# Geometry through Linear Algebra

**Teaching Learning Center**
IIT Hyderabad

G V V Sharma*

## Contents

*Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.

## 1 PAIR OF STRAIGHT LINES

1.1. Find the value of $h$ so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \tag{1.1.1}$$

may represent two straight lines.

**Solution:**

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \tag{1.1.2}$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{1.1.3}$$

$$f = 20 \tag{1.1.4}$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \tag{1.1.5}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

Expanding equation (1.1.5) along row 1 gives

$$\implies 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + 11 \times (\frac{31h}{2} - 132) = 0$$

$$\implies 20h^2 - 341h + \frac{2907}{2} = 0 \tag{1.1.6}$$

$$\implies \boxed{h = \frac{17}{2}} \tag{1.1.7}$$

$$\implies \boxed{h = \frac{171}{20}} \tag{1.1.8}$$

If $h = \frac{17}{2}$ or $h = \frac{171}{20}$, the equation given will represent two straight lines.

Sub $h = \frac{17}{2}$ in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \tag{1.1.9}$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{1.1.10}$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{1.1.11}$$

$$\mathbf{f} = 20 \tag{1.1.12}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \qquad (1.1.13)$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \qquad (1.1.14)$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \qquad (1.1.15)$$

$$(1.1.16)$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \qquad (1.1.17)$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \qquad (1.1.18)$$

$$\implies m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \qquad (1.1.19)$$

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \qquad (1.1.20)$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \qquad (1.1.21)$$

we know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \qquad (1.1.22)$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$
From equation (1.1.21)

$$\mathbf{n_1} = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \qquad (1.1.23)$$

$$\implies \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \qquad (1.1.24)$$

$\implies$ Equation (1.1.21) satisfies (1.1.22)

$c_1$ and $c_2$ can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \qquad (1.1.25)$$

Substituting (1.1.21) in (1.1.25), the augmented matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix}$$

$$(1.1.26)$$

$$\implies c_1 = 4, c_2 = 5 \qquad (1.1.27)$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\implies (-2x - 3y - 4)(3x - 4y - 5) = 0$$

$$\implies \boxed{(2x + 3y + 4)(3x + 4y + 5) = 0} \qquad (1.1.28)$$

Equation (1.1.28) represents equations of two straight lines.
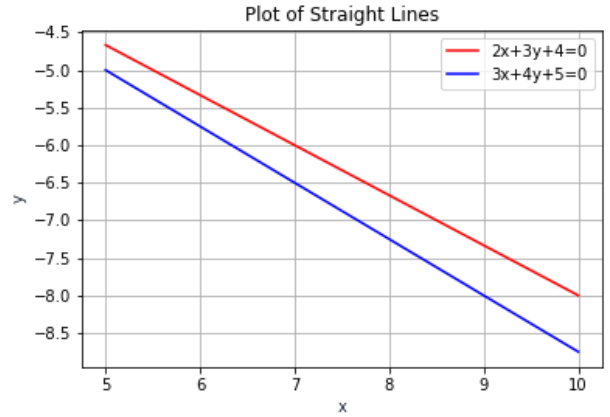


Fig. 1.1.1: Plot of Straight lines when $h = \frac{17}{2}$

Similarly, Sub $h = \frac{171}{20}$ in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \qquad (1.1.29)$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \qquad (1.1.30)$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{6} \\ \frac{310}{6} \end{pmatrix} \qquad (1.1.31)$$

$$\mathbf{f} = \frac{200}{3} \qquad (1.1.32)$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \qquad (1.1.33)$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \quad (1.1.34)$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \quad (1.1.35)$$

$$\implies m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \quad (1.1.36)$$

$$\mathbf{m_1} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (1.1.37)$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.38)$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$

From equation (1.1.38)

$$\mathbf{n_1} = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.39)$$

$$\implies \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.40)$$

$\implies$ Equation (1.1.38) satisfies (1.1.22)

$c_1$ and $c_2$ can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.1.41)$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 4R_2}{5}]{R_2 \leftarrow \frac{5R_2 - 8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \quad (1.1.42)$$

$$\implies c_1 = 10, c_2 = \frac{20}{3} \quad (1.1.43)$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\implies \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \quad (1.1.44)$$

Equation (1.1.44) represents equations of two straight lines.

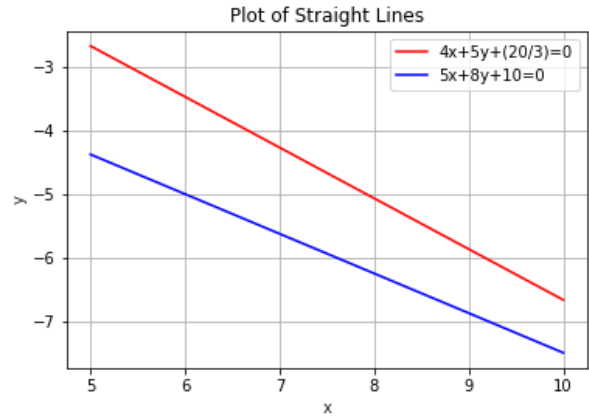1.2. Prove that the following equations represent two straight lines. Also find their point of in-



Fig. 1.1.2: Plot of Straight lines when $h = \frac{171}{20}$

tersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \quad (1.2.1)$$

**Solution:** $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \quad (1.2.2)$$

Expanding equation (1.2.2), we get zero. Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.2.3)$$

$$\implies 3m^2 - 8m - 3 = 0 \quad (1.2.4)$$

$$\text{Solving, } m = 3, -\frac{1}{3} \quad (1.2.5)$$

The normal vectors of the lines then become

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.2.6)$$

$$\mathbf{n_2} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.7)$$

Equations of the lines can therefore be written as

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = c \quad (1.2.8)$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = c_1, \quad (1.2.9)$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = c_2 \quad (1.2.10)$$

$$\implies \left[ \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - c_1 \right] \left[ \begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} - c_2 \right] \quad (1.2.11)$$

represents the equation specified in (1.2.1)

Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \tag{1.2.12}$$

$$\tag{1.2.13}$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xleftarrow{R_2 \leftarrow R_2 - 3 \times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \tag{1.2.14}$$

$$\xleftarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \tag{1.2.15}$$

$$\xleftarrow{R_1 \leftarrow R_1 + 3 \times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \tag{1.2.16}$$

$$\implies c_2 = 2 \text{ and } c_1 = -9 \tag{1.2.17}$$

The individual line equations therefore become

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = -9, \tag{1.2.18}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1.2.19}$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.2.20}$$

The LHS part of (1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.2.21}$$

The augmented matrix for the set of equations represented in (1.2.18), (1.2.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \tag{1.2.22}$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xleftarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \tag{1.2.23}$$

$$\xleftarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \tag{1.2.24}$$

$$\xleftarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \tag{1.2.25}$$

Hence, the intersection point is $\begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}$

$$\tag{1.2.26}$$

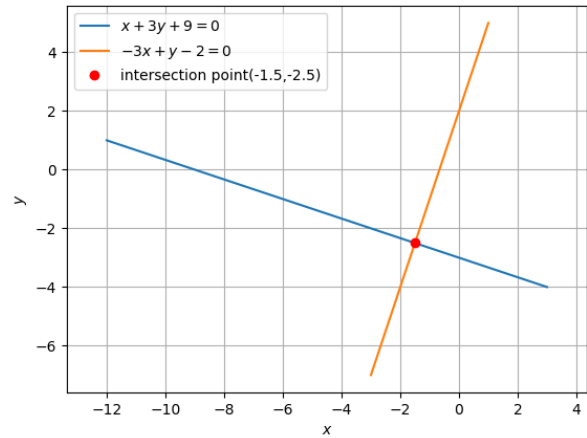Angle between two lines $\theta$ can be given by



Fig. 1.2.1: plot showing intersection of lines

$$\cos \theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} \tag{1.2.27}$$

$$\cos \theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \tag{1.2.28}$$

$$\implies \theta = 90° \tag{1.2.29}$$

1.3. Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \tag{1.3.1}$$

**Solution:**

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \qquad (1.3.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \qquad (1.3.3)$$

$$f = -2 \qquad (1.3.4)$$

$$\begin{vmatrix} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} \xleftrightarrow[R_1 \to R_1 + R_3]{R_1 \to R_1 - R_2} \begin{vmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} = 0 \qquad (1.3.5)$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$|V| < 0 \implies \begin{vmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{-9}{4} < 0 \qquad (1.3.6)$$

Let the pair of straight of lines be given by

$$\mathbf{n_1}^T \mathbf{x} = c_1 \qquad (1.3.7)$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \qquad (1.3.8)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \qquad (1.3.9)$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \qquad (1.3.10)$$

$$m_1 = 1, m_2 = -2 \qquad (1.3.11)$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} and \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad (1.3.12)$$

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \qquad (1.3.13)$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2\mathbf{u} \qquad (1.3.14)$$

$$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} & \frac{-1}{2} \end{pmatrix} \qquad (1.3.15)$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad (1.3.16)$$

Using row reduction we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \end{pmatrix} \qquad (1.3.17)$$

$$\xleftrightarrow[R_2 \leftarrow R_2 / -3]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad (1.3.18)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \qquad (1.3.19)$$

$$C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad (1.3.20)$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \qquad (1.3.21)$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \qquad (1.3.22)$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \qquad (1.3.23)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \qquad (1.3.24)$$

consider the augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \qquad (1.3.25)$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad (1.3.26)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + R_2]{R_1 \leftarrow R_1 / 3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad (1.3.27)$$

Therefore point of intersection is $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Angle between two lines $\theta$ can be given by

$$\cos \theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} \qquad (1.3.28)$$

$$\cos \theta = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} \qquad (1.3.29)$$

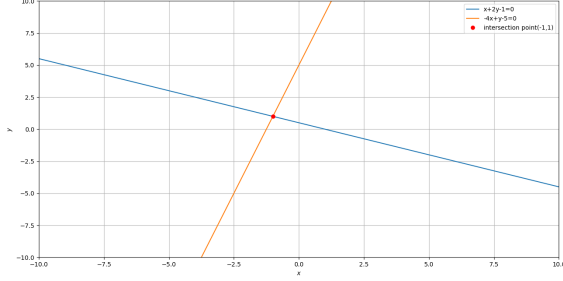$$\theta = \cos^{-1}(\frac{-1}{\sqrt{10}}) \implies \theta = tan^{-1}3 \quad (1.3.30)$$



Fig. 1.3.1: plot showing intersection of lines

1.4. **Solution:** Find the value of k such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \quad (1.4.1)$$

represent pairs of straight lines.
From (1.4.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \quad (1.4.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.4.3)$$

$$f = k \quad (1.4.4)$$

Compute the slopes of lines given by the roots of the polynomial $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.4.5)$$

$$\implies m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \quad (1.4.6)$$

$$\implies m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \quad (1.4.7)$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T\mathbf{x} = c_1 \quad (1.4.8)$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \quad (1.4.9)$$

Here,

$$\mathbf{n}_1 = k_1\begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = k_1\begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \quad (1.4.10)$$

$$\mathbf{n}_2 = k_2\begin{pmatrix} -m_2 \\ 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (1.4.11)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.4.12)$$

Substituting (1.4.10) and (1.4.11) in the above equation, we get

$$k_1\begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} * k_2\begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} \quad (1.4.13)$$

$$\implies k_1k_2 = -10 \quad (1.4.14)$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of $k_1$ and $k_2$ in (1.4.10) and (1.4.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (1.4.15)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.4.16)$$

Using Teoplitz matrix representation, the convolution of $\mathbf{n}_1$ with $\mathbf{n}_2$, is as follows:

$$\begin{pmatrix} 2 & 0 & 5 \\ 5 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.4.17)$$

Hence, $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfies (1.4.12). We have,

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\mathbf{u} \quad (1.4.18)$$

Substituting (1.4.15), (1.4.16) in (1.4.18), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.4.19)$$

Solving for $c_1$ and $c_2$, the augmented matrix is,

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & -2 & -31 \end{pmatrix} \xleftarrow[R_2 \leftarrow R_2 - 5R_1]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & \frac{-19}{2} & \frac{-57}{2} \end{pmatrix} \quad (1.4.20)$$

$$\xleftarrow[R_1 \leftarrow R_1 - \frac{3}{2}R_2]{R_2 \leftarrow \frac{R_2}{-19/2}} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.4.21)$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \quad (1.4.22)$$

We know that,

$$f = k = c_1 c_2 \qquad (1.4.23)$$

$$\implies \boxed{k = -15} \qquad (1.4.24)$$

Hence the solution. Using (1.4.8) and (1.4.9), the equation of pair of straight lines is given by,

$$\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 3 \qquad (1.4.25)$$

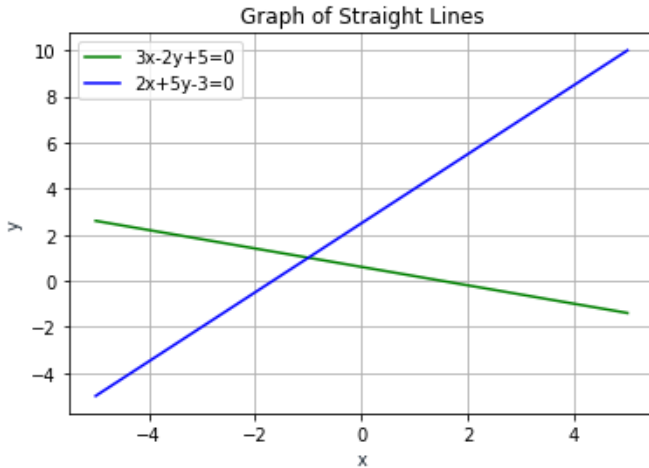$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -5 \qquad (1.4.26)$$

See Fig. 1.4.1



Fig. 1.4.1: Plot of two straight lines.

1.5. Find the value of k so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.5.1)$$

**Solution:**

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \qquad (1.5.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \qquad (1.5.3)$$

The equation (1.5.1) represents pair of straight

lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \qquad (1.5.4)$$

$$\implies \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \qquad (1.5.5)$$

$$\implies \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \qquad (1.5.6)$$

$$\implies 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11 \begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \qquad (1.5.7)$$

$$\implies 2k^2 + 55k + 350 = 0 \qquad (1.5.8)$$

$$\implies (10 + k)(2k + 35) = 0 \qquad (1.5.9)$$

$$\implies k = -10$$

$$k = -\frac{35}{2} \qquad (1.5.10)$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.
Now Lets find equation of lines for $k = -10$.
Substitute $k = -10$ in (1.5.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \qquad (1.5.11)$$

From (1.5.1), (1.5.2), (1.5.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \qquad (1.5.12)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \qquad (1.5.13)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \qquad (1.5.14)$$

$$\implies |\mathbf{V}| = -1 \qquad (1.5.15)$$

$$\implies |\mathbf{V}| < 0 \qquad (1.5.16)$$

Therefore the lines will intersect.
The equation of two lines is given by

$$\mathbf{n_1}^T \mathbf{x} = c_1 \qquad (1.5.17)$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \qquad (1.5.18)$$

Equating their product with (1.5.1)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2)$$
$$= \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (1.5.19)$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.5.20)$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = -2\begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.5.21)$$

$$c_1 c_2 = f = 2 \quad (1.5.22)$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.5.23)$$
$$\implies 2m^2 - 10m + 12 = 0 \quad (1.5.24)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.5.25)$$

$$\implies m_i = \frac{5 \pm \sqrt{1}}{2} \quad (1.5.26)$$

$$\implies m_1 = 3 \quad (1.5.27)$$
$$m_2 = 2 \quad (1.5.28)$$

The normal vector for two lines is given by

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.5.29)$$

$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.5.30)$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.5.31)$$

Substituting (1.5.30),(1.5.31) in (1.5.20). we get

$$k_1 k_2 = 2 \quad (1.5.32)$$

The possible combinations of $(k_1,k_2)$ are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1, k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.5.33)$$

$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (1.5.34)$$

We verify obtained $\mathbf{n_1},\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.5.35)$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.36)$$

Therefore the obtained $\mathbf{n_1},\mathbf{n_2}$ are correct. Substitute (1.5.33), (1.5.34) in (1.5.21) and calculate for $c_1$ and $c_2$

$$c_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.5.37)$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.5.38)$$

$$\xrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.5.39)$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.5.40)$$

$$\implies \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \quad (1.5.41)$$

$$\implies c_1 = -13 \quad (1.5.42)$$
$$c_2 = 21 \quad (1.5.43)$$

Substituting (1.5.33),(1.5.34),(1.5.42),(1.5.43) in (1.5.17) and (1.5.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} = -13 \quad (1.5.44)$$
$$\begin{pmatrix} -4 & 2 \end{pmatrix}\mathbf{x} = 21 \quad (1.5.45)$$

The plot of these two lines is shown in Fig. 1.5.1.
Now Lets find equation of lines for $k = -\frac{35}{2}$.
Substitute $k = -\frac{35}{2}$ in (1.5.1). We get equation of pair of straight lines as:

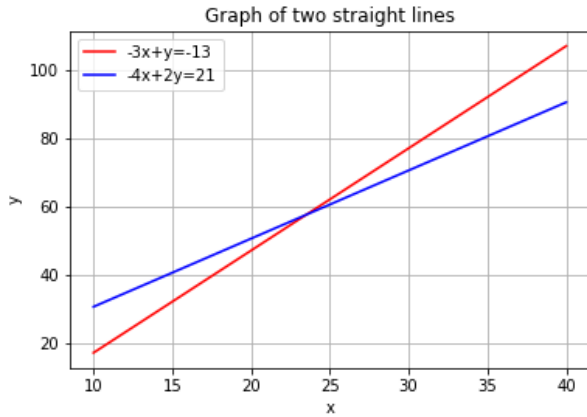$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0$$
$$(1.5.46)$$

Fig. 1.5.1: Pair of straight lines for $k = -10$

From (1.5.1), (1.5.2), (1.5.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \quad (1.5.47)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.5.48)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \quad (1.5.49)$$

$$\implies |\mathbf{V}| = -\frac{841}{16} \quad (1.5.50)$$

$$\implies |\mathbf{V}| < 0 \quad (1.5.51)$$

Therefore the lines will intersect.
Now from (1.5.20),

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (1.5.52)$$

The slopes of the lines are given by roots of equation (1.5.23)

$$\implies 2m^2 - \frac{35}{2}m + 12 = 0 \quad (1.5.53)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.5.54)$$

$$\implies m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \quad (1.5.55)$$

$$\implies m_1 = 8 \quad (1.5.56)$$

$$m_2 = \frac{3}{4} \quad (1.5.57)$$

The normal vector for two lines is given by (1.5.29)

$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.5.58)$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.5.59)$$

Substituting (1.5.58),(1.5.59) in (1.5.52). we get

$$k_1 k_2 = 2 \quad (1.5.60)$$

The possible combinations of $(k_1,k_2)$ are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1, k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.5.61)$$

$$\mathbf{n_2} = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \quad (1.5.62)$$

We verify obtained $\mathbf{n_1},\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (1.5.63)$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.64)$$

Therefore the obtained $\mathbf{n_1},\mathbf{n_2}$ are correct.
Substitute (1.5.61), (1.5.62) in (1.5.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.5.65)$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.5.66)$$

$$\xleftarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.5.67)$$

$$\xleftarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.5.68)$$

$$\xleftarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.5.69)$$

$$\implies \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \quad (1.5.70)$$

$$\implies c_1 = -\frac{102}{29} \quad (1.5.71)$$

$$c_2 = \frac{59}{29} \quad (1.5.72)$$

Substituting (1.5.61),(1.5.62),(1.5.71),(1.5.72) in (1.5.17) and (1.5.18). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \quad (1.5.73)$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \quad (1.5.74)$$

1.6. Find the value of $k$ so that the following equation may represent a pair of straight lines -

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \quad (1.6.1)$$

**Solution:** The given second degree equation is, Comparing coefficients of (1.6.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \quad (1.6.2)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.6.3)$$

$$f = -35 \quad (1.6.4)$$

The given second degree equation (1.6.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \quad (1.6.5)$$

Expanding the determinant,

$$k + 12 = 0 \quad (1.6.6)$$

$$\implies k = -12 \quad (1.6.7)$$

Hence, from (1.6.7) we find that for $k = -12$, the given second degree equation (1.6.1) represents pair of straight lines. For the appropriate value of $k$, (1.6.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \quad (1.6.8)$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n_1}^T \mathbf{x} = c_1 \quad (1.6.9)$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \quad (1.6.10)$$

The pair of straight lines is given by,

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.6.11)$$

Putting the values of $\mathbf{V}$ and $\mathbf{u}$ we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \quad (1.6.12)$$

Hence, from (1.6.12) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.6.13)$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.6.14)$$

$$c_1 c_2 = -35 \quad (1.6.15)$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \quad (1.6.16)$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.6.17)$$

$$\mathbf{n_i} = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.6.18)$$

Substituting the values in above equations (1.6.16) we get,

$$-12m^2 + m + 6 = 0 \quad (1.6.19)$$

$$\implies m_i = \frac{-\frac{1}{2} \pm \sqrt{-(-\frac{289}{4})}}{-12} \quad (1.6.20)$$

Solving equation (1.6.20) we get ,

$$m_1 = -\frac{2}{3} \qquad (1.6.21)$$

$$m_2 = \frac{3}{4} \qquad (1.6.22)$$

Hence putting the values of $m_1$ and $m_2$ in (1.6.18) we get

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \qquad (1.6.23)$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \qquad (1.6.24)$$

Putting values of $\mathbf{n_1}$ and $\mathbf{n_2}$ in (1.6.13) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \qquad (1.6.25)$$

$$\implies \begin{pmatrix} -\frac{1}{2}k_1 k_2 \\ -\frac{1}{12}k_1 k_2 \\ k_1 k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \qquad (1.6.26)$$

Thus, from (1.6.26), $k_1 k_2 = -12$. Possible combinations of $(k_1, k_2)$ are (6,-2), (-6,2), (3,-4), (-3,4) Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad (1.6.27)$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \qquad (1.6.28)$$

From equation (1.6.14) we get

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \qquad (1.6.29)$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \qquad (1.6.30)$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \qquad (1.6.31)$$

$$3c_2 - 4c_1 = -43 \qquad (1.6.32)$$

The augmented matrix of (1.6.31) ,(1.6.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} \xleftarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix}$$
$$(1.6.33)$$

$$\xleftarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix}$$
$$(1.6.34)$$

$$\xleftarrow{R_2 = -\frac{2}{17}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \qquad (1.6.35)$$

$$\xleftarrow{R_1 = R_1 - \frac{3}{2}R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix}$$
$$(1.6.36)$$

$$(1.6.37)$$

Hence we get,

$$c_1 = -5 \qquad (1.6.38)$$

$$c_2 = 7 \qquad (1.6.39)$$

Hence (1.6.9), (1.6.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \qquad (1.6.40)$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \qquad (1.6.41)$$

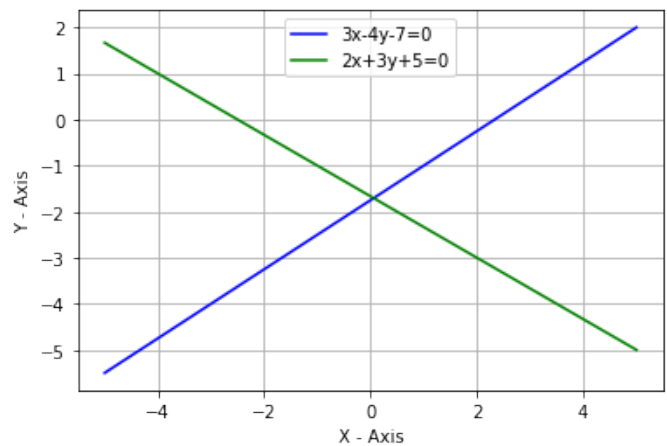The figure below corresponds to the pair of straight lines represented by (1.6.40) and (1.6.41).



Fig. 1.6.1: Pair of Straight Lines

1.7. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \qquad (1.7.1)$$

represent pairs of straight lines. **Solution:**

From (1.7.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \quad (1.7.2)$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \quad (1.7.3)$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & k \end{vmatrix} = 0 \quad (1.7.4)$$

$$\implies \left( k - \left( \frac{49}{4} \right) \right) - \frac{5}{3} \left( \frac{5}{3}k - \frac{35}{4} \right)$$

$$- \frac{5}{2} \left( \frac{-35}{6} + \frac{5}{2} \right) = 0 \quad (1.7.5)$$

$$\implies \frac{64}{k}36 - \frac{128}{12} = 0 \quad (1.7.6)$$

$$\implies \boxed{k = 6} \quad (1.7.7)$$

Substituting (1.7.7) in (1.7.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \quad (1.7.8)$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (1.7.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & 6 \end{vmatrix} \quad (1.7.9)$$

Simplyfying the above determinant , we get

$$\delta = 0 \quad (1.7.10)$$

(1.7.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \quad (1.7.11)$$

Since $\det(V) < 0$ lines would intersect each other

$$\mathbf{n_1} * \mathbf{n_2} = \{1, \frac{10}{3}, 1\} \quad (1.7.12)$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \quad (1.7.13)$$

$$c_1c_2 = 6 \quad (1.7.14)$$
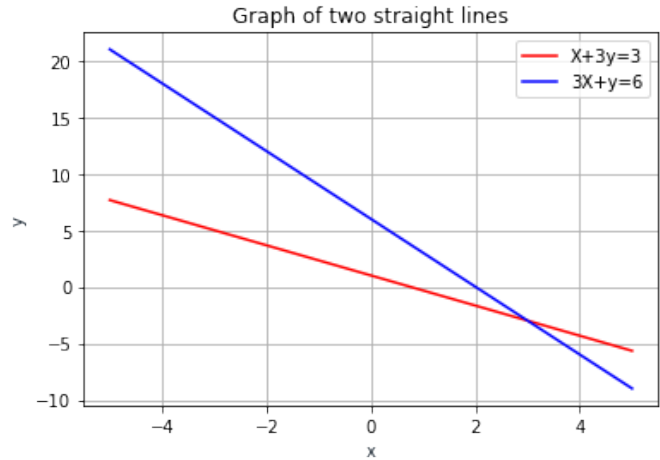
The slopes of the lines are given by the roots

of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.7.15)$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.7.16)$$

$$\mathbf{n_i} = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.7.17)$$

Substituting in above equations (1.7.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \quad (1.7.18)$$

$$\implies m_i = \frac{\frac{-10}{3} \pm \sqrt{-(\frac{-16}{9})}}{1} \quad (1.7.19)$$

Solving equation (1.7.19) we have ,

$$m_1 = \frac{-1}{3} \quad (1.7.20)$$

$$m_2 = -3 \quad (1.7.21)$$

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.7.22)$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.7.23)$$

Substituting equations (1.7.22), (1.7.23) in equation (1.7.12) we get

$$k_1k_2 = 1 \quad (1.7.24)$$

Possible combination of $(k_1, k_2)$ is (1,1) Lets



Fig. 1.7.1: Pair of straight lines

assume $k_1 = 1$, $k_2 = 1$, we get

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \qquad (1.7.25)$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad (1.7.26)$$

we have:

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \qquad (1.7.27)$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a teoplitz matrix and multiplying with $\mathbf{n_2}$

From equation (1.7.25) and (1.7.26)

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad (1.7.28)$$

$$\implies \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \qquad (1.7.29)$$

$c_1$ and $c_2$ can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \qquad (1.7.30)$$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \qquad (1.7.31)$$

Substituting (1.7.25) and (1.7.26) in (1.7.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xleftarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \qquad (1.7.32)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xleftarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \qquad (1.7.33)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xleftarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \qquad (1.7.34)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xleftarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \qquad (1.7.35)$$

From above we get

$$c_1 = 1 \qquad (1.7.36)$$

$$c_2 = 6 \qquad (1.7.37)$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \qquad (1.7.38)$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \qquad (1.7.39)$$

1.8. Find the value of k so that the following equation may represent the pair of staright lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \qquad (1.8.1)$$

**Solution:** We need to find the value of k for which (1.8.1) represents a pair of straight lines. Converting (1.8.1) into vector form, we get

$$\mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 = 0 \quad (1.8.2)$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \qquad (1.8.3)$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \qquad (1.8.4)$$

$$f = -9 \qquad (1.8.5)$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \qquad (1.8.6)$$

Since (1.8.1) represents a pair of straight lines, then by (1.8.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \qquad (1.8.7)$$

By solving, above determinant we get

$$2(9-9) + \frac{-1}{2}(\frac{-9}{2} + \frac{-3k}{2}) + \frac{k}{2}(\frac{3}{2} + \frac{k}{2}) = 0 \qquad (1.8.8)$$

$$\frac{(9+3k)}{4} + \frac{k(3+k)}{4} = 0 \qquad (1.8.9)$$

$$k^2 + 6k + 9 = 0 \qquad (1.8.10)$$

$$(k+3)^2 = 0 \qquad (1.8.11)$$

$$k = -3 \qquad (1.8.12)$$

Hence by (1.8.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \qquad (1.8.13)$$

represents family of straight lines for $k = -3$. To find the staright lines, we write each of thrm

in their vector form as

$$\mathbf{n_1}^T\mathbf{x} = c_1 \qquad (1.8.14)$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \qquad (1.8.15)$$

Equating the product of above with (1.8.2), we have

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) =$$
$$\mathbf{x}^T\begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} k/2 \\ 3 \end{pmatrix}\mathbf{x} - 9 \quad (1.8.16)$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad (1.8.17)$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_1} = -2\begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \qquad (1.8.18)$$

$$c_1 c_2 = -9 \qquad (1.8.19)$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \qquad (1.8.20)$$

$$m^2 - m - 2 = 0 \qquad (1.8.21)$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} \qquad (1.8.22)$$

$$m_1 = \frac{1+3}{2} = 2 \qquad (1.8.23)$$

$$m_2 = \frac{1-3}{2} = -1 \qquad (1.8.24)$$

$$n_1 = k_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad (1.8.25)$$

$$n_2 = k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (1.8.26)$$

Substituing (1.8.25) and (1.8.26) in (1.8.17), we get

$$k_1 k_2 = -1 \qquad (1.8.27)$$

Taking $k_1 = -1$ and $k_2 = 1$, we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad (1.8.28)$$

$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (1.8.29)$$

Substituting in (1.8.18) for above values of $n_1$

and $n_2$

$$\begin{pmatrix} n_1 n_2 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \qquad (1.8.30)$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \qquad (1.8.31)$$

Solving (1.8.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xleftrightarrow{r_2 = r_2 + 2r_1}$$
$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \quad (1.8.32)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xleftrightarrow{r_2 = r_2/3}$$
$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad (1.8.33)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \xleftrightarrow{r_1 = r_1 - r_2}$$
$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \quad (1.8.34)$$

Hence, we found out

$$c_1 = -3 \qquad (1.8.35)$$

$$c_2 = 3 \qquad (1.8.36)$$

Thus, pair of staright lines are

$$\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = -3 \qquad (1.8.37)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \qquad (1.8.38)$$

where

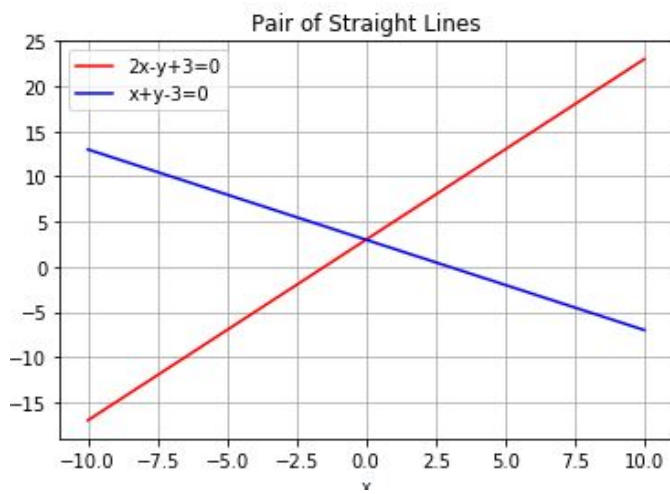$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad (1.8.39)$$

The plot of above is shown below

Fig. 1.8.1: Pair of Straight Lines