# Linear Algebra and Matrices

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CONTENTS



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***Abstract*—This book provides a simple introduction to linear algebra and matrix analysis. The content and exercises are based on NCERT textbooks from Class 6-12.**

## 1 Line

1.1. Any point $\mathbf{P}$ in the 2-D plane can be expressed in terms of its coordinates $(p_1, p_2)$ as the column vector

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.1}$$

1.2. The *direction vector* of the line joining $\mathbf{P}, \mathbf{Q}$ is defined as

$$\mathbf{m} = \mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{1.2.1}$$

$$= (p_1 - q_1) \begin{pmatrix} 1 \\ \frac{p_2 - q_2}{p_1 - q_1} \end{pmatrix} = (p_1 - q_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.2}$$

where

$$m = \frac{p_2 - q_2}{p_1 - q_1}. \tag{1.2.3}$$

*Without loss of generality, $k\mathbf{m}$, for any real scalar $k$ is also a direction vector. In the rest of the paper, $\mathbf{m}$ and $k\mathbf{m}$ are interchanged for* *computational simplicity*. Thus, if $m$ be the slope of the line $PQ$,

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.4}$$

1.3. Let $\mathbf{P}, \mathbf{Q}$ be two points on a line. The vector equation of the line is given by

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{m}, \quad \lambda \in \mathbb{R} \tag{1.3.1}$$
$$\mathbf{m} = \mathbf{P} - \mathbf{Q} \tag{1.3.2}$$

(1.3.1) can be used in 3D as well.

1.4. The *normal vector* $\mathbf{n}$ to a line is orthogonal to the direction vector $\mathbf{m}$ so that

$$\mathbf{m}^T \mathbf{n} = 0 \tag{1.4.1}$$

If $\mathbf{P}$ be a point on the line, the equation of the line can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{P}) = 0 \tag{1.4.2}$$
$$\text{or, } \mathbf{n}^T \mathbf{x} = c, \tag{1.4.3}$$

where

$$c = \mathbf{n}^T \mathbf{P} \tag{1.4.4}$$

which is the desired equation of the straight line. By subsuming the $c$ in (1.4.3) within $\mathbf{n}$, the equation of a line can also be expressed as

$$\mathbf{n}^T \mathbf{x} = 1 \tag{1.4.5}$$

Note that in 3D, (1.4.2) and (1.4.3) are used to represent the equation of a plane.

1.5. *Orthogonality:* Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.5.1}$$

are the vertices of a right angled triangle.
**Solution:** Let

$$\mathbf{v}_1 = \mathbf{A} - \mathbf{C} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \tag{1.5.2}$$

$$\mathbf{v}_2 = \mathbf{B} - \mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \tag{1.5.3}$$

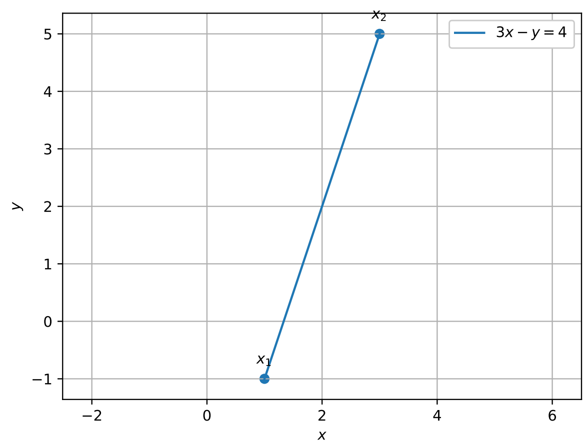


Fig. 1.6: Line obtained in Problem 1.6.

Then

$$\mathbf{v}_1^T \mathbf{v}_2 = \begin{pmatrix} -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 \tag{1.5.4}$$
$$\implies AC \perp BC \tag{1.5.5}$$

and $\mathbf{v}_1$ and $\mathbf{v}_2$ are said to be orthogonal.

1.6. Find the equation of the line through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ with slope - 4
**Solution:** From (1.2.4), the direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{1.6.1}$$

and from (1.4.1), the normal vector is

$$\mathbf{n} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.6.2}$$

Using (1.4.2), the equation of the line is

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} = 0 \tag{1.6.3}$$
$$\implies \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = -5 \tag{1.6.4}$$

Fig. 1.6 shows the line passing through the given point.

1.7. Write the equation of the line through the points $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

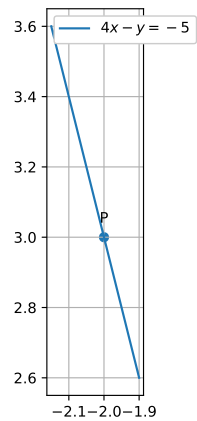


Fig. 1.7: Line obtained in Problem 1.7.

**Solution:** From (1.4.5),

$$\mathbf{n}^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \tag{1.7.1}$$

$$\mathbf{n}^T \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \tag{1.7.2}$$

resulting in the the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.7.3}$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \tag{1.7.4}$$

Performing row reduction,

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \tag{1.7.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 8 & -2 \end{pmatrix} \tag{1.7.6}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -1 \end{pmatrix} \tag{1.7.7}$$

$$\xleftrightarrow{R_1 \leftarrow 4R_1 + R_2} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 4 & -1 \end{pmatrix} \tag{1.7.8}$$

$$\xleftrightarrow[R_1 \leftarrow \frac{R_1}{4}]{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \tag{1.7.9}$$

From (1.7.9),

$$\mathbf{n} = \frac{1}{4} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{1.7.10}$$

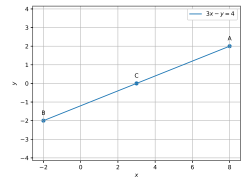

Fig. 1.8: Points on a line and points forming a triangle in Example 1.8.

Thus the equation of the desired line is

$$\frac{1}{4} \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{1.7.11}$$

$$\text{or, } \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{1.7.12}$$

Fig. 1.7 shows the line passing through the given points.

1.8. (*Linear Dependence*) Prove that the three points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \end{pmatrix}$ are collinear

**Solution:** Let

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \end{pmatrix} \end{aligned} \tag{1.8.1}$$

Then, the given points are collinear if

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = 0 \tag{1.8.2}$$

has a nontrivial solution as well, i.e.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \mathbf{0} \tag{1.8.3}$$

Substituting (1.8.1) in (1.8.2) results in the matrix equation

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \mathbf{x} = 0 \tag{1.8.4}$$

Performing row operations on the matrix,

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_1 - 5R_2} \begin{pmatrix} 5 & -10 \\ 0 & 0 \end{pmatrix} \tag{1.8.5}$$

which can be expressed as

$$\begin{pmatrix} 5 & -10 \end{pmatrix} \mathbf{x} = 0 \tag{1.8.6}$$

$$\text{or, } \mathbf{x} = x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.8.7}$$

Thus, there are infinite solutions. The vectors $\mathbf{v}_1, \mathbf{v}_2$ are are linearly dependent and the given

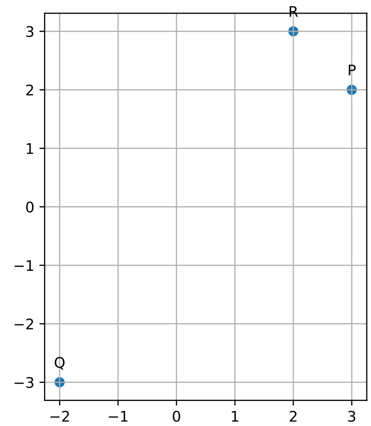


Fig. 1.10: Points on a triangle in Problem 1.10.

points lie on a straight line.

1.9. Alternatively, if the given points are collinear, from (1.4.5),

$$\begin{pmatrix} 3 & 0 \\ -2 & -2 \\ 8 & 2 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.9.1}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & -2 & 1 \\ 8 & 2 & 1 \end{pmatrix} \tag{1.9.2}$$

$$\xleftrightarrow[R_2 \leftarrow 3R_2 + 2R_1]{R_3 \leftarrow 3R_3 - 8R_1} \begin{pmatrix} 3 & 0 & 1 \\ 0 & -6 & 5 \\ 0 & 6 & -5 \end{pmatrix} \tag{1.9.3}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.9.4}$$

The above matrix has a zero row in echelon form, hence (1.9.1) is consistent and the given points are on a straight line. Also,

$$\mathbf{n} = \frac{1}{6} \begin{pmatrix} 2 \\ -5 \end{pmatrix} \tag{1.9.5}$$

1.10. (*Linear Independence*) Do the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle?

**Solution:** In this case

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{1.10.1}$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \tag{1.10.2}$$

Thus,

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = 0 \tag{1.10.3}$$

$$\implies \begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix} \mathbf{x} = 0 \tag{1.10.4}$$

Using row operations,

$$\begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 5 & -4 \\ 0 & 2 \end{pmatrix} \tag{1.10.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \tag{1.10.6}$$

resulting in a *full rank* matrix. Hence,

$$\mathbf{x} = 0 \tag{1.10.7}$$

and $\mathbf{v}_1$ and $\mathbf{v}_2$ are *linearly independent*. The points lie on a triangle.

1.11. Alternatively, from (1.4.5), row reducing the augmented matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{1.11.1}$$

The above matrix has a nonzero row in echelon form, hence the given points do not lie on a straight line. So they lie on a triangle.

1.12. Find the angle between the lines

$$\begin{aligned} \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} &= 5 \\ \begin{pmatrix} \sqrt{3} & -1 \end{pmatrix} \mathbf{x} &= -6. \end{aligned} \tag{1.12.1}$$

**Solution:** The angle between the lines can be expressed in terms of the normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \tag{1.12.2}$$

as

$$\cos\theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \tag{1.12.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30^\circ \tag{1.12.4}$$

1.13. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{1.13.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{1.13.2}$$

**Solution:** If the angle between the vectors be $\theta$, the projection is defined as

$$\mathbf{proj_b a} = (\|\mathbf{a}\| \cos\theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right)}{\|\mathbf{b}\|^2}\mathbf{b} \tag{1.13.3}$$

Substituting the values from (1.13.1) and (1.13.2),

$$\mathbf{proj_b a} = \frac{5}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1.13.4}$$

1.14. (*Reflection* ) Assuming that straight lines work as a plane mirror for a point, find the image of the point $\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the line

$$L: \quad \begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} = -4. \tag{1.14.1}$$

**Solution:** From the given equation, the line parameters are

$$\mathbf{n} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, c = -4, \mathbf{m} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{1.14.2}$$

Let $\mathbf{R}$ be the reflection of $\mathbf{P}$ such that $PR$ bisects the line $L$ at $\mathbf{Q}$. Then $\mathbf{Q}$ bisects $PR$. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{1.14.3}$$

$$\mathbf{n}^T\mathbf{Q} = c \quad \because \mathbf{Q} \text{ lies on the given line} \tag{1.14.4}$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \quad \because \mathbf{m} \perp \mathbf{P} - \mathbf{R} \tag{1.14.5}$$

From (1.14.3) and (1.14.4),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{1.14.6}$$

From (1.14.6) and (1.14.5),

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{R} = \begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix}^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{1.14.7}$$

Letting

$$\mathbf{V} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \tag{1.14.8}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{1.14.9}$$

Noting that

$$\begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.14.10}$$

(1.14.7) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{1.14.11}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{V}^{-1}\right]^T \mathbf{P} + \mathbf{V}\begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{1.14.12}$$

$$= \mathbf{V}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{1.14.13}$$

upon substituting from (1.14.8) in (1.14.13). It can be verified that the reflection is also given by

$$\mathbf{R} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{P} + 2c\mathbf{n} \tag{1.14.14}$$

$$= \begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix}\begin{pmatrix} \mathbf{m}^T \\ \mathbf{n}^T \end{pmatrix}\mathbf{P} + 2c\mathbf{n} \tag{1.14.15}$$

$$\implies \mathbf{R} = \left(\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T\right)\mathbf{P} + 2c\mathbf{n} \tag{1.14.16}$$

If $\mathbf{m}, \mathbf{n}$ are not orthonormal, (1.14.16) can be expressed as

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{1.14.17}$$

1.15. (*Gram-schmidt orthogonalization* ) Let

$$\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \tag{1.15.1}$$

$$\beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \tag{1.15.2}$$

Find $\beta_1, \beta_2$ such that

$$\beta = \beta_1 + \beta_2, \quad \beta_1 \parallel \alpha, \beta_2 \perp \alpha \tag{1.15.3}$$

**Solution:** Let $\beta_1 = k\alpha$. Then, $\beta_1 \parallel \alpha$ and

$$\beta = k\alpha + \beta_2 \tag{1.15.4}$$

$$\implies \alpha^T\beta = k\left\|\alpha\right\|^2 + k\beta_1^T\beta_2 \tag{1.15.5}$$

$$\text{or, } k = \frac{\alpha^T\beta}{\left\|\alpha\right\|^2}, \quad \because \beta_1 \perp \beta_2 \tag{1.15.6}$$

Thus,

$$\beta_1 = \frac{\alpha^T\beta}{\left\|\alpha\right\|^2}\alpha = \frac{1}{2}\begin{pmatrix}3\\-1\\0\end{pmatrix} \tag{1.15.7}$$

$$\beta_2 = \beta - \beta_1 = \begin{pmatrix}2\\1\\-3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}3\\-1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\3\\-6\end{pmatrix} \tag{1.15.8}$$

Thus, any given set of vectors can be expressed as a linear combination of another set of orthogonal vectors.

## 2 Plane

2.1. Find the equation of a plane passing through the points $\mathbf{a} = \begin{pmatrix}2\\5\\-3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-2\\-3\\5\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}5\\3\\-3\end{pmatrix}$

**Solution:** The equation of plane is also given by (1.4.5) in 3D. Following the approach in the previous example results in the matrix equation,

$$\begin{pmatrix}2 & 5 & -3\\-2 & -3 & 5\\5 & 3 & -3\end{pmatrix}\mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{2.1.1}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}2 & 5 & -3 & 1\\-2 & -3 & 5 & 1\\5 & 3 & -3 & 1\end{pmatrix} \tag{2.1.2}$$

$$\xleftrightarrow[R_3 \leftarrow 2R_3 - 5R_1]{R_2 \leftarrow \frac{R_2+R_1}{2}} \begin{pmatrix}2 & 5 & -3 & 1\\0 & 1 & 1 & 1\\0 & -19 & 9 & -3\end{pmatrix} \tag{2.1.3}$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3+19R_2}{4}]{R_1 \leftarrow R_1 - 5R_2} \begin{pmatrix}2 & 0 & -8 & -4\\0 & 1 & 1 & 1\\0 & 0 & 7 & 4\end{pmatrix} \tag{2.1.4}$$

$$\xleftrightarrow[R_3 \leftarrow 7R2 - R_3]{R_1 \leftarrow \frac{7R_1+8R_3}{2}} \begin{pmatrix}7 & 0 & 0 & 2\\0 & 7 & 0 & 3\\0 & 0 & 7 & 4\end{pmatrix} \tag{2.1.5}$$

$$\implies \mathbf{n} = \frac{1}{7}\begin{pmatrix}2\\3\\4\end{pmatrix} \tag{2.1.6}$$

Thus, the equation of the plane passing through the given points is

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 7 \tag{2.1.7}$$

2.2. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{2.2.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7 \tag{2.2.2}$$

**Solution:** The angle between two planes is the same as the angle between their normal vectors. For

$$\mathbf{n}_1 = \begin{pmatrix}2\\1\\-2\end{pmatrix}\mathbf{n}_2 = \begin{pmatrix}3\\-6\\-2\end{pmatrix} \tag{2.2.3}$$

using (1.12.3),

$$\cos\theta = \frac{6 - 6 + 4}{\sqrt{9}\sqrt{49}} = \frac{4}{21} \tag{2.2.4}$$

## 3 Pseudo Inverse

3.1. To find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{3.1.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\\2\end{pmatrix} \tag{3.1.2}$$

3.2. If the two lines intersect,

$$\mathbf{x}_1 + \lambda_1\mathbf{m}_1 = \mathbf{x}_2 + \lambda_2\mathbf{m}_2 \tag{3.2.1}$$

$$\implies \begin{pmatrix}\mathbf{m}_1 & \mathbf{m}_2\end{pmatrix}\begin{pmatrix}\lambda_1 \\ -\lambda_2\end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \tag{3.2.2}$$

$$\text{or, } \mathbf{M}\begin{pmatrix}\lambda_1 \\ -\lambda_2\end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \tag{3.2.3}$$

where

$$\mathbf{x}_1 = \begin{pmatrix}1\\2\\1\end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix}2\\-1\\-1\end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}2\\1\\2\end{pmatrix}. \tag{3.2.4}$$

$$\mathbf{M} = \begin{pmatrix}\mathbf{m}_1 & \mathbf{m}_2\end{pmatrix} \tag{3.2.5}$$

(3.2.3) can be expressed as the matrix equation

$$\begin{pmatrix}1 & 2\\-1 & 1\\1 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{3.2.6}$$

From the augmented matrix in (3.2.3),

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{3.2.7}$$

$$\begin{pmatrix}1 & -2 & 1\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix}0 & 0 & 3\\-1 & -1 & -3\\1 & -2 & -2\end{pmatrix} \tag{3.2.8}$$

The above matrix has a *rank* = 3. Hence the lines do not intersect.

3.3. Let

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1\mathbf{m}_1 \tag{3.3.1}$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2\mathbf{m}_2 \tag{3.3.2}$$

be the closest points on $L_1$ and $L_2$ respectively. Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines $L_1$,$L_2$ and passing through $A$ and $B$. Thus,

$$\mathbf{m_1}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{3.3.3}$$

$$\mathbf{m_2}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{3.3.4}$$

$$\implies \mathbf{M}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{3.3.5}$$

From (3.3.2) and (3.2.5)

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \mathbf{M}\begin{pmatrix}\lambda_1 \\ -\lambda_2\end{pmatrix} \tag{3.3.6}$$

and using (3.3.5), in the above,

$$\mathbf{M}^T\mathbf{M}\begin{pmatrix}\lambda_1 \\ -\lambda_2\end{pmatrix} = \mathbf{M}^T(\mathbf{x}_2 - \mathbf{x}_1) \tag{3.3.7}$$

3.4. Substituting the values from (3.2.4) in (3.3.7) and forming the augmented matrix,

$$\begin{pmatrix}3 & 3 & 2\\3 & 9 & -5\end{pmatrix} \tag{3.4.1}$$

$$\begin{pmatrix}3 & 3 & 2\\3 & 9 & -5\end{pmatrix} \xleftrightarrow{R_2=R_2-R_1} \begin{pmatrix}3 & 3 & 2\\0 & 6 & -7\end{pmatrix} \tag{3.4.2}$$

$$\begin{pmatrix}3 & 3 & 2\\0 & 6 & -7\end{pmatrix} \xleftrightarrow{R_1=2R_1-R_2} \begin{pmatrix}6 & 0 & 11\\0 & 6 & -7\end{pmatrix} \tag{3.4.3}$$

$$\begin{pmatrix}6 & 0 & 11\\0 & 6 & -7\end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{6},R_2=\frac{R_2}{6}} \begin{pmatrix}1 & 0 & \frac{11}{6}\\0 & 1 & \frac{-7}{6}\end{pmatrix} \tag{3.4.4}$$

$$\lambda_1 = \frac{11}{6}, \lambda_2 = \frac{7}{6} \tag{3.4.5}$$

yielding

$$A = \frac{1}{6}\begin{pmatrix}17\\1\\17\end{pmatrix}, B = \frac{1}{6}\begin{pmatrix}26\\1\\8\end{pmatrix}. \tag{3.4.6}$$

3.5. The distance is then obtained as

$$\|\mathbf{B} - \mathbf{A}\| = \frac{3}{\sqrt{2}} \tag{3.5.1}$$

Fig. 3.5 shows the various points and distances between the lines.

## 4 Quadratic Forms

4.1. The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{4.1.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.1.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b\\b & c\end{pmatrix} \tag{4.1.3}$$

$$\mathbf{u} = \begin{pmatrix}d & e\end{pmatrix} \tag{4.1.4}$$

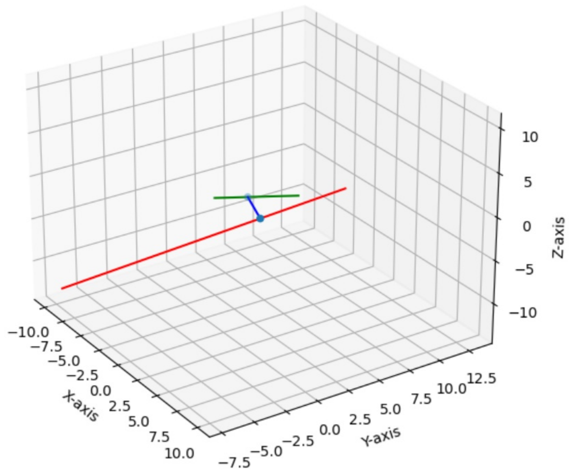


Fig. 3.5: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

4.2. *(Affine Transformation and Eigenvalue Decompostion)* Using

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{4.2.1}$$

such that

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{4.2.2}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{4.2.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^T = \mathbf{P}^{-1} \tag{4.2.4}$$

(4.1.2) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \tag{4.2.5}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \qquad |V| = 0 \tag{4.2.6}$$

with

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{4.2.7}$$

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad |V| = 0 \tag{4.2.8}$$

$$\text{where } \eta = \mathbf{n}^T\mathbf{p}_1 \tag{4.2.9}$$

**Solution:** Proofs for (4.2.5), (4.2.6), (4.2.7) and (4.2.8) are available in Appendix A.

4.3. *(Centre/Vertex)* The centre/vertex of the conic section in (4.1.2) is given by $\mathbf{c}$ in (4.2.7) or (4.2.8). This is because from (4.2.1),

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{4.3.1}$$

and

$$\mathbf{y} = \mathbf{0} \implies \mathbf{x} = \mathbf{c} \tag{4.3.2}$$

4.4. *(Circle)* For a circle,

$$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I} \tag{4.4.1}$$

and the centre is obtained from (4.2.7), (4.3.2) as

$$\mathbf{c} = -\mathbf{u} \tag{4.4.2}$$

(4.2.5) becomes

$$\mathbf{y}^T\mathbf{y} = \|\mathbf{y}\|^2 = \left(\sqrt{\mathbf{u}^T\mathbf{u} - f}\right)^2 \tag{4.4.3}$$

and the radius is

$$\sqrt{\mathbf{u}^T\mathbf{u} - f} \tag{4.4.4}$$

4.5. *(Ellipse)* For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{4.5.1}$$

and (4.2.5) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{4.5.2}$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}. \tag{4.5.3}$$

The centre is obtained from (4.3.2) as (4.2.7).

4.6. *(Hyperbola)* For

$$|\mathbf{V}| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \tag{4.6.1}$$

and (4.2.5) becomes

$$\lambda_1 y_1^2 - (-\lambda_2)\, y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{4.6.2}$$

with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}}, \tag{4.6.3}$$

The centre is obtained from (4.3.2) as (4.2.7).

4.7. (*Pair of straight lines:*) In (4.6.2), if

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0, \tag{4.7.1}$$

$$\begin{pmatrix} \sqrt{\lambda_1} & \pm\sqrt{\lambda_2} \end{pmatrix}\mathbf{y} = 0 \tag{4.7.2}$$

and the hyperbola reduces to a pair of straight lines given by

$$\left(\sqrt{\lambda_1} \quad \pm\sqrt{\lambda_2}\right)\mathbf{P}^T\left(\mathbf{x}-\mathbf{c}\right)=0 \tag{4.7.3}$$

and $\mathbf{c}$ is the point of intersection of the lines. Apart from (4.7.1), another condition for (4.1.2) to represent a pair of straight lines is

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix}=0 \tag{4.7.4}$$

4.8. *(Parabola)* For

$$\left|\mathbf{V}\right|=0, \quad \text{or, } \lambda_1=0. \tag{4.8.1}$$

The vertex of the parabola is obtained using (4.2.8) and the focal length is

$$\left|\frac{2\mathbf{p}_1^T\mathbf{u}}{\lambda_2}\right| \tag{4.8.2}$$

## 5 Tangents and Normals

5.1. *Secant:* The points of intersection of the line

$$L: \quad \mathbf{x}=\mathbf{q}+\mu\mathbf{m} \quad \mu\in\mathbb{R} \tag{5.1.1}$$

with the conic section in (4.1.2) are given by

$$\mathbf{x}_i=\mathbf{q}+\mu_i\mathbf{m} \tag{5.1.2}$$

where

$$\mu_i=\frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right) \\ \pm\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2-\left(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\Big) \tag{5.1.3}$$

**Solution:** Substituting (5.1.1) in (4.1.2),

$$\begin{aligned}\left(\mathbf{q}+\mu\mathbf{m}\right)^T\mathbf{V}\left(\mathbf{q}+\mu\mathbf{m}\right)+2\mathbf{u}^T\left(\mathbf{q}+\mu\mathbf{m}\right)+f=0\\ \implies \mu^2\mathbf{m}^T\mathbf{V}\mathbf{m}+2\mu\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\\ +\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f=0\end{aligned} \tag{5.1.4}$$

Solving the above quadratic in (5.1.4) yields (5.1.3).

5.2. *Tangent:* If $L$ in (5.1.1) touches (4.1.2) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)=0 \tag{5.2.1}$$

**Solution:** In this case, (5.1.4) has exactly one root. Hence, in (5.1.3)

$$\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2 \\ -\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\left(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f\right)=0 \tag{5.2.2}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (4.1.2) and

$$\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f=0 \tag{5.2.3}$$

Substituting (5.2.3) in (5.2.2) and simplifying, we obtain (5.2.1).

5.3. The normal vector is obtained from (5.2.1) and (1.4.1) as

$$\mathbf{n}=\mathbf{V}\mathbf{q}+\mathbf{u} \tag{5.3.1}$$

5.4. Given the point of contact $\mathbf{q}$, the equation of a tangent is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\mathbf{x}+\mathbf{u}^T\mathbf{q}+f=0 \tag{5.4.1}$$

**Solution:** From (5.3.1) and (1.4.2), the equation of the tangent is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\left(\mathbf{x}-\mathbf{q}\right)=0 \tag{5.4.2}$$

$$\implies \left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\mathbf{x}-\mathbf{q}^T\mathbf{V}\mathbf{q}-\mathbf{u}^T\mathbf{q}=0 \tag{5.4.3}$$

which, upon substituting from (5.2.3) and simplifying yields (5.1.1).

5.5. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (4.1.2) are given by

$$\mathbf{q}_i=\mathbf{V}^{-1}\left(\kappa_i\mathbf{n}-\mathbf{u}\right), i=1,2 \tag{5.5.1}$$

$$\text{where } \kappa_i=\pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{5.5.2}$$

**Solution:** From (5.3.1),

$$\mathbf{q}=\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right), \quad \kappa\in\mathbb{R} \tag{5.5.3}$$

Substituting (5.5.3) in (5.2.3),

$$\begin{aligned}&\left(\kappa\mathbf{n}-\mathbf{u}\right)^T\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right)\\ &\quad+2\mathbf{u}^T\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right)+f=0\\ &\implies \kappa^2\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}+f=0\\ &\text{or, } \kappa=\pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}}\end{aligned} \tag{5.5.4}$$

Substituting (5.5.4) in (5.5.3) yields (5.5.2).

5.6. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (4.1.2) is given by the matrix equation

$$\begin{pmatrix}\mathbf{u}+\kappa\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q}=\begin{pmatrix}-f\\ \kappa\mathbf{n}-\mathbf{u}\end{pmatrix} \quad (5.6.1)$$

$$\text{where } \kappa=\frac{\mathbf{p}_1^T\mathbf{u}}{\mathbf{p}_1^T\mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1=0 \quad (5.6.2)$$

**Solution:** If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1=0 \quad (5.6.3)$$

From (5.3.1),

$$\kappa\mathbf{n}=\mathbf{V}\mathbf{q}+\mathbf{u}, \quad \kappa\in\mathbb{R} \quad (5.6.4)$$

$$\implies \kappa\mathbf{p}_1^T\mathbf{n}=\mathbf{p}_1^T\mathbf{V}\mathbf{q}+\mathbf{p}_1^T\mathbf{u} \quad (5.6.5)$$

$$\text{or, } \kappa\mathbf{p}_1^T\mathbf{n}=\mathbf{p}_1^T\mathbf{u}, \quad \because \mathbf{p}_1^T\mathbf{V}=0, \quad (5.6.6)$$

$$\text{from (5.6.3)} \quad (5.6.7)$$

yielding $\kappa$ in (5.6.2). From (5.6.4),

$$\kappa\mathbf{q}^T\mathbf{n}=\mathbf{q}^T\mathbf{V}\mathbf{q}+\mathbf{q}^T\mathbf{u} \quad (5.6.8)$$

$$\implies \kappa\mathbf{q}^T\mathbf{n}=-f-\mathbf{q}^T\mathbf{u} \quad \text{from (5.2.3)}, \quad (5.6.9)$$

$$\text{or, } (\kappa\mathbf{n}+\mathbf{u})\,\mathbf{q}=-f \quad (5.6.10)$$

(5.6.4) can be expressed as

$$\mathbf{V}\mathbf{q}=\kappa\mathbf{n}-\mathbf{u}. \quad (5.6.11)$$

(5.6.10) and (5.6.11) clubbed together result in (5.6.1).

5.7. All the results related to conics are summarized in Table 5.7.

## 6 Circle

6.1. Find the centre and radius of the circle

$$x^2+y^2+8x+10y-8=0 \quad (6.1.1)$$

**Solution:** (6.1.1) can be expressed as

$$\mathbf{x}^T\mathbf{x}+2\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x}-8=0 \quad (6.1.2)$$

which is of the form (4.1.2) with

$$\mathbf{u}=\begin{pmatrix}4\\5\end{pmatrix}, f=-8 \quad (6.1.3)$$

From Table 5.7, the center and radius are given by

$$\mathbf{c}=-\mathbf{u}=\begin{pmatrix}-4\\-5\end{pmatrix}, r=\sqrt{\|u\|^2-f}=7 \quad (6.1.4)$$

6.2. Find the equation of a circle which passes through the points $\mathbf{P}=\begin{pmatrix}2\\-2\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}3\\4\end{pmatrix}$ and whose centre lies on the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}=2 \quad (6.2.1)$$

**Solution:** From (4.1.2) and Table 5.7, the equation of a circle can be expressed as

$$\mathbf{x}^T\mathbf{x}-2\mathbf{c}^T\mathbf{x}+f=0 \quad (6.2.2)$$

where $\mathbf{c}$ is the centre. Substituting the given points in (6.2.2) and using (6.2.1), the following equations are obtained

$$2\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{c}-f=8 \quad (6.2.3)$$

$$2\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{c}-f=25 \quad (6.2.4)$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{c}=2 \quad (6.2.5)$$

which can be expressed as the matrix equation

$$\begin{pmatrix}1 & 1 & 0\\4 & -4 & -1\\6 & 8 & -1\end{pmatrix}\begin{pmatrix}\mathbf{c}\\f\end{pmatrix}=\begin{pmatrix}2\\8\\25\end{pmatrix} \quad (6.2.6)$$

Row reducing the augmented matrix

$$\begin{pmatrix}1 & 1 & 0 & 2\\4 & -4 & -1 & 8\\6 & 8 & -1 & 25\end{pmatrix} \quad (6.2.7)$$

$$\xleftrightarrow[R_3\leftarrow R_3-6R_1]{R_2\leftarrow -R_2+4R_1}\begin{pmatrix}1 & 1 & 0 & 2\\0 & 8 & 1 & 0\\0 & 2 & -1 & 13\end{pmatrix} \quad (6.2.8)$$

$$\xleftrightarrow[R_3\leftarrow -\frac{4R_3-R_2}{2}]{R_1\leftarrow 8R_1-R_3}\begin{pmatrix}8 & 0 & -1 & 16\\0 & 8 & 1 & 0\\0 & 0 & 5 & -52\end{pmatrix} \quad (6.2.9)$$

$$\xleftrightarrow[R_2\leftarrow \frac{5R_2-R_3}{4}]{R_1\leftarrow \frac{5R_1+R_3}{4}}\begin{pmatrix}10 & 0 & 0 & 7\\0 & 10 & 0 & 13\\0 & 0 & 5 & -52\end{pmatrix} \quad (6.2.10)$$

Thus,

$$\mathbf{c}=\frac{1}{10}\begin{pmatrix}7\\13\end{pmatrix} \quad (6.2.11)$$

$$f=-\frac{52}{5} \quad (6.2.12)$$

which give the desired equation of the circle. From Table 5.7,

$$r=\sqrt{\|\mathbf{c}\|^2-f}=\frac{1}{10}\sqrt{1258} \quad (6.2.13)$$

Fig. 6.2 verifies the above results.

| Conic | Property | Standard Form | Standard Parameters | Point(s) of Contact |
|---|---|---|---|---|
| **Circle** | $\mathbf{V} = \mathbf{I}$ | $\frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f} = 1$ <br> $\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ <br> $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ <br> $\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}$ | $\mathbf{c} = -\mathbf{u}$, <br> $r = \sqrt{\mathbf{u}^T\mathbf{u} - f}$ | $\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right)$ <br> $\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}}$ |
| **Ellipse** | $\lvert\mathbf{V}\rvert > 0$ <br> $\lambda_1 > 0, \lambda_2 < 0$ | | $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$, <br> $axes = \begin{cases}\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}\\ \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_2}}\end{cases}$ | |
| **Hyperbola** | $\lvert\mathbf{V}\rvert < 0$ <br> $\lambda_1 > 0, \lambda_2 < 0$ | | $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$, <br> $axes = \begin{cases}\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}\\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}\end{cases}$ | |
| **Parabola** | $\lvert\mathbf{V}\rvert = 0$ <br> $\lambda_1 = 0$ | $\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y}$ | focal length $= \left\lvert\frac{\eta}{\lambda_2}\right\rvert$ <br> $\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p}_1^T\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p}_1 - \mathbf{u}\end{pmatrix}$ <br> $\eta = 2\mathbf{p}_1^T\mathbf{u}$ | $\begin{pmatrix}\mathbf{u} + \kappa\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f\\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix}$ <br> $\kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}}$ |

TABLE 5.7: $\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0$ can be expressed in the above standard form for various conics. $\mathbf{c}$ represents the centre/vertex of the conic. $\mathbf{q}$ is/are the point(s) of contact for the tangent(s).

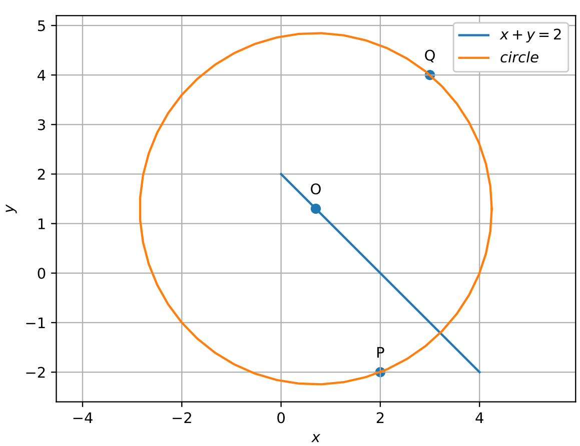


Fig. 6.2: Circle passing through $\begin{pmatrix}2\\-2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}$. Center is on line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.

6.3. Find the points on the curve

$$x^2 + y^2 - 2x - 3 = 0 \tag{6.3.1}$$

at which the tangents are parallel to the $x$-axis.

**Solution:** (6.3.1) can be expressed as

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x} - 3 = 0 \tag{6.3.2}$$

$$\implies \mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix}-1\\0\end{pmatrix}, f = -3 \tag{6.3.3}$$

From Table 5.7, the centre and radius are

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix}-1\\0\end{pmatrix}, r = \sqrt{\lVert\mathbf{u}\rVert^2 - f} = 2 \tag{6.3.4}$$

$\because$ the tangents are parallel to the $x$-axis, their direction and normal vectors are respectively,

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{n} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{6.3.5}$$

From Table 5.7,

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{u} - f}{\mathbf{n}^T\mathbf{n}}} = \pm\sqrt{\frac{4}{1}} = \pm 2 \tag{6.3.6}$$

and the desired points of contact are

$$\mathbf{q}_1, \mathbf{q}_2 = \begin{pmatrix}1\\0\end{pmatrix} \pm 2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix} \tag{6.3.7}$$

Fig. 6.2 verifies the above results.

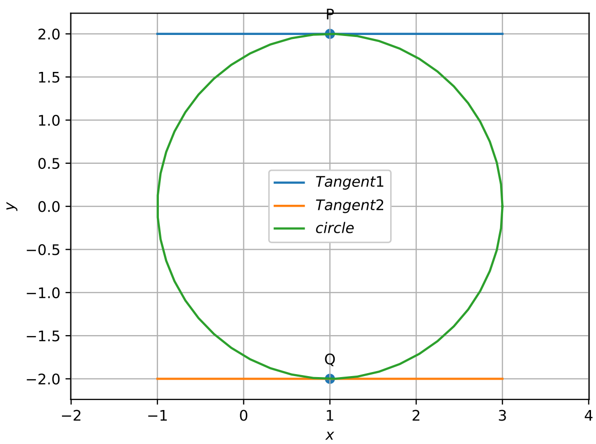


Fig. 6.3: Tangents are parallel to the $x$-axis.

## 7 Ellipse

7.1. Find $\frac{dy}{dx}$ if

$$E_1 : x^2 + xy + y^2 = 100 \tag{7.1.1}$$

**Solution:** Expressing (7.1.1) as (4.1.2),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \mathbf{u} = \mathbf{0}, f = -100. \tag{7.1.2}$$

$$\because |V| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} > 0, \tag{7.1.3}$$

(7.1.1) is the equation of an ellipse. To verify that this is indeed the case, we do the following exercise. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & \frac{1}{2} \\ \frac{1}{2} & \lambda - 1 \end{vmatrix} = 0 \tag{7.1.4}$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{7.1.5}$$

The eigenvalues are the roots of (7.1.5) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \tag{7.1.6}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{7.1.7}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{7.1.8}$$

where $\lambda$ is the eigenvalue. For $\lambda_1 = \frac{3}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{7.1.9}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7.1.10}$$

such that $\|\mathbf{p}_1\| = 1$. Similarly, the eigenvector corresponding to $\lambda_2$ can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{7.1.11}$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{7.1.12}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{7.1.13}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{7.1.14}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{7.1.15}$$

From Table 5.7, ellipse parameters are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \mathbf{0} \tag{7.1.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = 10\sqrt{\frac{2}{3}} \tag{7.1.17}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 10\sqrt{2} \tag{7.1.18}$$

In Fig. 7.1 the actual ellipse ellipse in (7.1.1) is obtained from (4.2.5) using (4.2.1). The anticlockwise 45°rotation is due to the fact that (7.1.14) can be expressed as

$$\mathbf{P} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \tag{7.1.19}$$

Coming back to the original question of finding $\frac{dy}{dx}$, if the point of contact

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{7.1.20}$$

from (7.1.2), (1.2.4) and (5.2.1),

$$\begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = 0 \quad (7.1.21)$$

$$\implies \begin{pmatrix}1+\frac{m}{2} & \frac{1}{2}+m\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = 0 \quad (7.1.22)$$

$$\implies \frac{m}{2}\left(q_1+2q_2\right)+q_1+\frac{q_2}{2} = 0 \quad (7.1.23)$$

$$\text{or, } m = \frac{dy}{dx} = -\frac{2q_1+q_2}{q_1+2q_2} \quad (7.1.24)$$

$\because \frac{dy}{dx}$ is the slope of the tangent. Note that no results from differential calculus were used to obtain (7.1.24).

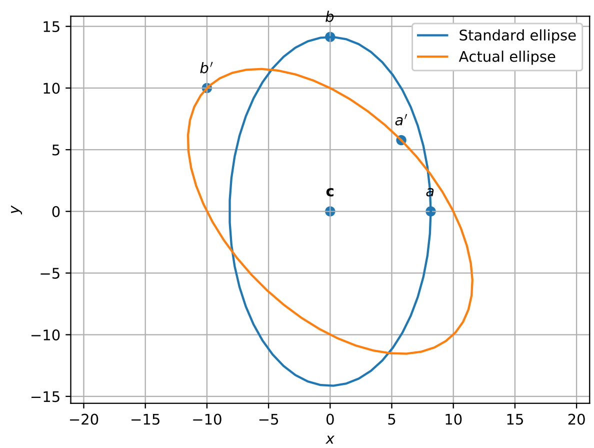


Fig. 7.1: Actual ellipse and transformed ellipse.

7.2. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\mathbf{a} = \begin{pmatrix}4\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}-1\\4\end{pmatrix}$

**Solution:** This is a standard ellipse given by

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \lambda_1, \lambda_2 > 0 \quad (7.2.1)$$

$\because \mathbf{a}, \mathbf{b}$ satisfy (7.2.1),

$$\mathbf{a}^T\mathbf{D}\mathbf{a} = 1, \quad (7.2.2)$$

$$\mathbf{b}^T\mathbf{D}\mathbf{b} = 1 \quad (7.2.3)$$

which can be expressed as

$$\begin{aligned}\mathbf{a}^T\mathbf{A}\mathbf{d} &= 1,\\ \mathbf{b}^T\mathbf{B}\mathbf{d} &= 1\end{aligned} \quad (7.2.4)$$

where

$$\mathbf{d} = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, \mathbf{A} = \begin{pmatrix}4 & 0\\ 0 & 3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1 & 0\\ 0 & 4\end{pmatrix}. \quad (7.2.5)$$

(7.2.4) can then be expressed as the matrix equation

$$\begin{pmatrix}\mathbf{a}^T\mathbf{A}\\ \mathbf{b}^T\mathbf{B}\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix}, \quad (7.2.6)$$

which, after substituing the appropriate values can be expressed as

$$\begin{pmatrix}16 & 9\\ 1 & 16\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\1\end{pmatrix}, \quad (7.2.7)$$

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix}16 & 9 & 1\\ 1 & 16 & 1\end{pmatrix} \xleftrightarrow[R_2\leftarrow -R_2]{R_2\leftarrow R_1} \begin{pmatrix}1 & 16 & 1\\ 0 & 247 & 15\end{pmatrix} \quad (7.2.8)$$

$$\xleftrightarrow{R_1\leftarrow 247R_1-16R_2} \begin{pmatrix}247 & 0 & 7\\ 0 & 247 & 15\end{pmatrix} \quad (7.2.9)$$

$$\implies \mathbf{d} = \frac{1}{247}\begin{pmatrix}7\\15\end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247}\begin{pmatrix}7 & 0\\ 0 & 15\end{pmatrix} \quad (7.2.10)$$

The ellipse parameters are obtained from Table 5.7 as

$$\mathbf{c} = \mathbf{0}, \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \quad (7.2.11)$$

Fig. 7.2 verifies the above results.

## 8 Hyperbola

8.1. Find the equation of all lines having slope 2 and being tangent to the curve

$$y + \frac{2}{x-3} = 0 \quad (8.1.1)$$

**Solution:** (8.1.1) can be expressed as

$$xy - 3y + 2 = 0 \quad (8.1.2)$$

which is of the same form as (4.1.2) with

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \mathbf{u} = -\frac{3}{2}\begin{pmatrix}0\\1\end{pmatrix}, f = 2 \quad (8.1.3)$$

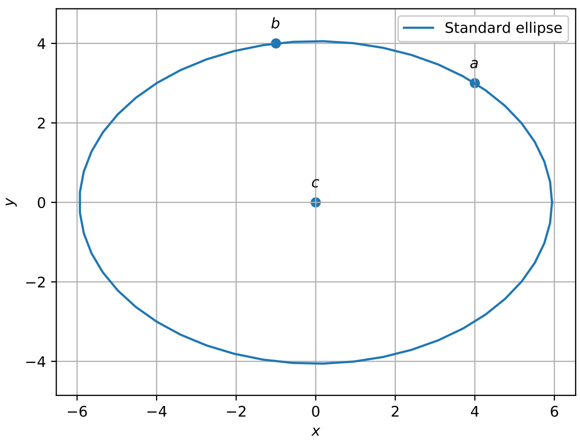


Fig. 7.2: Ellipse through the given points $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Using the approach in Example 7.1,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{8.1.4}$$

$$\because \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = -2 < 0, \tag{8.1.5}$$

the major and minor axis are swapped and from Table 5.7 the hyperbola parameters are given by

$$\mathbf{c} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 2, \tag{8.1.6}$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = 2 \tag{8.1.7}$$

with the standard hyperbola equation becoming

$$\frac{y_2^2}{4} - \frac{y_1^2}{4} = 1, \tag{8.1.8}$$

Fig. 8.1 shows the actual hyperbola in (8.1.1) obtained from (8.1.8) using (4.2.1). The direction and normal vectors of the tangent with slope 2 are given by (1.2.4) and (1.4.1) as

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{8.1.9}$$

From (5.5.2) and (6.3.3), using (8.1.3),

$$\kappa = \frac{1}{2}, \mathbf{q}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{8.1.10}$$

The desired tangents are

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} = 0 \implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 2 \tag{8.1.11}$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right\} = 0 \implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 10 \tag{8.1.12}$$

All the above results are verified in Fig. 8.1. As we can see, the hyperbola in (8.1.1) is obtained by rotating the standard hyperbola by $\mathbf{P}$ and then translating it by $\mathbf{c}$.

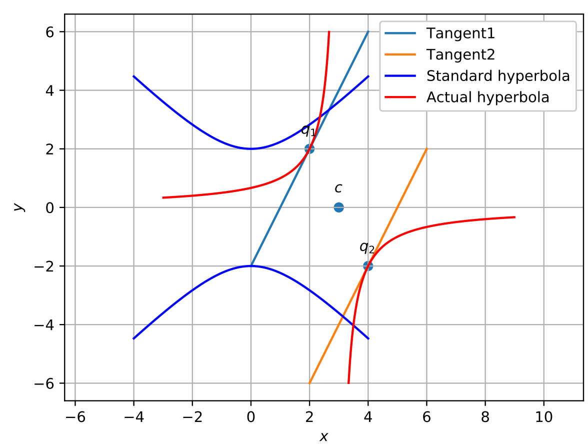


Fig. 8.1: Standard and actual hyperbola.

## 9 Parabola

9.1. Find the point at which the tangent to the curve

$$y = \sqrt{4x - 3} - 1 \tag{9.1.1}$$

has slope $\frac{2}{3}$.
**Solution:** (9.1.1) can be expressed as

$$(y + 1)^2 = 4x - 3 \tag{9.1.2}$$

$$\text{or, } y^2 - 4x + 2y + 4 = 0 \tag{9.1.3}$$

which has the form (4.1.2) with parameters

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, f = 4. \tag{9.1.4}$$

Thus, the given curve is a parabola. $\because$ $\mathbf{V}$ is diagonal and in standard form,

$$\mathbf{P} = \mathbf{I} \implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9.1.5}$$

From Table 5.7, the focus is 4 and the vertex $\mathbf{c}$ is

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \tag{9.1.6}$$

$$\implies \begin{pmatrix} -4 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{9.1.7}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \tag{9.1.8}$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}. \tag{9.1.9}$$

Also,

$$\mathbf{V}\mathbf{p} = \mathbf{0} \tag{9.1.10}$$

$$\implies \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9.1.11}$$

From (5.6.2), (9.1.9) and (9.1.11),

$$\kappa = -1 \tag{9.1.12}$$

which, upon substitution in (5.6.1) and simplification yields the matrix equation

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \tag{9.1.13}$$

$$\implies \begin{pmatrix} -4 & 4 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{9.1.14}$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{9.1.15}$$

Fig. 9.1 verifies the above results.

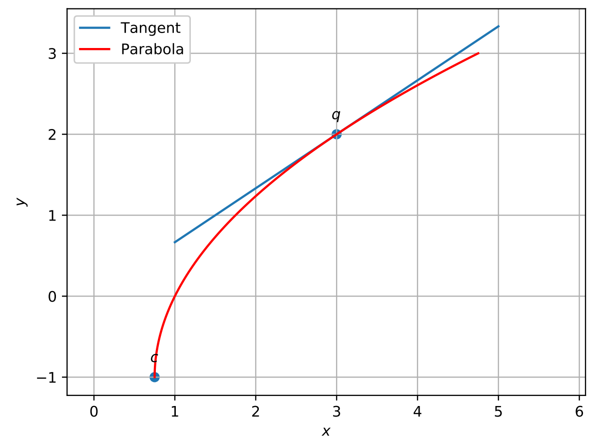


Fig. 9.1: Tangent to parabola in (9.1.1) with slope $\frac{2}{3}$.

9.2. Find a point on the curve

$$y = (x-2)^2 \tag{9.2.1}$$

at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).
**Solution:** (9.2.1) can be expressed as

$$x^2 - 4x - y + 4 = 0 \tag{9.2.2}$$

which has the form (4.1.2) with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = 4. \tag{9.2.3}$$

Using eigenvalue decomposition,

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{9.2.4}$$

Hence, the eigenvector of $\mathbf{V}$ corresponding to the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.2.5}$$

Substituting the above parameters in the equation for the vertex of the parabola in Table 5.7,

$$\begin{pmatrix} -2 & -\frac{5}{2} \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \tag{9.2.6}$$

$$\implies \begin{pmatrix} -1 & -\frac{5}{2} \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{9.2.7}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{9.2.8}$$

The direction vector is

$$\mathbf{m} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9.2.9}$$

and normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{9.2.10}$$

From the equation for the point of contact for the parabola in Table 5.7,

$$\kappa = \frac{1}{2} \tag{9.2.11}$$

resulting in the matrix equation

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{9.2.12}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{9.2.13}$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{9.2.14}$$

Fig. 9.2 verifies the above results. Note that $\mathbf{P}$ rotates the standard parabola by 90°.

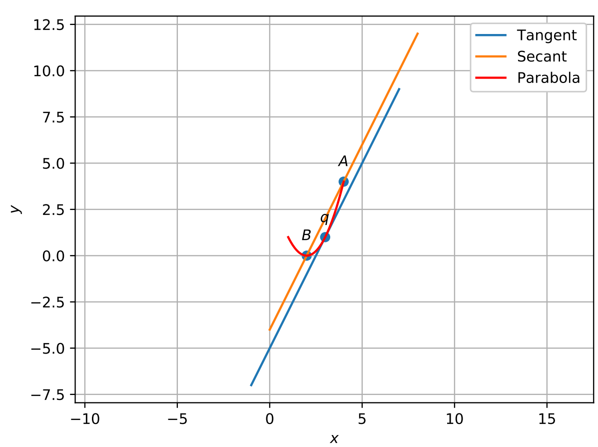


Fig. 9.2: Tangent to parabola in (9.2.1) is parallel to the line joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

## 10 Vector Calculus

10.1. *Definition:* Let $\mathbf{x} \in \mathbb{R}^2, f(\mathbf{x}) \in \mathbb{R}$. Then,

$$\frac{df(\mathbf{x})}{d\mathbf{x}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \end{pmatrix} \tag{10.1.1}$$

10.2.

$$\begin{aligned} \frac{d\mathbf{x}}{dx_1} &= \begin{pmatrix} \frac{dx_1}{dx_1} \\ \frac{dx_1}{dx_1} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ m \end{pmatrix} = \mathbf{m} \end{aligned} \tag{10.2.1}$$

10.3. Show that

$$\begin{aligned} \frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}} &= \mathbf{u} \\ \frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}} &= 2\mathbf{V}^T\mathbf{x} \end{aligned} \tag{10.3.1}$$

10.4. Differentiating (4.1.2) with respect to $x_1$,

$$\left[\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}}\right]^T \frac{d\mathbf{x}}{dx_1} + 2\frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{\mathbf{x}}\frac{d\mathbf{x}}{dx_1} = 0 \tag{10.4.1}$$

$$\implies 2\left(\mathbf{V}^T\mathbf{x} + \mathbf{u}\right)\mathbf{m} = 0 \tag{10.4.2}$$

from (10.2.1) and (10.3.1). Substituting the point of contact $\mathbf{x} = \mathbf{q}$ and simplifying results in

$$(\mathbf{V}\mathbf{q} + \mathbf{u})\,\mathbf{m} = 0 \tag{10.4.3}$$

which, upon taking the transpose, yields (5.2.1).

## 11 Vector Inequalities

11.1. *(Cauchy-Schwarz Inequality:)* Show that

$$\left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\| \tag{11.1.1}$$

*Proof.* Using the definition of the inner product,

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{11.1.2}$$

$$\because |\cos\theta| \le 1, \left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\| \tag{11.1.3}$$

*(Triangle Inequality:)* Show that

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{11.1.4}$$

*Proof.* Let $\mathbf{O}$ be the origin. In the triangle formed by $\mathbf{O}, \mathbf{a}$ and $-\mathbf{b}$, the lengths of the sides are

$$\|\mathbf{a}\|, \|\mathbf{b}\|, \|\mathbf{a} + \mathbf{b}\| \tag{11.1.5}$$

$\because$ the sum of two sides of a triangle is always greater than the third side,

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{11.1.6}$$

## 12 Convolution

12.1. Let the pair of straight lines be given by

$$\mathbf{n}_1^T\mathbf{x} = c_1 \quad (12.1.1)$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \quad (12.1.2)$$

Equating their product with (4.1.2),

$$\left(\mathbf{n}_1^T\mathbf{x} - c_1\right)\left(\mathbf{n}_2^T\mathbf{x} - c_2\right) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (12.1.3)$$

$$\implies \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (12.1.4)$$

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\mathbf{u} \quad (12.1.5)$$

$$c_1c_2 = f \quad (12.1.6)$$

where $*$ represents convolution.

12.2. The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (12.2.1)$$

$$\implies m_i = \frac{-b \pm \sqrt{-|V|}}{c} \quad (12.2.2)$$

and

$$\mathbf{n}_i = k_i\begin{pmatrix} -m_i \\ 1 \end{pmatrix}, \quad i = 1, 2. \quad (12.2.3)$$

12.3. From (12.1.5),

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (12.3.1)$$

$c_1, c_2$ can be obtained such that they satisfy (12.1.6).

12.4. Given,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (12.4.1)$$

it is easy to verify that

$$\begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} = 0 \quad (12.4.2)$$

Hence, (12.4.1) represents a pair of straight lines.

12.5. (12.4.1) can be expressed as (4.1.2) with

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (12.5.1)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (12.5.2)$$

$$f = -35 \quad (12.5.3)$$

From (12.4.1) and (12.2.1),

$$\implies m_i = \frac{-7 \pm \sqrt{49 + 480}}{-20} \quad (12.5.4)$$

$$\implies m_1 = \frac{3}{2}, m_2 = -\frac{4}{5} \quad (12.5.5)$$

Thus,

$$\mathbf{m}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (12.5.6)$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (12.5.7)$$

12.6. Using the Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 3 & 0 \\ -2 & 3 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (12.6.1)$$

which matches the corresponding coefficients in (12.4.1)

Substituting from (12.5.7) in (12.3.1), the augmented matrix is

$$\begin{pmatrix} 3 & 4 & -13 \\ -2 & 5 & -45 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{R_1 - 4R_2}{3}]{R_2 \leftarrow \frac{2R_1 + 3R_2}{23}} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \end{pmatrix} \quad (12.6.2)$$

$$\implies c_1 = -7, c_2 = 5 \quad (12.6.3)$$

Fig. 12.6 plots the lines in (12.4.1)

12.7. From (12.5.7) the angle between the two straight lines is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|}\right) \quad (12.7.1)$$

$$\mathbf{n_1}^T\mathbf{n_2} = \begin{pmatrix} 3 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = 2 \quad (12.7.2)$$

$$\|\mathbf{n_1}\| = \sqrt{3^2 + (-2)^2} = \sqrt{13} \quad (12.7.3)$$

$$\|\mathbf{n_2}\| = \sqrt{4^2 + 5^2} = \sqrt{41} \quad (12.7.4)$$

Substituting equations (12.7.2), (12.7.3)

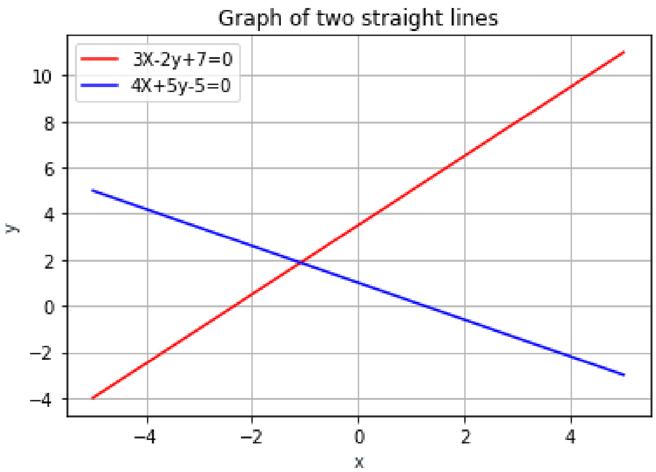


Fig. 12.6: Pair of straight lines

,(12.7.4) in equation (12.7.1), we get

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{13}\sqrt{41}}\right) \tag{12.7.5}$$

$$\theta = 85^\circ \tag{12.7.6}$$

## 13 QR Decomposition

13.1. Revisiting Problem (1.15),

$$\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \tag{13.1.1}$$

we can express

$$\begin{aligned} \alpha &= k_1 \mathbf{u}_1 \\ \beta &= r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \end{aligned} \tag{13.1.2}$$

where

$$k_1 = \|\alpha\|, \mathbf{u}_1 = \frac{\alpha}{k_1} \tag{13.1.3}$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2}, \mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \tag{13.1.4}$$

$$k_2 = \mathbf{u}_2^T \beta \tag{13.1.5}$$

From (13.1.2),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{13.1.6}$$

This is known as **QR** decomposition, where

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{13.1.7}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \tag{13.1.8}$$

Note that **R** is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I}. \tag{13.1.9}$$

13.2. From (13.1.1),

$$k_1 = \sqrt{10}, \mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \tag{13.2.1}$$

$$r_1 = \frac{1}{2}, \mathbf{u}_2 = \frac{1}{\sqrt{46}} \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \tag{13.2.2}$$

$$k_2 = \sqrt{\frac{23}{2}} \tag{13.2.3}$$

Thus, we obtain the **QR** decompositon

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{46}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{46}} \\ 0 & \frac{-6}{\sqrt{46}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{1}{2} \\ 0 & \sqrt{\frac{23}{2}} \end{pmatrix} \tag{13.2.4}$$

## 14 Singular Value Decomposition

14.1. We revisit (3.2.6)

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{14.1.1}$$

## Appendix A
## Proofs for the Conic Sections

A.1. Substituting (4.2.1) in (4.1.2)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{A.1.1}$$

which can be expressed as

$$\begin{aligned} &\mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2 (\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} \\ &\qquad + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \end{aligned} \tag{A.1.2}$$

From (A.1.2) and (4.2.2),

$$\begin{aligned} &\mathbf{y}^T \mathbf{D} \mathbf{y} + 2 (\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} \\ &\qquad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \end{aligned} \tag{A.1.3}$$

When $\mathbf{V}^{-1}$ exists,

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{A.1.4}$$

and substituting (A.1.4) in (A.1.3) yields (4.2.5).

A.2. When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{A.2.1}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$such that (4.2.2)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{A.2.2}$$

Substituting (A.2.2) in (A.1.3),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0$$

$$\implies \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\begin{pmatrix} \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_1 & \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0$$

$$\implies \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\begin{pmatrix} \mathbf{u}^T\mathbf{p}_1 & \left(\lambda_2\mathbf{c}^T + \mathbf{u}^T\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0$$

from (A.2.1)

$$\implies \lambda_2 y_2^2 + 2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1 + 2y_2\left(\lambda_2\mathbf{c} + \mathbf{u}\right)^T\mathbf{p}_2 + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{A.2.3}$$

which is the equation of a parabola. From (A.2.3), by comparing the coefficients of $y_2^2$ and $y_1$, the focal length of the parabola is obtained as

$$\left|\frac{2\mathbf{u}^T\mathbf{p}_1}{\lambda_2}\right|. \tag{A.2.4}$$

Thus, (A.2.3) can be expressed as (4.2.6) by choosing

$$\eta = 2\mathbf{u}^T\mathbf{p}_1 \tag{A.2.5}$$

and $\mathbf{c}$ in (A.1.3) such that

$$\mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{A.2.6}$$

$$\mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{A.2.7}$$

Multiplying (A.2.6) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta\mathbf{p}_1, \tag{A.2.8}$$

which, upon substituting in (A.2.7) results in

$$\eta\mathbf{c}^T\mathbf{p}_1 + \mathbf{u}^T\mathbf{c} + f = 0 \tag{A.2.9}$$

(A.2.8) and (A.2.9) can be clubbed together to obtain (4.2.8).