Geometry through Linear Algebra

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Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.**

1 Planes and Lines

1.1 Distance from a plane to a point

1.1.1. Solve the following

a) Find the foot of perpendicular from the point $\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ on the plane $\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\mathbf{x} = 2$.

Solution: Consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.1.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \tag{1.1.1.2}$$

$$\implies 3a + 2b - 6c = 0 \tag{1.1.1.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.1.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \tag{1.1.1.5}$$

Solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.1.6}$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \tag{1.1.1.7}$$

Solving (1.1.1.7) using Singular Value De-

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composition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U\Sigma V}^T \tag{1.1.1.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. We have,

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \tag{1.1.1.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \tag{1.1.1.10}$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U\Sigma V}^T\mathbf{x} = \mathbf{b} \tag{1.1.1.11}$$

$$\implies \mathbf{x} = \mathbf{V\Sigma^{-1}U^Tb} \tag{1.1.1.12}$$

Where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$ and is obtained by inversing only non-zero elements in $\mathbf{\Sigma}$
Calculating eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.13}$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda \end{vmatrix} = 0 \tag{1.1.1.14}$$

$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \tag{1.1.1.15}$$

From the characteristic equation (1.1.1.15), the eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{1.1.1.16}$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{3} \\ 1 \end{pmatrix} \tag{1.1.1.17}$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-7}{12} \\ \frac{-7}{18} \\ \frac{7}{6} \end{pmatrix} \tag{1.1.1.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \tag{1.1.1.19}$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.1.20}$$

Calculating eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.21}$$

$$\implies \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9}-\lambda \end{vmatrix} = 0 \tag{1.1.1.22}$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \tag{1.1.1.23}$$

From the characteristic equation, the eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \tag{1.1.1.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.1.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.27}$$

From (1.1.1.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.1.28}$$

And, the Moore-Penrose Pseudo inverse of $\mathbf{\Sigma}$ is given by,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.1.29}$$

From (1.1.1.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \quad (1.1.1.30)$$

$$\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.31)$$

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.32)$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{Mx} = \mathbf{b} \implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \quad (1.1.1.33)$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T\mathbf{Mx} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.34)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.35)$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xleftrightarrow{R_1=\frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \quad (1.1.1.36)$$

$$\xleftrightarrow{R_2=R_2-\frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{49}{45} & \frac{8}{3} \end{pmatrix} \quad (1.1.1.37)$$

$$\xleftrightarrow{R_2=\frac{45}{49}R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.38)$$

$$\xleftrightarrow{R_1=R_1-\frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.39)$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.40)$$

From the equations (1.1.1.32) and (1.1.1.40), the solution $\mathbf{x}$ is verified.

b) Find the foot of perpendicular from point $B = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ to the plane $\begin{pmatrix} 2 & 3 & -4 \end{pmatrix}\mathbf{x} = -5$.

Solution: Let us consider orthogonal vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ to the given normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then,

$$\mathbf{m}^T\mathbf{n} = 0 \quad (1.1.1.41)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \quad (1.1.1.42)$$

$$\implies 2a + 3b - 4c = 0 \quad (1.1.1.43)$$

Let $a = 1,\ \ b = 0$, so that

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.44)$$

and $a = 0,\ \ b = 1$, so that

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \quad (1.1.1.45)$$

We, now, solve the equation

$$\mathbf{Mx} = \mathbf{b} \quad (1.1.1.46)$$

which, upon substitution, becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad (1.1.1.47)$$

Any $m \times n$ matrix $\mathbf{M}$ can be factorized in SVD form as

$$\mathbf{M} = \mathbf{USV}^T \quad (1.1.1.48)$$

where $\mathbf{U}$ and $\mathbf{V}$ are matrices of eigen vectors which are orthogonal. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is the diagonal matrix of singular values of $\mathbf{M}$ of the eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \quad (1.1.1.49)$$

Putting (1.1.1.48) into (1.1.1.46), we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \quad (1.1.1.50)$$

$$\implies \mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} \quad (1.1.1.51)$$

where $\mathbf{S}_+$ is the Moore-Penrose Pseudoinverse of $\mathbf{S}$.

The eigenvalues of $\mathbf{M}^T\mathbf{M}$:

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.52}$$

$$\implies \begin{vmatrix} \frac{10}{8} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{vmatrix} = 0 \tag{1.1.1.53}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{116}{64} = 0 \tag{1.1.1.54}$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \tag{1.1.1.55}$$

$$\lambda_2 = 1 \tag{1.1.1.56}$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector $\mathbf{v_1}$ can be calculated using row reduction as :

$$\begin{pmatrix} -\frac{9}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{16}{9}R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \tag{1.1.1.57}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \tag{1.1.1.58}$$

Hence,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.59}$$

Similarly, for $\lambda_2 = 1$,

$$\mathbf{v_2} = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.60}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.61}$$

Now,

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.1.62}$$

Now, calculating eigenvalues of $\mathbf{MM}^T$

$$\begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \lambda \end{vmatrix} = 0 \tag{1.1.1.63}$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \tag{1.1.1.64}$$

$$\lambda_2 = 1 \tag{1.1.1.65}$$

$$\lambda_3 = 0 \tag{1.1.1.66}$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector can be computed as:

$$\begin{pmatrix} 1 - \frac{29}{16} & 0 & \frac{1}{2} \\ 0 & 1 - \frac{29}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \frac{29}{16} \end{pmatrix} \tag{1.1.1.67}$$

$$\leftrightarrow \begin{pmatrix} -\frac{13}{16} & 0 & \frac{1}{2} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \tag{1.1.1.68}$$

$$\xleftrightarrow{R_1 \leftarrow -\frac{16}{13}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \tag{1.1.1.69}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \tag{1.1.1.70}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{16}{13}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \tag{1.1.1.71}$$

$$\xleftrightarrow{R_2 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.72}$$

Hence, the eigen vector $\mathbf{u}_1$:

$$\mathbf{u_1} = \begin{pmatrix} \frac{8}{\sqrt{377}} \\ \frac{12}{\sqrt{377}} \\ \frac{13}{\sqrt{377}} \end{pmatrix} \tag{1.1.1.73}$$

For $\lambda_2 = 1$, the eigen vector is:

$$\begin{pmatrix} 1-1 & 0 & \frac{1}{2} \\ 0 & 1-1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - 1 \end{pmatrix} \tag{1.1.1.74}$$

$$\leftrightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{3}{16} \end{pmatrix} \tag{1.1.1.75}$$

Hence, the eigen vector $\mathbf{u}_2$:

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{1.1.1.76}$$

Similarly, for $\lambda_3 = 0$, the eigen vector is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.1.77}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1 - \frac{3}{4}R_2} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.78}$$

Hence, the eigen vector $\mathbf{u}_3$:

$$\mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.1.79}$$

So, the orthonormal matrix $\mathbf{U}$ of eigen vectors is:

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.1.80}$$

The matrix of singular values of $\mathbf{M}$ is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.81}$$

The Moore-Penrose pseudoinverse of $\mathbf{S}$ is computed as

$$\mathbf{S}_+ = (\mathbf{S}\mathbf{S}^T)^{-1}\mathbf{S}^T \tag{1.1.1.82}$$

$$= \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.1.83}$$

To solve for $\mathbf{x}$ in (1.1.1.51), noting that $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \\ 0 \end{pmatrix} \tag{1.1.1.84}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{1.1.1.85}$$

Thus, the foot of perpendicular is:

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}\begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{1.1.1.86}$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.87}$$

The solution can be verified using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.1.88}$$

The LHS gives

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.89}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.90}$$

Now, finding $\mathbf{x}$ from

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.91}$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} & -3 \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{3}{10}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \tag{1.1.1.92}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & \frac{29}{20} & \frac{58}{20} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & 1 & 2 \end{pmatrix} \tag{1.1.1.93}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.1.1.94}$$

Hence, the solution is given by

$$\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.95}$$

Comparing the results in Eq.(1.1.1.87) and (1.1.1.95), it is concluded that the solution is verified.

1.1.2. Solve the following

a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.1}$$

to the plane,

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.1.2.2}$$

Solution: The equation of plane is given as,

$$\mathbf{n}^T\mathbf{x} = c \tag{1.1.2.3}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.4}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.2.5}$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \tag{1.1.2.6}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.7}$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.8}$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.9}$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.2.10}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.11}$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.12}$$

To solve (1.1.2.10) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.2.13}$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.13) to (1.1.2.10),

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.2.14}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} \tag{1.1.2.15}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.16}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.17}$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \tag{1.1.2.18}$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \tag{1.1.2.19}$$

$$\therefore \lambda_1 = 14 \tag{1.1.2.20}$$

$$\lambda_2 = 1 \tag{1.1.2.21}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.2.22}$$

$$\mathbf{v_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.2.23}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{1.1.2.24}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.1.2.25}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \tag{1.1.2.26}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.2.27}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.28}$$

Eigenvalues corresponding to $\mathbf{MM}^T$ is,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.2.29)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\implies -\lambda(\lambda-1)(\lambda-14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v_5} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}}\begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}\begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}}\begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

$\therefore$ from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}}\begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix}\begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \tag{1.1.2.51}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.52}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.53}$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \tag{1.1.2.54}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \tag{1.1.2.55}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{17}{5} \end{pmatrix} \tag{1.1.2.56}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \tag{1.1.2.57}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \tag{1.1.2.58}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \tag{1.1.2.59}$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.

b) Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.60}$$

to the plane,

$$\tag{1.1.2.61}$$

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.1.2.62}$$

Solution: The equation of plane is give

$$\mathbf{n}^T\mathbf{x} = c \tag{1.1.2.63}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.64}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.2.65}$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \tag{1.1.2.66}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.67}$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.68}$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.69}$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.1.2.70}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.71}$$

To solve (1.1.2.70) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.1.2.72}$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.72) to (1.1.2.70),

$$\implies \mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.1.2.73}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{1.1.2.74}$$

where, $\mathbf{S}_+$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{M}\mathbf{M}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.75}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.76}$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \tag{1.1.2.77}$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \tag{1.1.2.78}$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \tag{1.1.2.79}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.2.80}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{1.1.2.81}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.1.2.82}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \tag{1.1.2.83}$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.2.84}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.85}$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.86}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \tag{1.1.2.87}$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \tag{1.1.2.88}$$

$$\implies -\lambda(\lambda - 1)(\lambda - 14) = 0 \tag{1.1.2.89}$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1 \tag{1.1.2.90}$$

$$\lambda_5 = 0 \tag{1.1.2.91}$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_5} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.92}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \tag{1.1.2.93}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{1.1.2.94}$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \tag{1.1.2.95}$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \tag{1.1.2.96}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.97}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.98}$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}}\begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}\begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \tag{1.1.2.99}$$

Now substituting the values of (1.1.2.83), (1.1.2.98), (1.1.2.96) and (1.1.2.71) in

(1.1.2.74),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.100}$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \tag{1.1.2.101}$$

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.102}$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \tag{1.1.2.103}$$

$\therefore$ from equation (1.1.2.74),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \tag{1.1.2.104}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.2.105}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.106}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.107}$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{1.1.2.108}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix} \tag{1.1.2.109}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix} \tag{1.1.2.110}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.2.111}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.2.112}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.2.113}$$

Hence from equations (1.1.2.105) and (1.1.2.113) we conclude that the solution is verified.

c) Find the foot of the perpendicular from $\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$ on the plane $\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}\mathbf{x} = 0$

Solution: Let orthogonal vectors be $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.2.114}$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.115}$$

$$\implies -5a + b + 3c = 0 \tag{1.1.2.116}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.117}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.118}$$

From (1.1.2.117) and (1.1.2.118),

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.119}$$

Now solving the equation

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.2.120}$$

Substituting the given point and (1.1.2.119) in (1.1.2.120)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.121}$$

Using the Singular value decomposition to solve (1.1.2.121) as follows,

$$\mathbf{M} = \mathbf{U \Sigma V}^T \tag{1.1.2.122}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{\Sigma}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.123}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.124}$$

Substituting (1.1.2.122) in (1.1.2.120)

$$\mathbf{U \Sigma V}^T \mathbf{x} = \mathbf{b} \tag{1.1.2.125}$$

$$\mathbf{x} = \mathbf{V \Sigma^{-1} U^T b} \tag{1.1.2.126}$$

where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$.
Now finding the eigen values of $\mathbf{MM}^T$

$$\left|\mathbf{MM}^T - \lambda \mathbf{I}\right| = 0 \tag{1.1.2.127}$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0 \tag{1.1.2.128}$$

$$\implies \lambda^3 - 15\lambda^2 + 14\lambda = 0 \tag{1.1.2.129}$$

Hence eigen values of $\mathbf{MM}^T$,

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad \lambda_3 = 0 \tag{1.1.2.130}$$

Therefore eigen vectors of $\mathbf{MM}^T$,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.131}$$

Normalizing the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.132}$$

Hence from the above we get,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{-3}{\sqrt{14}} \\ 0 & \frac{13}{\sqrt{182}} & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.133}$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 14 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.134}$$

Now calculating eigen values of $\mathbf{M}^T\mathbf{M}$

$$\left|\mathbf{M}^T\mathbf{M} - \lambda I\right| = 0 \tag{1.1.2.135}$$

$$\begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0 \tag{1.1.2.136}$$

$$\implies \lambda^2 - 15\lambda + 14 = 0 \tag{1.1.2.137}$$

hence the eigen values of $\mathbf{M}^T\mathbf{M}$

$$\lambda_1 = 1 \quad \lambda_2 = 14 \tag{1.1.2.138}$$

Therefore eigen vectors $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.2.139}$$

Normalizing the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.140}$$

Hence $\mathbf{V}$ is given as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.141}$$

Moore Pseudo inverse of Σ is,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \quad (1.1.2.142)$$

Substituting (1.1.2.133), (1.1.2.141) and (1.1.2.142) in (1.1.2.126),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{182}} & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{1}} \end{pmatrix} \quad (1.1.2.143)$$

$$\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} \quad (1.1.2.144)$$

$$\mathbf{V}\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.145)$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.146)$$

Now verifying (1.1.2.146) using (1.1.2.120)

$$\mathbf{Mx} = \mathbf{b} \implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \quad (1.1.2.147)$$

Substituting (1.1.2.119), (1.1.2.123) and given point in (1.1.2.147)

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 \\ 10 \end{pmatrix} \quad (1.1.2.148)$$

$$(1.1.2.149)$$

Solving the augmented matrix.

$$\begin{pmatrix} 5 & -6 & -11 \\ -6 & 10 & 10 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ -6 & 10 & 10 \end{pmatrix} \quad (1.1.2.150)$$

$$\xleftrightarrow{R_2=R_2+6R_1} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & \frac{14}{5} & \frac{-16}{5} \end{pmatrix} \quad (1.1.2.151)$$

$$\xleftrightarrow{R_2=\frac{5R_2}{14}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \quad (1.1.2.152)$$

$$\xleftrightarrow{R_1=R_1+\frac{6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{-25}{7} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \quad (1.1.2.153)$$

From (1.1.2.153) we get,

$$\mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.154)$$

Hence from (1.1.2.146) and (1.1.2.154) the $\mathbf{x}$ is verified

d) Find the coordinates of foot of perpendicular from $\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ to the plane

$$2x - 3y + z = 0 \quad (1.1.2.155)$$

using SVD

Solution: First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given plane $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$\implies 2a - 3b + c = 0 \quad (1.1.2.156)$$

By substituting $a = 1; b = 0$ in (1.1.2.156),

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.157)$$

By substituting $a = 0; b = 1$ in (1.1.2.156),

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.158)$$

Now $\mathbf{M}$ can be written as,

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.159)$$

such that solving $\mathbf{Mx} = \mathbf{b}$ gives the required solution.

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad (1.1.2.160)$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{USV}^T \quad (1.1.2.161)$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.162}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.163}$$

From (1.1.2.160) and (1.1.2.161),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$
$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{1.1.2.164}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^3 + 15\lambda^2 - 14\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = 14; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{1.1.2.165}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.166}$$

Normalizing the above eigenvectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.167}$$

From (1.1.2.167) we obtain $\mathbf{U}$ as,

$$\mathbf{U} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.168}$$

Using values from (1.1.2.165),

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.169}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - 15\lambda + 14 = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 14; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Normalizing the above eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.170}$$

From(1.1.2.170) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.171}$$

From (1.1.2.161) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.2.172}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.173}$$

From (1.1.2.164),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{2}}{\sqrt{91}} \\ \frac{3}{\sqrt{13}} \\ \frac{2\sqrt{2}}{7} \end{pmatrix}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.1.2.174}$$

To verify the solution obtained from (1.1.2.174),

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.175}$$

Substituting the values from (1.1.2.162) in (1.1.2.175),

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

Converting the above equation into augmented form and solving for $\mathbf{x}$,

$$\begin{pmatrix} 5 & -6 & -3 \\ -6 & 10 & 6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{5R_2+6R_1}{14}} \begin{pmatrix} 5 & -6 & -3 \\ 0 & 1 & \frac{6}{7} \end{pmatrix}$$
$$\xleftrightarrow{R_1 \leftarrow \frac{R_1+6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \tag{1.1.2.176}$$

From (1.1.2.176) it can be observed that,

$$\mathbf{x} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.1.2.177}$$

1.1.3. a) Find the foot of the perpendicular to the plane

$$2x + 3y - 2z + 4 = 0 \tag{1.1.3.1}$$

from the point $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ using SVD. **Solution:** The given plane equation is

$$\begin{pmatrix} 2 & 3 & -2 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.3.2}$$
$$\tag{1.1.3.3}$$

The equation of plane is

$$\mathbf{n}^T \mathbf{x} = c \tag{1.1.3.4}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \tag{1.1.3.5}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.3.6}$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \tag{1.1.3.7}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 0 \tag{1.1.3.8}$$

Let, a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.3.9}$$

Let, a=0 and b=1,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{1.1.3.10}$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.3.11}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.1.3.12}$$

To solve (1.1.3.11) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.3.13}$$

substituting the value of $\mathbf{M}$ from equation (1.1.3.13) to (1.1.3.11),

$$\implies \mathbf{USV}^T \mathbf{x} = \mathbf{b} \tag{1.1.3.14}$$
$$\implies \mathbf{x} = \mathbf{VS_+U}^T \mathbf{b} \tag{1.1.3.15}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$.
Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.
First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} \end{pmatrix} \tag{1.1.3.16}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left| \mathbf{M}^T\mathbf{M} - \lambda \mathbf{I} \right| = 0 \tag{1.1.3.17}$$
$$\implies \begin{pmatrix} 2-\lambda & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4}-\lambda \end{pmatrix} \tag{1.1.3.18}$$
$$\implies (\lambda - \frac{17}{4})(\lambda - 1) = 0 \tag{1.1.3.19}$$
$$\therefore \lambda_1 = \frac{17}{4}, \lambda_2 = 1, \tag{1.1.3.20}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \tag{1.1.3.21}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.1.3.22}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.3.23}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \tag{1.1.3.24}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \tag{1.1.3.25}$$

$$\implies \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{13}{4} \end{pmatrix} \tag{1.1.3.26}$$

Eigenvalues corresponding to $\mathbf{MM}^T$ is,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.3.27}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{13}{4}-\lambda \end{pmatrix} \tag{1.1.3.28}$$

$$\implies \lambda(\lambda-1)(\lambda-\frac{17}{4}) = 0 \tag{1.1.3.29}$$

$$\therefore \lambda_3 = \frac{17}{4}, \lambda_4 = 1, \lambda_5 = 0 \tag{1.1.3.30}$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{4}{13} \\ \frac{6}{13} \\ 1 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} \frac{-3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_5} = \begin{pmatrix} -1 \\ \frac{-3}{2} \\ 1 \end{pmatrix} \tag{1.1.3.31}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \begin{pmatrix} \frac{4}{\sqrt{221}} \\ \frac{6}{\sqrt{221}} \\ \frac{13}{\sqrt{221}} \end{pmatrix} \tag{1.1.3.32}$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{1.1.3.33}$$

$$\mathbf{v_5} = \begin{pmatrix} \frac{-2}{\sqrt{17}} \\ \frac{-3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix} \tag{1.1.3.34}$$

$$\implies \mathbf{U} = \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix} \tag{1.1.3.35}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\frac{17}{4}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.3.36}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.3.37}$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{17}{4}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}^T \tag{1.1.3.38}$$

Now substituting the values of (1.1.3.24), (1.1.3.37), (1.1.3.35) and (1.1.3.12) in (1.1.3.15),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.1.3.39}$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \tag{1.1.3.40}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \quad (1.1.3.41)$$

$$\implies \mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \quad (1.1.3.42)$$

$$\mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \quad (1.1.3.43)$$

$$\implies \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.1.3.44)$$

$\therefore$ from equation (1.1.3.15),

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.1.3.45)$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.3.46)$$

$$\implies \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad (1.1.3.47)$$

$$\implies \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.1.3.48)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 2 & \frac{3}{2} & 3 \\ \frac{3}{2} & \frac{13}{4} & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} & -2 \end{pmatrix} \quad (1.1.3.49)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ 0 & \frac{17}{8} & -\frac{17}{4} \end{pmatrix} \quad (1.1.3.50)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{8}{17}R_2} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ 0 & 1 & -2 \end{pmatrix} \quad (1.1.3.51)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \quad (1.1.3.52)$$

$$\implies \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.1.3.53)$$

Hence from equations (1.1.3.45) and (1.1.3.53) we conclude that the solution is verified.

a) Determine the distance from the Y-axis to the plane $5x - 2z - 3 = 0$
Solution: Equation of plane can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \quad (1.1.3.54)$$

Rewriting given equation of plane in (1.1.3.54) form

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \quad (1.1.3.55)$$

where : $\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 3$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \quad (1.1.3.56)$$

Here p is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Find two vectors that are $\perp$ to $\mathbf{n}$

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (1.1.3.57)$$

Put $a = 0$ and $b = 1$ in (1.1.3.56), $\implies c = 0$
Put $a = 1$ and $b = 0$ in (1.1.3.56), $\implies c = \frac{5}{2}$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, y = 0$ in (1.1.3.55), we get $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Since given plane is parallel to y-axis, we can use any point P on y-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.1.3.58)$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (1.1.3.57) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (1.1.3.59)$$

Equation **P** and **Q**, and computing pseudo inverse using SVD should give the value of λ_1 and λ_2 (since plane and y-axis never intersect pseudo inverse should give the points which are closest)

$$\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda_1\begin{pmatrix}1\\0\\\frac{5}{2}\end{pmatrix}+\lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \tag{1.1.3.60}$$

$$\lambda_1\begin{pmatrix}1\\0\\\frac{5}{2}\end{pmatrix}+\lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\-1\end{pmatrix} \tag{1.1.3.61}$$

$$\begin{pmatrix}1&0\\0&1\\\frac{5}{2}&0\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=\begin{pmatrix}-1\\0\\-1\end{pmatrix} \tag{1.1.3.62}$$

$$\mathbf{Mx}=\mathbf{b} \tag{1.1.3.63}$$

$$\mathbf{x}=\mathbf{M}^{+}\mathbf{b} \tag{1.1.3.64}$$

where $\mathbf{M}=\begin{pmatrix}1&0\\0&1\\\frac{5}{2}&0\end{pmatrix}$, $\mathbf{x}=\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-1\\0\\-1\end{pmatrix}$

Diagonalize $\mathbf{MM}^T$

$$\mathbf{MM}^T=\begin{pmatrix}1&0\\0&1\\\frac{5}{2}&0\end{pmatrix}\begin{pmatrix}1&0&\frac{5}{2}\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&\frac{5}{2}\\0&1&0\\\frac{5}{2}&0&\frac{25}{4}\end{pmatrix} \tag{1.1.3.65}$$

$$=\begin{pmatrix}0&\frac{2}{5}&-\frac{5}{2}\\1&0&0\\0&1&1\end{pmatrix}\begin{pmatrix}\frac{29}{4}&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\\frac{2}{5}&0&1\\-\frac{5}{2}&0&1\end{pmatrix} \tag{1.1.3.66}$$

$$=\mathbf{U}\Sigma^T\Sigma\mathbf{U}^T \tag{1.1.3.67}$$

Verify (1.1.3.66) from,

codes/diagonalize1.py

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M}=\begin{pmatrix}1&0&\frac{5}{2}\\0&1&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\\\frac{5}{2}&0\end{pmatrix}=\begin{pmatrix}\frac{29}{4}&0\\0&1\end{pmatrix} \tag{1.1.3.68}$$

$$=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&\frac{29}{4}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} \tag{1.1.3.69}$$

$$=\mathbf{V}\Sigma^T\Sigma\mathbf{V}^T \tag{1.1.3.70}$$

Verify (1.1.3.69) from,

codes/diagonalize2.py

Compute SVD of **M** from (1.1.3.66) and (1.1.3.71),

$$\mathbf{M}=\mathbf{U}\Sigma\mathbf{V}^T \tag{1.1.3.71}$$

$$\begin{pmatrix}1&0\\0&1\\\frac{5}{2}&0\end{pmatrix}=\begin{pmatrix}0&\frac{2}{5}&-\frac{5}{2}\\1&0&0\\0&1&1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{29}}{2}&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{1.1.3.72}$$

$$\mathbf{M}^{+}=\mathbf{V}\Sigma^T\mathbf{U}^T \tag{1.1.3.73}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{29}}{2}&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\\frac{2}{5}&0&1\\-\frac{5}{2}&0&1\end{pmatrix} \tag{1.1.3.74}$$

$$=\begin{pmatrix}\frac{4}{29}&0&\frac{10}{29}\\0&1&0\end{pmatrix} \tag{1.1.3.75}$$

Verify (1.1.3.75) from,

codes/pseudo_inverse.py

Substitute (1.1.3.75) in (1.1.3.64),

$$\mathbf{x}=\begin{pmatrix}\frac{4}{29}&0&\frac{10}{29}\\0&1&0\end{pmatrix}\begin{pmatrix}-1\\0\\-1\end{pmatrix}=\begin{pmatrix}-\frac{14}{29}\\0\end{pmatrix}=\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \tag{1.1.3.76}$$

Substituting λ_1, λ_2 in (1.1.3.59)

$$\mathbf{Q}=\begin{pmatrix}\frac{15}{29}\\0\\-\frac{6}{29}\end{pmatrix} \tag{1.1.3.77}$$

Distance between point **P** and **Q** is

$$\|\mathbf{P}-\mathbf{Q}\|=\sqrt{\left(\frac{15}{29}\right)^2+0+\left(-\frac{6}{29}\right)^2}=\frac{3}{\sqrt{29}} \tag{1.1.3.78}$$

Hence, distance from y-axis to $5x-2z-3=0$ is $\frac{3}{\sqrt{29}}$.

Verifying solution to (1.1.3.63) by least squares method

$$\mathbf{M}^T(\mathbf{b}-\mathbf{Mx})=0 \tag{1.1.3.79}$$

$$\implies \mathbf{M}^T\mathbf{Mx}=\mathbf{M}^T\mathbf{b} \tag{1.1.3.80}$$

Substituting $\mathbf{M},\mathbf{b}$ from (1.1.3.62) in

(1.1.3.80)

$$\begin{pmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\\frac{5}{2} & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\0\\-1\end{pmatrix} \tag{1.1.3.81}$$

$$\begin{pmatrix}\frac{29}{4} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}-\frac{7}{2}\\0\end{pmatrix} \tag{1.1.3.82}$$

$$\implies \frac{29}{4}\lambda_1 = -\frac{7}{2} \tag{1.1.3.83}$$

$$\lambda_1 = -\frac{7}{2}\times\frac{4}{29} = -\frac{14}{29} \tag{1.1.3.84}$$

$$\text{and } \lambda_2 = 0 \tag{1.1.3.85}$$

$$\mathbf{x} = \begin{pmatrix}-\frac{14}{29}\\0\end{pmatrix} \tag{1.1.3.86}$$

Comparing (1.1.3.76) and (1.1.3.86) solution is verified.

b) Determine the distance from the Z-axis to the plane 5x - 12y - 8 = 0

Solution: Equation of plane can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \tag{1.1.3.87}$$

Rewriting given equation of plane in (1.1.3.87) form

$$\begin{pmatrix}5 & -12 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 8 \tag{1.1.3.88}$$

where the value of

$$\mathbf{n} = \begin{pmatrix}5\\-12\\0\end{pmatrix} \tag{1.1.3.89}$$

$$\mathbf{x} = \begin{pmatrix}x\\y\\z\end{pmatrix} \tag{1.1.3.90}$$

$$c = 8 \tag{1.1.3.91}$$

We need to represent the equation of plane in parametric form,

$$\mathbf{Q} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \tag{1.1.3.92}$$

Here p is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Now, we need to find these two vectors $\mathbf{q}$ and $\mathbf{r}$ which are $\perp$ to $\mathbf{n}$

$$\begin{pmatrix}5 & -12 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \implies 5a - 12b = 0 \tag{1.1.3.93}$$

Put $a = 0$ and $c = 1$ in (1.1.3.93), $\implies b = 0$
Put $a = 1$ and $c = 0$ in (1.1.3.93), $\implies b = \frac{5}{12}$

Hence $\mathbf{q} = \begin{pmatrix}1\\\frac{5}{12}\\0\end{pmatrix}, \mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, z = 0$ in (1.1.3.88), we get $\mathbf{p} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

Since given plane is parallel to Z-axis, we can use any point P on Z-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{1.1.3.94}$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (1.1.3.93) form as

$$\mathbf{Q} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}1\\\frac{5}{12}\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{1.1.3.95}$$

Computation of Pseudo Inverse using SVD in order to determine the value of λ_1 and λ_2 :

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}1\\\frac{5}{12}\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{1.1.3.96}$$

$$\lambda_1\begin{pmatrix}1\\\frac{5}{12}\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\0\end{pmatrix} \tag{1.1.3.97}$$

$$\begin{pmatrix}1 & 0\\\frac{5}{12} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}-1\\-1\\0\end{pmatrix} \tag{1.1.3.98}$$

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.3.99}$$

$$\implies \mathbf{x} = \mathbf{M}^{+}\mathbf{b} \tag{1.1.3.100}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.3.101}$$

$$\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{1.1.3.102}$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{1.1.3.103}$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.1.3.104}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of Singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.3.105}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1.3.106}$$

As we know that,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{1.1.3.107}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} 1-\lambda & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144}-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^3 - \frac{313}{144}\lambda^2 + \frac{169}{144}\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{169}{144}; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{1.1.3.108}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} -\frac{5}{12} \\ 1 \\ 0 \end{pmatrix} \tag{1.1.3.109}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.3.110}$$

Using values from (1.1.3.108),

$$\mathbf{S} = \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.3.111}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} \frac{169}{144}-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 - \frac{313}{144}\lambda + \frac{169}{144} = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{169}{144}; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.1.3.112}$$

From (1.1.3.112) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.3.113}$$

Now, we can compute *SVD* of $\mathbf{M}$:

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.1.3.114}$$

$$= \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.3.115}$$

$$\mathbf{M}^+ = \mathbf{V}\mathbf{S}^T\mathbf{U}^T \tag{1.1.3.116}$$

$$= \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1.3.117}$$

Substitute (1.1.3.117) in (1.1.3.100),

$$\mathbf{x} = \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{1.1.3.118}$$

$$\mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{1.1.3.119}$$

$$\implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{1.1.3.120}$$

Substituting λ_1, λ_2 in (1.1.3.95)

$$\mathbf{Q} = \begin{pmatrix} -\frac{204}{169} \\ -\frac{85}{169} \\ 0 \end{pmatrix} \tag{1.1.3.121}$$

Distance between point $\mathbf{P}$ and $\mathbf{Q}$ is

$$\|\mathbf{P}-\mathbf{Q}\| = \sqrt{\left(-\frac{204}{169}\right)^2 + \left(-\frac{85}{169}\right)^2 + 0} \tag{1.1.3.122}$$

$$\|\mathbf{P}-\mathbf{Q}\| = \frac{17}{13} \tag{1.1.3.123}$$

Hence, the distance from the Z-axis to the plane $5x - 12y - 8 = 0$ is $\frac{17}{13}$. Now, we can verify the solution using Least Squares Method,

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \tag{1.1.3.124}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.3.125}$$

Substituting $\mathbf{M}, \mathbf{b}$ from (1.1.3.98) in (1.1.3.125)

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{1.1.3.126}$$

$$\begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{17}{12} \\ 0 \end{pmatrix} \tag{1.1.3.127}$$

$$\implies \frac{169}{144}\lambda_1 = -\frac{17}{12} \tag{1.1.3.128}$$

$$\lambda_1 = -\frac{17}{12} \times \frac{144}{169} = -\frac{204}{169} \tag{1.1.3.129}$$

$$\text{and } \lambda_2 = 0 \tag{1.1.3.130}$$

$$\implies \mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{1.1.3.131}$$

Comparing (1.1.3.118) and (1.1.3.131) solution is verified.

1.1.4. Find the foot of the perpendicular using svd drawn from $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ to the plane

$$\begin{pmatrix} 2 & -1 & -2 \end{pmatrix}\mathbf{x} + 4 = 0 \tag{1.1.4.1}$$

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.4.2}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0 \tag{1.1.4.3}$$

$$\implies 2a - b - 2c = 0 \tag{1.1.4.4}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.4.5}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \tag{1.1.4.6}$$

Let us solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.1.4.7}$$

Substituting (1.1.4.5) and (1.1.4.6) in (1.1.4.7),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \tag{1.1.4.8}$$

To solve (1.1.4.8), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.1.4.9}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \tag{1.1.4.10}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \tag{1.1.4.11}$$

Substituting (1.1.4.9) in (1.1.4.7),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.1.4.12}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U^T}\mathbf{b} \tag{1.1.4.13}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.

Let us calculate eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.4.14)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \quad (1.1.4.15)$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \quad (1.1.4.16)$$

From equation (1.1.4.16) eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.4.17)$$

The eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ -1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 1 \end{pmatrix} \quad (1.1.4.18)$$

Normalizing the eigen vectors in equation (1.1.4.18)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \quad (1.1.4.19)$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \quad (1.1.4.20)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.4.21)$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (1.1.4.22)$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \quad (1.1.4.23)$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (1.1.4.24)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad (1.1.4.25)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \quad (1.1.4.26)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.1.4.27)$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.1.4.28)$$

From (1.1.4.7), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (1.1.4.29)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.4.30)$$

From (1.1.4.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{11}{3} \end{pmatrix} \quad (1.1.4.31)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{8}{9\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (1.1.4.32)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \quad (1.1.4.33)$$

Verifying the solution of (1.1.4.33) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.4.34)$$

Evaluating the R.H.S in (1.1.4.34) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (1.1.4.35)$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (1.1.4.36)$$

Solving the augmented matrix of (1.1.4.36) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \tag{1.1.4.37}$$

$$\xleftrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{9}{8} & -\frac{1}{4} \end{pmatrix} \tag{1.1.4.38}$$

$$\xleftrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \tag{1.1.4.39}$$

$$\xleftrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & -\frac{5}{9} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \tag{1.1.4.40}$$

From equation (1.1.4.40), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \tag{1.1.4.41}$$

Comparing results of $\mathbf{x}$ from (1.1.4.33) and (1.1.4.41), we can say that the solution is verified.

1.1.5. Find the foot of the perpendicular to the given plane

$$2x + 3y - 4z + 5 = 0 \tag{1.1.5.1}$$

from

a)

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.1.5.2}$$

Solution: The given equation of plane can be represented as

$$\begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \mathbf{x} = -5 \tag{1.1.5.3}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \tag{1.1.5.4}$$

We need to find two vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{1.1.5.5}$$

Put $a = 1$ and $b = 0$ in (1.1.5.5),we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.5.6}$$

Put $a = 0$ and $b = 1$ in (1.1.5.5),we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \tag{1.1.5.7}$$

Now, solving the equation

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.5.8}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{1.1.5.9}$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.1.5.10}$$

Now, to solve equation (1.1.5.8), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.5.11}$$

Substituting the value of $\mathbf{M}$ from equation (1.1.5.11) to(1.1.5.8),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.5.12}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} \tag{1.1.5.13}$$

Where, $\mathbf{S_+}$ is the Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \tag{1.1.5.14}$$

Eigen values corresponding to $\mathbf{M}^T\mathbf{M}$ are given by,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.5.15}$$

$$\implies \left|\begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{pmatrix}\right| = 0 \tag{1.1.5.16}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{29}{16} = 0 \tag{1.1.5.17}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{29}{16} \tag{1.1.5.18}$$

$$\lambda_2 = 1 \tag{1.1.5.19}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \tag{1.1.5.20}$$

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.5.21}$$

Normalizing the eigen vectors, we obtain $\mathbf{V}$ of (1.1.5.11) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.5.22}$$

$\mathbf{S}$ of the diagonal matrix of (1.1.5.11) is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.5.23}$$

Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.5.24}$$

Eigen values corresponding to $\mathbf{M}\mathbf{M}^T$ are given by

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.5.25}$$

$$\implies \left|\begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{pmatrix}\right| = 0 \tag{1.1.5.26}$$

$$\implies \lambda^3 - \frac{45}{16}\lambda^2 + \frac{29}{16}\lambda = 0 \tag{1.1.5.27}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_3 = \frac{29}{16} \tag{1.1.5.28}$$

$$\lambda_4 = 1 \tag{1.1.5.29}$$

$$\lambda_5 = 0 \tag{1.1.5.30}$$

Hence we obtain $\mathbf{U}$ of (1.1.5.11) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{29} \\ \sqrt{\frac{13}{29}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.5.31}$$

Finally from (1.1.5.11) we get the Singular Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{29} \\ \sqrt{\frac{13}{29}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.5.32}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.5.33}$$

Substituting the values of (1.1.5.31),(1.1.5.22),(1.1.5.33) in (1.1.5.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \tag{1.1.5.34}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \tag{1.1.5.35}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.5.36}$$

Verifying the solution of (1.1.5.36) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.5.37}$$

Evaluating the R.H.S in (1.1.5.37) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.5.38}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.5.39}$$

The augmented matrix of (1.1.5.39) is,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & 3 \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \tag{1.1.5.40}$$

Solving the augmented matrix into Row reduced echelon form of (1.1.5.40) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & 3 \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \tag{1.1.5.41}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ 0 & \frac{29}{20} & -\frac{29}{10} \end{pmatrix} \tag{1.1.5.42}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ 0 & 1 & -2 \end{pmatrix} \tag{1.1.5.43}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \tag{1.1.5.44}$$

Therefore,

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.5.45}$$

Comparing results of $\mathbf{x}$ from (1.1.5.36) and (1.1.5.45) we conclude that the solution is verified.

b)

$$\mathbf{c} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \tag{1.1.5.46}$$

Solution:
The given equation of plane can be represented as

$$\begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \mathbf{x} = -5 \tag{1.1.5.47}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \tag{1.1.5.48}$$

We need to find two vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{1.1.5.49}$$

Put $a = 1$ and $b = 0$ in (1.1.5.49),we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.5.50}$$

Put $a = 0$ and $b = 1$ in (1.1.5.49),we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \tag{1.1.5.51}$$

Now, solving the equation

$$\mathbf{Mx} = \mathbf{c} \tag{1.1.5.52}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{1.1.5.53}$$

$$\mathbf{c} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \tag{1.1.5.54}$$

Now, to solve equation (1.1.5.52), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.5.55}$$

Substituting the value of $\mathbf{M}$ from equation (1.1.5.55) to(1.1.5.52),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{c} \tag{1.1.5.56}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{c} \tag{1.1.5.57}$$

Where, $\mathbf{S_+}$ is the Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \tag{1.1.5.58}$$

Eigen values corresponding to $\mathbf{M}^T\mathbf{M}$ are given by,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.5.59}$$

$$\implies \left|\begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{pmatrix}\right| = 0 \tag{1.1.5.60}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{29}{16} = 0 \tag{1.1.5.61}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{29}{16} \tag{1.1.5.62}$$

$$\lambda_2 = 1 \tag{1.1.5.63}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \tag{1.1.5.64}$$

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.5.65}$$

Normalizing the eigen vectors, we obtain $\mathbf{V}$ of (1.1.5.55) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.5.66}$$

$\mathbf{S}$ of the diagonal matrix of (1.1.5.55) is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.5.67}$$

Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.5.68}$$

Eigen values corresponding to $\mathbf{M}\mathbf{M}^T$ are given by

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.5.69}$$

$$\implies \left|\begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{pmatrix}\right| = 0 \tag{1.1.5.70}$$

$$\implies \lambda^3 - \frac{45}{16}\lambda^2 + \frac{29}{16}\lambda = 0 \tag{1.1.5.71}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_3 = \frac{29}{16} \tag{1.1.5.72}$$

$$\lambda_4 = 1 \tag{1.1.5.73}$$

$$\lambda_5 = 0 \tag{1.1.5.74}$$

Hence we obtain $\mathbf{U}$ of (1.1.5.55) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.5.75}$$

Finally from (1.1.5.55) we get the Singular Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.5.76}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.5.77}$$

Substituting the values of (1.1.5.75),(1.1.5.66),(1.1.5.77) in (1.1.5.57) we get,

$$\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{125}{\sqrt{377}} \\ \frac{-3}{\sqrt{13}} \\ \frac{5}{\sqrt{29}} \end{pmatrix} \tag{1.1.5.78}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{500}{29\sqrt{13}} \\ -\frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.5.79}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{97}{29} \\ \frac{102}{29} \end{pmatrix} \tag{1.1.5.80}$$

Verifying the solution of (1.1.5.80) using,

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{c} \tag{1.1.5.81}$$

Evaluating the R.H.S in (1.1.5.81) we get,

$$\mathbf{M}^T\mathbf{c} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ \frac{27}{4} \end{pmatrix} \tag{1.1.5.82}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{11}{2} \\ \frac{27}{4} \end{pmatrix} \tag{1.1.5.83}$$

The augmented matrix of (1.1.5.83) is,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & \frac{11}{2} \\ \frac{3}{8} & \frac{25}{16} & \frac{27}{4} \end{pmatrix} \tag{1.1.5.84}$$

Solving the augmented matrix into Row reduced echelon form of (1.1.5.84) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & \frac{11}{2} \\ \frac{3}{8} & \frac{25}{16} & \frac{27}{4} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ \frac{3}{8} & \frac{25}{16} & \frac{27}{4} \end{pmatrix} \quad (1.1.5.85)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ 0 & \frac{29}{20} & \frac{51}{10} \end{pmatrix} \quad (1.1.5.86)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ 0 & 1 & \frac{102}{29} \end{pmatrix} \quad (1.1.5.87)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & \frac{97}{29} \\ 0 & 1 & \frac{102}{29} \end{pmatrix} \quad (1.1.5.88)$$

Therefore,

$$\mathbf{x} = \begin{pmatrix} \frac{97}{29} \\ \frac{102}{29} \end{pmatrix} \quad (1.1.5.89)$$

Comparing results of $\mathbf{x}$ from (1.1.5.80) and (1.1.5.89), Hence, the solution is verified.

1.2 Two planes

1.2.1. Set up the equation of a plane through the point A $(-2, -3, 4)$ and perpendicular to the line

$$\frac{x}{4} = \frac{y-3}{6} = \frac{z+2}{-12} \quad (1.2.1.1)$$

Solution: Let the equation of plane is

$$ax + by + cz + d = 0 \quad (1.2.1.2)$$

Direction ratio of the line (1.2.1.1) is given as

$$\mathbf{D} = \begin{pmatrix} 4 \\ 6 \\ -12 \end{pmatrix} \quad (1.2.1.3)$$

Now let consider

$$\mathbf{A} = \begin{pmatrix} -2 & -3 & 4 \end{pmatrix} \mathbf{AD} + d = 0 \quad (1.2.1.4)$$

$$\implies d = 37 \quad (1.2.1.5)$$

Since plane is passing through the point A (-2, -3, 4) and perpendicular to the line (1.2.1.1). Hence equation of the plane is

$$2x + 3y - 6z + 37 = 0 \quad (1.2.1.6)$$

$$\implies 2x + 3y - 6z = -37 \quad (1.2.1.7)$$

equation (1.2.1.7) can written as :

$$\begin{pmatrix} 2 & 3 & -6 \end{pmatrix} \mathbf{x} = -37 \quad (1.2.1.8)$$

$$(1.2.1.9)$$

For foot perpendicular we need to find the distance between the plane and point P $(0, 3, -2)$.

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \quad (1.2.1.10)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = 0 \quad (1.2.1.11)$$

$$\implies 2a + 3b - 6c = 0 \quad (1.2.1.12)$$

$$(1.2.1.13)$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{3} \end{pmatrix} \quad (1.2.1.14)$$

$$(1.2.1.15)$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \quad (1.2.1.16)$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \quad (1.2.1.17)$$

$$(1.2.1.18)$$

Putting values in (1.2.1.17),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \quad (1.2.1.19)$$

Now, to solve (1.2.1.19), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \quad (1.2.1.20)$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{9} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{4} \end{pmatrix} \tag{1.2.1.21}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{36} \end{pmatrix} \tag{1.2.1.22}$$

From (1.2.1.17) putting (1.2.1.20) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.2.1.23}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{1.2.1.24}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.2.1.25}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{1}{3} \\ 0 & 1-\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{36}-\lambda \end{pmatrix} = 0 \tag{1.2.1.26}$$

$$\implies \lambda(\lambda-1)(\lambda-\frac{49}{36}) = 0 \tag{1.2.1.27}$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \tag{1.2.1.28}$$

$$\lambda_2 = 1 \tag{1.2.1.29}$$

$$\lambda_3 = 0 \tag{1.2.1.30}$$

$$\tag{1.2.1.31}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{12}{13} \\ \frac{18}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{2} \\ 1 \end{pmatrix} \tag{1.2.1.32}$$

Normalizing the eigen vectors we get,

$$\mathbf{u_1} = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{18}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-2}{7} \\ \frac{-3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.2.1.33}$$

Hence we obtain $\mathbf{U}$ of (1.2.1.20) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{12}{7\sqrt{13}} & \frac{-3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{\sqrt{13}}{7} & 0 & \frac{6}{7} \end{pmatrix} \tag{1.2.1.34}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (1.2.1.20) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.2.1.35}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.2.1.36}$$

$$\implies \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9}-\lambda \end{vmatrix} = 0 \tag{1.2.1.37}$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \tag{1.2.1.38}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \tag{1.2.1.39}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \tag{1.2.1.40}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.2.1.41}$$

Hence we obtain $\mathbf{V}$ of (1.2.1.20) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.2.1.42}$$

Finally from (1.2.1.20) we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{12}{7\sqrt{13}} & \frac{-3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{\sqrt{13}}{7} & 0 & \frac{6}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.2.1.43}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.1.44}$$

From (1.2.1.24) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{\sqrt{13}} \\ \frac{6}{\sqrt{13}} \\ -3 \end{pmatrix} \quad (1.2.1.45)$$

$$\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{24}{7\sqrt{13}} \\ \frac{6}{\sqrt{13}} \end{pmatrix} \quad (1.2.1.46)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} \quad = \begin{pmatrix} \frac{-6}{7} \\ \frac{12}{7} \end{pmatrix} \quad (1.2.1.47)$$

Verifying the solution of (1.2.1.47) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.2.1.48)$$

Evaluating the R.H.S in (1.2.1.48) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{-2}{3} \\ 2 \end{pmatrix} \quad (1.2.1.49)$$

$$\implies \begin{pmatrix} \frac{10}{9} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-2}{3} \\ 2 \end{pmatrix} \quad (1.2.1.50)$$

Solving the augmented matrix of (1.2.1.50) we get,

$$\begin{pmatrix} \frac{10}{9} & \frac{1}{6} & \frac{-2}{3} \\ \frac{1}{6} & \frac{5}{4} & 2 \end{pmatrix} \xleftrightarrow{R_1=\frac{9R_1}{10}} \begin{pmatrix} 1 & \frac{3}{20} & \frac{-3}{5} \\ \frac{1}{6} & \frac{5}{4} & 2 \end{pmatrix} \quad (1.2.1.51)$$

$$\xleftrightarrow{R_2=R_2-\frac{R_1}{6}} \begin{pmatrix} 1 & \frac{3}{20} & \frac{-3}{5} \\ 0 & \frac{49}{40} & \frac{21}{10} \end{pmatrix} \quad (1.2.1.52)$$

$$\xleftrightarrow{R_2=\frac{40}{49}R_2} \begin{pmatrix} 1 & \frac{3}{20} & \frac{-3}{5} \\ 0 & 1 & \frac{12}{7} \end{pmatrix} \quad (1.2.1.53)$$

$$\xleftrightarrow{R_1=R_1-\frac{3R_2}{20}} \begin{pmatrix} 1 & 0 & \frac{-6}{7} \\ 0 & 1 & \frac{12}{7} \end{pmatrix} \quad (1.2.1.54)$$

Hence, Solution of (1.2.1.48) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-6}{7} \\ \frac{12}{7} \end{pmatrix} \quad (1.2.1.55)$$

Comparing results of $\mathbf{x}$ from (1.2.1.47) and (1.2.1.55) we conclude that the solution is verified.

1.3 The Pencil of Planes. The Bundle of Planes

1.3.1. Write the equation of a plane through the point A $(-3, 4, -1)$ and perpendicular to the line

$$\frac{x+2}{-3} = \frac{y-2}{1} = \frac{z-4}{2} \quad (1.3.1.1)$$

Solution:
Let the equation of plane is

$$ax + by + cz + d = 0 \quad (1.3.1.2)$$

Direction ratio of the line (1.3.1.1) is given as

$$\mathbf{D} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad (1.3.1.3)$$

Now let consider

$$\mathbf{A} = \begin{pmatrix} -3 & 4 & -1 \end{pmatrix} \quad (1.3.1.4)$$

Since plane is passing through the point A (-3, 4, -1) and perpendicular to the line (1.3.1.1), hence

$$\mathbf{A}\mathbf{D} + d = 0 \quad (1.3.1.5)$$

$$\implies d = -11 \quad (1.3.1.6)$$

Hence equation of the plane is

$$-3x + y + 2z - 11 = 0 \quad (1.3.1.7)$$

$$\implies -3x + y + 2z = 11 \quad (1.3.1.8)$$

equation (1.3.1.8) can written as :

$$\begin{pmatrix} -3 & 1 & 2 \end{pmatrix}\mathbf{x} = 11 \quad (1.3.1.9)$$

For foot perpendicular we need to find the distance between the plane and point P $(-2, 2, 4)$. First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^{\mathbf{T}}\mathbf{n} = 0 \quad (1.3.1.10)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 0 \quad (1.3.1.11)$$

$$\implies -3a + b + 2c = 0 \quad (1.3.1.12)$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} \quad (1.3.1.13)$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \quad (1.3.1.14)$$

Now we solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.3.1.15}$$

Putting values in (1.3.1.15),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \tag{1.3.1.16}$$

Now, to solve (1.3.1.16), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.3.1.17}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{13}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \tag{1.3.1.18}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \tag{1.3.1.19}$$

From (1.3.1.15) putting (1.3.1.17) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.3.1.20}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{1.3.1.21}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.3.1.22}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{3}{2} \\ 0 & 1-\lambda & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{5}{2}-\lambda \end{pmatrix} = 0 \tag{1.3.1.23}$$

$$\implies \lambda(\lambda-1)(\lambda-\frac{7}{2}) = 0 \tag{1.3.1.24}$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{7}{2} \tag{1.3.1.25}$$

$$\lambda_2 = 1 \tag{1.3.1.26}$$

$$\lambda_3 = 0 \tag{1.3.1.27}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}, \tag{1.3.1.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{\sqrt{35}} \\ -\frac{1}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix} \tag{1.3.1.29}$$

Hence we obtain $\mathbf{U}$ of (1.3.1.17) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \end{pmatrix} \tag{1.3.1.30}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (1.3.1.17) as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\frac{7}{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.3.1.31}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.3.1.32}$$

$$\implies \begin{pmatrix} \frac{13}{4}-\lambda & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{1.3.1.33}$$

$$\implies \lambda^2 - \frac{9}{2}\lambda + \frac{7}{2} = 0 \tag{1.3.1.34}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{7}{2} \tag{1.3.1.35}$$

$$\lambda_5 = 1 \tag{1.3.1.36}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.3.1.37}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \tag{1.3.1.38}$$

Hence we obtain $\mathbf{V}$ of (1.3.1.17) as follows,

$$\mathbf{V} = \begin{pmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \tag{1.3.1.39}$$

Finally from (1.3.1.17) we get the Singualr

Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{7}{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}^T \tag{1.3.1.40}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \sqrt{\frac{2}{7}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.3.1.41}$$

From (1.3.1.21) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{\sqrt{35}} \\ \frac{2\sqrt{2}}{\sqrt{5}} \\ \frac{8\sqrt{2}}{\sqrt{7}} \end{pmatrix} \tag{1.3.1.42}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{10}}{35} \\ \frac{2\sqrt{10}}{5} \end{pmatrix} \tag{1.3.1.43}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{10}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.3.1.44}$$

Verifying the solution of (1.3.1.44) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.3.1.45}$$

Evaluating the R.H.S in (1.3.1.45) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{1.3.1.46}$$

$$\implies \begin{pmatrix} \frac{13}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.3.1.47}$$

Solving the augmented matrix of (1.3.1.47) we get,

$$\begin{pmatrix} \frac{13}{4} & -\frac{3}{4} & 4 \\ -\frac{3}{4} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1 = \frac{4}{13}R_1} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ -\frac{3}{4} & \frac{5}{4} & 0 \end{pmatrix} \tag{1.3.1.48}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{3}{4}R_1} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ 0 & \frac{14}{13} & \frac{12}{13} \end{pmatrix} \tag{1.3.1.49}$$

$$\xleftrightarrow{R_2 = \frac{13}{14}R_2} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \tag{1.3.1.50}$$

$$\xleftrightarrow{R_1 = R_1 + \frac{3}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \tag{1.3.1.51}$$

Hence, Solution of (1.3.1.45) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{10}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.3.1.52}$$

Comparing results of $\mathbf{x}$ from (1.3.1.44) and (1.3.1.52) we conclude that the solution is verified.

1.3.2. Write the equation of the line through $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and perpendicular to the plane $2x - y + 2z - 5 = 0$. Determine the coordinates of the point in which the plane is met by this line.

Solution: Given a point $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and a plane $\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\mathbf{x} = 5$. We know that the equation of a plane is given by

$$\mathbf{n^T}\mathbf{x} = c \tag{1.3.2.1}$$

Hence, normal vector $\mathbf{n}$ is given by

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{1.3.2.2}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be two vectors that are normal to normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then if

$$\mathbf{n^T}\mathbf{m} = 0 \tag{1.3.2.3}$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{1.3.2.4}$$

Taking $a = 1$, $b = 0$, we get $c = -1$, and hence

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1.3.2.5}$$

Take $a = 0$ and $b = 1$, we get $c = \frac{1}{2}$, and hence

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \tag{1.3.2.6}$$

Since foot of perpendicular is the point where the plane is met by a line perpendicular to the same plane. So, to get foot of perpendicular, we solve

$$\mathbf{Mx} = \mathbf{b} \tag{1.3.2.7}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}, b = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{1.3.2.8}$$

To solve (1.3.2.7), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \tag{1.3.2.9}$$

Substituting the value of $\mathbf{M}$ from (1.3.2.9) in (1.3.2.7), we get

$$\mathbf{USV^Tx} = \mathbf{b} \tag{1.3.2.10}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.3.2.11}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigen-values of $\mathbf{M^TM}$. First calculating the eigenvectors corresponding to $\mathbf{M^TM}$.

$$\mathbf{M^TM} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \tag{1.3.2.12}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left|\mathbf{M^TM} - \lambda\mathbf{I}\right| = 0 \tag{1.3.2.13}$$

$$\left|\begin{pmatrix} 2-\lambda & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4}-\lambda \end{pmatrix}\right| = 0 \tag{1.3.2.14}$$

$$\left(\frac{5}{4} - \lambda\right)(2-\lambda) - \frac{1}{4} = 0 \tag{1.3.2.15}$$

$$\left(\lambda - \frac{9}{4}\right)(\lambda - 1) = 0 \tag{1.3.2.16}$$

Hence,

$$\lambda_1 = \frac{9}{4}, \lambda_2 = 1 \tag{1.3.2.17}$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.3.2.18}$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.3.2.19}$$

Normalizing, the eigen vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, we get

$$\mathbf{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.3.2.20}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.3.2.21}$$

Hence,

$$\mathbf{V} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \tag{1.3.2.22}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM^T}$

$$\mathbf{MM^T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \tag{1.3.2.23}$$

Eigen values of $\mathbf{MM^T}$ can be found out as

$$\left|\mathbf{MM^T} - \lambda\mathbf{I}\right| = 0 \tag{1.3.2.24}$$

$$\left|\begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix}\right| = 0 \tag{1.3.2.25}$$

$$(1-\lambda)\left((1-\lambda)\left(\frac{5}{4} - \lambda\right) - \frac{1}{4}\right) - 1 + \lambda = 0 \tag{1.3.2.26}$$

$$\lambda\left(\lambda - \frac{9}{4}\right)(\lambda - 1) = 0 \tag{1.3.2.27}$$

Hence,

$$\lambda_3 = 0, \lambda_4 = 1, \lambda_5 = \frac{9}{4} \tag{1.3.2.28}$$

Eigen-vector corresponding to $\lambda = 0$,

$$\mathbf{v_3} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{1.3.2.29}$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_4} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \tag{1.3.2.30}$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v_5} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} \tag{1.3.2.31}$$

Normalizing, the eigen vectors $\mathbf{v_3}$, $\mathbf{v_4}$ and $\mathbf{v_5}$, we get

$$\mathbf{v_3} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.3.2.32}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{1.3.2.33}$$

$$\mathbf{v_5} = \frac{1}{3\sqrt{5}}\begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \\ \frac{-5}{3\sqrt{5}} \end{pmatrix} \tag{1.3.2.34}$$

Hence,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \tag{1.3.2.35}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_5, λ_4 and λ_3 is as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.3.2.36}$$

Now, Moore-Pen-Rose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.3.2.37}$$

Hence, we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{1.3.2.38}$$

Substituting values of (1.3.2.8), (1.3.2.22), (1.3.2.35) and (1.3.2.36) into (1.3.2.11), we get

$$\mathbf{U^T b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{-5}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{1.3.2.39}$$

$$\implies \mathbf{U^T b} = \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{1.3.2.40}$$

Now,

$$\mathbf{VS}_+ = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.3.2.41}$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{-2}{3} & 2 & 0 \end{pmatrix} \tag{1.3.2.42}$$

Now, by (1.3.2.11), we have

$$\mathbf{x} = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{-2}{3} & 2 & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{1.3.2.43}$$

$$\implies \mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.3.2.44}$$

Now, we verify our solution using

$$\mathbf{M^T M x} = \mathbf{M^T b} \tag{1.3.2.45}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \tag{1.3.2.46}$$

$$\implies \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 4 \\ \frac{7}{2} \end{pmatrix} \tag{1.3.2.47}$$

Solving the augumented matrix, we get

$$\begin{pmatrix} 2 & \frac{-1}{2} & 4 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \xleftrightarrow{r_1=(1/2)*(r_1)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \tag{1.3.2.48}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \xleftrightarrow{r_2=r_2+(1/2)*(r_1)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & \frac{9}{8} & \frac{9}{2} \end{pmatrix} \tag{1.3.2.49}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & \frac{9}{8} & \frac{9}{2} \end{pmatrix} \xleftrightarrow{r_2=(8/9)*(r_2)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.3.2.50}$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & 1 & 4 \end{pmatrix} \xleftrightarrow{r_1=r_1+(-1/4)*(r_2)} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.3.2.51}$$

Thus,

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.3.2.52}$$

verifying the result from SVD.
Now, we solve for third coordinate of foot of perpendicular by,

$$\mathbf{n^T x} = 5 \tag{1.3.2.53}$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ z \end{pmatrix} = 5 \tag{1.3.2.54}$$

$$z = \frac{-5}{2} \tag{1.3.2.55}$$

Normalizing z, we get

$$z = \frac{\left(\frac{-5}{2}\right)}{3} \implies z = \frac{-5}{6} \tag{1.3.2.56}$$

Hence, coordinate of foot of perpendicular is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ \frac{-5}{6} \end{pmatrix} \tag{1.3.2.57}$$

Now, we try to find equation of straight line through $\mathbf{P} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and having direction cosines as $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{1.3.2.58}$$