Coordinate Geometry Exercises

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Contents

***Abstract*—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.**

1 Conics

1.1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.1.1}$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.2}$$
$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.1.3}$$
$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.1.4}$$
$$f_1 = -4 \tag{1.1.5}$$
$$\mathbf{O_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.1.6}$$

Taking equation of the second circle to be,

$$\left\|\mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\|^2 = 2^2 \tag{1.1.7}$$
$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{1.1.8}$$
$$\mathbf{u_2} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.1.9}$$
$$f_2 = 0 \tag{1.1.10}$$
$$\mathbf{O_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.1.11}$$

Now,Subtracting equation (1.1.8) from (1.1.3) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{1.1.12}$$
$$2\mathbf{u}^T\mathbf{x} = -4 \tag{1.1.13}$$
$$\begin{pmatrix} -4 & 0 \end{pmatrix}\mathbf{x} = -4 \tag{1.1.14}$$

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Which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1 \tag{1.1.15}$$

$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{1.1.16}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.17}$$

$$\mathbf{q} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.1.18}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.1.19}$$

Substituting (1.1.17) in (1.1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.20}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.21}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.22}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.23}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.24}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.25}$$

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.26}$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.27}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.1.28}$$

$$\lambda^2 = 3 \tag{1.1.29}$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{1.1.30}$$

Substituting the value of λ in(1.1.17)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.31}$$

$$\mathbf{A} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{1.1.32}$$

$$\mathbf{B} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{1.1.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix} \tag{1.1.34}$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}-1\\\sqrt{3}\end{pmatrix} \tag{1.1.35}$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{1.1.36}$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{1.1.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \tag{1.1.38}$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.1.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.1.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.1.41}$$

$$\theta_1 = 120^\circ \tag{1.1.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \tag{1.1.43}$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.1.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.1.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.1.46}$$

$$\theta_2 = 120^\circ \tag{1.1.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.48}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.50}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.52}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \tag{1.1.53}$$

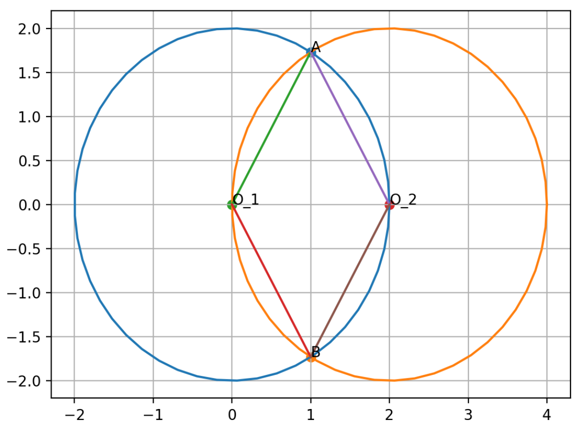

Fig. 1.1: Figure depicting intersection points of circle

1.2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre(h,k) is given by,

$$x^T x + 2u^T x + f = 0 \tag{1.2.1}$$

where,

$$f = \mathbf{u}^T \mathbf{u} - r^2 \tag{1.2.2}$$

The radius and centre are respectively given by,

$$r = 5 \tag{1.2.3}$$

$$\mathbf{c} = -u = k\mathbf{e} \tag{1.2.4}$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.5}$$

$$\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.2.6}$$

From the given data , we modify equation 1.2.1 as,

$$\mathbf{x_1}^T \mathbf{x_1} + 2\begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \tag{1.2.7}$$

$$\|\mathbf{x_1}\|^2 + 2\left(k^2\right) + f = 0 \tag{1.2.8}$$

$$2k^2 + f = -\|\mathbf{x_1}\|^2 \tag{1.2.9}$$

Subsituting $\mathbf{u}$ in equation 1.2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \tag{1.2.10}$$

$$f = \left(k^2\right) - r^2 \tag{1.2.11}$$

$$k^2 - f = r^2 \tag{1.2.12}$$

From equations 1.2.9 and 1.2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x_1}\|^2 \\ r^2 \end{pmatrix} \tag{1.2.13}$$

Here ,$\|\mathbf{x_1}\|$ is given by ,

$$\|\mathbf{x_1}\| = \sqrt{2^2 + 3^2} \tag{1.2.14}$$

$$\|\mathbf{x_1}\| = \sqrt{13} \tag{1.2.15}$$

Subsituting equation 1.2.6,1.2.3 in equation 1.2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \tag{1.2.16}$$

The augumented matrix of 1.2.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \tag{1.2.17}$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \tag{1.2.18}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \tag{1.2.19}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \tag{1.2.20}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array}\right) \tag{1.2.21}$$

Equation 1.2.16 can we rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \tag{1.2.22}$$

Expanding the above equation 1.2.22 we get ,

$$k^2 = 4 \tag{1.2.23}$$
$$k = \pm 2 \tag{1.2.24}$$
$$f = -21 \tag{1.2.25}$$

To get the centre substitute equation 1.2.24 in equation 1.2.4 To verify the above results we plot the circle with centre $\mathbf{c}$ as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$,

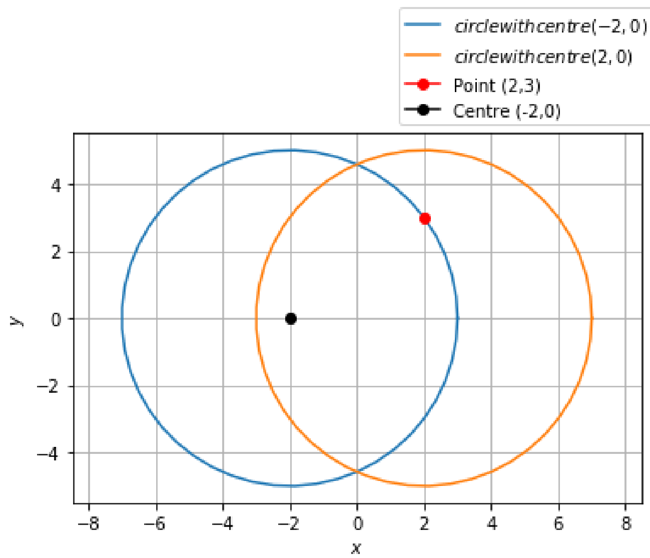

Fig. 1.2: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

qFrom the above figure 1.2 it is clear that circle with centre $\mathbf{c}$=$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ passes through the point $\mathbf{x_1}$ Desired equation of circle is given by ,

$$c = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.2.26}$$
$$f = -21 \tag{1.2.27}$$

1.3. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.

1.4. Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Solution: he general equation of a circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.4.1}$$
$$\text{If } r \text{ is radius, } f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.4.2}$$
$$\text{center } \mathbf{c} = -\mathbf{u} \tag{1.4.3}$$

Given centre is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{1.4.4}$$
$$\implies \mathbf{u} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{1.4.5}$$

Equation (1.4.1) becomes

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -4 & -4 \end{pmatrix}\mathbf{x} + f = 0 \tag{1.4.6}$$

This passes through point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Substituting $\mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ in (1.4.6)

$$\begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 & -4 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} + f = 0 \tag{1.4.7}$$
$$\implies f = -5 \tag{1.4.8}$$

Also, radius can be determined as follows

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.4.9}$$
$$\implies -5 = \begin{pmatrix} -2 & -2 \end{pmatrix}\begin{pmatrix} -2 \\ -2 \end{pmatrix} - r^2 \tag{1.4.10}$$
$$\implies -5 = 8 - r^2 \tag{1.4.11}$$
$$\implies r = \sqrt{13} \tag{1.4.12}$$

The equation of required circle is

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -4 & -4 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{1.4.13}$$

See Fig. 1.4

1.5. Find the locus of all the unit vectors in the xy-plane.

1.6. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.

Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.6.1}$$

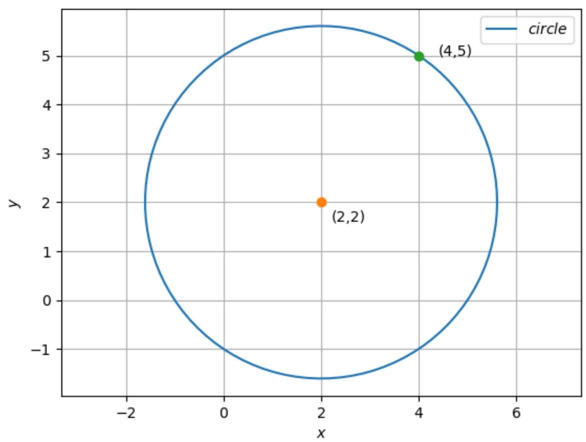

Fig. 1.4: plot showing the circle

The centre and the radius can be obtained as,

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.2}$$

$$f = -3 \tag{1.6.3}$$

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.6.4}$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \tag{1.6.5}$$

$\because$ The tangents are parallel to the x-axis, their direction and normal vectors, $\mathbf{m}$ and $\mathbf{n}$ are respectively,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.6.6}$$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.6.7}$$

For a circle, given the normal vector $\mathbf{n}$, the tangent points of contact to circle given by equation (1.6.1) are given by

$$\mathbf{q_i} = (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \tag{1.6.8}$$

where

$$\kappa_i = \pm \sqrt{\frac{\mathbf{u^T u} - f}{\mathbf{n^T n}}} \tag{1.6.9}$$

$$\kappa = \pm \sqrt{\frac{\begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} - (-3)}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{1.6.10}$$

$$\implies \kappa = \pm \sqrt{\frac{4}{1}} \tag{1.6.11}$$

$$\implies \kappa = \pm 2 \tag{1.6.12}$$

and from (1.6.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_1} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.13}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.6.14}$$

$$\mathbf{q_2} = -2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.15}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.6.16}$$

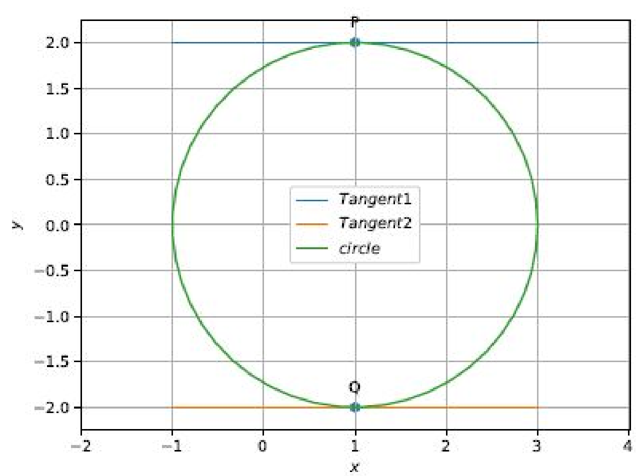

Fig. 1.6: Figure depicting tangents of circle parallel to x-axis

1.7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 0$ and the circle $\mathbf{x}^T \mathbf{x} = 4$.

Solution: The equation of a circle can be expressed as,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \tag{1.7.1}$$

where $\mathbf{c}$ is the center.
Comparing equation (1.7.1) with the circle equation given,

$$\mathbf{x}^T\mathbf{x} = 4 \tag{1.7.2}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad f = -4 \tag{1.7.3}$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \tag{1.7.4}$$

$$\implies \boxed{r = 2} \tag{1.7.5}$$

From equation (1.7.5), the point at which circle touches x-axis is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The direction vector of x-axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
The direction vector of the given line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ is $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$.
The angle that the line makes with the x-axis is given by,

$$\cos\theta = \frac{\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\|\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 & 0 \end{pmatrix}\right\|} = \frac{\sqrt{3}}{2} \tag{1.7.6}$$

$$\implies \boxed{\theta = 30^\circ} \tag{1.7.7}$$

Using equation (1.7.5) and (1.7.7),the area of the sector is obtained as,

$$\implies \boxed{\frac{\theta}{360^\circ}\pi r^2 = \frac{30^\circ}{360^\circ}\pi(2)^2 = \frac{\pi}{3}} \tag{1.7.8}$$

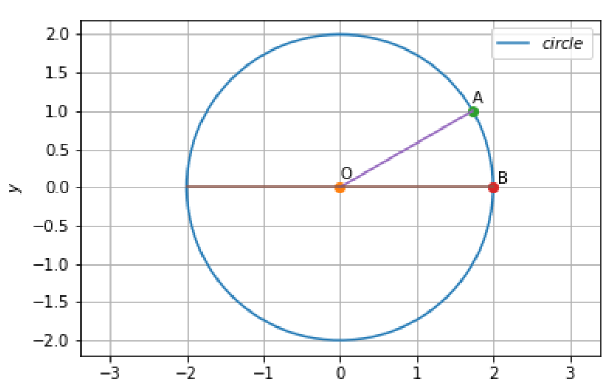

Fig. 1.7: Region enclosed by x-axis, line and circle

To find points $\mathbf{A}$ and $\mathbf{B}$,

The parametric form of x-axis is,

$$\mathbf{B} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.9}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.7.10}$$

From the intersection of circle and line, the value of λ can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.11}$$

$$= \frac{4 - 0}{1} = 4 \tag{1.7.12}$$

$$\implies \lambda = \pm 2 \tag{1.7.13}$$

Sub equation (1.7.13) in (1.7.10),

$$\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.7.14}$$

As given in question as first quadrant,

$$\implies \boxed{\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}} \tag{1.7.15}$$

Similarly, to find point $\mathbf{A}$, The parametric form of line is,

$$\mathbf{A} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.16}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{1.7.17}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.18}$$

$$= \frac{4 - 0}{4} = 1 \tag{1.7.19}$$

$$\implies \lambda = \pm 1 \tag{1.7.20}$$

$$\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \tag{1.7.21}$$

$$\implies \boxed{\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \tag{1.7.22}$$

1.8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
1.9. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.
1.10. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
Solution:
General equation of circle is $\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} +$ f $= 0$

Taking equation of the first curve to be,

$$\left\|\mathbf{x}-\begin{pmatrix}1\\0\end{pmatrix}\right\|^2 = 1^2 \tag{1.10.1}$$

$$\mathbf{x}^T\mathbf{x}+2\mathbf{u_1}^T\mathbf{x}=0 \tag{1.10.2}$$

$$\mathbf{u_1}=\begin{pmatrix}-1\\0\end{pmatrix} \tag{1.10.3}$$

$$f_1=0 \tag{1.10.4}$$

$$\mathbf{O_1}=\begin{pmatrix}1\\0\end{pmatrix} \tag{1.10.5}$$

Taking equation of the second curve to be,

$$\|\mathbf{x}\|^2+2\mathbf{u}_2^T\mathbf{x}+f_2=0 \tag{1.10.6}$$

$$\mathbf{x}^T\mathbf{x}-1=0 \tag{1.10.7}$$

$$\mathbf{u_2}=\begin{pmatrix}0\\0\end{pmatrix} \tag{1.10.8}$$

$$f_2=-1 \tag{1.10.9}$$

$$\mathbf{O_2}=\begin{pmatrix}0\\0\end{pmatrix} \tag{1.10.10}$$

Now, subtracting equation (1.10.2) from (1.10.7) We get,

$$\mathbf{x}^T\mathbf{x}+2\mathbf{u_1}^T\mathbf{x}-\mathbf{x}^T\mathbf{x}-f_2=0 \tag{1.10.11}$$

$$2\mathbf{u}^T\mathbf{x}=-1 \tag{1.10.12}$$

$$\begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x}=-1 \tag{1.10.13}$$

which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}=1/2 \tag{1.10.14}$$

$$\mathbf{x}=\begin{pmatrix}1/2\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{1.10.15}$$

$$\mathbf{x}=\mathbf{q}+\lambda\mathbf{m} \tag{1.10.16}$$

$$\mathbf{q}=\begin{pmatrix}1/2\\0\end{pmatrix} \tag{1.10.17}$$

$$\mathbf{m}=\begin{pmatrix}0\\1\end{pmatrix} \tag{1.10.18}$$

Substituting (1.10.16) in (1.10.6)

$$\|\mathbf{x}\|^2+2\mathbf{u}_2^T\mathbf{x}+f_2=0 \tag{1.10.19}$$

$$\|\mathbf{q}+\lambda\mathbf{m}\|^2+f_2=0 \tag{1.10.20}$$

$$(\mathbf{q}+\lambda\mathbf{m})^T(\mathbf{q}+\lambda\mathbf{m})+f_2=0 \tag{1.10.21}$$

$$\mathbf{q}^T(\mathbf{q}+\lambda\mathbf{m})+\lambda\mathbf{m}^T(\mathbf{q}+\lambda\mathbf{m})+f_2=0 \tag{1.10.22}$$

$$\|\mathbf{q}\|^2+\lambda\mathbf{q}^T\mathbf{m}+\lambda\mathbf{m}^T\mathbf{q}+\lambda^2\|\mathbf{m}\|^2+f_2=0 \tag{1.10.23}$$

$$\|\mathbf{q}\|^2+2\lambda\mathbf{q}^T\mathbf{m}+\lambda^2\|\mathbf{m}\|^2+f_2=0 \tag{1.10.24}$$

Taking λ as common :

$$\lambda(\lambda\|\mathbf{m}\|^2+2\mathbf{q}^T\mathbf{m})=-f_2-\|\mathbf{q}\|^2 \tag{1.10.25}$$

$$\lambda^2\|\mathbf{m}\|^2=-f_2-\|\mathbf{q}\|^2 \tag{1.10.26}$$

$$\lambda^2=\frac{-f_2-\|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.10.27}$$

$$\lambda^2=\frac{3}{4} \tag{1.10.28}$$

$$\lambda=+\sqrt{\frac{3}{4}},-\sqrt{\frac{3}{4}} \tag{1.10.29}$$

$$\lambda=+\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2} \tag{1.10.30}$$

Substituting the value of λ in (1.10.16)

$$\mathbf{x}=\mathbf{q}+\lambda\mathbf{m} \tag{1.10.31}$$

$$\mathbf{A}=\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.32}$$

$$\mathbf{B}=\begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A}=\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.34}$$

$$\mathbf{m}_{O_1B}=\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.35}$$

$$\mathbf{m}_{O_2A}=\begin{pmatrix}0\\0\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.36}$$

$$\mathbf{m}_{O_2B}=\begin{pmatrix}0\\0\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \tag{1.10.38}$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.10.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.10.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.10.41}$$

$$\theta_1 = 120^\circ \tag{1.10.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \tag{1.10.43}$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.10.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.10.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.10.46}$$

$$\theta_2 = 120^\circ \tag{1.10.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\pi\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.10.48}$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \tag{1.10.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.10.50}$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \tag{1.10.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \tag{1.10.52}$$

$$= \frac{2\pi}{3} - 2\sqrt{3} \tag{1.10.53}$$

1.11. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.

1.12. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

Solution:

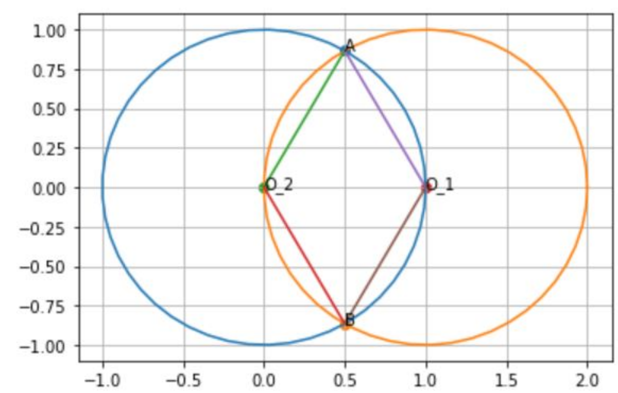

Fig. 1.10: Figure depicting intersection points of circle

$$y = \frac{x-1}{x-2} \tag{1.12.1}$$

Equation (1.12.1) can be expressed as

$$y(x-2) = x-1 \tag{1.12.2}$$

$$yx - 2y - x + 1 = 0 \tag{1.12.3}$$

From above we can say,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{1.12.4}$$

$$\mathbf{u} = \begin{pmatrix}-\frac{1}{2} & -1\end{pmatrix} \tag{1.12.5}$$

$$f = 1 \tag{1.12.6}$$

Now,

$$\because |V| = \begin{vmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{vmatrix} < 0, \tag{1.12.7}$$

(1.12.1) is the equation of a hyperbola. To verify that this we will find the the characteristic equation of $\mathbf{V}$.

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda & \frac{1}{2}\\ \frac{1}{2} & \lambda\end{vmatrix} = 0 \tag{1.12.8}$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{1.12.9}$$

The eigenvalues are the roots of (1.12.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \tag{1.12.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{1.12.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{1.12.12}$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.12.13}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.12.14}$$

Now,λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.12.15}$$

$$\implies \mathbf{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.12.16}$$

From Equations,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{1.12.17}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{1.12.18}$$

We can say that

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.12.19}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{1.12.20}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f > 0$,there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-}1\mathbf{u} \tag{1.12.21}$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{1.12.22}$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{1.12.23}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.12.24}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{1.12.25}$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \tag{1.12.26}$$

Let us assume slope to be l,now finding the direction vector and normal vector of the tangent with slope l.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \tag{1.12.27}$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \tag{1.12.28}$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{1.12.29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.12.30}$$

By using (1.12.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \tag{1.12.31}$$

Now substituting this κ in (1.12.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{\frac{-l}{4}} + 1 \end{pmatrix} \tag{1.12.32}$$

We know that x=10.

$$-2\sqrt{-\frac{1}{4l}}+2=10 \tag{1.12.33}$$

$$-2\sqrt{-\frac{1}{4l}}=8 \tag{1.12.34}$$

$$\sqrt{-\frac{1}{4l}}=4 \tag{1.12.35}$$

$$-\frac{1}{4l}=16 \tag{1.12.36}$$

$$l=-\frac{1}{64} \tag{1.12.37}$$

The slope of the tangent to the curve y=$\frac{x-1}{x-2}$, x≠2 at x=10 is $\frac{1}{64}$. So,from the above we can say that κ=4,-4 and from equation (1.12.27) and (1.12.28) direction and normal vectors will come out to be

$$\mathbf{m}=\begin{pmatrix}1\\-\frac{1}{64}\end{pmatrix} \tag{1.12.38}$$

$$\mathbf{n}=\begin{pmatrix}-\frac{1}{64}\\-1\end{pmatrix} \tag{1.12.39}$$

Now using equation (1.12.29)

$$\mathbf{q}_1=\mathbf{V}^{-1}\left(\kappa_1\mathbf{n}-\mathbf{u}\right) \tag{1.12.40}$$

$$\mathbf{q}_1=\begin{pmatrix}0&2\\2&0\end{pmatrix}\left(-4\begin{pmatrix}-\frac{1}{64}\\-1\end{pmatrix}-\begin{pmatrix}-\frac{1}{2}\\-1\end{pmatrix}\right) \tag{1.12.41}$$

$$\mathbf{q}_1=\begin{pmatrix}10\\\frac{9}{8}\end{pmatrix} \tag{1.12.42}$$

$$\mathbf{q}_2=\mathbf{V}^{-1}\left(\kappa_2\mathbf{n}-\mathbf{u}\right) \tag{1.12.43}$$

$$\mathbf{q}_2=\begin{pmatrix}0&2\\2&0\end{pmatrix}\left(4\begin{pmatrix}-\frac{1}{64}\\-1\end{pmatrix}-\begin{pmatrix}-\frac{1}{2}\\-1\end{pmatrix}\right) \tag{1.12.44}$$

$$\mathbf{q}_2=\begin{pmatrix}-6\\\frac{7}{8}\end{pmatrix} \tag{1.12.45}$$

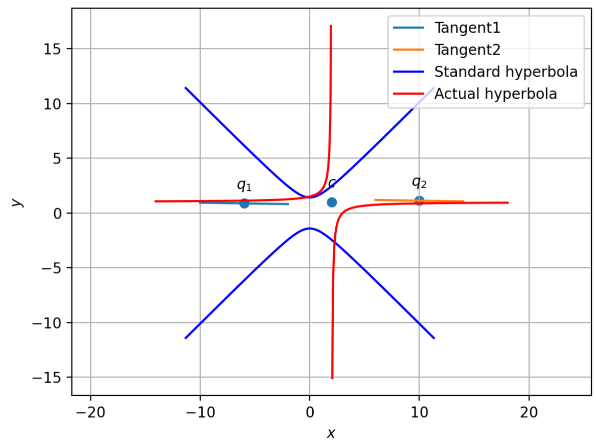

Fig. 1.12: Tangent 2 shows the tangent

1.13. Find a point on the curve $y=(x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix}2\\0\end{pmatrix}$ and $\begin{pmatrix}4\\4\end{pmatrix}$.

Solution: $y=(x-2)^2$ can be written as,

$$x^2-4x-y+4=0 \tag{1.13.1}$$

From (1.13.1),

$$\mathbf{V}=\begin{pmatrix}1&0\\0&0\end{pmatrix};\mathbf{u}=\begin{pmatrix}-2\\-\frac{1}{2}\end{pmatrix};f=4 \tag{1.13.2}$$

$$\left|V\right|=\begin{vmatrix}1&0\\0&0\end{vmatrix}=0 \tag{1.13.3}$$

(1.13.3) implies that the curve is a parabola. Now, finding the eigen values corresponding to the **V**,

$$\left|V-\lambda I\right|=0$$

$$\begin{vmatrix}1-\lambda&0\\0&-\lambda\end{vmatrix}=0$$

$$\implies \lambda=0,1 \tag{1.13.4}$$

Calculating the eigenvectors corresponding to $\lambda=0,1$ respectively,

$$\mathbf{Vx}=\lambda\mathbf{x}$$

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\mathbf{x}=0;\implies\mathbf{p}_1=\begin{pmatrix}0\\1\end{pmatrix} \tag{1.13.5}$$

$$\begin{pmatrix}0&0\\0&-1\end{pmatrix}\mathbf{x}=0;\implies\mathbf{p}_2=\begin{pmatrix}1\\0\end{pmatrix} \tag{1.13.6}$$

By Eigen decomposition on **V**,

$$\mathbf{V}=\mathbf{PDP}^T$$

$$where,\mathbf{P}=\begin{pmatrix}\mathbf{p}_1&\mathbf{p}_2\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix} \tag{1.13.7}$$

$$\mathbf{D}=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix} \tag{1.13.8}$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{1.13.9}$$

$$where, \eta = \mathbf{u}^T \mathbf{p}_1 = -\frac{1}{2} \tag{1.13.10}$$

Substituting values from (1.13.2), (1.13.5) and (1.13.10) in (1.13.9),

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \tag{1.13.11}$$

Removing last row and representing (1.13.11) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -2 & -1 & -4 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-2R_2)} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.13.12}$$

From (1.13.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.13.13}$$

Direction vector of the chord joining A(4,4) and B(2,0) can be calculated as,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.13.14}$$

We know that,

$$\mathbf{m}^T \mathbf{n} = 0; \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.13.15}$$

To find the point of contact $\mathbf{q}$, which is intersection point for normal of the chord AB and also tangent of the curve,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.13.16}$$

$$where, \kappa = \frac{\mathbf{p_1}^T \mathbf{u}}{\mathbf{p_1}^T \mathbf{n}} = \frac{1}{2} \tag{1.13.17}$$

Substituting the values from (1.13.2),(1.13.15) and (1.13.17) in (1.13.16),

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{1.13.18}$$

Removing last row and representing (1.13.18) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -1 & -1 & -4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow (-R_1)} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-R_2)} \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.13.19}$$

From (1.13.19), it can be observed,

$$\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.13.20}$$

which is the required point of contact

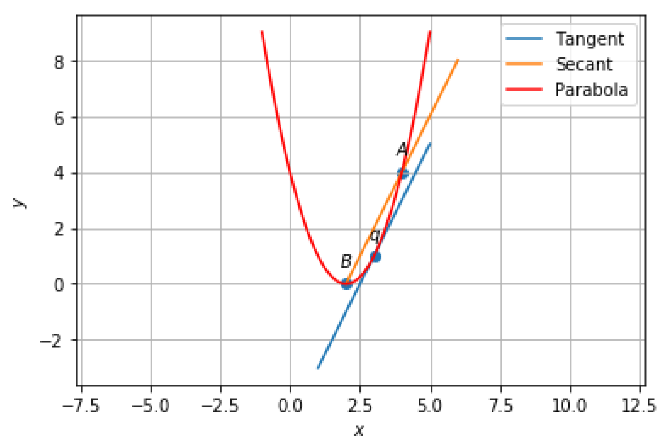

Fig. 1.13: Parabola with AB as chord, a tangent parallel to the chord

1.14. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$
Solution: The given curve

$$y = \frac{1}{x-1} \tag{1.14.1}$$

can be expressed as

$$xy - y - 1 = 0 \tag{1.14.2}$$

Hence, we have

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix}, f = -1 \tag{1.14.3}$$

Since $\left|\mathbf{V}\right| < 0$, the equation (1.14.2) represents hyperbola. To find the values of λ_1 and λ_2, consider the characteristic equation,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{1.14.4}$$

$$\implies \left|\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right| = 0 \tag{1.14.5}$$

$$\implies \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \tag{1.14.6}$$

$$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.14.7}$$

In addition, given the slope -1, the direction and normal vectors are given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.14.8}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.14.9}$$

The parameters of hyperbola are as follows:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{1.14.10}$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \tag{1.14.11}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.14.12}$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} = \sqrt{2} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \end{cases} \tag{1.14.13}$$

which represents the standard hyperbola equation,

$$\frac{x^2}{2} - \frac{x^2}{2} = 1 \tag{1.14.14}$$

The points of contact are given by

$$K = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} = \pm\frac{1}{2} \tag{1.14.15}$$

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{1.14.16}$$

$$\mathbf{q_1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \tag{1.14.17}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.14.18}$$

$$\mathbf{q_2} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{-1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \tag{1.14.19}$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1.14.20}$$

$\therefore$ The tangents are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 \tag{1.14.21}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = 0 \tag{1.14.22}$$

The desired equations of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$ are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{1.14.23}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = -1 \tag{1.14.24}$$

The above results are verified in the following figure.

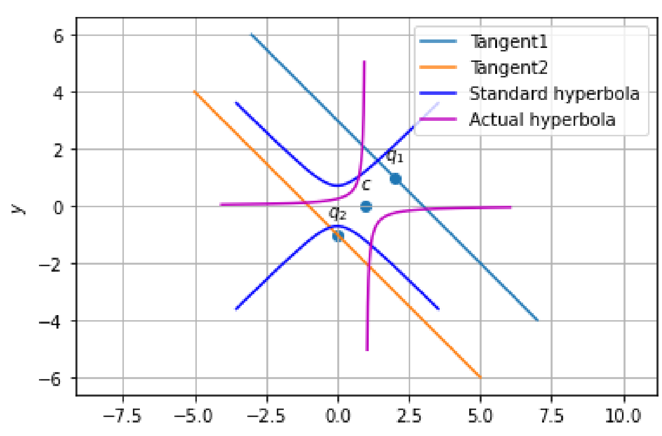

Fig. 1.14: The standard and actual hyperbola.

1.15. Find the equation of all lines having slope -2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.

Solution: Given the curve,

$$y = \frac{1}{x-3} \tag{1.15.1}$$

$$\implies xy - 3y - 1 = 0 \tag{1.15.2}$$

From (1.15.2) we get,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \mathbf{u} = \frac{-3}{2}\begin{pmatrix}0\\1\end{pmatrix}, f = -1 \tag{1.15.3}$$

Now,

$$\because \left|V\right| = \begin{vmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{vmatrix} = \frac{-1}{2} < 0 \tag{1.15.4}$$

(1.15.1) is equation of hyperbola. Now,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda & \frac{-1}{2}\\ \frac{-1}{2} & \lambda\end{vmatrix} = 0 \tag{1.15.5}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{1.15.6}$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.15.7}$$

The eigen vector $\mathbf{p}$ is given by,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{1.15.8}$$

For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix} \tag{1.15.9}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{1.15.10}$$

Similarly for λ_2,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{-1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{-1}{2}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_- R_1} \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \tag{1.15.11}$$

$$\implies \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \tag{1.15.12}$$

Now,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} \tag{1.15.13}$$

$$\mathbf{D} = \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{-1}{2}\end{pmatrix} \tag{1.15.14}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{1.15.15}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0$, there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = 3\begin{pmatrix}1\\0\end{pmatrix} \tag{1.15.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.15.17}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = \sqrt{2} \tag{1.15.18}$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \tag{1.15.19}$$

The direction and normal vectors of the tangent with slope -2 are given as,

$$\mathbf{m} = \begin{pmatrix}1\\-2\end{pmatrix}, \mathbf{n} = \begin{pmatrix}2\\1\end{pmatrix} \tag{1.15.20}$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{1.15.21}$$

$$k = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.15.22}$$

Thus,

$$\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n} = 8 \tag{1.15.23}$$

$$k = \pm\frac{1}{2\sqrt{2}} \tag{1.15.24}$$

$$\mathbf{q_1} = \begin{pmatrix}\frac{1+3\sqrt{2}}{\sqrt{2}}\\ \sqrt{2}\end{pmatrix} \tag{1.15.25}$$

$$\mathbf{q_2} = \begin{pmatrix}\frac{-1+3\sqrt{2}}{\sqrt{2}}\\ -\sqrt{2}\end{pmatrix} \tag{1.15.26}$$

The desired tangents are,

$$\begin{pmatrix}2 & 1\end{pmatrix}\left\{\mathbf{x}-\begin{pmatrix}\frac{1+3\sqrt{2}}{\sqrt{2}}\\ \sqrt{2}\end{pmatrix}\right\}=0 \tag{1.15.27}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}=6+2\sqrt{2} \tag{1.15.28}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\left\{\mathbf{x}-\begin{pmatrix}\frac{-1+3\sqrt{2}}{\sqrt{2}}\\ -\sqrt{2}\end{pmatrix}\right\}=0 \tag{1.15.29}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}=6-2\sqrt{2} \tag{1.15.30}$$

Below figure corresponds to the tangents on the hyperbola, represented by (1.15.28) and (1.15.30) each having slope of −2.

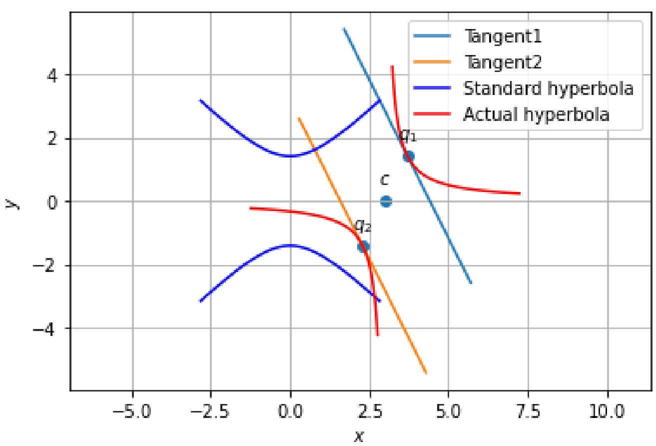

Fig. 1.15: Tangents to the hyperbola

1.16. Find points on the curve $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{1}{16}\end{pmatrix}\mathbf{x}=1$ at which tangents are
  a) parallel to x-axis
  b) parallel to y-axis.

Solution:
General equation of conics is

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0 \tag{1.16.1}$$

Comparing with the equation given,

$$\mathbf{V}=\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{1}{16}\end{pmatrix} \tag{1.16.2}$$

$$\mathbf{u}=\mathbf{0} \tag{1.16.3}$$

$$f=-1 \tag{1.16.4}$$

$$\left|\mathbf{v}\right|=\left|\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{1}{16}\end{pmatrix}\right|>0 \tag{1.16.5}$$

$\because |\mathbf{V}|>0$, the given equation is of ellipse.
a)The tangents are parallel to the x-axis, hence, their direction and normal vectors, $\mathbf{m_1}$ and $\mathbf{n_1}$ are respectively,

$$\mathbf{m_1}=\begin{pmatrix}1\\0\end{pmatrix} \tag{1.16.6}$$

$$\mathbf{n_1}=\begin{pmatrix}0\\1\end{pmatrix} \tag{1.16.7}$$

For an ellipse, given the normal vector $\mathbf{n}$, the tangent points of contact to the ellipse are given by

$$\mathbf{q}=\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u})=\mathbf{V}^{-1}\kappa\mathbf{n} \tag{1.16.8}$$

where

$$\kappa=\pm\sqrt{\frac{\mathbf{u^T}\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n^T}\mathbf{V}^{-1}\mathbf{n}}} \tag{1.16.9}$$

$$=\pm\sqrt{\frac{-f}{\mathbf{n^T}\mathbf{V}^{-1}\mathbf{n}}} \tag{1.16.10}$$

$$\mathbf{V}^{-1}=\begin{pmatrix}9 & 0\\ 0 & 16\end{pmatrix} \tag{1.16.11}$$

$$\kappa_1=\pm\sqrt{\frac{-(-1)}{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}9 & 0\\ 0 & 16\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}}} \tag{1.16.12}$$

$$\implies \kappa_1=\pm\sqrt{\frac{1}{16}} \tag{1.16.13}$$

$$\implies \kappa_1=\pm\frac{1}{4} \tag{1.16.14}$$

From (1.16.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_1}=\begin{pmatrix}9 & 0\\ 0 & 16\end{pmatrix}\frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix} \tag{1.16.15}$$

$$=\begin{pmatrix}9 & 0\\ 0 & 16\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{4}\end{pmatrix} \tag{1.16.16}$$

$$=\begin{pmatrix}0\\4\end{pmatrix} \tag{1.16.17}$$

$$\mathbf{q_2}=\begin{pmatrix}9 & 0\\ 0 & 16\end{pmatrix}\left(-\frac{1}{4}\right)\begin{pmatrix}0\\1\end{pmatrix} \tag{1.16.18}$$

$$=\begin{pmatrix}9 & 0\\ 0 & 16\end{pmatrix}\begin{pmatrix}0\\ -\frac{1}{4}\end{pmatrix} \tag{1.16.19}$$

$$=\begin{pmatrix}0\\-4\end{pmatrix} \tag{1.16.20}$$

b) The tangents are parallel to the y-axis, hence, their direction and normal vectors, $\mathbf{m_2}$

and $\mathbf{n_2}$ are respectively,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16.21}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.22}$$

Using equation (1.16.9), the values of κ for this case are

$$\kappa_2 = \pm \sqrt{\frac{-(-1)}{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}} \tag{1.16.23}$$

$$\implies \kappa_2 = \pm \sqrt{\frac{1}{9}} \tag{1.16.24}$$

$$\implies \kappa_2 = \pm \frac{1}{3} \tag{1.16.25}$$

and from (1.16.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_3} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.26}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \tag{1.16.27}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.16.28}$$

$$\mathbf{q_4} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.29}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} \tag{1.16.30}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.16.31}$$

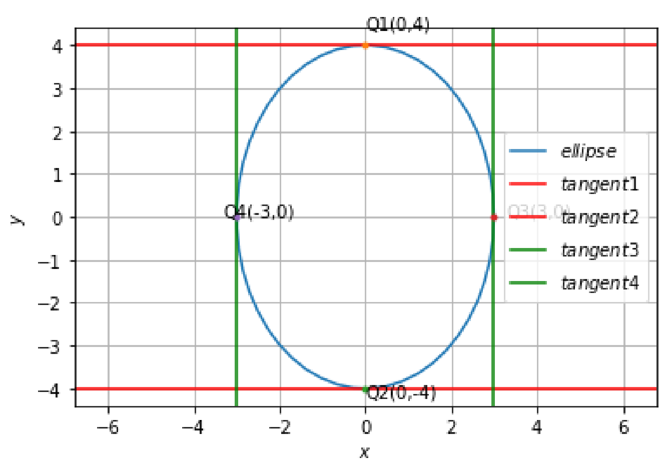

Fig. 1.16: Figure depicting point of contact of tangents of ellipse parallel to x-axis and y-axis

1.17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

1.18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$
   a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$
   b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.

1.19. Find the equation of the tangent to the curve,

$$y = \sqrt{3x - 2} \tag{1.19.1}$$

which is parallel to the line,

$$\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.19.2}$$

Solution: The equation (1.19.1) can be written as,

$$y^2 - 3x + 2 = 0 \tag{1.19.3}$$

Comparing it with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.19.4}$$

$\therefore$ a = b = e = 0, d = $\frac{-3}{2}$, c = 1, f = 2.

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.19.5}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix} \tag{1.19.6}$$

$$\text{Now,} \left| V \right| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \tag{1.19.7}$$

$\implies$ that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left| \mathbf{V} - \lambda \mathbf{I} \right| = 0 \tag{1.19.8}$$

$$\begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \tag{1.19.9}$$

$$\implies \lambda = 0, 1. \tag{1.19.10}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{Vx} = \lambda \mathbf{x} \tag{1.19.11}$$

$$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.19.12}$$

$$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.19.13}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{PDP^T} \tag{1.19.14}$$

$$\text{where,}\mathbf{P} = \begin{pmatrix}\mathbf{p_1}\mathbf{p_2}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{1.19.15}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{1.19.16}$$

Hence equation (1.19.14) becomes,

$$\mathbf{V} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{1.19.17}$$

$$\mathbf{V} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{1.19.18}$$

Now the tangent to parabola is parallel to the line equation (1.19.2), Hence the direction vectors ($\mathbf{m}$) and normal ($\mathbf{n}$) vectors are,

$$\mathbf{m} = \begin{pmatrix}1\\-2\end{pmatrix} \tag{1.19.19}$$

$$\mathbf{n} = \begin{pmatrix}2\\1\end{pmatrix} \tag{1.19.20}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix}\mathbf{u}^T + \kappa\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f\\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix} \tag{1.19.21}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = \frac{-3}{4} \tag{1.19.22}$$

Hence substituting the values of (1.19.22), (1.19.20), (1.19.14) and (1.19.6) in equation (1.19.21) we get,

$$\begin{pmatrix}-3 & \frac{-3}{4}\\ 0 & 0\\ 0 & 1\end{pmatrix}\mathbf{q} = \begin{pmatrix}-2\\ 0\\ \frac{-3}{4}\end{pmatrix} \tag{1.19.23}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (1.19.23) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix}-3 & \frac{-3}{4} & -2\\ 0 & 1 & \frac{-3}{4}\end{pmatrix} \xleftrightarrow{R_1\leftarrow\left(\frac{-R_1}{3}\right)} \begin{pmatrix}1 & \frac{1}{4} & \frac{2}{3}\\ 0 & 1 & \frac{-3}{4}\end{pmatrix} \tag{1.19.24}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{1}{4}R_2} \begin{pmatrix}1 & 0 & \frac{41}{48}\\ 0 & 1 & \frac{-3}{4}\end{pmatrix} \tag{1.19.25}$$

Hence from equation (1.19.25) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix}\frac{41}{48}\\ \frac{-3}{4}\end{pmatrix} \tag{1.19.26}$$

Now $\mathbf{q}$ is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \tag{1.19.27}$$

where c is,

$$c = \mathbf{n}^T\mathbf{q} = \begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}\frac{41}{48}\\ \frac{-3}{4}\end{pmatrix} = \frac{23}{24} \tag{1.19.28}$$

Hence equation of tangent to the curve (1.19.1) parallel to (1.19.2) is given by substituting the value of c and $\mathbf{n}$ from equation (1.19.28) and (1.19.20) respectively to the equation (1.19.27),

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = \frac{23}{24} \tag{1.19.29}$$

Figure 1.19 verifies that the $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = \frac{23}{24}$ is a tangent to parabola $y = \sqrt{3x-2}$

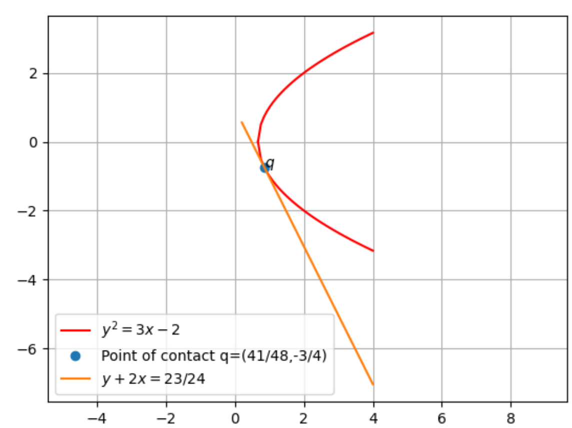

Fig. 1.19: Tangent to parabola $y = \sqrt{3x-2}$

1.20. Find the point at which the line $\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 1$

is a tangent to the curve $y^2 = 4x$.

1.21. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.

1.22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$

1.23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

1.24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

1.25. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

1.26. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

1.27. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.28. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

1.30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

1.31. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.

1.32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

1.33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.

1.34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.

1.35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.

1.36. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.

1.37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.

1.38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.

1.39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.

1.40. Find the area between $y = x^2$ and $y = x$.

1.41. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

1.42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.

1.43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$

1.44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.

1.45. Find the area bounded by the curves

$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{1.45.1}$$

1.46. Find the area of the region

$$\left\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{1.46.1}$$

1.47. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.

2 QR Decomposition

2.1. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution: Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.1.1}$$

$$\beta = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.1.2}$$

We can express these as

$$\alpha = k_1\mathbf{u}_1 \tag{2.1.3}$$

$$\beta = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.1.4}$$

where

$$k_1 = \|\alpha\| \tag{2.1.5}$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\beta}{\|\mathbf{u}_1\|^2} \tag{2.1.7}$$

$$\mathbf{u}_2 = \frac{\beta - r_1\mathbf{u}_1}{\|\beta - r_1\mathbf{u}_1\|} \tag{2.1.8}$$

$$k_2 = \mathbf{u}_2^T\beta \tag{2.1.9}$$

From (2.1.3) and (2.1.4),

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.1.10}$$

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \mathbf{QR} \tag{2.1.11}$$

From above we can see that $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.1.12}$$

Now by using equations (2.1.5) to (2.1.9)

$$k_1 = \sqrt{5} \tag{2.1.13}$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}}\begin{pmatrix}1 \\ 2\end{pmatrix}, \tag{2.1.14}$$

$$r_1 = \sqrt{5} \tag{2.1.15}$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}}\begin{pmatrix}-2 \\ 1\end{pmatrix} \tag{2.1.16}$$

$$k_2 = \sqrt{5} \tag{2.1.17}$$

Thus obtained QR decomposition is

$$\begin{pmatrix}1 & -1 \\ 2 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5}\end{pmatrix} \tag{2.1.18}$$

2.2. $\begin{pmatrix}2 & 1 \\ 1 & 1\end{pmatrix}$

2.3. $\begin{pmatrix}1 & 3 \\ 2 & 7\end{pmatrix}$

2.4. $\begin{pmatrix}2 & 3 \\ 5 & 7\end{pmatrix}$

2.5. $\begin{pmatrix}2 & 1 \\ 7 & 4\end{pmatrix}$

2.6. $\begin{pmatrix}2 & 5 \\ 1 & 3\end{pmatrix}$

2.7. $\begin{pmatrix}3 & 1 \\ 5 & 2\end{pmatrix}$

2.8. $\begin{pmatrix}4 & 5 \\ 3 & 4\end{pmatrix}$

2.9. $\begin{pmatrix}3 & 10 \\ 2 & 7\end{pmatrix}$

2.10. $\begin{pmatrix}3 & -1 \\ -4 & 2\end{pmatrix}$

2.11. $\begin{pmatrix}2 & -6 \\ 1 & -2\end{pmatrix}$

2.12. $\begin{pmatrix}6 & -3 \\ -2 & 1\end{pmatrix}$

2.13. $\begin{pmatrix}2 & -3 \\ -1 & 2\end{pmatrix}$

2.14. $\begin{pmatrix}2 & 1 \\ 4 & 2\end{pmatrix}$

2.15. Find QR decomposition of $\begin{pmatrix}2 & 3 \\ 3 & -4\end{pmatrix}$

Solution: Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix.

$$\mathbf{a} = \begin{pmatrix}2 \\ 3\end{pmatrix} \tag{2.15.1}$$

$$\mathbf{b} = \begin{pmatrix}3 \\ -4\end{pmatrix} \tag{2.15.2}$$

The column vectors can be expressed as follows,

$$\mathbf{a} = k_1\mathbf{u}_1 \tag{2.15.3}$$

$$\mathbf{b} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.15.4}$$

Here,

$$k_1 = \|\mathbf{a}\| \tag{2.15.5}$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \tag{2.15.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\mathbf{b}}{\|\mathbf{u}_1\|^2} \tag{2.15.7}$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1\mathbf{u}_1}{\|\mathbf{b} - r_1\mathbf{u}_1\|} \tag{2.15.8}$$

$$k_2 = \mathbf{u}_2^T\mathbf{b} \tag{2.15.9}$$

The (2.15.3) and (2.15.4) can be written as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.15.10}$$

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \mathbf{QR} \tag{2.15.11}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.15.12}$$

Now using equations (2.15.5) to (2.15.9) we get,

$$k_1 = \sqrt{2^2+3^2} = \sqrt{13} \tag{2.15.13}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{13}}\begin{pmatrix}2\\3\end{pmatrix} \tag{2.15.14}$$

$$r_1 = \begin{pmatrix}\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix} = -\frac{6}{\sqrt{13}} \tag{2.15.15}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\-2\end{pmatrix} \tag{2.15.16}$$

$$k_2 = \begin{pmatrix}\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix} = \frac{17}{\sqrt{13}} \tag{2.15.17}$$

Thus putting the values from (2.15.13) to (2.15.17) in (2.15.11) we obtain QR decomposition,

$$\begin{pmatrix}2 & 3\\3 & -4\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}\sqrt{13} & -\frac{6}{\sqrt{13}}\\ 0 & \frac{17}{\sqrt{13}}\end{pmatrix} \tag{2.15.18}$$

2.16. Find the QR decomposition of $\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix}$

Solution:
Let $\mathbf{c_1}$ and $\mathbf{c_2}$ be the column vectors of the given matrix.

$$\mathbf{c_1} = \begin{pmatrix}3\\1\end{pmatrix} \tag{2.16.1}$$

$$\mathbf{c_2} = \begin{pmatrix}2\\4\end{pmatrix} \tag{2.16.2}$$

The column vectors can be represented as,

$$\mathbf{c_1} = k_1\mathbf{u}_1 \tag{2.16.3}$$

$$\mathbf{c_2} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.16.4}$$

where,

$$k_1 = \|\mathbf{c_1}\| \tag{2.16.5}$$

$$\mathbf{u_1} = \frac{\mathbf{c_1}}{k_1} \tag{2.16.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\mathbf{c_2}}{\|\mathbf{u}_1\|^2} \tag{2.16.7}$$

$$\mathbf{u_2} = \frac{\mathbf{c_2} - r_1\mathbf{u_1}}{\|\mathbf{c_2} - r_1\mathbf{u_1}\|} \tag{2.16.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{c_2} \tag{2.16.9}$$

From (2.16.3) and (2.16.4),

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{2.16.10}$$

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \mathbf{QR} \tag{2.16.11}$$

Where $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.16.12}$$

Using equations (2.16.5) to (2.16.9) we get,

$$k_1 = \sqrt{3^2+1^2} = \sqrt{10} \tag{2.16.13}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{pmatrix} \tag{2.16.14}$$

$$r_1 = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \sqrt{10} \tag{2.16.15}$$

$$\mathbf{u_2} = \begin{pmatrix}\frac{-1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{pmatrix} \tag{2.16.16}$$

$$k_2 = \begin{pmatrix}\frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \sqrt{10} \tag{2.16.17}$$

Now putting the values from (2.16.13) to (2.16.17), we obtain the QR decomposition of given matrix,

$$\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & \sqrt{10}\\ 0 & \sqrt{10}\end{pmatrix} \tag{2.16.18}$$

2.17. Find QR decomposition of $\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$\mathbf{A} = \mathbf{QR} \tag{2.17.1}$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix Given

$$\mathbf{A} = \begin{pmatrix}4 & 3\\5 & -2\end{pmatrix} \tag{2.17.2}$$

Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix

$$\mathbf{a} = \begin{pmatrix}4\\5\end{pmatrix} \tag{2.17.3}$$

$$\mathbf{b} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{2.17.4}$$

The above column vectors (2.17.3) ,(2.17.4) can be expressed as ,

$$\mathbf{a} = t_1\mathbf{u}_1 \quad (2.17.5)$$

$$\mathbf{b} = s_1\mathbf{u}_1 + t_2\mathbf{u}_2 \quad (2.17.6)$$

Where,

$$t_1 = \|\mathbf{a}\| \quad (2.17.7)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{t_1} \quad (2.17.8)$$

$$s_1 = \frac{\mathbf{u}_1^T\mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.17.9)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - s_1\mathbf{u}_1}{\|\mathbf{b} - s_1\mathbf{u}_1\|} \quad (2.17.10)$$

$$t_2 = \mathbf{u}_2^T\mathbf{b} \quad (2.17.11)$$

The (2.17.5) and (2.17.6) can be written as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}t_1 & s_1 \\ 0 & t_2\end{pmatrix} \quad (2.17.12)$$

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \mathbf{QR} \quad (2.17.13)$$

Here, $\mathbf{R}$ is an upper triangular matrix and $\mathbf{Q}$ is an orthogonal matrix such that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad (2.17.14)$$

Now using equations from (2.17.7) to (2.17.11) we get,

$$t_1 = \sqrt{4^2 + 5^2} = \sqrt{41} \quad (2.17.15)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{41}}\begin{pmatrix}4 \\ 5\end{pmatrix} \quad (2.17.16)$$

$$s_1 = \begin{pmatrix}\frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}3 \\ -2\end{pmatrix} = \frac{2}{\sqrt{41}} \quad (2.17.17)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{41}}\begin{pmatrix}5 \\ -4\end{pmatrix} \quad (2.17.18)$$

$$t_2 = \begin{pmatrix}\frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}3 \\ -2\end{pmatrix} = \frac{23}{\sqrt{41}} \quad (2.17.19)$$

Substituting the values from (2.17.15) to (2.17.19) in (2.17.13) we obtain QR decomposition as,

$$\begin{pmatrix}4 & 3 \\ 5 & -2\end{pmatrix} = \begin{pmatrix}\frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \\ \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}\sqrt{41} & \frac{2}{\sqrt{41}} \\ 0 & \frac{23}{\sqrt{41}}\end{pmatrix} \quad (2.17.20)$$

2.18. Perform the QR decomposition of matrix

$$\mathbf{A} = \begin{pmatrix}1 & 2 \\ 3 & 1\end{pmatrix} \quad (2.18.1)$$

Solution:
If α and β are the columns of a (2×2) matrix $\mathbf{A}$,
then $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \mathbf{QR} \quad (2.18.2)$$

$$\text{where, } \mathbf{U} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}, \quad (2.18.3)$$

$$\text{uppertriangular matrix } \mathbf{R} = \begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \quad (2.18.4)$$

$$k_1 = \|\alpha\|, \mathbf{u_1} = \frac{\alpha}{k_1} \quad (2.18.5)$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} \quad (2.18.6)$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|}, k_2 = \mathbf{u_2}^T\beta \quad (2.18.7)$$

$$\alpha = \begin{pmatrix}1 \\ 3\end{pmatrix}, \beta = \begin{pmatrix}2 \\ 1\end{pmatrix} \quad (2.18.8)$$

$$\text{From, (2.18.5), } k_1 = \|\alpha\| = \sqrt{10} \quad (2.18.9)$$

$$\text{and } \mathbf{u_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 \\ 3\end{pmatrix} \quad (2.18.10)$$

$$\text{From (2.18.6), } r_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 \\ 1\end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.11)$$

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix}2 \\ 1\end{pmatrix} - \frac{5}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{pmatrix}1 \\ 3\end{pmatrix} \quad (2.18.12)$$

$$= \begin{pmatrix}\frac{3}{2} \\ \frac{-1}{2}\end{pmatrix} \quad (2.18.13)$$

$$\text{From (2.18.7), } \mathbf{u_2} = \frac{\begin{pmatrix}\frac{3}{2} \\ \frac{-1}{2}\end{pmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4}}} \quad (2.18.14)$$

$$\implies \mathbf{u_2} = \begin{pmatrix}\frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}}\end{pmatrix}, \quad (2.18.15)$$

$$k_2 = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2 \\ 1\end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.16)$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} = \mathbf{I} \quad (2.18.17)$$

The matrix **A** can now be rewritten using (2.18.2) as

$$\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & \frac{5}{\sqrt{10}}\\ 0 & \frac{5}{\sqrt{10}}\end{pmatrix} \tag{2.18.18}$$

2.19. Find the QR decomposition of the given matrix.

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \tag{2.19.1}$$

Solution: QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \tag{2.19.2}$$

where **Q** is an orthogonal matrix and **R** is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \tag{2.19.3}$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix} \quad \mathbf{b} = \begin{pmatrix}2\\-2\end{pmatrix} \tag{2.19.4}$$

Equation (2.19.3) can be written in form of (2.19.4) as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}u_1 & u_3\\0 & u_2\end{pmatrix} = \mathbf{QR} \tag{2.19.5}$$

where,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{2.19.6}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix} \tag{2.19.7}$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \frac{-2}{\sqrt{5}} \tag{2.19.8}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ -\frac{1}{\sqrt{5}}\end{pmatrix} \tag{2.19.9}$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \frac{6}{\sqrt{5}} \tag{2.19.10}$$

Substituting equation (2.19.6) to (2.19.10) in (2.19.5),

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & -\frac{2}{\sqrt{5}}\\ 0 & \frac{6}{\sqrt{5}}\end{pmatrix} \tag{2.19.11}$$

The QR decomposition is,

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & -\frac{2}{\sqrt{5}}\\ 0 & \frac{6}{\sqrt{5}}\end{pmatrix} \tag{2.19.12}$$

2.20. Find the QR decomposition on a given 2×2 matrix.

$$\begin{pmatrix}2 & 1\\1 & -2\end{pmatrix} \tag{2.20.1}$$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \tag{2.20.2}$$

Here **Q** is an orthogonal matrix and **R** is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix}2 & 1\\1 & -2\end{pmatrix} \tag{2.20.3}$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix}2\\1\end{pmatrix} \quad \mathbf{b} = \begin{pmatrix}1\\-2\end{pmatrix} \tag{2.20.4}$$

Equation (2.20.3) can be written in **QR** form as:

$$\mathbf{QR} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}u_1 & u_3\\0 & u_2\end{pmatrix} \tag{2.20.5}$$

Now,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{2.20.6}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix} \tag{2.20.7}$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = 0 \tag{2.20.8}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}}\end{pmatrix} \tag{2.20.9}$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = \sqrt{5} \tag{2.20.10}$$

Substituting equation (2.20.6) to (2.20.10) in (2.20.5),to obtain the QR Decomposition of the given matrix as:

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{2.20.11}$$

In equation (2.20.11) **R** is diagonal because the columns and rows are orthogonal to each other.

2.21. Perform QR decomposition on matrix **A**

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} \tag{2.21.1}$$

Solution:
The columns of matrix **A** can be represented in α and β as

$$\implies \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.21.2}$$

$$\beta = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{2.21.3}$$

For QR decomposition, matrix **A** can be expressed as

$$\mathbf{A} = \mathbf{QR} \tag{2.21.4}$$

where, **Q** and **R** are expressed as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.21.5}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.21.6}$$

Note that **R** is an upper triangular matrix.
Now,we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \tag{2.21.7}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.21.8}$$

$$r_1 = \frac{\mathbf{u_1}^T \beta}{\|\mathbf{u_1}\|^2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{2.21.9}$$

$$\implies r_1 = -\frac{11}{\sqrt{10}} \tag{2.21.10}$$

$$\mathbf{u_2} = \frac{\beta - r_1 \mathbf{u_1}}{\|\beta - r_1 \mathbf{u_1}\|} \tag{2.21.11}$$

Consider

$$\beta - r_1 \mathbf{u_1} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} + \frac{11}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.21.12}$$

$$\implies \beta - r_1 \mathbf{u_1} = \begin{pmatrix} \frac{51}{10} \\ -\frac{17}{10} \end{pmatrix} \tag{2.21.13}$$

$$\|\beta - r_1 \mathbf{u_1}\| = \frac{17}{\sqrt{10}} \tag{2.21.14}$$

Substitute (2.21.13),(2.21.14) in (2.21.11), we get

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix} \tag{2.21.15}$$

$$k_2 = \mathbf{u_2}^T \beta = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{2.21.16}$$

$$\implies k_2 = \frac{17}{\sqrt{10}} \tag{2.21.17}$$

Therefore, from (2.21.5) and (2.21.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \tag{2.21.18}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \tag{2.21.19}$$

Note that,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{2.21.20}$$

Now matrix **A** can be written as (2.21.4)

$$\begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \tag{2.21.21}$$

2.22. Perform QR decomposition on matrix **A**

$$\mathbf{A} = \begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} \tag{2.22.1}$$

Solution: The columns of matrix **A** can be represented in α and β as

$$\implies \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.22.2}$$

$$\beta = \begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{2.22.3}$$

For QR decomposition, matrix **A** can be expressed as

$$\mathbf{A} = \mathbf{QR} \tag{2.22.4}$$

where, **Q** and **R** are expressed as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.22.5}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.22.6}$$

Note that **R** is an upper triangular matrix.

Now,we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \tag{2.22.7}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \tag{2.22.8}$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}-7\\1\end{pmatrix} \tag{2.22.9}$$

$$\implies r_1 = -\frac{4}{\sqrt{10}} \tag{2.22.10}$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{2.22.11}$$

Consider

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix}-7\\1\end{pmatrix} + \frac{4}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \tag{2.22.12}$$

$$\implies \beta - r_1\mathbf{u_1} = \begin{pmatrix}\frac{-66}{10}\\\frac{22}{10}\end{pmatrix} \tag{2.22.13}$$

$$\|\beta - r_1\mathbf{u_1}\| = \frac{22}{\sqrt{10}} \tag{2.22.14}$$

Substitute (2.22.13),(2.22.14) in (2.22.11), we get

$$\mathbf{u_2} = \begin{pmatrix}-\frac{3}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\end{pmatrix} \tag{2.22.15}$$

$$k_2 = \mathbf{u_2}^T\beta = \begin{pmatrix}-\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}-7\\1\end{pmatrix} \tag{2.22.16}$$

$$\implies k_2 = \frac{22}{\sqrt{10}} \tag{2.22.17}$$

Therefore, from (2.22.5) and (2.22.6)

$$\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix} \tag{2.22.18}$$

$$\mathbf{R} = \begin{pmatrix}\sqrt{10} & -\frac{4}{\sqrt{10}}\\0 & \frac{22}{\sqrt{10}}\end{pmatrix} \tag{2.22.19}$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\-\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \tag{2.22.20}$$

Now matrix **A** can be written as (2.22.4)

$$\begin{pmatrix}1 & -7\\3 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & -\frac{4}{\sqrt{10}}\\0 & \frac{22}{\sqrt{10}}\end{pmatrix} \tag{2.22.21}$$

2.23. Given a matrix $\mathbf{A} = \begin{pmatrix}3 & -2\\4 & -2\end{pmatrix}$, find its **QR** decomposition **Solution:**
Given

$$\mathbf{A} = \begin{pmatrix}3 & -2\\4 & -2\end{pmatrix} \tag{2.23.1}$$

Let us use the Gram-Schmidt approach to obtain QR decomposition of **A**. Consider column vectors say $\mathbf{a}_1$ and $\mathbf{a}_2$ of **A** which is given by

$$\mathbf{a}_1 = \begin{pmatrix}3\\4\end{pmatrix} \tag{2.23.2}$$

$$\mathbf{a}_2 = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{2.23.3}$$

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{pmatrix}3\\4\end{pmatrix} \tag{2.23.4}$$

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix} \tag{2.23.5}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - \left(\mathbf{a}_2^T.\mathbf{e}_1\right)\mathbf{e}_1 \tag{2.23.6}$$

$$= \begin{pmatrix}-2\\-2\end{pmatrix} - \left(-\frac{14}{5}\right)\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix} \tag{2.23.7}$$

$$= \begin{pmatrix}-2\\-2\end{pmatrix} - \begin{pmatrix}-\frac{42}{25}\\-\frac{56}{25}\end{pmatrix} = \begin{pmatrix}-\frac{8}{25}\\\frac{6}{25}\end{pmatrix} \tag{2.23.8}$$

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{pmatrix}-\frac{4}{5}\\\frac{3}{5}\end{pmatrix} \tag{2.23.9}$$

The matrix **Q** and **R** is given by,

$$\mathbf{Q} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\\frac{4}{5} & \frac{3}{5}\end{pmatrix} \tag{2.23.10}$$

$$\mathbf{R} = \begin{pmatrix}\mathbf{a}_1^T.\mathbf{e}_1 & \mathbf{a}_2^T.\mathbf{e}_1\\0 & \mathbf{a}_2^T.\mathbf{e}_2\end{pmatrix} = \begin{pmatrix}5 & -\frac{14}{5}\\0 & \frac{2}{5}\end{pmatrix} \tag{2.23.11}$$

Hence, the **QR** decomposition of matrix **A** is as follows:

$$\begin{pmatrix}3 & -2\\4 & -2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}5 & -\frac{14}{5}\\0 & \frac{2}{5}\end{pmatrix} \tag{2.23.12}$$

2.24. Perform the QR decomposition of the matrix $\mathbf{A} = \begin{pmatrix}3 & -1\\-4 & 2\end{pmatrix}$. **Solution:**
Let **a** and **b** are the columns of matrix **A**. The matrix **A** can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \tag{2.24.1}$$

$$\mathbf{U} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \tag{2.24.2}$$

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.24.3}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.24.4}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.24.5}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|u_1\|^2} \tag{2.24.6}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.24.7}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{2.24.8}$$

Then the given matrix can be represented as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.24.9}$$

The the columns of matrix $\mathbf{A} = \begin{pmatrix}3 & -1 \\ -4 & 2\end{pmatrix}$ are $\mathbf{a}$ and $\mathbf{b}$ where

$$\mathbf{a} = \begin{pmatrix}3 \\ -4\end{pmatrix} \tag{2.24.10}$$

$$\mathbf{b} = \begin{pmatrix}-1 \\ 2\end{pmatrix} \tag{2.24.11}$$

Now for the given matrix, From (2.24.4) and (2.24.5)

$$k_1 = \|\mathbf{a}\| = 5 \tag{2.24.12}$$

$$\mathbf{u_1} = \frac{1}{5}\begin{pmatrix}3 \\ -4\end{pmatrix} \tag{2.24.13}$$

From (2.24.6)

$$r_1 = \frac{1}{5}\begin{pmatrix}3 & -4\end{pmatrix}\begin{pmatrix}-1 \\ 2\end{pmatrix} = \frac{-11}{5} \tag{2.24.14}$$

From (2.24.7)

$$\mathbf{b} - r_1\mathbf{u_1} = \begin{pmatrix}-1 \\ 2\end{pmatrix} - \frac{-11}{5}\begin{pmatrix}\frac{3}{5} \\ \frac{-4}{5}\end{pmatrix} \tag{2.24.15}$$

$$\|\mathbf{b} - r_1\mathbf{u_1}\| = \frac{2}{5} \tag{2.24.16}$$

$$\implies \mathbf{u_2} = \frac{5}{2}\begin{pmatrix}\frac{8}{25} \\ \frac{6}{25}\end{pmatrix} \tag{2.24.17}$$

From (2.24.8)

$$k_2 = \mathbf{u_2}^T\mathbf{b} = \begin{pmatrix}\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}-1 \\ 2\end{pmatrix} = \frac{2}{5} \tag{2.24.18}$$

Now we can observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = I$

$$\begin{pmatrix}\frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}\frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5}\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} \tag{2.24.19}$$

From (2.24.9), The matrix $\mathbf{A}$ can now be written as,

$$\mathbf{A} = \begin{pmatrix}3 & -1 \\ -4 & 2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}-5 & \frac{-11}{5} \\ 0 & \frac{2}{5}\end{pmatrix} \tag{2.24.20}$$

2.25. Perform QR decomposition on matrix A given by

$$A = \begin{pmatrix}3 & -4 \\ -4 & 3\end{pmatrix}$$

Solution: Representing matrix A in terms of its column vectors as

$$A = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} \tag{2.25.1}$$

Let

$$\mathbf{q_1} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{2.25.2}$$

An orthonormal vector to $\mathbf{q_1}$ can be obtained by subtracting the projection of $\mathbf{b}$ on $\mathbf{q_1}$ from $\mathbf{b}$. Thus

$$\mathbf{q_2} = \frac{\mathbf{b} - k\mathbf{q_1}}{\|\mathbf{b} - k\mathbf{q_1}\|} \tag{2.25.3}$$

where

$$k = \frac{\mathbf{b}^T\mathbf{q_1}}{\|\mathbf{q_1}\|^2} \tag{2.25.4}$$

From (2.25.2) and (2.25.3)

$$\mathbf{a} = \|\mathbf{a}\|\,\mathbf{q_1} \tag{2.25.5}$$

$$\mathbf{b} = k\mathbf{q_1} + \|\mathbf{b} - k\mathbf{q_1}\|\,\mathbf{q_2} \tag{2.25.6}$$

$$\implies \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}\|\mathbf{a}\| & k \\ 0 & \|\mathbf{b} - k\mathbf{q_1}\|\end{pmatrix} \tag{2.25.7}$$

$$\implies \mathbf{A} = \mathbf{QR} \tag{2.25.8}$$

QR decomposition of a matrix A is essentially representation of column vectors of matrix A in terms of linear combination of orthonormal basis of column space of A. For matrix A

$$\mathbf{a} = \begin{pmatrix}3 \\ -4\end{pmatrix},\ \mathbf{b} = \begin{pmatrix}-4 \\ 3\end{pmatrix} \tag{2.25.9}$$

$$\tag{2.25.10}$$

Let

$$\mathbf{q_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{2.25.11}$$

$$\tag{2.25.12}$$

From (2.25.3) and (2.25.4)

$$\mathbf{q_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{2.25.13}$$

$$\implies \mathbf{Q} = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix} \tag{2.25.14}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{2.25.15}$$

Therefore the matrix A can be decomposed as

$$\mathbf{A} = \begin{pmatrix} \frac{3}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \\ -\frac{4}{\sqrt{5}} & \frac{3}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{2.25.16}$$

2.26. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix} \tag{2.26.1}$$

2.27. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \tag{2.27.1}$$

2.28. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 6 & -2 \end{pmatrix} \tag{2.28.1}$$

Solution: Let c_1 and c_2 be the column vectors of given matrix **A**

$$c_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.28.2}$$

$$c_2 = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \tag{2.28.3}$$

We can express the matrix **A** as,

$$\mathbf{A} = \mathbf{QR} \tag{2.28.4}$$

Where, **Q** is an orthogonal matrix given as,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.28.5}$$

and **R** is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.28.6}$$

Now, we can express α and β as,

$$c_1 = k_1\mathbf{u_1} \tag{2.28.7}$$

$$c_2 = r_1\mathbf{u_1} + k_2\mathbf{u_2} \tag{2.28.8}$$

$$\text{where, } k_1 = \|c_1\| = \sqrt{2^2 + (1^2} = \sqrt{5} \tag{2.28.9}$$

Solving equation (2.28.7) for $\mathbf{u_1}$,

$$\mathbf{u_1} = \frac{c_1}{k_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.28.10}$$

$$\text{Now, } r_1 = \frac{\mathbf{u_1}^T c_2}{\|\mathbf{u_1}\|^2} \tag{2.28.11}$$

$$\implies \frac{\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} -6 \\ -2 \end{pmatrix}}{1} \tag{2.28.12}$$

$$\text{Hence, } r_1 = -\frac{-14}{\sqrt{5}} \tag{2.28.13}$$

$$\mathbf{u_2} = \frac{c_2 - r_1\mathbf{u_1}}{\|c_2 - r_1\mathbf{u_1}\|} \tag{2.28.14}$$

$$\implies \frac{\begin{pmatrix} -6 \\ -2 \end{pmatrix} - \left(\frac{-14}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)}{\left\|\begin{pmatrix} -6 \\ -2 \end{pmatrix} - \left(-\frac{14}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)\right\|} \tag{2.28.15}$$

$$\implies \mathbf{u_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.28.16}$$

$$\text{Now, } k_2 = u_2^T c_2 \tag{2.28.17}$$

$$\implies \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \end{pmatrix}\begin{pmatrix} -6 \\ -2 \end{pmatrix} \tag{2.28.18}$$

$$\implies k_2 = \frac{2}{\sqrt{5}} \tag{2.28.19}$$

Hence substituting the values of unknown parameter from equations (2.28.9), (2.28.19), (2.28.10), (2.28.16) and (2.28.13) to equation (2.28.5) and (2.28.6) we get,

$$\mathbf{Q} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{2.28.20}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & \frac{-14}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.28.21}$$

2.29. **Solution:** If $\mathbf{A} \in \mathbf{R}^{m\times n}$ has linearly independent columns then it can be factored as

$$\mathbf{A} = \mathbf{QR}$$

where $\mathbf{Q}$ is a orthogonal matrix and $\mathbf{R}$ is a upper triangular matrix with non zero diagonal elements

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \tag{2.29.1}$$

The column vectors of $\mathbf{A}$ are,

$$\mathbf{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{2.29.2}$$

(2.29.1) can be written as,

$$\mathbf{QR} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \tag{2.29.3}$$

Now,

$$u_1 = \|a\| = \sqrt{4^2 + 7^2} = \sqrt{65} \tag{2.29.4}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{4}{\sqrt{65}} \\ \frac{7}{\sqrt{65}} \end{pmatrix} \tag{2.29.5}$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix} \frac{4}{\sqrt{65}} & \frac{7}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{47}{\sqrt{65}} \tag{2.29.6}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix} \frac{7}{\sqrt{65}} \\ -\frac{4}{\sqrt{65}} \end{pmatrix} \tag{2.29.7}$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix} \frac{7}{\sqrt{65}} & -\frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{65}} \tag{2.29.8}$$

Substituting (2.29.4) to (2.29.8) in (2.29.3),

$$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{65}} & \frac{7}{\sqrt{65}} \\ \frac{7}{\sqrt{65}} & -\frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} \sqrt{65} & \frac{47}{\sqrt{65}} \\ 0 & \frac{1}{\sqrt{65}} \end{pmatrix} \tag{2.29.9}$$

Which can also be written as,

$$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{4}{\sqrt{65}} & -\frac{7}{\sqrt{65}} \\ -\frac{7}{\sqrt{65}} & \frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} -\sqrt{65} & -\frac{47}{\sqrt{65}} \\ 0 & -\frac{1}{\sqrt{65}} \end{pmatrix} \tag{2.29.10}$$

3 Singular Value Decomposition

3.1. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{3.1.1}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.2}$$

Solution:
The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.3}$$

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.1.4}$$

$$\mathbf{Mx} = \mathbf{b} \tag{3.1.5}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{USV}^T \tag{3.1.6}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}^T\mathbf{A}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{AA}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{3.1.7}$$

$$\mathbf{MM}^T = \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{3.1.8}$$

Calculating eigen value of $\mathbf{M}^T\mathbf{M}$.

$$\begin{vmatrix} 6-\lambda & 13 \\ 13 & 38-\lambda \end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \tag{3.1.9}$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22 \tag{3.1.10}$$

Eigen vectors of $\mathbf{MM}^T$.

$$\begin{vmatrix} 13-\lambda & -17 & 8 \\ 17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \tag{3.1.11}$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \tag{3.1.12}$$

Hence,The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{5} \\ \frac{-3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \tag{3.1.13}$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12-\sqrt{17}}{5}\right)^2 + \left(\frac{1-3\sqrt{17}}{5}\right)^2 + 1^2} \tag{3.1.14}$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} \end{pmatrix} \tag{3.1.15}$$

$$\tag{3.1.16}$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \tag{3.1.17}$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+30\sqrt{7}}} \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \tag{3.1.18}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} \\ \frac{5}{\sqrt{340+30\sqrt{17}}} \end{pmatrix} \tag{3.1.19}$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \tag{3.1.20}$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}} \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \end{pmatrix} \tag{3.1.21}$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.22}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.23}$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{3.1.24}$$

Now, $\mathbf{V} = \mathbf{M}^T \frac{\mathbf{u_i}}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.25}$$

So,from equation (3.1.6)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} = \tag{3.1.26}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.27}$$

$$\begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{3.1.28}$$

$$\begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}^T \tag{3.1.29}$$

Now, Finding Moore-Penrose Pseudo inverse of **S**

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \tag{3.1.30}$$

We,know that, $\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+0\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix} \tag{3.1.31}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.32}$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.33}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.34}$$

$$\mathbf{x} = \begin{pmatrix} \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \\ \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \end{pmatrix} \tag{3.1.35}$$

Simplifying the values of x_1 and x_2

$$x_2 = \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \tag{3.1.36}$$

$$= \frac{-702100}{591700} \tag{3.1.37}$$

$$= -\frac{7}{59} \tag{3.1.38}$$

$$x_1 = \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \tag{3.1.39}$$

$$= \frac{2507500}{591700} \tag{3.1.40}$$

$$= \frac{25}{59} \tag{3.1.41}$$

Now, Verifying the values using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.1.42}$$

Solving R.H.S

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.1.43}$$

Now using equation (3.1.7) in (3.1.43)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.1.44}$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xleftrightarrow{R_2-\frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \tag{3.1.45}$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \tag{3.1.46}$$

$$6x_1 + 13x_2 = 1 \tag{3.1.47}$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \tag{3.1.48}$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{3.1.49}$$

3.2. Find the distance of the point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane $\begin{pmatrix} 6 & -3 & 2 \end{pmatrix}\mathbf{x} = 4$

Solution:

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{3.2.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \tag{3.2.2}$$

$$\implies 6a - 3b + 2c = 0 \tag{3.2.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \tag{3.2.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{3.2.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{3.2.6}$$

Putting values in (3.2.6),

$$\begin{pmatrix}1 & 0\\0 & 1\\3 & \frac{3}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\5\\-3\end{pmatrix} \tag{3.2.7}$$

Now, to solve (3.2.7), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{3.2.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}10 & \frac{9}{2}\\ \frac{9}{2} & \frac{13}{4}\end{pmatrix} \tag{3.2.9}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & 3\\0 & 1 & \frac{3}{2}\\3 & \frac{3}{2} & \frac{45}{4}\end{pmatrix} \tag{3.2.10}$$

From (3.2.6) putting (3.2.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.2.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{3.2.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{3.2.13}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & 3\\0 & 1-\lambda & \frac{3}{2}\\3 & \frac{3}{2} & \frac{45}{4}-\lambda\end{pmatrix} = 0 \tag{3.2.14}$$

$$\implies \lambda^3 - \frac{53}{4}\lambda^2 + \frac{49}{4}\lambda = 0 \tag{3.2.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{4} \tag{3.2.16}$$

$$\lambda_2 = 1 \tag{3.2.17}$$

$$\lambda_3 = 0 \tag{3.2.18}$$

Hence the eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{4}{15}\\ \frac{2}{15}\\ 1\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{2}\\ 1\\ 0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-3\\ -\frac{3}{2}\\ 1\end{pmatrix} \tag{3.2.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{4}{7\sqrt{5}}\\ \frac{2}{7\sqrt{5}}\\ \frac{3\sqrt{5}}{7}\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\\ 0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-\frac{6}{7}\\ -\frac{3}{7}\\ \frac{2}{7}\end{pmatrix} \tag{3.2.20}$$

Hence we obtain $\mathbf{U}$ of (3.2.8) as follows,

$$\mathbf{U} = \begin{pmatrix}\frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7}\\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7}\\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7}\end{pmatrix} \tag{3.2.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (3.2.8) as follows,

$$\mathbf{S} = \begin{pmatrix}\frac{7}{2} & 0\\0 & 1\\0 & 0\end{pmatrix} \tag{3.2.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{3.2.23}$$

$$\implies \begin{pmatrix}10-\lambda & \frac{9}{2}\\ \frac{9}{2} & \frac{13}{4}-\lambda\end{pmatrix} = 0 \tag{3.2.24}$$

$$\implies \lambda^2 - \frac{53}{4}\lambda + \frac{49}{4} = 0 \tag{3.2.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{49}{4} \tag{3.2.26}$$

$$\lambda_5 = 1 \tag{3.2.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix} \tag{3.2.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix} \tag{3.2.29}$$

Hence we obtain $\mathbf{V}$ of (3.2.8) as follows,

$$\mathbf{V} = \begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix} \tag{3.2.30}$$

Finally from (3.2.8) we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix}\frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7}\\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7}\\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7}\end{pmatrix}\begin{pmatrix}\frac{7}{2} & 0\\0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}^T \tag{3.2.31}$$

Now, Moore-Penrose Pseudo inverse of **S** is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.2.32}$$

From (3.2.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{27}{7\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \\ -\frac{33}{7} \end{pmatrix} \tag{3.2.33}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{54}{49\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \end{pmatrix} \tag{3.2.34}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{3.2.35}$$

Verifying the solution of (3.2.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.2.36}$$

Evaluating the R.H.S in (3.2.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{3.2.37}$$

$$\implies \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{3.2.38}$$

Solving the augmented matrix of (3.2.38) we get,

$$\begin{pmatrix} 10 & \frac{9}{2} & -7 \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{10}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \tag{3.2.39}$$

$$\xleftrightarrow{R_2=R_2-\frac{9}{2}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & \frac{49}{40} & \frac{73}{20} \end{pmatrix} \tag{3.2.40}$$

$$\xleftrightarrow{R_2=\frac{40}{49}R_2} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{3.2.41}$$

$$\xleftrightarrow{R_1=R_1-\frac{9}{20}R_1} \begin{pmatrix} 1 & 0 & -\frac{100}{49} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{3.2.42}$$

Hence, Solution of (3.2.36) is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{3.2.43}$$

Comparing results of **x** from (3.2.35) and (3.2.43) we conclude that the solution is verified.

3.3. Check whether the given line equations intersect. If they do not intersect find the closest points on the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.3.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.2}$$

Solution:
Given

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.3.3}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.4}$$

The above equations (3.3.3), (3.3.4) are in the form

$$L_1 : \quad \mathbf{x} = \mathbf{a_1} + \lambda_1\mathbf{b_1} \tag{3.3.5}$$

$$L_2 : \quad \mathbf{x} = \mathbf{a_2} + \lambda_2\mathbf{b_2} \tag{3.3.6}$$

Here ,

$$\mathbf{a_1} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \tag{3.3.7}$$

$$\mathbf{a_2} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} \tag{3.3.8}$$

$$\mathbf{b_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.3.9}$$

$$\mathbf{b_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.10}$$

Now let us assume the lines L_1 and L_2 are intersecting at a point. Therefore ,

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.11}$$

$$\lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.3.12}$$

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.3.13}$$

The augumented matrix of (3.3.13) is given by

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array}\right) \tag{3.3.14}$$

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array}\right) \xleftrightarrow{R_2=R_2-\frac{2}{3}R_1} \left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array}\right) \tag{3.3.15}$$

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array}\right) \xleftrightarrow{R_3=R_3-2R_1} \left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{array}\right) \tag{3.3.16}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.Hence our assumptions is wrong
Equation (3.3.13) can be expressed as

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.3.17}$$

By singular value decomposition $\mathbf{M}$ can be expressed as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{3.3.18}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix} \tag{3.3.19}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \tag{3.3.20}$$

The characteristic equation of $\mathbf{M}^T\mathbf{M}$ is obtained by evaluating the determinant

$$\begin{vmatrix} 49-\lambda & -19 \\ -19 & 9-\lambda \end{vmatrix} = 0 \tag{3.3.21}$$

$$\implies \lambda^2 - 58\lambda + 80 = 0 \tag{3.3.22}$$

The eigenvalues are the roots of equation 3.3.22 is given by

$$\lambda_{11} = 29 + \sqrt{761} \tag{3.3.23}$$

$$\lambda_{12} = 29 - \sqrt{761} \tag{3.3.24}$$

The eigen vectors comes out to be ,

$$\mathbf{u_{11}} = \begin{pmatrix} \frac{-20-\sqrt{761}}{19} \\ 1 \end{pmatrix}, \mathbf{u_{12}} = \begin{pmatrix} \frac{-20+\sqrt{761}}{19} \\ 1 \end{pmatrix} \tag{3.3.25}$$

Normalising the eigen vectors,

$$l_{11} = \sqrt{\left(\frac{-20-\sqrt{761}}{19}\right)^2 + 1^2} \tag{3.3.26}$$

$$\implies l_{11} = \frac{\sqrt{1522 + 40\sqrt{761}}}{19} \tag{3.3.27}$$

$$\mathbf{u_{11}} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} \end{pmatrix} \tag{3.3.28}$$

$$l_{12} = \sqrt{\left(\frac{-20+\sqrt{761}}{19}\right)^2 + 1^2} \tag{3.3.29}$$

$$\implies l_{12} = \frac{\sqrt{1522 - 40\sqrt{761}}}{19} \tag{3.3.30}$$

$$\mathbf{u_{12}} = \begin{pmatrix} \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \tag{3.3.31}$$

$$\mathbf{V} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} & \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} & \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \tag{3.3.32}$$

$\mathbf{S}$ is given by

$$\mathbf{S} = \begin{pmatrix} \sqrt{29+\sqrt{761}} & 0 \\ 0 & \sqrt{29-\sqrt{761}} \\ 0 & 0 \end{pmatrix} \tag{3.3.33}$$

The characteristic equation of $\mathbf{M}\mathbf{M}^T$ is obtained by evaluating the determinant

$$\begin{vmatrix} 10-\lambda & 8 & 20 \\ 8 & 8-\lambda & 16 \\ 20 & 16 & 40-\lambda \end{vmatrix} = 0 \tag{3.3.34}$$

$$\implies \lambda^3 - 58\lambda^2 + 80\lambda = 0 \tag{3.3.35}$$

The eigenvalues are the roots of equation

3.3.35 is given by

$$\lambda_{21} = 29 + \sqrt{761} \tag{3.3.36}$$
$$\lambda_{22} = 29 - \sqrt{761} \tag{3.3.37}$$
$$\lambda_{23} = 0 \tag{3.3.38}$$

The eigen vectors comes out to be ,

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-\sqrt{761}+21}{16} \\ -1 \end{pmatrix}, \mathbf{u_{22}} = \begin{pmatrix} \frac{1}{2} \\ \frac{-\sqrt{761}-21}{16} \\ 1 \end{pmatrix}, \mathbf{u_{23}} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \tag{3.3.39}$$

Normalising the eigen vectors,

$$l_{21} = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{21-\sqrt{761}}{16}\right)^2 + (-1)^2} \tag{3.3.40}$$

$$\implies l_{21} = \frac{\sqrt{1522 - 42\sqrt{761}}}{16} \tag{3.3.41}$$

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \tag{3.3.42}$$

$$l_{22} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-21-\sqrt{761}}{16}\right)^2 + 1^2} \tag{3.3.43}$$

$$\implies l_{22} = \frac{\sqrt{1522 + 42\sqrt{761}}}{16} \tag{3.3.44}$$

$$\mathbf{u_{22}} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} \end{pmatrix} \tag{3.3.45}$$

$$l_{23} = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \tag{3.3.46}$$

$$\mathbf{u_{23}} = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.3.47}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.3.48}$$

From equation (3.3.18) we rewrite $\mathbf{M}$ as follows,

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.3.49}$$

$$\begin{pmatrix} \sqrt{29+\sqrt{7\ldots}} \\ 0 \\ 0 \end{pmatrix} \tag{3.3.50}$$

$$\begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{76\ldots}}} \\ \frac{19}{\sqrt{1522+40\sqrt{76\ldots}}} \end{pmatrix} \tag{3.3.51}$$

By substituting the equation (3.3.18) in equation (3.3.17) we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.3.52}$$
$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{3.3.53}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$

$$\mathbf{S_+} = \begin{pmatrix} \frac{1}{\sqrt{29+\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29-\sqrt{761}}} & 0 \end{pmatrix} \tag{3.3.54}$$

From (3.3.53) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{761}-45}{\sqrt{1522-42\sqrt{761}}} \\ \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ -\frac{11}{\sqrt{5}} \end{pmatrix} \tag{3.3.55}$$

$$\mathbf{S_+U}^T\mathbf{b} = \begin{pmatrix} \frac{761\sqrt{15}-761-45\sqrt{11415}+45\sqrt{761}}{10654} \\ \frac{45\sqrt{11415}+45\sqrt{761}+761\sqrt{15}+761}{10654} \end{pmatrix} \tag{3.3.56}$$

$$\mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{3.3.57}$$

Verifying the solution of (3.3.57) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.3.58}$$

Evaluating the R.H.S in (3.3.58) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{3.3.59}$$

$$\implies \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{3.3.60}$$

Solving the augmented matrix of (3.3.60) we get,

$$\begin{pmatrix} 49 & -19 & 7 \\ -19 & 9 & -1 \end{pmatrix} \xleftrightarrow{R_2=R_2+\frac{19}{49}R_1} \begin{pmatrix} 49 & -19 & 7 \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{3.3.61}$$

$$\xleftrightarrow{R_1=\frac{1}{49}R_1} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{3.3.62}$$

$$\xleftrightarrow{R_2=\frac{80}{49}R_2} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{3.3.63}$$

$$\xleftrightarrow{R_1=R_1+\frac{19}{49}R_2} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{3.3.64}$$

Hence, Solution of (3.3.58) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{3.3.65}$$

Comparing results of $\mathbf{x}$ from (3.3.57) and (3.3.65) we conclude that the solution is verified.

3.4. Check if the lines L_1, L_2 are skew.If so, find the closest points on those lines using Singular Value Decomposition(SVD)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.4.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.4.2}$$

Solution:
The matrix $\mathbf{M}$ of dimensions $(m \times n)$ can be decomposed using SVD as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{3.4.3}$$

where,columns of $\mathbf{U}_{(m\times m)}$ are eigen vectors of $\mathbf{M}\mathbf{M}^T$
columns of $\mathbf{V}_{(n\times n)}$ are eigen vectors of $\mathbf{M}^T\mathbf{M}$
$\mathbf{S}$ is a diagonal matrix containing singular values of $\mathbf{M}$. Also, $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices

$$\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{3.4.4}$$

$$\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{3.4.5}$$

Given line equations intersect if

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.4.6}$$

This can be written as

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.4.7}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.4.8}$$

$$\text{where, } \mathbf{x} = \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{3.4.9}$$

The augmented matrix is

$$\begin{pmatrix} 3 & 1 & 5 \\ 2 & 2 & -1 \\ 6 & 2 & -1 \end{pmatrix} \xleftrightarrow[R_2\leftarrow R_2-R_1\times\frac{2}{3}]{R_3\leftarrow R_3-2\times R_1} \begin{pmatrix} 3 & 1 & 5 \\ 0 & \frac{5}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{pmatrix} \tag{3.4.10}$$

So, the given pair of lines do not intersect and also their direction vectors are not parallel. Hence they are skew lines.
To find $\mathbf{U}$,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \tag{3.4.11}$$

To calculate its Eigen values,

$$\begin{vmatrix} 10-\lambda & 8 & 20 \\ 8 & 8-\lambda & 16 \\ 20 & 16 & 40-\lambda \end{vmatrix} = 0 \tag{3.4.12}$$

$$\implies \lambda^3 + 58\lambda^2 + 80\lambda = 0 \tag{3.4.13}$$

$$\lambda_1 = 29 - \sqrt{761}, \lambda_2 = 29 + \sqrt{761}, \lambda_3 = 0 \tag{3.4.14}$$

with corresponding Eigen vectors as

$$\mathbf{u_1} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ \frac{-21-\sqrt{761}}{16} \\ 1 \end{pmatrix} \tag{3.4.15}$$

$$\mathbf{u_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{-21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ \frac{-21+\sqrt{761}}{16} \\ 1 \end{pmatrix} \tag{3.4.16}$$

$$\mathbf{u_3} = \frac{1}{\sqrt{(-2)^2 + 1}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \tag{3.4.17}$$

Solving, the $\mathbf{U}$ matrix becomes

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.4.18}$$

Also, from the obtained Eigen values,the $\mathbf{S}$ matrix becomes

$$\mathbf{S} = \begin{pmatrix} \sqrt{29-\sqrt{761}} & 0 \\ 0 & \sqrt{29+\sqrt{761}} \\ 0 & 0 \end{pmatrix} \tag{3.4.19}$$

The Moore-Penrose pseudo inverse of $\mathbf{S}$ is given by

$$\mathbf{S_+} = \begin{pmatrix} \frac{1}{\sqrt{29-\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29+\sqrt{761}}} & 0 \end{pmatrix} \tag{3.4.20}$$

Now to find $\mathbf{V}$,
Rewriting (3.4.3)

$$\mathbf{V} = (\mathbf{M}^T\mathbf{U})\mathbf{S_+}^T \tag{3.4.21}$$

$\mathbf{M}^T\mathbf{U}$ becomes

$$\begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} \tag{3.4.22}$$

$$\begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.4.23}$$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \end{pmatrix} \tag{3.4.24}$$

Therefore from (3.4.20),(3.4.21),(3.4.24),

$$\mathbf{V} = \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.4.25}$$

Now, to calculate $\mathbf{x}$

$$\mathbf{Mx} = \mathbf{b} \tag{3.4.26}$$

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.4.27}$$

$$\implies \mathbf{SV}^T\mathbf{x} = \mathbf{U}^T\mathbf{b} \tag{3.4.28}$$

$$\implies \mathbf{x} = \mathbf{V}(\mathbf{S_+}(\mathbf{U}^T\mathbf{b})) \tag{3.4.29}$$

Calculating $\mathbf{U}^T\mathbf{b}$, we have

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \tag{3.4.30}$$

$$\mathbf{S_+}(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.4.31}$$

$\mathbf{V}(\mathbf{S_+}(\mathbf{U}^T\mathbf{b}))$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.4.32}$$

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{3.4.33}$$

Solving,

$$\mathbf{x} = \begin{pmatrix} \frac{8371}{15220} \\ \frac{-15981}{15220} \end{pmatrix} = \begin{pmatrix} \frac{11}{20} \\ \frac{-21}{20} \end{pmatrix} \tag{3.4.34}$$

Verifying the solution,

$$\mathbf{Mx} = \mathbf{b} \tag{3.4.35}$$

$$\implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{3.4.36}$$

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.4.37}$$

$$= \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{3.4.38}$$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \tag{3.4.39}$$

$$= \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix} \tag{3.4.40}$$

$$\text{From, (3.4.36)} \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{3.4.41}$$

Solving for $\mathbf{x}$

$$\begin{pmatrix} 49 & 19 & 7 \\ 19 & 9 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1 \times \frac{19}{49}} \begin{pmatrix} 49 & 19 & 7 \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{3.4.42}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 \times \frac{1}{49}} \begin{pmatrix} 1 & \frac{19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{3.4.43}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2 \times \frac{19}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{3.4.44}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{49}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & 1 & \frac{-21}{20} \end{pmatrix} \tag{3.4.45}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{-21}{20} \end{pmatrix} \tag{3.4.46}$$

3.5. Find the point on the plane closest to the point $\begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$ and the plane is determined by the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$$

Solution: The equation of plane is given by,

$$\mathbf{n}^T\mathbf{x} = c \tag{3.5.1}$$

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = \mathbf{n}^T\mathbf{C} = c \tag{3.5.2}$$

$$\implies \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} \end{pmatrix}^T \mathbf{n} = 0 \tag{3.5.3}$$

Using row reduction on above matrix,

$$\begin{pmatrix} -2 & -3 & -2 \\ 6 & 3 & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ 6 & 3 & -2 \end{pmatrix} \tag{3.5.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 6R_1} \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ 0 & -6 & -8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ 0 & 3 & 4 \end{pmatrix} \tag{3.5.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \end{pmatrix} \tag{3.5.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix} 1 \\ \frac{-4}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} \tag{3.5.7}$$

$$c = \mathbf{n}^T\mathbf{A} = 19 \tag{3.5.8}$$

Thus the equation of the plane is,

$$\begin{pmatrix} 3 & -4 & 3 \end{pmatrix}\mathbf{x} = 19 \tag{3.5.9}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be the two orthogonal vectors to the given normal. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0 \tag{3.5.10}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} = 0 \tag{3.5.11}$$

$$\implies 3a - 4b + 3c = 0 \tag{3.5.12}$$

Let $a = 1$, $b = 0$ we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.5.13}$$

Let $a = 0$, $b = 1$ we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{4}{3} \end{pmatrix} \tag{3.5.14}$$

Solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.5.15}$$

Putting the values in (3.5.15),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{4}{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix} \tag{3.5.16}$$

To solve (3.5.16), we perform Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{3.5.17}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} \end{pmatrix} \tag{3.5.18}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{4}{3} \\ -1 & \frac{4}{3} & \frac{25}{9} \end{pmatrix} \tag{3.5.19}$$

Putting (3.5.17) in (3.5.15) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{3.5.20}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{3.5.21}$$

Where $\mathbf{S}_{+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Now, calculating eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{3.5.22}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{4}{3} \\ -1 & \frac{4}{3} & \frac{25}{9}-\lambda \end{pmatrix} = 0 \tag{3.5.23}$$

$$\implies \lambda^3 - \frac{43}{9}\lambda^2 + \frac{34}{9}\lambda = 0 \tag{3.5.24}$$

Thus the eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{34}{9} \tag{3.5.25}$$

$$\lambda_2 = 1 \tag{3.5.26}$$

$$\lambda_3 = 0 \tag{3.5.27}$$

The eigen vectors comes out to be,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-9}{25} \\ \frac{12}{25} \\ 1 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} 1 \\ \frac{-4}{3} \\ 1 \end{pmatrix} \tag{3.5.28}$$

Normalising the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} \\ \frac{12}{5\sqrt{34}} \\ \frac{5}{\sqrt{34}} \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} \frac{3}{\sqrt{34}} \\ \frac{-4}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{pmatrix} \tag{3.5.29}$$

Hence we obtain $\mathbf{U}$ matrix as,

$$\mathbf{U} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \tag{3.5.30}$$

Now,

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.5.31}$$

Calculating the eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{3.5.32}$$

$$\implies \begin{pmatrix} 2-\lambda & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9}-\lambda \end{pmatrix} = 0 \tag{3.5.33}$$

$$\implies \lambda^2 - \frac{43}{9}\lambda + \frac{34}{9} = 0 \tag{3.5.34}$$

The eigen values are,

$$\lambda_1 = \frac{34}{9} \tag{3.5.35}$$

$$\lambda_2 = 1 \tag{3.5.36}$$

The eigen vectors are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \tag{3.5.37}$$

Normalising the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{3.5.38}$$

Hence we obtain $\mathbf{V}$ matrix as,

$$\mathbf{V} = \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \tag{3.5.39}$$

Thus we get the Singular Value Decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}^T \tag{3.5.40}$$

The Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.5.41}$$

From (3.5.21) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{231}{5\sqrt{34}} \\ \frac{39}{5} \\ \frac{25}{\sqrt{34}} \end{pmatrix} \tag{3.5.42}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{693}{170} \\ \frac{39}{5} \end{pmatrix} \tag{3.5.43}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \tag{3.5.44}$$

Verifying the solution of (3.5.44) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.5.45}$$

Evaluating the R.H.S in (3.5.45) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \tag{3.5.46}$$

$$\implies \begin{pmatrix} 2 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \tag{3.5.47}$$

Solving the augmented matrix of (3.5.47) we get,

$$\begin{pmatrix} 2 & \frac{-4}{3} & -3 \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \tag{3.5.48}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{4}{3}R-1} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \tag{3.5.49}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \tag{3.5.50}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{9}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & 1 & \frac{135}{17} \end{pmatrix} \tag{3.5.51}$$

Hence, solution of (3.5.45) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \tag{3.5.52}$$

Comparing results of $\mathbf{x}$ from (3.5.44) and (3.5.52) we conclude that the solution is verified. Find the foot of the perpendicular using svd drawn from $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ to the plane

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\mathbf{x} + 3 = 0 \tag{3.5.53}$$

Solution:

3.6. Find the distance of the given point $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ from the plane $\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\mathbf{x} = 3$.

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{3.6.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0 \tag{3.6.2}$$

$$\implies 2a - b + 2c = 0 \tag{3.6.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.6.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \tag{3.6.5}$$

Let us solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.6.6}$$

Substituting (3.6.4) and (3.6.5) in (3.6.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{3.6.7}$$

To solve (3.6.7), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{3.6.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \tag{3.6.9}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \tag{3.6.10}$$

Substituting (3.6.8) in (3.6.6),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{3.6.11}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{3.6.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.
Let us calculate eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{3.6.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{3.6.14}$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \tag{3.6.15}$$

From equation (3.6.15) eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{3.6.16}$$

The eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \tag{3.6.17}$$

Normalizing the eigen vectors in equation (3.6.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{3.6.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \tag{3.6.19}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.6.20}$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{3.6.21}$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{3.6.22}$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{3.6.23}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \tag{3.6.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{3.6.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{3.6.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{3.6.27}$$

From (3.6.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \tag{3.6.28}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.6.29}$$

From (3.6.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{11}{3\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ \frac{10}{3} \end{pmatrix} \tag{3.6.30}$$

$$\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{22}{9\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.6.31}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \tag{3.6.32}$$

Verifying the solution of (3.6.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.6.33}$$

Evaluating the R.H.S in (3.6.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \tag{3.6.34}$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \tag{3.6.35}$$

Solving the augmented matrix of (3.6.35) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & 2 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \tag{3.6.36}$$

$$\xleftrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{9}{8} & -1 \end{pmatrix} \tag{3.6.37}$$

$$\xleftrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \tag{3.6.38}$$

$$\xleftrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \tag{3.6.39}$$

From equation (3.6.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \tag{3.6.40}$$

4 Solution

Comparing results of **x** from (3.6.32) and (3.6.40), we can say that the solution is verified.