Geometry through Linear Algebra

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Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1 Pair of Straight Lines

1.1. Find the value of h so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \tag{1.1.1}$$

may represent two straight lines.
Solution:

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \tag{1.1.2}$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{1.1.3}$$

$$f = 20 \tag{1.1.4}$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \tag{1.1.5}$$

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Expanding equation (1.1.5) along row 1 gives

$$\implies 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + 11 \times (\frac{31h}{2} - 132) = 0$$

$$\implies 20h^2 - 341h + \frac{2907}{2} = 0 \tag{1.1.6}$$

$$\implies \boxed{h = \frac{17}{2}} \tag{1.1.7}$$

$$\implies \boxed{h = \frac{171}{20}} \tag{1.1.8}$$

If $h = \frac{17}{2}$ or $h = \frac{171}{20}$, the equation given will represent two straight lines.

Sub $h = \frac{17}{2}$ in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \tag{1.1.9}$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{1.1.10}$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{1.1.11}$$

$$\mathbf{f} = 20 \tag{1.1.12}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \tag{1.1.13}$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \tag{1.1.14}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.1.15}$$

$$\tag{1.1.16}$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \tag{1.1.17}$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \tag{1.1.18}$$

$$\implies m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \tag{1.1.19}$$

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{1.1.20}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.21}$$

we know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.22}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$
From equation (1.1.21)

$$\mathbf{n_1} = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.23}$$

$$\implies \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.24}$$

$\implies$ Equation (1.1.21) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.25}$$

Substituting (1.1.21) in (1.1.25), the augmented matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.1.26}$$

$$\implies c_1 = 4, c_2 = 5 \tag{1.1.27}$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\implies (-2x - 3y - 4)(3x - 4y - 5) = 0$$

$$\implies \boxed{(2x + 3y + 4)(3x + 4y + 5) = 0} \tag{1.1.28}$$

Equation (1.1.28) represents equations of two straight lines.

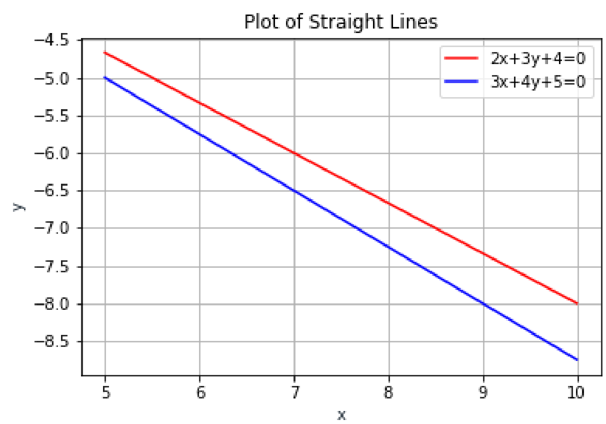

Fig. 1.1.1: Plot of Straight lines when $h = \frac{17}{2}$

Similarly, Sub $h = \frac{171}{20}$ in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \tag{1.1.29}$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \tag{1.1.30}$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{6} \\ \frac{310}{6} \end{pmatrix} \tag{1.1.31}$$

$$\mathbf{f} = \frac{200}{3} \tag{1.1.32}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \tag{1.1.33}$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \tag{1.1.34}$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \tag{1.1.35}$$

$$\implies m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \tag{1.1.36}$$

$$\mathbf{m_1} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \tag{1.1.37}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{1.1.38}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$
From equation (1.1.38)

$$\mathbf{n_1} = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{1.1.39}$$

$$\implies \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.40}$$

$\implies$ Equation (1.1.38) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.41}$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1-4R_2}{5}]{R_2 \leftarrow \frac{5R_2-8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \tag{1.1.42}$$

$$\implies c_1 = 10, c_2 = \frac{20}{3} \tag{1.1.43}$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\implies \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \tag{1.1.44}$$

Equation (1.1.44) represents equations of two straight lines.

1.2. Prove that the following equations represent two straight lines.Also find their point of intersection and the angle between them

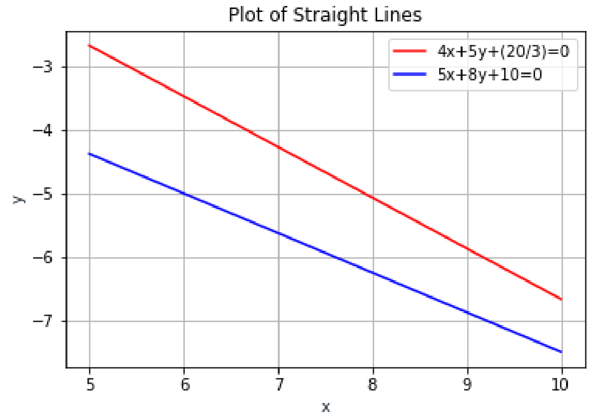

Fig. 1.1.2: Plot of Straight lines when $h = \frac{171}{20}$

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \tag{1.2.1}$$

Solution: $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \tag{1.2.2}$$

Expanding equation (1.2.2), we get zero.
Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \tag{1.2.3}$$

$$\implies 3m^2 - 8m - 3 = 0 \tag{1.2.4}$$

$$\text{Solving, } m = 3, -\frac{1}{3} \tag{1.2.5}$$

The normal vectors of the lines then become

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.2.6}$$

$$\mathbf{n_2} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.2.7}$$

Equations of the lines can therefore be written as

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = c \tag{1.2.8}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = c_1, \tag{1.2.9}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{1.2.10}$$

$$\implies \left[\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - c_1 \right] \left[\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} - c_2 \right] \tag{1.2.11}$$

represents the equation specified in (1.2.1)

Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \tag{1.2.12}$$

$$\tag{1.2.13}$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3 \times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \tag{1.2.14}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \tag{1.2.15}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3 \times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \tag{1.2.16}$$

$$\implies c_2 = 2 \text{ and } c_1 = -9 \tag{1.2.17}$$

The individual line equations therefore become

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = -9, \tag{1.2.18}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1.2.19}$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.2.20}$$

The LHS part of (1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.2.21}$$

The augmented matrix for the set of equations represented in (1.2.18), (1.2.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \tag{1.2.22}$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \tag{1.2.23}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \tag{1.2.24}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \tag{1.2.25}$$

$$\text{Hence, the intersection point is} \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.2.26}$$

Angle between two lines θ can be given by

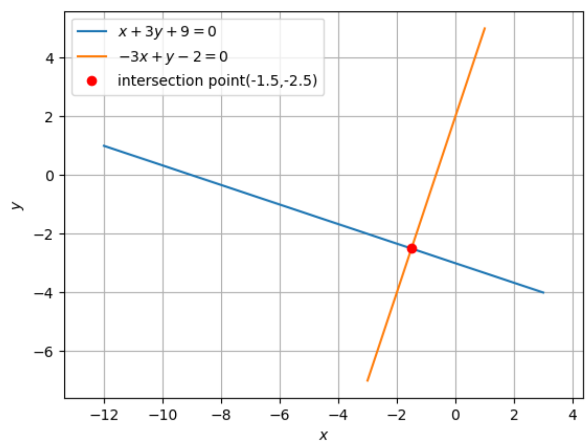

Fig. 1.2.1: plot showing intersection of lines

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{1.2.27}$$

$$\cos\theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \tag{1.2.28}$$

$$\implies \theta = 90° \tag{1.2.29}$$

1.3. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \tag{1.3.1}$$

represent pairs of straight lines. **Solution:**

From (1.3.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \tag{1.3.2}$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \tag{1.3.3}$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & k \end{vmatrix} = 0 \tag{1.3.4}$$

$$\implies \left(k - \left(\frac{49}{4}\right)\right) - \frac{5}{3}\left(\frac{5}{3}k - \frac{35}{4}\right)$$

$$-\frac{5}{2}\left(\frac{-35}{6} + \frac{5}{2}\right) = 0 \tag{1.3.5}$$

$$\implies \frac{64}{k}36 - \frac{128}{12} = 0 \tag{1.3.6}$$

$$\implies \boxed{k = 6} \tag{1.3.7}$$

Substituting (1.3.7) in (1.3.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \tag{1.3.8}$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (1.3.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & 6 \end{vmatrix} \tag{1.3.9}$$

Simplyfying the above determinant , we get

$$\delta = 0 \tag{1.3.10}$$

(1.3.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \tag{1.3.11}$$

Since $\det(V) < 0$ lines would intersect each other

$$\mathbf{n_1} * \mathbf{n_2} = \{1, \frac{10}{3}, 1\} \tag{1.3.12}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.3.13}$$

$$c_1c_2 = 6 \tag{1.3.14}$$

The slopes of the lines are given by the roots

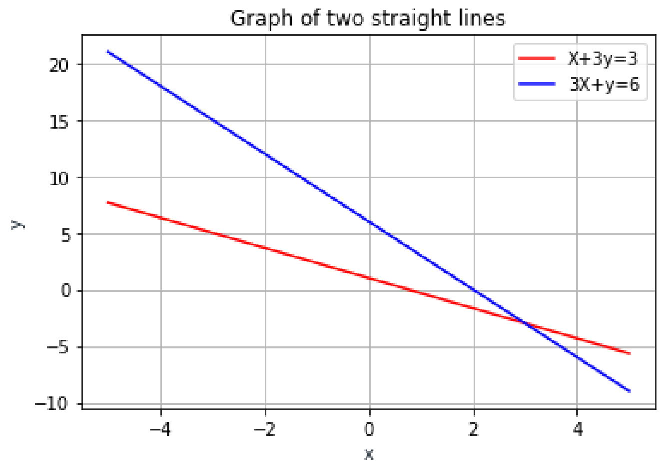

Fig. 1.3.1: Pair of straight lines

of the polynomial

$$cm^2 + 2bm + a = 0 \tag{1.3.15}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.3.16}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.3.17}$$

Substituting in above equations (1.3.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \tag{1.3.18}$$

$$\implies m_i = \frac{\frac{-10}{3} \pm \sqrt{-(\frac{-16}{9})}}{1} \tag{1.3.19}$$

Solving equation (1.3.19) we have ,

$$m_1 = \frac{-1}{3} \tag{1.3.20}$$

$$m_2 = -3 \tag{1.3.21}$$

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.3.22}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.3.23}$$

Substituting equations (1.3.22), (1.3.23) in equation (1.3.12) we get

$$k_1k_2 = 1 \tag{1.3.24}$$

Possible combination of (k_1, k_2) is (1,1) Lets assume $k_1 = 1$, $k_2 = 1$, we get

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.3.25}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.3.26}$$

we have:

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.3.27}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a teoplitz matrix and multiplying with $\mathbf{n_2}$

From equation (1.3.25) and (1.3.26)

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.3.28}$$

$$\implies \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.3.29}$$

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.3.30}$$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.3.31}$$

Substituting (1.3.25) and (1.3.26) in (1.3.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \tag{1.3.32}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \tag{1.3.33}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.3.34}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.3.35}$$

From above we get

$$c_1 = 1 \tag{1.3.36}$$

$$c_2 = 6 \tag{1.3.37}$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \tag{1.3.38}$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{1.3.39}$$