Coordinate Geometry Exercises

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Contents

***Abstract*—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.**

1 Conics

1.1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.1.1}$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.2}$$
$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.1.3}$$
$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.1.4}$$
$$f_1 = -4 \tag{1.1.5}$$
$$\mathbf{O_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.1.6}$$

Taking equation of the second circle to be,

$$\left\|\mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\|^2 = 2^2 \tag{1.1.7}$$
$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{1.1.8}$$
$$\mathbf{u_2} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.1.9}$$
$$f_2 = 0 \tag{1.1.10}$$
$$\mathbf{O_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.1.11}$$

Now,Subtracting equation (1.1.8) from (1.1.3) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{1.1.12}$$
$$2\mathbf{u}^T\mathbf{x} = -4 \tag{1.1.13}$$
$$\begin{pmatrix} -4 & 0 \end{pmatrix}\mathbf{x} = -4 \tag{1.1.14}$$

Which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 1 \tag{1.1.15}$$
$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.1.16}$$
$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.17}$$
$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.1.18}$$
$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.1.19}$$

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Substituting (1.1.17) in (1.1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.20}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.21}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.22}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.23}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.24}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.25}$$

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.26}$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.27}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.1.28}$$

$$\lambda^2 = 3 \tag{1.1.29}$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{1.1.30}$$

Substituting the value of λ in(1.1.17)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.31}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.1.32}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{1.1.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \tag{1.1.34}$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \tag{1.1.35}$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{1.1.36}$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.1.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \tag{1.1.38}$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.1.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.1.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.1.41}$$

$$\theta_1 = 120^\circ \tag{1.1.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \tag{1.1.43}$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.1.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.1.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.1.46}$$

$$\theta_2 = 120^\circ \tag{1.1.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.48}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.50}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.52}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \tag{1.1.53}$$

1.2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre(h,k) is given by,

$$x^Tx + 2u^Tx + f = 0 \tag{1.2.1}$$

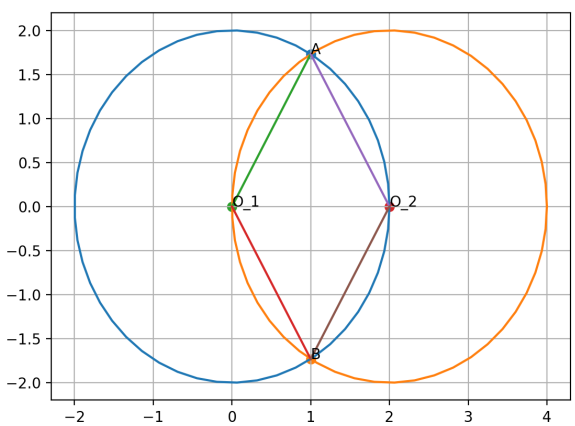

Fig. 1.1: Figure depicting intersection points of circle

where,

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.2.2}$$

The radius and centre are respectively given by,

$$r = 5 \tag{1.2.3}$$

$$\mathbf{c} = -u = k\mathbf{e} \tag{1.2.4}$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.5}$$

$$\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.2.6}$$

From the given data , we modify equation 1.2.1 as,

$$\mathbf{x_1}^T\mathbf{x_1} + 2\begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \tag{1.2.7}$$

$$\|\mathbf{x_1}\|^2 + 2\left(k^2\right) + f = 0 \tag{1.2.8}$$

$$2k^2 + f = -\|\mathbf{x_1}\|^2 \tag{1.2.9}$$

Subsituting $\mathbf{u}$ in equation 1.2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \tag{1.2.10}$$

$$f = \left(k^2\right) - r^2 \tag{1.2.11}$$

$$k^2 - f = r^2 \tag{1.2.12}$$

From equations 1.2.9 and 1.2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x_1}\|^2 \\ r^2 \end{pmatrix} \tag{1.2.13}$$

Here ,$\|\mathbf{x_1}\|$ is given by ,

$$\|\mathbf{x_1}\| = \sqrt{2^2 + 3^2} \tag{1.2.14}$$

$$\|\mathbf{x_1}\| = \sqrt{13} \tag{1.2.15}$$

Subsituting equation 1.2.6,1.2.3 in equation 1.2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \tag{1.2.16}$$

The augumented matrix of 1.2.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \tag{1.2.17}$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \tag{1.2.18}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \tag{1.2.19}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \tag{1.2.20}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array}\right) \tag{1.2.21}$$

Equation 1.2.16 can we rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \tag{1.2.22}$$

Expanding the above equation 1.2.22 we get ,

$$k^2 = 4 \tag{1.2.23}$$

$$k = \pm 2 \tag{1.2.24}$$

$$f = -21 \tag{1.2.25}$$

To get the centre substitute equation 1.2.24 in equation 1.2.4 To verify the above results we plot the circle with centre $\mathbf{c}$ as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, qFrom the above figure 1 it is clear that circle with centre $\mathbf{c}$=$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ passes through the point $\mathbf{x_1}$

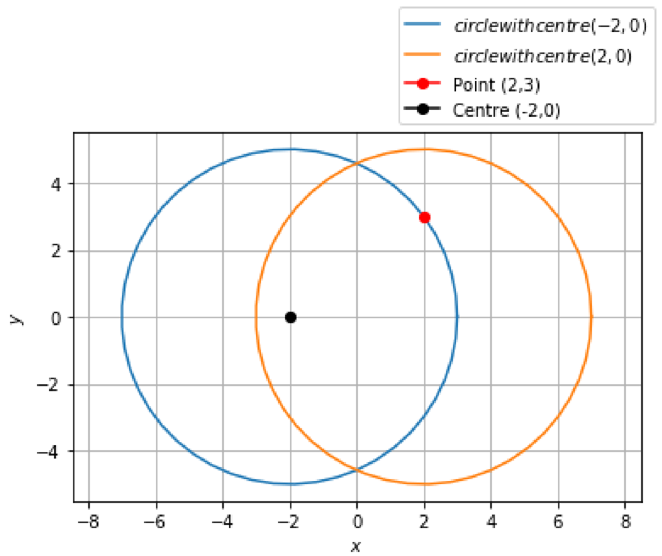

Fig. 1: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

Desired equation of circle is given by ,

$$c = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.2.26}$$

$$f = -21 \tag{1.2.27}$$

1.3. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.
1.4. Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.
1.5. Find the locus of all the unit vectors in the xy-plane.
1.6. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
1.7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^T\mathbf{x} = 4$.

Solution: The equation of a circle can be expressed as,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{1.7.1}$$

where $\mathbf{c}$ is the center.

Comparing equation (1.7.1) with the circle equation given,

$$\mathbf{x}^T\mathbf{x} = 4 \tag{1.7.2}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad f = -4 \tag{1.7.3}$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \tag{1.7.4}$$

$$\implies \boxed{r = 2} \tag{1.7.5}$$

From equation (1.7.5), the point at which circle touches x-axis is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The direction vector of x-axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
The direction vector of the given line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ is $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$.
The angle that the line makes with the x-axis is given by,

$$\cos\theta = \frac{\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\|\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 & 0 \end{pmatrix}\right\|} = \frac{\sqrt{3}}{2} \tag{1.7.6}$$

$$\implies \boxed{\theta = 30^\circ} \tag{1.7.7}$$

Using equation (1.7.5) and (1.7.7),the area of the sector is obtained as,

$$\implies \boxed{\frac{\theta}{360^\circ}\pi r^2 = \frac{30^\circ}{360^\circ}\pi(2)^2 = \frac{\pi}{3}} \tag{1.7.8}$$

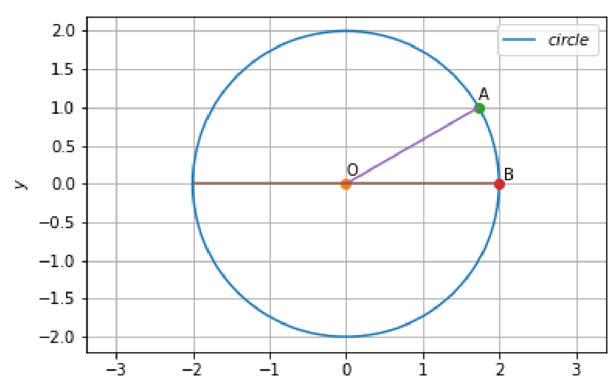

Fig. 3: Region enclosed by x-axis, line and circle

To find points $\mathbf{A}$ and $\mathbf{B}$, The parametric form of x-axis is,

$$\mathbf{B} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.9}$$
$$= \begin{pmatrix}0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\end{pmatrix} \tag{1.7.10}$$

From the intersection of circle and line, the value of λ can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.11}$$
$$= \frac{4-0}{1} = 4 \tag{1.7.12}$$
$$\implies \lambda = \pm 2 \tag{1.7.13}$$

Sub equation (1.7.13) in (1.7.10),

$$\mathbf{B} = \begin{pmatrix}2\\0\end{pmatrix} \quad \mathbf{B} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.7.14}$$

As given in question as first quadrant,

$$\implies \boxed{\mathbf{B} = \begin{pmatrix}2\\0\end{pmatrix}} \tag{1.7.15}$$

Similarly, to find point $\mathbf{A}$, The parametric form of line is,

$$\mathbf{A} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.16}$$
$$= \begin{pmatrix}0\\0\end{pmatrix} + \lambda\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} \tag{1.7.17}$$
$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.18}$$
$$= \frac{4-0}{4} = 1 \tag{1.7.19}$$
$$\implies \lambda = \pm 1 \tag{1.7.20}$$
$$\mathbf{A} = \begin{pmatrix}\sqrt{3}\\1\end{pmatrix} \quad \mathbf{A} = \begin{pmatrix}-\sqrt{3}\\-1\end{pmatrix} \tag{1.7.21}$$
$$\implies \boxed{\mathbf{A} = \begin{pmatrix}\sqrt{3}\\1\end{pmatrix}} \tag{1.7.22}$$

1.8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
1.9. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.
1.10. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
1.11. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.
1.12. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

Solution:

$$y = \frac{x-1}{x-2} \tag{1.12.1}$$

Equation (1.12.1) can be expressed as

$$y(x-2) = x - 1 \tag{1.12.2}$$
$$yx - 2y - x + 1 = 0 \tag{1.12.3}$$

From above we can say,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{1.12.4}$$
$$\mathbf{u} = \begin{pmatrix}-\frac{1}{2} & -1\end{pmatrix} \tag{1.12.5}$$
$$f = 1 \tag{1.12.6}$$

Now,

$$\because |V| = \begin{vmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{vmatrix} < 0, \tag{1.12.7}$$

(1.12.1) is the equation of a hyperbola. To verify that this we will find the the characteristic equation of $\mathbf{V}$.

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda & \frac{1}{2}\\ \frac{1}{2} & \lambda\end{vmatrix} = 0 \tag{1.12.8}$$
$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{1.12.9}$$

The eigenvalues are the roots of (1.12.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \tag{1.12.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{1.12.11}$$
$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{1.12.12}$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \tag{1.12.13}$$
$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{1.12.14}$$

Now,λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.12.15}$$

$$\implies \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.12.16}$$

From Equations,

$$\mathbf{V} = \mathbf{PDP}^{-1} = \mathbf{PDP}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{1.12.17}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{VP} \tag{1.12.18}$$

We can say that

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.12.19}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{1.12.20}$$

$\because$ $\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f > 0$,there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-}1\mathbf{u} \tag{1.12.21}$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{1.12.22}$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{1.12.23}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.12.24}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{1.12.25}$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \tag{1.12.26}$$

Let us assume slope to be l,now finding the direction vector and normal vector of the tangent with slope l.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \tag{1.12.27}$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \tag{1.12.28}$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{1.12.29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.12.30}$$

By using (1.12.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \tag{1.12.31}$$

Now substituting this κ in (1.12.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{\frac{-l}{4}} + 1 \end{pmatrix} \tag{1.12.32}$$

We know that x=10.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \tag{1.12.33}$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \tag{1.12.34}$$

$$\sqrt{-\frac{1}{4l}} = 4 \tag{1.12.35}$$

$$-\frac{1}{4l} = 16 \tag{1.12.36}$$

$$l = -\frac{1}{64} \tag{1.12.37}$$

The slope of the tangent to the curve y=$\frac{x-1}{x-2}$, x≠2 at x=10 is $\frac{1}{64}$. So,from the above we can say that κ=4,-4 and from equation (1.12.27) and (1.12.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \tag{1.12.38}$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \tag{1.12.39}$$

Now using equation (1.12.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1}\left(\kappa_1 \mathbf{n} - \mathbf{u}\right) \tag{1.12.40}$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(-4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.12.41}$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{1.12.42}$$

$$\mathbf{q}_2 = \mathbf{V}^{-1}\left(\kappa_2 \mathbf{n} - \mathbf{u}\right) \tag{1.12.43}$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.12.44}$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \tag{1.12.45}$$

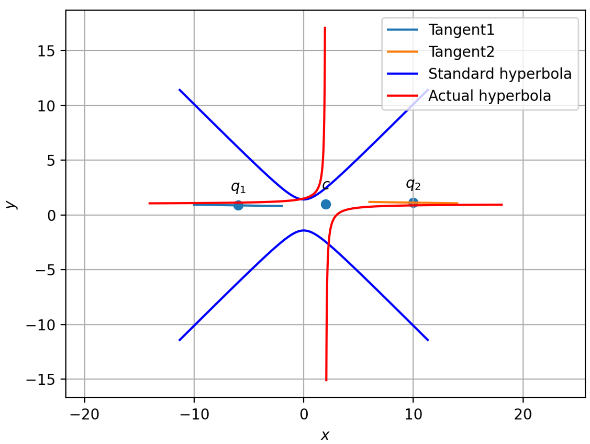

Fig. 4: Tangent 2 shows the tangent

1.13. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

1.14. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

1.15. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.

1.16. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at which tangents are

a) parallel to x-axis

b) parallel to y-axis.

1.17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

1.18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$

a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$

b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.

1.19. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0$.

1.20. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.

1.21. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.

1.22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$

1.23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

1.24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

1.25. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

1.26. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

1.27. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.28. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

1.30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

1.31. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.

1.32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

1.33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.

1.34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.

1.35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.

1.36. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and

$OB = 6$. Find the area between the arc AB and the chord AB.

1.37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.

1.38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.

1.39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.

1.40. Find the area between $y = x^2$ and $y = x$.

1.41. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

1.42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix}\mathbf{x} = 12$.

1.43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$

1.44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.

1.45. Find the area bounded by the curves

$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{1.45.1}$$

1.46. Find the area of the region

$$\left\{(x, y) : y^2 \le 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{1.46.1}$$

1.47. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.

2 QR Decomposition

2.1. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution: Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.1.1}$$

$$\beta = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.1.2}$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \tag{2.1.3}$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{2.1.4}$$

where

$$k_1 = \|\alpha\| \tag{2.1.5}$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \tag{2.1.7}$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \tag{2.1.8}$$

$$k_2 = \mathbf{u}_2^T \beta \tag{2.1.9}$$

From (2.1.3) and (2.1.4),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.10}$$

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \mathbf{QR} \tag{2.1.11}$$

From above we can see that $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \tag{2.1.12}$$

Now by using equations (2.1.5) to (2.1.9)

$$k_1 = \sqrt{5} \tag{2.1.13}$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{2.1.14}$$

$$r_1 = \sqrt{5} \tag{2.1.15}$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{2.1.16}$$

$$k_2 = \sqrt{5} \tag{2.1.17}$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \tag{2.1.18}$$

2.2. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

2.3. $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

2.4. $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

2.5. $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$

2.6. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

2.7. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

2.8. $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

2.9. $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

2.10. $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2.11. $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$

2.12. $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

2.13. $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

2.14. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$