



# Linear Algebra and Matrices



G V V Sharma\*

## CONTENTS

1	Line	1
2	Plane	6
3	Pseudo Inverse	6
4	Quadratic Forms	7
5	Tangents and Normals	9
6	Circle	10
7	Ellipse	12
8	Hyperbola	14
9	Parabola	16
10	Vector Calculus	18
11	Vector Inequalities	19
12	QR Decomposition	19
13	Singular Value Decomposition	19

## Appendix A: Proofs for the Conic Sections 20

*Abstract*—This book provides a simple introduction to linear algebra and matrix analysis. The content and exercises are based on NCERT textbooks from Class 6-12.

### 1 LINE

1.1. Any point  $\mathbf{P}$  in the 2-D plane can be expressed in terms of its coordinates  $(p_1, p_2)$  as the column vector

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad (1.1.1)$$

1.2. The *direction vector* of the line joining  $\mathbf{P}, \mathbf{Q}$  is defined as

$$\mathbf{m} = \mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \quad (1.2.1)$$

$$= (p_1 - q_1) \begin{pmatrix} 1 \\ \frac{p_2 - q_2}{p_1 - q_1} \end{pmatrix} = (p_1 - q_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.2.2)$$

where

$$m = \frac{p_2 - q_2}{p_1 - q_1}. \quad (1.2.3)$$

\*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

*Without loss of generality,  $k\mathbf{m}$ , for any real scalar  $k$  is also a direction vector. In the rest of the paper,  $\mathbf{m}$  and  $k\mathbf{m}$  are interchanged for*

computational simplicity. Thus, if  $m$  be the slope of the line  $PQ$ ,

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.2.4)$$

1.3. Let  $\mathbf{P}, \mathbf{Q}$  be two points on a line. The vector equation of the line is given by

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{m}, \quad \lambda \in \mathbb{R} \quad (1.3.1)$$

$$\mathbf{m} = \mathbf{P} - \mathbf{Q} \quad (1.3.2)$$

(1.3.1) can be used in 3D as well.

1.4. The *normal vector*  $\mathbf{n}$  to a line is orthogonal to the direction vector  $\mathbf{m}$  so that

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.4.1)$$

If  $\mathbf{P}$  be a point on the line, the equation of the line can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{P}) = 0 \quad (1.4.2)$$

$$\text{or, } \mathbf{n}^T \mathbf{x} = c, \quad (1.4.3)$$

where

$$c = \mathbf{n}^T \mathbf{P} \quad (1.4.4)$$

which is the desired equation of the straight line. By subsuming the  $c$  in (1.4.3) within  $\mathbf{n}$ , the equation of a line can also be expressed as

$$\mathbf{n}^T \mathbf{x} = 1 \quad (1.4.5)$$

Note that in 3D, (1.4.2) and (1.4.3) are used to represent the equation of a plane.

1.5. *Orthogonality*: Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.5.1)$$

are the vertices of a right angled triangle.

**Solution:** Let

$$\mathbf{v}_1 = \mathbf{A} - \mathbf{C} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad (1.5.2)$$

$$\mathbf{v}_2 = \mathbf{B} - \mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \quad (1.5.3)$$

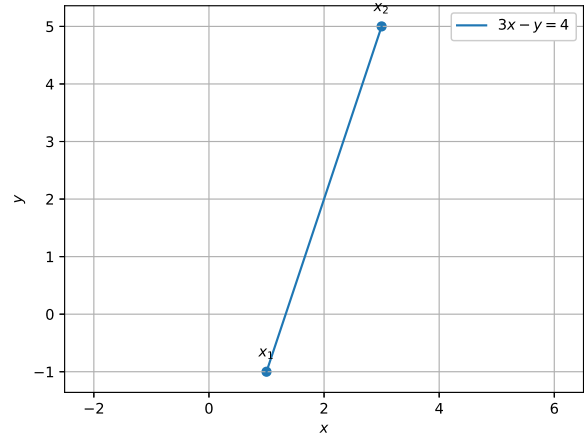


Fig. 1.6: Line obtained in Problem 1.6.

Then

$$\mathbf{v}_1^T \mathbf{v}_2 = \begin{pmatrix} -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 \quad (1.5.4)$$

$$\Rightarrow AC \perp BC \quad (1.5.5)$$

and  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are said to be orthogonal.

1.6. Find the equation of the line through  $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$  with slope - 4

**Solution:** From (1.2.4), the direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad (1.6.1)$$

and from (1.4.1), the normal vector is

$$\mathbf{n} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (1.6.2)$$

Using (1.4.2), the equation of the line is

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} = 0 \quad (1.6.3)$$

$$\Rightarrow \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = -5 \quad (1.6.4)$$

Fig. 1.6 shows the line passing through the given point.

1.7. Write the equation of the line through the points  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  and  $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ .

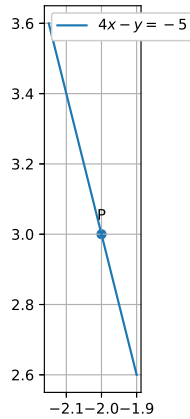


Fig. 1.7: Line obtained in Problem 1.7.

**Solution:** From (1.4.5),

$$\mathbf{n}^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \quad (1.7.1)$$

$$\mathbf{n}^T \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \quad (1.7.2)$$

resulting in the the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.7.3)$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \quad (1.7.4)$$

Performing row reduction,

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \quad (1.7.5)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 8 & -2 \end{pmatrix} \quad (1.7.6)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{8}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \quad (1.7.7)$$

$$\xleftrightarrow{R_1 \leftarrow 4R_1 + R_2} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \quad (1.7.8)$$

$$\xleftrightarrow{\begin{matrix} R_2 \leftarrow \frac{R_2}{4} \\ R_1 \leftarrow \frac{R_1}{4} \end{matrix}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \quad (1.7.9)$$

From (1.7.9),

$$\mathbf{n} = \frac{1}{4} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad (1.7.10)$$

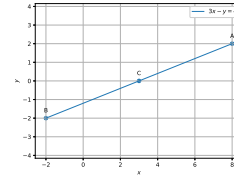


Fig. 1.8: Points on a line and points forming a triangle in Example 1.8.

Thus the equation of the desired line is

$$\frac{1}{4} \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 1 \quad (1.7.11)$$

$$\text{or, } \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 4 \quad (1.7.12)$$

Fig. 1.7 shows the line passing through the given points.

- 1.8. (*Linear Dependence*) Prove that the three points  $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \end{pmatrix}$  are collinear

**Solution:** Let

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (1.8.1)$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \end{pmatrix}$$

Then, the given points are collinear if

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = \mathbf{0} \quad (1.8.2)$$

has a nontrivial solution as well, i.e.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \mathbf{0} \quad (1.8.3)$$

Substituting (1.8.1) in (1.8.2) results in the matrix equation

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (1.8.4)$$

Performing row operations on the matrix,

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_1 - 5R_2} \begin{pmatrix} 5 & -10 \\ 0 & 0 \end{pmatrix} \quad (1.8.5)$$

which can be expressed as

$$\begin{pmatrix} 5 & -10 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (1.8.6)$$

$$\text{or, } \mathbf{x} = x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (1.8.7)$$

Thus, there are infinite solutions. The vectors  $\mathbf{v}_1, \mathbf{v}_2$  are linearly dependent and the given

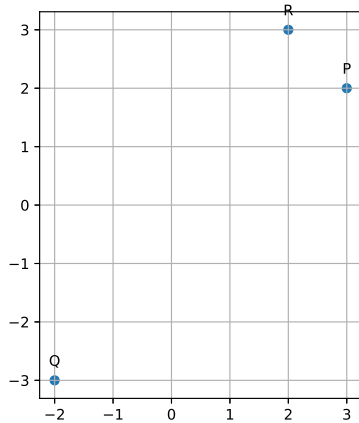


Fig. 1.10: Points on a triangle in Problem 1.10.

points lie on a straight line.

- 1.9. Alternatively, if the given points are collinear, from (1.4.5),

$$\begin{pmatrix} 3 & 0 \\ -2 & -2 \\ 8 & 2 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.9.1)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & -2 & 1 \\ 8 & 2 & 1 \end{pmatrix} \quad (1.9.2)$$

$$\xrightarrow[R_2 \leftarrow -3R_2 + 2R_1]{R_3 \leftarrow -3R_3 - 8R_1} \begin{pmatrix} 3 & 0 & 1 \\ 0 & -6 & 5 \\ 0 & 6 & -5 \end{pmatrix} \quad (1.9.3)$$

$$\xrightarrow{R_3 \leftarrow -R_3 + R_2} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.9.4)$$

The above matrix has a zero row in echelon form, hence (1.9.1) is consistent and the given points are on a straight line. Also,

$$\mathbf{n} = \frac{1}{6} \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix} \quad (1.9.5)$$

- 1.10. (Linear Independence) Do the points  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$  form a triangle?

**Solution:** In this case

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad (1.10.1)$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \quad (1.10.2)$$

Thus,

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = 0 \quad (1.10.3)$$

$$\Rightarrow \begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix} \mathbf{x} = 0 \quad (1.10.4)$$

Using row operations,

$$\begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 5 & -4 \\ 0 & 2 \end{pmatrix} \quad (1.10.5)$$

$$\xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \quad (1.10.6)$$

resulting in a *full rank* matrix. Hence,

$$\mathbf{x} = 0 \quad (1.10.7)$$

and  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are *linearly independent*. The points lie on a triangle.

- 1.11. Alternatively, from (1.4.5), row reducing the augmented matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow -R_3 + R_2} \begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad (1.11.1)$$

The above matrix has a nonzero row in echelon form, hence the given points do not lie on a straight line. So they lie on a triangle.

- 1.12. Find the angle between the lines

$$(1 - \sqrt{3})\mathbf{x} = 5 \quad (1.12.1)$$

$$(\sqrt{3} - 1)\mathbf{x} = -6.$$

**Solution:** The angle between the lines can be expressed in terms of the normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \quad (1.12.2)$$

as

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.12.3)$$

$$= \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ \quad (1.12.4)$$

1.13. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad (1.13.1)$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \quad (1.13.2)$$

**Solution:** If the angle between the vectors be  $\theta$ , the projection is defined as

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\|\mathbf{a}\| \cos \theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{(\mathbf{a}^T \mathbf{b})}{\|\mathbf{b}\|^2} \mathbf{b} \quad (1.13.3)$$

Substituting the values from (1.13.1) and (1.13.2),

$$\text{proj}_{\mathbf{b}} \mathbf{a} = \frac{5}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad (1.13.4)$$

1.14. (*Reflection*) Assuming that straight lines work as a plane mirror for a point, find the image of the point  $\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  in the line

$$L: (1 \ -3)\mathbf{x} = -4. \quad (1.14.1)$$

**Solution:** From the given equation, the line parameters are

$$\mathbf{n} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, c = -4, \mathbf{m} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad (1.14.2)$$

Let  $\mathbf{R}$  be the reflection of  $\mathbf{P}$  such that  $PR$  bisects the line  $L$  at  $\mathbf{Q}$ . Then  $\mathbf{Q}$  bisects  $PR$ . This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (1.14.3)$$

$$\mathbf{n}^T \mathbf{Q} = c \quad \because \mathbf{Q} \text{ lies on the given line} \quad (1.14.4)$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \quad \because \mathbf{m} \perp \mathbf{P} - \mathbf{R} \quad (1.14.5)$$

From (1.14.3) and (1.14.4),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \quad (1.14.6)$$

From (1.14.6) and (1.14.5),

$$(\mathbf{m} \ \mathbf{n})^T \mathbf{R} = (\mathbf{m} \ -\mathbf{n})^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.14.7)$$

Letting

$$\mathbf{V} = (\mathbf{m} \ \mathbf{n}) \quad (1.14.8)$$

with the condition that  $\mathbf{m}, \mathbf{n}$  are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (1.14.9)$$

Noting that

$$(\mathbf{m} \ -\mathbf{n}) = (\mathbf{m} \ \mathbf{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.14.10)$$

(1.14.7) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[ \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.14.11)$$

$$\Rightarrow \mathbf{R} = \left[ \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.14.12)$$

$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c\mathbf{n} \quad (1.14.13)$$

upon substituting from (1.14.8) in (1.14.13). It can be verified that the reflection is also given by

$$\mathbf{R} = (\mathbf{m} \ \mathbf{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\mathbf{m} \ \mathbf{n})^T \mathbf{P} + 2c\mathbf{n} \quad (1.14.14)$$

$$= (\mathbf{m} \ -\mathbf{n}) \begin{pmatrix} \mathbf{m}^T \\ \mathbf{n}^T \end{pmatrix} \mathbf{P} + 2c\mathbf{n} \quad (1.14.15)$$

$$\Rightarrow \mathbf{R} = (\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T) \mathbf{P} + 2c\mathbf{n} \quad (1.14.16)$$

If  $\mathbf{m}, \mathbf{n}$  are not orthonormal, (1.14.16) can be expressed as

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T \mathbf{m} + \mathbf{n}^T \mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (1.14.17)$$

1.15. (*Gram-schmidt orthogonalization*) Let

$$\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \quad (1.15.1)$$

$$\beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \quad (1.15.2)$$

Find  $\beta_1, \beta_2$  such that

$$\beta = \beta_1 + \beta_2, \quad \beta_1 \parallel \alpha, \beta_2 \perp \alpha \quad (1.15.3)$$

**Solution:** Let  $\beta_1 = k\alpha$ . Then,  $\beta_1 \parallel \alpha$  and

$$\beta = k\alpha + \beta_2 \quad (1.15.4)$$

$$\Rightarrow \alpha^T \beta = k \|\alpha\|^2 + k\beta_1^T \beta_2 \quad (1.15.5)$$

$$\text{or, } k = \frac{\alpha^T \beta}{\|\alpha\|^2}, \quad \because \beta_1 \perp \beta_2 \quad (1.15.6)$$

Thus,

$$\beta_1 = \frac{\alpha^T \beta}{\|\alpha\|^2} \alpha = \frac{1}{2} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \quad (1.15.7)$$

$$\beta_2 = \beta - \beta_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \quad (1.15.8)$$

Thus, any given set of vectors can be expressed as a linear combination of another set of orthogonal vectors.

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{pmatrix} \quad (2.1.2)$$

$$\xleftrightarrow[R_3 \leftarrow 2R_3 - 5R_1]{R_2 \leftarrow \frac{R_2 + R_1}{2}} \begin{pmatrix} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -19 & 9 & -3 \end{pmatrix} \quad (2.1.3)$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3 + 19R_2}{4}]{R_1 \leftarrow R_1 - 5R_2} \begin{pmatrix} 2 & 0 & -8 & -4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 7 & 4 \end{pmatrix} \quad (2.1.4)$$

$$\xleftrightarrow[R_3 \leftarrow 7R_2 - R_3]{R_1 \leftarrow \frac{7R_1 + 8R_3}{2}} \begin{pmatrix} 7 & 0 & 0 & 2 \\ 0 & 7 & 0 & 3 \\ 0 & 0 & 7 & 4 \end{pmatrix} \quad (2.1.5)$$

$$\Rightarrow \mathbf{n} = \frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad (2.1.6)$$

Thus, the equation of the plane passing through the given points is

$$(2 \ 3 \ 4)\mathbf{x} = 7 \quad (2.1.7)$$

2.2. Find the angle between the two planes

$$(2 \ 1 \ -2)\mathbf{x} = 5 \quad (2.2.1)$$

$$(3 \ -6 \ -2)\mathbf{x} = 7 \quad (2.2.2)$$

**Solution:** The angle between two planes is the same as the angle between their normal vectors. For

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} \quad (2.2.3)$$

using (1.12.3),

$$\cos \theta = \frac{6 - 6 + 4}{\sqrt{9} \sqrt{49}} = \frac{4}{21} \quad (2.2.4)$$

## 2 PLANE

2.1. Find the equation of a plane passing through

the points  $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$  and  $\mathbf{c} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$

**Solution:** The equation of plane is also given by (1.4.5) in 3D. Following the approach in the previous example results in the matrix equation,

$$\begin{pmatrix} 2 & 5 & -3 \\ -2 & -3 & 5 \\ 5 & 3 & -3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.1)$$

## 3 PSEUDO INVERSE

3.1. To find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.1.1)$$

$$L_2: \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (3.1.2)$$

3.2. If the two lines intersect,

$$\mathbf{x}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (3.2.1)$$

$$\Rightarrow (\mathbf{m}_1 \quad \mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \quad (3.2.2)$$

$$\text{or, } \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \quad (3.2.3)$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}. \quad (3.2.4)$$

$$\mathbf{M} = (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (3.2.5)$$

(3.2.3) can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (3.2.6)$$

From the augmented matrix in (3.2.3),

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (3.2.7)$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \xrightarrow{R_1=R_1-R_2} \begin{pmatrix} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (3.2.8)$$

The above matrix has a *rank* = 3. Hence the lines do not intersect.

3.3. Let

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (3.3.1)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (3.3.2)$$

be the closest points on  $L_1$  and  $L_2$  respectively. Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines  $L_1, L_2$  and passing through A and B. Thus,

$$\mathbf{m}_1^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.3.3)$$

$$\mathbf{m}_2^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.3.4)$$

$$\Rightarrow \mathbf{M}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.3.5)$$

From (3.3.2) and (3.2.5)

$$\mathbf{A} - \mathbf{B} = \mathbf{x}_1 - \mathbf{x}_2 + \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (3.3.6)$$

and using (3.3.5), in the above,

$$\mathbf{M}^T \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M}^T (\mathbf{x}_2 - \mathbf{x}_1) \quad (3.3.7)$$

3.4. Substituting the values from (3.2.4) in (3.3.7) and forming the augmented matrix,

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \quad (3.4.1)$$

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \xrightarrow{R_2=R_2-R_1} \begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \xrightarrow{R_1=2R_1-R_2} \begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \quad (3.4.3)$$

$$\begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \xrightarrow{R_1=\frac{R_1}{6}, R_2=\frac{R_2}{6}} \begin{pmatrix} 1 & 0 & \frac{11}{6} \\ 0 & 1 & -\frac{7}{6} \end{pmatrix} \quad (3.4.4)$$

$$\lambda_1 = \frac{11}{6}, \lambda_2 = \frac{7}{6} \quad (3.4.5)$$

yielding

$$\mathbf{A} = \frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix}, \mathbf{B} = \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}. \quad (3.4.6)$$

3.5. The distance is then obtained as

$$\|\mathbf{B} - \mathbf{A}\| = \frac{3}{\sqrt{2}} \quad (3.5.1)$$

Fig. 3.5 shows the various points and distances between the lines.

#### 4 QUADRATIC FORMS

4.1. The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.1)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (4.1.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (4.1.4)$$

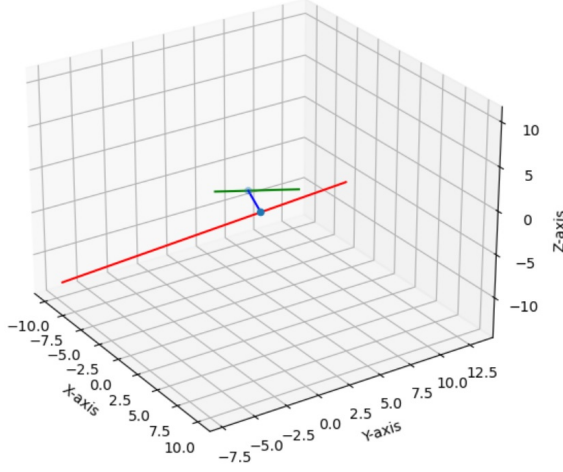


Fig. 3.5: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

4.2. (*Affine Transformation and Eigenvalue Decomposition*) Using

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (\text{Affine Transformation}) \quad (4.2.1)$$

such that

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D}. \quad (\text{Eigenvalue Decomposition}) \quad (4.2.2)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (4.2.3)$$

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad \mathbf{P}^T = \mathbf{P}^{-1} \quad (4.2.4)$$

(4.1.2) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |V| \neq 0 \quad (4.2.5)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad |V| = 0 \quad (4.2.6)$$

with

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |V| \neq 0 \quad (4.2.7)$$

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad |V| = 0 \quad (4.2.8)$$

$$\text{where } \eta = \mathbf{n}^T \mathbf{p}_1 \quad (4.2.9)$$

**Solution:** Proofs for (4.2.5), (4.2.6), (4.2.7) and (4.2.8) are available in Appendix A.

4.3. (*Centre/Vertex*) The centre/vertex of the conic section in (4.1.2) is given by  $\mathbf{c}$  in (4.2.7) or (4.2.8). This is because from (4.2.1),

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (4.3.1)$$

and

$$\mathbf{y} = \mathbf{0} \implies \mathbf{x} = \mathbf{c} \quad (4.3.2)$$

4.4. (*Circle*) For a circle,

$$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I} \quad (4.4.1)$$

and the centre is obtained from (4.2.7), (4.3.2) as

$$\mathbf{c} = -\mathbf{u} \quad (4.4.2)$$

(4.2.5) becomes

$$\mathbf{y}^T \mathbf{y} = \|\mathbf{y}\|^2 = \left( \sqrt{\mathbf{u}^T \mathbf{u} - f} \right)^2 \quad (4.4.3)$$

and the radius is

$$\sqrt{\mathbf{u}^T \mathbf{u} - f} \quad (4.4.4)$$

4.5. (*Ellipse*) For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \quad (4.5.1)$$

and (4.2.5) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (4.5.2)$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}, \quad \sqrt{\frac{\lambda_2}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}. \quad (4.5.3)$$

The centre is obtained from (4.3.2) as (4.2.7).

4.6. (*Hyperbola*) For

$$|\mathbf{V}| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \quad (4.6.1)$$

and (4.2.5) becomes

$$\lambda_1 y_1^2 - (-\lambda_2) y_2^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (4.6.2)$$

with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}, \quad \sqrt{\frac{\lambda_2}{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}}, \quad (4.6.3)$$

The centre is obtained from (4.3.2) as (4.2.7).

4.7. (*Pair of straight lines:*) The *asymptotes* of the hyperbola (4.1.2) are defined as the pair of intersecting straight lines

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + K = 0 \quad (4.7.1)$$



such that

$$K = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} \quad (4.7.2)$$

$$|\mathbf{V}| < 0 \quad (4.7.3)$$

From (4.6.2) and (4.7.2) the equation of the asymptotes for (4.6.2) is

$$\left( \sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|} \right) \mathbf{y} = 0 \quad (4.7.4)$$

and the asymptotes for the hyperbola are obtained using (4.2.1) as

$$\left( \sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|} \right) \mathbf{P}^T (\mathbf{x} - \mathbf{c}) = 0 \quad (4.7.5)$$

Thus,  $\mathbf{c}$  is the point of intersection of the lines and the normal vectors of the lines in (4.7.5) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \quad (4.7.6)$$

4.8. The angle between the asymptotes is given by

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (4.8.1)$$

The orthogonal matrix  $\mathbf{P}$  preserves the norm, i.e.

$$\|\mathbf{n}_1\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \quad (4.8.2)$$

$$= \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n}_2\| \quad (4.8.3)$$

It is easy to verify that

$$\mathbf{n}_1^T \mathbf{n}_2 = |\lambda_1| - |\lambda_2| \quad (4.8.4)$$

Thus, the angle between the asymptotes is obtained from (4.8.1) as

$$\cos \theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \quad (4.8.5)$$

4.9. Apart from (4.7.2), another condition for (4.1.2) to represent a pair of straight lines is

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.9.1)$$

4.10. (*Parabola*) For

$$|\mathbf{V}| = 0, \quad \text{or, } \lambda_1 = 0. \quad (4.10.1)$$

The vertex of the parabola is obtained using (4.2.8) and the focal length is

$$\left| \frac{2\mathbf{p}_1^T \mathbf{u}}{\lambda_2} \right| \quad (4.10.2)$$

## 5 TANGENTS AND NORMALS

5.1. *Secant*: The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (5.1.1)$$

with the conic section in (4.1.2) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (5.1.2)$$

where

$$\begin{aligned} \mu_i &= \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left( -\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \right. \\ &\quad \left. \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \end{aligned} \quad (5.1.3)$$

**Solution:** Substituting (5.1.1) in (4.1.2),

$$\begin{aligned} (\mathbf{q} + \mu \mathbf{m})^T \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) + 2\mathbf{u}^T (\mathbf{q} + \mu \mathbf{m}) + f &= 0 \\ \implies \mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \\ &\quad + \mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \end{aligned} \quad (5.1.4)$$

Solving the above quadratic in (5.1.4) yields (5.1.3).

5.2. *Tangent*: If  $L$  in (5.1.1) touches (4.1.2) at exactly one point  $\mathbf{q}$ ,

$$\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (5.2.1)$$

**Solution:** In this case, (5.1.4) has exactly one root. Hence, in (5.1.3)

$$\begin{aligned} &\left[ \mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \right]^2 \\ &\quad - \left( \mathbf{m}^T \mathbf{V} \mathbf{m} \right) \left( \mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f \right) = 0 \end{aligned} \quad (5.2.2)$$

$\because \mathbf{q}$  is the point of contact,  $\mathbf{q}$  satisfies (4.1.2) and

$$\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \quad (5.2.3)$$

Substituting (5.2.3) in (5.2.2) and simplifying, we obtain (5.2.1).

5.3. The normal vector is obtained from (5.2.1) and (1.4.1) as

$$\mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u} \quad (5.3.1)$$

5.4. Given the point of contact  $\mathbf{q}$ , the equation of a tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \quad (5.4.1)$$

**Solution:** From (5.3.1) and (1.4.2), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T (\mathbf{x} - \mathbf{q}) = 0 \quad (5.4.2)$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} - \mathbf{q}^T \mathbf{V}\mathbf{q} - \mathbf{u}^T \mathbf{q} = 0 \quad (5.4.3)$$

which, upon substituting from (5.2.3) and simplifying yields (5.1.1).

5.5. If  $\mathbf{V}^{-1}$  exists, given the normal vector  $\mathbf{n}$ , the tangent points of contact to (4.1.2) are given by

$$\mathbf{q}_i = \mathbf{V}^{-1} (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \quad (5.5.1)$$

$$\text{where } \kappa_i = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (5.5.2)$$

**Solution:** From (5.3.1),

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \quad (5.5.3)$$

Substituting (5.5.3) in (5.2.3),

$$\begin{aligned} & (\kappa \mathbf{n} - \mathbf{u})^T \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \\ & + 2\mathbf{u}^T \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + f = 0 \\ \implies & \kappa^2 \mathbf{n}^T \mathbf{V}^{-1} \mathbf{n} - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} + f = 0 \\ \text{or, } \kappa = & \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (5.5.4) \end{aligned}$$

Substituting (5.5.4) in (5.5.3) yields (5.5.2).

5.6. If  $\mathbf{V}$  is not invertible, given the normal vector  $\mathbf{n}$ , the point of contact to (4.1.2) is given by the matrix equation

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (5.6.1)$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \quad \mathbf{V} \mathbf{p}_1 = 0 \quad (5.6.2)$$

**Solution:** If  $\mathbf{V}$  is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is  $\mathbf{p}_1$ , then,

$$\mathbf{V} \mathbf{p}_1 = 0 \quad (5.6.3)$$

From (5.3.1),

$$\kappa \mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \quad (5.6.4)$$

$$\implies \kappa \mathbf{p}_1^T \mathbf{n} = \mathbf{p}_1^T \mathbf{V}\mathbf{q} + \mathbf{p}_1^T \mathbf{u} \quad (5.6.5)$$

$$\text{or, } \kappa \mathbf{p}_1^T \mathbf{n} = \mathbf{p}_1^T \mathbf{u}, \quad \because \mathbf{p}_1^T \mathbf{V} = 0, \quad (5.6.6)$$

$$\text{from (5.6.3)} \quad (5.6.7)$$

yielding  $\kappa$  in (5.6.2). From (5.6.4),

$$\kappa \mathbf{q}^T \mathbf{n} = \mathbf{q}^T \mathbf{V}\mathbf{q} + \mathbf{q}^T \mathbf{u} \quad (5.6.8)$$

$$\implies \kappa \mathbf{q}^T \mathbf{n} = -f - \mathbf{q}^T \mathbf{u} \quad \text{from (5.2.3),} \quad (5.6.9)$$

$$\text{or, } (\kappa \mathbf{n} + \mathbf{u}) \mathbf{q} = -f \quad (5.6.10)$$

(5.6.4) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa \mathbf{n} - \mathbf{u}. \quad (5.6.11)$$

(5.6.10) and (5.6.11) clubbed together result in (5.6.1).

5.7. All the results related to conics are summarized in Table 5.7.

## 6 CIRCLE

6.1. Find the centre and radius of the circle

$$x^2 + y^2 + 8x + 10y - 8 = 0 \quad (6.1.1)$$

**Solution:** (6.1.1) can be expressed as

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 8 = 0 \quad (6.1.2)$$

which is of the form (4.1.2) with

$$\mathbf{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, f = -8 \quad (6.1.3)$$

From Table 5.7, the center and radius are given by

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}, r = \sqrt{\|\mathbf{u}\|^2 - f} = 7 \quad (6.1.4)$$

6.2. Find the equation of a circle which passes through the points  $\mathbf{P} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$  and  $\mathbf{Q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$  and whose centre lies on the line

$$(1 \ 1) \mathbf{x} = 2 \quad (6.2.1)$$

**Solution:** From (4.1.2) and Table 5.7, the equation of a circle can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (6.2.2)$$

Conic	Property	Standard Form	Standard Parameters	Point(s) of Contact
Circle	$\mathbf{V} = \mathbf{I}$	$\frac{\mathbf{y}^T \mathbf{D} \mathbf{y}}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f} = 1$	$\mathbf{c} = -\mathbf{u},$ $r = \sqrt{\mathbf{u}^T \mathbf{u} - f}$	$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u})$
Ellipse	$ \mathbf{V}  > 0$ $\lambda_1 > 0, \lambda_2 < 0$	$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T$ $\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2)$	$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u},$ $axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} \end{cases}$	$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}}$
Hyperbola	$ \mathbf{V}  < 0$ $\lambda_1 > 0, \lambda_2 < 0$		$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u},$ $axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases}$	
Parabola	$ \mathbf{V}  = 0$ $\lambda_1 = 0$	$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta(1 \quad 0) \mathbf{y}$	focal length = $\left  \frac{\eta}{\lambda_2} \right $ $\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{v} \end{pmatrix} \mathbf{c}$ $= \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix}$ $\eta = 2 \mathbf{p}_1^T \mathbf{u}$	$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{v} \end{pmatrix} \mathbf{q}$ $= \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix}$ $\kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}$

TABLE 5.7:  $\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0$  can be expressed in the above standard form for various conics.  $\mathbf{c}$  represents the centre/vertex of the conic.  $\mathbf{q}$  is/are the point(s) of contact for the tangent(s).

where  $\mathbf{c}$  is the centre. Substituting the given points in (6.2.2) and using (6.2.1), the following equations are obtained

$$2 \begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{c} - f = 8 \quad (6.2.3)$$

$$2 \begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{c} - f = 25 \quad (6.2.4)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{c} = 2 \quad (6.2.5)$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 4 & -4 & -1 \\ 6 & 8 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 25 \end{pmatrix} \quad (6.2.6)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 4 & -4 & -1 & 8 \\ 6 & 8 & -1 & 25 \end{pmatrix} \quad (6.2.7)$$

$$\xrightarrow[R_3 \leftarrow R_3 - 6R_1]{R_2 \leftarrow R_2 - 4R_1} \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 8 & -1 & 0 \\ 0 & 2 & -1 & 13 \end{pmatrix} \quad (6.2.8)$$

$$\xrightarrow[R_3 \leftarrow \frac{4R_3 - R_2}{2}]{R_1 \leftarrow 8R_1 - R_3} \begin{pmatrix} 8 & 0 & -1 & 16 \\ 0 & 8 & -1 & 0 \\ 0 & 0 & 5 & -52 \end{pmatrix} \quad (6.2.9)$$

$$\xrightarrow[R_2 \leftarrow \frac{5R_2 - R_3}{4}]{R_1 \leftarrow \frac{5R_1 + R_3}{4}} \begin{pmatrix} 10 & 0 & 0 & 7 \\ 0 & 10 & 0 & 13 \\ 0 & 0 & 5 & -52 \end{pmatrix} \quad (6.2.10)$$

Thus,

$$\mathbf{c} = \frac{1}{10} \begin{pmatrix} 7 \\ 13 \end{pmatrix} \quad (6.2.11)$$

$$f = -\frac{52}{5} \quad (6.2.12)$$

which give the desired equation of the circle. From Table 5.7,

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \frac{1}{10} \sqrt{1258} \quad (6.2.13)$$

Fig. 6.2 verifies the above results.

6.3. Find the points on the curve

$$x^2 + y^2 - 2x - 3 = 0 \quad (6.3.1)$$

at which the tangents are parallel to the  $x$ -axis.

**Solution:** (6.3.1) can be expressed as

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} - 3 = 0 \quad (6.3.2)$$

$$\Rightarrow \mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, f = -3 \quad (6.3.3)$$

From Table 5.7, the centre and radius are

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \quad (6.3.4)$$

$\therefore$  the tangents are parallel to the  $x$ -axis, their

## 7 ELLIPSE

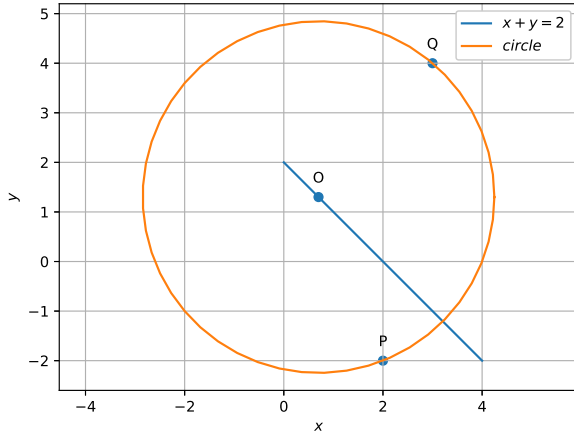


Fig. 6.2: Circle passing through  $\begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ . Center is on line  $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$ .

direction and normal vectors are respectively,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (6.3.5)$$

From Table 5.7,

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\mathbf{n}^T \mathbf{n}}} = \pm \sqrt{\frac{4}{1}} = \pm 2 \quad (6.3.6)$$

and the desired points of contact are

$$\mathbf{q}_1, \mathbf{q}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (6.3.7)$$

Fig. 6.2 verifies the above results.

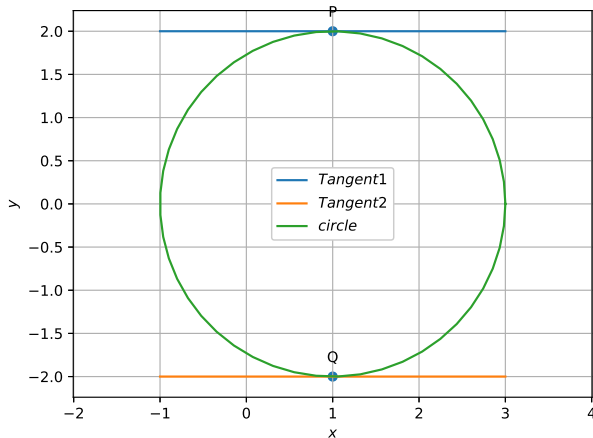


Fig. 6.3: Tangents are parallel to the x-axis.

7.1. Find  $\frac{dy}{dx}$  if

$$E_1 : x^2 + xy + y^2 = 100 \quad (7.1.1)$$

**Solution:** Expressing (7.1.1) as (4.1.2),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \mathbf{u} = \mathbf{0}, f = -100. \quad (7.1.2)$$

$$\because |V| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} > 0, \quad (7.1.3)$$

(7.1.1) is the equation of an ellipse. To verify that this is indeed the case, we do the following exercise. The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 1 & \frac{1}{2} \\ \frac{1}{2} & \lambda - 1 \end{vmatrix} = 0 \quad (7.1.4)$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (7.1.5)$$

The eigenvalues are the roots of (7.1.5) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \quad (7.1.6)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (7.1.7)$$

$$\implies (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = \mathbf{0} \quad (7.1.8)$$

where  $\lambda$  is the eigenvalue. For  $\lambda_1 = \frac{3}{2}$ ,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (7.1.9)$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (7.1.10)$$

such that  $\|\mathbf{p}_1\| = 1$ . Similarly, the eigenvector corresponding to  $\lambda_2$  can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (7.1.11)$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (7.1.12)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (7.1.13)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad (7.1.14)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad (7.1.15)$$

From Table 5.7, ellipse parameters are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \mathbf{0} \quad (7.1.16)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = 10 \sqrt{\frac{2}{3}} \quad (7.1.17)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 10 \sqrt{2} \quad (7.1.18)$$

In Fig. 7.1 the actual ellipse in (7.1.1) is obtained from (4.2.5) using (4.2.1). The anticlockwise  $45^\circ$  rotation is due to the fact that (7.1.14) can be expressed as

$$\mathbf{P} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \quad (7.1.19)$$

Coming back to the original question of finding  $\frac{dy}{dx}$ , if the point of contact

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad (7.1.20)$$

from (7.1.2), (1.2.4) and (5.2.1),

$$\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \quad (7.1.21)$$

$$\Rightarrow \left(1 + \frac{m}{2} \quad \frac{1}{2} + m\right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \quad (7.1.22)$$

$$\Rightarrow \frac{m}{2} (q_1 + 2q_2) + q_1 + \frac{q_2}{2} = 0 \quad (7.1.23)$$

$$\text{or, } m = \frac{dy}{dx} = -\frac{2q_1 + q_2}{q_1 + 2q_2} \quad (7.1.24)$$

$\therefore \frac{dy}{dx}$  is the slope of the tangent. Note that no results from differential calculus were used to obtain (7.1.24).

- 7.2. Find the equation of the ellipse, with major axis along the x-axis and passing through the points  $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

**Solution:** This is a standard ellipse given by

$$\mathbf{x}^T \mathbf{D} \mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \quad (7.2.1)$$

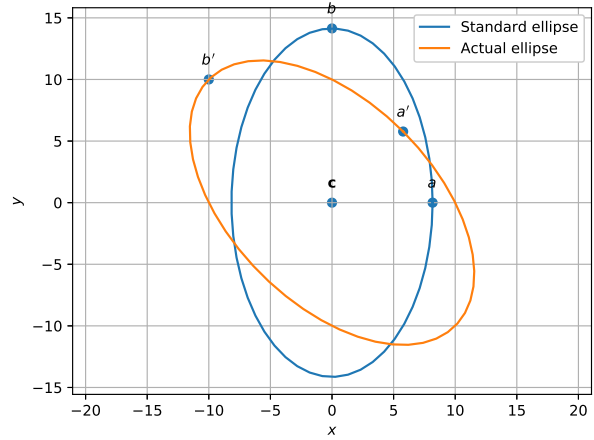


Fig. 7.1: Actual ellipse and transformed ellipse.

$\therefore \mathbf{a}, \mathbf{b}$  satisfy (7.2.1),

$$\mathbf{a}^T \mathbf{D} \mathbf{a} = 1, \quad (7.2.2)$$

$$\mathbf{b}^T \mathbf{D} \mathbf{b} = 1 \quad (7.2.3)$$

which can be expressed as

$$\mathbf{a}^T \mathbf{A} \mathbf{d} = 1, \quad (7.2.4)$$

$$\mathbf{b}^T \mathbf{B} \mathbf{d} = 1$$

where

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}. \quad (7.2.5)$$

(7.2.4) can then be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{a}^T \mathbf{A} \\ \mathbf{b}^T \mathbf{B} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (7.2.6)$$

which, after substituting the appropriate values can be expressed as

$$\begin{pmatrix} 16 & 9 \\ 1 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (7.2.7)$$

Forming the augmented matrix and performing

row reduction,

$$\begin{pmatrix} 16 & 9 & 1 \\ 1 & 16 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow -R_2]{R_2 \leftarrow R_1} \begin{pmatrix} 1 & 16 & 1 \\ 0 & 247 & 15 \end{pmatrix} \quad (7.2.8)$$

$$\xleftrightarrow{R_1 \leftarrow 247R_1 - 16R_2} \begin{pmatrix} 247 & 0 & 7 \\ 0 & 247 & 15 \end{pmatrix} \quad (7.2.9)$$

$$\Rightarrow \mathbf{d} = \frac{1}{247} \begin{pmatrix} 7 \\ 15 \end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247} \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \quad (7.2.10)$$

The ellipse parameters are obtained from Table 5.7 as

$$\mathbf{c} = \mathbf{0}, \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \quad (7.2.11)$$

Fig. 7.2 verifies the above results.

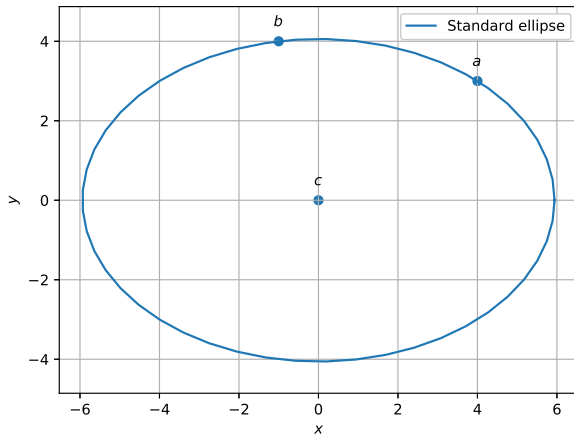


Fig. 7.2: Ellipse through the given points  $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ .

## 8 HYPERBOLA

8.1. Find the equation of all lines having slope 2 and being tangent to the curve

$$y + \frac{2}{x-3} = 0 \quad (8.1.1)$$

**Solution:** (8.1.1) can be expressed as

$$xy - 3y + 2 = 0 \quad (8.1.2)$$

which is of the same form as (4.1.2) with

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = -\frac{3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = 2 \quad (8.1.3)$$

Using the approach in Example 7.1,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (8.1.4)$$

$$\therefore \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = -2 < 0, \quad (8.1.5)$$

the major and minor axis are swapped and from Table 5.7 the hyperbola parameters are given by

$$\mathbf{c} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 2, \quad (8.1.6)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = 2 \quad (8.1.7)$$

with the standard hyperbola equation becoming

$$\frac{y_2^2}{4} - \frac{y_1^2}{4} = 1, \quad (8.1.8)$$

Fig. 8.1 shows the actual hyperbola in (8.1.1) obtained from (8.1.8) using (4.2.1). The direction and normal vectors of the tangent with slope 2 are given by (1.2.4) and (1.4.1) as

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (8.1.9)$$

From (5.5.2) and (6.3.3), using (8.1.3),

$$\kappa = \frac{1}{2}, \mathbf{q}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \quad (8.1.10)$$

The desired tangents are

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} = 0 \Rightarrow \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 2 \quad (8.1.11)$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right\} = 0 \Rightarrow \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 10 \quad (8.1.12)$$

All the above results are verified in Fig. 8.1. As we can see, the hyperbola in (8.1.1) is obtained by rotating the standard hyperbola by  $\mathbf{P}$  and then translating it by  $\mathbf{c}$ .

8.2. Find the asymptotes of the hyperbola given below and also the equations to their conjugate

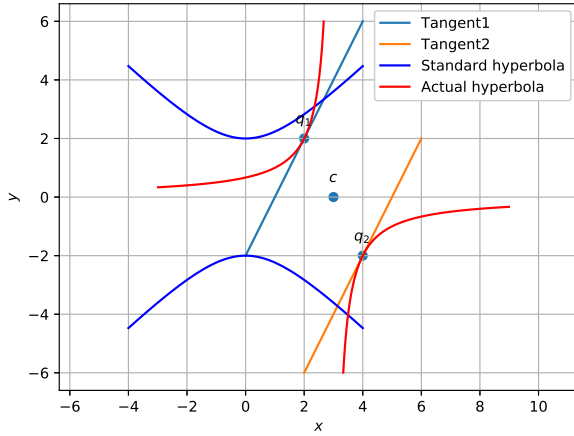


Fig. 8.1: Standard and actual hyperbola.

hyperbolas.

$$8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0 \quad (8.2.1)$$

a) (8.2.1) can be expressed as (4.1.2) with

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \quad (8.2.2)$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (8.2.3)$$

$$f = -2 \quad (8.2.4)$$

Expanding the Determinant of  $\mathbf{V}$ ,

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} = -49 < 0 \quad (8.2.5)$$

Hence from (4.6.1) and (8.2.5), (8.2.1) represents a hyperbola. The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|V - \lambda \mathbf{I}| = 0 \quad (8.2.6)$$

$$\begin{vmatrix} 8 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = 0 \quad (8.2.7)$$

$$(8 - \lambda)(-3 - \lambda) - 25 = 0 \quad (8.2.8)$$

$$\Rightarrow \lambda^2 - 5\lambda - 49 = 0 \quad (8.2.9)$$

$$\lambda_1 = \frac{5 + \sqrt{221}}{2} \quad (8.2.10)$$

$$\lambda_2 = \frac{5 - \sqrt{221}}{2} \quad (8.2.11)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (8.2.12)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (8.2.13)$$

For  $\lambda_1 = \frac{5 + \sqrt{221}}{2}$ ,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (8.2.14)$$

By row reduction,

$$\begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (8.2.15)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{11 + \sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11 - \sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \quad (8.2.16)$$

Substituting (8.2.16) in (8.2.13) we get

$$\begin{pmatrix} 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \mathbf{p}_1 = \mathbf{0} \quad (8.2.17)$$

$$\Rightarrow \mathbf{p}_1 = k \begin{pmatrix} \frac{11 + \sqrt{221}}{2} \\ 5 \end{pmatrix} \quad (8.2.18)$$

For  $\lambda_2 = \frac{5 - \sqrt{221}}{2}$ ,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{11 + \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 + \sqrt{221}}{2} \end{pmatrix} \quad (8.2.19)$$

By row reduction,

$$\begin{pmatrix} \frac{11 + \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 + \sqrt{221}}{2} \end{pmatrix} \quad (8.2.20)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{11 - \sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11 + \sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \quad (8.2.21)$$

Substituting (8.2.21) in (8.2.13) we get

$$\begin{pmatrix} \frac{11 + \sqrt{221}}{2} & 5 \end{pmatrix} \mathbf{p}_2 = \mathbf{0} \quad (8.2.22)$$

$$\Rightarrow \mathbf{p}_2 = k \begin{pmatrix} -5 \\ \frac{11 + \sqrt{221}}{2} \end{pmatrix} \quad (8.2.23)$$

Thus, we obtain

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = k \begin{pmatrix} \frac{11 + \sqrt{221}}{2} & -5 \\ 5 & \frac{11 + \sqrt{221}}{2} \end{pmatrix} \quad (8.2.24)$$

For

$$\mathbf{P}^T \mathbf{P} = \mathbf{I}, k = \sqrt{\frac{221 + 11\sqrt{221}}{2}} \quad (8.2.25)$$

$$\Rightarrow \mathbf{P} = \sqrt{\frac{2}{221 + 11\sqrt{221}}} \times \begin{pmatrix} \frac{11+\sqrt{221}}{2} & -5 \\ 5 & \frac{11+\sqrt{221}}{2} \end{pmatrix} \quad (8.2.26)$$

and

$$\mathbf{V} = \mathbf{PDP}^T \quad (8.2.27)$$

where

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \quad (8.2.28)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (8.2.29)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (8.2.30)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (8.2.31)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (8.2.32)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 > 0 \quad (8.2.33)$$

there isn't a need to swap axes In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (8.2.34)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{2}{5 + \sqrt{221}}} \quad (8.2.35)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{\frac{2}{5 - \sqrt{221}}} \quad (8.2.36)$$

The equation of the hyperbola at the origin is then given by (4.2.5) as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (8.2.37)$$

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \mathbf{y} = 1 \quad (8.2.38)$$

b) (*Asymptotes of hyperbola:*) The equation for the asymptotes of (8.2.1) is given by (4.7.1)

with

$$K = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} \quad (8.2.39)$$

$$= \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1 \quad (8.2.40)$$

From (4.7.6), (8.2.10) and (8.2.11),

$$\mathbf{n}_1 = \sqrt{\frac{2}{221 + 11\sqrt{221}}} \times \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ -5 & \frac{11+\sqrt{221}}{2} \end{pmatrix} \times \begin{pmatrix} \sqrt{\frac{\sqrt{221}+5}{2}} \\ \sqrt{\frac{\sqrt{221}-5}{2}} \end{pmatrix} \quad (8.2.41)$$

which can be expressed as

$$\mathbf{n}_1 = \frac{1}{\sqrt{(\lambda + 3)^2 + 5^2}} \times \begin{pmatrix} \lambda_1 + 3 & 5 \\ -5 & \lambda_1 + 3 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} \\ \frac{7}{\sqrt{\lambda_1}} \end{pmatrix} \quad (8.2.42)$$

which is equivalent to

$$\mathbf{n}_1 = \begin{pmatrix} \lambda_1 + 3 & 5 \\ -5 & \lambda_1 + 3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ 7 \end{pmatrix} \quad (8.2.43)$$

$$= \begin{pmatrix} \lambda_1^2 + 3\lambda_1 + 35 \\ 2\lambda_1 + 21 \end{pmatrix} \quad (8.2.44)$$

$$= \begin{pmatrix} 8\lambda_1 + 84 \\ 2\lambda_1 + 21 \end{pmatrix} \quad (8.2.45)$$

using (8.2.9), which is equivalent to

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (8.2.46)$$

Similarly, it can be shown that

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (8.2.47)$$

Fig. 8.2 plots the hyperbola in (8.2.1) along with the asymptotes obtained using (4.7.5).

## 9 PARABOLA

9.1. Find the point at which the tangent to the curve

$$y = \sqrt{4x - 3} - 1 \quad (9.1.1)$$



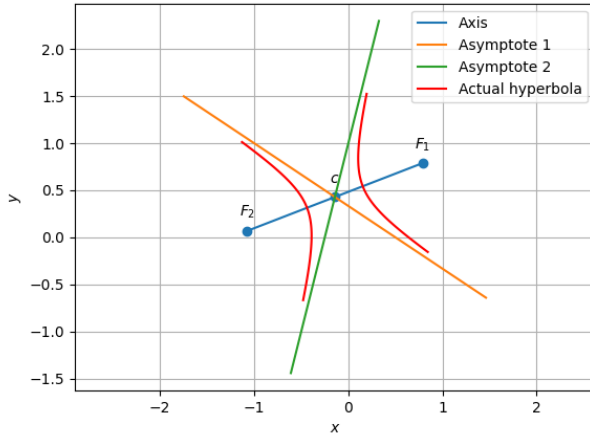


Fig. 8.2: Hyperbola with asymptotes and its conjugate

has slope  $\frac{2}{3}$ .

**Solution:** (9.1.1) can be expressed as

$$(y + 1)^2 = 4x - 3 \quad (9.1.2)$$

$$\text{or, } y^2 - 4x + 2y + 4 = 0 \quad (9.1.3)$$

which has the form (4.1.2) with parameters

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, f = 4. \quad (9.1.4)$$

Thus, the given curve is a parabola.  $\therefore \mathbf{V}$  is diagonal and in standard form,

$$\mathbf{P} = \mathbf{I} \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (9.1.5)$$

From Table 5.7, the focus is 4 and the vertex  $\mathbf{c}$  is

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \quad (9.1.6)$$

$$\Rightarrow \begin{pmatrix} -4 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \quad (9.1.7)$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \quad (9.1.8)$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}. \quad (9.1.9)$$

Also,

$$\mathbf{V}\mathbf{p} = \mathbf{0} \quad (9.1.10)$$

$$\Rightarrow \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (9.1.11)$$

From (5.6.2), (9.1.9) and (9.1.11),

$$\kappa = -1 \quad (9.1.12)$$

which, upon substitution in (5.6.1) and simplification yields the matrix equation

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \quad (9.1.13)$$

$$\Rightarrow \begin{pmatrix} -4 & 4 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (9.1.14)$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (9.1.15)$$

Fig. 9.1 verifies the above results.

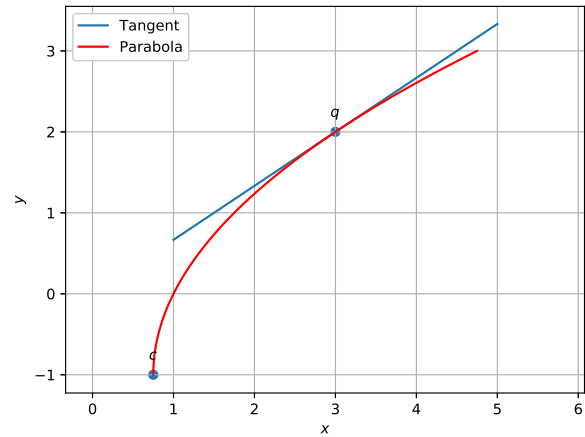


Fig. 9.1: Tangent to parabola in (9.1.1) with slope  $\frac{2}{3}$ .

9.2. Find a point on the curve

$$y = (x - 2)^2 \quad (9.2.1)$$

at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

**Solution:** (9.2.1) can be expressed as

$$x^2 - 4x - y + 4 = 0 \quad (9.2.2)$$

which has the form (4.1.2) with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = 4. \quad (9.2.3)$$

Using eigenvalue decomposition,

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (9.2.4)$$

Hence, the eigenvector of  $\mathbf{V}$  corresponding to the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (9.2.5)$$

Substituting the above parameters in the equation for the vertex of the parabola in Table 5.7,

$$\begin{pmatrix} -2 & -\frac{5}{2} \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \quad (9.2.6)$$

$$\Rightarrow \begin{pmatrix} -1 & -\frac{5}{2} \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (9.2.7)$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (9.2.8)$$

The direction vector is

$$\mathbf{m} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad (9.2.9)$$

and normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (9.2.10)$$

From the equation for the point of contact for the parabola in Table 5.7,

$$\kappa = \frac{1}{2} \quad (9.2.11)$$

resulting in the matrix equation

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \quad (9.2.12)$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (9.2.13)$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (9.2.14)$$

Fig. 9.2 verifies the above results. Note that  $\mathbf{P}$  rotates the standard parabola by  $90^\circ$ .

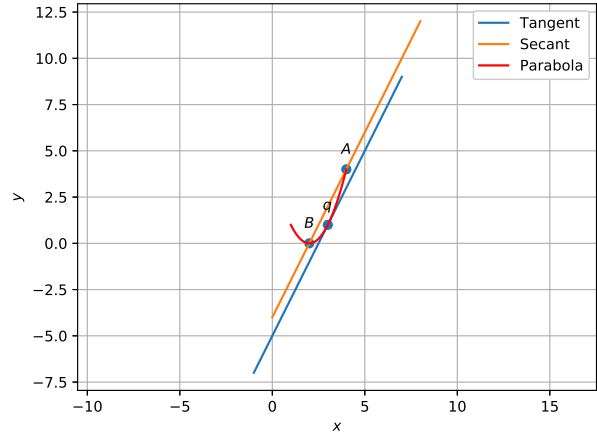


Fig. 9.2: Tangent to parabola in (9.2.1) is parallel to the line joining the points  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ .

## 10 VECTOR CALCULUS

10.1. *Definition:* Let  $\mathbf{x} \in \mathbb{R}^2$ ,  $f(\mathbf{x}) \in \mathbb{R}$ . Then,

$$\frac{df(\mathbf{x})}{d\mathbf{x}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \end{pmatrix} \quad (10.1.1)$$

10.2.

$$\begin{aligned} \frac{d\mathbf{x}}{dx_1} &= \begin{pmatrix} \frac{dx_1}{dx_1} \\ \frac{dx_2}{dx_1} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ m \end{pmatrix} = \mathbf{m} \end{aligned} \quad (10.2.1)$$

10.3. Show that

$$\begin{aligned} \frac{d(\mathbf{u}^T \mathbf{x})}{d\mathbf{x}} &= \mathbf{u} \\ \frac{d(\mathbf{x}^T \mathbf{V} \mathbf{x})}{d\mathbf{x}} &= 2\mathbf{V}^T \mathbf{x} \end{aligned} \quad (10.3.1)$$

10.4. Differentiating (4.1.2) with respect to  $x_1$ ,

$$\left[ \frac{d(\mathbf{x}^T \mathbf{V} \mathbf{x})}{d\mathbf{x}} \right]^T \frac{d\mathbf{x}}{dx_1} + 2 \frac{d(\mathbf{u}^T \mathbf{x})}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} = 0 \quad (10.4.1)$$

$$\Rightarrow 2(\mathbf{V}^T \mathbf{x} + \mathbf{u}) \mathbf{m} = 0 \quad (10.4.2)$$

from (10.2.1) and (10.3.1). Substituting the point of contact  $\mathbf{x} = \mathbf{q}$  and simplifying results

in

$$(\mathbf{V}\mathbf{q} + \mathbf{u})\mathbf{m} = 0 \quad (10.4.3)$$

which, upon taking the transpose, yields (5.2.1).

## 11 VECTOR INEQUALITIES

11.1. (*Cauchy-Schwarz Inequality*;) Show that

$$|\mathbf{a}^T \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad (11.1.1)$$

*Proof.* Using the definition of the inner product,

$$\cos \theta = \frac{\mathbf{a}^T \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \quad (11.1.2)$$

$$\because |\cos \theta| \leq 1, |\mathbf{a}^T \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad (11.1.3)$$

(*Triangle Inequality*;) Show that

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \quad (11.1.4)$$

*Proof.* Let  $\mathbf{O}$  be the origin. In the triangle formed by  $\mathbf{O}$ ,  $\mathbf{a}$  and  $-\mathbf{b}$ , the lengths of the sides are

$$\|\mathbf{a}\|, \|\mathbf{b}\|, \|\mathbf{a} + \mathbf{b}\| \quad (11.1.5)$$

$\therefore$  the sum of two sides of a triangle is always greater than the third side,

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \quad (11.1.6)$$

## 12 QR DECOMPOSITION

12.1. Revisiting Problem (1.15),

$$\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \quad (12.1.1)$$

we can express

$$\begin{aligned} \alpha &= k_1 \mathbf{u}_1 \\ \beta &= r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \end{aligned} \quad (12.1.2)$$

where

$$k_1 = \|\alpha\|, \mathbf{u}_1 = \frac{\alpha}{k_1} \quad (12.1.3)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2}, \mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (12.1.4)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (12.1.5)$$

From (12.1.2),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (12.1.6)$$

This is known as **QR** decomposition, where

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (12.1.7)$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \quad (12.1.8)$$

Note that  $\mathbf{R}$  is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I}. \quad (12.1.9)$$

12.2. From (12.1.1),

$$k_1 = \sqrt{10}, \mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \quad (12.2.1)$$

$$r_1 = \frac{1}{2}, \mathbf{u}_2 = \frac{1}{\sqrt{46}} \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \quad (12.2.2)$$

$$k_2 = \sqrt{\frac{23}{2}} \quad (12.2.3)$$

Thus, we obtain the **QR** decomposition

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{46}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{46}} \\ 0 & \frac{-6}{\sqrt{46}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{1}{2} \\ 0 & \sqrt{\frac{23}{2}} \end{pmatrix} \quad (12.2.4)$$

## 13 SINGULAR VALUE DECOMPOSITION

13.1. We revisit (3.2.6)

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (13.1.1)$$

13.2. Find  $\mathbf{M}^T \mathbf{M}$  and  $\mathbf{M} \mathbf{M}^T$ .

13.3. Obtain the eigen decomposition

$$\mathbf{M}^T \mathbf{M} = \mathbf{P}_1 \mathbf{D}_1 \mathbf{P}_1^T \quad (13.3.1)$$

and

$$\mathbf{M} \mathbf{M}^T = \mathbf{P}_2 \mathbf{D}_2 \mathbf{P}_2^T \quad (13.3.2)$$

13.4. Perform the *QR* decompositions

$$\mathbf{P}_1 = \mathbf{U} \mathbf{R}_1 \mathbf{P}_2 = \mathbf{V} \mathbf{R}_2 \quad (13.4.1)$$

13.5. The singular value decomposition is the given

by

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T, \quad (13.5.1)$$

where  $\mathbf{\Sigma}$  has the same shape as  $\mathbf{M}$  and

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (13.5.2)$$

13.6. (3.2.6) can then be expressed as

$$\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (13.6.1)$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T \mathbf{b} \quad (13.6.2)$$

where  $\mathbf{\Sigma}^{-1}$  is obtained by inverting only the non-zero elements of  $\mathbf{\Sigma}$ .

13.7. The relevant codes are available at

codes/line/skew\_builtin.py  
codes/line/skew\_svd.py

## APPENDIX A

### PROOFS FOR THE CONIC SECTIONS

A.1. Substituting (4.2.1) in (4.1.2)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \quad (A.1.1)$$

which can be expressed as

$$\mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V}\mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (A.1.2)$$

From (A.1.2) and (4.2.2),

$$\mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.1.3)$$

When  $\mathbf{V}^{-1}$  exists,

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or, } \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \quad (A.1.4)$$

and substituting (A.1.4) in (A.1.3) yields (4.2.5).

A.2. When  $|\mathbf{V}| = 0$ ,  $\lambda_1 = 0$  and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \quad (A.2.1)$$

where  $\mathbf{p}_1, \mathbf{p}_2$  are the eigenvectors of  $\mathbf{V}$  such that (4.2.2)

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad (A.2.2)$$

Substituting (A.2.2) in (A.1.3),

$$\begin{aligned} & \mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T)(\mathbf{p}_1 \quad \mathbf{p}_2) \mathbf{y} \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \implies \mathbf{y}^T \mathbf{D} \mathbf{y} \\ & \quad + 2((\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_1 \quad (\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \implies \mathbf{y}^T \mathbf{D} \mathbf{y} \\ & \quad + 2(\mathbf{u}^T \mathbf{p}_1 \quad (\lambda_2 \mathbf{c}^T + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \quad \text{from (A.2.1)} \\ & \implies \lambda_2 y_2^2 + 2(\mathbf{u}^T \mathbf{p}_1) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^T \mathbf{p}_2 \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.2.3) \end{aligned}$$

which is the equation of a parabola. From (A.2.3), by comparing the coefficients of  $y_2^2$  and  $y_1$ , the focal length of the parabola is obtained as

$$\left| \frac{2\mathbf{u}^T \mathbf{p}_1}{\lambda_2} \right|. \quad (A.2.4)$$

Thus, (A.2.3) can be expressed as (4.2.6) by choosing

$$\eta = 2\mathbf{u}^T \mathbf{p}_1 \quad (A.2.5)$$

and  $\mathbf{c}$  in (A.1.3) such that

$$\mathbf{P}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (A.2.6)$$

$$\mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.2.7)$$

Multiplying (A.2.6) by  $\mathbf{P}$  yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \mathbf{p}_1, \quad (A.2.8)$$

which, upon substituting in (A.2.7) results in

$$\eta \mathbf{c}^T \mathbf{p}_1 + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.2.9)$$

(A.2.8) and (A.2.9) can be clubbed together to obtain (4.2.8).