Geometry through Linear Algebra

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Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.**

1 Planes and Lines

1.1 Distance from a plane to a point

1.1.1. Find the foot of perpendicular from the point $\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ on the plane $\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\mathbf{x} = 2$.
Solution: Consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.1.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \tag{1.1.1.2}$$

$$\implies 3a + 2b - 6c = 0 \tag{1.1.1.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.1.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \tag{1.1.1.5}$$

Solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.1.6}$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \tag{1.1.1.7}$$

Solving (1.1.1.7) using Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{1.1.1.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. We

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have,

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \tag{1.1.1.9}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \tag{1.1.1.10}$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U\Sigma V}^T\mathbf{x} = \mathbf{b} \tag{1.1.1.11}$$

$$\implies \mathbf{x} = \mathbf{V\Sigma^{-1}U^Tb} \tag{1.1.1.12}$$

Where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$ and is obtained by inversing only non-zero elements in $\mathbf{\Sigma}$

Calculating eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.13}$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda \end{vmatrix} = 0 \tag{1.1.1.14}$$

$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \tag{1.1.1.15}$$

From the characteristic equation (1.1.1.15), the eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{1.1.1.16}$$

The eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{3} \\ 1 \end{pmatrix} \tag{1.1.1.17}$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-7}{12} \\ \frac{-7}{18} \\ \frac{7}{6} \end{pmatrix} \tag{1.1.1.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \tag{1.1.1.19}$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.1.20}$$

Calculating eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.21}$$

$$\implies \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9}-\lambda \end{vmatrix} = 0 \tag{1.1.1.22}$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \tag{1.1.1.23}$$

From the characteristic equation, the eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \tag{1.1.1.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.1.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.27}$$

From (1.1.1.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.1.28}$$

And, the Moore-Penrose Pseudo inverse of $\mathbf{\Sigma}$ is given by,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.1.29}$$

From (1.1.1.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \tag{1.1.1.30}$$

$$\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.31}$$

$$\mathbf{x} = \mathbf{V\Sigma^{-1}U}^T\mathbf{b} \quad = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \tag{1.1.1.32}$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{Mx} = \mathbf{b} \implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{1.1.1.33}$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T\mathbf{Mx} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \tag{1.1.1.34}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \tag{1.1.1.35}$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xleftrightarrow{R_1=\frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \tag{1.1.1.36}$$

$$\xleftrightarrow{R_2=R_2-\frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{49}{45} & \frac{8}{3} \end{pmatrix} \tag{1.1.1.37}$$

$$\xleftrightarrow{R_2=\frac{45}{49}R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \tag{1.1.1.38}$$

$$\xleftrightarrow{R_1=R_1-\frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \tag{1.1.1.39}$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \tag{1.1.1.40}$$

From the equations (1.1.1.32) and (1.1.1.40), the solution $\mathbf{x}$ is verified.