



Coordinate Geometry Exercises



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Abstract—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.

1 CONICS

1.1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T \mathbf{x} = 4$ and $\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.1.1)$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (1.1.2)$$

$$\mathbf{x}^T \mathbf{x} - 4 = 0 \quad (1.1.3)$$

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.1.4)$$

$$f_1 = -4 \quad (1.1.5)$$

$$\mathbf{O}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.1.6)$$

Taking equation of the second circle to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\|^2 = 2^2 \quad (1.1.7)$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}_2^T \mathbf{x} = 0 \quad (1.1.8)$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.1.9)$$

$$f_2 = 0 \quad (1.1.10)$$

$$\mathbf{O}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (1.1.11)$$

Now, Subtracting equation (1.1.8) from (1.1.3)
We get,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{u}_2^T \mathbf{x} + f_1 - \mathbf{x}^T \mathbf{x} = 0 \quad (1.1.12)$$

$$2\mathbf{u}_2^T \mathbf{x} = -4 \quad (1.1.13)$$

$$\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} = -4 \quad (1.1.14)$$

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Which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \quad (1.1.15)$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.16)$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (1.1.17)$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.1.18)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.19)$$

Substituting (1.1.17) in (1.1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (1.1.20)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 + f_1 = 0 \quad (1.1.21)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (1.1.22)$$

$$\mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (1.1.23)$$

$$\|\mathbf{q}\|^2 + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (1.1.24)$$

$$\|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (1.1.25)$$

$$\lambda(\lambda \|\mathbf{m}\|^2 + 2\mathbf{q}^T \mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \quad (1.1.26)$$

$$\lambda^2 \|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \quad (1.1.27)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.1.28)$$

$$\lambda^2 = 3 \quad (1.1.29)$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \quad (1.1.30)$$

Substituting the value of λ in (1.1.17)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (1.1.31)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (1.1.32)$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (1.1.33)$$

Now finding the direction vector \mathbf{m}_{O_1A} , \mathbf{m}_{O_1B} , \mathbf{m}_{O_2A} and \mathbf{m}_{O_2B} .

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \quad (1.1.34)$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad (1.1.35)$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (1.1.36)$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (1.1.37)$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B} = \|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\| \cos \theta_1 \quad (1.1.38)$$

$$\frac{\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B}}{\|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\|} = \cos \theta_1 \quad (1.1.39)$$

$$\frac{-2}{4} = \cos \theta_1 \quad (1.1.40)$$

$$\frac{-1}{2} = \cos \theta_1 \quad (1.1.41)$$

$$\theta_1 = 120^\circ \quad (1.1.42)$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B} = \|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\| \cos \theta_2 \quad (1.1.43)$$

$$\frac{\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B}}{\|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\|} = \cos \theta_2 \quad (1.1.44)$$

$$\frac{-2}{4} = \cos \theta_2 \quad (1.1.45)$$

$$\frac{-1}{2} = \cos \theta_2 \quad (1.1.46)$$

$$\theta_2 = 120^\circ \quad (1.1.47)$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2 \sqrt{3} \quad (1.1.48)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} \quad (1.1.49)$$

$$A_{O_2AB} = \frac{\pi \theta_2}{360} r^2 - \frac{1}{2} 2 \sqrt{3} \quad (1.1.50)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} \quad (1.1.51)$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \quad (1.1.52)$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \quad (1.1.53)$$

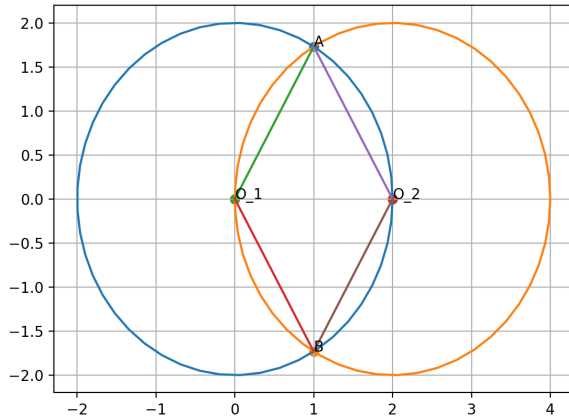


Fig. 1.1: Figure depicting intersection points of circle

- 1.2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre (h,k) is given by,

$$x^T x + 2u^T x + f = 0 \quad (1.2.1)$$

where,

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (1.2.2)$$

The radius and centre are respectively given by,

$$r = 5 \quad (1.2.3)$$

$$\mathbf{c} = -\mathbf{u} = k\mathbf{e} \quad (1.2.4)$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.2.5)$$

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.2.6)$$

From the given data , we modify equation 1.2.1

as,

$$\mathbf{x}_1^T \mathbf{x}_1 + 2 \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \quad (1.2.7)$$

$$\|\mathbf{x}_1\|^2 + 2(k^2) + f = 0 \quad (1.2.8)$$

$$2k^2 + f = -\|\mathbf{x}_1\|^2 \quad (1.2.9)$$

Substituting \mathbf{u} in equation 1.2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \quad (1.2.10)$$

$$f = (k^2) - r^2 \quad (1.2.11)$$

$$k^2 - f = r^2 \quad (1.2.12)$$

From equations 1.2.9 and 1.2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x}_1\|^2 \\ r^2 \end{pmatrix} \quad (1.2.13)$$

Here $\|\mathbf{x}_1\|$ is given by ,

$$\|\mathbf{x}_1\| = \sqrt{2^2 + 3^2} \quad (1.2.14)$$

$$\|\mathbf{x}_1\| = \sqrt{13} \quad (1.2.15)$$

Substituting equation 1.2.6, 1.2.3 in equation 1.2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \quad (1.2.16)$$

The augmented matrix of 1.2.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \quad (1.2.17)$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \quad (1.2.18)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \quad (1.2.19)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \quad (1.2.20)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array} \right) \quad (1.2.21)$$

Equation 1.2.16 can be rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \quad (1.2.22)$$

Expanding the above equation 1.2.22 we get ,

$$k^2 = 4 \quad (1.2.23)$$

$$k = \pm 2 \quad (1.2.24)$$

$$f = -21 \quad (1.2.25)$$

To get the centre substitute equation 1.2.24 in equation 1.2.4 To verify the above results we plot the circle with centre \mathbf{c} as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$,



Fig. 1: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

From the above figure 1 it is clear that circle with centre $\mathbf{c} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ passes through the point \mathbf{x}_1 . Desired equation of circle is given by ,

$$\mathbf{c} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.2.26)$$

$$f = -21 \quad (1.2.27)$$

1.3. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.

1.4. Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Solution: The general equation of a circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.4.1)$$

$$\text{If } r \text{ is radius, } f = \mathbf{u}^T \mathbf{u} - r^2 \quad (1.4.2)$$

$$\text{center } \mathbf{c} = -\mathbf{u} \quad (1.4.3)$$

Given centre is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (1.4.4)$$

$$\Rightarrow \mathbf{u} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (1.4.5)$$

Equation (1.4.1) becomes

$$\mathbf{x}^T \mathbf{x} + (-4 \ -4) \mathbf{x} + f = 0 \quad (1.4.6)$$

This passes through point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Substituting $\mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ in (1.4.6)

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} + (-4 \ -4) \begin{pmatrix} 4 \\ 5 \end{pmatrix} + f = 0 \quad (1.4.7)$$

$$\Rightarrow f = -5 \quad (1.4.8)$$

Also, radius can be determined as follows

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (1.4.9)$$

$$\Rightarrow -5 = (-2 \ -2) \begin{pmatrix} -2 \\ -2 \end{pmatrix} - r^2 \quad (1.4.10)$$

$$\Rightarrow -5 = 8 - r^2 \quad (1.4.11)$$

$$\Rightarrow r = \sqrt{13} \quad (1.4.12)$$

The equation of required circle is

$$\mathbf{x}^T \mathbf{x} + (-4 \ -4) \mathbf{x} - 5 = 0 \quad (1.4.13)$$

See Fig. 1

1.5. Find the locus of all the unit vectors in the xy-plane.

1.6. Find the points on the curve $\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.

1.7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 0$ and the circle $\mathbf{x}^T \mathbf{x} = 4$.

Solution: The equation of a circle can be expressed as,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (1.7.1)$$

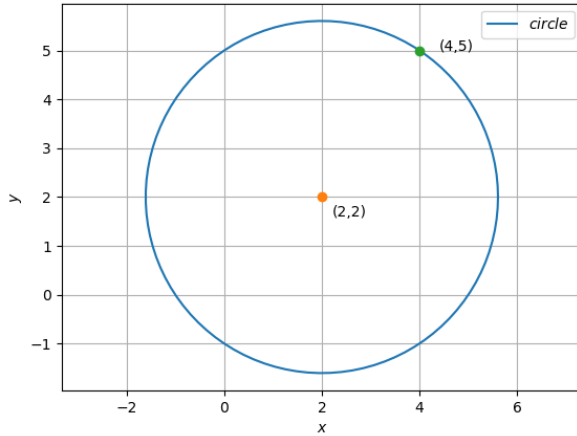


Fig. 1: plot showing the circle

where \mathbf{c} is the center.

Comparing equation (1.7.1) with the circle equation given,

$$\mathbf{x}^T \mathbf{x} = 4 \quad (1.7.2)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad f = -4 \quad (1.7.3)$$

$$r = \sqrt{\mathbf{c}^T \mathbf{c} - f} = \sqrt{4} \quad (1.7.4)$$

$$\Rightarrow \boxed{r = 2} \quad (1.7.5)$$

From equation (1.7.5), the point at which circle touches x -axis is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The direction vector of x -axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The direction vector of the given line

$$(1 - \sqrt{3})\mathbf{x} = 0 \text{ is } \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}.$$

The angle that the line makes with the x -axis is given by,

$$\cos \theta = \frac{(\sqrt{3} \ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\|(\sqrt{3} \ 1)\| \|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\|} = \frac{\sqrt{3}}{2} \quad (1.7.6)$$

$$\Rightarrow \boxed{\theta = 30^\circ} \quad (1.7.7)$$

Using equation (1.7.5) and (1.7.7), the area of the sector is obtained as,

$$\Rightarrow \boxed{\frac{\theta}{360^\circ} \pi r^2 = \frac{30^\circ}{360^\circ} \pi (2)^2 = \frac{\pi}{3}} \quad (1.7.8)$$

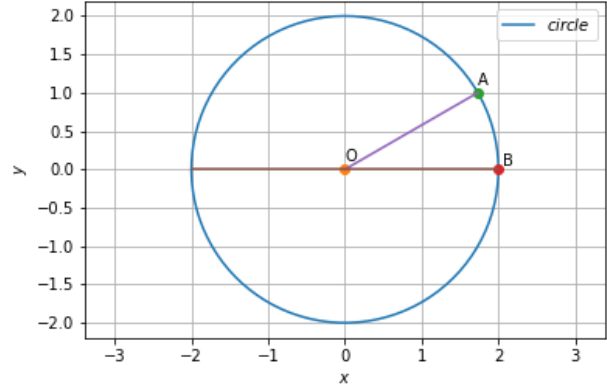


Fig. 4: Region enclosed by x -axis, line and circle

To find points **A** and **B**,

The parametric form of x -axis is,

$$\mathbf{B} = \mathbf{q} + \lambda \mathbf{m} \quad (1.7.9)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.7.10)$$

From the intersection of circle and line, the value of λ can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.7.11)$$

$$= \frac{4 - 0}{1} = 4 \quad (1.7.12)$$

$$\Rightarrow \lambda = \pm 2 \quad (1.7.13)$$

Sub equation (1.7.13) in (1.7.10),

$$\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.7.14)$$

As given in question as first quadrant,

$$\Rightarrow \boxed{\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}} \quad (1.7.15)$$

Similarly, to find point **A**, The parametric form

of line is,

$$\mathbf{A} = \mathbf{q} + \lambda \mathbf{m} \quad (1.7.16)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad (1.7.17)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.7.18)$$

$$= \frac{4-0}{4} = 1 \quad (1.7.19)$$

$$\Rightarrow \lambda = \pm 1 \quad (1.7.20)$$

$$\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \quad (1.7.21)$$

$$\Rightarrow \boxed{\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \quad (1.7.22)$$

1.8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.

1.9. Find the area of the circle $4\mathbf{x}^T \mathbf{x} = 9$.

1.10. Find the area bounded by curves $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$ and $\|\mathbf{x}\| = 1$

1.11. Find the smaller area enclosed by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$.

1.12. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Solution:

$$y = \frac{x-1}{x-2} \quad (1.12.1)$$

Equation (1.12.1) can be expressed as

$$y(x-2) = x-1 \quad (1.12.2)$$

$$yx - 2y - x + 1 = 0 \quad (1.12.3)$$

From above we can say,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.12.4)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} & -1 \end{pmatrix} \quad (1.12.5)$$

$$f = 1 \quad (1.12.6)$$

Now,

$$\therefore |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0, \quad (1.12.7)$$

(1.12.1) is the equation of a hyperbola. To verify that this we will find the characteristic

equation of \mathbf{V} .

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} = 0 \quad (1.12.8)$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (1.12.9)$$

The eigenvalues are the roots of (1.12.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (1.12.10)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \quad (1.12.11)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (1.12.12)$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.12.13)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.12.14)$$

Now, λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.12.15)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.12.16)$$

From Equations,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \therefore \mathbf{P}^{-1} = \mathbf{P}^T \quad (1.12.17)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (1.12.18)$$

We can say that

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (1.12.19)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (1.12.20)$$

$\therefore \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f > 0$, there isn't a need to swap

axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (1.12.21)$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (1.12.22)$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad (1.12.23)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (1.12.24)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \quad (1.12.25)$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \quad (1.12.26)$$

Let us assume slope to be 1, now finding the direction vector and normal vector of the tangent with slope 1.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \quad (1.12.27)$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \quad (1.12.28)$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (1.12.29)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (1.12.30)$$

By using (1.12.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \quad (1.12.31)$$

Now substituting this κ in (1.12.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{-\frac{1}{4l}} + 1 \end{pmatrix} \quad (1.12.32)$$

We know that $x=10$.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \quad (1.12.33)$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \quad (1.12.34)$$

$$\sqrt{-\frac{1}{4l}} = 4 \quad (1.12.35)$$

$$-\frac{1}{4l} = 16 \quad (1.12.36)$$

$$l = -\frac{1}{64} \quad (1.12.37)$$

The slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x=10$ is $\frac{1}{64}$. So, from the above we can say that $\kappa=4, -4$ and from equation (1.12.27) and (1.12.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \quad (1.12.38)$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \quad (1.12.39)$$

Now using equation (1.12.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1}(\kappa_1 \mathbf{n} - \mathbf{u}) \quad (1.12.40)$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left(-4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (1.12.41)$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \quad (1.12.42)$$

$$\mathbf{q}_2 = \mathbf{V}^{-1}(\kappa_2 \mathbf{n} - \mathbf{u}) \quad (1.12.43)$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left(4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (1.12.44)$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \quad (1.12.45)$$

1.13. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Solution: $y = (x-2)^2$ can be written as,

$$x^2 - 4x - y + 4 = 0 \quad (1.13.1)$$

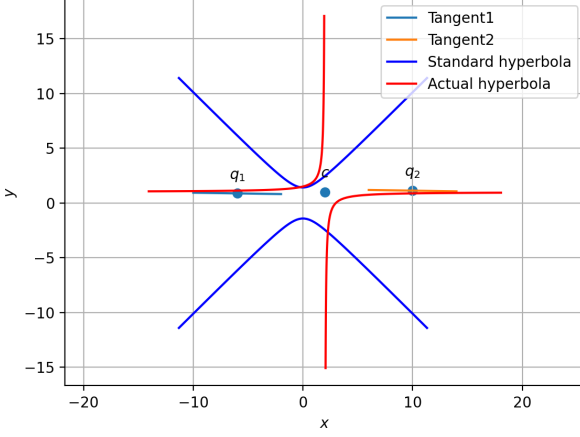


Fig. 5: Tangent 2 shows the tangent

From (1.13.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \mathbf{u} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}; f = 4 \quad (1.13.2)$$

$$|V| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \quad (1.13.3)$$

(1.13.3) implies that the curve is a parabola. Now, finding the eigen values corresponding to the \mathbf{V} ,

$$\begin{aligned} |V - \lambda I| &= 0 \\ \begin{vmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda &= 0, 1 \end{aligned} \quad (1.13.4)$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0; \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.13.5)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0; \Rightarrow \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.13.6)$$

By Eigen decomposition on \mathbf{V} ,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$$

$$\text{where, } \mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.13.7)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.13.8)$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (1.13.9)$$

$$\text{where, } \eta = \mathbf{u}^T \mathbf{p}_1 = -\frac{1}{2} \quad (1.13.10)$$

Substituting values from (1.13.2), (1.13.5) and (1.13.10) in (1.13.9),

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \quad (1.13.11)$$

Removing last row and representing (1.13.11) as augmented matrix and then converting the matrix to echelon form,

$$\begin{aligned} \begin{pmatrix} -2 & -1 & -4 \\ 1 & 0 & 2 \end{pmatrix} &\xrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \\ \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} &\xrightarrow{R_2 \leftarrow (-2R_2)} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}} \\ &\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \quad (1.13.12)$$

From (1.13.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \quad (1.13.13)$$

Direction vector of the chord joining A(4,4) and B(2,0) can be calculated as,

$$\begin{aligned} \mathbf{m} &= \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \Rightarrow \mathbf{m} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned} \quad (1.13.14)$$

We know that,

$$\mathbf{m}^T \mathbf{n} = 0; \Rightarrow \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.13.15)$$

To find the point of contact \mathbf{q} , which is intersection point for normal of the chord AB and also tangent of the curve,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (1.13.16)$$

$$\text{where, } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}} = \frac{1}{2} \quad (1.13.17)$$

Substituting the values from (1.13.2), (1.13.15)

and (1.13.17) in (1.13.16),

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \quad (1.13.18)$$

Removing last row and representing (1.13.18) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -1 & -1 & -4 \\ 1 & 0 & 3 \end{pmatrix} \xrightarrow{R_1 \leftarrow (-R_1)} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow (-R_2)} \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.13.19)$$

From (1.13.19), it can be observed,

$$\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.13.20)$$

which is the required point of contact

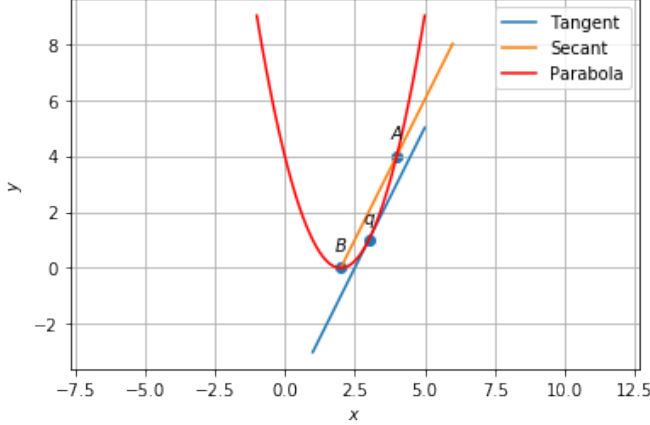


Fig. 6: Parabola with AB as chord, a tangent parallel to the chord

1.14. Find the equation of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

Solution: The given curve

$$y = \frac{1}{x-1} \quad (1.14.1)$$

can be expressed as

$$xy - y - 1 = 0 \quad (1.14.2)$$

Hence, we have

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix}, f = -1 \quad (1.14.3)$$

Since $|\mathbf{V}| < 0$, the equation (1.14.2) represents hyperbola. To find the values of λ_1 and λ_2 , consider the characteristic equation,

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (1.14.4)$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \right| = 0 \quad (1.14.5)$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \lambda \end{pmatrix} \right| = 0 \quad (1.14.6)$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (1.14.7)$$

In addition, given the slope -1 , the direction and normal vectors are given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.14.8)$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.14.9)$$

The parameters of hyperbola are as follows:

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (1.14.10)$$

$$= - \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \quad (1.14.11)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.14.12)$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \end{cases} \quad (1.14.13)$$

which represents the standard hyperbola equation,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (1.14.14)$$

The points of contact are given by

$$K = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} = \pm \frac{1}{2} \quad (1.14.15)$$

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \quad (1.14.16)$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left[\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix} \right] \quad (1.14.17)$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.14.18)$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left[\frac{-1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix} \right] \quad (1.14.19)$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (1.14.20)$$

\therefore The tangents are given by

$$(1 \ 1) \left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (1.14.21)$$

$$(1 \ 1) \left(\mathbf{x} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) = 0 \quad (1.14.22)$$

The desired equations of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$ are given by

$$(1 \ 1)\mathbf{x} = 3 \quad (1.14.23)$$

$$(1 \ 1)\mathbf{x} = -1 \quad (1.14.24)$$

The above results are verified in the following figure.

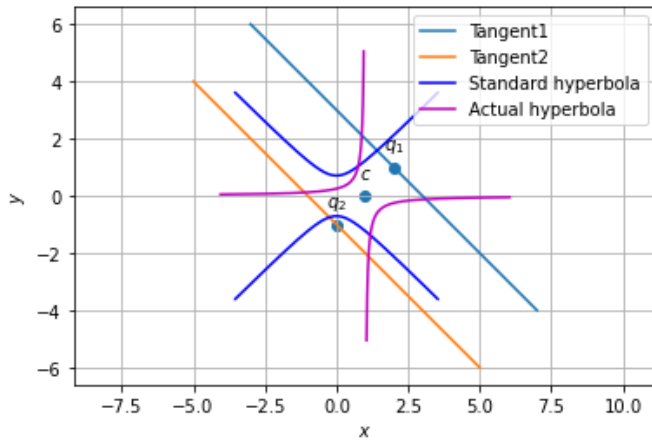


Fig. 7: The standard and actual hyperbola.

Solution: Given the curve,

$$y = \frac{1}{x-3} \quad (1.15.1)$$

$$\Rightarrow xy - 3y - 1 = 0 \quad (1.15.2)$$

From (1.15.2) we get,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = -1 \quad (1.15.3)$$

Now,

$$\because |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = \frac{-1}{2} < 0 \quad (1.15.4)$$

(1.15.1) is equation of hyperbola. Now,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \quad (1.15.5)$$

$$\Rightarrow \lambda^2 - \frac{1}{4} = 0 \quad (1.15.6)$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \quad (1.15.7)$$

The eigen vector \mathbf{p} is given by,

$$(\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (1.15.8)$$

For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (1.15.9)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.15.10)$$

Similarly for λ_2 ,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow -R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.15.11)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.15.12)$$

1.15. Find the equation of all lines having slope -2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.

Now,

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (1.15.13)$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (1.15.14)$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 \quad (1.15.15)$$

$\because \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 > 0$, there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.15.16)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (1.15.17)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = \sqrt{2} \quad (1.15.18)$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (1.15.19)$$

The direction and normal vectors of the tangent with slope -2 are given as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.15.20)$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \quad (1.15.21)$$

$$k = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (1.15.22)$$

Thus,

$$\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n} = 8 \quad (1.15.23)$$

$$k = \pm \frac{1}{2\sqrt{2}} \quad (1.15.24)$$

$$\mathbf{q}_1 = \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \quad (1.15.25)$$

$$\mathbf{q}_2 = \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \quad (1.15.26)$$

The desired tangents are,

$$(2 \ -1) \left\{ \mathbf{x} - \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \right\} = 0 \quad (1.15.27)$$

$$\Rightarrow (2 \ -1) \mathbf{x} = 6 + 2\sqrt{2} \quad (1.15.28)$$

$$(2 \ -1) \left\{ \mathbf{x} - \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \right\} = 0 \quad (1.15.29)$$

$$\Rightarrow (2 \ -1) \mathbf{x} = 6 - 2\sqrt{2} \quad (1.15.30)$$

Below figure corresponds to the tangents on the hyperbola, represented by (1.15.28) and (1.15.30) each having slope of -2 .

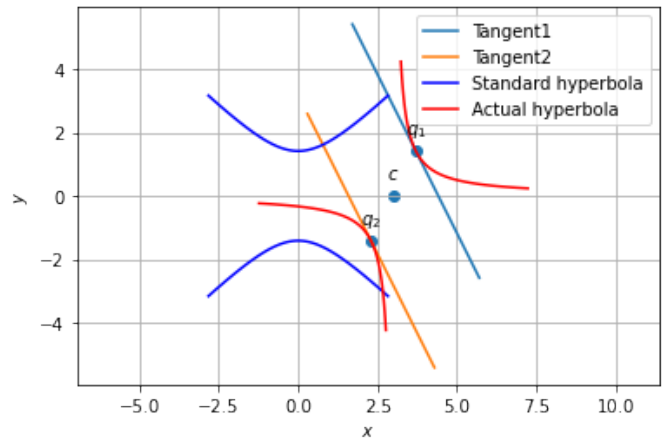


Fig. 8: Tangents to the hyperbola

1.16. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at

which tangents are

- parallel to x-axis
- parallel to y-axis.

1.17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

1.18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$

- parallel to the line $(2 \ -1) \mathbf{x} = -9$
- perpendicular to the line $(-15 \ 5) \mathbf{x} = 13$.

1.19. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $(4 \ 2) \mathbf{x} + 5 = 0$.

1.20. Find the point at which the line $(-1 \ 1) \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.

1.21. The line $(-m \ 1) \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m .

- 1.22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$
- 1.23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- 1.24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.
- 1.25. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.
- 1.26. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.
- 1.27. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
- 1.28. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
- 1.29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a .
- 1.30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
- 1.31. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.
- 1.32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
- 1.33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
- 1.34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
- 1.35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T \mathbf{x} - 8 \begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.
- 1.36. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB .
- 1.37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
- 1.38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
- 1.39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
- 1.40. Find the area between $y = x^2$ and $y = x$.
- 1.41. Find the area of the region lying in the first

quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

- 1.42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.

- 1.43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$

- 1.44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.

- 1.45. Find the area bounded by the curves

$$\{(x, y) : y > x^2, y = |x|\} \quad (1.45.1)$$

- 1.46. Find the area of the region

$$\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T \mathbf{x} = 9\} \quad (1.46.1)$$

- 1.47. Find the area of the circle $\mathbf{x}^T \mathbf{x} = 16$ exterior to the parabola $y^2 = 6x$.

2 QR DECOMPOSITION

2.1. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution: Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.1.1)$$

$$\beta = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (2.1.2)$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \quad (2.1.3)$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (2.1.4)$$

where

$$k_1 = \|\alpha\| \quad (2.1.5)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \quad (2.1.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (2.1.7)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (2.1.8)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (2.1.9)$$

From (2.1.3) and (2.1.4),

$$(\alpha \ \beta) = (\mathbf{u}_1 \ \mathbf{u}_2) \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.1.10) \quad 2.13. \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$(\alpha \ \beta) = \mathbf{QR} \quad (2.1.11) \quad 2.14. \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

From above we can see that \mathbf{R} is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (2.1.12)$$

Now by using equations (2.1.5) to (2.1.9)

$$k_1 = \sqrt{5} \quad (2.1.13)$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (2.1.14)$$

$$r_1 = \sqrt{5} \quad (2.1.15)$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (2.1.16)$$

$$k_2 = \sqrt{5} \quad (2.1.17)$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \quad (2.1.18)$$

$$2.2. \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$2.3. \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$$

$$2.4. \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$$

$$2.5. \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$$

$$2.6. \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

$$2.7. \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

$$2.8. \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$$

$$2.9. \begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$$

$$2.10. \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$$

$$2.11. \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$$

$$2.12. \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$$

$$2.15. \text{ Find QR decomposition of } \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$$

Solution: Let \mathbf{a} and \mathbf{b} be the column vectors of the given matrix.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (2.15.1)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (2.15.2)$$

The column vectors can be expressed as follows,

$$\mathbf{a} = k_1 \mathbf{u}_1 \quad (2.15.3)$$

$$\mathbf{b} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (2.15.4)$$

Here,

$$k_1 = \|\mathbf{a}\| \quad (2.15.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (2.15.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.15.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (2.15.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (2.15.9)$$

The (2.15.3) and (2.15.4) can be written as,

$$(\mathbf{a} \ \mathbf{b}) = (\mathbf{u}_1 \ \mathbf{u}_2) \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.15.10)$$

$$(\mathbf{a} \ \mathbf{b}) = \mathbf{QR} \quad (2.15.11)$$

Now, \mathbf{R} is an upper triangular matrix and also,

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (2.15.12)$$

Now using equations (2.15.5) to (2.15.9) we

get,

$$k_1 = \sqrt{2^2 + 3^2} = \sqrt{13} \quad (2.15.13)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (2.15.14)$$

$$r_1 = \left(\frac{2}{\sqrt{13}} \quad \frac{3}{\sqrt{13}} \right) \begin{pmatrix} 3 \\ -4 \end{pmatrix} = -\frac{6}{\sqrt{13}} \quad (2.15.15)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.15.16)$$

$$k_2 = \left(\frac{3}{\sqrt{13}} \quad -\frac{2}{\sqrt{13}} \right) \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{17}{\sqrt{13}} \quad (2.15.17)$$

Thus putting the values from (2.15.13) to (2.15.17) in (2.15.11) we obtain QR decomposition,

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \sqrt{13} & -\frac{6}{\sqrt{13}} \\ 0 & \frac{17}{\sqrt{13}} \end{pmatrix} \quad (2.15.18)$$

2.16. Find the QR decomposition of $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

Solution:

Let \mathbf{c}_1 and \mathbf{c}_2 be the column vectors of the given matrix.

$$\mathbf{c}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.16.1)$$

$$\mathbf{c}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad (2.16.2)$$

The column vectors can be represented as,

$$\mathbf{c}_1 = k_1 \mathbf{u}_1 \quad (2.16.3)$$

$$\mathbf{c}_2 = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (2.16.4)$$

where,

$$k_1 = \|\mathbf{c}_1\| \quad (2.16.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{c}_1}{k_1} \quad (2.16.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{c}_2}{\|\mathbf{u}_1\|^2} \quad (2.16.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{c}_2 - r_1 \mathbf{u}_1}{\|\mathbf{c}_2 - r_1 \mathbf{u}_1\|} \quad (2.16.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{c}_2 \quad (2.16.9)$$

From (2.16.3) and (2.16.4),

$$\begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.16.10)$$

$$\begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (2.16.11)$$

Where \mathbf{R} is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (2.16.12)$$

Using equations (2.16.5) to (2.16.9) we get,

$$k_1 = \sqrt{3^2 + 1^2} = \sqrt{10} \quad (2.16.13)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \quad (2.16.14)$$

$$r_1 = \left(\frac{3}{\sqrt{10}} \quad \frac{1}{\sqrt{10}} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \sqrt{10} \quad (2.16.15)$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{-1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \quad (2.16.16)$$

$$k_2 = \left(\frac{-1}{\sqrt{10}} \quad \frac{3}{\sqrt{10}} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \sqrt{10} \quad (2.16.17)$$

Now putting the values from (2.16.13) to (2.16.17), we obtain the QR decomposition of given matrix,

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \sqrt{10} \\ 0 & \sqrt{10} \end{pmatrix} \quad (2.16.18)$$

2.17. Find QR decomposition of $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$\mathbf{A} = \mathbf{Q} \mathbf{R} \quad (2.17.1)$$

where \mathbf{Q} is an orthogonal matrix and \mathbf{R} is an upper triangular matrix Given

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} \quad (2.17.2)$$

Let \mathbf{a} and \mathbf{b} be the column vectors of the given matrix

$$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (2.17.3)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.17.4)$$

The above column vectors (2.17.3) ,(2.17.4) can be expressed as ,

$$\mathbf{a} = t_1 \mathbf{u}_1 \quad (2.17.5)$$

$$\mathbf{b} = s_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 \quad (2.17.6)$$

Where,

$$t_1 = \|\mathbf{a}\| \quad (2.17.7)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{t_1} \quad (2.17.8)$$

$$s_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.17.9)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - s_1 \mathbf{u}_1}{\|\mathbf{b} - s_1 \mathbf{u}_1\|} \quad (2.17.10)$$

$$t_2 = \mathbf{u}_2^T \mathbf{b} \quad (2.17.11)$$

The (2.17.5) and (2.17.6) can be written as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} t_1 & s_1 \\ 0 & t_2 \end{pmatrix} \quad (2.17.12)$$

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (2.17.13)$$

Here, \mathbf{R} is an upper triangular matrix and \mathbf{Q} is an orthogonal matrix such that

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (2.17.14)$$

Now using equations from (2.17.7) to (2.17.11) we get,

$$t_1 = \sqrt{4^2 + 5^2} = \sqrt{41} \quad (2.17.15)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{41}} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (2.17.16)$$

$$s_1 = \left(\frac{4}{\sqrt{41}} \quad \frac{5}{\sqrt{41}} \right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{2}{\sqrt{41}} \quad (2.17.17)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{41}} \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (2.17.18)$$

$$t_2 = \left(\frac{5}{\sqrt{41}} \quad \frac{-4}{\sqrt{41}} \right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{23}{\sqrt{41}} \quad (2.17.19)$$

Substituting the values from (2.17.15) to (2.17.19) in (2.17.13) we obtain QR decomposition as,

$$\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \\ \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}} \end{pmatrix} \begin{pmatrix} \sqrt{41} & \frac{2}{\sqrt{41}} \\ 0 & \frac{23}{\sqrt{41}} \end{pmatrix} \quad (2.17.20)$$

2.18. Perform the QR decomposition of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \quad (2.18.1)$$

Solution:

If α and β are the columns of a (2×2) matrix \mathbf{A} , then \mathbf{A} can be decomposed as

$$\mathbf{A} = \mathbf{Q} \mathbf{R} \quad (2.18.2)$$

$$\text{where, } \mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}, \quad (2.18.3)$$

$$\text{uppertriangular matrix } \mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.18.4)$$

$$k_1 = \|\alpha\|, \mathbf{u}_1 = \frac{\alpha}{k_1} \quad (2.18.5)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (2.18.6)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|}, k_2 = \mathbf{u}_2^T \beta \quad (2.18.7)$$

$$\alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.18.8)$$

$$\text{From, (2.18.5), } k_1 = \|\alpha\| = \sqrt{10} \quad (2.18.9)$$

$$\text{and } \mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.18.10)$$

$$\text{From (2.18.6), } r_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.11)$$

$$\beta - r_1 \mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{5}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.18.12)$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (2.18.13)$$

$$\text{From (2.18.7), } \mathbf{u}_2 = \frac{\begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{pmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4}}} \quad (2.18.14)$$

$$\Rightarrow \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}, \quad (2.18.15)$$

$$k_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.16)$$

Note that,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (2.18.17)$$

The matrix \mathbf{A} can now be rewritten using (2.18.2) as

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{5}{\sqrt{10}} \\ 0 & \frac{5}{\sqrt{10}} \end{pmatrix} \quad (2.18.18)$$

2.19. Find the QR decomposition of the given matrix.

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \quad (2.19.1)$$

Solution: QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \quad (2.19.2)$$

where \mathbf{Q} is an orthogonal matrix and \mathbf{R} is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \quad (2.19.3)$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (2.19.4)$$

Equation (2.19.3) can be written in form of (2.19.4) as,

$$(\mathbf{a} \quad \mathbf{b}) = (\mathbf{q}_1 \quad \mathbf{q}_2) \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} = \mathbf{QR} \quad (2.19.5)$$

where,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (2.19.6)$$

$$\mathbf{q}_1 = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.19.7)$$

$$u_3 = \frac{\mathbf{q}_1^T \mathbf{b}}{\|\mathbf{q}_1\|^2} = \left(\frac{1}{\sqrt{5}} \quad \frac{2}{\sqrt{5}} \right) \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{-2}{\sqrt{5}} \quad (2.19.8)$$

$$\mathbf{q}_2 = \frac{\mathbf{b} - u_3 \mathbf{q}_1}{\|\mathbf{b} - u_3 \mathbf{q}_1\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (2.19.9)$$

$$u_2 = \mathbf{q}_2^T \mathbf{b} = \left(\frac{2}{\sqrt{5}} \quad -\frac{1}{\sqrt{5}} \right) \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{6}{\sqrt{5}} \quad (2.19.10)$$

Substituting equation (2.19.6) to (2.19.10) in (2.19.5),

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{5}} \end{pmatrix} \quad (2.19.11)$$

The QR decomposition is,

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{5}} \end{pmatrix} \quad (2.19.12)$$

3 SINGULAR VALUE DECOMPOSITION

3.1. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.1.1)$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.1.2)$$

Solution:

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.1.3)$$

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.1.4)$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (3.1.5)$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{USV}^T \quad (3.1.6)$$

Where the columns of \mathbf{V} are the eigenvectors of $\mathbf{A}^T \mathbf{A}$, the columns of \mathbf{U} are the eigenvectors of $\mathbf{A} \mathbf{A}^T$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T \mathbf{A}$.

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \quad (3.1.7)$$

$$\mathbf{M} \mathbf{M}^T = \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \quad (3.1.8)$$

Calculating eigen value of $\mathbf{M}^T \mathbf{M}$.

$$\begin{vmatrix} 6 - \lambda & 13 \\ 13 & 38 - \lambda \end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \quad (3.1.9)$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22 \quad (3.1.10)$$

Eigen vectors of $\mathbf{M}\mathbf{M}^T$.

$$\begin{vmatrix} 13-\lambda & -17 & 8 \\ 17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \quad (3.1.11)$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \quad (3.1.12)$$

Hence, The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{5} \\ \frac{-3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \quad (3.1.13)$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12-\sqrt{17}}{5}\right)^2 + \left(\frac{1-3\sqrt{17}}{5}\right)^2 + 1^2} \quad (3.1.14)$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} \end{pmatrix} \quad (3.1.15)$$

$$(3.1.16)$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \quad (3.1.17)$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+30\sqrt{7}}} \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \quad (3.1.18)$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} \\ \frac{5}{\sqrt{340+30\sqrt{17}}} \end{pmatrix} \quad (3.1.19)$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \quad (3.1.20)$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}} \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \quad (3.1.21)$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \quad (3.1.22)$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \quad (3.1.23)$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \quad (3.1.24)$$

Now, $\mathbf{V} = \mathbf{M}^T \frac{\mathbf{u}_i}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (3.1.25)$$

So, from equation (3.1.6)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} =$$

(3.1.26)

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix}$$

(3.1.27)

$$\begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix}$$

(3.1.28)

$$\begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}^T$$

(3.1.29)

Now, Finding Moore-Penrose Pseudo inverse of \mathbf{S}

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \quad (3.1.30)$$

We, know that, $\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T \mathbf{b}))$

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix} \quad (3.1.31)$$

$$\mathbf{S}_+(\mathbf{U}^T \mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (3.1.32)$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (3.1.33)$$

$$\begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (3.1.34)$$

$$\mathbf{x} = \begin{pmatrix} \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \\ \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \end{pmatrix} \quad (3.1.35)$$

Simplifying the values of x_1 and x_2

$$x_2 = \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \quad (3.1.36)$$

$$= \frac{-702100}{591700} \quad (3.1.37)$$

$$= -\frac{7}{59} \quad (3.1.38)$$

$$x_1 = \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \quad (3.1.39)$$

$$= \frac{2507500}{591700} \quad (3.1.40)$$

$$= \frac{25}{59} \quad (3.1.41)$$

Now, Verifying the values using

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (3.1.42)$$

Solving R.H.S

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.1.43)$$

Now using equation (3.1.7) in (3.1.43)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.1.44)$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xleftrightarrow{R_2 - \frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \quad (3.1.45)$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \quad (3.1.46)$$

$$6x_1 + 13x_2 = 1 \quad (3.1.47)$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \quad (3.1.48)$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \quad (3.1.49)$$