



# Geometry through Linear Algebra



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## CONTENTS

### 1 Pair of Straight Lines 1

**Abstract**—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.

#### 1 PAIR OF STRAIGHT LINES

1.1. Find the value of  $h$  so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.1)$$

may represent two straight lines.

**Solution:**

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \quad (1.1.2)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.3)$$

$$f = 20 \quad (1.1.4)$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \quad (1.1.5)$$

Expanding equation (1.1.5) along row 1 gives

$$\begin{aligned} \Rightarrow 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + \\ 11 \times (\frac{31h}{2} - 132) = 0 \end{aligned}$$

$$\Rightarrow 20h^2 - 341h + \frac{2907}{2} = 0 \quad (1.1.6)$$

$$\Rightarrow h = \frac{17}{2} \quad (1.1.7)$$

$$\Rightarrow h = \frac{171}{20} \quad (1.1.8)$$

If  $h = \frac{17}{2}$  or  $h = \frac{171}{20}$ , the equation given will represent two straight lines.

Sub  $h = \frac{17}{2}$  in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.9)$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (1.1.10)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.11)$$

$$\mathbf{f} = 20 \quad (1.1.12)$$

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The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.13)$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \quad (1.1.14)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.1.15)$$

$$(1.1.16)$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \quad (1.1.17)$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \quad (1.1.18)$$

$$\Rightarrow m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \quad (1.1.19)$$

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.1.20)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.21)$$

we know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.22)$$

Convolution of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  can be done by converting  $\mathbf{n}_1$  into a toeplitz matrix and multiplying with  $\mathbf{n}_2$

From equation (1.1.21)

$$\mathbf{n}_1 = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.23)$$

$$\Rightarrow \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.24)$$

$\Rightarrow$  Equation (1.1.21) satisfies (1.1.22)

$c_1$  and  $c_2$  can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.1.25)$$

Substituting (1.1.21) in (1.1.25), the augmented

matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \quad (1.1.26)$$

$$\Rightarrow c_1 = 4, c_2 = 5 \quad (1.1.27)$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\begin{aligned} \Rightarrow (-2x - 3y - 4)(3x - 4y - 5) &= 0 \\ \Rightarrow (2x + 3y + 4)(3x + 4y + 5) &= 0 \end{aligned} \quad (1.1.28)$$

Equation (1.1.28) represents equations of two straight lines.

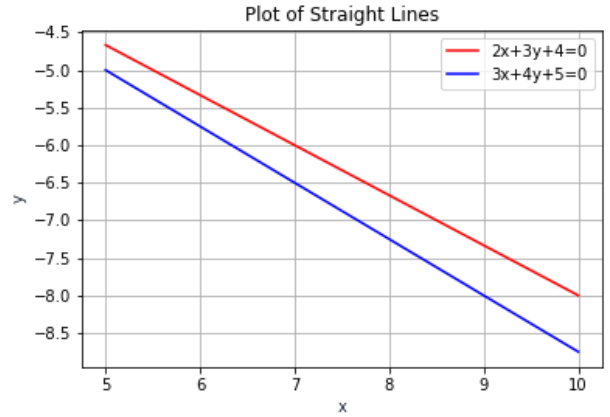


Fig. 1.1.1: Plot of Straight lines when  $h = \frac{17}{2}$

Similarly, Sub  $h = \frac{171}{20}$  in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \quad (1.1.29)$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \quad (1.1.30)$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{3} \\ \frac{310}{3} \end{pmatrix} \quad (1.1.31)$$

$$\mathbf{f} = \frac{200}{3} \quad (1.1.32)$$

The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.33)$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \quad (1.1.34)$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \quad (1.1.35)$$

$$\Rightarrow m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \quad (1.1.36)$$

$$\mathbf{m}_1 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (1.1.37)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.38)$$

Convolution of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  can be done by converting  $\mathbf{n}_1$  into a toeplitz matrix and multiplying with  $\mathbf{n}_2$

From equation (1.1.38)

$$\mathbf{n}_1 = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.39)$$

$$\Rightarrow \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.40)$$

$\Rightarrow$  Equation (1.1.38) satisfies (1.1.22)

$c_1$  and  $c_2$  can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -2\mathbf{u} \quad (1.1.41)$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 4R_2}{5}]{R_2 \leftarrow \frac{5R_2 - 8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \quad (1.1.42)$$

$$\Rightarrow c_1 = 10, c_2 = \frac{20}{3} \quad (1.1.43)$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\Rightarrow \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \quad (1.1.44)$$

Equation (1.1.44) represents equations of two straight lines.

1.2. Prove that the following equations represent two straight lines. Also find their point of in-

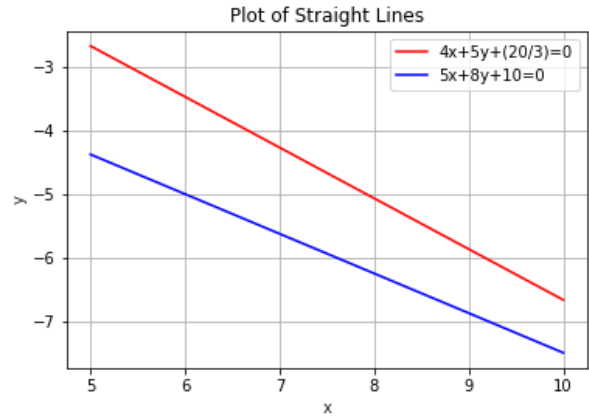


Fig. 1.1.2: Plot of Straight lines when  $h = \frac{171}{20}$

tersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \quad (1.2.1)$$

**Solution:**  $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$  of (1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \quad (1.2.2)$$

Expanding equation (1.2.2), we get zero.

Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.2.3)$$

$$\Rightarrow 3m^2 - 8m - 3 = 0 \quad (1.2.4)$$

$$\text{Solving, } m = 3, -\frac{1}{3} \quad (1.2.5)$$

The normal vectors of the lines then become

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.2.6)$$

$$\mathbf{n}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.7)$$

Equations of the lines can therefore be written as

$$\left(\frac{1}{3} \ 1\right) \mathbf{x} = c \quad (1.2.8)$$

$$\Rightarrow (1 \ 3) \mathbf{x} = c_1, \quad (1.2.9)$$

$$(-3 \ 1) \mathbf{x} = c_2 \quad (1.2.10)$$

$$\Rightarrow [(1 \ 3) \mathbf{x} - c_1][(-3 \ 1) \mathbf{x} - c_2] \quad (1.2.11)$$

represents the equation specified in (1.2.1)

Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \quad (1.2.12)$$

$$(1.2.13)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3 \times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \quad (1.2.14)$$

$$\xrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.15)$$

$$\xrightarrow{R_1 \leftarrow R_1 + 3 \times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.16)$$

$$\Rightarrow c_2 = 2 \text{ and } c_1 = -9 \quad (1.2.17)$$

The individual line equations therefore become

$$(1 \ 3)\mathbf{x} = -9, \quad (1.2.18)$$

$$(-3 \ 1)\mathbf{x} = 2 \quad (1.2.19)$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.20)$$

The LHS part of (1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.21)$$

The augmented matrix for the set of equations represented in (1.2.18), (1.2.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \quad (1.2.22)$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.23)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.24)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \quad (1.2.25)$$

$$\text{Hence, the intersection point is } \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.2.26)$$

Angle between two lines  $\theta$  can be given by

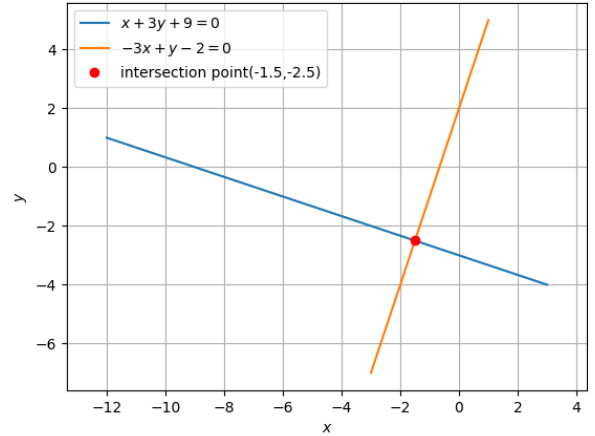


Fig. 1.2.1: plot showing intersection of lines

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.2.27)$$

$$\cos \theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \quad (1.2.28)$$

$$\Rightarrow \theta = 90^\circ \quad (1.2.29)$$

1.3. Find the value of  $k$  so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.3.1)$$

**Solution:**

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \quad (1.3.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.3.3)$$

The equation (1.3.1) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.3.4)$$

$$\Rightarrow \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \quad (1.3.5)$$

$$\Rightarrow \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \quad (1.3.6)$$

$$\Rightarrow 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11 \begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \quad (1.3.7)$$

$$\Rightarrow 2k^2 + 55k + 350 = 0 \quad (1.3.8)$$

$$\Rightarrow (10 + k)(2k + 35) = 0 \quad (1.3.9)$$

$$\Rightarrow k = -10$$

$$k = -\frac{35}{2} \quad (1.3.10)$$

Therefore, for  $k = -10$  and  $k = -\frac{35}{2}$  the given equation represents pair of straight lines.

Now Lets find equation of lines for  $k = -10$ . Substitute  $k = -10$  in (1.3.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.3.11)$$

From (1.3.1), (1.3.2), (1.3.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (1.3.12)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.3.13)$$

If  $|\mathbf{V}| < 0$  then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \quad (1.3.14)$$

$$\Rightarrow |\mathbf{V}| = -1 \quad (1.3.15)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.3.16)$$

Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.3.17)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.3.18)$$

Equating their product with (1.3.1)

$$\begin{aligned} &(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ &= \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (1.3.19)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.3.20)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} = -2 \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.3.21)$$

$$c_1 c_2 = f = 2 \quad (1.3.22)$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.3.23)$$

$$\Rightarrow 2m^2 - 10m + 12 = 0 \quad (1.3.24)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.3.25)$$

$$\Rightarrow m_i = \frac{5 \pm \sqrt{1}}{2} \quad (1.3.26)$$

$$\Rightarrow m_1 = 3 \quad (1.3.27)$$

$$m_2 = 2 \quad (1.3.28)$$

The normal vector for two lines is given by

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.3.29)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.3.30)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.3.31)$$

Substituting (1.3.30), (1.3.31) in (1.3.20). we get

$$k_1 k_2 = 2 \quad (1.3.32)$$

The possible combinations of  $(k_1, k_2)$  are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume  $k_1 = 1, k_2 = 2$  we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.3.33)$$

$$\mathbf{n}_2 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (1.3.34)$$

We verify obtained  $\mathbf{n}_1, \mathbf{n}_2$  using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.3.35)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.36)$$

Therefore the obtained  $\mathbf{n}_1, \mathbf{n}_2$  are correct.

Substitute (1.3.33), (1.3.34) in (1.3.21) and calculate for  $c_1$  and  $c_2$

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.3.37)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.3.38)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.3.39)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.3.40)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \quad (1.3.41)$$

$$\Rightarrow c_1 = -13 \quad (1.3.42)$$

$$c_2 = 21 \quad (1.3.43)$$

Substituting (1.3.33), (1.3.34), (1.3.42), (1.3.43) in (1.3.17) and (1.3.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = -13 \quad (1.3.44)$$

$$\begin{pmatrix} -4 & 2 \end{pmatrix} \mathbf{x} = 21 \quad (1.3.45)$$

The plot of these two lines is shown in Fig. 1.3.1.

Now Lets find equation of lines for  $k = -\frac{35}{2}$ . Substitute  $k = -\frac{35}{2}$  in (1.3.1). We get equation

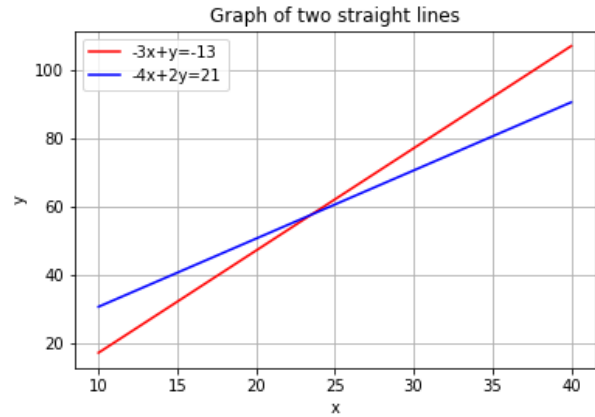


Fig. 1.3.1: Pair of straight lines for  $k = -10$

of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.3.46)$$

From (1.3.1), (1.3.2), (1.3.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \quad (1.3.47)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.3.48)$$

If  $|\mathbf{V}| < 0$  then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \quad (1.3.49)$$

$$\Rightarrow |\mathbf{V}| = -\frac{841}{16} \quad (1.3.50)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.3.51)$$

Therefore the lines will intersect.

Now from (1.3.20),

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (1.3.52)$$

The slopes of the lines are given by roots of

equation (1.3.23)

$$\Rightarrow 2m^2 - \frac{35}{2}m + 12 = 0 \quad (1.3.53)$$

$$m_i = \frac{-b \pm \sqrt{-|V|}}{c} \quad (1.3.54)$$

$$\Rightarrow m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \quad (1.3.55)$$

$$\Rightarrow m_1 = 8 \quad (1.3.56)$$

$$m_2 = \frac{3}{4} \quad (1.3.57)$$

The normal vector for two lines is given by (1.3.29)

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.3.58)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.3.59)$$

Substituting (1.3.58),(1.3.59) in (1.3.52). we get

$$k_1 k_2 = 2 \quad (1.3.60)$$

The possible combinations of  $(k_1, k_2)$  are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume  $k_1 = 1, k_2 = 2$  we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.3.61)$$

$$\mathbf{n}_2 = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \quad (1.3.62)$$

We verify obtained  $\mathbf{n}_1, \mathbf{n}_2$  using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (1.3.63)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.64)$$

Therefore the obtained  $\mathbf{n}_1, \mathbf{n}_2$  are correct.

Substitute (1.3.61), (1.3.62) in (1.3.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.3.65)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.3.66)$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.3.67)$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.3.68)$$

$$\xleftrightarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.3.69)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \quad (1.3.70)$$

$$\Rightarrow c_1 = -\frac{102}{29} \quad (1.3.71)$$

$$c_2 = \frac{59}{29} \quad (1.3.72)$$

Substituting (1.3.61),(1.3.62),(1.3.71),(1.3.72) in (1.3.17) and (1.3.18). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \quad (1.3.73)$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \quad (1.3.74)$$

The plot of these two lines is shown in Fig. 1.3.2.

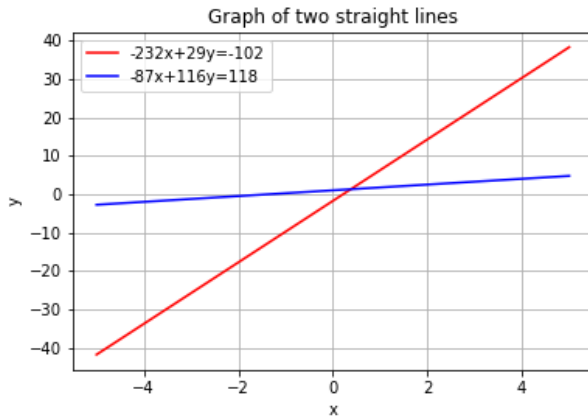


Fig. 1.3.2: Pair of straight lines for  $k = -\frac{35}{2}$

1.4. Find the value of  $k$  such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \quad (1.4.1)$$

represent pairs of straight lines. **Solution:** From (1.4.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \quad (1.4.2)$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{5}{2} & -\frac{7}{2} \end{pmatrix} \quad (1.4.3)$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & k \end{vmatrix} = 0 \quad (1.4.4)$$

$$\Rightarrow \left( k - \left( \frac{49}{4} \right) \right) - \frac{5}{3} \left( \frac{5}{3}k - \frac{35}{4} \right) - \frac{5}{2} \left( \frac{-35}{6} + \frac{5}{2} \right) = 0 \quad (1.4.5)$$

$$\Rightarrow \frac{64}{k}36 - \frac{128}{12} = 0 \quad (1.4.6)$$

$$\Rightarrow \boxed{k = 6} \quad (1.4.7)$$

Substituting (1.4.7) in (1.4.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \quad (1.4.8)$$

Hence value of  $k=6$  represents pair of straight lines. Substituting value of  $k=6$  in (1.4.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & 6 \end{vmatrix} \quad (1.4.9)$$

Simplifying the above determinant, we get

$$\delta = 0 \quad (1.4.10)$$

(1.4.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \quad (1.4.11)$$

Since  $\det(V) < 0$  lines would intersect each other

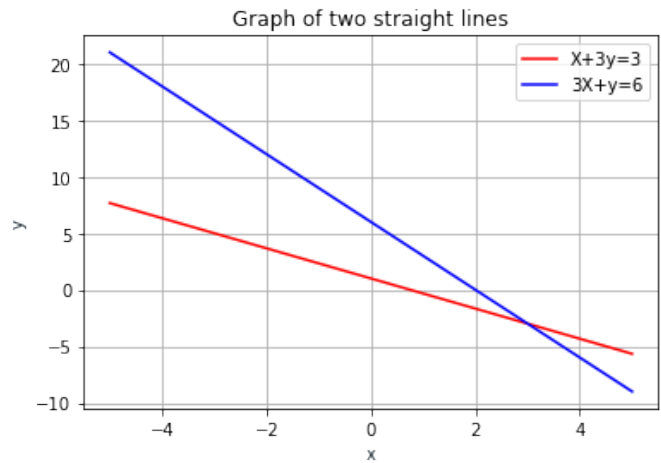


Fig. 1.4.1: Pair of straight lines

$$\mathbf{n}_1 * \mathbf{n}_2 = \left\{ 1, \frac{10}{3}, 1 \right\} \quad (1.4.12)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{5}{2} \\ \frac{7}{2} \end{pmatrix} \quad (1.4.13)$$

$$c_1 c_2 = 6 \quad (1.4.14)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.4.15)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.4.16)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.4.17)$$

Substituting in above equations (1.4.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \quad (1.4.18)$$

$$\Rightarrow m_i = \frac{\frac{-10}{3} \pm \sqrt{-\left(\frac{-16}{9}\right)}}{1} \quad (1.4.19)$$



Solving equation (1.4.19) we have ,

$$m_1 = \frac{-1}{3} \quad (1.4.20)$$

$$m_2 = -3 \quad (1.4.21)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.4.23)$$

Substituting equations (1.4.22), (1.4.23) in equation (1.4.12) we get

$$k_1 k_2 = 1 \quad (1.4.24)$$

Possible combination of  $(k_1, k_2)$  is (1,1) Lets assume  $k_1 = 1, k_2 = 1$ , we get

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.25)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.4.26)$$

we have:

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.4.27)$$

Convolution of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  can be done by converting  $\mathbf{n}_1$  into a teoplitz matrix and multiplying with  $\mathbf{n}_2$

From equation (1.4.25) and (1.4.26)

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.4.28)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.4.29)$$

$c_1$  and  $c_2$  can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.4.30)$$

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -5 \\ \frac{2}{7} \\ \frac{2}{2} \end{pmatrix} \quad (1.4.31)$$

Substituting (1.4.25) and (1.4.26) in (1.4.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \quad (1.4.32)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \quad (1.4.33)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.4.34)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.4.35)$$

From above we get

$$c_1 = 1 \quad (1.4.36)$$

$$c_2 = 6 \quad (1.4.37)$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \quad (1.4.38)$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (1.4.39)$$