 2b - 6c = 0 \tag{1.1.1.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.1.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \tag{1.1.1.5}$$

Solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.1.6}$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \tag{1.1.1.7}$$

Solving (1.1.1.7) using Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U\Sigma V}^T \tag{1.1.1.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. We

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have,

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \quad (1.1.1.9)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \quad (1.1.1.10)$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U\Sigma V}^T\mathbf{x} = \mathbf{b} \quad (1.1.1.11)$$

$$\implies \mathbf{x} = \mathbf{V\Sigma^{-1}U^Tb} \quad (1.1.1.12)$$

Where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$ and is obtained by inversing only non-zero elements in $\mathbf{\Sigma}$

Calculating eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.1.13)$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda \end{vmatrix} = 0 \quad (1.1.1.14)$$

$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \quad (1.1.1.15)$$

From the characteristic equation (1.1.1.15), the eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.1.16)$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{3} \\ 1 \end{pmatrix} \quad (1.1.1.17)$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-7}{12} \\ \frac{-7}{18} \\ \frac{7}{6} \end{pmatrix} \quad (1.1.1.18)$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \quad (1.1.1.19)$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.1.20)$$

Calculating eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (1.1.1.21)$$

$$\implies \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9}-\lambda \end{vmatrix} = 0 \quad (1.1.1.22)$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \quad (1.1.1.23)$$

From the characteristic equation, the eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad (1.1.1.24)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \quad (1.1.1.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.26)$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.27)$$

From (1.1.1.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \quad (1.1.1.28)$$

And, the Moore-Penrose Pseudo inverse of $\mathbf{\Sigma}$ is given by,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.1.29)$$

From (1.1.1.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \quad (1.1.1.30)$$

$$\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.31)$$

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.32)$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.1.33)$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.34)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.35)$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xleftrightarrow{R_1=\frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \quad (1.1.1.36)$$

$$\xleftrightarrow{R_2=R_2-\frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{49}{45} & \frac{8}{3} \end{pmatrix} \quad (1.1.1.37)$$

$$\xleftrightarrow{R_2=\frac{45}{49}R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.38)$$

$$\xleftrightarrow{R_1=R_1-\frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.39)$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.40)$$

From the equations (1.1.1.32) and (1.1.1.40), the solution $\mathbf{x}$ is verified.

b) Find the foot of perpendicular from point $B = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ to the plane $\begin{pmatrix} 2 & 3 & -4 \end{pmatrix}\mathbf{x} = -5$.

Solution: Let us consider orthogonal vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ to the given normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then,

$$\mathbf{m}^T\mathbf{n} = 0 \quad (1.1.1.41)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \quad (1.1.1.42)$$

$$\implies 2a + 3b - 4c = 0 \quad (1.1.1.43)$$

Let $a = 1, \; b = 0$, so that

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.44)$$

and $a = 0, \; b = 1$, so that

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \quad (1.1.1.45)$$

We, now, solve the equation

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.1.46)$$

which, upon substitution, becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad (1.1.1.47)$$

Any $m \times n$ matrix $\mathbf{M}$ can be factorized in SVD form as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.1.48)$$

where $\mathbf{U}$ and $\mathbf{V}$ are matrices of eigen vectors which are orthogonal. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is the diagonal matrix of singular values of $\mathbf{M}$ of the eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \quad (1.1.1.49)$$

Putting (1.1.1.48) into (1.1.1.46), we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (1.1.1.50)$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (1.1.1.51)$$

where $\mathbf{S}_+$ is the Moore-Penrose Pseudoinverse of $\mathbf{S}$.

The eigenvalues of $\mathbf{M}^T\mathbf{M}$:

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (1.1.1.52)$$

$$\implies \begin{vmatrix} \frac{10}{8} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{vmatrix} = 0 \quad (1.1.1.53)$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{116}{64} = 0 \quad (1.1.1.54)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (1.1.1.55)$$

$$\lambda_2 = 1 \quad (1.1.1.56)$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector $\mathbf{v_1}$ can be calculated using row reduction as :

$$\begin{pmatrix} -\frac{9}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{16}{9}R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \quad (1.1.1.57)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad (1.1.1.58)$$

Hence,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.59)$$

Similarly, for $\lambda_2 = 1$,

$$\mathbf{v_2} = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.60)$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.61)$$

Now,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \quad (1.1.1.62)$$

Now, calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$

$$\begin{vmatrix} 1 - \lambda & 0 & \frac{1}{2} \\ 0 & 1 - \lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \lambda \end{vmatrix} = 0 \quad (1.1.1.63)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (1.1.1.64)$$

$$\lambda_2 = 1 \quad (1.1.1.65)$$

$$\lambda_3 = 0 \quad (1.1.1.66)$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector can be computed as:

$$\begin{pmatrix} 1 - \frac{29}{16} & 0 & \frac{1}{2} \\ 0 & 1 - \frac{29}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \frac{29}{16} \end{pmatrix} \quad (1.1.1.67)$$

$$\leftrightarrow \begin{pmatrix} -\frac{13}{16} & 0 & \frac{1}{2} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (1.1.1.68)$$

$$\xleftrightarrow{R_1 \leftarrow -\frac{16}{13}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (1.1.1.69)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \quad (1.1.1.70)$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{16}{13}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \quad (1.1.1.71)$$

$$\xleftrightarrow{R_2 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.1.1.72)$$

Hence, the eigen vector $\mathbf{u}_1$:

$$\mathbf{u_1} = \begin{pmatrix} \frac{8}{\sqrt{377}} \\ \frac{12}{\sqrt{377}} \\ \frac{13}{\sqrt{377}} \end{pmatrix} \quad (1.1.1.73)$$

For $\lambda_2 = 1$, the eigen vector is:

$$\begin{pmatrix} 1 - 1 & 0 & \frac{1}{2} \\ 0 & 1 - 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - 1 \end{pmatrix} \quad (1.1.1.74)$$

$$\leftrightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{3}{16} \end{pmatrix} \quad (1.1.1.75)$$

Hence, the eigen vector $\mathbf{u}_2$:

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.1.76)$$

Similarly, for $\lambda_3 = 0$, the eigen vector is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.1.77}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1 - \frac{3}{4}R_2} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.78}$$

Hence, the eigen vector $\mathbf{u}_3$:

$$\mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.1.79}$$

So, the orthonormal matrix $\mathbf{U}$ of eigen vectors is:

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.1.80}$$

The matrix of singular values of $\mathbf{M}$ is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.81}$$

The Moore-Penrose pseudoinverse of $\mathbf{S}$ is computed as

$$\mathbf{S}_+ = (\mathbf{S}\mathbf{S}^T)^{-1}\mathbf{S}^T \tag{1.1.1.82}$$

$$= \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.1.83}$$

To solve for $\mathbf{x}$ in (1.1.1.51), noting that $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \\ 0 \end{pmatrix} \tag{1.1.1.84}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{1.1.1.85}$$

Thus, the foot of perpendicular is:

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{1.1.1.86}$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.87}$$

The solution can be verified using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.1.88}$$

The LHS gives

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.89}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.90}$$

Now, finding $\mathbf{x}$ from

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.91}$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} & -3 \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{3}{10}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \tag{1.1.1.92}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & \frac{29}{20} & \frac{58}{20} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & 1 & 2 \end{pmatrix} \tag{1.1.1.93}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.1.1.94}$$

Hence, the solution is given by

$$\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.95}$$

Comparing the results in Eq.(1.1.1.87) and (1.1.1.95), it is concluded that the solution is verified.

1.1.2. Solve the following

a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.1}$$

to the plane,

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.2.2}$$

Solution: The equation of plane is given as,

$$\mathbf{n}^T\mathbf{x} = c \tag{1.1.2.3}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.4}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.2.5}$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \tag{1.1.2.6}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.7}$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.8}$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.9}$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.2.10}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.11}$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.12}$$

To solve (1.1.2.10) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.2.13}$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.13) to (1.1.2.10),

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.2.14}$$

$$\implies \mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} \tag{1.1.2.15}$$

where, $\mathbf{S}_+$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.16}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.17}$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \tag{1.1.2.18}$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \tag{1.1.2.19}$$

$$\therefore \lambda_1 = 14 \tag{1.1.2.20}$$

$$\lambda_2 = 1 \tag{1.1.2.21}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.2.22}$$

$$\mathbf{v_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.2.23}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{1.1.2.24}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.1.2.25}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \tag{1.1.2.26}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.2.27}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.28}$$

Eigenvalues corresponding to $\mathbf{MM}^T$ is,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.2.29)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\implies -\lambda(\lambda-1)(\lambda-14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v_5} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}}\begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}\begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}}\begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

$\therefore$ from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}}\begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix}\begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.51)$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.2.52)$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.53)$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \quad (1.1.2.54)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \quad (1.1.2.55)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{17}{5} \end{pmatrix} \quad (1.1.2.56)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \quad (1.1.2.57)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \quad (1.1.2.58)$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.59)$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.

b) Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.60)$$

to the plane,

$$(1.1.2.61)$$

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}\mathbf{x} = 0 \quad (1.1.2.62)$$

Solution: The equation of plane is give

$$\mathbf{n}^T\mathbf{x} = c \quad (1.1.2.63)$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.64)$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.65)$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \quad (1.1.2.66)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.67)$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.68)$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.69)$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.70)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.71)$$

To solve (1.1.2.70) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.72)$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.72) to (1.1.2.70),

$$\implies \mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (1.1.2.73)$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (1.1.2.74)$$

where, $\mathbf{S}_+$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{M}\mathbf{M}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}1 & 0 & -2\\0 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\-2 & 3\end{pmatrix} = \begin{pmatrix}5 & -6\\-6 & 10\end{pmatrix} \tag{1.1.2.75}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.76}$$

$$\implies \begin{pmatrix}5-\lambda & -6\\-6 & 10-\lambda\end{pmatrix} \tag{1.1.2.77}$$

$$\implies (\lambda-14)(\lambda-1) = 0 \tag{1.1.2.78}$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \tag{1.1.2.79}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix}\frac{-2}{3}\\1\end{pmatrix}, \mathbf{v_2} = \begin{pmatrix}\frac{3}{2}\\1\end{pmatrix} \tag{1.1.2.80}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}}\begin{pmatrix}-2\\3\end{pmatrix} \tag{1.1.2.81}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix} \tag{1.1.2.82}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}}\begin{pmatrix}-2 & 3\\3 & 2\end{pmatrix} \tag{1.1.2.83}$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix}1 & 0\\0 & 1\\-2 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & -2\\0 & 1 & 3\end{pmatrix} \tag{1.1.2.84}$$

$$\implies \begin{pmatrix}1 & 0 & -2\\0 & 1 & 3\\-2 & 3 & 13\end{pmatrix} \tag{1.1.2.85}$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.86}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & -2\\0 & 1-\lambda & 3\\-2 & 3 & 13-\lambda\end{pmatrix} \tag{1.1.2.87}$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \tag{1.1.2.88}$$

$$\implies -\lambda(\lambda-1)(\lambda-14) = 0 \tag{1.1.2.89}$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1 \tag{1.1.2.90}$$

$$\lambda_5 = 0 \tag{1.1.2.91}$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix}\frac{-2}{13}\\\frac{3}{13}\\1\end{pmatrix}, \mathbf{v_4} = \begin{pmatrix}\frac{3}{2}\\1\\0\end{pmatrix}, \mathbf{v_5} = \begin{pmatrix}2\\-3\\1\end{pmatrix} \tag{1.1.2.92}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}}\begin{pmatrix}-2\\3\\13\end{pmatrix} = \begin{pmatrix}-\sqrt{\frac{2}{91}}\\\frac{3}{\sqrt{182}}\\\sqrt{\frac{13}{14}}\end{pmatrix} \tag{1.1.2.93}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{13}}\\\frac{2}{\sqrt{13}}\\0\end{pmatrix} \tag{1.1.2.94}$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{2}{7}}\\-\frac{3}{\sqrt{14}}\\\sqrt{\frac{1}{14}}\end{pmatrix} \tag{1.1.2.95}$$

$$\implies \mathbf{U} = \begin{pmatrix}-\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}}\\\frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}}\\\sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}}\end{pmatrix} \tag{1.1.2.96}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix}\sqrt{14} & 0\\0 & 1\\0 & 0\end{pmatrix} \tag{1.1.2.97}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S_+} = \begin{pmatrix}\frac{1}{\sqrt{14}} & 0 & 0\\0 & 1 & 0\end{pmatrix} \tag{1.1.2.98}$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}}\begin{pmatrix}-\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}}\\\frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}}\\\sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}}\end{pmatrix}\begin{pmatrix}\sqrt{14} & 0\\0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}-2 & 3\\3 & 2\end{pmatrix}^T \tag{1.1.2.99}$$

Now substituting the values of (1.1.2.83), (1.1.2.98), (1.1.2.96) and (1.1.2.71) in

(1.1.2.74),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.100}$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \tag{1.1.2.101}$$

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.102}$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \tag{1.1.2.103}$$

$\therefore$ from equation (1.1.2.74),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \tag{1.1.2.104}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.2.105}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.106}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.107}$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{1.1.2.108}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix} \tag{1.1.2.109}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix} \tag{1.1.2.110}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.2.111}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.2.112}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.2.113}$$

Hence from equations (1.1.2.105) and (1.1.2.113) we conclude that the solution is verified.

c) Find the foot of the perpendicular from $\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$ on the plane $\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \mathbf{x} = 0$
Solution: Let orthogonal vectors be $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.2.114}$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.115}$$

$$\implies -5a + b + 3c = 0 \tag{1.1.2.116}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.117}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.118}$$

From (1.1.2.117) and (1.1.2.118),

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.119}$$

Now solving the equation

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.2.120}$$

Substituting the given point and (1.1.2.119) in (1.1.2.120)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.121}$$

Using the Singular value decomposition to solve (1.1.2.121) as follows,

$$\mathbf{M} = \mathbf{U\Sigma V}^T \tag{1.1.2.122}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{\Sigma}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.123}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.124}$$

Substituting (1.1.2.122) in (1.1.2.120)

$$\mathbf{U\Sigma V}^T\mathbf{x} = \mathbf{b} \tag{1.1.2.125}$$

$$\mathbf{x} = \mathbf{V\Sigma^{-1}U^Tb} \tag{1.1.2.126}$$

where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$.
Now finding the eigen values of $\mathbf{MM}^T$

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.127}$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0 \tag{1.1.2.128}$$

$$\implies \lambda^3 - 15\lambda^2 + 14\lambda = 0 \tag{1.1.2.129}$$

Hence eigen values of $\mathbf{MM}^T$,

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad \lambda_3 = 0 \tag{1.1.2.130}$$

Therefore eigen vectors of $\mathbf{MM}^T$,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.131}$$

Normalizing the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.132}$$

Hence from the above we get,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{-3}{\sqrt{14}} \\ 0 & \frac{13}{\sqrt{182}} & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.133}$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 14 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.134}$$

Now calculating eigen values of $\mathbf{M}^T\mathbf{M}$

$$\left|\mathbf{M}^T\mathbf{M} - \lambda I\right| = 0 \tag{1.1.2.135}$$

$$\begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0 \tag{1.1.2.136}$$

$$\implies \lambda^2 - 15\lambda + 14 = 0 \tag{1.1.2.137}$$

hence the eigen values of $\mathbf{M}^T\mathbf{M}$

$$\lambda_1 = 1 \quad \lambda_2 = 14 \tag{1.1.2.138}$$

Therefore eigen vectors $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.2.139}$$

Normalizing the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.140}$$

Hence $\mathbf{V}$ is given as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.141}$$

Moore Pseudo inverse of Σ is,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \tag{1.1.2.142}$$

Substituting (1.1.2.133), (1.1.2.141) and (1.1.2.142) in (1.1.2.126),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{182}} & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{1}} \end{pmatrix} \tag{1.1.2.143}$$

$$\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} \tag{1.1.2.144}$$

$$\mathbf{V}\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{1.1.2.145}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{1.1.2.146}$$

Now verifying (1.1.2.146) using (1.1.2.120)

$$\mathbf{Mx} = \mathbf{b} \implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.147}$$

Substituting (1.1.2.119), (1.1.2.123) and given point in (1.1.2.147)

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 \\ 10 \end{pmatrix} \tag{1.1.2.148}$$

$$\tag{1.1.2.149}$$

Solving the augmented matrix.

$$\begin{pmatrix} 5 & -6 & -11 \\ -6 & 10 & 10 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ -6 & 10 & 10 \end{pmatrix} \tag{1.1.2.150}$$

$$\xleftrightarrow{R_2=R_2+6R_1} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & \frac{14}{5} & \frac{-16}{5} \end{pmatrix} \tag{1.1.2.151}$$

$$\xleftrightarrow{R_2=\frac{5R_2}{14}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \tag{1.1.2.152}$$

$$\xleftrightarrow{R_1=R_1+\frac{6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{-25}{7} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \tag{1.1.2.153}$$

From (1.1.2.153) we get,

$$\mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{1.1.2.154}$$

Hence from (1.1.2.146) and (1.1.2.154) the $\mathbf{x}$ is verified

d) Find the coordinates of foot of perpendicular from $\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ to the plane

$$2x - 3y + z = 0 \tag{1.1.2.155}$$

using SVD **Solution:** First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given plane $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$\implies 2a - 3b + c = 0 \tag{1.1.2.156}$$

By substituting $a = 1; b = 0$ in (1.1.2.156),

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.157}$$

By substituting $a = 0; b = 1$ in (1.1.2.156),

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.158}$$

Now $\mathbf{M}$ can be written as,

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.159}$$

such that solving $\mathbf{Mx} = \mathbf{b}$ gives the required solution.

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \tag{1.1.2.160}$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.2.161}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.162}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.163}$$

From (1.1.2.160) and (1.1.2.161),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{1.1.2.164}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^3 + 15\lambda^2 - 14\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = 14; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{1.1.2.165}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.166}$$

Normalizing the above eigenvectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.167}$$

From (1.1.2.167) we obtain $\mathbf{U}$ as,

$$\mathbf{U} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.168}$$

Using values from (1.1.2.165),

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.169}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 - 15\lambda + 14 = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 14; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Normalizing the above eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.170}$$

From(1.1.2.170) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.171}$$

From (1.1.2.161) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.2.172}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.173}$$

From (1.1.2.164),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{2}}{\sqrt{91}} \\ \frac{3}{\sqrt{13}} \\ \frac{2\sqrt{2}}{7} \end{pmatrix}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.1.2.174}$$

To verify the solution obtained from (1.1.2.174),

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.175}$$

Substituting the values from (1.1.2.162) in (1.1.2.175),

$$\begin{pmatrix}5 & -6\\ -6 & 10\end{pmatrix}\mathbf{x} = \begin{pmatrix}-3\\ 6\end{pmatrix}$$

Converting the above equation into augmented form and solving for $\mathbf{x}$,

$$\begin{pmatrix}5 & -6 & -3\\ -6 & 10 & 6\end{pmatrix} \xleftrightarrow{R_2\leftarrow\frac{5R_2+6R_1}{14}} \begin{pmatrix}5 & -6 & -3\\ 0 & 1 & \frac{6}{7}\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{R_1+6R_2}{5}} \begin{pmatrix}1 & 0 & \frac{3}{7}\\ 0 & 1 & \frac{6}{7}\end{pmatrix} \tag{1.1.2.176}$$

From (1.1.2.176) it can be observed that,

$$\mathbf{x} = \begin{pmatrix}\frac{3}{7}\\ \frac{6}{7}\end{pmatrix} \tag{1.1.2.177}$$

1.1.3. Find the foot of the perpendicular using svd drawn from $\begin{pmatrix}-3\\ 1\\ 2\end{pmatrix}$ to the plane

$$\begin{pmatrix}2 & -1 & -2\end{pmatrix}\mathbf{x} + 4 = 0 \tag{1.1.3.1}$$

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\ b\\ c\end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.3.2}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}2\\ -1\\ -2\end{pmatrix} = 0 \tag{1.1.3.3}$$

$$\implies 2a - b - 2c = 0 \tag{1.1.3.4}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\ 0\\ 1\end{pmatrix} \tag{1.1.3.5}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\ 1\\ -\frac{1}{2}\end{pmatrix} \tag{1.1.3.6}$$

Let us solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.3.7}$$

Substituting (1.1.3.5) and (1.1.3.6) in (1.1.3.7),

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & -\frac{1}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}-3\\ 1\\ 2\end{pmatrix} \tag{1.1.3.8}$$

To solve (1.1.3.8), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.3.9}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}2 & \frac{-1}{2}\\ \frac{-1}{2} & \frac{5}{4}\end{pmatrix} \tag{1.1.3.10}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -\frac{1}{2}\\ 1 & -\frac{1}{2} & \frac{5}{4}\end{pmatrix} \tag{1.1.3.11}$$

Substituting (1.1.3.9) in (1.1.3.7),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.3.12}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.1.3.13}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.
Let us calculate eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.3.14}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & 1\\ 0 & 1-\lambda & -\frac{1}{2}\\ 1 & -\frac{1}{2} & \frac{5}{4}-\lambda\end{pmatrix} = 0 \tag{1.1.3.15}$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \tag{1.1.3.16}$$

From equation (1.1.3.16) eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{1.1.3.17}$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix}-\frac{4}{5}\\ \frac{2}{5}\\ -1\end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{2}\\ -1\\ 0\end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix}-1\\ \frac{1}{2}\\ 1\end{pmatrix} \tag{1.1.3.18}$$

Normalizing the eigen vectors in equation

(1.1.3.18)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.1.3.19}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \tag{1.1.3.20}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.3.21}$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.3.22}$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{1.1.3.23}$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{1.1.3.24}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \tag{1.1.3.25}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \tag{1.1.3.26}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \tag{1.1.3.27}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \tag{1.1.3.28}$$

From (1.1.3.7), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}^T \tag{1.1.3.29}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.3.30}$$

From (1.1.3.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{11}{3} \end{pmatrix} \tag{1.1.3.31}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{8}{9\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.1.3.32}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \tag{1.1.3.33}$$

Verifying the solution of (1.1.3.33) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.3.34}$$

Evaluating the R.H.S in (1.1.3.34) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.1.3.35}$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.1.3.36}$$

Solving the augmented matrix of (1.1.3.36) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \tag{1.1.3.37}$$

$$\xleftrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{9}{8} & -\frac{1}{4} \end{pmatrix} \tag{1.1.3.38}$$

$$\xleftrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \tag{1.1.3.39}$$

$$\xleftrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & -\frac{5}{9} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \tag{1.1.3.40}$$

From equation (1.1.3.40), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \tag{1.1.3.41}$$

Comparing results of $\mathbf{x}$ from (1.1.3.33) and (1.1.3.41), we can say that the solution is verified.