



Geometry through Linear Algebra



G V V Sharma*

CONTENTS

1	Planes and Lines	1
1.1	Distance from a plane to a point	1

Abstract—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.

1 PLANES AND LINES

1.1 Distance from a plane to a point

1.1.1. Find the foot of perpendicular from the point

$$\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \text{ on the plane } (3 \ 2 \ -6)\mathbf{x} = 2.$$

Solution: Consider orthogonal vectors \mathbf{m}_1 and \mathbf{m}_2 to the given normal vector \mathbf{n} . Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.1.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \quad (1.1.1.2)$$

$$\Rightarrow 3a + 2b - 6c = 0 \quad (1.1.1.3)$$

Let $a=1$ and $b=0$ we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.4)$$

Let $a=0$ and $b=1$ we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \quad (1.1.1.5)$$

Solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.1.6)$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.1.7)$$

Solving (1.1.1.7) using Singular Value Decomposition on \mathbf{M} as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (1.1.1.8)$$

Where the columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T \mathbf{M}$, the columns of \mathbf{U} are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$. We

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

have,

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{5}{6} & \frac{1}{9} \\ \frac{4}{6} & \frac{10}{9} \end{pmatrix} \quad (1.1.1.9)$$

$$\mathbf{M} \mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \quad (1.1.1.10)$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.1.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V} \mathbf{\Sigma}^{-1} \mathbf{U}^T \mathbf{b} \quad (1.1.1.12)$$

Where $\mathbf{\Sigma}^{-1}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$ and is obtained by inverting only non-zero elements in $\mathbf{\Sigma}$

Calculating eigen values of $\mathbf{M} \mathbf{M}^T$,

$$|\mathbf{M} \mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (1.1.1.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} - \lambda \end{vmatrix} = 0 \quad (1.1.1.14)$$

$$\Rightarrow \lambda^3 - \frac{85}{36} \lambda^2 + \frac{49}{36} \lambda = 0 \quad (1.1.1.15)$$

From the characteristic equation (1.1.1.15), the eigen values of $\mathbf{M} \mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.1.16)$$

The eigen vectors of $\mathbf{M} \mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.1.1.17)$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{1}{7} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-7}{12} \\ \frac{18}{7} \\ \frac{1}{6} \end{pmatrix} \quad (1.1.1.18)$$

Hence we obtain \mathbf{U} as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{18}{7} \\ \frac{1}{7} & 0 & \frac{1}{6} \end{pmatrix} \quad (1.1.1.19)$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.1.20)$$

Calculating eigen values of $\mathbf{M}^T \mathbf{M}$,

$$|\mathbf{M}^T \mathbf{M} - \lambda \mathbf{I}| = 0 \quad (1.1.1.21)$$

$$\Rightarrow \begin{vmatrix} \frac{5}{4} - \lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} - \lambda \end{vmatrix} = 0 \quad (1.1.1.22)$$

$$\Rightarrow \lambda^2 - \frac{85}{36} \lambda + \frac{49}{36} = 0 \quad (1.1.1.23)$$

From the characteristic equation, the eigen values of $\mathbf{M}^T \mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad (1.1.1.24)$$

Hence the eigen vectors of $\mathbf{M}^T \mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \quad (1.1.1.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.26)$$

Hence we obtain \mathbf{V} as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.27)$$

From (1.1.1.6), the Singular Value Decomposition of \mathbf{M} is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{1}{7} & 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \quad (1.1.1.28)$$

And, the Moore-Penrose Pseudo inverse of $\mathbf{\Sigma}$ is given by,

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.1.29)$$

From (1.1.1.12) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \quad (1.1.1.30)$$

$$\Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.31)$$

$$\mathbf{x} = \mathbf{V} \Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.32)$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \implies \mathbf{M}^T \mathbf{M}\mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.1.33)$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T \mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.34)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.35)$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xrightarrow{R_1 = \frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \quad (1.1.1.36)$$

$$\xrightarrow{R_2 = R_2 - \frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{15}{45} & \frac{8}{3} \end{pmatrix} \quad (1.1.1.37)$$

$$\xrightarrow{R_2 = \frac{45}{45} R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.38)$$

$$\xrightarrow{R_1 = R_1 - \frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.39)$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.40)$$

From the equations (1.1.1.32) and (1.1.1.40), the solution \mathbf{x} is verified.

1.1.2. Solve the following

a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.1)$$

to the plane,

$$(2 \ -3 \ 1)\mathbf{x} = 0 \quad (1.1.2.2)$$

Solution: The equation of plane is given as,

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.3)$$

Hence the normal vector \mathbf{n} is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.4)$$

Let, the normal vectors \mathbf{m}_1 and \mathbf{m}_2 to the normal vector \mathbf{n} be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.5)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.6)$$

$$\implies (a \ b \ c) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.7)$$

Let, $a=0$ and $b=1$ we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.8)$$

Let, $a=1$ and $b=0$,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.9)$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.10)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.11)$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.12)$$

To solve (1.1.2.10) we perform singular value decomposition on \mathbf{M} given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.13)$$

substituting the value of \mathbf{M} from equation (1.1.2.13) to (1.1.2.10),

$$\implies \mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.14)$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+^{-1}\mathbf{U}^T \mathbf{b} \quad (1.1.2.15)$$

where, \mathbf{S}_+ is Moore-Pen-rose Pseudo-Inverse of \mathbf{S} . Columns of \mathbf{U} are eigenvectors of \mathbf{MM}^T , columns of \mathbf{V} are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.16)$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.2.17)$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \quad (1.1.2.18)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (1.1.2.19)$$

$$\therefore \lambda_1 = 14 \quad (1.1.2.20)$$

$$\lambda_2 = 1 \quad (1.1.2.21)$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.22)$$

$$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.23)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} \quad (1.1.2.24)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \quad (1.1.2.25)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (1.1.2.26)$$

Now calculating the eigenvectors corresponding to \mathbf{MM}^T

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.1.2.27)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.28)$$

Eigenvalues corresponding to \mathbf{MM}^T is,

$$|\mathbf{MM}^T - \lambda\mathbf{I}| = 0 \quad (1.1.2.29)$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\Rightarrow -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to λ_3 , λ_4 and λ_5 respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v}_4 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now \mathbf{S} corresponding to eigenvalues λ_3, λ_4

and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of \mathbf{M} as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & -\sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & -\sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\Rightarrow \mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

\therefore from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.51)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.52)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.53)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \quad (1.1.2.54)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \quad (1.1.2.55)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{7}{5} \end{pmatrix} \quad (1.1.2.56)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14} R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{7}{14} \end{pmatrix} \quad (1.1.2.57)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5} R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{7}{14} \end{pmatrix} \quad (1.1.2.58)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.59)$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.