

# Problems in Linear Algebra

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## 1 INTRODUCTION

### 1.1 Points

1. Find the distance between

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (1.1.1)$$

**Solution:** Two point are  $P(x_1, y_1)$  and  $Q(x_2, y_2)$ .  
The distance between both points is d.

$$\mathbf{Z} = \mathbf{P} - \mathbf{Q} \quad (1.1.1)$$

Then the distance between P and Q is given

by:

$$d = \|\mathbf{Z}\| \quad (1.1.1)$$

$$d = \|\mathbf{P} - \mathbf{Q}\| \quad (1.1.1)$$

So, the distance between given points P and Q is:

$$d = \sqrt{(-2 - 3)^2 + (4 - (-5))^2} \quad (1.1.1)$$

$$d = \sqrt{25 + 81} \quad (1.1.1)$$

$$d = \sqrt{106} \quad (1.1.1)$$

So, the distance between P(-2,4) and Q(3,-5) is :

$$d = \sqrt{106} \quad (1.1.1)$$

2. Find the length of PQ for

a)  $\mathbf{P} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  and  $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ;

b)  $\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$  and  $\mathbf{Q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ ;

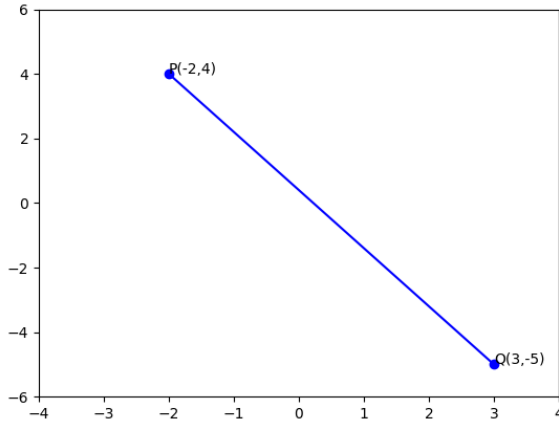


Fig. 1.1.1: Line between two points

c)  $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$  and  $\mathbf{Q} = \begin{pmatrix} -b \\ a \end{pmatrix}$ .

3. Using direction vectors, show that  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$  are the vertices of a parallelogram.
4. Using Baudhayana's theorem, show that the points  $\begin{pmatrix} -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$  are the vertices of a right-angled triangle. Repeat using orthogonality.
5. Plot the points  $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$  and prove that they are the vertices of a rectangle.
6. Show that  $\mathbf{B} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  are the vertices of an isosceles triangle.
7. In the last question, find the distance of the vertex  $\mathbf{A}$  of the triangle from the middle point of the base  $BC$ .
8. Prove that the points  $\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$  are the vertices of a square.
9. Prove that the points  $\mathbf{A} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$  and  $\mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  are the vertices of a parallelogram. Find  $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ , the mid points of  $AB, BC, CD, AD$  respectively. Show that  $EG$  and  $FH$  bisect each other.
10. Prove that the points  $\begin{pmatrix} 21 \\ -2 \end{pmatrix}, \begin{pmatrix} 15 \\ 10 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -12 \end{pmatrix}$  are the vertices of a rectangle, and find the coordinates of its centre.
11. Find the lengths of the medians of the triangle whose vertices are at the points  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ .
12. Find the coordinates of the points that divide the line joining the points  $\begin{pmatrix} -35 \\ -20 \end{pmatrix}$  and  $\begin{pmatrix} 5 \\ -10 \end{pmatrix}$  into four equal parts.
13. Find the coordinates of the points of trisection of the line joining the points  $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$  and  $\begin{pmatrix} 25 \\ 10 \end{pmatrix}$ .
14. Prove that the middle point of the line joining the points  $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$  and  $\begin{pmatrix} 9 \\ -2 \end{pmatrix}$  is a point of trisection of the line joining the points  $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$  and  $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$ .
15. The points  $\begin{pmatrix} 8 \\ 5 \end{pmatrix}, \begin{pmatrix} -7 \\ -5 \end{pmatrix}$  and  $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$  are three of the vertices of a parallelogram. Find the coordinates of the remaining vertex which is to be taken as opposite to  $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$ .
16. The point  $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$  is the intersection of the diagonals of a parallelogram two of whose vertices are at the points  $\begin{pmatrix} 7 \\ 16 \end{pmatrix}$  and  $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$ . Find the coordinates of the remaining vertices.
17. Find the area of the triangle whose vertices are the points  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 7 \end{pmatrix}$  and  $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ .
18. Find the coordinates of points which divide the join of  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix}$  externally in the ratio 2 : 3, and also externally in the ratio 3 : 2.
19. Prove the centroid of  $\triangle ABC$  is
 
$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.1.19)$$

## 1.2 Loci

1. A point moves so that its distance from the point  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  is double its distance from the point  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . Find the equation of its locus.
2. Find the equation of the perpendicular bisector of the line joining the points  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ .

- Find the equation of the circle of radius 5 with centre at  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ .
- A point moves so that its distance from the  $y$ -axis is equal to the distance from the point  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ . Find the equation of its locus.
- A point moves so that the sum of the squares of its distance from the points  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$  is constant. Find the equation of the locus.
- A point moves so that its distance from the axis of  $x$  is twice its distance from the point  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Find the equation of the locus.
- A point moves in such a way that with the points  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$  it forms a triangle of area 8.5. Show that its locus has an equation

$$\{(1 \ 5)\mathbf{x}\}\{(1 \ 5)\mathbf{x} - 34\} = 0 \quad (1.2.7)$$

## 2 THE STRAIGHT LINE

### 2.1 Intercepts

- Find the intercepts made on the axes by the straight lines whose equations are
  - $\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 2$
  - $\begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} = -5$
  - $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$
  - $\begin{pmatrix} \frac{1}{a+b} & \frac{1}{a-b} \end{pmatrix}\mathbf{x} = \frac{1}{a^2-b^2}$
  - $\begin{pmatrix} 1 & -m \end{pmatrix}\mathbf{x} = -c$
- Write down the equations of straight lines which make the following pairs of intercepts on the axes:
  - 3, -4
  - 5, 6
  - $\frac{1}{a}, \frac{1}{b}$
  - $2a, -2a$
- A straight line passes through a fixed point  $\begin{pmatrix} h \\ k \end{pmatrix}$  and cuts the axes in **A**, **B**. Parallels to the axes through **A** and **B** intersect in **P**. Find the equation of the locus of **P**.

### 2.2 Line Equation

- Find the equations of two straight lines at a distance 3 from the origin and making an angle of  $120^\circ$  with  $OX$ .

- Find the equation of a straight line making an angle of  $60^\circ$  with  $OX$  and passing through the point  $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ . Transform the equation to the form

$$(\cos \alpha \ \sin \alpha)\mathbf{x} = p \quad (2.2.2)$$

- Find the equation of the straight line that passes through the points  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ . What is its inclination to  $OX$ ?
- Find the equation of the straight line through the point  $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$  that makes equal intercepts on the axes.
- Find the equations of the sides of a triangle whose vertices are  $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ ,  $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ .
- For the same triangle find the equations of the medians
- Find the equation of a straight line passing through the point  $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$  parallel to the line  $\begin{pmatrix} 4 & -1 \end{pmatrix}\mathbf{x} + 7 = 0$ .
- Find the intercepts on the axes made by a straight line which passes through the point  $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$  and makes an angle of  $30^\circ$  with  $OX$ .
- Find the equation of the straight line through the points  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  and of the parallel line through  $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ .
- What is the distance from the origin of the line  $\begin{pmatrix} 4 & -1 \end{pmatrix}\mathbf{x} = 7$ ? Write down the equation of a parallel line at double the distance.
- Find the equation of the straight line through the point  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$  parallel to the line joining the origin to the point  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ .
- Write down the equation of the straight line which makes intercepts 2 and -7 on the axes, and of the parallel line through the point  $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ .
- Find the equations of the straight line joining the points  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ,  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$  and of the parallel line through the origin.
- $ABC$  is a triangle and **A**, **B** and **C** are the points

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ . Find the equation of the straight line through A parallel to BC.

15. Find the equation of a line parallel to  $(2 \ 5)\mathbf{x} = 11$  passing through the middle point of the join of the points  $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} 5 \\ -11 \end{pmatrix}$ .

**Solution:** General equation of straight line is given by:

$$\mathbf{n}^T \mathbf{x} = c \quad (2.2.15)$$

$\mathbf{n}$  will be same because both lines are parallel.

$$\mathbf{n} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (2.2.15)$$

Passing through mid point  $\mathbf{M}$  of  $\mathbf{A}, \mathbf{B}$ :

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (2.2.15)$$

$$\mathbf{n}^T (\mathbf{x} - \mathbf{M}) = 0 \quad (2.2.15)$$

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{M} \quad (2.2.15)$$

So, the mid point  $\mathbf{M}$  is :

$$\mathbf{M} = \frac{\begin{pmatrix} -7 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -11 \end{pmatrix}}{2} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (2.2.15)$$

$$\mathbf{n}^T \mathbf{x} = (2 \ 5) \begin{pmatrix} -1 \\ -4 \end{pmatrix} = -22 \quad (2.2.15)$$

So, the equation of line is :

$$(2 \ 5)\mathbf{x} = -22 \quad (2.2.15)$$

16. The base of a triangle passes through a fixed point  $\begin{pmatrix} f \\ g \end{pmatrix}$  and the sides are bisected at right angles by the axes. Prove that the locus of the vertex is the line

$$(g \ f)\mathbf{x} = 0 \quad (2.2.16)$$

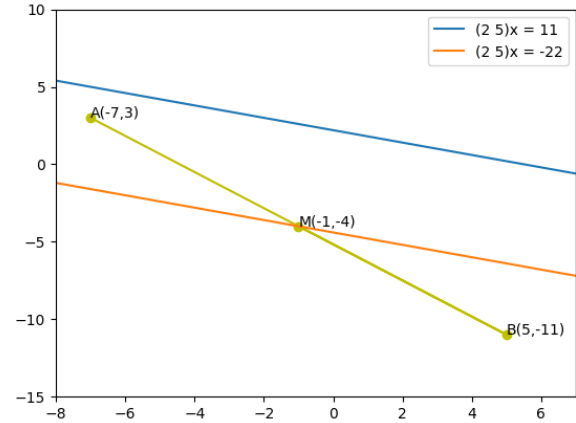


Fig. 2.2.15: Parallel Lines

### 2.3 Point of Intersection

1. Find the vertices of the triangle whose sides are

$$(3 \ 2)\mathbf{x} + 6 = 0, \quad (2.3.1.1)$$

$$(2 \ -5)\mathbf{x} + 4 = 0, \quad (2.3.1.2)$$

$$(1 \ -3)\mathbf{x} - 6 = 0 \quad (2.3.1.3)$$

2. Prove that the lines

$$(1 \ 1)\mathbf{x} + 25 = 0, \quad (2.3.2.1)$$

$$(2 \ 3)\mathbf{x} + 7 = 0 \quad (2.3.2.2)$$

$$(3 \ 5)\mathbf{x} = 11 \quad (2.3.2.3)$$

are concurrent, and find the coordinates of their common point.

3. Find the equation of a line parallel to the line

$$(2 \ -1)\mathbf{x} = 3 \quad (2.3.3.1)$$

and passing through the intersection of the lines

$$(3 \ 1)\mathbf{x} = 7 \quad (2.3.3.2)$$

$$(2 \ -3)\mathbf{x} = 5 \quad (2.3.3.3)$$

4. Find the equation of the line joining the origin to the point of intersection of the lines

$$(3 \ -5)\mathbf{x} = 11 \quad (2.3.4.1)$$

$$(2 \ 7)\mathbf{x} + 4 = 0 \quad (2.3.4.2)$$

5. Find the acute angle between the lines

$$(1 \ -1)\mathbf{x} = -7 \quad (2.3.5.1)$$

$$(2 + \sqrt{3} \ 1)\mathbf{x} = 11 \quad (2.3.5.2)$$

6. Find the angle between the lines

$$(-2 \ 1)\mathbf{x} = 5 \quad (2.3.6.1)$$

$$(2 \ 4)\mathbf{x} + 11 = 0 \quad (2.3.6.2)$$

7. Find the equation of a straight line through the point  $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$  at right angles to the line

$$(5 \ 7)\mathbf{x} + 12 = 0 \quad (2.3.7)$$

and find the point in which the lines intersect.

8. Find the equation of a straight line through the origin and at right angles to the line

$$(a \ b)\mathbf{x} + c = 0 \quad (2.3.8)$$

9. Find the equation of a straight line at right angles to the line

$$(5 \ -2)\mathbf{x} + 11 = 0 \quad (2.3.9.1)$$

and passing through the intersection of the lines

$$(1 \ 2)\mathbf{x} + 1 = 0, \quad (2.3.9.2)$$

$$(-1 \ 1)\mathbf{x} = 7. \quad (2.3.9.3)$$

10. The origin is a corner of a square and two of its sides have equations

$$(2 \ 1)\mathbf{x} = 0 \quad (2.3.10.1)$$

$$(2 \ 1)\mathbf{x} = 3. \quad (2.3.10.2)$$

Find the equations of the other two sides.

11. Write down the equations of the perpendiculars from the origin to the lines

$$(1 \ 5)\mathbf{x} = 13, \quad (2.3.11.1)$$

$$(5 \ 1)\mathbf{x} = 13 \quad (2.3.11.2)$$

and find the equation of the line joining the feet of the perpendiculars.

12. Prove that the line

$$(1 \ 1)\mathbf{x} = 11 \quad (2.3.12.1)$$

makes equal angles with the lines

$$(1 \ -(2 - \sqrt{3}))\mathbf{x} + 2 = 0, \quad (2.3.12.2)$$

$$((2 - \sqrt{3}) \ -1)\mathbf{x} + 5 = 0 \quad (2.3.12.3)$$

13. **A** is the point  $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$  and **B** is the point  $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ . Find the locus of a point **P** such that the angles  $APO$ ,  $OPB$  are equal, where **O** is the origin.

## 2.4 Perpendiculars and Bisectors

1. Find the distance of the point  $\begin{pmatrix} 4 \\ 2\sqrt{3} \end{pmatrix}$  from the line

$$(\cos 60^\circ \ \sin 60^\circ)\mathbf{x} = 6 \quad (2.4.1)$$

2. Find the distance of the point  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  from each of the straight lines

$$(5 \ 12)\mathbf{x} - 20 = 0 \quad (2.4.2.1)$$

$$(4 \ -3)\mathbf{x} + 11 = 0 \quad (2.4.2.2)$$

$$(3 \ 4)\mathbf{x} - 28 = 0. \quad (2.4.2.3)$$

3. Find the distance of the point  $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$  from the line joining the points  $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ ,  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ .

4. Are the points  $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$  on the same or on opposite sides of the line

$$(4 \ 5)\mathbf{x} = 10? \quad (2.4.4)$$

5. Find the equations of the bisectors of the angles between the lines

$$(-4 \ 2)\mathbf{x} = -9 \quad (2.4.5.1)$$

$$(-1 \ 2)\mathbf{x} = 4 \quad (2.4.5.2)$$

and state which equation refers to the angle which contains the origin.

6. Prove that the bisector of one of the angles between the lines

$$(5 \ 1)\mathbf{x} - 7 = 0 \quad (2.4.6.1)$$

$$(1 \ -5)\mathbf{x} + 7 = 0 \quad (2.4.6.2)$$

passes through the origin. What is the equation of the bisector of the other angle?

7. What is the condition that the point  $\begin{pmatrix} x \\ y \end{pmatrix}$  may be at unit distance from the line

$$(3 \ -4)\mathbf{x} + 10 = 0 \quad (2.4.7)$$

Write down the equations of two straight lines parallel to the given line and at unit distances from it, and state which of the two lies on the same side of the given line as the origin.

8. The sides  $AB$ ,  $BC$ ,  $CA$  of a triangle have equations

$$(4 \ -3)\mathbf{x} = 12 \quad (2.4.8.1)$$

$$(3 \ 4)\mathbf{x} = 24 \quad (2.4.8.2)$$

$$(0 \ 1)\mathbf{x} = 2. \quad (2.4.8.3)$$

Find the coordinates of the centres of the inscribed circle and of the escribed circle opposite to the vertex  $A$ .

9. Prove that the point  $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$  lies outside the triangle whose sides are the lines

$$(3 \ 4)\mathbf{x} = 24 \quad (2.4.9.1)$$

$$(5 \ -3)\mathbf{x} = 15 \quad (2.4.9.2)$$

$$(0 \ 1)\mathbf{x} = 0 \quad (2.4.9.3)$$

10. Find the equation of the line joining the origin to the point of intersection of the lines

$$(1 \ 7)\mathbf{x} - 11 = 0 \quad (2.4.10.1)$$

$$(-2 \ 1)\mathbf{x} = 3 \quad (2.4.10.2)$$

11. Find the equation of a line perpendicular to the line

$$(3 \ 5)\mathbf{x} + 11 = 0 \quad (2.4.11.1)$$

and passing through the intersection of the lines

$$(5 \ -6)\mathbf{x} = 1 \quad (2.4.11.2)$$

$$(3 \ 2)\mathbf{x} + 5 = 0 \quad (2.4.11.3)$$

12. Find the equation of a line through the intersection of the lines

$$(2 \ 5)\mathbf{x} = 1 \quad (2.4.12.1)$$

$$(-4 \ 1)\mathbf{x} = 9 \quad (2.4.12.2)$$

parallel to the line

$$(1 \ -1)\mathbf{x} = 1 \quad (2.4.12.3)$$

13. The vertices of a triangle are at the points

$$A, B, C \quad (2.4.13)$$

Find the equations of the medians and prove that they meet in a point. What are the coordinates of their point of intersection?

14. For what multiples  $k, l, m$  is the equation

$$k\{(2 \ 3)\mathbf{x} - 13\} + l\{(5 \ -y)\mathbf{x} - 7\} + m\{(1 \ -4)\mathbf{x} + 10\} = 0 \quad (2.4.14)$$

an identity? In what point do the lines given by equating the three terms to zero concur?

15. Find the equations of the diagonals of the parallelogram

$$(2 \ -1)\mathbf{x} + 7 = 0 \quad (2.4.15)$$

$$(2 \ -1)\mathbf{x} - 5 = 0, \quad (2.4.15)$$

$$(3 \ 2)\mathbf{x} - 5 = 0 \quad (2.4.15)$$

$$(3 \ 2)\mathbf{x} + 4 = 0 \quad (2.4.15)$$

16. The vertices of a triangle are at the points

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (2.4.16)$$

Find the equations of the perpendiculars to the sides through their middle points.

17. Work the same problem when the vertices of the triangle are at the points

$$A, B, C \quad (2.4.17.1)$$

and show that the perpendiculars meet in a point.

18. The line

$$(2 \ -8)\mathbf{x} - 4 = 0 \quad (2.4.18.1)$$

is the perpendicular bisector of the line  $AB$  and  $A$  is the point  $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ . What are the coordinates of  $B$ ?

## 2.5 Angle Between Lines

1. What lines are represented by the following equations:

- a)  $\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = 0$       d)  $\mathbf{x}^T \begin{pmatrix} -1 & -\tan \theta \\ \tan \theta & 1 \end{pmatrix} \mathbf{x} = 0$       7. Prove that the line joining the points  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix}$  makes with the axes a triangle of area  $\frac{49}{60}$ .
- b)  $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0$       e)  $x^3 + 3x^2y - 3xy^2 - y^3 = 0$       8.  $ABCD$  is a parallelogram and the coordinates of **A**, **B** and **C** are  $\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  Find the coordinates of **D**.
- c)  $\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \mathbf{x} = 0$

2. Find the angles between the pairs of straight lines represented by the following equations:

- a)  $\mathbf{x}^T \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0$       d)  $\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \mathbf{x} = 0$
- b)  $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0$       e)  $\mathbf{x}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \mathbf{x} = 0$
- c)  $\mathbf{x}^T \begin{pmatrix} 1 & -\frac{5}{2} \\ -\frac{5}{2} & 4 \end{pmatrix} \mathbf{x} = 0$

3. Prove that the equations

$$\mathbf{x}^T \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.5.3.1)$$

$$(1 \ 1) \mathbf{x} = 3 \quad (2.5.3.2)$$

are the sides of an equilateral triangle.

## 2.6 Miscellaneous

1. Find the locus of a point which is equidistant from the points  $\begin{pmatrix} 6 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 7 \end{pmatrix}$ .
2. Find the point on the line

$$(2 \ 5) \mathbf{x} + 7 = 0 \quad (2.6.2)$$

which is equidistant from the points  $\begin{pmatrix} 2 \\ -3 \end{pmatrix},$

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix}.$$

3. Find the coordinates of the circumcentre of the triangle whose corners are at the points  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ .

4. Find the equations of the lines through  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$  which are respectively parallel and perpendicular to the line joining the points  $\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ -6 \end{pmatrix}$ .

5. Find the locus of a point at which the join of the points  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$  subtends a right angle.

6. Find the orthocentre of a triangle whose corners are at the points  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ .

9. Find the area of the triangle formed by the lines

$$(3 \ -2) \mathbf{x} = 5 \quad (2.6.9.1)$$

$$(3 \ 4) \mathbf{x} = 7 \quad (2.6.9.2)$$

$$(0 \ 1) \mathbf{x} + 2 = 0 \quad (2.6.9.3)$$

10. Find the centre of the inscribed circle of the triangle whose sides are

$$(3 \ -4) \mathbf{x} = 0 \quad (2.6.10.1)$$

$$(12 \ -5) \mathbf{x} = 0, \quad (2.6.10.2)$$

$$(4 \ 3) \mathbf{x} = 8 \quad (2.6.10.3)$$

11. The ends of a diagonal of a square are on the coordinate axes at the points  $\begin{pmatrix} 2a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix}$ . Find the equations of the sides.

12. The sides of a triangle  $ABC$  are

$$AB = 3, BC = 5, CA = 4 \quad (2.6.12)$$

and **A**, **B** are on the axes  $OX, OY$  respectively, while  $AC$  makes an angle  $\theta$  with  $OX$ . Prove that the locus of **C**, as  $\theta$  varies, is given by the equation

$$\mathbf{x}^T \begin{pmatrix} 16 & -12 \\ -12 & 25 \end{pmatrix} \mathbf{x} = 256 \quad (2.6.12)$$

13. Prove that the locus of a point at which the join of the points  $\begin{pmatrix} a \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} -a \\ 0 \end{pmatrix}$  subtends an angle of  $45^\circ$  is

$$\mathbf{x}^T \mathbf{x} - 2a(0 \ 1) \mathbf{x} = a^2 \quad (2.6.13)$$

14. Prove that the line

$$\mathbf{n}^T \mathbf{x} + c = 0 \quad (2.6.14.1)$$

divides the line joining the points  $\mathbf{x}_1, \mathbf{x}_2$  in the ratio

$$-\frac{\mathbf{n}^T \mathbf{x}_1 + c}{\mathbf{n}^T \mathbf{x}_2 + c} \quad (2.6.14.2)$$

15. Find the equation of the line joining the point  $\mathbf{x}_1$ , to the point of intersection of the lines

$$\mathbf{n}^T \mathbf{x} + c = 0 \quad (2.6.15.1)$$

$$\mathbf{n}_1^T \mathbf{x} + c_1 = 0 \quad (2.6.15.2)$$

16. Find the equations of the diagonals of the parallelogram whose sides are

$$\mathbf{n}^T \mathbf{x} + c = 0 \quad (2.6.16.1)$$

$$\mathbf{n}^T \mathbf{x} + d = 0 \quad (2.6.16.2)$$

$$\mathbf{n}_1^T \mathbf{x} + c_1 = 0 \quad (2.6.16.3)$$

$$\mathbf{n}_1^T \mathbf{x} + d_1 = 0 \quad (2.6.16.4)$$

17. Prove that for all values of  $k$  the line

$$(2 + k \quad 1 - 2k) \mathbf{x} + 5 = 0 \quad (2.6.17)$$

passes through a fixed point, and find its coordinates.

18. Find the angle between the lines

$$\mathbf{x}^T \begin{pmatrix} 1 & -\sec \theta \\ -\sec \theta & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.6.18)$$

19. Prove that the pairs of straight lines represented by

$$\mathbf{x}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = 0 \quad (2.6.19.1)$$

$$\mathbf{x}^T \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} \mathbf{x} = 0 \quad (2.6.19.2)$$

are such that the angles between one pair are equal to the angles between the other pair.

20. Find the angles between the lines

$$x^3 - 3x^2y - 3xy^2 + y^3 = 0 \quad (2.6.20)$$

21. Find the area of the triangle whose sides are given by

$$\mathbf{x}^T \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \mathbf{x} = 0 \quad (2.6.21.1)$$

$$(3 \quad 4) \mathbf{x} = 7 \quad (2.6.21.2)$$

22. Show that the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \mathbf{x} + (-11 \quad 31) \mathbf{x} - 10 = 0 \quad (2.6.22)$$

represents two straight lines, and find the equations of the bisectors of the angles between them.

### Solution:

construction Any quadratic equation in terms of  $x, y$  of the form  $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$ , can be written as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.6.22)$$

$$\text{where, } \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.6.22)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (2.6.22)$$

The equation (2.6.22) represents two intersecting straight lines when

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.6.22)$$

$$|\mathbf{V}| < 0 \quad (2.6.22)$$

explanation From equation (2.6.22) we get

$$\mathbf{V} = \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \quad (2.6.22)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ \frac{31}{2} \end{pmatrix} \quad (2.6.22)$$

$$f = -10 \quad (2.6.22)$$

calculating the equation (2.6.22), we get

$$\begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ \frac{-11}{2} & \frac{31}{2} & -10 \end{pmatrix} \xrightarrow{R_3=R_3+R_2+R_1} \begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.6.22)$$

Therefore the determinant 0. And also the determinant of  $\mathbf{V}$  is

$$|\mathbf{V}| = \begin{vmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{vmatrix} \quad (2.6.22)$$

$$= -90.25 \quad (2.6.22)$$

$$< 0 \quad (2.6.22)$$

Therefore the given equation represents the equation of two straight lines which intersect. point of intersection The point of intersection of the straight lines is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (2.6.22)$$

The inverse of  $\mathbf{V}$  can be found by using rref of



augmented matrix of the matrices  $\mathbf{V}$  and  $\mathbf{I}$

$$\begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -15 & 0 & 1 \end{pmatrix} \xrightarrow{R_2=12R_2+R_1} \begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{361}{2} & 1 & 12 \end{pmatrix} \quad (2.6.22)$$

$$\begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \xrightarrow{R_1=R_1+\frac{R_2}{2}} \begin{pmatrix} 6 & 0 & \frac{360}{361} & -\frac{12}{361} \\ 0 & 1 & -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \quad (2.6.22)$$

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{60}{361} & -\frac{2}{361} \\ -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \quad (2.6.22)$$

$$c = -\begin{pmatrix} \frac{60}{361} & -\frac{2}{361} \\ -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \begin{pmatrix} -\frac{11}{2} \\ \frac{31}{2} \end{pmatrix} \quad (2.6.22)$$

$$c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.6.22)$$

Therefore the lines intersect at the point  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .  
eigenvectors The characteristic equation of the matrix  $\mathbf{V}$  is

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.6.22)$$

$$\begin{vmatrix} 6 - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -15 - \lambda \end{vmatrix} = 0 \quad (2.6.22)$$

$$\lambda^2 + 9\lambda - 90.25 = 0 \quad (2.6.22)$$

So the eigenvalues will be

$$\lambda_1 = \frac{-1}{2} (9 + \sqrt{442}) \quad (2.6.22)$$

$$\lambda_2 = \frac{-1}{2} (9 - \sqrt{442}) \quad (2.6.22)$$

The eigen vectors will be in the nullspace of  $\mathbf{V} - \lambda_1 \mathbf{I}$  and  $\mathbf{V} - \lambda_2 \mathbf{I}$ . The eigen vector corresponding to eigen value  $\lambda_1$  will be

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2} (9 + \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2} (9 + \sqrt{442}) \end{pmatrix} \quad (2.6.22)$$

$$= \begin{pmatrix} \frac{1}{2} (21 + \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} (-21 + \sqrt{442}) \end{pmatrix} \quad (2.6.22)$$

$$\xrightarrow{R_2=(21+\sqrt{442})R_2+R_1} \begin{pmatrix} \frac{1}{2} (21 + \sqrt{442}) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad (2.6.22)$$

The above reduced matrix has one free vari-

able. Let it be 1, then the eigen vector will be

$$p_1 = \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

normalizing  $p_1$ , we get

$$p_1 = \frac{1}{\sqrt{884 + 42\sqrt{442}}} \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

the eigen vector corresponding to eigen value  $\lambda_2$  will be

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2} (9 - \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2} (9 - \sqrt{442}) \end{pmatrix} \quad (2.6.22)$$

$$= \begin{pmatrix} \frac{1}{2} (21 - \sqrt{442}) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} (-21 - \sqrt{442}) \end{pmatrix} \quad (2.6.22)$$

$$\xrightarrow{R_2=(21-\sqrt{442})R_2+R_1} \begin{pmatrix} \frac{1}{2} (21 - \sqrt{442}) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad (2.6.22)$$

The above reduced matrix has one free variable. Let it be 1, then the eigen vector will be

$$p_2 = \begin{pmatrix} 1 \\ 21 - \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

normalizing  $p_2$ , we get

$$p_2 = \frac{1}{\sqrt{884 - 42\sqrt{442}}} \begin{pmatrix} 1 \\ 21 - \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

So the transformation matrix will be

$$\mathbf{P} = (p_1 \ p_2) = \begin{pmatrix} \frac{1}{\sqrt{884+42\sqrt{442}}} & \frac{1}{\sqrt{884-42\sqrt{442}}} \\ \frac{21+\sqrt{442}}{\sqrt{884+42\sqrt{442}}} & \frac{21-\sqrt{442}}{\sqrt{884-42\sqrt{442}}} \end{pmatrix} \quad (2.6.22)$$

affine transformation Doing the affine transformation on given quadratic equation, we get pair to intersecting straight lines passing through origin.

Let the affine transformation be  $\mathbf{x} = \mathbf{P}\mathbf{y} + c$ . The transformation will be

$$(\mathbf{P}\mathbf{y} + c)^T \mathbf{V} (\mathbf{P}\mathbf{y} + c) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + c) + f = 0 \quad (2.6.22)$$

$$(2.6.22)$$

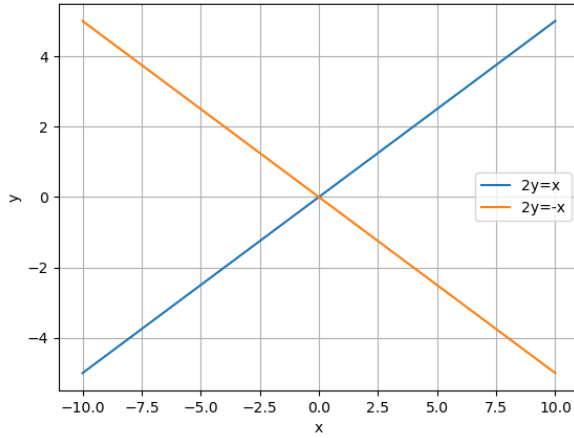


Fig. 2.6.22: straight lines after affine transformation passing through origin

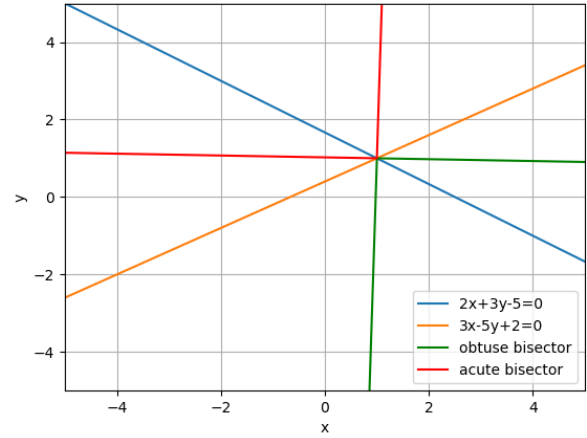


Fig. 2.6.22: Par of straight lines and their angular bisectors

$$\mathbf{y}^T (\mathbf{P}^T \mathbf{V} \mathbf{P}) \mathbf{y} + 2 (\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2 \mathbf{u}^T \mathbf{c} + f = 0 \quad (2.6.22)$$

if the point  $\mathbf{c}$  is taken as the point of intersection of the two lines.

$$\mathbf{c}^T \mathbf{V} \mathbf{c} + 2 \mathbf{u}^T \mathbf{c} + f = 0 \quad (2.6.22)$$

$$\mathbf{c}^T \mathbf{V} + \mathbf{u}^T = 0 \quad (2.6.22)$$

So the affine transformation of the given lines will be

$$\mathbf{y}^T (\mathbf{P}^T \mathbf{V} \mathbf{P}) \mathbf{y} = 0 \quad (2.6.22)$$

$$\mathbf{y}^T \begin{pmatrix} -1.5 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{y} = 0 \quad (2.6.22)$$

$$(x - 2y)(x + 2y) = 0 \quad (2.6.22)$$

Since the two lines are symmetric with respect to both  $X$ -axis and  $Y$ -axis, the axes themselves are the bisectors of the transformed pair of lines. So the bisectors will be  $x = 0$  and  $y = 0$ . Matrix notation will be of the form

$$\mathbf{y}^T \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \mathbf{y} = 0 \quad (2.6.22)$$

$$\mathbf{y}^T \mathbf{K} \mathbf{y} = 0 \quad (2.6.22)$$

$$\mathbf{K} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \quad (2.6.22)$$

bisectors Taking the inverse of the affine transformation of the equation  $xy = 0$ , will give the

angle bisectors.

$$(\mathbf{P}^{-1} \mathbf{x} - \mathbf{P}^{-1} \mathbf{c})^T \mathbf{K} (\mathbf{P}^{-1} \mathbf{x} - \mathbf{P}^{-1} \mathbf{c}) = 0 \quad (2.6.22)$$

$$\mathbf{x}^T \mathbf{P} \mathbf{K} \mathbf{P}^T \mathbf{x} - 2 \mathbf{c}^T \mathbf{P} \mathbf{K} \mathbf{P}^T \mathbf{x} + \mathbf{c}^T \mathbf{P} \mathbf{K} \mathbf{P}^T \mathbf{c} = 0 \quad (2.6.22)$$

Substituting the values we get

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{2\sqrt{442}} & \frac{21}{2\sqrt{442}} \\ \frac{21}{2\sqrt{442}} & -\frac{1}{2\sqrt{442}} \end{pmatrix} \mathbf{x} - \left( \frac{22}{\sqrt{442}} \quad \frac{20}{\sqrt{442}} \right) \mathbf{x} + \frac{21}{\sqrt{442}} = 0 \quad (2.6.22)$$

$$\frac{x^2}{2\sqrt{442}} + \frac{21}{\sqrt{442}}xy - \frac{y^2}{2\sqrt{442}} - \frac{22x}{\sqrt{442}} - \frac{20y}{\sqrt{442}} + \frac{21}{\sqrt{442}} = 0 \quad (2.6.22)$$

$$x^2 + 42xy - y^2 - 44x - 40y + 42 = 0 \quad (2.6.22)$$

Therefore the equation of bisectors of the given line in quadratic form is

$$\mathbf{x}^T \begin{pmatrix} 1 & 21 \\ 21 & -1 \end{pmatrix} \mathbf{x} - (44 \quad 40) \mathbf{x} + 42 = 0 \quad (2.6.22)$$

23. For what value of  $k$  does the equation

$$\mathbf{x}^T \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & k \end{pmatrix} \mathbf{x} + (13 \ -1) \mathbf{x} + 3 = 0 \quad (2.6.23)$$

represent two straight lines? What is the angle between them?

24. For what values of  $k$  does the equation

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{k}{2} \\ \frac{k}{2} & -3 \end{pmatrix} \mathbf{x} + (4 \ 5) \mathbf{x} - 2 = 0 \quad (2.6.24)$$

represent two straight lines?

### 3 CURVES

#### 3.1 Tangent

1. Find the equations of the tangents to the following curves at the points specified:

- |  |   |
|--|---|
| a) $y = x(x^2 - 1), x = 2$                     | f) $y = x^3 - x + 1, x = 3$                       |
| b) $y = x^2 + \frac{1}{x^2}, x = 1$            | g) $y = (x - a)^3, x = 2a$                        |
| c) $y = x^3 + 2x, x = 0$                       | h) $y = ax^2 + 2bx + c, (x_1, y_1)$               |
| d) $y = \left(x + \frac{1}{x}\right)^3, x = 2$ | i) $y = \frac{x^3}{a^2} + \frac{a^3}{x^3}, x = a$ |
| e) $y = (x^2 - 1)^2, x = 1$                    | j) $y = \frac{x^2}{a} + \frac{a^2}{x}, x = a$     |

2. Find the tangents to the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + (1 \ -1) \mathbf{x} = 0 \quad (3.1.2.1)$$

at the points where it is cut by the line

$$(1 \ -1) \mathbf{x} + 4 = 0 \quad (3.1.2.2)$$

and find the point of intersection of the tangents.

3. Prove that the line

$$(3 \ -4) \mathbf{x} + 4 = 0 \quad (3.1.3.1)$$

touches the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + 1 = 0. \quad (3.1.3.2)$$

4. Find the points on the curve

$$3y = x^3 + 3x \quad (3.1.4.1)$$

at which the tangent is parallel to the line

$$(5 \ -1) \mathbf{x} = 0 \quad (3.1.4.2)$$

5. Find at what points on the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + (0 \ -1) \mathbf{x} + 9 = 0 \quad (3.1.5)$$

the tangents pass through the origin.

6. Show that there are three points on the curve

$$3y = 3x^4 + 8x^3 - 6x^2 \quad (3.1.6.1)$$

at which the tangents are parallel to the line

$$(8 \ -1) \mathbf{x} = 0 \quad (3.1.6.2)$$

7. Show that the line

$$(0 \ 4) \mathbf{x} = 17 \quad (3.1.7.1)$$

meets the curve

$$y = x^2 + \frac{1}{x^2} \quad (3.1.7.2)$$

in four points and that two of the points of intersection of the tangents at these four points are on the line

$$(0 \ 4) \mathbf{x} + 1 = 0, \quad (3.1.7.3)$$

and two are on the line

$$(1 \ 0) \mathbf{x} = 0. \quad (3.1.7.4)$$

#### 3.2 More on Tangents

1. Find the equation of the tangents to the following curves at the points stated:

a)  $\mathbf{x}^T \mathbf{x} = 25, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

b)  $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 2, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

c)  $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(0 \ 1) \mathbf{x} = 0, \begin{pmatrix} a \\ 2a \end{pmatrix}$

d)  $\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2, \begin{pmatrix} a \cos \theta \\ b \sin \theta \end{pmatrix}$

e)  $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = a^2, \begin{pmatrix} a \sec \theta \\ b \tan \theta \end{pmatrix}$

f)  $\mathbf{x}^T \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \mathbf{x} = 4, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

g)  $\mathbf{x}^T \mathbf{x} + (2 \ 4) \mathbf{x} - 20 = 0, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

h)  $x^3 + y^3 - 3xy^2 + a^3 = 0, \begin{pmatrix} a \\ a \end{pmatrix}$

2. Find the equation of the tangent at the point  $\mathbf{p}$  on each of the following curves:

a)  $\mathbf{x}^T \mathbf{x} = a^2$

b)  $\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2$

c)  $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(1 \ 0) \mathbf{x} = 0$

- d)  $\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} - c^2 = 0$   
 e)  $y^2(x^2 - a^2) = a^2(x^2 + a^2)$   
 f)  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

### 3.3 Normal

1. Find the equations of the normals to the following curves at the given points

- a)  $\mathbf{x}^T \mathbf{x} - (2 \ 4) \mathbf{x} = 3, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$   
 b)  $\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 13, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$   
 c)  $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(1 \ 0) \mathbf{x} = 0, \begin{pmatrix} a \\ 2a \end{pmatrix}$   
 d)  $\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2, \begin{pmatrix} a \cos \theta \\ b \sin \theta \end{pmatrix}$   
 e)  $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x} = 4a^2, \begin{pmatrix} 2a \sec \theta \\ a \tan \theta \end{pmatrix}$   
 f)  $x^3 - y^3 - 3xy^2 + a^2 = 0, \begin{pmatrix} a \\ -a \end{pmatrix}$

2. Find the equation of the normal at the point  $\mathbf{p}$  on each of the following curves:

- a)  $\mathbf{x}^T \mathbf{x} = a^2$   
 b)  $\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2$   
 c)  $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(1 \ 0) \mathbf{x} = 0$   
 d)  $\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c^2$

3. Prove that for the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(1 \ 0) \mathbf{x} = 0 \quad (3.3.3)$$

the subnormal is of constant length.

4. Prove that the portion of any tangent to the curve  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$  intercepted by the axes is of length  $a$ .  
 5. Prove that for the curve  $ay^2 = x^3$  the subnormal varies as the square of the subtangent.  
 6. Prove that for the curve  $y = ae^{\frac{x}{b}}$  the subtangent is of length  $b$ .  
 7. Prove that the area of the triangle formed by the axes and any tangent to the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c^2 \quad (3.3.7)$$

is  $2c^2$ .

8. Prove that for the curve  $x^m y^n = e^{m+n}$  the portion of a tangent intercepted by the axes is divided at the point of contact in the ratio  $m : n$ .  
 9. Prove that, if  $N$  is the foot of the ordinate and  $NT$  is the subtangent at a point on the curve

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2, \begin{pmatrix} a \cos \theta \\ b \sin \theta \end{pmatrix} \quad (3.3.9)$$

then  $OT.ON = a^2$ .

10. Prove that the perpendicular from the foot of the ordinate to the tangent to a curve is of length  $\frac{y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$ . Show that for the curve  $y = c \cosh \frac{x}{c}$ , this perpendicular is of length  $c$ .  
 11. Find the equation of the tangent to the curve

$$2x^3 + 2y^3 - 9axy = 0 \quad (3.3.11.1)$$

at the point  $\begin{pmatrix} 2a \\ a \end{pmatrix}$ ; and show that the tangent meets the curve again where

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.3.11.2)$$

### 3.4 Affine Transformation: Exercises

1. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} - (4 \ 6) \mathbf{x} - 6 = 0 \quad (3.4.1)$$

become when the origin is moved to the point  $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ ?

**Solution:** Comparing the given equation with the form:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.4.1)$$

We get:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + 2(-2 \ -3) \mathbf{x} - 6 = 0 \quad (3.4.1)$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.4.1)$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (3.4.1)$$

$$f = -6 \quad (3.4.1)$$

Here,  $|\mathbf{V}| = -1$ . Since  $|\mathbf{V}| < 0$  the given equation represents a hyperbola with center:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad (3.4.1)$$

The characteristic equation of  $\mathbf{V}$  is:

$$|V - \lambda \mathbf{I}| = 0 \quad (3.4.1)$$

$$\begin{vmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = 0 \quad (3.4.1)$$

$$\implies \lambda^2 - 1 = 0 \quad (3.4.1)$$

$$\lambda_1 = 1, \lambda_2 = -1 \quad (3.4.1)$$

Finding the eigen vector matrix  $\mathbf{P}$  such that  $\mathbf{P}^T = \mathbf{P}^{-1}$ . For  $\lambda_1 = 1$

$$(\mathbf{V} - \lambda_1 \mathbf{I})\mathbf{p}_1 = 0 \quad (3.4.1)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{p}_1 = 0 \quad (3.4.1)$$

$$\implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.1)$$

$$(3.4.1)$$

For  $\lambda_2 = -1$

$$(\mathbf{V} - \lambda_2 \mathbf{I})\mathbf{p}_2 = 0 \quad (3.4.1)$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{p}_2 = 0 \quad (3.4.1)$$

$$\implies \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.4.1)$$

[Choosing Orthonormal eigen vectors]

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.4.1)$$

By affine transformation  $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$ , Equation (3.4.1) can be written in the form:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (3.4.1)$$

where,

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (3.4.1)$$

Thus, we can write:

$$\lambda_1 y_1^2 - (-\lambda_2) y_1^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (3.4.1)$$

The equation (3.4.1) represents a modified hyperbola, The equation of the asymptotes for (3.4.1) is:

$$(\sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|}) \mathbf{y} = 0 \quad (3.4.1)$$

Putting the values of  $\lambda_1$  and  $\lambda_2$  in equation (3.4.1), we get the two asymptotes for (3.4.1):

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{y} = 0 \quad (3.4.1)$$

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{y} = 0 \quad (3.4.1)$$

These are the asymptotes of our modified hyperbola. The asymptotes of our original hyperbola in equation (3.4.1) can be obtained using:

$$(\sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|}) \mathbf{P}^T (\mathbf{x} - \mathbf{c}) = 0 \quad (3.4.1)$$

Putting the values of  $\lambda_1$ ,  $\lambda_2$  and  $\mathbf{P}$  in equation (3.4.1), we get the equations of the asymptotes of the original hyperbola with center at  $\mathbf{c}$ :

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left( \mathbf{x} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right) = 0 \quad (3.4.1)$$

$$\implies \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 5} \quad (3.4.1)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left( \mathbf{x} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right) = 0 \quad (3.4.1)$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -1} \quad (3.4.1)$$

2. To what point must origin be shifted so that

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} + (10 \ -19) \mathbf{x} + 23 = 0 \quad (3.4.2)$$

is transformed to

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} = 1 \quad (3.4.2)$$

**Solution:** Given,

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} + (10 \ -19) \mathbf{x} + 23 = 0 \quad (3.4.2)$$

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.4.2)$$

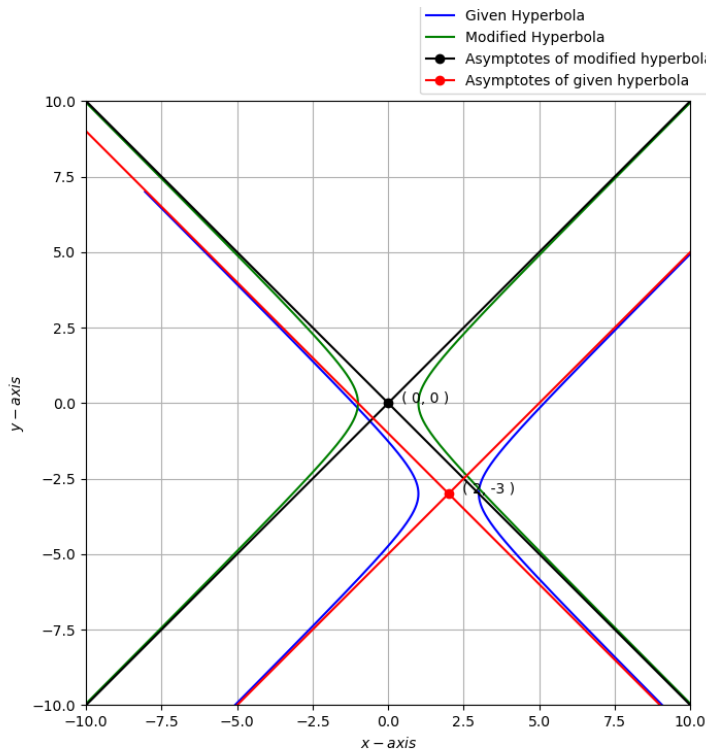


Fig. 3.4.1: Plot of the Asymptotes.

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} \quad (3.4.2)$$

$$f = 23 \quad (3.4.2)$$

Origin which is moved to the point is given by

**c** The above equation (3.4.2) can be modified as

$$(\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{x} + \mathbf{c}) + 23 = 0 \quad (3.4.2)$$

From equation (3.4.2) consider,

$$\Rightarrow (\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) \quad (3.4.2)$$

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{x}^T \mathbf{V} \mathbf{c} + \mathbf{c}^T \mathbf{V} \mathbf{c} \quad (3.4.2)$$

we know that

$$\mathbf{x}^T \mathbf{V} \mathbf{c} = \mathbf{c}^T \mathbf{V} \mathbf{x} \quad (3.4.2)$$

Substituting equation (3.4.2) in equation (3.4.2)

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} \quad (3.4.2)$$

Equation (3.4.2) is modified as

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{x} + 23 = 0 \quad (3.4.2)$$

Equating (3.4.2) and (3.4.2):

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{x} + 23 = \mathbf{x}^T \mathbf{V} \mathbf{x} - 1 \quad (3.4.2)$$

From above equation (3.4.2) we have,

$$2\mathbf{c}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} = 0 \quad (3.4.2)$$

and

$$2\mathbf{u}^T \mathbf{x} + \mathbf{v}^T \mathbf{c} = -22 \quad (3.4.2)$$

From (3.4.2)

$$\mathbf{c}^T \mathbf{V} \mathbf{x} = -\mathbf{u}^T \mathbf{x} \quad (3.4.2)$$

$$\mathbf{c}^T \mathbf{V} = -\mathbf{u}^T \quad (3.4.2)$$

$$\mathbf{c}^T = -\mathbf{V}^{-1} \mathbf{u}^T \quad (3.4.2)$$

Adjoining  $\mathbf{V}$  with identity matrix to compute inverse:

$$\begin{pmatrix} 2 & \frac{-3}{2} & 1 & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{3}{2} R_1} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & \frac{23}{8} & \frac{3}{4} & 1 \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & \frac{23}{8} & \frac{3}{4} & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{8}{23} R_2} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + \frac{3}{4} R_2} \begin{pmatrix} 1 & 0 & \frac{16}{23} & \frac{6}{23} \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{16}{23} & \frac{6}{23} \\ \frac{6}{23} & \frac{8}{23} \end{pmatrix} \quad (3.4.2)$$

From (3.4.2)

$$\mathbf{c}^T = \begin{pmatrix} \frac{-16}{23} & \frac{-6}{23} \end{pmatrix} \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} \quad (3.4.2)$$

From above we have :

$$\mathbf{c}^T = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (3.4.2)$$

Hence,

$$\mathbf{c} = \begin{pmatrix} -1 & 2 \end{pmatrix} \quad (3.4.2)$$

From (3.4.2) and (3.4.2) when the origin is moved to the point  $\mathbf{c} = \begin{pmatrix} -1 & 2 \end{pmatrix}$   $\mathbf{V}$  doesn't change

$$\det(\mathbf{V}) = 5.75 \quad (3.4.2)$$

Since  $\det(\mathbf{V}) > 0$  the given equation represents the ellipse. The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|V - \lambda \mathbf{I}| = 0 \quad (3.4.2)$$

$$\begin{vmatrix} 2 - \lambda & \frac{-3}{2} \\ \frac{-3}{2} & 4 - \lambda \end{vmatrix} = 0 \quad (3.4.2)$$

$$\implies 4\lambda^2 - 24\lambda + 23 = 0 \quad (3.4.2)$$

The eigenvalues are the roots of equation 3.4.2 is given by

$$\lambda_1 = \frac{6 + \sqrt{13}}{2} \quad (3.4.2)$$

$$\lambda_2 = \frac{6 - \sqrt{13}}{2} \quad (3.4.2)$$

Hence from above:

$$\mathbf{D} = \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \quad (3.4.2)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (3.4.2)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (3.4.2)$$

For  $\lambda_1 = \frac{6+\sqrt{13}}{2}$ ,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 \div \frac{-\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{p}_1 = \begin{pmatrix} \frac{-\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \quad (3.4.2)$$

For  $\lambda_2 = \frac{6-\sqrt{13}}{2}$ ,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 \div \frac{\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{p}_2 = \begin{pmatrix} \frac{\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \quad (3.4.2)$$

Again, for ellipse

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (3.4.2)$$

Where  $\mathbf{D}$  is a diagonal matrix, we get,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (3.4.2)$$

$$\implies \mathbf{P} = \begin{pmatrix} \frac{-\sqrt{13}+2}{2} & \frac{\sqrt{13}+2}{2} \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{D} = \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \quad (3.4.2)$$

Standard ellipse can be written in the form:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (3.4.2)$$

Simplifying we get:

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} = \begin{pmatrix} 5 & \frac{-19}{2} \end{pmatrix} \begin{pmatrix} \frac{16}{23} & \frac{6}{23} \\ \frac{6}{23} & \frac{8}{23} \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ \frac{-19}{2} \end{pmatrix} = 24 \quad (3.4.2)$$

substituting (3.4.2) in (3.4.2) we have :

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = 1 \quad (3.4.2)$$

To get  $\mathbf{y}$ ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (3.4.2)$$

$$\mathbf{y} = \begin{pmatrix} \frac{-\sqrt{13}+2}{2} & 1 \\ \frac{3}{2} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{-\sqrt{13}+2}{2} & 1 \\ \frac{3}{2} & 1 \end{pmatrix} \quad (3.4.2)$$

Substituting equation (3.4.2), in equation (3.4.2)

$$\mathbf{y}^T \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \mathbf{y} = 1 \quad (3.4.2)$$

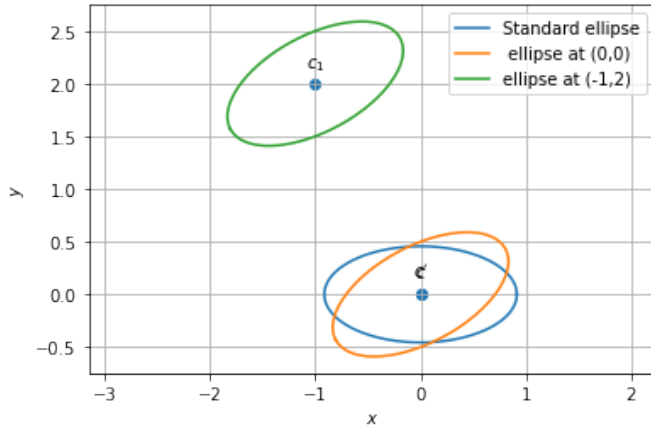


Fig. 3.4.2: Standard ellipse, Ellipse at (0,0) and ellipse at (-1,2)

3. Show that the equation

$$\mathbf{x}^T \mathbf{x} = a^2 \quad (3.4.3)$$

remains unaltered by any rotation of the axes.

4. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{x} = 2a^2 \quad (3.4.4)$$

become when the axes are turned through  $30^\circ$ ?

**Solution:** The general second degree equation is expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.4.4)$$

Comparing (3.4.4) and (3.4.4), we get

$$\mathbf{V} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad (3.4.4)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.4.4)$$

$$f = -2a^2 \quad (3.4.4)$$

Now we find

$$|\mathbf{V}| = \begin{vmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{vmatrix} \quad (3.4.4)$$

$$\Rightarrow |\mathbf{V}| = -4 \quad (3.4.4)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (3.4.4)$$

Therefore the given equation (3.4.4) represents hyperbola.

Now from affine transformations,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (3.4.4)$$

We have to rotate the axes by  $\theta = 30^\circ$ , Then using rotation matrix.

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (3.4.4)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} \quad (3.4.4)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.4.4)$$

From eigenvalue decomposition,

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D} \quad (3.4.4)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.4.4)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (3.4.4)$$

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (3.4.4)$$

$$\mathbf{P}^T = \mathbf{P}^{-1} \quad (3.4.4)$$

The equation (3.4.4) becomes as below due to affine transformation

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (3.4.4)$$

with

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (3.4.4)$$

Substitute (3.4.4),(3.4.4),(3.4.4),(3.4.4) in (3.4.4) and (3.4.4), we get

$$\mathbf{y}^T \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{y} = 2a^2 \quad (3.4.4)$$

with centre,

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.4.4)$$

Therefore the given equation (3.4.4) becomes (3.4.4) when the axes are turned through  $30^\circ$ .

The plot is shown in Fig 3.4.4 for  $a=2$ .



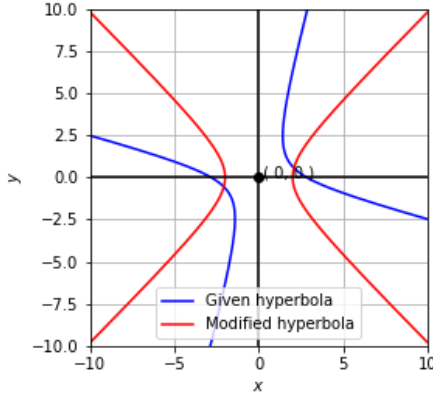


Fig. 3.4.4: For  $a = 2$ , Hyperbola rotated by  $30^\circ$

5. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

become when the axes are turned through  $45^\circ$ ?

**Solution:**

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow |\mathbf{V}| = 0, \mathbf{u} = \begin{pmatrix} -2\sqrt{2}a \\ -2\sqrt{2}a \end{pmatrix} \quad (3.4.5)$$

The characteristics equation of  $\mathbf{V}$

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{vmatrix} = 0 \quad (3.4.5)$$

$$\Rightarrow \lambda^2 - 2\lambda = 0 \quad (3.4.5)$$

The eigen values are

$$\lambda_1 = 0, \lambda_2 = 2 \quad (3.4.5)$$

The (3.4.5) is equation of parabola as  $\lambda_1 = 0$  and  $|\mathbf{V}| = 0$ . The eigen vector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (3.4.5)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (3.4.5)$$

for  $\lambda_1 = 0$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.4.5)$$

such that  $\|\mathbf{p}_1\| = 1$  similarly the eigen vector for  $\lambda_2 = 2$  can be find

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad (3.4.5)$$

The parabola parameters are given by

$$f = \frac{|\eta|}{|\lambda_2|}; \eta = 2\mathbf{p}_1^T \mathbf{u} \quad (3.4.5)$$

$$f = \frac{8a}{2} = 4a \quad (3.4.5)$$

$$\begin{pmatrix} 0 & -6\sqrt{2}a \\ 0 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4a \\ -2\sqrt{2}a \\ 0 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ where } \mathbf{c} \text{ is vertex} \quad (3.4.5)$$

The axes are turned around  $45^\circ$  then

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mathbf{x} \quad (3.4.5)$$

$$\mathbf{P} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \mathbf{x} \quad (3.4.5)$$

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} \quad (3.4.5)$$

when  $\mathbf{P}$  passes through the (3.4.5) we get

$$\mathbf{x}^T \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow |\mathbf{V}| = 0 \quad (3.4.5)$$

Therefore it is parabola..

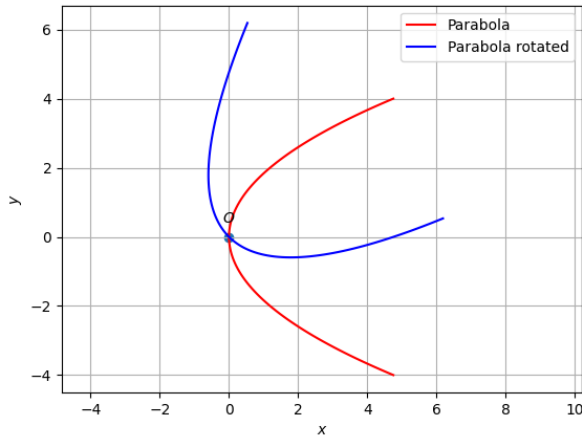


Fig. 3.4.5: parabola with  $a = 1$

6. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -4 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.6)$$

may become

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = 1 \quad (3.4.6)$$

and through what angle must the axes be turned in order to obtain

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1 \quad (3.4.6)$$

**Solution:** The general second order equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.4.6)$$

Comparing (3.4.6) with (3.4.6),

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \quad (3.4.6)$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (3.4.6)$$

$$f = 0 \quad (3.4.6)$$

Let the point to which the origin is moved be

$\mathbf{c}$

The above equation (3.4.6) can be modified as

$$(\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{x} + \mathbf{c}) = 0 \quad (3.4.6)$$

From equation (3.4.6) consider,

$$\Rightarrow (\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) \quad (3.4.6)$$

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{x}^T \mathbf{V} \mathbf{c} + \mathbf{c}^T \mathbf{V} \mathbf{c} \quad (3.4.6)$$

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} \quad (3.4.6)$$

Substituting (3.4.6) in equation (3.4.6)

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T (\mathbf{x} + \mathbf{c}) = 0 \quad (3.4.6)$$

Comparing equations (3.4.6) and (3.4.6), we can write as,

$$\Rightarrow 2\mathbf{c}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} = 0 \quad (3.4.6)$$

$$\Rightarrow \mathbf{c}^T \mathbf{V} = -\mathbf{u}^T \quad (3.4.6)$$

$$\Rightarrow \mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = -\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (3.4.6)$$

$$\Rightarrow \boxed{\mathbf{c} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}} \quad (3.4.6)$$

From (3.4.6), when the origin is moved to point  $\mathbf{c}$ , the equation (3.4.6) becomes (3.4.6).

From equations (3.4.6) and (3.4.6),  $\mathbf{V}$  doesn't change

$$\det(\mathbf{V}) = -6 \quad (3.4.6)$$

Since  $\det(\mathbf{V}) < 0$  the given equation represents the hyperbola

From equation (3.4.6), the equation is of the form,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + f = 0 \quad (3.4.6)$$

The matrix  $\mathbf{V}$  can be decomposed as,

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (3.4.6)$$

where  $\lambda_1$  and  $\lambda_2$  are Eigen values of  $\mathbf{V}$ , and  $\mathbf{P}$  contains the Eigen vectors corresponding to the Eigen values  $\lambda_1$  and  $\lambda_2$ . The affine transformation is given by,

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (3.4.6)$$

where,  $\mathbf{P}$  indicates the rotation of axes and  $\mathbf{c}$

indicates the shift of origin.

Eigen values of  $\mathbf{V}$  are,

$$|\lambda \mathbf{I} - \mathbf{v}| = 0 \quad (3.4.6)$$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda + 2 \end{vmatrix} = 0 \quad (3.4.6)$$

$$\Rightarrow \lambda^2 + \lambda - 6 = 0 \quad (3.4.6)$$

$$\Rightarrow \lambda_1 = -3, \lambda_2 = 2 \quad (3.4.6)$$

$$\mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \quad (3.4.6)$$

Eigen vector for  $\lambda_1 = -3$ ,

$$\lambda_1 \mathbf{I} - \mathbf{v} = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{R_1}{2}} \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix} \quad (3.4.6)$$

$$\Rightarrow \mathbf{P}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix} \quad (3.4.6)$$

Eigen vector for  $\lambda_2 = 2$ ,

$$\lambda_2 \mathbf{I} - \mathbf{v} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \quad (3.4.6)$$

$$\Rightarrow \mathbf{P}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.4.6)$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.4.6)$$

Therefore  $\mathbf{V}$  can be written as,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.4.6)$$

Equation (3.4.6) can be written as,

$$\mathbf{x}^T \left[ \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \right] \mathbf{x} = 1 \quad (3.4.6)$$

$$\left[ \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} \right]^T \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \left[ \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} \right] = 1 \quad (3.4.6)$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \quad (3.4.6)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{y} \quad (3.4.6)$$

$$\mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^{-1}\mathbf{P}\mathbf{y} \quad (3.4.6)$$

$$\Rightarrow \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \quad (3.4.6)$$

$$\text{But, } \mathbf{P}^{-1} = \mathbf{P}^T \quad (3.4.6)$$

$$\Rightarrow \mathbf{y} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} \quad (3.4.6)$$

Using (3.4.6) in (3.4.6), the equation can be rewritten as

$$\mathbf{y}^T \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = 1 \quad (3.4.6)$$

Equation (3.4.6) is same as (3.4.6) with  $p = -3$  and  $q = 2$ .

From equation (3.4.6), the orthogonal matrix represents the rotation matrix in form of,

$$\mathbf{P} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (3.4.6)$$

Comparing (3.4.6) and (3.4.6),

$$\cos \theta = \frac{1}{\sqrt{5}} \quad (3.4.6)$$

$$\Rightarrow \boxed{\theta = 63.43^\circ} \quad (3.4.6)$$

From equation (3.4.6), if the axes is turned by  $\theta$  then the equation obtained would be (3.4.6).

7. Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \quad (3.4.7)$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c \quad (3.4.7)$$

where  $c$  is a constant.

8. Show that, by changing the origin, the equation

$$2\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 13 = 0 \quad (3.4.8)$$

can be transformed to

$$8\mathbf{x}^T \mathbf{x} = 89 \quad (3.4.8)$$

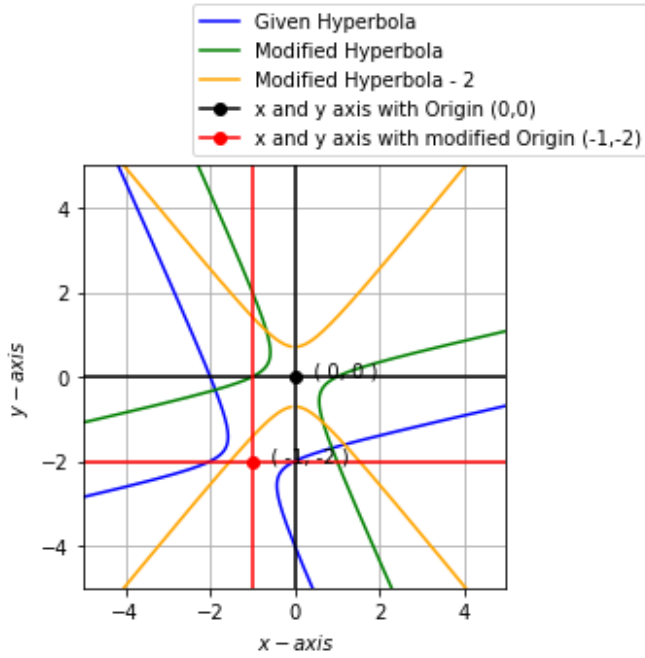


Fig. 3.4.6: Hyperbola plot when origin is shifted and rotated

**Solution:** Eq (3.4.8) can be written as

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} \frac{7}{2} & \frac{5}{2} \end{pmatrix} \mathbf{x} - \frac{13}{2} = 0 \quad (3.4.8)$$

$$\Rightarrow \mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} \frac{7}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x} - \frac{13}{2} = 0 \quad (3.4.8)$$

The above eq (3.4.8) can be compared with the circle equation gives as

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.4.8)$$

then

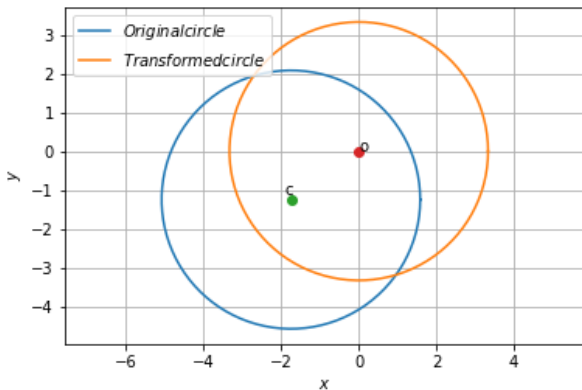


Fig. 3.4.8: Figure depicting transformation of circle

$$\mathbf{u} = \begin{pmatrix} \frac{7}{4} \\ \frac{5}{4} \end{pmatrix} \quad (3.4.8)$$

$$\Rightarrow \text{centre, } \mathbf{c} = \begin{pmatrix} -\frac{7}{4} \\ -\frac{5}{4} \end{pmatrix} \quad (3.4.8)$$

$$\|\mathbf{u}\|^2 - r^2 = f \quad (3.4.8)$$

$$\Rightarrow r^2 = \|\mathbf{u}\|^2 - f \quad (3.4.8)$$

$$\Rightarrow r^2 = \left(\frac{7}{4}\right)^2 + \left(\frac{5}{4}\right)^2 + \frac{13}{2} \quad (3.4.8)$$

$$\Rightarrow \text{radius, } r = \sqrt{\frac{89}{8}} \quad (3.4.8)$$

The eq (3.4.8) can be written by changing the origin as

$$(\mathbf{x} + \mathbf{c})^T (\mathbf{x} + \mathbf{c}) = \frac{89}{8} \quad (3.4.8)$$

$$\Rightarrow \mathbf{x}^T \mathbf{x} + \mathbf{x}^T \mathbf{c} + \mathbf{c}^T \mathbf{x} + \mathbf{c}^T \mathbf{c} = \frac{89}{8} \quad (3.4.8)$$

We know that

$$\mathbf{x}^T \mathbf{c} = \mathbf{c}^T \mathbf{x} \quad (3.4.8)$$

by substituting (3.4.8) in (3.4.8)

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{c}^T \mathbf{x} + \mathbf{c}^T \mathbf{c} = \frac{89}{8} \quad (3.4.8)$$

substituting the origin of (3.4.8) in above eq (3.4.8)

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} \frac{7}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\frac{7}{4} & -\frac{5}{4} \end{pmatrix} \begin{pmatrix} -\frac{7}{4} \\ -\frac{5}{4} \end{pmatrix} = \frac{89}{8} \quad (3.4.8)$$

$$\Rightarrow \mathbf{x}^T \mathbf{x} + \begin{pmatrix} \frac{7}{2} & \frac{5}{2} \end{pmatrix} \mathbf{x} + \frac{74}{16} - \frac{89}{8} = 0 \quad (3.4.8)$$

$$\Rightarrow \mathbf{x}^T \mathbf{x} + \begin{pmatrix} \frac{7}{2} & \frac{5}{2} \end{pmatrix} \mathbf{x} - \frac{13}{2} = 0 \quad (3.4.8)$$

$$\Rightarrow 2\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 13 = 0 \quad (3.4.8)$$

$\therefore$  It is proved that by changing the origin in (3.4.8) we obtained (3.4.8).

9. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 3 & \frac{7}{2} \\ \frac{7}{2} & -3 \end{pmatrix} \mathbf{x} = 1 \quad (3.4.9)$$

can be reduced to

$$\sqrt{85} \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 2 \quad (3.4.9)$$

10. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \quad (3.4.10)$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (3.4.10)$$

**Solution:**

The given equation is of the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + f = 0 \quad (3.4.10)$$

The matrix  $\mathbf{V}$  can be decomposed as

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (3.4.10)$$

$$\text{where } \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (3.4.10)$$

$\lambda_1$  and  $\lambda_2$  are Eigen values of  $\mathbf{V}$ , and  $\mathbf{P}$  contains the Eigen vectors corresponding to the Eigen values  $\lambda_1$  and  $\lambda_2$

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (3.4.10)$$

indicates the linear transformation where  $\mathbf{P}$  indicates the rotation of axes and  $\mathbf{c}$  gives the shift of origin.

$$\mathbf{V} = \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \quad (3.4.10)$$

$$\det(\mathbf{V}) = \begin{vmatrix} 41 & 12 \\ 12 & 34 \end{vmatrix} > 0 \quad (3.4.10)$$

So, the given equation represents an ellipse

To find the Eigen values of  $\mathbf{V}$

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (3.4.10)$$

$$\Rightarrow \begin{vmatrix} \lambda - 41 & -12 \\ -12 & \lambda - 34 \end{vmatrix} = 0 \quad (3.4.10)$$

$$\Rightarrow \lambda^2 - 75\lambda + 1250 = 0 \quad (3.4.10)$$

$$\Rightarrow \lambda_1 = 50, \lambda_2 = 25 \quad (3.4.10)$$

$$\mathbf{D} = \begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} \quad (3.4.10)$$

Finding Eigen vector  $\mathbf{p}_1$ ,

$$\lambda_1 \mathbf{I} - \mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2/4]{R_1 \leftarrow R_1/3} \begin{pmatrix} 3 & -4 \\ -3 & 4 \end{pmatrix} \quad (3.4.10)$$

$$\xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \quad (3.4.10)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{4^2 + 3^2}} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \quad (3.4.10)$$

Similarly,

$$\lambda_2 \mathbf{I} - \mathbf{V} = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2/-3]{R_1 \leftarrow R_1/-4} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} \quad (3.4.10)$$

$$\xrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix} \quad (3.4.10)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{4^2 + 3^2}} \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix} \quad (3.4.10)$$

$$\text{Therefore, } \mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \quad (3.4.10)$$

From (3.4.10)  $\mathbf{V}$  can be rewritten as

$$\mathbf{V} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \quad (3.4.10)$$

(3.4.10) can be now rewritten as

$$25 \left[ \mathbf{x}^T \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \right] = 75 \quad (3.4.10)$$

$$\left[ \left( \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \right)^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix} \mathbf{x} \right) \right] = 3 \quad (3.4.10)$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \quad (3.4.10)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 4 & -3 \\ 3 & 5 \\ 5 & 5 \end{pmatrix} \mathbf{y} \quad (3.4.10)$$

$$\mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^{-1}\mathbf{P}\mathbf{y} \quad (3.4.10)$$

$$\Rightarrow \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \quad (3.4.10)$$

$$\text{But, } \mathbf{P}^{-1} = \mathbf{P}^T \quad (3.4.10)$$

$$\Rightarrow \mathbf{y} = \begin{pmatrix} 4 & 3 \\ 3 & 5 \\ 5 & 5 \end{pmatrix} \mathbf{x} \quad (3.4.10)$$

Using (3.4.10) in (3.4.10), the ellipse equation can be rewritten as

$$\mathbf{y}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} = 3 \quad (3.4.10)$$

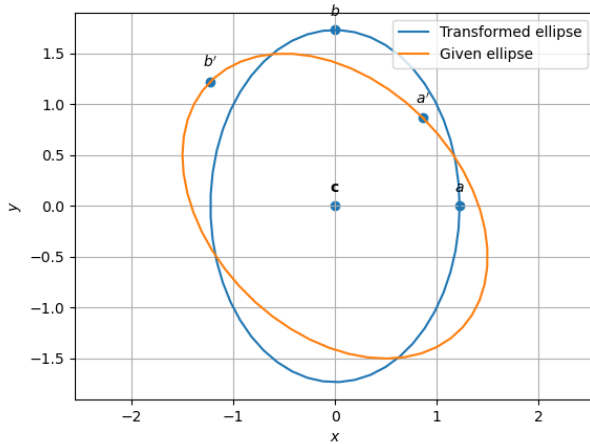


Fig. 3.4.10: plot showing the original and rotated ellipse

11. Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - (14 \ 22) \mathbf{x} + 27 = 0 \quad (3.4.11)$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \quad (3.4.11)$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \quad (3.4.11)$$

**Solution:** The general second order equation

can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.4.11)$$

Comparing (3.4.11) with (3.4.11),

$$\mathbf{V} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \quad (3.4.11)$$

$$\mathbf{u} = \begin{pmatrix} -7 \\ -11 \end{pmatrix} \quad (3.4.11)$$

$$f = 27 \quad (3.4.11)$$

Let  $\mathbf{c}$  be the change in the origin. The equation (3.4.11) can be modified as

$$(\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{x} + \mathbf{c}) + f = 0 \quad (3.4.11)$$

Considering (3.4.11)

$$\Rightarrow (\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) \quad (3.4.11)$$

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{x}^T \mathbf{V} \mathbf{c} + \mathbf{c}^T \mathbf{V} \mathbf{c} \quad (3.4.11)$$

In the above equation

$$\mathbf{c}^T \mathbf{V} \mathbf{x} = \mathbf{x}^T \mathbf{V} \mathbf{c} \quad (3.4.11)$$

From (3.4.11) and (3.4.11) then (3.4.11) becomes

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (3.4.11)$$

Comparing (3.4.11) and (3.4.11)

$$2\mathbf{c}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2\mathbf{u}^T \mathbf{P} \mathbf{y} = 0 \quad (3.4.11)$$

$$\mathbf{c}^T \mathbf{V} \mathbf{P} \mathbf{y} = -\mathbf{u}^T \mathbf{P} \mathbf{y} \quad (3.4.11)$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (3.4.11)$$

Substituting (3.4.11) and (3.4.11) in (3.4.11)

$$\mathbf{c} = \frac{-1}{24} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} -7 \\ -11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.4.11)$$

Hence (3.4.11) becomes

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (3.4.11)$$

Substituting (3.4.11), (3.4.11) and (3.4.11) the above equation becomes

$$\begin{aligned} \mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} + (1 \ 2) \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} -7 & -11 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ + 27 = 0 \end{aligned} \quad (3.4.11)$$

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} + 29 - 58 + 27 = 0 \quad (3.4.11)$$

$$\mathbf{x}^T \mathbf{V} \mathbf{x} - 2 = 0 \quad (3.4.11)$$

With change in the origin to point  $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  but the  $\mathbf{V}$  doesn't change.

$$|\mathbf{V}| = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24 \quad (3.4.11)$$

As  $|\mathbf{V}| > 0$  it represents a ellipse. Hence  $\mathbf{V}$  can be written as,

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (3.4.11)$$

The characteristic equation of  $\mathbf{V}$  is given by

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.4.11)$$

$$\begin{vmatrix} 5 - \lambda & 1 \\ 1 & 5 - \lambda \end{vmatrix} = 0 \quad (3.4.11)$$

$$\implies \lambda^2 - 10\lambda + 24 = 0 \quad (3.4.11)$$

Hence the eigen vales are,

$$\lambda_1 = 4 \quad (3.4.11)$$

$$\lambda_2 = 6 \quad (3.4.11)$$

Hence diagonal vector is given by,

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \quad (3.4.11)$$

The eigen vector  $\mathbf{p}$  is given by

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (3.4.11)$$

$$(\mathbf{V} - \lambda \mathbf{I}) \mathbf{p} = 0 \quad (3.4.11)$$

For  $\lambda_1 = 4$  the eigenvector is,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.4.11)$$

$$\mathbf{p}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.4.11)$$

For  $\lambda_1 = 6$  the eigenvector is,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.4.11)$$

$$\mathbf{p}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.4.11)$$

Hence,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.4.11)$$

Substituting (3.4.11) and (3.4.11) in (3.4.11)

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.4.11)$$

Hence substituting (3.4.11) in (3.4.11)

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{x} = 2 \quad (3.4.11)$$

$$\mathbf{y}^T \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{y} = 2 \quad (3.4.11)$$

$$\mathbf{y}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{y} = 1 \quad (3.4.11)$$

where  $\mathbf{y}$  is given by Affine transformation

$$\mathbf{x} = \mathbf{P} \mathbf{y} \quad (3.4.11)$$

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} \quad (3.4.11)$$

The rotation matrix  $\mathbf{P}$  can be given by,

$$\mathbf{P} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (3.4.11)$$

Comparing (3.4.11) and (3.4.11)

$$\cos \theta = \frac{1}{\sqrt{2}} \quad (3.4.11)$$

$$\theta = \frac{\pi}{4} \quad (3.4.11)$$

But given the direction of coordinate axes changes so,

$$\theta = \pi + \frac{\pi}{4} \quad (3.4.11)$$

Substituting (3.4.11) in (3.4.11) we get

$$\mathbf{P} = \begin{pmatrix} \cos \left( \pi + \frac{\pi}{4} \right) & \sin \left( \pi + \frac{\pi}{4} \right) \\ -\sin \left( \pi + \frac{\pi}{4} \right) & \cos \left( \pi + \frac{\pi}{4} \right) \end{pmatrix} \quad (3.4.11)$$

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.4.11)$$

From (3.4.11) we find the diagonal matrix

$$\mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (3.4.11)$$

$$\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \quad (3.4.11)$$

Hence using (3.4.11), (3.4.11) and (3.4.11) in (3.4.11) we get.

$$\mathbf{x}^T \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \mathbf{x} = 2 \quad (3.4.11)$$

using (3.4.11) the above equation becomes,

$$\mathbf{y}^T \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{y} = 2 \quad (3.4.11)$$

$$\mathbf{y}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = 1 \quad (3.4.11)$$

Hence from (3.4.11) and (3.4.11) proved that change of origin and the directions of the coordinate axes (3.4.11) can be transformed to (3.4.11) or (3.4.11)

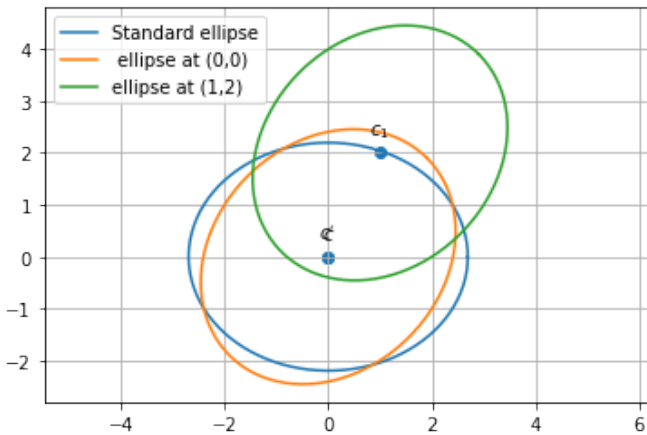


Fig. 3.4.11: Ellipse

## 4 CIRCLE

### 4.1 Equation

- Find the radius and the coordinates of the centre of each of the following circles:

a)  $3\mathbf{x}^T \mathbf{x} + (-12 \ 6)\mathbf{x} + 11 = 0$

**Solution:**

The general equation of circle can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (4.1.1)$$

where  $\mathbf{c}$  is the centre of the circle and radius of the circle is given as

$$r = \sqrt{\|\mathbf{c}\|^2 - f} \quad (4.1.1)$$

Given equation is

$$3\mathbf{x}^T \mathbf{x} + (-12 \ 6)\mathbf{x} + 11 = 0 \quad (4.1.1)$$

$$\mathbf{x}^T \mathbf{x} + (-4 \ 2)\mathbf{x} + \frac{11}{3} = 0 \quad (4.1.1)$$

$$\mathbf{x}^T \mathbf{x} - 2(2 \ -1)\mathbf{x} + \frac{11}{3} = 0 \quad (4.1.1)$$

Compare Eq (4.1.1) and Eq (4.1.1)

$$\Rightarrow \mathbf{c}^T = (2 \ -1) \quad (4.1.1)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (4.1.1)$$

$$f = \frac{11}{3} \quad (4.1.1)$$

From Eq (4.1.1),

$$\Rightarrow r = \sqrt{(2^2 + (-1)^2) - \frac{11}{3}} \quad (4.1.1)$$

$$= \sqrt{4 + 1 - \frac{11}{3}} \quad (4.1.1)$$

$$= \sqrt{\frac{4}{3}} \quad (4.1.1)$$

From Eq (4.1.1) and Eq (4.1.1)

$$\mathbf{c} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (4.1.1)$$

$$r = \sqrt{\frac{4}{3}} \quad (4.1.1)$$

b)  $\mathbf{x}^T \mathbf{x} = a^2 + b^2$

c)

$$2\mathbf{x}^T \mathbf{x} + (16 \ -4)\mathbf{x} + 33 = 0 \quad (4.1.1)$$

**Solution:**

The general equation of a circle is given by

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (4.1.1)$$

where  $\mathbf{c}$  is the centre of the circle and  $r$



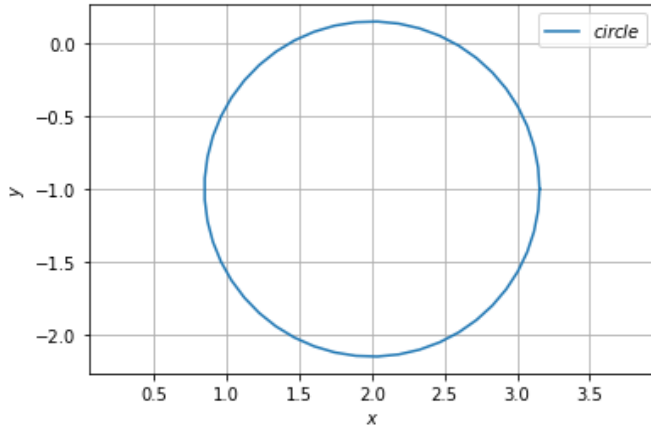


Fig. 4.1.1: Circle with radius 1.154 and center coordinates (2,-1)

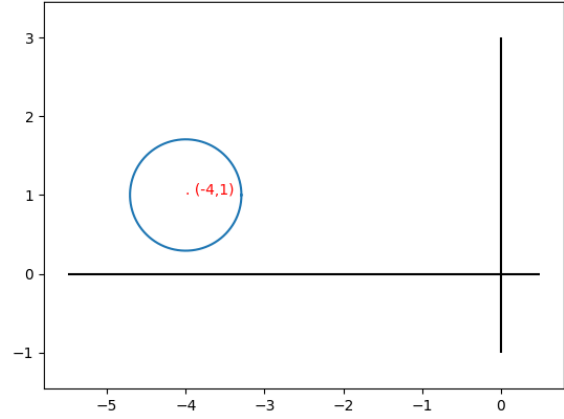


Fig. 4.1.1: Graph of  $2x^2 + 2y^2 + 16x - 4y + 33 = 0$

$= \sqrt{\|\mathbf{c}\|^2 - f}$  is the radius of the circle. Dividing (4.1.1) by 2 and rearranging terms (4.1.1) can be rewritten as

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} -4 \\ 1 \end{pmatrix}^T \mathbf{x} + \frac{33}{2} = 0 \quad (4.1.1)$$

Comparing (4.1.1) and (4.1.1) we get

$$\mathbf{c} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad (4.1.1)$$

$$f = \frac{33}{2} \quad (4.1.1)$$

Then centre of the circle (4.1.1) is  $\mathbf{c} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$  and radius

$$r = \sqrt{\|\mathbf{c}\|^2 - f} \quad (4.1.1)$$

$$= \sqrt{(-4)^2 + 1^2 - \frac{33}{2}} \quad (4.1.1)$$

$$= \sqrt{\frac{1}{2}} \quad (4.1.1)$$

$$= 0.7071 \quad (4.1.1)$$

d)  $36\mathbf{x}^T \mathbf{x} - (36 \ 24)\mathbf{x} - 131 = 0$

2. Find the equation of the circle that passes through the points  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

**Solution:** The equation of circle can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (4.1.2)$$

$\mathbf{c}$  is the centre and substituting the points in the equation of circle we get

$$2 \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{c} - f = 5 \quad (4.1.2)$$

$$2 \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{c} - f = 5 \quad (4.1.2)$$

$$2 \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{c} - f = 0 \quad (4.1.2)$$

can be expressed in matrix form

$$\begin{pmatrix} 2 & 4 & -1 \\ 4 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix} \quad (4.1.2)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 2 & 4 & -1 & 5 \\ 4 & 2 & -1 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_1 - R_2} \begin{pmatrix} 2 & 4 & -1 & 5 \\ 0 & 6 & -1 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (4.1.2)$$

$$\xleftrightarrow{\begin{matrix} R_2 \leftarrow R_2 - R_3 \\ R_1 \leftarrow R_1 - R_3 \end{matrix}} \begin{pmatrix} 2 & 4 & 0 & 5 \\ 0 & 6 & 0 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (4.1.2)$$

$$\xleftrightarrow{R_1 \leftarrow 3R_1 - 2R_2} \begin{pmatrix} 6 & 0 & 0 & 5 \\ 0 & 6 & 0 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (4.1.2)$$

$$\mathbf{c} = \begin{pmatrix} \frac{5}{6} \\ \frac{5}{6} \end{pmatrix} \quad (4.1.2)$$

$$f = 0 \quad (4.1.2)$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{\frac{50}{36}} \quad (4.1.2)$$

The required equation of circle is

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} \frac{5}{6} & \frac{5}{6} \end{pmatrix} \mathbf{x} = 0 \quad (4.1.2)$$

See Fig. 4.1.2

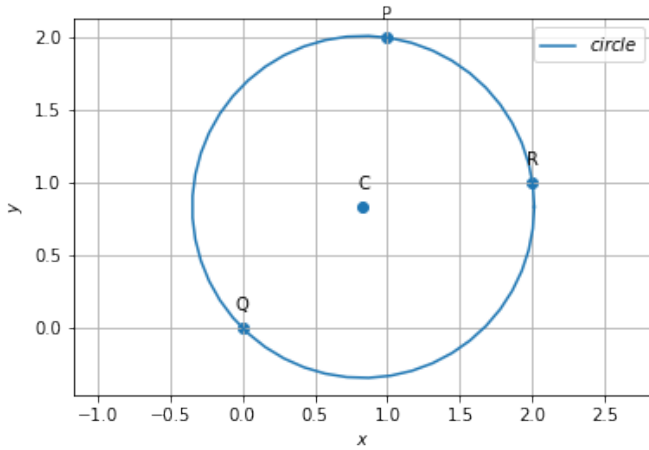


Fig. 4.1.2: Circle passing through point P,Q,R with centre C.

3. Find the equation of the circle that passes through the points  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ .

**Solution:** The general equation of circle is represented as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (4.1.3)$$

where  $\mathbf{c}$  is the center of the circle. Substituting the given points in the equation (4.1.3), we obtain

$$2 \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{c} - f = 13 \quad (4.1.3)$$

$$2 \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{c} - f = 13 \quad (4.1.3)$$

$$2 \begin{pmatrix} 5 & 1 \end{pmatrix} \mathbf{c} - f = 36 \quad (4.1.3)$$

can be expressed in matrix form as

$$\begin{pmatrix} 4 & 6 & -1 \\ 6 & 4 & -1 \\ 10 & 2 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \\ 36 \end{pmatrix} \quad (4.1.3)$$

The augmented matrix for (4.1.3) can be row

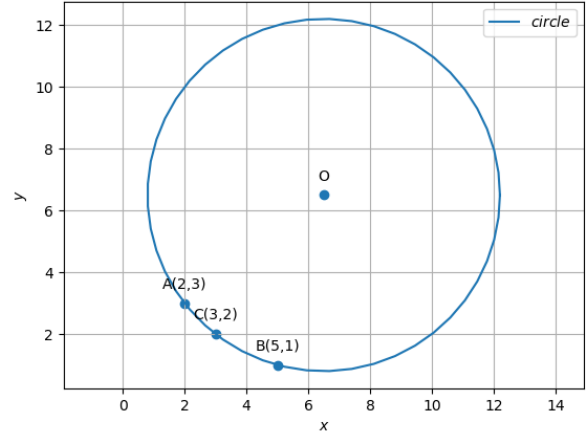


Fig. 4.1.3: Circle passing through the points A,B,C with center O

reduced as follows

$$\begin{pmatrix} 4 & 6 & -1 & 13 \\ 6 & 4 & -1 & 13 \\ 10 & 2 & -1 & 26 \end{pmatrix} \quad (4.1.3)$$

$$\begin{matrix} R_3 \leftarrow 4R_3 - 10R_1 \\ R_2 \leftarrow 4R_2 - 6R_1 \end{matrix} \begin{pmatrix} 4 & 6 & -1 & 13 \\ 0 & -20 & 2 & -26 \\ 0 & -52 & 6 & -26 \end{pmatrix} \quad (4.1.3)$$

$$\begin{matrix} R_3 \leftarrow 5R_3 - 13R_2 \end{matrix} \begin{pmatrix} 4 & 6 & -1 & 13 \\ 0 & -20 & 2 & -26 \\ 0 & 0 & 4 & 208 \end{pmatrix} \quad (4.1.3)$$

$$\begin{matrix} R_2 \leftarrow 2R_2 - R_3 \\ R_1 \leftarrow 4R_1 + R_3 \end{matrix} \begin{pmatrix} 16 & 24 & 0 & 260 \\ 0 & -40 & 0 & -260 \\ 0 & 0 & 4 & 208 \end{pmatrix} \quad (4.1.3)$$

$$\begin{matrix} R_1 \leftarrow 5R_1 + 3R_2 \end{matrix} \begin{pmatrix} 80 & 0 & 0 & 520 \\ 0 & -40 & 0 & -260 \\ 0 & 0 & 4 & 208 \end{pmatrix} \quad (4.1.3)$$

$$\begin{matrix} R_2 \leftarrow \frac{R_2}{-20}, R_3 \leftarrow \frac{R_3}{4} \\ R_1 \leftarrow \frac{R_1}{40} \end{matrix} \begin{pmatrix} 2 & 0 & 0 & 13 \\ 0 & 2 & 0 & 13 \\ 0 & 0 & 1 & 52 \end{pmatrix} \quad (4.1.3)$$

From the matrix (4.1.3),

$$\mathbf{c} = \begin{pmatrix} \frac{13}{2} \\ \frac{13}{2} \end{pmatrix} \quad (4.1.3)$$

$$k = 52 \quad (4.1.3)$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = 11 \quad (4.1.3)$$

Hence the circle equation can be written as,

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} \frac{13}{2} & \frac{13}{2} \end{pmatrix}^T \mathbf{x} + 52 = 0 \quad (4.1.3)$$

4. Find the equation of the circle that passes through the points  $\begin{pmatrix} 2a \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 2b \end{pmatrix}$ ,  $\begin{pmatrix} a+b \\ a+b \end{pmatrix}$ .

**Solution:** The equation of circle can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (4.1.4)$$

$\mathbf{c}$  is the centre and substituting the points in the equation of circle we get

$$2(2a \ 0)\mathbf{c} - f = 4a^2 \quad (4.1.4)$$

$$2(0 \ 2b)\mathbf{c} - f = 4b^2 \quad (4.1.4)$$

$$2(a+b \ a+b)\mathbf{c} - f = 2(a+b)^2 \quad (4.1.4)$$

which can be expressed in matrix form

$$\begin{pmatrix} 4a & 0 & -1 \\ 0 & 4b & -1 \\ 2(a+b) & 2(a+b) & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 4a^2 \\ 4b^2 \\ 2(a+b)^2 \end{pmatrix} \quad (4.1.4)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 4a & 0 & -1 & 4a^2 \\ 0 & 4b & -1 & 4b^2 \\ 2(a+b) & 2(a+b) & -1 & 2(a+b)^2 \end{pmatrix} \quad (4.1.4)$$

$$\xleftarrow[R_3 \leftarrow R_3 - 2(a+b)R_1]{R_1 \leftarrow \frac{R_1}{4a}} \begin{pmatrix} 1 & 0 & -\frac{1}{4a} & a \\ 0 & 4b & -1 & 4b^2 \\ 0 & 2(a+b) & \frac{-a+b}{2a} & 2b(a+b) \end{pmatrix} \quad (4.1.4)$$

$$\xleftarrow[R_2 \leftarrow \frac{R_2}{4b}]{R_3 \leftarrow R_3 - 2(a+b)R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{4a} & a \\ 0 & 1 & -\frac{1}{4b} & b \\ 0 & 0 & \frac{a}{2b} + \frac{b}{2a} & 0 \end{pmatrix} \quad (4.1.4)$$

$$\xleftarrow[R_3 \leftarrow \frac{R_3}{\frac{a}{2b} + \frac{b}{2a}}]{R_2 \leftarrow \frac{R_2}{4b}} \begin{pmatrix} 1 & 0 & -\frac{1}{4a} & a \\ 0 & 1 & -\frac{1}{4b} & b \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (4.1.4)$$

$$\xleftarrow[R_1 \leftarrow R_1 - (-\frac{1}{4a})R_3]{R_2 \leftarrow R_2 - (-\frac{1}{4b})R_3} \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (4.1.4)$$

$$\mathbf{c} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (4.1.4)$$

$$f = 0 \quad (4.1.4)$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{a^2 + b^2} \quad (4.1.4)$$

The required equation of circle is

$$\mathbf{x}^T \mathbf{x} - 2(a \ b)\mathbf{x} = 0 \quad (4.1.4)$$

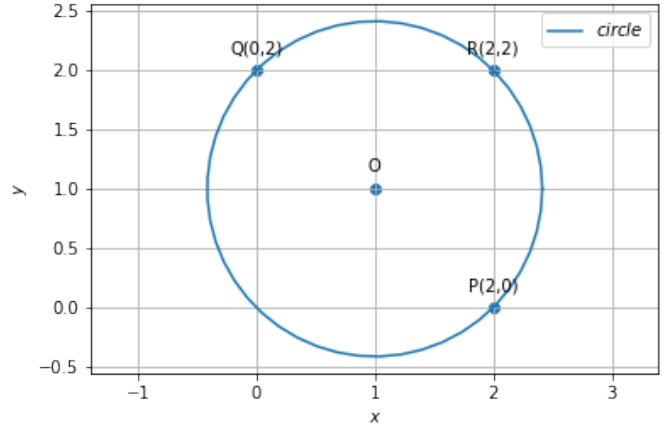


Fig. 4.1.4: Circle passing through point P and Q and R

5. A circle has its centre on the line  $x = 2y$  and passes through the points  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ . Find the coordinates of the centre and the equation of the circle.

**Solution:** The equation of circle can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (4.1.5)$$

$\mathbf{c}$  is the centre and substituting the points in the equation of circle we get

$$2(-1 \ 2)\mathbf{c} - f = 5 \quad (4.1.5)$$

$$2(3 \ -2)\mathbf{c} - f = 13 \quad (4.1.5)$$

$$(1 \ -2)\mathbf{c} = 0 \quad (4.1.5)$$

can be expressed in matrix form

$$\begin{pmatrix} 1 & -2 & 0 \\ 6 & -4 & -1 \\ -2 & 4 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \\ 5 \end{pmatrix} \quad (4.1.5)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -2 & 0 & 0 \\ 6 & -4 & -1 & 13 \\ -2 & 4 & -1 & 5 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 + 2R_1]{R_2 \leftarrow R_2 - 6R_1} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 8 & -1 & 13 \\ 0 & 0 & -1 & 5 \end{pmatrix} \quad (4.1.5)$$

$$\xrightarrow[R_1 \leftarrow R_1 + R_2]{R_2 \leftarrow R_2/4} \begin{pmatrix} 1 & 0 & -1 & 13 \\ 0 & 2 & -1/4 & 13/4 \\ 0 & 0 & -1 & 5 \end{pmatrix} \quad (4.1.5)$$

$$\xrightarrow[R_1 \leftarrow R_1 - R_3/4]{R_2 \leftarrow R_2 - R_3/4} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -1 & 5 \end{pmatrix} \quad (4.1.5)$$

$$\mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.1.5)$$

$$f = -5 \quad (4.1.5)$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{10} \quad (4.1.5)$$

The required equation of circle is

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} + 5 = 0 \quad (4.1.5)$$

See Fig. 4.1.5

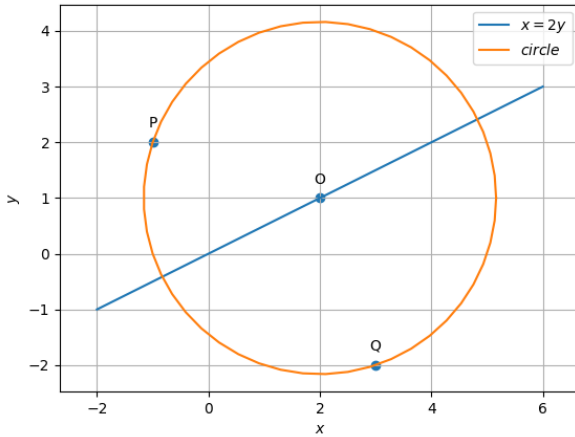


Fig. 4.1.5: Circle passing through point P and Q also centre lie on the line  $x=2y$

6. Find the locus of the centre of a circle which touches the line

$$(\cos \alpha \quad \sin \alpha) \mathbf{x} = p \quad (4.1.6.1)$$

and the circle

$$\|\mathbf{x} - \mathbf{c}\| = r \quad (4.1.6.2)$$

## 4.2 Tangent and Normal

1. Without drawing a figure, determine whether the points  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$  lie outside, on the circumference, or inside the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -5 & 2 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (4.2.1.1)$$

**Solution:** Without drawing a figure, determine whether the points  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$  lie outside, on the circumference, or inside the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -5 & 2 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (4.2.1.2)$$

The equation of circle with center  $\mathbf{c}$  can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (4.2.1.3)$$

Comparing (4.2.1.3) with (4.2.1.2)

$$\mathbf{c} = \begin{pmatrix} \frac{5}{2} \\ -1 \end{pmatrix}, f = -5 \quad (4.2.1.4)$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{\frac{49}{4}} \quad (4.2.1.5)$$

a) Let  $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\|\mathbf{a} - \mathbf{c}\| = \sqrt{\frac{49}{4} + 9} = \sqrt{\frac{84}{4}} \Rightarrow \|\mathbf{a} - \mathbf{c}\| > r \quad (4.2.1.6)$$

Point a is outside the circle

b) Let  $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\|\mathbf{b} - \mathbf{c}\| = \sqrt{\frac{25}{4} + 1} = \sqrt{\frac{29}{4}} \Rightarrow \|\mathbf{b} - \mathbf{c}\| < r \quad (4.2.1.7)$$

Point b is inside the circle.

c) Let  $\mathbf{d} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$$\|\mathbf{d} - \mathbf{c}\| = \sqrt{\frac{1}{4} + 9} = \sqrt{\frac{37}{4}} \Rightarrow \|\mathbf{d} - \mathbf{c}\| < r \quad (4.2.1.8)$$

Point d is inside the circle.

2. Find the points of intersection of the line

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 12 \quad (4.2.2.1)$$

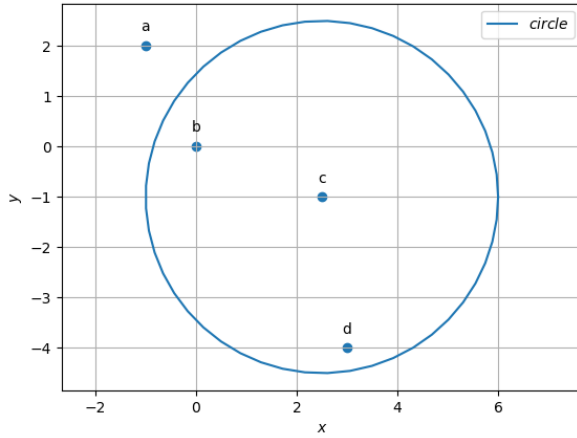


Fig. 4.2.1: Points a,b,d in the circle with center c

and the circle

$$\|\mathbf{x}\|^2 = 13 \quad (4.2.2.2)$$

and for what values of  $c$  the line

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = c \quad (4.2.2.3)$$

touches the circle.

**Solution:** If  $\mathbf{P}$  be a point on the line and  $\mathbf{n}$  is the normal vector, the equation of the line can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{P}) = 0 \quad (4.2.2.4)$$

$$\implies \mathbf{n}^T \mathbf{x} = c \quad (4.2.2.5)$$

where

$$c = \mathbf{n}^T \mathbf{P} \quad (4.2.2.6)$$

From (4.2.2.1) and (4.2.2.5),

$$\mathbf{n}^T = \begin{pmatrix} 3 & 2 \end{pmatrix} \quad (4.2.2.7)$$

We know,

$$\mathbf{m}^T \mathbf{n} = 0 \quad (4.2.2.8)$$

$$\implies \mathbf{m}^T \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 0 \quad (4.2.2.9)$$

$$\implies \mathbf{m}^T = \begin{pmatrix} -2 & 3 \end{pmatrix} \quad (4.2.2.10)$$

Now,

$$\mathbf{n}^T \mathbf{P} = c \quad (4.2.2.11)$$

$$\implies \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{P} = 12 \quad (4.2.2.12)$$

$\mathbf{P}$  can be

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (4.2.2.13)$$

Let us take

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \mathbf{q} \quad (4.2.2.14)$$

The circle equation:

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.2.2.15)$$

From (4.2.2.2),

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (4.2.2.16)$$

$$f = -13 \quad (4.2.2.17)$$

The points of intersection of the line

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m}, \mu \in \mathbb{R} \quad (4.2.2.18)$$

with the conic section

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.2.2.19)$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (4.2.2.20)$$

where,

$$\begin{aligned} \mu_i &= \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left( -\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \right. \\ &\quad \left. \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \end{aligned} \quad (4.2.2.21)$$

For circle,

$$\mathbf{V} = \mathbf{I} \quad (4.2.2.22)$$

$$\therefore \mu_i = \frac{1}{13} \left( -5 \pm \sqrt{25 - (13 - 13)13} \right) \quad (4.2.2.23)$$

$$= \frac{1}{13} (-5 \pm 5) \quad (4.2.2.24)$$

$$= 0, -\frac{10}{13} \quad (4.2.2.25)$$

Using (4.2.2.20), the points of intersection are given by

$$\mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} \frac{46}{13} \\ \frac{9}{13} \end{pmatrix} \quad (4.2.2.26)$$

Points of contact are given by

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (4.2.2.27)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (4.2.2.28)$$

Since for circle,

$$\mathbf{V} = \mathbf{I} \quad (4.2.2.29)$$

$$\therefore \mathbf{V}^{-1} = \mathbf{I} \quad \because \mathbf{I}^{-1} = \mathbf{I} \quad (4.2.2.30)$$

$$\therefore \kappa = \pm \sqrt{\frac{-f}{\mathbf{n}^T \mathbf{n}}} \quad \because \mathbf{u}^T \mathbf{u} = 0 \quad (4.2.2.31)$$

$$= \pm \sqrt{\frac{13}{(3 \ 2) \begin{pmatrix} 3 \\ 2 \end{pmatrix}}} \quad (4.2.2.32)$$

$$= \pm \sqrt{\frac{13}{13}} \quad (4.2.2.33)$$

$$= \pm 1 \quad (4.2.2.34)$$

$$\therefore \mathbf{q} = \pm 1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (4.2.2.35)$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad (4.2.2.36)$$

From (4.2.2.3),

$$c = (3 \ 2) \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 13, \quad (4.2.2.37)$$

$$(3 \ 2) \begin{pmatrix} -3 \\ -2 \end{pmatrix} = -13 \quad (4.2.2.38)$$

The line (4.2.2.3) touches the circle for  $c = 13, -13$ .

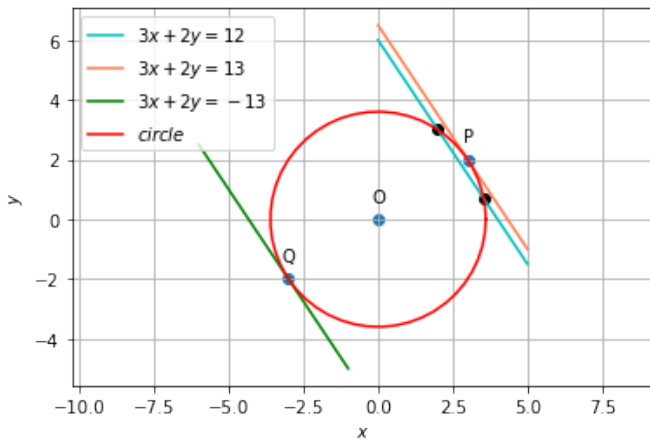


Fig. 4.2.2: Circle with tangent and intersection lines

3. Prove that the line

$$(3 \ 2) \mathbf{x} = 30 \quad (4.2.3.1)$$

touches the circle

$$\mathbf{x}^T \mathbf{x} - (10 \ 2) \mathbf{x} + 13 = 0 \quad (4.2.3.2)$$

and find the coordinates of the point of contact.

**Solution:**

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (4.2.3.3)$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.2.3.4)$$

Comparing (4.2.3.2) with (4.2.3.4)

$$\mathbf{u} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, f = 13 \quad (4.2.3.5)$$

If  $\mathbf{n}$  is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T \mathbf{x} = c \quad (4.2.3.6)$$

Comparing (4.2.3.1) with (4.2.3.6)

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (4.2.3.7)$$

The point of contact  $\mathbf{q}$ , of a line with a normal vector  $\mathbf{n}$  to the conic in (4.2.3.4) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (4.2.3.8)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (4.2.3.9)$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \quad (4.2.3.10)$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \quad (4.2.3.11)$$

$$\mathbf{I} \mathbf{x} = \mathbf{x} \quad (4.2.3.12)$$

Solving for the point of contact using the above

equations we get,

$$\kappa = \pm \sqrt{\frac{(-5 \ -1) \begin{pmatrix} -5 \\ -1 \end{pmatrix} - 13}{(3 \ 2) \begin{pmatrix} 3 \\ 2 \end{pmatrix}}} \quad (4.2.3.13)$$

$$= \pm \sqrt{\frac{26 - 13}{13}} \quad (4.2.3.14)$$

$$= \pm \sqrt{1} \quad (4.2.3.15)$$

$$\mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ -1 \end{pmatrix} \quad (4.2.3.16)$$

$$= \begin{pmatrix} 8 \\ 3 \end{pmatrix} \quad (4.2.3.17)$$

If the line in (4.2.3.3) touches (4.2.3.4) at exactly one point  $\mathbf{q}$ , then

$$\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \quad (4.2.3.18)$$

It can be seen that for the given line

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} \quad (4.2.3.19)$$

Solving (4.2.3.18) for given line and circle, we get

$$= (1 \ -1.5) \left( \begin{pmatrix} 8 \\ 3 \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \end{pmatrix} \right) \quad (4.2.3.20)$$

$$= (1 \ -1.5) \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (4.2.3.21)$$

$$= 0 \quad (4.2.3.22)$$

Hence, it is proved that the given line touches the given circle at only one point and so it is a tangent.

4. For what values of  $m$  does the line

$$(m \ -1)\mathbf{x} = 0 \quad (4.2.4.1)$$

touch the circle

$$\mathbf{x}^T \mathbf{x} - (6 \ 2)\mathbf{x} + 8 = 0 \quad (4.2.4.2)$$

**Solution:** The general equation of a circle is

$$\Rightarrow \mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.2.4.3)$$

$$\text{If } r \text{ is radius, } f = \mathbf{u}^T \mathbf{u} - r^2 \quad (4.2.4.4)$$

$$\text{center } \mathbf{c} = -\mathbf{u} \quad (4.2.4.5)$$

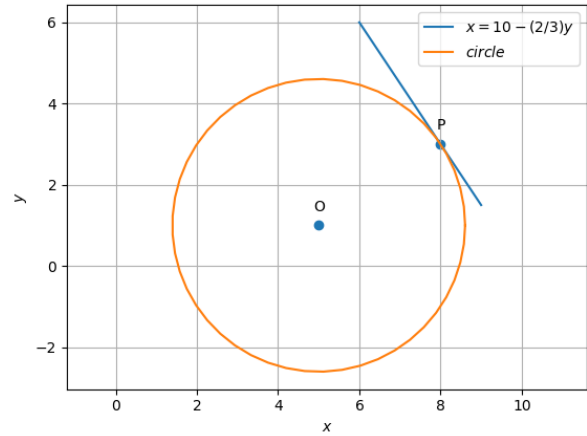


Fig. 4.2.3: Circle with center (5 1) and having the line (3 2)x = 30 as tangent with (8 3) as point of contact.

From equations 4.2.4.2 and 4.2.4.3,

$$\mathbf{u} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \quad (4.2.4.6)$$

$$\Rightarrow \text{Center of the circle } \mathbf{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (4.2.4.7)$$

Also, radius can be determined as follows from 4.2.4.4

$$\Rightarrow 8 = (-3 \ -1) \begin{pmatrix} -3 \\ -1 \end{pmatrix} - r^2 \quad (4.2.4.8)$$

$$\Rightarrow 8 = 10 - r^2 \quad (4.2.4.9)$$

$$\Rightarrow r = \sqrt{2} \quad (4.2.4.10)$$

Given equation of the line is

$$(m \ -1)\mathbf{x} = 0 \quad (4.2.4.11)$$

It can be expressed as:-

$$L: \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (4.2.4.12)$$

$$(4.2.4.13)$$

The normal vector to the line is obtained as

$$\mathbf{n} = \mathbf{q} + \mathbf{u} \quad (4.2.4.14)$$

$$\Rightarrow \mathbf{q} = \mathbf{n} - \mathbf{u} \quad (4.2.4.15)$$

$$\mathbf{n} = (m \ -1)^T \text{ and } \mathbf{u} = (-3 \ -1)^T \quad (4.2.4.16)$$

$$\mathbf{q} = \begin{pmatrix} m+3 \\ 0 \end{pmatrix} \quad (4.2.4.17)$$

The point  $\mathbf{q}$  also satisfies the equation of the

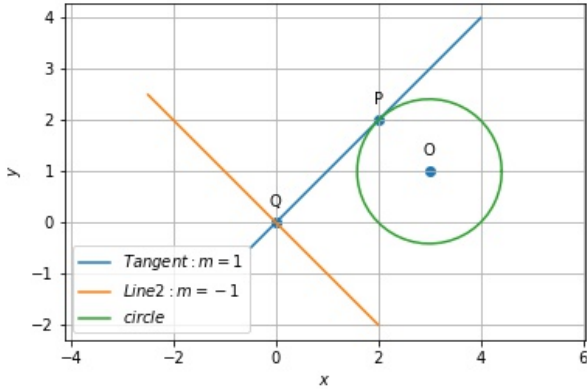


Fig. 4.2.4: Circle with tangent

circle at 4.2.4.2.

$$\mathbf{q}^T \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \quad (4.2.4.18)$$

$$(4.2.4.19)$$

$$(\mathbf{n} - \mathbf{u})^T (\mathbf{n} - \mathbf{u}) + 2\mathbf{u}^T (\mathbf{n} - \mathbf{u}) + f = 0 \quad (4.2.4.20)$$

$$\|\mathbf{n}\|^2 - \mathbf{n}^T \mathbf{u} - \mathbf{u}^T \mathbf{n} + \|\mathbf{u}\|^2 + 2\mathbf{u}^T \mathbf{n} - 2\|\mathbf{u}\|^2 + f = 0 \quad (4.2.4.21)$$

$$\|\mathbf{n}\|^2 - \mathbf{n}^T \mathbf{u} + \mathbf{n}^T \mathbf{u} - \|\mathbf{u}\|^2 + f = 0 \quad (4.2.4.22)$$

$$\|\mathbf{n}\|^2 - \|\mathbf{u}\|^2 + f = 0 \quad (4.2.4.23)$$

$$(m^2 + 1) - 10 + 8 = 0 \quad (4.2.4.24)$$

$$m^2 - 1 = 0 \quad (4.2.4.25)$$

$$m = \pm 1 \quad (4.2.4.26)$$

For the line 4.2.4.1 to be a tangent to circle at equation 4.2.4.2, the values of  $m$  are  $\pm 1$

5. Prove that the circle

$$\mathbf{x}^T \mathbf{x} - 2a(1 \ 1)\mathbf{x} + a^2 = 0 \quad (4.2.5)$$

touches the coordinate axes. **Solution:** Let  $r$  be the radius of the circle. Now for proving that the circle touches the co-ordinate axes we have to prove that it touches  $x$  axis and  $y$  axes at points such that:

$$\mathbf{Point}_1 = \begin{pmatrix} r \\ 0 \end{pmatrix} \quad (4.2.5)$$

$$\mathbf{Point}_2 = \begin{pmatrix} 0 \\ r \end{pmatrix} \quad (4.2.5)$$

The general equation of a circle is given by

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{O}^T \mathbf{x} + \mathbf{f} = 0 \quad (4.2.5)$$

Where  $\mathbf{O}$  is the centre and  $r$  is the radius of the circle.

Substituting (4.2.5) in (4.2.5), we rewrite (4.2.5) as:

$$\begin{pmatrix} r & 0 \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} - 2 \begin{pmatrix} a & a \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\Rightarrow \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\Rightarrow \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\Rightarrow (\mathbf{r} - \mathbf{a})^2 = 0 \quad (4.2.5)$$

$$\Rightarrow \mathbf{r} = \mathbf{a} \quad (4.2.5)$$

Similarly, substituting (4.2.5) in (4.2.5), we rewrite (4.2.5) as:

$$\begin{pmatrix} 0 & r \end{pmatrix} \begin{pmatrix} 0 \\ r \end{pmatrix} - 2 \begin{pmatrix} a & a \end{pmatrix} \begin{pmatrix} 0 \\ r \end{pmatrix} + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\Rightarrow \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\Rightarrow \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\Rightarrow (\mathbf{r} - \mathbf{a})^2 = 0 \quad (4.2.5)$$

$$\Rightarrow \mathbf{r} = \mathbf{a} \quad (4.2.5)$$

Therefore, the circle touches  $x$  axis at  $\mathbf{Point}_1$  i.e.  $\begin{pmatrix} a \\ 0 \end{pmatrix}$  and  $y$  axis at  $\mathbf{Point}_2$  i.e.  $\begin{pmatrix} 0 \\ a \end{pmatrix}$

Hence, it is proved that the circle touches the co-ordinate axes

6. Show that two circles can be drawn to pass through the point  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and touch the coordinate axes, and find their equations. **Solution:** Let us consider we have a circle which passes through the point  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and touches  $x$  - axis at point  $\begin{pmatrix} r \\ 0 \end{pmatrix}$  and  $y$  - axis at  $\begin{pmatrix} 0 \\ r \end{pmatrix}$ . Radius of the circle is  $r$  since it touches both axes. Hence we have 3



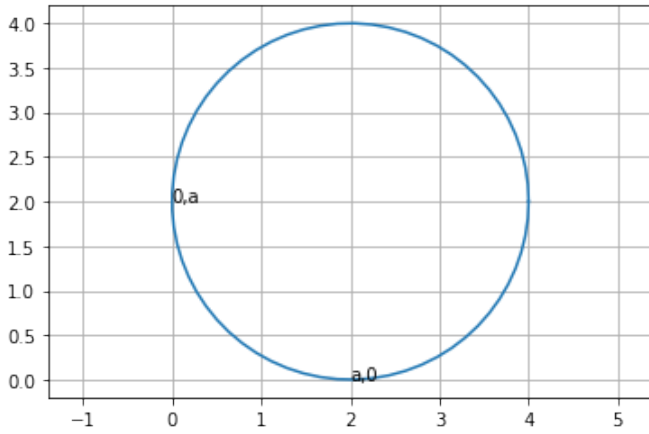


Fig. 4.2.5: Circle touching the co-ordinate axes  
a=2

points which are :

$$\mathbf{P}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (4.2.6)$$

$$\mathbf{P}_2 = \begin{pmatrix} r \\ 0 \end{pmatrix} \quad (4.2.6)$$

$$\mathbf{P}_3 = \begin{pmatrix} 0 \\ r \end{pmatrix} \quad (4.2.6)$$

The general equation of circle is :

$$\|\mathbf{x} - \mathbf{O}\| = r \quad (4.2.6)$$

Substituting the given coordinates:

$$\|\mathbf{P}_2 - \mathbf{O}\|^2 = r^2 \quad (4.2.6)$$

$$\|\mathbf{P}_3 - \mathbf{O}\|^2 = r^2 \quad (4.2.6)$$

$$\|\mathbf{P}_1 - \mathbf{O}\|^2 = r^2 \quad (4.2.6)$$

From equation 4.2.6, 4.2.6 and 4.2.6 we have

$$\|\mathbf{P}_2 - \mathbf{O}\|^2 - \|\mathbf{P}_1 - \mathbf{O}\|^2 = 0 \quad (4.2.6)$$

And,

$$\|\mathbf{P}_3 - \mathbf{O}\|^2 - \|\mathbf{P}_1 - \mathbf{O}\|^2 = 0 \quad (4.2.6)$$

Simplifying 4.2.6 and 4.2.6,

$$(\mathbf{P}_2 - \mathbf{O})^T (\mathbf{P}_2 - \mathbf{O}) - (\mathbf{P}_1 - \mathbf{O})^T (\mathbf{P}_1 - \mathbf{O}) = 0 \quad (4.2.6)$$

$$\Rightarrow \|\mathbf{P}_2\|^2 - 2\mathbf{P}_2^T \mathbf{O} - \|\mathbf{P}_1\|^2 + 2\mathbf{P}_1^T \mathbf{O} = 0 \quad (4.2.6)$$

Substituting the value of  $\|\mathbf{P}_1\|$  and  $\|\mathbf{P}_2\|$  and other values then rearranging it, we get :

$$(2 - 2r \quad 4)(\mathbf{O}) = 5 - r^2 \quad (4.2.6)$$

Similarly,

$$(\mathbf{P}_3 - \mathbf{O})^T (\mathbf{P}_3 - \mathbf{O}) - (\mathbf{P}_1 - \mathbf{O})^T (\mathbf{P}_1 - \mathbf{O}) = 0 \quad (4.2.6)$$

$$\Rightarrow \|\mathbf{P}_3\|^2 - 2\mathbf{P}_3^T \mathbf{O} - \|\mathbf{P}_1\|^2 + 2\mathbf{P}_1^T \mathbf{O} = 0 \quad (4.2.6)$$

Substituting the value of  $\|\mathbf{P}_1\|$  and  $\|\mathbf{P}_3\|$  and other values then rearranging it, we get :

$$(2 \quad 4 - 2r)(\mathbf{O}) = 5 - r^2 \quad (4.2.6)$$

Combining 4.2.6 and 4.2.6

$$\begin{pmatrix} 2 - 2r & 4 \\ 2 & 4 - 2r \end{pmatrix} (\mathbf{O}) = \begin{pmatrix} 5 - r^2 \\ 5 - r^2 \end{pmatrix} \quad (4.2.6)$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2 - 2r & 4 & 5 - r^2 \\ 2 & 4 - 2r & 5 - r^2 \end{pmatrix} \quad (4.2.6)$$

$$\begin{pmatrix} 2 - 2r & 4 & 5 - r^2 \\ 2 & 4 - 2r & 5 - r^2 \end{pmatrix} \xrightarrow{R1 \leftarrow \frac{R1}{2-2r}}$$

$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 2 & 4 - 2r & 5 - r^2 \end{pmatrix} \xrightarrow{R2 \leftarrow R2 - 2R1}$$

$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & \frac{2r(r-3)}{r-1} & \frac{r(r^2-5)}{r-1} \end{pmatrix} \xrightarrow{R2 \leftarrow \left(\frac{1-r}{2r(r-3)}\right)R2}$$

$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \xrightarrow{R1 \leftarrow R1 - \left(\frac{-2}{r-1}\right)R2}$$

$$\begin{pmatrix} 1 & 0 & \frac{r^2-5}{2(r-3)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \quad (4.2.6)$$

So,

$$\mathbf{O} = \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} \quad (4.2.6) \quad \Rightarrow \quad r = \frac{r^2-5}{2(r-3)} \quad (4.2.6)$$

Now substituting the 4.2.6 in 4.2.6, we have

$$\|\mathbf{P}_3 - \mathbf{O}\|^2 = r^2 \quad (4.2.6) \quad \Rightarrow \quad r^2 - 6r + 5 = 0 \quad (4.2.6)$$

Substituting the value of  $\mathbf{O}$  in 4.2.6 and simplify,

$$(\mathbf{P}_3 - \mathbf{O})^T (\mathbf{P}_3 - \mathbf{O}) = r^2 \quad (4.2.6) \quad \Rightarrow \quad (r-1)(r-5) = 0 \quad (4.2.6)$$

$$\Rightarrow \|\mathbf{P}_3\|^2 - \mathbf{P}_3^T \mathbf{O} - \mathbf{P}_3 \mathbf{O}^T + \|\mathbf{O}\|^2 = r^2 \quad (4.2.6)$$

Putting the values of  $\mathbf{O}$  from 4.2.6 and  $\|\mathbf{P}_3\|^2$

$$\Rightarrow -\mathbf{P}_3^T \mathbf{O} - \mathbf{P}_3 \mathbf{O}^T = -\|\mathbf{O}\|^2 \quad (4.2.6)$$

$$\Rightarrow -\begin{pmatrix} 0 \\ r \end{pmatrix}^T \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} - \begin{pmatrix} 0 \\ r \end{pmatrix} \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix}^T = -\|\mathbf{O}\|^2 \quad (4.2.6)$$

Substituting the value of  $\|\mathbf{O}\|^2$  and simplify it,

$$\Rightarrow 2r \left( \frac{r^2-5}{2(r-3)} \right) = 2 \left( \frac{r^2-5}{2(r-3)} \right)^2 \quad (4.2.6)$$

Hence,

$$\mathbf{O}_1 = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \text{ and } \mathbf{O}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.2.6)$$

Hence equation of circles are :

$$\left\| \mathbf{x} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right\| = 5 \quad (4.2.6)$$

And,

$$\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 1 \quad (4.2.6)$$

7. Find the length of the tangent from the point  $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$  to the circle

$$\mathbf{x}^T \mathbf{x} - (4 \ 6) \mathbf{x} + 12 = 0 \quad (4.2.7)$$

**Solution:**

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.2.7)$$

Let the equation of the tangent be

$$(-m \ 1) \mathbf{x} = c \quad (4.2.7)$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \quad (4.2.7)$$

$$\mathbf{c} = -\mathbf{u} \quad (4.2.7)$$

Comparing the equation (4.2.7) and (4.2.7) we

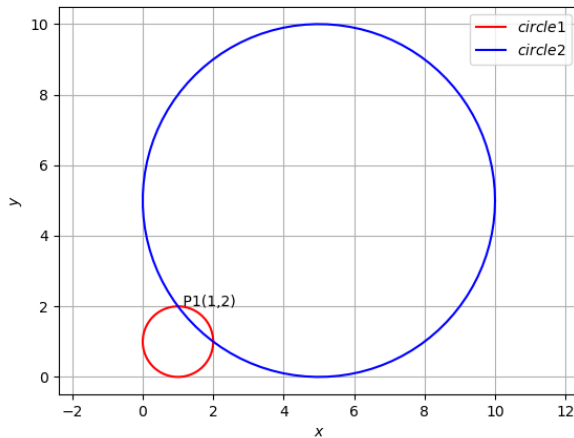


Fig. 4.2.6: Two circles passing through a common point

get

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, f = 12 \quad (4.2.7)$$

$$\mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.2.7)$$

The normal vector to the line is obtained as

$$\mathbf{n} = \mathbf{q} + \mathbf{u} \quad (4.2.7)$$

$$\mathbf{q} = \mathbf{n} - \mathbf{u} \quad (4.2.7)$$

Comparing the equation (4.2.7)

$$\mathbf{n} = (-m \ 1)^T \quad (4.2.7)$$

Given

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (4.2.7)$$

$$\Rightarrow \mathbf{q} = \begin{pmatrix} -m + 2 \\ 4 \end{pmatrix} \quad (4.2.7)$$

The point q also satisfies the equation of the circle at (4.2.7)

$$\mathbf{q}^T \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \quad (4.2.7)$$

$$(\mathbf{n} - \mathbf{u})^T (\mathbf{n} - \mathbf{u}) + 2\mathbf{u}^T (\mathbf{n} - \mathbf{u}) + f = 0 \quad (4.2.7)$$

$$\|\mathbf{n}\|^2 - \mathbf{n}^T \mathbf{u} - \mathbf{u}^T \mathbf{n} + \|\mathbf{u}\|^2 + 2\mathbf{u}^T \mathbf{n} - 2\|\mathbf{u}\|^2 + f = 0 \quad (4.2.7)$$

$$\|\mathbf{n}\|^2 - \|\mathbf{u}\|^2 + f = 0 \quad (4.2.7)$$

$$m^2 + 1 - 13 + 12 = 0 \quad (4.2.7)$$

$$m^2 = 0 \quad (4.2.7)$$

$$m = 0 \quad (4.2.7)$$

Simplifying (4.2.7) and (4.2.7) we get

$$\mathbf{q} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad (4.2.7)$$

Let  $\mathbf{p} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$  The length of tangent is

$$\|\mathbf{p} - \mathbf{q}\| = \sqrt{(7-2)^2 + (4-4)^2} \quad (4.2.7)$$

$$= \sqrt{25} \quad (4.2.7)$$

$$= 5 \quad (4.2.7)$$

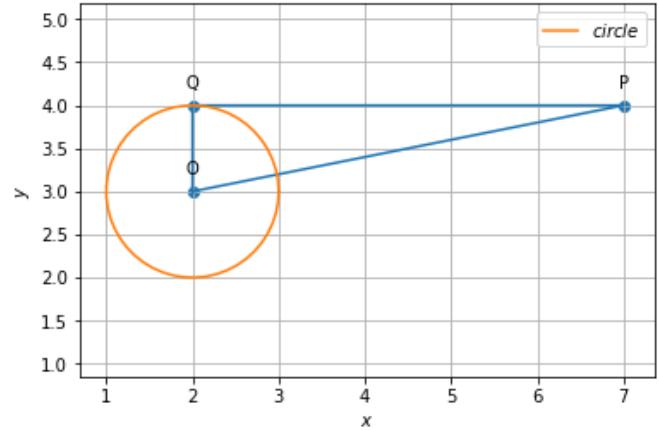


Fig. 4.2.7: Perpendicular Line

8. Find the equations of tangents to the circle

$$\mathbf{x}^T \mathbf{x} - (4 \ 3) \mathbf{x} + 5 = 0 \quad (4.2.8.1)$$

that are parallel to the line

$$(1 \ 1) \mathbf{x} = 0 \quad (4.2.8.2)$$

**Solution:** The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (4.2.8.3)$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.2.8.4)$$

Comparing (4.2.8.1) with (4.2.8.4)

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3/2 \end{pmatrix}, f = 5 \quad (4.2.8.5)$$

If  $\mathbf{n}$  is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T \mathbf{x} = c \quad (4.2.8.6)$$

Comparing (4.2.8.2) with (4.2.8.6)

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.2.8.7)$$

Since it is mentioned that the tangent is parallel to given line it will have same normal vector. The point of contact  $\mathbf{q}$ , of a line with a normal vector  $\mathbf{n}$  to the conic in (4.2.8.4) is given by:

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \quad (4.2.8.8)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (4.2.8.9)$$

The point of contact  $\mathbf{q}$ , of a line with a normal vector  $\mathbf{n}$  to the conic in (4.2.8.4) is given by:

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \quad (4.2.8.10)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (4.2.8.11)$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \quad (4.2.8.12)$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \quad (4.2.8.13)$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \quad (4.2.8.14)$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm \sqrt{\frac{\begin{pmatrix} -2 & -1.5 \end{pmatrix} \begin{pmatrix} -2 \\ -1.5 \end{pmatrix} - 5}{\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}} \quad (4.2.8.15)$$

$$= \pm \sqrt{\frac{6.25 - 5}{2}} \quad (4.2.8.16)$$

$$= \pm \sqrt{\frac{5}{8}} \quad (4.2.8.17)$$

$$(4.2.8.18)$$

So there are two tangents to a circle which are parallel to given line which touch circle at two different points  $\mathbf{q}_1$  and  $\mathbf{q}_2$

$$\mathbf{q}_1 = \begin{pmatrix} \sqrt{\frac{5}{8}} \\ \sqrt{\frac{5}{8}} \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} \quad (4.2.8.19)$$

$$= \begin{pmatrix} \frac{279}{100} \\ \frac{229}{100} \end{pmatrix} \quad (4.2.8.20)$$

$$\mathbf{q}_2 = \begin{pmatrix} -\sqrt{\frac{5}{8}} \\ \sqrt{\frac{5}{8}} \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} \quad (4.2.8.21)$$

$$= \begin{pmatrix} \frac{121}{100} \\ \frac{71}{100} \end{pmatrix} \quad (4.2.8.22)$$

Since points  $\mathbf{q}_1$  and  $\mathbf{q}_2$  lie on tangent they satisfy the line equation of tangents, there are two different tangents with same normal vector

$$\mathbf{n}^T \mathbf{q}_1 = c_1 \quad (4.2.8.23)$$

$$\mathbf{n}^T \mathbf{q}_2 = c_2 \quad (4.2.8.24)$$

$c_1$  and  $c_2$  are some constants

$$c_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{279}{100} \\ \frac{229}{100} \end{pmatrix} = \frac{127}{25} \quad (4.2.8.25)$$

$$c_2 = \begin{pmatrix} \frac{121}{100} \\ \frac{71}{100} \end{pmatrix} = \frac{48}{25} \quad (4.2.8.26)$$

So line equations of tangents to the given circle which are parallel to line are

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{127}{25} \quad (4.2.8.27)$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{48}{25} \quad (4.2.8.28)$$

9. Find the equations of the tangents to the circle

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} + 18 = 0 \quad (4.2.9)$$

at the points  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ , showing that they are parallel.

**Solution:**

The general equation of a second degree can be expressed as:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.2.9)$$

Comparing 1.0.1 with 2.0.1

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix} \frac{-7}{2} \\ \frac{-5}{2} \end{pmatrix}, f = 18 \quad (4.2.9)$$

The vector equation of a line can be expressed

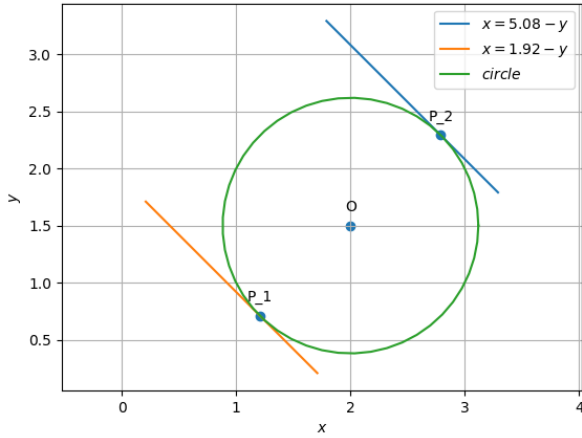


Fig. 4.2.8: Circle with center (2 1.5) and having the lines (1 1)x = 5.08 and (1 1)x = 1.92 as tangents with (2.79 2.29) and (1.21 0.71) as point of contact.

as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (4.2.9)$$

Tangent-1 Given point

$$\mathbf{q}_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (4.2.9)$$

The direction vector of the line joining the point  $\mathbf{q}_1$  and the centre  $\mathbf{c}$  expressed as:

$$\mathbf{n}_1 = \mathbf{q}_1 - \mathbf{c} \quad (4.2.9)$$

$$\Rightarrow \mathbf{n}_1 = \mathbf{q}_1 + \mathbf{u} \quad (4.2.9)$$

where,

$$\mathbf{c} = -\mathbf{u} \quad (4.2.9)$$

The vector  $\mathbf{n}_1$  is normal to the tangent drawn at  $\mathbf{q}_1$ . From (2.0.2) and (2.1.1) we get,

$$\mathbf{n}_1^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (4.2.9)$$

We know,

$$\mathbf{m}^T \mathbf{n}_1 = 0 \quad (4.2.9)$$

$$\Rightarrow \mathbf{m}^T \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = 0 \quad (4.2.9)$$

$$\Rightarrow \mathbf{m}^T = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (4.2.9)$$

If  $\mathbf{q}_1$  be a point on the line and  $\mathbf{n}_1$  is the normal

vector then the equation of the line can be expressed From(2.0.3) is :

$$\mathbf{n}_1^T (\mathbf{x} - \mathbf{q}_1) = 0 \quad (4.2.9)$$

$$\Rightarrow \mathbf{n}_1^T \mathbf{x} = c \quad (4.2.9)$$

where

$$c = \mathbf{n}_1^T \mathbf{q}_1 \quad (4.2.9)$$

Using the equations (2.1.1) and (2.1.5),

$$\Rightarrow c = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{7}{2} \quad (4.2.9)$$

From (2.1.10), Line equation of Tangent-1 is:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{7}{2} \quad (4.2.9)$$

$$\Rightarrow \boxed{(1 \ 1) \mathbf{x} = 7} \quad (4.2.9)$$

Tangent-2 Now,

$$\mathbf{q}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (4.2.9)$$

The direction vector of the line joining the point  $\mathbf{q}_2$  and the centre  $\mathbf{c}$  expressed as:

$$\mathbf{n}_2 = \mathbf{q}_2 - \mathbf{c} \quad (4.2.9)$$

$$\Rightarrow \mathbf{n}_2 = \mathbf{q}_2 + \mathbf{u} \quad (4.2.9)$$

where,

$$\mathbf{c} = -\mathbf{u} \quad (4.2.9)$$

The vector  $\mathbf{n}_2$  is normal to the tangent drawn at  $\mathbf{q}_2$ . From (2.0.2) and (2.2.1),

$$\mathbf{n}_2^T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (4.2.9)$$

We know,

$$\mathbf{m}^T \mathbf{n}_2 = 0 \quad (4.2.9)$$

$$\Rightarrow \mathbf{m}^T \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = 0 \quad (4.2.9)$$

$$\Rightarrow \mathbf{m}^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (4.2.9)$$

If  $\mathbf{q}_2$  be a point on the line and  $\mathbf{n}_2$  is the normal vector, the equation of the line can be

expressed From (2.0.3) is:

$$\mathbf{n}_2^T(\mathbf{x} - \mathbf{q}_2) = 0 \quad (4.2.9)$$

$$\Rightarrow \mathbf{n}_2^T \mathbf{x} = c \quad (4.2.9)$$

where

$$c = \mathbf{n}_2^T \mathbf{q}_2 \quad (4.2.9)$$

Using the equations 2.2.1 and 2.2.5,

$$\Rightarrow c = \begin{pmatrix} -1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{-5}{2} \quad (4.2.9)$$

From (2.2.10), Line equation of Tangent-2 is:

$$\begin{pmatrix} -1 & -1 \end{pmatrix} \mathbf{x} = \frac{-5}{2} \quad (4.2.9)$$

$$\Rightarrow \boxed{(1 \ 1) \mathbf{x} = 5} \quad (4.2.9)$$

Result From the equations (2.1.14) and (2.2.14), normal vectors of Tangent-1 and Tangent-2 are equal.

Hence, the two tangents are parallel.

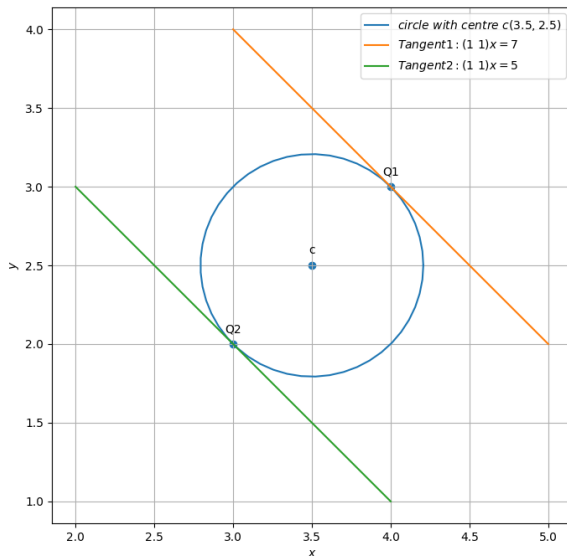


Fig. 4.2.9: Tangents to the circle at given points

10. Find the equations of the tangent and normal to the circle

$$\mathbf{x}^T \mathbf{x} + (-6 \ 4) \mathbf{x} - 12 = 0 \quad (4.2.10)$$

at the point  $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ .

11. Prove that the line

$$(1 \ 1) \mathbf{x} = 1 \quad (4.2.11.1)$$

touches the circle

$$\mathbf{x}^T \mathbf{x} - (8 \ 6) \mathbf{x} + 7 = 0 \quad (4.2.11.2)$$

and find the equations of the parallel and perpendicular tangents.

12. Find the equation of the tangent at the origin to the circle

$$\mathbf{x}^T \mathbf{x} + 2(g \ f) \mathbf{x} = 0 \quad (4.2.12)$$

13. Prove that the line

$$(\cos \alpha \ \sin \alpha) \mathbf{x} = p \quad (4.2.13.1)$$

touches the circle

$$\left\| \mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right\| = r \quad (4.2.13.2)$$

if

$$r = \pm(p - a \cos \alpha - b \sin \alpha) \quad (4.2.13.3)$$

14. Find the points of contact of the tangents to the circle

$$\|\mathbf{x}\| = 5 \quad (4.2.14)$$

that pass through the point  $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$  and write down the equations of the tangents.

15. Prove that the tangent to the circle

$$\|\mathbf{x}\|^2 = 5 \quad (4.2.15.1)$$

at the point  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  also touches the circle

$$\mathbf{x}^T \mathbf{x} + (-8 \ 6) \mathbf{x} + 20 = 0 \quad (4.2.15.2)$$

and find the coordinates of the point of contact.

16. Find the equations of the circles that touch the lines

$$(0 \ 1) \mathbf{x} = 0 \quad (4.2.16.1)$$

$$(0 \ 1) \mathbf{x} = 4 \quad (4.2.16.2)$$

$$(2 \ 1) \mathbf{x} = 2 \quad (4.2.16.3)$$

17. Find the coordinates of the middle point of

the chord

$$(1 \ 7)\mathbf{x} = 25 \quad (4.2.17.1)$$

of the circle

$$\|\mathbf{x}\| = 5 \quad (4.2.17.2)$$

18. Find the equation of the chord of the circle

$$\mathbf{x}^T \mathbf{x} - (6 \ 4)\mathbf{x} - 23 = 0 \quad (4.2.18)$$

which has the point  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  as its middle point.

19. Prove that the circle

$$\mathbf{x}^T \mathbf{x} - (6 \ 4)\mathbf{x} + 9 = 0 \quad (4.2.19)$$

subtends an angle  $\tan^{-1} \frac{12}{5}$  at the origin.

20. Find the condition that the line

$$(l \ m)\mathbf{x} + n = 0 \quad (4.2.20.1)$$

should touch the circle

$$\left\| \mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right\| = r \quad (4.2.20.2)$$

21. Verify that the perpendicular bisector of the chord joining two points  $\mathbf{x}_1, \mathbf{x}_2$  on the circle

$$\mathbf{x}^T \mathbf{x} + 2(g \ f)\mathbf{x} + c = 0 \quad (4.2.21)$$

passes through the centre.

3. Show that the points  $\begin{pmatrix} 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix}$  are conjugate with regard to the circle

$$\|\mathbf{x}\|^2 = 24 \quad (4.3.3)$$

4. Prove that the lines

$$(5 \ 3)\mathbf{x} = 40 \quad (4.3.4.1)$$

$$(7 \ -5)\mathbf{x} = 10 \quad (4.3.4.2)$$

are conjugate with regard to the circle

$$\|\mathbf{x}\|^2 = 20 \quad (4.3.4.3)$$

5. Find the polar of the point  $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$  with regard to the circle

$$\mathbf{x}^T \mathbf{x} - (4 \ 3)\mathbf{x} - 8 = 0 \quad (4.3.5)$$

6. Find the pole of

$$(l \ m)\mathbf{x} + n = 0 \quad (4.3.6)$$

with regard to

a)  $\|\mathbf{x}\| = a$

b)  $\mathbf{x}^T \mathbf{x} + 2(g \ f)\mathbf{x} + c = 0$

7. Prove that if two lines at right angles are conjugate with regard to a circle one of them must pass through the centre.

8. Prove that, if the chords of contact of pairs of tangents to a circle from  $\mathbf{P}$  and  $\mathbf{Q}$  intersect in  $\mathbf{R}$ , then the line joining  $\mathbf{R}$  to the centre is perpendicular to  $PQ$ .

### 4.3 Pole and Polar

1. Write down the equations of the polars of the following points with regard to the circle

$$\|\mathbf{x}\|^2 = 6 \quad (4.3.1)$$

a)  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

b)  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

2. Find the poles of the following lines with regard to the circle

$$\|\mathbf{x}\| = 3 \quad (4.3.2)$$

a)  $(3 \ 4)\mathbf{x} = 7$

b)  $(5 \ -1)\mathbf{x} + 6 = 0$

and verify that the polar of their point of intersection is the line joining their poles.

### 4.4 Systems of Circles

1. Find the equation of a circle which passes through the points  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$  and cuts orthogonally the circle

$$\mathbf{x}^T \mathbf{x} + (-2 \ 3)\mathbf{x} - 5 = 0 \quad (4.4.1)$$

2. Find the equation of a circle which cuts orthogonally the three circles

$$\mathbf{x}^T \mathbf{x} + (4 \ -5)\mathbf{x} + 6 = 0 \quad (4.4.2.1)$$

$$\mathbf{x}^T \mathbf{x} + (5 \ -6)\mathbf{x} + 7 = 0 \quad (4.4.2.2)$$

$$\mathbf{x}^T \mathbf{x} - (1 \ 1)\mathbf{x} - 1 = 0 \quad (4.4.2.3)$$

3. Find the equation of a circle which cuts orthog-

onally the two circles

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (4.4.3.1)$$

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -3 & 6 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (4.4.3.2)$$

and passes through the point  $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ .

4. Write down the equations of the radical axes of the following pairs of circles:

a)

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 4 & -5 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (4.4.4.1)$$

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 5 & -6 \end{pmatrix} \mathbf{x} = 0 \quad (4.4.4.2)$$

b)

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (4.4.4.3)$$

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} + 2 = 0 \quad (4.4.4.4)$$

c)

$$\mathbf{x}^T \mathbf{x} + 2g \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + c = 0 \quad (4.4.4.5)$$

$$\mathbf{x}^T \mathbf{x} + 2f \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + c = 0 \quad (4.4.4.6)$$

5. Find the equation of a circle coaxal with

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (4.4.5.1)$$

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (4.4.5.2)$$

and passing through the point  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

6. Find the coordinates of the point from which the tangents to the three circles

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 4 & 4 \end{pmatrix} \mathbf{x} + 7 = 0 \quad (4.4.6.1)$$

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (4.4.6.2)$$

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} = 0 \quad (4.4.6.3)$$

are of equal length.

7. Find the limiting points of the circles

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (4.4.7.1)$$

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} - 10 = 0 \quad (4.4.7.2)$$

8. Prove that if a point moves so that the tangent from it to the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 4 & -5 \end{pmatrix} \mathbf{x} + 6 = 0 \quad (4.4.8.1)$$

is double the length of the tangent to the circle

$$\|\mathbf{x}\| = 2, \quad (4.4.8.2)$$

its locus is the circle

$$3\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 4 & -5 \end{pmatrix} \mathbf{x} - 22 = 0 \quad (4.4.8.3)$$

9. Prove that the locus of a point such that the lengths of the tangents from it to two given circles are in a constant ratio is a circle coaxal with the given circles.

10. Find the equations of the two circles coaxal with

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 8 & -10 \end{pmatrix} \mathbf{x} + 2 = 0 \quad (4.4.10.1)$$

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (4.4.10.2)$$

that touch the line

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 3 = 0 \quad (4.4.10.3)$$

11. Find the centre and radius of the circle which cuts orthogonally the three circles

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 6 & 4 \end{pmatrix} \mathbf{x} + 12 = 0 \quad (4.4.11.1)$$

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (4.4.11.2)$$

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} + 4 = 0 \quad (4.4.11.3)$$

12. The line

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} + 2 = 0 \quad (4.4.12.1)$$

is the radical axis of a family of coaxal circles of which one circle is

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} - 1 = 0. \quad (4.4.12.2)$$

Find the equation of the member of the family that passes through the point  $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ .

#### 4.5 Miscellaneous

- Find the locus of a point which moves so that the sum of the squares of its distances from the sides of an equilateral triangle is constant.
- Find the locus of a point which moves so that the sum of the squares of its distances from  $n$  fixed points is constant.
- Find the locus of a point at which two given circles subtend equal angles.
- A circle passes through the four points  $\begin{pmatrix} a \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} b \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ c \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ d \end{pmatrix}$ . By what relation are  $a$ ,  $b$ ,  $c$ ,  $d$  connected? Find the equation of the circle and



show that the tangent at the point  $\begin{pmatrix} a \\ c+d \end{pmatrix}$  is

$$(a-b \ c+d)\mathbf{x} - a(a-b) - (c+d)^2 = 0 \quad (4.5.4)$$

5. Write down the equations of the tangents to the circles

$$\mathbf{x}^T \mathbf{x} + (-2a \ 0)\mathbf{x} - 5 = 0 \quad (4.5.5.1)$$

$$\mathbf{x}^T \mathbf{x} + (0 \ -2b)\mathbf{x} - 5 = 0 \quad (4.5.5.2)$$

at their points of intersection and verify that they cut at right angles.

6. Find the equation of the tangent to the circle

$$\|\mathbf{x}\| = a \quad (4.5.6.1)$$

at the point  $\begin{pmatrix} a \cos \theta \\ a \sin \theta \end{pmatrix}$  and show that the length of the tangent intercepted by the lines

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (4.5.6.2)$$

is  $\pm 2a \sec \theta$ .

7. **A** and **B** are two fixed points  $\begin{pmatrix} c \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} -c \\ 0 \end{pmatrix}$ , and **P** moves so that  $PA = k.PB$ . Find the locus of **P** and prove that it is cut orthogonally by any circle through **A** and **B**.

8. Show that the common chord of the circles

$$\mathbf{x}^T \mathbf{x} - (6 \ 4)\mathbf{x} + 9 = 0 \quad (4.5.8.1)$$

$$\mathbf{x}^T \mathbf{x} - (8 \ 6)\mathbf{x} + 23 = 0 \quad (4.5.8.2)$$

is a diameter of the latter circle and find the angle at which the circles cut.

9. Prove analytically that the tangents to a circle at the ends of a chord are equally inclined to the chord.
10. Prove that for different values of  $a$  the equation

$$\mathbf{x}^T \mathbf{x} + (-2a \operatorname{cosec} \alpha \ 0)\mathbf{x} + a^2 \cot^2 \alpha = 0 \quad (4.5.10.1)$$

represents a family of circles touching the lines

$$(\pm \tan \alpha \ 1)\mathbf{x} = 0 \quad (4.5.10.2)$$

Prove also that the locus of the poles of the line

$$(l \ m)\mathbf{x} = 0 \quad (4.5.10.3)$$

with regard to the circles is the line

$$(m \sin^2 \alpha \ l \cos^2 \alpha)\mathbf{x} = 0 \quad (4.5.10.4)$$

11. Find the coordinates of the middle point of the chord

$$(l \ m)\mathbf{x} = 1 \quad (4.5.11.1)$$

of the circle

$$\mathbf{x}^T \mathbf{x} + 2(g \ f)\mathbf{x} + c = 0 \quad (4.5.11.2)$$

12. Prove that the points of intersection of the line

$$(l \ m)\mathbf{x} = 1 \quad (4.5.12.1)$$

and the circle

$$\mathbf{x}^T \mathbf{x} + 2(g \ f)\mathbf{x} + c = 0 \quad (4.5.12.2)$$

subtend a right angle at the origin if

$$c(l^2 + m^2) + 2gl + 2fm + 2 = 0 \quad (4.5.12.3)$$

13. Prove that the equation of the circle having for diameter the portion of the line

$$(\cos \alpha \ \sin \alpha)\mathbf{x} = p \quad (4.5.13.1)$$

intercepted by the circle

$$\|\mathbf{x}\| = a \quad (4.5.13.2)$$

is

$$\mathbf{x}^T \mathbf{x} - 2p(\cos \alpha \ \sin \alpha)\mathbf{x} + 2p^2 - a^2 = 0 \quad (4.5.13.3)$$

14. Prove that if a chord of the circle

$$\|\mathbf{x}\| = a \quad (4.5.14.1)$$

subtends a right angle at a fixed point  $\mathbf{x}_1$ , the locus of the middle point of the chord is

$$2\mathbf{x}^T \mathbf{x} - 2\mathbf{x}_1^T \mathbf{x} + \|\mathbf{x}_1\|^2 - a^2 = 0 \quad (4.5.14.2)$$

15. Prove that the equation of any tangent to the circle

$$\left\| \mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right\| = r \quad (4.5.15.1)$$

may be written in the form

$$(\cos \theta \ \sin \theta) \left( \mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right) = r \quad (4.5.15.2)$$

Deduce that the equation of the tangents from

$\mathbf{x}_1$  to the circle is

$$r^2 \|\mathbf{x} - \mathbf{x}_1\|^2 = \left[ \left\{ \mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right\} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left\{ \mathbf{x}_1 - \begin{pmatrix} a \\ b \end{pmatrix} \right\} \right]^2 \quad (4.5.15.3)$$

16. Prove that the distances of two points from the centre of a circle are proportional to the distance of each point from the polar of the other.
17. Prove that the tangents to the circles of a coaxial system drawn from a limiting point are bisected by the radical axis.
18. Show that a common tangent to the two circles is bisected by their radical axis and subtends a right angle at either limiting point.
19. Prove that if a point moves so that the difference of the squares of the tangents from it to two given circles is constant its locus is a straight line parallel to the radical axis of the circles.
20. Prove that the polars of a fixed point with regard to a family of coaxial circles all pass through another fixed point.
21. The circles

$$\mathbf{x}^T \mathbf{x} + (-2a \sec \alpha \ 0) \mathbf{x} - a^2 = 0 \quad (4.5.21.1)$$

$$\mathbf{x}^T \mathbf{x} + (0 \ -2a \operatorname{cosec} \alpha) \mathbf{x} - a^2 = 0 \quad (4.5.21.2)$$

where  $\alpha$  is a given angle, both cut orthogonally every member of a coaxial family of circles. Find the radical axis and the limiting points of the family.

22. Prove that, if two points  $\mathbf{P}$ ,  $\mathbf{Q}$  are conjugate with regard to a circle, the circle on  $PQ$  as diameter cuts the first circle orthogonally.
23. Prove that if  $\mathbf{P}$ ,  $\mathbf{Q}$  are conjugate points with regard to a circle, the circles with  $\mathbf{P}$ ,  $\mathbf{Q}$  as centres which cut the given circle orthogonally are orthogonal to one another.
24. Prove that, if  $PQ$  is a diameter of a circle, then  $\mathbf{P}$ ,  $\mathbf{Q}$  are conjugate points with regard to any circle which cuts the given circle orthogonally.
25. Prove that if  $\mathbf{P}$ ,  $\mathbf{Q}$  are conjugate points with regard to a circle, the square on  $PQ$  is equal to the sum of the squares on the tangents from  $\mathbf{P}$ ,  $\mathbf{Q}$  to the circle.

26. The equation

$$\mathbf{x}^T \mathbf{x} + (-2g \ 0) \mathbf{x} + 2g - 5 = 0 \quad (4.5.26)$$

where  $g$  is a variable parameter, represents a family of coaxial circles. Show that the radius of the smallest circle of the family is 2.

27. Prove that, if perpendiculars are drawn from a fixed point  $\mathbf{P}$  to the polars of  $\mathbf{P}$  with regard to a family of coaxial circles, then the locus of the feet of these perpendiculars is a circle whose centre lies on the radical axis of the family.
28. Prove that, if the points in which the line

$$(l \ m) \mathbf{x} + n = 0 \quad (4.5.28.1)$$

meets the circle,

$$\mathbf{x}^T \mathbf{x} + 2(g \ f) \mathbf{x} + c = 0 \quad (4.5.28.2)$$

and those in which the line

$$(l_1 \ m_1) \mathbf{x} + n_1 = 0 \quad (4.5.28.3)$$

meets

$$\mathbf{x}^T \mathbf{x} + 2(g_1 \ f_1) \mathbf{x} + c_1 = 0 \quad (4.5.28.4)$$

lie on a circle, then

$$2(g - g_1)(mn_1 - m_1n) + 2(f - f_1)(nl_1 - n_1l) + (c - c_1)(lm_1 - l_1m) = 0 \quad (4.5.28.5)$$

29. Show that, if a diameter of a circle is the portion of the line

$$(l \ m) \mathbf{x} = 1 \quad (4.5.29.1)$$

intercepted by the lines

$$\mathbf{x}^T \begin{pmatrix} a & h \\ h & b \end{pmatrix} \mathbf{x} = 0 \quad (4.5.29.2)$$

then the equation of the circle is

$$(am^2 - 2hlm + bl^2) \mathbf{x}^T \mathbf{x} + 2((hm - bl) \ (hl - am)) \mathbf{x} + a + b = 0 \quad (4.5.29.3)$$

30. Prove that, as  $k$  varies, the equation

$$\mathbf{x}^T \mathbf{x} + 2(a \ b) \mathbf{x} + c + 2k\{(a \ -b) \mathbf{x} + 1\} = 0 \quad (4.5.30.1)$$

represents a system of coaxial circles. Also



prove that, if this tangent meets the axis in  $T$  and  $PN$  is the ordinate of  $P$  and  $A$  is the vertex, then

$$TA \cdot AN = a^2 \quad (5.1.12.2)$$

13. Prove that the circle

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} g & f \end{pmatrix} \mathbf{x} + c = 0 \quad (5.1.13.1)$$

cuts the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (5.1.13.2)$$

in four points the sum of whose ordinates is zero; and conversely that if four points on a parabola be such that the sum of their ordinates is zero then the four points lie on a circle.

14. Prove that the orthocentre of a triangle whose sides all touch a parabola lies on the directrix.

15. A chord  $POQ$  of a parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (5.1.15.1)$$

cuts the axis in a fixed point  $O$ .  $PN$ ,  $QM$  are the ordinates of  $P$  and  $Q$ , and  $A$  is the vertex. Prove that

$$NP \cdot MQ + 4a \cdot AO = 0 \quad (5.1.15.2)$$

16. From a point  $P = \begin{pmatrix} at_1^2 \\ 2at_1 \end{pmatrix}$  on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (5.1.16.1)$$

two chords  $PQ$ ,  $PR$  are drawn normal to the curve at  $Q$  and  $R$ . Prove that, if  $Q$ ,  $R$  are the points with parameters  $t_2$ ,  $t_3$  on the curve, then  $t_2 t_3 = 2$ , and the equation of  $QR$  is

$$(2 - t_1) \mathbf{x} + 4a = 0 \quad (5.1.16.2)$$

17. Prove that the normals to a parabola at the ends of a chord whose inclination to the axis is  $\theta$  meet on the normal whose inclination is  $\tan^{-1}(2 \cot \theta)$ .
18. Prove that, if two parabolas are on the same side of the same directrix and have their axes in the same line, then they intersect at a distance from the directrix equal to one-quarter of the sum of their latera recta.

## 5.2 Miscellaneous

In the following problems, the equation of a parabola is assumed to be the standard parabola and capital letters refer to Fig. 5.1.0 unless the contrary is stated.

1. Prove that as  $P$  moves along the curve  $GP^2$  varies as  $SG$ .
2. Prove that, if  $PP_1$  is a double ordinate and  $PX$  meets the curve in  $Q$ , then  $P_1Q$  passes through  $S$ .
3. Prove that, if  $PSP_1$  is a focal chord and  $AP$ ,  $AP_1$  meet the latus rectum in  $Q$ ,  $Q_1$ , the  $SQ$ ,  $SQ_1$  are equal to the ordinates of  $P_1$  and  $P$ .
4. Prove that, if the tangents at  $P$ ,  $Q$  intersect in  $T$ , then

$$ST^2 = SP \cdot SQ. \quad (5.2.4)$$

5. Prove that, if the tangent at the end  $Q$  of a focal chord  $PSQ$  meets the latus rectum in  $R$ , then  $PGR$  is a right angle.
6. Tangents at  $P$ ,  $Q$ ,  $R$  on a parabola form a triangle  $UVW$ . Show that the centroids of the triangles  $PQR$  and  $UVW$  lie on the same diameter.
7. Prove that, if the difference of the ordinates of two points on a parabola is constant, then the locus of the point of intersection of the tangents at these points is an equal parabola.
8. Prove that, if two tangents intercept a fixed length on the tangent at the vertex, the locus of their intersections is an equal parabola.
9. The chord of contact of tangents from any point  $Q$  meets the tangent at the vertex in  $R$ . Prove that the tangent of the angle which  $AQ$  makes with the axis is  $\frac{2a}{AR}$ .
10. The parameters,  $t$ ,  $t_1$  of two points on a parabola are connected by the relation  $t = k^2 t_1$ , prove that the tangents at the points intersect on the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \left(k + \frac{1}{k}\right)^2 a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (5.2.10)$$

11. Show that the length of the normal chord at the point of parameter  $t$  is

$$\frac{4a}{t^2} (1 + t^2)^{\frac{3}{2}} \quad (5.2.11)$$

12. Prove that the locus of intersection of tangents

at the ends of a normal chord is

$$(x + 2a)y^2 + 4a^2 = 0. \quad (5.2.12)$$

13. Prove that the locus of the point of intersection of perpendicular normals is the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 3a^2 = 0 \quad (5.2.13)$$

14. Prove that if the tangents at two points on the parabola intersect in the point  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ , the corresponding normals intersect in the point

$$\begin{pmatrix} 2a - x_1 + \frac{y_1^2}{a} \\ -\frac{x_1 y_1}{a} \end{pmatrix}. \quad (5.2.14)$$

15. Show that, if the tangent at **P** meets the latus rectum in **K**, then  $SK$  is a mean proportional between the segments of the focal chord through **P**.
16. Show that, if the tangents from **Q** to the parabola form with the tangent at the vertex, a triangle of constant area  $c^2$ , then the locus of **Q** is the curve

$$x^2(y^2 - 4ax) = 4c^4. \quad (5.2.16)$$

17. Show that the normals at the ends of each of a series of parallel chords of a parabola intersect on a fixed straight line, itself a normal to the parabola.
18. **P**, **Q** are points on the parabola subtending a constant angle  $\alpha$  at the vertex. Show that the locus of the intersection of the tangents at **P**, **Q** is the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \left(4a + \frac{\tan^2 \alpha}{4} \right) \mathbf{x} = a^2 \tan^2 \alpha \quad (5.2.18)$$

19. Prove that the exterior angle between two tangents to a parabola is equal to the angle which either of them subtends at the focus.
20. Two perpendicular focal chords of a parabola meet the directrix in **T** and **T**<sub>1</sub>. Show that the tangents to the parabola which are parallel to these chords intersect in the middle point of  $TT_1$ .
21. Prove that, if the tangents at the points **Q**, **R** intersect at **P**, then

$$PQ^2 : PR^2 = SQ : SR \quad (5.2.21)$$

22. The tangents at any two points **P**, **Q** meet at **T** and the normals meet at **N**. Prove that the projection of  $TN$  on the axis is equal to the sum of the distances of **P** and **Q** from the directrix.
23. Prove that the circumscribing circle of the triangle formed by three tangents to a parabola passes through the focus.
24.  $PQ$  is a chord of a parabola normal at **P**; the circle on  $PQ$  as diameter cuts the parabola again in **R**. Prove that the projection of  $QR$  on the axis is twice the latus rectum.
25. Prove that the distance between a tangent and the parallel normal is  $a \operatorname{cosec} \theta \sec^2 \theta$ , where  $\theta$  is the angle which either makes with the axis.
26. Prove that, if the normals at **P** and **Q** intersect on the curve, then  $PQ$  cuts the axis in a fixed point.
27. Prove that, if the normals at **P** and **Q** meet at the point **R**  $\begin{pmatrix} x_1, y_1 \end{pmatrix}$  on the parabola, and the tangents at **P** and **Q** meet at **T**, then

$$TP.TQ = \frac{1}{2} (x_1 - 8a) \sqrt{y_1^2 + 4a^2}. \quad (5.2.27)$$

28. Show that, in the last problem, as **R** moves along the parabola, the middle point of  $PQ$  always lies on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 2a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4a^2 \quad (5.2.28)$$

29. Prove that the area of the triangle formed by the tangents at the points  $t_1, t_2$ , and their chord of contact is

$$\frac{1}{2} a^2 (t_1 - t_2)^2 \quad (5.2.29)$$

30. Prove that the area of the triangle formed by three points  $t_1, t_2, t_3$  on the parabola is

$$a^2 (t_2 - t_3) (t_3 - t_1) (t_1 - t_2) \quad (5.2.30)$$

and that this is double the area of the triangle formed by the tangents at these points.

31. Prove that, if a line through any point  $P \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  making an angle  $\theta$  with the axis meets the parabola at **Q** and **R**, then

$$PQ.PR = (y_1^2 - 4ax_1) \operatorname{cosec}^2 \theta. \quad (5.2.31)$$

32. Two chords  $QR, Q_1R_1$  of a parabola meet at **O**, and the diameters bisecting them meet the

curve at  $\mathbf{P}$  and  $P_1$ . Prove that

$$QO.OR : Q_1O.OR_1 = SP : SP_1 \quad (5.2.32)$$

33. Show that, if  $\mathbf{P}$  is on the parabola, the length of the chord through  $\mathbf{P}$  that makes an angle  $\theta$  with the axis is

$$4a \sin(\alpha - \theta) \operatorname{cosec}^2 \theta \operatorname{cosec} \alpha \quad (5.2.33)$$

where  $\alpha$  is the inclination of the tangent at  $\mathbf{P}$  to the axis.

34. Show that the locus of the middle point of a chord which passes through the fixed point  $\begin{pmatrix} h \\ k \end{pmatrix}$  is the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - (2a \ k) \mathbf{x} + 2ah = 0 \quad (5.2.34)$$

35. A tangent to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4b(1 \ 0) \mathbf{x} = 0 \quad (5.2.35)$$

meets the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(1 \ 0) \mathbf{x} = 0 \quad (5.2.35)$$

at  $\mathbf{P}, \mathbf{Q}$ . Prove that the locus of the middle point of  $PQ$  is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 2a+b \end{pmatrix} \mathbf{x} - 4a^2(1 \ 0) \mathbf{x} = 0 \quad (5.2.35)$$

36. Prove that the polar of the focus of a parabola is the directrix.  
37. Prove that, if a chord of the parabola subtends a right angle at the vertex, the locus of its pole is

$$(1 \ 0) \mathbf{x} + 4a = 0 \quad (5.2.37)$$

38. Show that, if parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(1 \ 0) \mathbf{x} = 0 \quad (5.2.38)$$

are drawn corresponding to different values of  $a$ , the feet of the perpendiculars from a fixed point on its polar lines all lie on a circle passing through the point.

39. Prove that, if from a point  $\mathbf{Q} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  a perpendicular be drawn to the polar of  $\mathbf{Q}$  with regard to the parabola cutting it in  $\mathbf{R}$  and the axis in

$G$ , then

$$SG = SR = x_1 + a \quad (5.2.39)$$

40. Prove that, if the diameter through a point  $\mathbf{P}$  of a parabola meets any chord in  $\mathbf{O}$  and the tangents at the ends of the chord in  $\mathbf{T}, \mathbf{T}_1$ , then

$$PO^2 = PT.PT_1 \quad (5.2.40)$$

41.  $QQ_1$  is a chord of a parabola and  $TOR$  is a diameter which meets the tangent at  $\mathbf{Q}$  in  $\mathbf{T}$ , the curve in  $\mathbf{O}$  and  $QQ_1$  in  $\mathbf{R}$ . Prove that

$$TO : OR = QR : RQ_1 \quad (5.2.41)$$

42. Prove that if the normals at three points  $\mathbf{P}, \mathbf{Q}, \mathbf{R}$  on a parabola concur, then the points  $\mathbf{P}, \mathbf{Q}, \mathbf{R}$  and the vertex of the parabola are concyclic.  
43. Prove that in general, two members of the family of parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a(1 \ 0) \mathbf{x} = 4a^2 \quad (5.2.43)$$

, where  $a$  is the parameter specifying members of the family, pass through any assigned point of the plane, and that these two parabolas cut orthogonally at  $\mathbf{P}$ .

## 6 THE ELLIPSE

### 6.1 Properties

1. Find the length of the latus rectum, the eccentricity and the coordinates of the foci of the ellipses

a)  $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 36$

b)  $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 36$

c)  $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 8$

2. Find the equation of the ellipse whose foci are the points  $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix}$  and eccentricity  $\frac{1}{2}$ . What are the equations of the directrices?

3. Find the equation of the ellipse of eccentricity  $\frac{3}{4}$ , which has its centre at the point  $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$  and touches the axis of  $y$  at the origin. What is the length of its latus rectum?

4. An ellipse has the axis of  $y$  for directrix and its centre at the point  $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ . Find its equation if its eccentricity is  $\frac{3}{4}$ .

5. Find the equation of the ellipse which has a focus at  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ , corresponding directrix the line

$$(1 \ 0)\mathbf{x} = 6 \quad (6.1.5.1)$$

and eccentricity  $\frac{1}{2}$ . What are the lengths of its axes?

6. An ellipse has its centre at  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  and a focus at  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$  and its eccentricity is  $\frac{1}{2}$ . Find its equation.

7. Are the points  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$  inside or outside the ellipse

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{x} = 20 \quad (6.1.7.1)$$

8. Prove that the line  $y = x - 5$  touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \mathbf{x} = 144 \quad (6.1.8.1)$$

and find the coordinates of the point of contact.

9. The line

$$(2 \ 3)\mathbf{x} = c \quad (6.1.9.1)$$

touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 1 \quad (6.1.9.2)$$

Find the values of  $c$ .

10. Find the equation of the tangent to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 8 \quad (6.1.10.1)$$

at the point  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ , and also the equations of the two tangents perpendicular to this.

11. Find the equations of the tangents to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 2 \quad (6.1.11.1)$$

which makes an angle of  $60^\circ$  with the major axis.

12. Find at what point the line

$$(e \ 1)\mathbf{x} = a \quad (6.1.12.1)$$

touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = a^2(1 - e^2) \quad (6.1.12.2)$$

13. Prove that the normal at an end of the latus rectum meets the major axis at a distance  $ae^3$  from the centre.

14. Prove that the normal at an end  $L$  of the latus rectum meets the minor axis in  $g$ , and  $l$  is the projection of  $L$  on the minor axis, then

$$gl = a \quad (6.1.14.1)$$

15. Prove that the normal at  $\begin{pmatrix} x \\ y \end{pmatrix}$  divides the major axis into segments of lengths  $a - e^2x$  and  $a + e^2x$ .

16. Prove that if the normal at  $\mathbf{P}$  meets the major axis in  $\mathbf{G}$ , and the minor axis in  $\mathbf{G}_1$ , then

a)  $SG = eSP$ ;

b)  $PG : PG_1 = b^2 : a^2$

## 6.2 Pole and Polar

1. Find the polar of the point  $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$  with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 6 \quad (6.2.1.1)$$

2. Find the pole of the line

$$(3 \ 4)\mathbf{x} = 5 \quad (6.2.2.1)$$

with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 5 \quad (6.2.2.2)$$

3. Find the poles of the lines

$$(2 \ -1)\mathbf{x} = 1 \quad (6.2.3.1)$$

$$(1 \ 3)\mathbf{x} = 4 \quad (6.2.3.2)$$

with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 6, \quad (6.2.3.3)$$

and verify that the line joining the poles is the polar of the point of intersection of the lines.

4. Prove that the points  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  are conjugate with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 5 \quad (6.2.4.1)$$

5. Find the equation of a line through the point  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$  and conjugate to the line

$$(1 \ 1) \mathbf{x} = 1 \quad (6.2.5.1)$$

with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 2 \quad (6.2.5.2)$$

6. Prove that, if the polar of a point is at right angles to the line joining the point to the centre of an ellipse, the point must lie on one of the axes.
7. Prove that, if  $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$  is the pole of the normal at  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ , then

$$\frac{x_1 x_2}{a^4} = -\frac{y_1 y_2}{b^4} = \frac{1}{a^2 - b^2} \quad (6.2.7.1)$$

8. Prove that a directrix of an ellipse is the polar of the corresponding focus.

### 6.3 Eccentric Angles

The following problems refer to the ellipse whose equation is

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (8)$$

- Show that the eccentric angle of one end of a latus rectum is  $\cos^{-1} e$ .
- If  $\alpha$  is the eccentric angle of a point  $\mathbf{P}$  on an ellipse, what are the coordinates of the corresponding point  $\mathbf{Q}$  on the auxiliary circle? Write down the equations of the tangents at  $\mathbf{P}$  and  $\mathbf{Q}$  and show that they intersect on the major axis.
- Prove that, if  $CP$ ,  $CD$  are conjugate radii of an ellipse and  $\omega$  the angle between them, then  $\sin^2 \omega$  varies as  $CP^{-2} + CD^{-2}$ .
- Prove that if a parallelogram is inscribed in an ellipse, its sides are parallel to the conjugate diameters.

5. Prove that if  $QQ_1$  is a chord of an ellipse parallel to the tangent at  $\mathbf{P}$  the eccentric angles of  $\mathbf{Q}$  and  $\mathbf{Q}_1$  differ from the eccentric angle at  $\mathbf{P}$  by equal amounts.

6. Prove that the equation of the perpendicular bisector of the chord joining the points  $\mathbf{Q}$ ,  $\mathbf{Q}_1$  whose eccentric angles are  $\alpha + \beta$ ,  $\alpha - \beta$  is

$$(a \sec \alpha \ -b \operatorname{cosec} \alpha) \mathbf{x} = (a^2 - b^2) \cos \beta \quad (6.3.6.1)$$

and deduce the equation of the normal at the point whose eccentric angle is  $\alpha$ .

7. Prove that, if the same chord passes through a focus, then

$$\cos \beta = \pm e \cos \alpha \quad (6.3.7.1)$$

8. Prove that the equation of a focal chord parallel to the tangent at the point whose eccentric angle is  $\alpha$  is

$$\left( \frac{\cos \alpha}{a} \ \frac{\sin \alpha}{b} \right) \mathbf{x} = \pm e \cos \alpha \quad (6.3.8.1)$$

9. Prove that, if  $\alpha$  is variable and  $\beta$  constant, the chord joining the points whose eccentric angles are  $\alpha + \beta$  and  $\alpha - \beta$  touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \cos^2 \beta \quad (6.3.9.1)$$

and that the locus of the poles of the chord is

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \sec^2 \beta \quad (6.3.9.2)$$

10. Prove that the tangents at the points whose eccentric angles are  $\alpha$ ,  $\alpha + \frac{\pi}{2}$  intersect on the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = 2a^2 b^2 \quad (6.3.10.1)$$

11. Prove that, if  $\mathbf{P}$ ,  $\mathbf{Q}$  are corresponding points on an ellipse and its auxiliary circle and the normals at  $\mathbf{P}$ ,  $\mathbf{Q}$  intersect in  $R$ , then

$$CR = a + b \quad (6.3.11.1)$$

12. Prove that, if the line joining the ends of two equal conjugate radii of an ellipse passes through a focus the eccentricity is  $\frac{1}{\sqrt{2}}$ .
13. Prove that the chords that join the ends of



conjugate radii all touch the ellipse

$$2\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (6.3.13.1)$$

#### 6.4 Miscellaneous

1. The following problems refer to the ellipse whose equation is

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (6.4.1.1)$$

and that  $C$  is its centre and  $S, S_1$  its foci.

2. Prove that, if the tangent and normal at a point  $P$  on an ellipse meet the major axis in  $T, G$ , then the tangent from either end of the minor axis to the circle  $TPG$  is equal in length to half the major axis.
3. Show that if  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  are the coordinates of a point of intersection of the ellipses

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (6.4.3.1)$$

and

$$\mathbf{x}^T \begin{pmatrix} b_1^2 & 0 \\ 0 & a_1^2 \end{pmatrix} \mathbf{x} = a_1^2 b_1^2 \quad (6.4.3.2)$$

the equations of their common tangents are

$$\left( \pm \frac{x_1}{aa_1} \pm \frac{y_1}{bb_1} \right) \mathbf{x} = 1 \quad (6.4.3.3)$$

4. The normal at any point  $P$  of the ellipse meets the axis in  $G$ ; a point  $Q$  is taken in the tangent so that  $PQ = \lambda.PG$ , where  $\lambda$  is constant; prove that the locus of  $Q$  is the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \frac{a^2 + \lambda b^2}{a^2} \quad (6.4.4.1)$$

5. Prove that the line

$$\left( \frac{a}{k^2 - 1} \quad \frac{b}{2k} \right) \mathbf{x} + \frac{a^2 - b^2}{k^2 + 1} = 0 \quad (6.4.5.1)$$

is a normal to the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (6.4.5.2)$$

for all values of  $k$ .

6. Prove that the foot of the focal perpendicular on the normal at any point of an ellipse is at a distance from the centre equal to the difference between the semi-major axis and the focal

radius vector to the point at which the normal is drawn.

7. Prove that the locus of the poles of normal chords of the ellipse is the curve

$$\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2. \quad (6.4.7.1)$$

8.  $P$  is a point  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  on the ellipse and  $PS, PS_1$  meet the curve again in  $Q, R$ . Prove that the equation of  $QR$  is

$$\left( \frac{x_1}{a^2} - \frac{y_1}{b^2} \frac{1+e^2}{1-e^2} \right) \mathbf{x} + 1 = 0, \quad (6.4.8.1)$$

where  $e$  is the eccentricity.

9. Show that two parallel tangents to an ellipse are met by any other tangent in points which lie on conjugate diameters.
10. Prove that if  $CP, CD$  be any two conjugate semi-diameters of an ellipse and  $PF$  be drawn perpendicular to  $CD$  and produced both ways to  $E, E_1$  so that  $PE = PE_1 = CD$ , then  $CE.CE_1 = CS^2$ , where  $S$  is a focus.
11. Tangents are drawn from points on the ellipse to the circle

$$\|\mathbf{x}\| = r \quad (6.4.11.1)$$

, show that the chords of contact touch the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = r^4 \quad (6.4.11.2)$$

12. Tangents  $TP, TQ_1$  are drawn to an ellipse so that  $SP, S_1Q$  are parallel. Prove that  $CT$  is parallel to  $SP$  or  $S_1Q$ .
13. the normal to an ellipse at a point  $P$  passes through one end of the minor axis, and  $CD$  is the semidiameter conjugate to  $CP$ . The perpendicular from  $C$  to  $CD$  meets the auxiliary circle in  $E$ . Prove that  $DE$  is equal to half the distance between the directrices.
14. Prove that, if  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  be the middle point of a chord of the ellipse, the equation of the chord is

$$\left( \frac{x_1}{a^2} - \frac{y_1}{b^2} \right) \mathbf{x} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \quad (6.4.14.1)$$

15. If  $Q$  be the pole of the chord of the ellipse which is normal at a point  $P$ , and if  $CR$  drawn through the centre  $C$  perpendicular to  $CQ$  meet

the normal at  $P$  in  $R$ , prove that the locus of  $R$  is

$$\mathbf{x}^T \begin{pmatrix} a^6 & 0 \\ 0 & b^6 \end{pmatrix} \mathbf{x} = a^4 b^4 \quad (6.4.15.1)$$

16. Prove that, if the point  $P$  lies on the ellipse

$$\mathbf{x}^T \begin{pmatrix} b_1^2 & 0 \\ 0 & a_1^2 \end{pmatrix} \mathbf{x} = a_1^2 b_1^2 \quad (6.4.16.1)$$

its polar with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (6.4.16.2)$$

touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} a_1^2 b^4 & 0 \\ 0 & b_1^2 a^4 \end{pmatrix} \mathbf{x} = a^4 b^4 \quad (6.4.16.3)$$

17. Show that the polar with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^6 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (6.4.17.1)$$

of a point on the circle

$$\|\mathbf{x}\| = c \quad (6.4.17.2)$$

touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^4 & 0 \\ 0 & a^4 \end{pmatrix} \mathbf{x} = \frac{a^4 b^4}{c^2} \quad (6.4.17.3)$$

18. Prove that, if the tangents to an ellipse at  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  and  $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$  meet at  $\begin{pmatrix} x \\ y \end{pmatrix}$  and the normals at  $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ , then  $a^2 \xi = e^2 x x_1 x_2$  and  $b^2 \eta = -e^2 y y_1 y_2$ , where  $e$  is the eccentricity.

19. Show that, if  $\begin{pmatrix} x \\ y \end{pmatrix}$  is the middle point of a chord of the ellipse, and the tangents at the ends of the chord intersect in  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  and the normals in  $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ , then

$$\frac{a^2 x_2}{x_1} + \frac{b^2 y_2}{y_1} = (a^2 - b^2) \left( \frac{x x_1}{a^2} - \frac{y y_1}{b^2} \right). \quad (6.4.19.1)$$

20. Prove that, if the normals at two points  $P, Q$  on an ellipse intersect on the diameter that bisects  $PQ$ , then the two normals are at right angles.

21. Prove that if a chord of the ellipse subtends

a right angle at the centre then it touches the circle

$$\|\mathbf{x}\| = \frac{ab}{\sqrt{a^2 + b^2}} \quad (6.4.21.1)$$

22. The locus of middle points of chords of the ellipse which subtend a right angle at its centre is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{a^2 + b^2}{a^2 b^2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 \quad (6.4.22.1)$$

23. Show that the tangents at the extremities of all chords of the ellipse which subtend a right angle at the centre, intersect on the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^4 & 0 \\ 0 & a^4 \end{pmatrix} \mathbf{x} = a^2 b^2 (a^2 + b^2) \quad (6.4.23.1)$$

24. If  $P$  be any point on the ellipse whose axes are  $AA_1, BB_1$  and if the parallel lines  $AP, BQ$  be drawn,  $Q$  being on the ellipse;  $Q$  will be one extremity of the diameter conjugate to that drawn from  $P$ .

25. From a any point  $T$  on one of the equiconjugate diameters of a conic whose centre is  $O$ , tangents  $TP, TQ$  are drawn to the conic. Show that  $O, P, Q, T$  are concyclic.

26. Prove that, if two conjugate radii of an ellipse cut the director circle in  $T, T_1$  then  $TT_1$  touches the ellipse.

27. If  $PS P_1, QS Q_1$  be two focal chords and if  $PQ$  be parallel to the major axis, show that  $P_1 Q_1$  bisects the distance between  $S$  and the nearer directrix.

28. Tangents are drawn at the extremities of conjugate diameters of an ellipse, and meet in  $O$ . Prove that the perpendicular from  $O$  on the focal radius to a point of contact is half the minor axis.

29. Show that the area of the rectangle formed by two parallel tangents and the corresponding normals to an ellipse is never greater than half the square on the line joining the foci.

30. Prove that the angle between the normal and the central radius at a point on an ellipse is greatest when the point is the end of one of the equiconjugate diameters.

31.  $P, Q$  are two points on the ellipse, and  $PS, QS_1$  intersect on the curve, prove that the locus of

the pole of  $PQ$  is

$$\mathbf{x}^T \begin{pmatrix} (2a^2 - b^2)^2 & 0 \\ 0 & a^2 b^2 \end{pmatrix} \mathbf{x} = a^2 (2a^2 - b^2)^2 \quad (6.4.31.1)$$

32. Two conjugate diameters of an ellipse meet a fixed straight line

$$(l \ m) \mathbf{x} = 1 \quad (6.4.32.1)$$

in  $P$ ,  $Q$ , and the straight lines through  $P$ ,  $Q$  perpendicular to these diameters intersect in  $R$ ; prove that the locus of  $R$  is the straight line

$$(a^2 l \ b^2 m) \mathbf{x} = a^2 + b^2 \quad (6.4.32.2)$$

33. Prove that if  $\alpha$ ,  $\beta$  are the eccentric angles of two points  $P$ ,  $P_1$  on an ellipse such that the focal distances  $SP$ ,  $S_1 P_1$  are parallel, then

$$\tan \frac{\alpha}{2} : \tan \frac{\beta}{2} = 1 \pm e : 1 \mp e, \quad (6.4.33.1)$$

and  $PP_1$  touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^4 & 0 \\ 0 & a^4 \end{pmatrix} \mathbf{x} = a^2 b^4 \quad (6.4.33.2)$$

34. Prove that the square of the sum of two conjugate radii is greatest when the radii are equal.  
 35. Prove that, if on the inward normal to an ellipse at  $P$  a length  $PQ$  be taken equal to the conjugate radius  $CD$ , the locus of  $Q$  is a circle of radius  $a - b$ .  
 36. Prove that, if  $P$ ,  $Q$  are corresponding points on an ellipse and its auxiliary circle, and the normal at  $P$  to the ellipse meets the normal at  $Q$  to the circle in  $R$ , then the locus of  $R$  is a circle of radius  $a + b$ .  
 37. Prove that, if lines drawn from any point on an ellipse to the ends of a diameter  $PCP_1$  meet the conjugate diameter  $DCD_1$  in  $M$ ,  $M_1$ , then  $CM.CM_1 = CD^2$ .  
 38. Prove that, if an ellipse slides between two straight lines at right angles to one another, the locus of its centre is a circle.

## 6.5 Hyperbola

1. Show that the eccentricity of a rectangular hyperbola is  $\sqrt{2}$
2. Prove that the locus of the centre of a circle which touches two given non-concentric circles of unequal radii is an ellipse or a hyperbola.
3.  $ABC$  is a triangle in which  $C$  is a right angle. On  $CA$ ,  $CB$  points  $P$ ,  $Q$  are taken such that  $CP.CQ = CA.CB$ . Prove that the locus of the centroid of the triangle  $CPQ$  is a rectangular hyperbola.
4. Show that the tangents to a rectangular hyperbola at the extremities of its latera recta pass through the vertices of the conjugate hyperbola.
5.  $P$  is a point on a rectangular hyperbola whose centre is  $C$  and a line is drawn through  $C$  perpendicular to  $CP$ . Through  $Q$ , any point on the curve, lines are drawn parallel to the asymptotes meeting this line in  $L$ ,  $M$ , show that  $LPM$  is a right angle.
6. Prove that the intercept on any tangent to a hyperbola, made by its asymptotes, subtends a constant angle at either focus.
7. Prove that the line  $lx + my = n$  touches the rectangular hyperbola  $xy = c^2$ , if
 
$$n^2 = 4lmc^2$$
8. A tangent to a hyperbola of foci  $S$ ,  $S_1$  meets the asymptotes in  $L$ ,  $L_1$ . Prove that the points  $S$ ,  $L$ ,  $S_1$ ,  $L_1$  are concyclic.
9. The tangents at two points  $P$ ,  $P_1$  on a rectangular hyperbola meet an asymptote in  $L$ ,  $L_1$  and  $P$ ,  $P_1$  meet it in  $K$ . Prove that
 
$$LK = KL_1$$
10. Prove that conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.
11. Prove that the polars of a point with regard to two conjugate hyperbolas are parallel and equidistant from the centre.
12. Prove that in a rectangular hyperbola any chord  $PP_1$  subtends at the ends of a diameter  $AA_1$  angles which are either equal or supplementary.
13. Prove that if  $CP$ ,  $CD$  are conjugate radii of a hyperbola the orthocentre of the triangle  $CPD$  lies on the line  $ax = by$ .
14. Prove that, if the tangent at  $P$  to a rectangular hyperbola meets the asymptotes in  $L$ ,  $L_1$  and the normal at  $P$  meets the transverse axis in  $G$ , then  $LGL_1$  is a right angle.
15. Prove that if an ellipse and a hyperbola have the same foci they cut one another at right angles.
16. The perpendiculars from a point  $P$  to the axes meet them in  $M$ ,  $N$ , and the perpendicular

bisector of  $MN$  passes through a fixed point  $C$  on one of the axes. Prove that the locus of  $P$  is a rectangular hyperbola with centre at  $C$ .

17. Show that the locus of the middle point of a chord of the rectangular hyperbola  $xy = c^2$  of constant length  $2a$  is

$$(xy - c^2)(x^2 + y^2) = a^2 xy$$

18. Prove that, if the position of a point on a rectangular hyperbola is determined by the variable  $\theta$  where  $x = c \tan \theta$ ,  $y = c \cot \theta$ , the locus of the intersection of tangents at the points  $\theta$ ,  $\theta + \alpha$  is

$$4(c^2 - xy) = (x + y)^2 \tan^2 \alpha$$

$\alpha$  being a constant angle.

19. Tangents at right angles are drawn to a rectangular hyperbola and its conjugate. Show that they cut either asymptote in two points  $K$ ,  $K_1$  such that the rectangle  $CK.CK_1$  is equal to twice the square on the semi-axis, where  $C$  is the centre.
20. Prove that the line joining the feet of the perpendiculars drawn to a pair of conjugate diameters of a rectangular hyperbola from any point  $P$  on the hyperbola is parallel to the normal at  $P$ .
21. The normal at  $P_1$  on the hyperbola  $xy - c^2 = 0$  meets the curve again at  $P_2$ , the normal at  $P_2$  meets the curve again at  $P_3$  and so on. Prove that if  $y_1, y_2, \dots, y_{n+1}$  are the ordinates of these points respectively,

$$y_1^2 y_2 = y_2^2 y_3 = \dots = y_n^2 y_{n+1} = c^4.$$

22. If  $PN$  be the ordinate and  $PG$  the normal of a point  $P$  of a hyperbola, whose centre is  $C$ , and the tangent at  $P$  intersect the asymptotes at  $L$  and  $L_1$ , show that half the sum of  $CL$  and  $CL_1$  is the mean proportional between  $CN$  and  $CG$ .
23. At the point of intersection of the rectangular hyperbola  $xy = k^2$ , and of the parabola  $y^2 = 4ax$ , the tangents to the hyperbola and parabola make angles  $\theta$  and  $\phi$  respectively with the axis of  $x$ . Prove that

$$\tan \theta = -2 \tan \phi.$$

24. Prove that, if  $A, B, C$  are three points on a rectangular hyperbola, the curve passes through the orthocentre of the triangle  $ABC$ .
25. Prove that, in a rectangular hyperbola, the

product of the focal distances of a point is equal to the square of the distance of the point from the centre.

26. Prove that, if  $(c \tan \theta, c \cot \theta)$  and  $(c \tan \theta_1, c \cot \theta_1)$  are two points on the hyperbola  $xy = c^2$ , and  $\theta + \theta_1$  is constant, then the chord joining the points passes through a fixed point on the conjugate axis of the hyperbola.

27. Show that the point whose coordinates are

$$\frac{a}{2} \left( t + \frac{1}{t} \right), \frac{b}{2} \left( t - \frac{1}{t} \right)$$

lies on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . Prove that, if  $C$  is the centre of the hyperbola and  $S$  is either focus and if the tangent at the above point meets the asymptote  $\frac{x}{a} = \frac{y}{b}$  at  $X$  and meets the asymptote  $\frac{x}{a} = -\frac{y}{b}$  at  $Y$ , then

$$t = \frac{CX}{CS} = \frac{CS}{CY}$$

28. From any point on the normal at a given point  $A$  on a rectangular hyperbola the other three normals to the curve are drawn. Show that the centroid of the feet of these three normals lies on the diameter of the hyperbola parallel to the normal at  $A$ .
29. A circle is drawn passing through any point  $P$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  and through the ends  $A, A_1$  of the transverse axis. The ordinate  $NP$  is produced to meet the circle again in  $Q$ . Prove that, as the position of  $P$  on the hyperbola varies, the locus of  $Q$  is the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b_1^2} = 1$ , where  $b_1 = \frac{a^2}{b}$ .

## 7 POLAR COORDINATES

1. Find the polar equations of the curves whose cartesian equations are:
- $x^2 + y^2 + 2gx + 2hy + c = 0$
  - $x^2 + y^2 = ax$
  - $x^2 - y^2 = c^2$
  - $xy = \frac{c^2}{2}$
  - $x^3 - 8xy^2 = a^3$ .
2. Find the Cartesian equations corresponding to the following equations in polar coordinates:
- $r \sin \theta + a = 0$
  - $r = a \sin \theta + b \cos \theta$
  - $r^2 \cos 2\theta = a^2$
  - $\frac{1}{r} = 1 + e \cos \theta$

- e)  $r^3 \sin 3\theta = a^3$   
 f)  $r^2 = a^2 + b^2 \cos 2\theta$

- Using the polar equation of a conic with focus as pole, show that the semi-latus rectum is a harmonic mean between the segments of a focal chord.
- Draw the curve  $r = a \cos 2\theta$ , noting carefully when  $r$  changes sign and showing that the curve forms a double figure of eight.
- Draw the curve

$$r = a(2 + \cos \theta)$$

- Draw the curve  $r = a(1 + 2 \cos \theta)$ , showing that it consists of two loops one within the other.
- Deduce from the polar equations of the curves the pedal equations for the circle, parabola with focus as pole, ellipse with focus as pole and ellipse with centre as pole.
- Find the pedal equation of:
  - The rectangular hyperbola  $xy = c^2$
  - the lemniscate  $r^2 = a^2 \cos 2\theta$
  - the curve  $r^n = a^n \cos n\theta$
  - the spiral  $r\theta = a$ .
- Prove that, for the curve  $r\theta = a$ , if a perpendicular through the pole be drawn to the radius vector at any point, the length intercepted on it by the tangent is constant.
- Prove that, for the curve  $r = a \cos(\theta - \alpha)$ ,  $\phi = \frac{\pi}{2} + \theta - \alpha$ .
- Show that, for the curve  $r = a(1 + \cos \theta)$ , the locus of the foot of the perpendicular from the pole to the tangent is  $r = 2a \cos^3 \frac{\theta}{3}$ .
- Prove that, for the curve  $r = a \sec^3 \frac{\theta}{3}$ , the locus of the foot of the perpendicular from the pole to the tangent is a parabola.
- Trace the curve  $r^2 = a^2 \cos \theta$ .
- Prove that the pedal equation of the curve  $r \cos n\theta = a$  is

$$\frac{1}{p^2} = \frac{1 - n^2}{r^2} + \frac{n^2}{a^2}$$

## 8 EASY PROBLEMS

### 8.1 The Straight Line

- Find the distance between the points  $A$ ,  $B$  whose coordinates are  $(3, 7)$  and  $(11, 13)$ . Also find the coordinates of the point which divides  $AB$  in the ratio  $3 : 1$ .
- Find the equations of the sides of a triangle whose corners are at the points  $(3, 4)$ ,  $(-3, 2)$ ,  $(2, -4)$ .
- In the last problem, find the equations of the perpendiculars from the corners of the triangle to the opposite sides and verify that they meet in a point.
- Find the distance between the points  $A(2, 3)$  and  $B(14, 8)$ . Find also the equation of the line  $AB$  and the distance from it of the point  $C(5, 6)$ . What is the area of the triangle  $ABC$ ?
- The equations of the sides of a triangle are  $x + y = 5$ ,  $2x - y + 4 = 0$  and  $x - 4y + 4 = 0$ . Find the tangents of the angles of the triangle.
- Find the equation of a line parallel to the line  $5x + 4y = 9$  and making an intercept  $-5$  on the  $x$ -axis.
- Find the equations of two straight lines which pass through the intersection of the lines  $3x + 4y = 7$ ,  $5x - 2y = 3$  and are parallel and perpendicular respectively to the line  $2x + 7y = 3$ .
- Find the value of  $k$  if the three lines  $2x + 5y = 12$ ,  $7x + 2y = 11$  and  $kx - 3y = 10$  meet in a point.
- Prove that the points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $\left(\frac{x_1 + kx_2}{1 + k}, \frac{y_1 + ky_2}{1 + k}\right)$  are collinear.
- Show that the points  $(2, 3)$  and  $(-1, 2)$  are on opposite sides of the line  $2x - 3y = 7$ . What are their distances from the line?
- Two straight lines  $AB$ ,  $CD$  bisect one another at right angles. The coordinates of  $A$ ,  $B$  and  $C$  are  $(1, 5)$ ,  $(3, 1)$  and  $(-1, 1\frac{1}{2})$ . Find the coordinates of  $D$  and the area  $ABCD$ .
- Prove that the line  $2x + 3y = 5$  divides the join of the points  $(3, -5)$ ,  $(2, 1)$  in the ratio  $7 : 1$ .
- Find the equations of the two straight lines which pass through the point  $(3, 2)$  and make angles of  $45^\circ$  with the line  $4x - 5y = 6$ .
- Find the equation of the perpendicular bisector of the join of the points  $(3, 5)$ ,  $(-6, 7)$ .
- From a point  $A(2, 5)$  a perpendicular  $AB$  is drawn to the line  $3x - 4y = 8$ .  $AB$  is then produced to  $C$  so that  $BC = 3AB$ . Find the coordinates of  $C$ .
- The sides  $BC$ ,  $CA$ ,  $AB$  of a triangle have equations  $3x + 4y = 9$ ,  $4x - 3y + 6 = 0$ ,

$5x - 12y + 12 = 0$ . Find the equations of the bisectors of the interior angles of the triangle and the coordinates of their point of concurrence.

17.  $ABC$  and  $DBC$  are two triangles such that  $DB = AB$  and  $DC = AC$ . The coordinates of  $A, B, C$  are  $(3, 1), (2, 2), (-1, 3)$ . Find the coordinates of  $D$ . Also find the equations of  $AD$  and  $BC$  and verify that these lines are at right angles.
18. Find the centre of the inscribed circle of the triangle formed by the lines  $3x - 4y + 8 = 0$ ,  $4x + 3y - 9 = 0$ ,  $y = 6$ .
19. Find the equations of two lines passing through the intersection of the lines  $x + 3y = 5$  and  $4x - y + 2 = 0$ , and perpendicular and parallel respectively to the line  $5x - 3y + 3 = 0$ .
20. Find the equation of the line which passes through the intersection of the lines  $2x + 3y = 4$  and  $3x - y + 2 = 0$  and also through the intersection of the lines  $x + y = 0$ ,  $4x - y + 3 = 0$ .
21. Prove that the points  $A(4, 2)$  and  $C(-3, 1)$  subtend a right angle at  $B(-\frac{17}{13}, -\frac{20}{13})$ . Find the coordinates of  $D$ , the remaining corner of the rectangle  $ABCD$ , and verify that  $DB = AC$ .
22. The points  $A(4, 2)$ ,  $B(1, -2)$ ,  $C(-3, 1)$  and  $D$  are the corners of a parallelogram. Find the coordinates of  $D$  and prove that the parallelogram is equal in area to the square on  $OD$  where  $O$  is the origin.
23. Show that the equation  $6x^2 + 7xy - 3y^2 + x + 7y - 2 = 0$  represents two straight lines, and find the tangents of the angles between them.
24. Find the separate equations of the two straight lines represented by the equation  $x^2 - 2xy \csc \theta + y^2 = 0$ . Also find the angles between the lines, and show that these angles are bisected by the lines  $x^2 - y^2 = 0$ .

## 8.2 Curves and Loci

1. Find the equations of the tangents to the curves  $y = x^3 - x$  and  $x^2 - 3x + 2$  at their point of intersection, and the angle at which the curves cut.
2. Prove that the line  $3y = 9x - 16$  is a tangent to the curve  $3y = x^3 - 3x$ , and find the coordinates of the point of contact.
3. Prove that the tangents to the curve  $y = x^3 - 6x^2 + 13x - 6$  at the points  $(1, 2)$  and  $(3, 6)$  are

parallel, and that the tangent at the point  $(2, 4)$  is inclined to them at an angle  $\tan^{-1} \frac{3}{5}$ .

4. Prove that the line  $y = 5x$  is a tangent to the curve

$$y = x^3 - 2x^2 + x + 8$$

and find the coordinates of the point of contact. In what other point does the line  $y = 5x$  meet the curve?

5. Find the equation of the tangent to the curve  $y = 3x^2 - 2x - 1$  at the point  $(2, 7)$ . Also find the coordinates of the point on the curve the tangent at which is perpendicular to the tangent at  $(2, 7)$ .
6. Find the equation of the normal to the curve  $y = 2x^2 - 3x - 5$  at the point  $(2, -3)$ , and find the length of the subnormal at this point.
7. Find the equation of the tangent to the curve

$$3y = x^3 - 6x^2 - 12x + 1$$

at the point  $(-1, 2)$ , and find the coordinates of another point on the curve where the tangent is parallel to that at  $(-1, 2)$ .

8. Prove that at every point of the curve  $y = x^3 - 3x^2 + 3x + 7$  the gradient is positive. Table the values of  $y$  and  $\frac{dy}{dx}$  at the points  $x = -1, 0, 1, 2, 3$  and make a rough drawing of the curve.
9. Find the equation of the tangent and normal to the curve  $y = (x - 1)^4$  at the point  $(2, 1)$ . Find the area of the triangle which the tangent and normal make with the  $x$ -axis.
10. Show that the curve  $y = 1 - 2x + x^3$  cuts the  $x$ -axis in three real points. Find the gradient of the curve at the point  $(1, 0)$ ; also find the points at which the gradient vanishes and make a rough drawing of the curve.
11. Find the gradients of the curve  $y = (x - 1)^2(x - 2)$  at the points where it crosses the axes. Find also the points at which the gradient vanishes and make a rough drawing of the curve.
12. Find the points on the curve  $y = x^3 - x^2 + 1$  at which the gradient is unity. Find the equations of the tangents and normals at these points.
13. Find the points in which the curve  $y = 1 + 2x - 3x^2$  cuts the coordinate axes and find the gradients of the curve at these points. At what point does the gradient vanish? Make a rough drawing of the curve.

14. Show that the curve  $y = (x^2 - 1)^2$  is symmetrical about the  $y$ -axis. Find its gradient and the points at which the gradient vanishes. In what points does the line  $y = 9$  cut the curve? Make a rough drawing of the curve.
15. Show that the curve  $y = x^4 - 4x^2$  is symmetrical about the  $y$ -axis. Find the gradients of the curve at its intersections with the  $x$ -axis. find also the points where the gradient vanishes and make a rough drawing of the curve.
16. Show that the curve  $y = x^4 - 4x^2$  is symmetrical about the  $y$ -axis. Find the gradients of the curve at its intersections with the  $x$ -axis. Find also the points where the gradient vanishes and make a rough drawing of the curve.
17. Find the gradient of the curve  $y = x^2 - 3x - 4$ , and the point on the curve at which the gradient vanishes. Also find the gradient at the points where the curve crosses the  $x$ -axis and make a rough drawing of the curve. To what point on the  $x$ -axis must the origin be moved so that the curve may become symmetrical about the  $y$ -axis, and what will be the equation of the curve when the position of the origin is so changed?
18. Show that the curve whose equation is  $27y = x^3 - 27x$  is symmetrical in opposite quadrants. Prove that the tangent to the curve at the origin is  $x + y = 0$ . Find the gradient at the other intersections with the  $x$ -axis. Also find at what points the tangent is parallel to the  $x$ -axis, and make a rough drawing of the curve.
19. Find the equation of the locus of a point which moves so that its distance from the point  $(2, 1)$  is twice its distance from the line  $3x + 4y = 2$ .
20. Find the locus of a point which moves so that its distance from the origin is half the distance from the line  $3x + 4y = 2$ .
21. A point  $P$  moves so that  $PA^2 + PB^2 = AB^2$ , where  $A, B$  are the points  $(2, 0), (0, 3)$ . Find the equation of the locus of  $P$ .
22. Find the locus of a point whose distance from the point  $A(3, 0)$  is three times its distance from the point  $B(-1, 0)$ . Prove that the tangents to the locus at the points  $(0, 0)$  and  $(-3, 0)$  are parallel to the  $y$ -axis.
23. The coordinates of a point are given in terms of a variable parameter  $t$  by the relations  $x = \frac{t}{t+1}, y = \frac{t}{t-1}$ . Find the equation of the locus of the point, and prove that the equation of the

tangent at any point  $t$  on the locus is  $(t + 1)^2 x + (t - 1)^2 y = 2t^2$ .

24. The coordinates of a point are given by the relations  $x = t + \frac{1}{t}, y = t - \frac{1}{t}$  and  $t$  is a variable parameter. Find the equation of the locus of the point for different values of  $t$ , and also the equations of the tangent at the point denoted by  $t$ .

### 8.3 The Circle

1. Prove that the circle of centre  $(3, 4)$  and radius 5 passes through the origin. Find the equations of the tangent at the origin and of the parallel tangent. IN what other points does the circle cut the axes?
2. find the equation of a circle having the points  $(2, 1)$  and  $(1, -3)$  for the ends of a diameter. find the coordinates of the ends of the perpendicular diameter.
3. Find the equation of the tangent to the circle  $x^2 + y^2 = 1$  at the point  $(\frac{4}{5}, \frac{3}{5})$ . Find the two points in which this tangent cuts the axes and also find the equations of the other tangents to the circle that can be drawn from the last two points.
4. For what values of  $k$  does the line  $y = kx$  touch the circle  $x^2 + y^2 + 2x + 4y + 1 = 0$ ? find the coordinates of the points of contact.
5. Show that the points  $(5, 8)$  and  $(-1, 0)$  are the opposite ends of a diameter of the circle  $x^2 + y^2 - 4x - 8y - 5 = 0$ . Find the equations of the tangents at these points.
6. Find the equation of the circle through the points  $(3, 0), (4, 0)$  and  $(1, 2)$ . Find the equation of the tangent at the point  $(3, 13)$ .
7. Find the equation of the circle which passes through the points  $(3, 0), (0, 4), (0, 6)$ . In what other points does the circle cut the  $x$ -axis?
8. Find where the line  $x + y = 5$  cuts the circle

$$3x^2 + 3y^2 - 11x - 11y + 16 = 0? \quad (8.3.8.1)$$

What are the equations of the tangents parallel to the line?

9. Find the coordinates of the points in which the line  $x + 2y = 7$  cuts the circle  $x^2 + y^2 - 13x - 13y + 52 = 0$ . Find the equations of the tangents at these points and the coordinates of the point in which these tangents intersect.

10. Prove that if  $(x_1, y_1)$  is the middle point of a chord of a circle  $x^2 + y^2 = a^2$ , then the equation of the chord is  $xx_1 + yy_1 = x_1^2 + y_1^2$ .
11. Find the coordinates of the middle point of the chord of the circle  $x^2 + y^2 = 25$  which lies along the line  $3x - 4y = 7$ . Also find the points of contact of the parallel tangents.
12. Find the coordinates of the middle point of the chord of the circle  $x^2 + y^2 - 4x + 6y + 1 = 0$  which lies along the line  $2x - 3y = 12$ .
13. Prove that the line  $2x + y - 7 = 0$  is the equation of a diameter of the circle  $x^2 + y^2 - 6x - 2y + 5 = 0$ , and find the equations of the tangents at the ends of this diameter.
14. Find the angle which the circle  $x^2 + y^2 - 6x - 4y + 9 = 0$  subtends at the origin.
15. Find the equations of the circles which pass through the point  $(-2, 1)$  and touch both the coordinate axes.
16. Prove that the locus of points at which the circle

$$x^2 + y^2 - 4x - 6y + 4 = 0 \quad (8.3.16.1)$$

subtends a right angle is the circle  $x^2 + y^2 - 4x - 6y - 5 = 0$ .

17. Find the equations of the tangents to the circle

$$x^2 + y^2 - 4x + 2y - 8 = 0 \quad (8.3.17.1)$$

which are parallel to the line  $2x + 3y = c$ .

18. Find the equation of the chord of the circle

$$x^2 + y^2 - 4x + 2y - 4 = 0 \quad (8.3.18.1)$$

which has  $(3, 1)$  as its middle point.

19. Show that the points  $(0, 0)$ ,  $(2, 1)$ ,  $(3, 3)$  and  $(1, 2)$  are the corners of a rhombus, and find the equation of its inscribed circle.
20. Find the equation of the circle which has the same centre as the circle  $x^2 + y^2 + 6x - 10y + 18 = 0$  and passes through the point  $(6, 7)$ . Show that the origin is outside the given circle and inside the other.
21. Find the locus of the centre of a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (8.3.21.1)$$

which is such that the tangents drawn to the circle from the origin are always at right angles.

#### 8.4 The Parabola

1. Find the value of  $c$  if the parabola  $y^2 = 4ax$  intercepts a length  $4a$  on the line  $y = x + c$ .
2. What is the equation of a parabola which is symmetrical about the  $x$ -axis, touches the  $y$ -axis at the origin and has a latus rectum of length 8? What are the coordinates of the focus? What are the coordinates of a point on the curve at a distance 20 from the focus?
3. The parabola  $y^2 = 4ax$  passes through the point  $P(6, 6)$ . If  $S$  is the focus find the coordinates of the other point in which  $PS$  meets the curve.
4. Find the value of  $a$  if the parabola  $y^2 = 4ax$  touches the line  $y = 3x + 4$ . What are the coordinates of the point of contact?
5. Prove that the line  $ty = x + at^2$  touches the parabola  $y^2 = 4ax$ , and find the coordinates of the point of contact. Also find the coordinates of the foot of the perpendicular to this tangent from the focus of the parabola.
6. Prove that, if  $P$  is a point on a parabola  $y^2 = 4ax$  whose vertex is  $A$ , and  $PL$  at right angles to  $AP$  meets the  $x$ -axis in  $L$ , and  $PN$  is the ordinate of  $P$ , then  $NL = 4a$ .
7.  $PN$  is the ordinate at a point  $P$  on a parabola and the normal at  $P$  meets the axis in  $G$ . Prove that  $PG$  is equal to the ordinate that passes through the middle point of  $NG$ .
8. Find the locus of a point  $P$  which moves so that its distance from the point  $(1, 0)$  is equal to its distance from the line  $x + 1 = 0$ . Also find the coordinates of the middle point of the chord of this locus which lies along the line  $3y = 2x + 4$ .
9. The normal at  $P$  to a parabola meets the axis in  $G$  and  $S$  is the focus. Prove that, if the triangle  $SPG$  is equilateral, then  $SP$  is equal to the latus rectum.
10.  $A$  is the vertex of a parabola  $y^2 = 4ax$  and  $LL_1$  is the latus rectum. Prove that the diameter of the circle  $LAL_1$  is  $5a$ .
11. The chord joining the points  $(x_1, y_1)$ ,  $(x_2, y_2)$  on the parabola  $y^2 = 4ax$  cuts the axis at  $C$ . Prove that, if  $A$  is the vertex,  $x_1x_2 = AC^2$ .
12.  $C$  is a fixed point  $(0, c)$  on the axis  $Oy$  and  $Q$  is a variable point on the line through  $C$  parallel to  $Ox$ . A point  $P$  is taken on  $OQ$  so that the ordinate of  $P$  is equal to  $CQ$ . Prove that the locus of  $P$  is the parabola  $y^2 = cx$ .



13. the chord  $PQ$  of a parabola passes through the focus. If  $P$  is the point  $(at^2, 2at)$ , what are the coordinates of  $Q$ ?
14. The chord  $PQ$  of a parabola is the normal at  $P$ . If  $P$  is the point  $(at^2, 2at)$ , what are the coordinates of  $Q$ ?
15. Find the value of  $a$  if the parabola  $y^2 = 4ax$  touches the line  $3y = x + 8$ . Find the equation of the normal at the point of contact, and the coordinates of the second point of intersection of the normal and the curve.
16. Find the equations of the tangents to the parabola  $y^2 = 16x$  at the points  $(36, 24)$ ,  $(\frac{4}{9}, -\frac{8}{3})$  and verify that they intersect on the directrix.
17. Prove that the line  $x - 6y + 36 = 0$  touches the parabola  $y^2 = 4x$  and find the coordinates of the point of contact. Find also the coordinates of the foot of the perpendicular drawn to this tangent from the focus of the parabola.
18. Prove that the lines  $wy = 6x + 1$  and  $8y = 32x + 3$  both touch the parabola  $y^2 = 6x$ , and find the equation of the line joining the points of contact.
19. Find the value of  $c$  if the parabola  $y^2 = 8x$  intercepts a length 8 on the line  $y = x + c$ .
20. Points on a parabola are represented parametrically by the relations  $x = at^2$ ,  $y = 2at$ , where  $t$  is a variable parameter. Normals are drawn at the points  $t = 2$  and  $t = 1$ . Prove that they intersect on the curve.
21. Prove that, if the points  $(x_2, y_1)$ ,  $x_2, y_2$  are the ends of a chord passing through the focus of a parabola  $y^2 = 4ax$ , then  $x_1x_2 = a^2$  and  $y_1y_2 = -4a^2$ .
22. From the focus  $S$  of a parabola a line is drawn parallel to the tangent at  $P(at^2, 2at)$  meeting the line  $y = 2at$  in  $Q$ . Prove that the locus of  $Q$  is the parabola  $y^2 = 2a(x - a)$ .
23. Prove that the tangents and normal to a parabola at the points  $(at^2, 2at)$ ,  $(\frac{a}{t^2}, \frac{-2a}{t})$  enclose a rectangle of area  $a^2(\frac{t+1}{t^2})$ .
3. Find the equation of the ellipse which touches the line  $2x + 3y = 9$  and has the points  $(-2, 0)$ ,  $(2, 0)$  as its foci. Find also the coordinates of the point of contact of the line and the ellipse.
4. Show that the distance of the point  $(a \cos \theta, b \sin \theta)$  from the focus  $(ae, 0)$  of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $a(1 - e \cos \theta)$ .
5. Prove that if the normals at  $P(6, 4)$  and  $Q(-8, 3)$  on the ellipse  $\frac{x^2}{100} + \frac{y^2}{25} = 1$  meet at  $G$ , then the diameter through  $G$  is perpendicular to  $PA$ .
6. Find the equation of an ellipse of eccentricity  $\frac{1}{2}$  which has a focus at  $(3, 0)$  and  $x = 1$  for corresponding directrix.
7. Find the equations of the tangents to the ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$  which are parallel to the line  $x = 2y$ .
8. Find the equation of the ellipse of eccentricity  $\frac{1}{2}$  which has its foci at the points  $(-1, 0)$ ,  $(1, 0)$ . Find also the length of the latus rectum and verify that the tangent at either end of the latus rectum cuts the major axis on the directrix.
9. Find the equation of an ellipse which has the point  $(2, 3)$  as an end of a latus rectum and its axes along the coordinate axes. At what point does the line  $x - 2y = 8$  touch the ellipse?
10. Find the equation of an ellipse of eccentricity 0.8 which has its centre at the origin and the lines  $x = \pm 25$  as directrices. Verify that the ellipse touches the line  $9x + 20y = 300$ .
11. The axes of an ellipse are the coordinate axes, its directrices pass through the points  $(\pm 5\frac{1}{3}, 0)$  and it touches the line  $3x + 4y = 16$ . Find its equation.
12. The major and minor axes of an ellipse lie along the lines  $3x - 4y + 6 = 0$  and  $4x + 3y - 17 = 0$  and the lengths of the semi-axes are 5 and 4. Find the eccentricity and the coordinates of the centre and foci.
13. Find the equation of an ellipse which has its axes along the coordinate axes and the line  $3x - 2y = 5$  as the normal at the point  $(15, 20)$ .
14. Prove that, if the tangent at an end of the minor axis of an ellipse cuts the latus rectum produced in  $D$ , and  $C$  is the centre, then a perpendicular to  $CD$  through  $D$  cuts the major axis on the directrix.
15. Prove that the tangent at the ends of the latera recta of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  form a quadrilateral of area  $\frac{2a^2}{e}$ , where  $e$  is the eccentricity.

### 8.5 The Ellipse

1. An ellipse has its foci at the points  $(2, 0)$ ,  $(-2, 0)$  and passes through the point  $(2, 3)$ . Find its equation.
2. Prove that the line  $x + 2y = 8$  touches the ellipse  $3x^2 + 4y^2 = 48$ , and find the coordinates of the point of contact.

16. Prove that, if a series of ellipses have the same major axis, the tangents at the ends of the latera recta pass through one or other of the two fixed points on the minor axis.
17. Find the equations of the tangents to the ellipse  $4x^2 + 9y^2 = 180$  at the points  $P(6, 2)$  and  $P_1(-6, -2)$ . Find also the equations of the tangents that are parallel to the line  $PP_1$ , and the coordinates of their points of contact.
18. Prove that the line  $x+3y = 9$  touches the ellipse  $\frac{x^2}{9} + \frac{y^2}{8} = 1$ , and find the coordinates of the point of contact. Find the coordinates of the foci of the ellipse, and verify that the product of the distances of the foci from the above tangent is equal to the square on the minor axis.
19. An ellipse has its foci at the point  $(\pm 3, 0)$  and passes through the point  $P(2, 2\sqrt{6})$ . Prove that its eccentricity is  $\frac{1}{2}$  and that the normal at  $P$  passes through the point  $(\frac{1}{2}, 0)$ .
20. An ellipse has a focus at the point  $(3, 0)$ , the  $y$ -axis is the corresponding directrix and the point  $(6, 4)$  lies on the curve. Prove that the axes are in the ratio  $6 : \sqrt{11}$ .
21. Prove that, if the line  $lx+my+n = 0$  is a normal to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , then  $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)}{n^2}$ .
22. Find the coordinates of the points on the ellipse  $8x^2 + 25y^2 = 200$  at which the normals make angles of  $60^\circ$  with the major axis.
23. Find the values of  $c$  for which the line  $5x-2y = c$  is normal to the ellipse  $x^2 = 5y^2 = 9$ .
24. Find the coordinates of the four points on the ellipse

$$9x^2 + 16y^2 = 1 \quad (8.5.24.1)$$

the tangents at which are equally inclined to the coordinate axes; and prove that the normals at these points form a square of area  $\frac{49}{1800}$ .

25. Find the equation of the normal at the point  $(2, 3)$  on the ellipse  $3x^2 + 4y^2 = 48$  and the coordinates of the point in which the normal again cuts the curve. Show that the middle point of this normal chord is at a distance  $\frac{\sqrt{73}}{19}$  from the centre of the ellipse.
26. Find the equation of an ellipse of eccentricity  $\frac{1}{3}$  which touches the line  $2x + 3y = 5$  and has its axes along the coordinate axes. find the coordinates of the point of contact of the ellipse with the given line.
27. Find the equations of the two ellipses which have their axes along the coordinate axes, pass through the point  $(2, 1)$  and touch the line  $6x + 12y = 25$ .
28. Prove that, if an ordinate  $NP$  to an ellipse is produced to meet the tangent at the end of the latus rectum in  $Q$ , then  $QN = SP$ , where  $S$  is the corresponding focus.
29. The tangent is drawn at the point  $P(2, 1)$  on the ellipse  $4x^2 + 9y^2 = 25$  whose centre is  $O$ , and the diameter  $DOD_1$  of the ellipse is parallel to the tangent at  $P$ . Find the coordinates of  $D$  and  $D_1$  and prove that the tangents at these points are parallel to the radius  $OP$ .
30. The tangent at  $P$  to an ellipse meets a directrix in  $T$  and  $S$  is the corresponding focus. Prove that  $PST$  is a right angle.
31. The foci of an ellipse are the points  $(0, 0)$  and  $(8, 6)$  and the eccentricity is  $\frac{4}{5}$ . Find the coordinates of the centre and the equations and lengths of the major and minor axes. Find also the equations of the directrices.
32. Show that, if the normal at a point  $P(x_1, y_1)$  on an ellipse of focus  $S$  and eccentricity  $e$  meets the major axis in  $G$  and  $GL$  is the perpendicular to  $SP$ , then  $GL = ey_1$  and  $PL =$  the semi-latus rectum.
33.  $P$  denotes any point on an ellipse of which the major axis is  $AA_1$ . Prove that, if  $AP$ ,  $A_1P$  cut the minor axis in  $M$ ,  $M_1$ , then the tangent at  $P$  bisects  $MM_1$ .

## 8.6 The Hyperbola

1. Prove that, if a variable line moves with its ends on the coordinate axes so as to enclose with them a constant area, then the locus of the middle point of the line is a rectangular hyperbola.
2. Prove that, if a straight line moves with its ends on the coordinate axes so as to form with them a triangle of constant area  $c^2$ , then the line touches the rectangular hyperbola  $2xy = c^2$ .
3. Find the value of  $c^2$  if the rectangular hyperbola  $xy = c^2$  touches the line  $3x + 5y = 80$ .
4. Show that the equation of the tangent at  $(x_1, y_1)$  on the rectangular hyperbola  $xy = c^2$  may be put in the form  $\frac{x}{x_1} + \frac{y}{y_1} = 2$ .
5. Prove that the line  $3x + 4y = 24c$  touches the hyperbola  $xy = 12c^2$ . What is the area of the

triangle whose sides are the asymptotes and this tangent?

6. Find the points in which the line  $2x + y = 3c$  cuts the hyperbola  $xy = c^2$ , showing that one of the points of intersection lies on the transverse axis of the hyperbola.
7. Prove that, if  $a$  and  $b$  are real numbers of opposite signs, the straight line  $ax + by = 1$  cannot touch the rectangular hyperbola  $xy = c^2$ .
8. Prove that the straight line  $x + t^2y = 2ct$  is a tangent to the rectangular hyperbola  $xy = c^2$ , and that no perpendicular line can touch the curve.
9. Prove that, if through any point on the curve  $xy = c^2$  two perpendicular lines are drawn, their other intersections with the curve will lie on opposite branches.
10. The normal at  $P(3, 4)$  on the rectangular hyperbola  $xy = 12$  meets the curve again at  $Q$ . Prove that  $PQ = \frac{125}{12}$ .
11. Find the equations of the tangent and normal at the point  $x = 4ct$ ,  $y = \frac{3c}{t}$  on the rectangular hyperbola  $xy = 12c^2$ . Verify that the length of the tangent intercepted by the asymptotes is bisected at the point of contact.
12. Find the equation of the tangent at the point  $x = 2ct$ ,  $y = \frac{5c}{t}$  on the hyperbola  $xy = 10c^2$ , and show that the area of the triangle the tangent forms with the asymptotes is independent of  $t$ .
13. Find the equation of the chord joining the points  $(ct_1, \frac{c}{t_1})$  and  $(ct_2, \frac{c}{t_2})$  on the rectangular hyperbola  $xy = c^2$ . Find the equation of the tangent parallel to the chord, and the coordinates of its point of contact.
14. Show that the hyperbola  $x^2 - y^2 = 20$  and  $xy = 24$  cut at right angles at all the common points.
15. If the line  $2x - ky = 8$  touches a rectangular hyperbola  $xy = 9$ , what is the value of  $k$ , and what are the coordinates of the point of contact?
16. Find the coordinates of the foci and the equations of the directrices of the hyperbola  $xy = c^2$ , referred to the asymptotes as axes.
17. A circle cuts the rectangular hyperbola  $xy = 1$  in the points  $(x_r, y_r)$ ,  $r = 1, 2, 3, 4$ . Prove that  $x_1x_2x_3x_4 = y_1y_2y_3y_4 = 1$ .
18. Find the points of intersection of the line  $8y - 2x = 15c$  and the rectangular hyperbola  $xy = c^2$ . Prove that the line is a normal to the curve

at one of these intersections.

19. Prove that the normal at the point  $P(ct, \frac{c}{t})$  on the rectangular hyperbola  $xy = c^2$  meets the curve again at  $Q$  so that

$$PQ = c \left( t^2 + \frac{1}{t^2} \right)^{\frac{3}{2}} \quad (8.6.19.1)$$

20. Four points are taken on a rectangular hyperbola  $xy = c^2$ . Find the condition that the chord joining two of the points should be perpendicular to the chord joining the other two. Prove that, if the condition is satisfied for one pair of such chords, then it is true for all three pairs.
21. Prove that the locus of the middle points of chords of the rectangular hyperbola  $xy = c^2$  which pass through the point  $(2c, 2c)$  is an equal hyperbola and find the coordinates of its vertices.
22. The radius  $OP$  from the centre to a point  $P$  on the rectangular hyperbola  $xy = c^2$  makes an angle  $\theta$  with the  $x$ -axis, and the normal at  $P$  cuts the axes of  $x$  and  $y$  in  $G$  and  $G_1$ . Prove that  $\frac{PG}{PG_1} = \tan^2 \theta$ .