Geometry through Linear Algebra

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Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1 Pair of Straight Lines

1.1. Find the value of h so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.1)$$

may represent two straight lines.
Solution:

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \quad (1.1.2)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.3)$$

$$f = 20 \quad (1.1.4)$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \quad (1.1.5)$$

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Expanding equation (1.1.5) along row 1 gives

$$\implies 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + 11 \times (\frac{31h}{2} - 132) = 0$$

$$\implies 20h^2 - 341h + \frac{2907}{2} = 0 \quad (1.1.6)$$

$$\implies \boxed{h = \frac{17}{2}} \quad (1.1.7)$$

$$\implies \boxed{h = \frac{171}{20}} \quad (1.1.8)$$

If $h = \frac{17}{2}$ or $h = \frac{171}{20}$, the equation given will represent two straight lines.

Sub $h = \frac{17}{2}$ in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.9)$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (1.1.10)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.11)$$

$$\mathbf{f} = 20 \quad (1.1.12)$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \tag{1.1.13}$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \tag{1.1.14}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.1.15}$$

$$\tag{1.1.16}$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \tag{1.1.17}$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \tag{1.1.18}$$

$$\implies m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \tag{1.1.19}$$

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{1.1.20}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.21}$$

we know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.22}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$
From equation (1.1.21)

$$\mathbf{n_1} = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.23}$$

$$\implies \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.24}$$

$\implies$ Equation (1.1.21) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.25}$$

Substituting (1.1.21) in (1.1.25), the augmented matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.1.26}$$

$$\implies c_1 = 4, c_2 = 5 \tag{1.1.27}$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\implies (-2x - 3y - 4)(3x - 4y - 5) = 0$$
$$\implies \boxed{(2x + 3y + 4)(3x + 4y + 5) = 0} \tag{1.1.28}$$

Equation (1.1.28) represents equations of two straight lines.

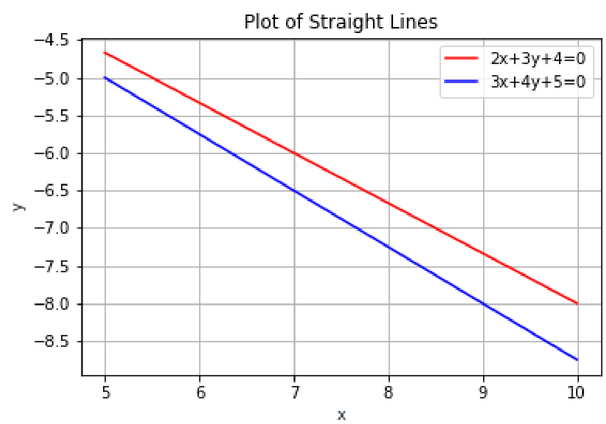

Fig. 1.1.1: Plot of Straight lines when $h = \frac{17}{2}$

Similarly, Sub $h = \frac{171}{20}$ in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \tag{1.1.29}$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \tag{1.1.30}$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{6} \\ \frac{310}{6} \end{pmatrix} \tag{1.1.31}$$

$$\mathbf{f} = \frac{200}{3} \tag{1.1.32}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \tag{1.1.33}$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \tag{1.1.34}$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \tag{1.1.35}$$

$$\implies m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \tag{1.1.36}$$

$$\mathbf{m_1} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \tag{1.1.37}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{1.1.38}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$

From equation (1.1.38)

$$\mathbf{n_1} = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{1.1.39}$$

$$\implies \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.40}$$

$\implies$ Equation (1.1.38) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.41}$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1-4R_2}{5}]{R_2 \leftarrow \frac{5R_2-8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \tag{1.1.42}$$

$$\implies c_1 = 10, c_2 = \frac{20}{3} \tag{1.1.43}$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\implies \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \tag{1.1.44}$$

Equation (1.1.44) represents equations of two straight lines.

1.2. Prove that the following equations represent two straight lines.Also find their point of intersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \tag{1.2.1}$$

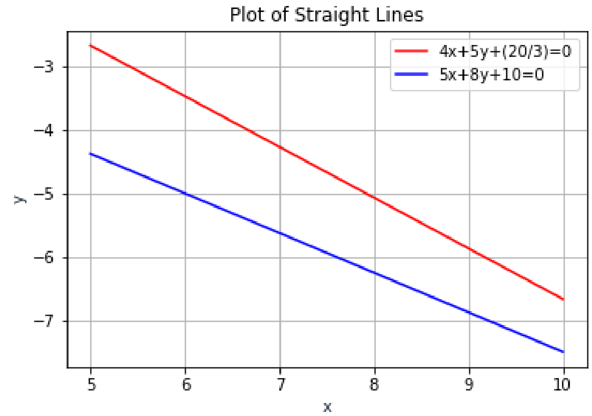

Fig. 1.1.2: Plot of Straight lines when $h = \frac{171}{20}$