Solutions to Plane Coordinate Geometry by S L Loney

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***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1 Pair of Straight Lines

1.1 13

1.1.1. Prove that the following equations represent two straight lines; and also find their point of intersection and the angle between them

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

Solution: Proving that given equation represents two straight lines The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \quad (1.1.1.1)$$

Comparing this to the standard equation,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \quad (1.1.1.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (1.1.1.3)$$

$$f = -2 \quad (1.1.1.4)$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (1.1.1.5)$$

Equation (1.1.1.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.1.1.6)$$

$$\delta = \begin{vmatrix} 1 & \frac{-5}{2} & \frac{1}{2} \\ \frac{-5}{2} & 4 & 1 \\ \frac{1}{2} & 1 & -2 \end{vmatrix} \quad (1.1.1.7)$$

$$= 0 \quad (1.1.1.8)$$

Hence, proved that given equation represents two straight lines. Finding point of intersection

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between the straight lines

$$\det V = \begin{vmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{vmatrix} \tag{1.1.1.9}$$

$$= \frac{-9}{4} < 0 \tag{1.1.1.10}$$

Thus, the two straight lines intersect. Let the equation of the straight lines be given as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \tag{1.1.1.11}$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \tag{1.1.1.12}$$

with their slopes as $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively. Then the equation of the pair of straight lines is

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \tag{1.1.1.13}$$

Using (1.1.1.5) and (1.1.1.13),

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 \tag{1.1.1.14}$$

Comparing both sides,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{1.1.1.15}$$

$$c_1 c_2 = -2 \tag{1.1.1.16}$$

Slopes of the lines are roots of the equation

$$cm^2 + 2bm + a = 0 \tag{1.1.1.17}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{1.1.1.18}$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.1.1.19}$$

Substituting (1.1.1.1) in (1.1.1.17),

$$4m^2 - 5m + 1 = 0 \tag{1.1.1.20}$$

$$\implies m_i = \frac{\frac{5}{2} \pm \frac{3}{2}}{4} \tag{1.1.1.21}$$

$$\implies m_1 = 1, m_2 = \frac{1}{4} \tag{1.1.1.22}$$

Therefore,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.1.1.23}$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} \tag{1.1.1.24}$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.1.25}$$

$$k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{1.1.1.26}$$

$$\implies k_1 k_2 = 4 \tag{1.1.1.27}$$

Taking $k_1 = 1$, $k_2 = 4$, we get

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{1.1.1.28}$$

For verifying values of $\mathbf{n}_1$ and $\mathbf{n}_2$, we compute the convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \tag{1.1.1.29}$$

Now, obtaining c_1 and c_2 using (1.1.1.28) and (1.1.1.15)

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{1.1.1.30}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{1.1.1.31}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 4 & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & -2 \end{pmatrix} \tag{1.1.1.32}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \tag{1.1.1.33}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{1.1.1.34}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c_1 = -1 \tag{1.1.1.35}$$

$$c_2 = 2 \tag{1.1.1.36}$$

Thus, equation of lines can be written as

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{1.1.1.37}$$

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} = 2 \tag{1.1.1.38}$$

Augmented matrix for these set of equations is

$$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 4 & 2 \end{pmatrix} \tag{1.1.1.39}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.1.40}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.1.41}$$

Thus, the point of intersection is $\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

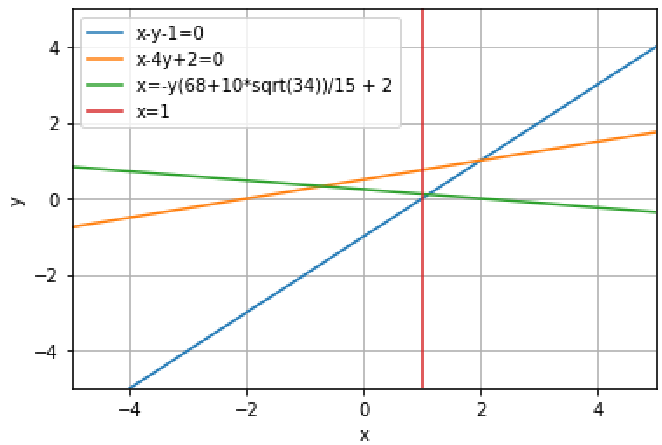

Fig. 1.1.1.1: Intersection of pair of original pair of straight lines and the pair of straight lines after affine transform

Using (1.1.1.28) and (1.1.1.36) in (1.1.1.13), equation of the pair of straight lines is

$$(x - y - 1)(x - 4y + 2) = 0 \tag{1.1.1.42}$$

Angle between lines Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|}\right) \tag{1.1.1.43}$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = 5 \tag{1.1.1.44}$$

$$\|\mathbf{n}_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \tag{1.1.1.45}$$

$$\|\mathbf{n}_2\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} \tag{1.1.1.46}$$

Substituting these values (1.1.1.43)

$$\theta = 30.9^\circ \tag{1.1.1.47}$$

Hence, angle between the given pair of straight lines is 30.9° Affine Transformation and Eigen Value decomposition First, verifying if $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 0$. To do this, finding V^{-1} by augmenting with identity matrix and row reducing as follows :

$$\begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ \frac{-5}{2} & 4 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{5}{2} R_1} \begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ 0 & \frac{-9}{4} & \frac{5}{2} & 1 \end{pmatrix} \tag{1.1.1.48}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-4}{9} R_2} \begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \tag{1.1.1.49}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{5}{2} R_2} \begin{pmatrix} 1 & 0 & \frac{-16}{9} & \frac{-10}{9} \\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \tag{1.1.1.50}$$

$$\implies \mathbf{V}^{-1} = \begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \tag{1.1.1.51}$$

$$u^T V^{-1} u - f = \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - (-2) \tag{1.1.1.52}$$

$$= 0 \tag{1.1.1.53}$$

The characteristic equation of $\mathbf{V}$ is given as :

$$\left|\lambda \mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & \frac{5}{2} \\ \frac{5}{2} & \lambda - 4 \end{vmatrix} = 0 \tag{1.1.1.54}$$

$$\implies (\lambda - 1)(\lambda - 4) - \frac{25}{4} = 0 \tag{1.1.1.55}$$

$$\implies 4\lambda^2 - 20\lambda - 9 = 0 \tag{1.1.1.56}$$

The roots of (1.1.1.56), i.e. the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = \frac{5 + \sqrt{34}}{2}, \lambda_2 = \frac{5 - \sqrt{34}}{2} \tag{1.1.1.57}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \tag{1.1.1.58}$$

$$\implies (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{1.1.1.59}$$

For $\lambda_1 = \dfrac{5+\sqrt{34}}{2}$

$$(\lambda_1\mathbf{I}-\mathbf{V}) = \begin{pmatrix} \frac{3+\sqrt{34}}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} \end{pmatrix} \tag{1.1.1.60}$$

To find $\mathbf{p}_1$, let's look at Augmented form of $(\lambda_1\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix} \frac{3+\sqrt{34}}{2} & \frac{5}{2} & 0 \\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0 \end{pmatrix} \tag{1.1.1.61}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{2}{3+\sqrt{34}}R_1} \begin{pmatrix} 1 & \frac{-3+\sqrt{34}}{5} & 0 \\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0 \end{pmatrix} \tag{1.1.1.62}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2}{5}R_2 - R_1} \begin{pmatrix} 1 & \frac{-3+\sqrt{34}}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.63}$$

So we get

$$x_1 + \left(\frac{-3+\sqrt{34}}{5}\right)x_2 = 0 \tag{1.1.1.64}$$

Thus, our eigenvector corresponding to λ_1

$$\mathbf{p}_1 = \begin{pmatrix} \frac{3-\sqrt{34}}{5} \\ 1 \end{pmatrix} \tag{1.1.1.65}$$

For $\lambda_2 = \dfrac{5-\sqrt{34}}{2}$

$$(\lambda_2\mathbf{I}-\mathbf{V}) = \begin{pmatrix} \frac{3-\sqrt{34}}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} \end{pmatrix} \tag{1.1.1.66}$$

To find $\mathbf{p}_2$, let's look at Augmented form of $(\lambda_2\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix} \frac{3-\sqrt{34}}{2} & \frac{5}{2} & 0 \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0 \end{pmatrix} \tag{1.1.1.67}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{2}{3-\sqrt{34}}R_1} \begin{pmatrix} 1 & \frac{-3-\sqrt{34}}{5} & 0 \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0 \end{pmatrix} \tag{1.1.1.68}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2}{5}R_2 - R_1} \begin{pmatrix} 1 & \frac{-3-\sqrt{34}}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.69}$$

So we get

$$x_1 + \left(\frac{-3-\sqrt{34}}{5}\right)x_2 = 0 \tag{1.1.1.70}$$

Thus, our eigenvector corresponding to λ_2

$$\mathbf{p}_2 = \begin{pmatrix} \frac{3+\sqrt{34}}{5} \\ 1 \end{pmatrix} \tag{1.1.1.71}$$

We know $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where $\mathbf{P}$ and the diagonal matrix $\mathbf{D}$ are given as:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{1.1.1.72}$$

$$= \begin{pmatrix} \frac{5+\sqrt{34}}{2} & 0 \\ 0 & \frac{5-\sqrt{34}}{2} \end{pmatrix} \tag{1.1.1.73}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{1.1.1.74}$$

$$= \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \tag{1.1.1.75}$$

So, the equation of the pair of straight lines is given by :

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{1.1.1.76}$$

$$\mathbf{y}^T \begin{pmatrix} \dfrac{5+\sqrt{34}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{34}}{2} \end{pmatrix} \mathbf{y} = 0 \tag{1.1.1.77}$$

$$\implies \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \dfrac{5+\sqrt{34}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{34}}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{1.1.1.78}$$

$$\implies (5+\sqrt{34})y_1^2 + (5-\sqrt{34})y_2^2 = 0 \tag{1.1.1.79}$$

So we get the equation of the pair of straight lines, as we can see this passes through the origin $(0,0)$. The corresponding image is shown in Fig. 1.1.1.2

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{1.1.1.80}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.1.1.81}$$

And,

$$\mathbf{P}^T = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & 1 \\ \frac{3+\sqrt{34}}{5} & 1 \end{pmatrix} \tag{1.1.1.82}$$

Using affine transformation, we can express the equation as

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{1.1.1.83}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.1.1.84}$$

The corresponding image is shown in Fig. 1.1.1.1

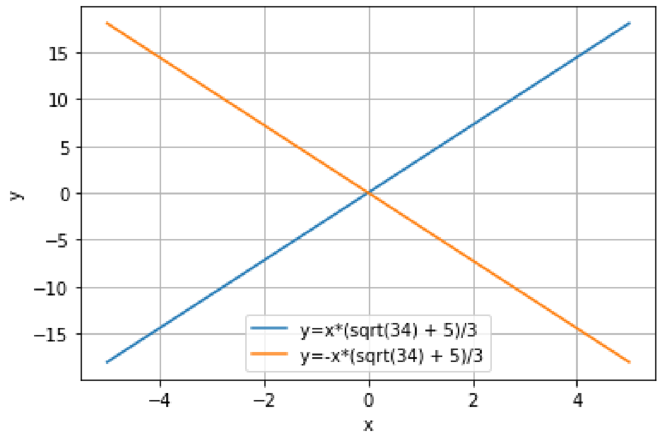

Fig. 1.1.1.2: Pair of straight lines passing through origin after eigenvalue decomposition

1.1.2. Prove that the following equations represent two straight lines.Also find their point of intersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \tag{1.1.2.1}$$

Solution: $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (1.1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \tag{1.1.2.2}$$

Expanding equation (1.1.2.2), we get zero. Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \tag{1.1.2.3}$$

$$\implies 3m^2 - 8m - 3 = 0 \tag{1.1.2.4}$$

$$\text{Solving, } m = 3, -\frac{1}{3} \tag{1.1.2.5}$$

The normal vectors of the lines then become

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.1.2.6}$$

$$\mathbf{n_2} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.1.2.7}$$

Equations of the lines can therefore be written as

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = c \tag{1.1.2.8}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = c_1, \tag{1.1.2.9}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{1.1.2.10}$$

$$\implies \left[\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - c_1\right]\left[\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} - c_2\right] \tag{1.1.2.11}$$

represents the equation specified in (1.1.2.1) Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \tag{1.1.2.12}$$

$$\tag{1.1.2.13}$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3\times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \tag{1.1.2.14}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \tag{1.1.2.15}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3\times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \tag{1.1.2.16}$$

$$\implies c_2 = 2 \text{ and } c_1 = -9 \tag{1.1.2.17}$$

The individual line equations therefore become

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = -9, \tag{1.1.2.18}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1.1.2.19}$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.2.20}$$

The LHS part of (1.1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.2.21}$$

The augmented matrix for the set of equations represented in (1.1.2.18), (1.1.2.19) is

$$\begin{pmatrix}1 & 3 & -9\\ -3 & 1 & 2\end{pmatrix} \tag{1.1.2.22}$$

Row reducing the matrix

$$\begin{pmatrix}1 & 3 & -9\\ -3 & 1 & 2\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2+3\times R_1} \begin{pmatrix}1 & 3 & -9\\ 0 & 10 & -25\end{pmatrix} \tag{1.1.2.23}$$

$$\xleftrightarrow{R_1 \leftarrow R_1-\frac{3}{10}\times R_2} \begin{pmatrix}1 & 0 & -\frac{3}{2}\\ 0 & 10 & -25\end{pmatrix} \tag{1.1.2.24}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix}1 & 0 & -\frac{3}{2}\\ 0 & 1 & -\frac{5}{2}\end{pmatrix} \tag{1.1.2.25}$$

$$\text{Hence, the intersection point is} \begin{pmatrix}-\frac{3}{2}\\ -\frac{5}{2}\end{pmatrix} \tag{1.1.2.26}$$

Angle between two lines θ can be given by

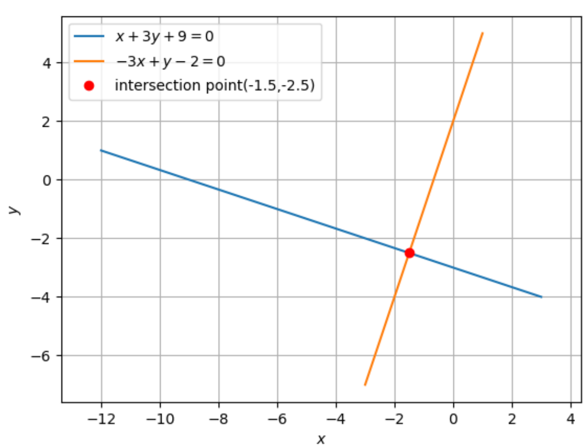

Fig. 1.1.2.1: plot showing intersection of lines

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{1.1.2.27}$$

$$\cos\theta = \frac{\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}}{\sqrt{(3)^2+1} \times \sqrt{(-3)^2+1}} = 0 \tag{1.1.2.28}$$

$$\implies \theta = 90^\circ \tag{1.1.2.29}$$

1.1.3. Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \tag{1.1.3.1}$$

Solution:

$$\mathbf{V} = \begin{pmatrix}a & b\\ b & c\end{pmatrix} = \begin{pmatrix}-2 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix} \tag{1.1.3.2}$$

$$\mathbf{u} = \begin{pmatrix}d\\ e\end{pmatrix} = \begin{pmatrix}\frac{-5}{2}\\ \frac{-1}{2}\end{pmatrix} \tag{1.1.3.3}$$

$$f = -2 \tag{1.1.3.4}$$

$$\begin{vmatrix}-2 & \frac{1}{2} & \frac{-5}{2}\\ \frac{1}{2} & 1 & \frac{-1}{2}\\ \frac{-5}{2} & \frac{-1}{2} & -2\end{vmatrix} \xleftrightarrow[R_1 \rightarrow R_1+R_3]{R_1 \rightarrow R_1-R_2} \begin{vmatrix}0 & 0 & 0\\ \frac{1}{2} & 1 & \frac{-1}{2}\\ \frac{-5}{2} & \frac{-1}{2} & -2\end{vmatrix} = 0 \tag{1.1.3.5}$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$\left|V\right| < 0 \implies \begin{vmatrix}-2 & \frac{1}{2}\\ \frac{1}{2} & 1\end{vmatrix} = \frac{-9}{4} < 0 \tag{1.1.3.6}$$

Let the pair of straight of lines be given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{1.1.3.7}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{1.1.3.8}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{1.1.3.9}$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \tag{1.1.3.10}$$

$$m_1 = 1, m_2 = -2 \tag{1.1.3.11}$$

$$\implies \mathbf{n_1} = \begin{pmatrix}-1\\ 1\end{pmatrix} and \mathbf{n_2} = \begin{pmatrix}2\\ 1\end{pmatrix} \tag{1.1.3.12}$$

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f \tag{1.1.3.13}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} \tag{1.1.3.14}$$

$$c_2\begin{pmatrix}-1\\ 1\end{pmatrix} + c_1\begin{pmatrix}2\\ 1\end{pmatrix} = -2\begin{pmatrix}\frac{-5}{2} & \frac{-1}{2}\end{pmatrix} \tag{1.1.3.15}$$

$$\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}1\\ 5\end{pmatrix} \tag{1.1.3.16}$$

Using row reduction we get

$$\begin{pmatrix}1 & 1 & 1\\ 2 & -1 & 5\end{pmatrix} \tag{1.1.3.17}$$

$$\xleftrightarrow[R_2\leftarrow R_2/-3]{R_2\leftarrow R_2-2R_1}\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -1\end{pmatrix} \tag{1.1.3.18}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2}\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -1\end{pmatrix} \tag{1.1.3.19}$$

$$C = \begin{pmatrix}2\\ -1\end{pmatrix} \tag{1.1.3.20}$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix}-1\\ 1\end{pmatrix} * \begin{pmatrix}2\\ 1\end{pmatrix} = \begin{pmatrix}a\\ 2b\\ c\end{pmatrix} \tag{1.1.3.21}$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix}-1 & 0\\ 1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix} = \begin{pmatrix}-2\\ 1\\ 1\end{pmatrix} = \begin{pmatrix}a\\ 2b\\ c\end{pmatrix} \tag{1.1.3.22}$$

Therefore the equation of lines is given by

$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 2 \tag{1.1.3.23}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = -1 \tag{1.1.3.24}$$

consider the augmented matrix

$$\begin{pmatrix}-1 & 1 & 2\\ 2 & 1 & -1\end{pmatrix} \tag{1.1.3.25}$$

$$\xleftrightarrow[R_2\leftarrow R_2-2R_1]{R_1\leftarrow -R_1}\begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 1\end{pmatrix} \tag{1.1.3.26}$$

$$\xleftrightarrow[R_1\leftarrow R_1+R_2]{R_1\leftarrow R_1/3}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\end{pmatrix} \tag{1.1.3.27}$$

Therefore point of intersection is $\mathbf{A} = \begin{pmatrix}-1\\ 1\end{pmatrix}$.
Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{1.1.3.28}$$

$$\cos\theta = \frac{\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}}{\sqrt{(1)^2+1}\times\sqrt{(2)^2+1}} \tag{1.1.3.29}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right) \implies \theta = tan^{-1}3 \tag{1.1.3.30}$$

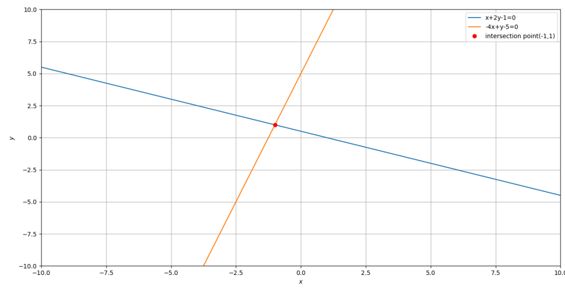

Fig. 1.1.3.1: plot showing intersection of lines

1.1.4. Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \tag{1.1.4.1}$$

represents two parallel lines.
Solution: The given equation (1.1.4.1) can be written as

$$\mathbf{x}^T\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}2 & 6\end{pmatrix}\mathbf{x} - 5 = 0 \tag{1.1.4.2}$$

$$\mathbf{V} = \begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix} \quad \mathbf{u} = \begin{pmatrix}2\\ 6\end{pmatrix} \quad f = -5 \tag{1.1.4.3}$$

Equation (1.1.4.1) represents pair of straight line as,

$$D = \begin{vmatrix}1 & 3 & 2\\ 3 & 9 & 6\\ 2 & 6 & -5\end{vmatrix} = 0 \tag{1.1.4.4}$$

Vector form of straight lines,

$$\mathbf{n_1}^T\mathbf{x} = \mathbf{c_1} \tag{1.1.4.5}$$

$$\mathbf{n_2}^T\mathbf{x} = \mathbf{c_2} \tag{1.1.4.6}$$

Equating their product with (1.1.4.2)

$$(\mathbf{n_1}^T\mathbf{x} - \mathbf{c_1})(\mathbf{n_2}^T\mathbf{x} - \mathbf{c_2}) = \mathbf{x}^T\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}2 & 6\end{pmatrix}\mathbf{x} - 5 \tag{1.1.4.7}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix}1\\ 6\\ 9\end{pmatrix} \tag{1.1.4.8}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix}2\\ 6\end{pmatrix} \tag{1.1.4.9}$$

$$c_1c_2 = -5 \tag{1.1.4.10}$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \tag{1.1.4.11}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{1.1.4.12}$$

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.1.4.13}$$

From (1.1.4.2) equation (1.1.4.11) becomes

$$9m^2 + 6m + 1 = 0 \tag{1.1.4.14}$$

Using (1.1.4.3),

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \tag{1.1.4.15}$$

Substituting the values in (1.1.4.12),

$$m_i = \frac{-3 \pm 0}{9} \tag{1.1.4.16}$$

$$m_1 = m_2 = \frac{-1}{3} \tag{1.1.4.17}$$

Substituting values in (1.1.4.13)

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.1.4.18}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.1.4.19}$$

Using the above values in (1.1.4.8),

$$k_1 k_2 = 9 \tag{1.1.4.20}$$

Taking $k_1 = 3$ and $k_2 = 3$ we get

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.4.21}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.4.22}$$

Verifying $\mathbf{n_1}$ and $\mathbf{n_2}$ by computing the convolution by representing $\mathbf{n_1}$ as Toeplitz matrix,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \tag{1.1.4.23}$$

Finding the Angle between the lines,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|}\right) \tag{1.1.4.24}$$

$$\mathbf{n_1}^T \mathbf{n_2} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \tag{1.1.4.25}$$

$$\|\mathbf{n_1}\| = \sqrt{10} \quad \|\mathbf{n_2}\| = \sqrt{10} \tag{1.1.4.26}$$

Substituting (1.1.4.25) and (1.1.4.26) in (1.1.4.24) we get,

$$\theta = \cos^{-1}(1) \tag{1.1.4.27}$$

$$\theta = 0^\circ \tag{1.1.4.28}$$

From (1.1.4.17) and (1.1.4.28) shows the given equation (1.1.4.1) represents two parallel lines. Hence proved.

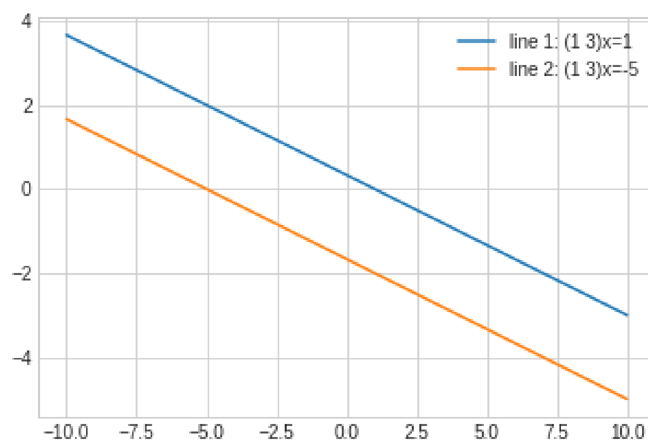

Fig. 1.1.4.1: Pair of straight lines plot generated using python

1.1.5. **Solution:** Find the value of k such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \tag{1.1.5.1}$$

represent pairs of straight lines.
From (1.1.5.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \tag{1.1.5.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \tag{1.1.5.3}$$

$$f = k \tag{1.1.5.4}$$

Compute the slopes of lines given by the roots of the polynomial $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{1.1.5.5}$$

$$\implies m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \tag{1.1.5.6}$$

$$\implies m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \tag{1.1.5.7}$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \tag{1.1.5.8}$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \tag{1.1.5.9}$$

Here,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \tag{1.1.5.10}$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -m_2 \\ 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \tag{1.1.5.11}$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.5.12}$$

Substituting (1.1.5.10) and (1.1.5.11) in the above equation, we get

$$k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} \tag{1.1.5.13}$$

$$\implies k_1 k_2 = -10 \tag{1.1.5.14}$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of k_1 and k_2 in (1.1.5.10) and (1.1.5.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \tag{1.1.5.15}$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.5.16}$$

Using Teoplitz matrix representation, the convolution of $\mathbf{n}_1$ with $\mathbf{n}_2$, is as follows:

$$\begin{pmatrix} 2 & 0 & 5 \\ 5 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.5.17}$$

Hence, $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfies (1.1.5.12). We have,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \tag{1.1.5.18}$$

Substituting (1.1.5.15), (1.1.5.16) in (1.1.5.18), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \tag{1.1.5.19}$$

Solving for c_1 and c_2, the augmented matrix is,

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & -2 & -31 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 5R_1]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & \frac{-19}{2} & \frac{-57}{2} \end{pmatrix} \tag{1.1.5.20}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{3}{2}R_2]{R_2 \leftarrow \frac{R_2}{-19/2}} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.5.21}$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \tag{1.1.5.22}$$

We know that,

$$f = k = c_1 c_2 \tag{1.1.5.23}$$

$$\implies \boxed{k = -15} \tag{1.1.5.24}$$

Hence the solution. Using (1.1.5.8) and (1.1.5.9), the equation of pair of straight lines is given by,

$$\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 3 \tag{1.1.5.25}$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -5 \tag{1.1.5.26}$$

See Fig. 1.1.5.1

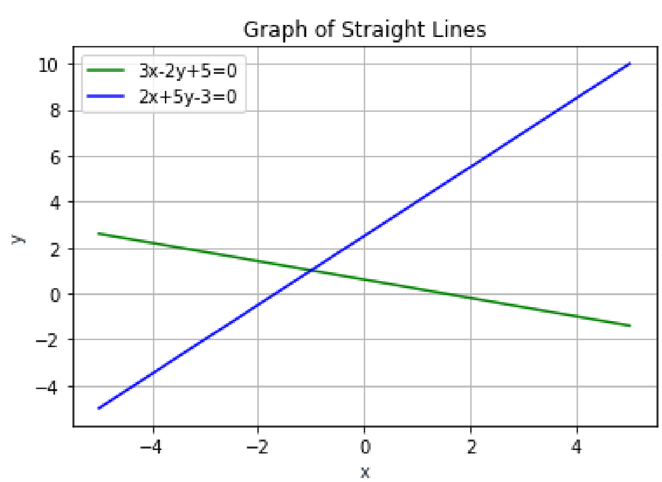

Fig. 1.1.5.1: Plot of two straight lines.

1.1.6. Find the value of k so that following equation may represent pairs of straight lines,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0 \tag{1.1.6.1}$$

Solution: The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.1.6.2}$$

In vector from the equation (1.1.6.2) can be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{1.1.6.3}$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{1.1.6.4}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{1.1.6.5}$$

Now, comparing (1.1.6.2) to (1.1.6.1) we get, a =12, b=-5, c = 2, d = $\frac{11}{2}$,e = -$\frac{5}{2}$, f = k. Hence, substituting these values in (1.1.6.4) and (1.1.6.5) we get,

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{1.1.6.6}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.1.6.7}$$

(1.1.6.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.1.6.8}$$

$$\begin{vmatrix} 12 & -5 & \frac{11}{2} \\ -5 & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & k \end{vmatrix} = 0 \tag{1.1.6.9}$$

$$\implies k = 2 \tag{1.1.6.10}$$

Lines Intercept if

$$|\mathbf{V}| < 0 \tag{1.1.6.11}$$

$$|\mathbf{V}| = -1 < 0 \tag{1.1.6.12}$$

Hence Line intercept.
Let (α,β) be their point of intersection, then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \tag{1.1.6.13}$$

Substituting in (1.1.6.13)

$$\begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \tag{1.1.6.14}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.1.6.15}$$

Spectral Decomposition of $\mathbf{V}$ is given as

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{1.1.6.16}$$

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{1.1.6.17}$$

$$\mathbf{P} = \begin{pmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{pmatrix} \tag{1.1.6.18}$$

$$\mathbf{D} = \begin{pmatrix} 7 + 5\sqrt{2} & 0 \\ 0 & 7 - 5\sqrt{2} \end{pmatrix} \tag{1.1.6.19}$$

Using Spectral decomposition concept and substution

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x - \alpha) + v_2(y - \beta)) \tag{1.1.6.20}$$

Substituting (1.1.6.15), (1.1.6.18) and (1.1.6.19) in (1.1.6.20)

$$\begin{aligned} &\left(-1 - \sqrt{2}\right)\left(x - \frac{-3}{2}\right) + \left(y - \frac{-5}{2}\right) \\ &= \pm \sqrt{-\frac{7 + 5\sqrt{2}}{7 - 5\sqrt{2}}} \left(\left(-1 + \sqrt{2}\right)\left(x - \frac{-3}{2}\right) + \left(y - \frac{-5}{2}\right)\right) \end{aligned} \tag{1.1.6.21}$$

Simplifying (1.1.6.21),

$$-6x + 2y - 4 = 0 \text{ and } -2x + y - \frac{1}{2} = 0 \tag{1.1.6.22}$$

$$\implies (-6x + 2y - 4)\left(-2x + y - \frac{1}{2}\right) = 0 \tag{1.1.6.23}$$

Thus the equation of lines are

$$\begin{pmatrix} -6 & 2 \end{pmatrix} \mathbf{x} = 4 \tag{1.1.6.24}$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \tag{1.1.6.25}$$

Hence, Plot is shown below

1.1.7. Find the value of k so that the following

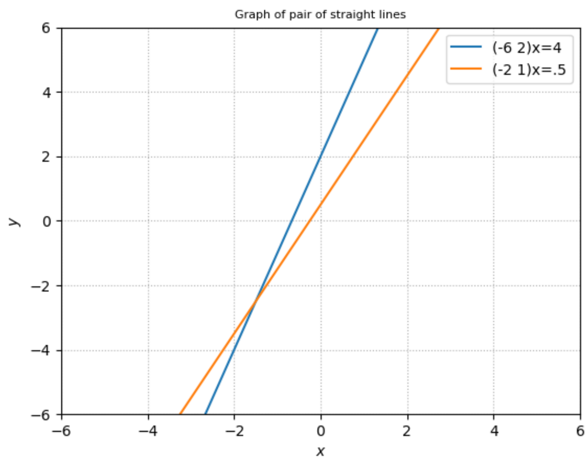

Fig. 1.1.6.1: Pair of lines

equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \tag{1.1.7.1}$$

Solution:

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \tag{1.1.7.2}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.1.7.3}$$

The equation (1.1.7.1) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.1.7.4}$$

$$\implies \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \tag{1.1.7.5}$$

$$\implies \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \tag{1.1.7.6}$$

$$\implies 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11 \begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \tag{1.1.7.7}$$

$$\implies 2k^2 + 55k + 350 = 0 \tag{1.1.7.8}$$

$$\implies (10 + k)(2k + 35) = 0 \tag{1.1.7.9}$$

$$\implies k = -10$$

$$k = -\frac{35}{2} \tag{1.1.7.10}$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.
Now Lets find equation of lines for $k = -10$. Substitute $k = -10$ in (1.1.7.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \tag{1.1.7.11}$$

From (1.1.7.1), (1.1.7.2), (1.1.7.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{1.1.7.12}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.1.7.13}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \tag{1.1.7.14}$$

$$\implies \left|\mathbf{V}\right| = -1 \tag{1.1.7.15}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{1.1.7.16}$$

Therefore the lines will intersect.
The equation of two lines is given by

$$\mathbf{n_1}^T \mathbf{x} = c_1 \tag{1.1.7.17}$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \tag{1.1.7.18}$$

Equating their product with (1.1.7.1)

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{1.1.7.19}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{1.1.7.20}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = -2\begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.1.7.21}$$

$$c_1 c_2 = f = 2 \tag{1.1.7.22}$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \tag{1.1.7.23}$$
$$\implies 2m^2 - 10m + 12 = 0 \tag{1.1.7.24}$$
$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \tag{1.1.7.25}$$
$$\implies m_i = \frac{5 \pm \sqrt{1}}{2} \tag{1.1.7.26}$$
$$\implies m_1 = 3 \tag{1.1.7.27}$$
$$m_2 = 2 \tag{1.1.7.28}$$

The normal vector for two lines is given by

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.1.7.29}$$
$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.1.7.30}$$
$$\mathbf{n_2} = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.7.31}$$

Substituting (1.1.7.30),(1.1.7.31) in (1.1.7.20). we get

$$k_1 k_2 = 2 \tag{1.1.7.32}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.1.7.33}$$
$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{1.1.7.34}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{1.1.7.35}$$
$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.7.36}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (1.1.7.33), (1.1.7.34) in (1.1.7.21) and calculate for c_1 and c_2

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{1.1.7.37}$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \tag{1.1.7.38}$$
$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \tag{1.1.7.39}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \tag{1.1.7.40}$$
$$\implies \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \tag{1.1.7.41}$$
$$\implies c_1 = -13 \tag{1.1.7.42}$$
$$c_2 = 21 \tag{1.1.7.43}$$

Substituting (1.1.7.33),(1.1.7.34),(1.1.7.42),(1.1.7.43) in (1.1.7.17) and (1.1.7.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = -13 \tag{1.1.7.44}$$
$$\begin{pmatrix} -4 & 2 \end{pmatrix} \mathbf{x} = 21 \tag{1.1.7.45}$$

The plot of these two lines is shown in Fig. 1.1.7.1.

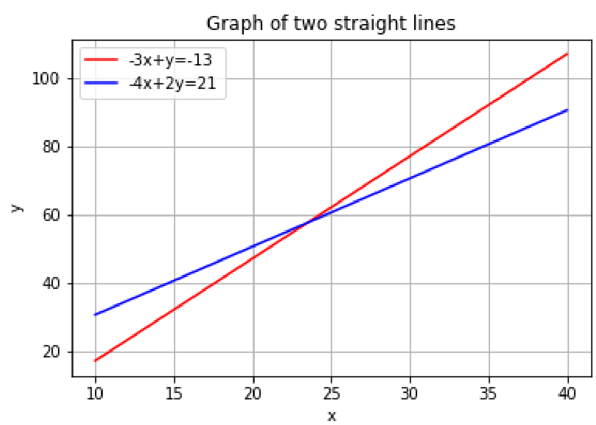

Fig. 1.1.7.1: Pair of straight lines for $k = -10$

Now Lets find equation of lines for $k = -\frac{35}{2}$. Substitute $k = -\frac{35}{2}$ in (1.1.7.1). We get equation of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \tag{1.1.7.46}$$

From (1.1.7.1), (1.1.7.2), (1.1.7.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \tag{1.1.7.47}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.1.7.48}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \tag{1.1.7.49}$$

$$\implies \left|\mathbf{V}\right| = -\frac{841}{16} \tag{1.1.7.50}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{1.1.7.51}$$

Therefore the lines will intersect.
Now from (1.1.7.20),

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{1.1.7.52}$$

The slopes of the lines are given by roots of equation (1.1.7.23)

$$\implies 2m^2 - \frac{35}{2}m + 12 = 0 \tag{1.1.7.53}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{1.1.7.54}$$

$$\implies m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \tag{1.1.7.55}$$

$$\implies m_1 = 8 \tag{1.1.7.56}$$

$$m_2 = \frac{3}{4} \tag{1.1.7.57}$$

The normal vector for two lines is given by (1.1.7.29)

$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{1.1.7.58}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \tag{1.1.7.59}$$

Substituting (1.1.7.58),(1.1.7.59) in (1.1.7.52). we get

$$k_1 k_2 = 2 \tag{1.1.7.60}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{1.1.7.61}$$

$$\mathbf{n_2} = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \tag{1.1.7.62}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{1.1.7.63}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.7.64}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (1.1.7.61), (1.1.7.62) in (1.1.7.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{1.1.7.65}$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \tag{1.1.7.66}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \tag{1.1.7.67}$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \tag{1.1.7.68}$$

$$\xleftrightarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \tag{1.1.7.69}$$

$$\implies \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \tag{1.1.7.70}$$

$$\implies c_1 = -\frac{102}{29} \tag{1.1.7.71}$$

$$c_2 = \frac{59}{29} \tag{1.1.7.72}$$

Substituting (1.1.7.61),(1.1.7.62),(1.1.7.71),(1.1.7.72) in (1.1.7.17) and (1.1.7.18). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \tag{1.1.7.73}$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \tag{1.1.7.74}$$

1.1.8. Find the value of k so that the following equation may represent a pair of straight lines

-

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \tag{1.1.8.1}$$

Solution: The given second degree equation is, Comparing coefficients of (1.1.8.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \tag{1.1.8.2}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{1.1.8.3}$$

$$f = -35 \tag{1.1.8.4}$$

The given second degree equation (1.1.8.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \tag{1.1.8.5}$$

Expanding the determinant,

$$k + 12 = 0 \tag{1.1.8.6}$$

$$\implies k = -12 \tag{1.1.8.7}$$

Hence, from (1.1.8.7) we find that for $k = -12$, the given second degree equation (1.1.8.1) represents pair of straight lines. For the appropriate value of k, (1.1.8.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \tag{1.1.8.8}$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{1.1.8.9}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{1.1.8.10}$$

The pair of straight lines is given by,

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u^T}\mathbf{x} + f = 0 \tag{1.1.8.11}$$

Putting the values of $\mathbf{V}$ and $\mathbf{u}$ we get,

$$\mathbf{x}^T\begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix}\mathbf{x} - 35 = 0 \tag{1.1.8.12}$$

Hence, from (1.1.8.12) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \tag{1.1.8.13}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{1.1.8.14}$$

$$c_1c_2 = -35 \tag{1.1.8.15}$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \tag{1.1.8.16}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.1.8.17}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.1.8.18}$$

Substituting the values in above equations (1.1.8.16) we get,

$$-12m^2 + m + 6 = 0 \tag{1.1.8.19}$$

$$\implies m_i = \frac{-\frac{1}{2} \pm \sqrt{-(-\frac{289}{4})}}{-12} \tag{1.1.8.20}$$

Solving equation (1.1.8.20) we get ,

$$m_1 = -\frac{2}{3} \tag{1.1.8.21}$$

$$m_2 = \frac{3}{4} \tag{1.1.8.22}$$

Hence putting the values of m_1 and m_2 in (1.1.8.18) we get

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \tag{1.1.8.23}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \tag{1.1.8.24}$$

Putting values of $\mathbf{n_1}$ and $\mathbf{n_2}$ in (1.1.8.13) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \tag{1.1.8.25}$$

$$\implies \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \tag{1.1.8.26}$$

Thus, from (1.1.8.26), $k_1k_2 = -12$. Possible combinations of (k_1, k_2) are (6,-2), (-6,2), (3,-4), (-3,4) Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.1.8.27}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.1.8.28}$$

From equation (1.1.8.14) we get

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.8.29}$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{1.1.8.30}$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \tag{1.1.8.31}$$

$$3c_2 - 4c_1 = -43 \tag{1.1.8.32}$$

The augmented matrix of (1.1.8.31) ,(1.1.8.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix} \tag{1.1.8.33}$$

$$\xleftrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \tag{1.1.8.34}$$

$$\xleftrightarrow{R_2 = -\frac{2}{17}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \tag{1.1.8.35}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix} \tag{1.1.8.36}$$

$$\tag{1.1.8.37}$$

Hence we get,

$$c_1 = -5 \tag{1.1.8.38}$$

$$c_2 = 7 \tag{1.1.8.39}$$

Hence (1.1.8.9), (1.1.8.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \tag{1.1.8.40}$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \tag{1.1.8.41}$$

The figure below corresponds to the pair of straight lines represented by (1.1.8.40) and (1.1.8.41).

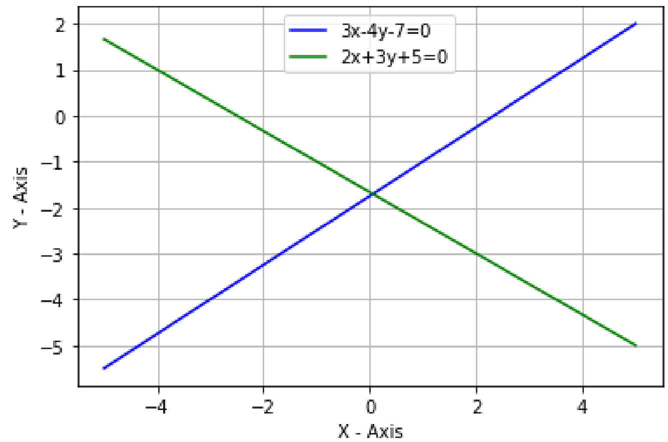

Fig. 1.1.8.1: Pair of Straight Lines

1.1.9. Find the value of k so that following equation may represent pairs of straight lines,

$$kxy - 8x + 9y - 12 = 0 \tag{1.1.9.1}$$

Solution: The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.1.9.2}$$

In vector from the equation (1.1.9.2) canb be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{1.1.9.3}$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{1.1.9.4}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{1.1.9.5}$$

Now, comparing equation (1.1.9.2) to (1.1.9.1) we get, a = c = 0, b = $\left(\frac{k}{2}\right)$, d = -4, e = $\left(\frac{9}{2}\right)$, f = -12. Hence, substituting these values in equation (1.1.9.4) and (1.1.9.5) we get,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} \tag{1.1.9.6}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix} \tag{1.1.9.7}$$

Now equation (1.1.9.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.1.9.8}$$

$$\begin{vmatrix} 0 & \frac{k}{2} & -4 \\ \frac{k}{2} & 0 & \frac{9}{2} \\ -4 & \frac{9}{2} & -12 \end{vmatrix} = 0 \tag{1.1.9.9}$$

$$\implies k = 0, k = 6 \tag{1.1.9.10}$$

Substituting (1.1.9.10) in (1.1.9.1) we get,

$$6xy - 8x + 9y - 12 = 0 \tag{1.1.9.11}$$

$$-8x + 9y - 12 = 0 \tag{1.1.9.12}$$

Hence value of k = 6 represents pair of straight lines. Also it can be verified that the pair of lines intersect as,

$$\left|\mathbf{V}\right| = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} < 0 \tag{1.1.9.13}$$

Let the pair of straight lines is given by,

$$\mathbf{n_1}^T\mathbf{x} = c1 \tag{1.1.9.14}$$

$$\mathbf{n_2}^T\mathbf{x} = c2 \tag{1.1.9.15}$$

Now equating the product of equation (1.1.9.14) and (1.1.9.15) with (1.1.9.3) we get,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = \tag{1.1.9.16}$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -4 & \frac{9}{2} \end{pmatrix} \mathbf{x} - 12 \tag{1.1.9.17}$$

$$\implies n_1 * n_2 = \{0, 6, 0\} \tag{1.1.9.18}$$

$$c_1 n_1 + c_2 n_2 = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \tag{1.1.9.19}$$

$$c_1 c_2 = -12. \tag{1.1.9.20}$$

Now the slopes of line is given by roots of polynomial,

$$cm^2 + 2bm + a = 0 \tag{1.1.9.21}$$

$$\implies 2bm = 0 \tag{1.1.9.22}$$

$$\implies m = 0 \tag{1.1.9.23}$$

Also

$$m_i = \frac{-b \pm \sqrt{-|V|}}{c} \tag{1.1.9.24}$$

$$\implies m_i = \frac{-0 \pm \sqrt{9}}{0} \tag{1.1.9.25}$$

$$\therefore m_1 = 0 \tag{1.1.9.26}$$

$$m_2 = \infty \tag{1.1.9.27}$$

The normal vector to the two lines is given by,

$$n_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.1.9.28}$$

$$\implies n_1 = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.1.9.29}$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.1.9.30}$$

Also,

$$k_1 k_2 = 6 \tag{1.1.9.31}$$

Let k_1 = 2 and k_2 =3

$$\implies n_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{1.1.9.32}$$

$$n_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.1.9.33}$$

We verify obtained n_1 and n_2 using Toeplitz matrix,

$$n_1 * n_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \tag{1.1.9.34}$$

Hence (1.1.9.18) and (1.1.9.34) are same. Hence verified.
Now substituting it in (1.1.9.19) we get,

$$c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \tag{1.1.9.35}$$

Solve using Row reduction Technique we get,

$$\implies \begin{pmatrix} 3 & 0 & 8 \\ 0 & 2 & -9 \end{pmatrix} \tag{1.1.9.36}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 2 & -9 \end{pmatrix} \tag{1.1.9.37}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & -9/2 \end{pmatrix} \tag{1.1.9.38}$$

$$\implies c_1 = \frac{8}{3} \tag{1.1.9.39}$$

$$c_2 = \frac{-9}{2} \tag{1.1.9.40}$$

substituting the values of c_1, c_2 and equation

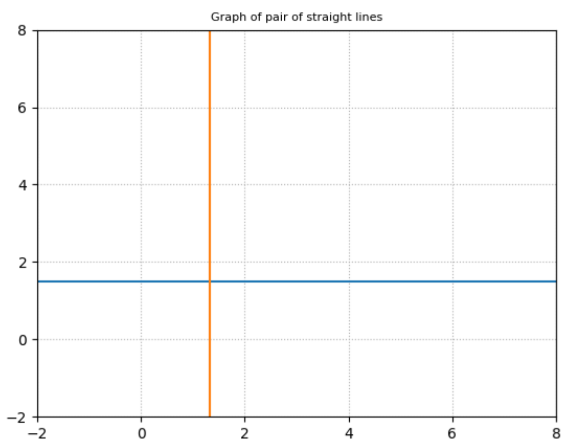

Fig. 1.1.9.1: Intersection of 2 lines

(1.1.9.32) and (1.1.9.33) to equation (1.1.9.14) and (1.1.9.15) we get equation of two straight lines.

$$\implies \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} = \frac{8}{3} \tag{1.1.9.41}$$

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} = \frac{-9}{2} \tag{1.1.9.42}$$

Hence the equation of pair of straight lines are,

$$\left(\begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} - \frac{8}{3}\right)\left(\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - \frac{-9}{2}\right) = 0 \tag{1.1.9.43}$$

Hence, Plot of the equation (1.1.9.43) is shown in Figure.1.1.9.1 Now for value of k = 0 does not represent pair of straight lines.as,

$$\left|\mathbf{V}\right| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \nless 0 \tag{1.1.9.44}$$

Hence, Plot of the equation $\begin{pmatrix} -8 & 9 \end{pmatrix} \mathbf{x} = 12$ is shown in figure 1.1.9.2,

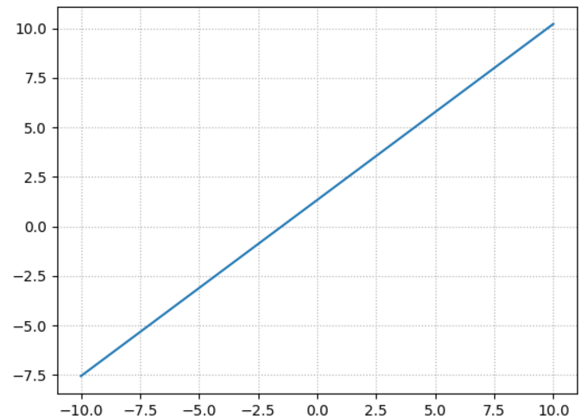

Fig. 1.1.9.2: Intersection of 2 lines

1.1.10. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \tag{1.1.10.1}$$

represent pairs of straight lines.
Solution: From (1.1.10.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \tag{1.1.10.2}$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \tag{1.1.10.3}$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & k \end{vmatrix} = 0 \tag{1.1.10.4}$$

$$\implies \left(k - \left(\frac{49}{4}\right)\right) - \frac{5}{3}\left(\frac{5}{3}k - \frac{35}{4}\right) - \frac{5}{2}\left(\frac{-35}{6} + \frac{5}{2}\right) = 0 \tag{1.1.10.5}$$

$$\implies \frac{64}{k}36 - \frac{128}{12} = 0 \tag{1.1.10.6}$$

$$\implies \boxed{k = 6} \tag{1.1.10.7}$$

Substituting (1.1.10.7) in (1.1.10.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \tag{1.1.10.8}$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (1.1.10.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & 6 \end{vmatrix} \tag{1.1.10.9}$$

Simplyfying the above determinant , we get

$$\delta = 0 \tag{1.1.10.10}$$

(1.1.10.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \tag{1.1.10.11}$$

Since $\det(V) < 0$ lines would intersect each other

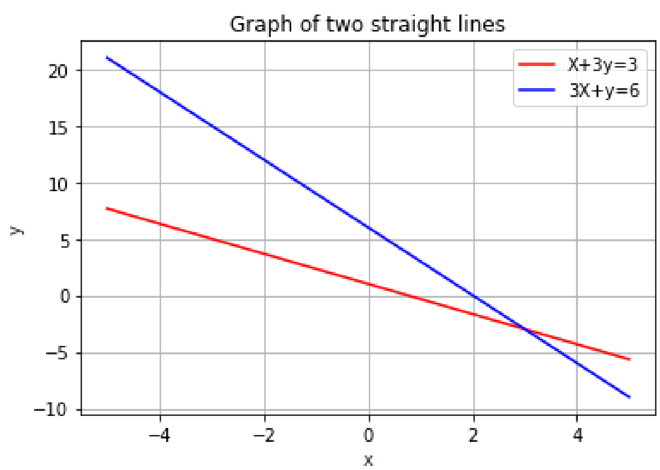

Fig. 1.1.10.1: Pair of straight lines

$$\mathbf{n_1} * \mathbf{n_2} = \{1, \frac{10}{3}, 1\} \tag{1.1.10.12}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.10.13}$$

$$c_1c_2 = 6 \tag{1.1.10.14}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{1.1.10.15}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.1.10.16}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.1.10.17}$$

Substituting in above equations (1.1.10.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \tag{1.1.10.18}$$

$$\implies m_i = \frac{\frac{-10}{3} \pm \sqrt{-(\frac{-16}{9})}}{1} \tag{1.1.10.19}$$

Solving equation (1.1.10.19) we have ,

$$m_1 = \frac{-1}{3} \tag{1.1.10.20}$$

$$m_2 = -3 \tag{1.1.10.21}$$

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.1.10.22}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.1.10.23}$$

Substituting equations (1.1.10.22), (1.1.10.23) in equation (1.1.10.12) we get

$$k_1k_2 = 1 \tag{1.1.10.24}$$

Possible combination of (k_1, k_2) is (1,1) Lets assume $k_1 = 1$, $k_2 = 1$, we get

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.1.10.25}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.1.10.26}$$

we have:

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.10.27}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a teoplitz matrix and multiplying with $\mathbf{n_2}$
From equation (1.1.10.25) and (1.1.10.26)

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.1.10.28}$$

$$\implies \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.10.29}$$

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.10.30}$$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.10.31}$$

Substituting (1.1.10.25) and (1.1.10.26) in (1.1.10.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow 3\times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \quad (1.1.10.32)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \quad (1.1.10.33)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.10.34)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 9\times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.10.35)$$

From above we get

$$c_1 = 1 \quad (1.1.10.36)$$
$$c_2 = 6 \quad (1.1.10.37)$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \quad (1.1.10.38)$$
$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (1.1.10.39)$$

1.1.11. Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (1.1.11.1)$$

represents two straight lines and find the angle between the lines.

Solution: The above equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.1.11.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (1.1.11.3)$$
$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (1.1.11.4)$$
$$f = -35 \quad (1.1.11.5)$$

(1.1.11.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.1.11.6)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (1.1.11.7)$$

$$\implies 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \quad (1.1.11.8)$$

(1.1.11.9)

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \quad (1.1.11.10)$$
$$\left|\mathbf{V}\right| = -\frac{529}{4} < 0 \quad (1.1.11.11)$$

From (1.1.11.8) and (1.1.11.11) it can be concluded that the given equation represents a pair of intersecting lines. Let the equations of lines be

$$\mathbf{n_1}^T \mathbf{x} = c_1 \quad (1.1.11.12)$$
$$\mathbf{n_2}^T \mathbf{x} = c_2 \quad (1.1.11.13)$$

Since (1.1.11.2) represents a pair of straight lines it must satisfy

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_1}^T \mathbf{x} - c_1) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.1.11.14)$$

where

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (1.1.11.15)$$
$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2\mathbf{u} \quad (1.1.11.16)$$
$$c_1 c_2 = f \quad (1.1.11.17)$$

Slopes of the lines can be obtained by solving

$$cm^2 + 2bm + a = 0 \quad (1.1.11.18)$$
$$-10m^2 + 7m + 12 = 0 \quad (1.1.11.19)$$
$$\implies m_1 = \frac{-4}{5}, m_2 = \frac{3}{2} \quad (1.1.11.20)$$

The normal vectors can be expressed in terms of corresponding slopes of lines as

$$\mathbf{n} = k\begin{pmatrix} -m \\ 1 \end{pmatrix} \quad (1.1.11.21)$$

$$\implies \mathbf{n_1} = k_1\begin{pmatrix} \frac{4}{5} \\ 1 \end{pmatrix} \quad (1.1.11.22)$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \quad (1.1.11.23)$$

Substituing (1.1.11.22) and (1.1.11.23) in (1.1.11.15) we get

$$k_1 k_2 = -10 \quad (1.1.11.24)$$

Assuming $k_1 = 5$ and $k_2 = -2$

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.1.11.25)$$

Verification using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 4 & 0 \\ 5 & 4 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (1.1.11.26)$$

From (1.1.11.16) we have

$$c_2\begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_1\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ -45 \end{pmatrix} \quad (1.1.11.27)$$

Solving the augmented matrix

$$\begin{pmatrix} 4 & 3 & -13 \\ 5 & -2 & -45 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 4R_2 - 5R_1} \begin{pmatrix} 4 & 3 & -13 \\ 0 & -23 & -115 \end{pmatrix} \quad (1.1.11.28)$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{R_2}{23}} \begin{pmatrix} 4 & 3 & -13 \\ 0 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 4 & 0 & -28 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.1.11.29)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.1.11.30)$$

$$\implies c_1 = -7, \ c_2 = 5 \quad (1.1.11.31)$$

Thus the equation of lines are

$$\begin{pmatrix} 4 & 5 \end{pmatrix}\mathbf{x} = 5 \quad (1.1.11.32)$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix}\mathbf{x} = -7 \quad (1.1.11.33)$$

The angle between the lines can be expressed interms of normal vectors

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.1.11.34)$$

as

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \quad (1.1.11.35)$$

$$\implies \theta = \cos^{-1}(\frac{2}{\sqrt{533}}) = \tan^{-1}(\frac{23}{2}) \quad (1.1.11.36)$$

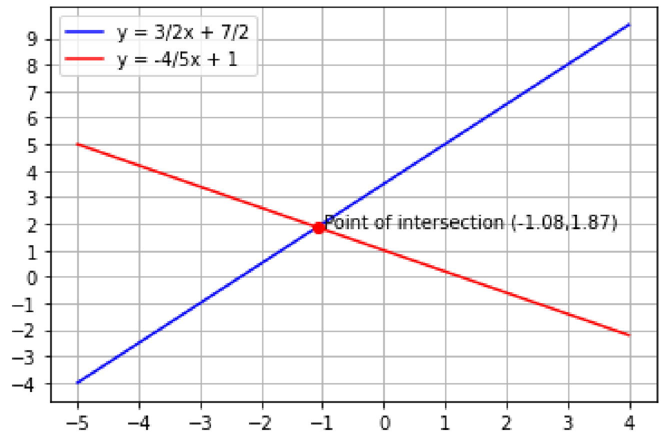

Fig. 1.1.11.1

1.1.12. Find the value of k so that the following equation may represent the pair of staright lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \quad (1.1.12.1)$$

Solution: We need to find the value of k for which (1.1.12.1) represents a pair of straight lines.
Converting (1.1.12.1) into vector form, we get

$$\mathbf{x}^T\begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} k/2 \\ 3 \end{pmatrix}\mathbf{x} - 9 = 0 \quad (1.1.12.2)$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \quad (1.1.12.3)$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \quad (1.1.12.4)$$

$$f = -9 \quad (1.1.12.5)$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.1.12.6)$$

Since (1.1.12.1) represents a pair of straight lines, then by (1.1.12.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \tag{1.1.12.7}$$

By solving, above determinant we get

$$2(9-9) + \frac{-1}{2}(\frac{-9}{2} + \frac{-3k}{2}) + \frac{k}{2}(\frac{3}{2} + \frac{k}{2}) = 0 \tag{1.1.12.8}$$

$$\frac{(9+3k)}{4} + \frac{k(3+k)}{4} = 0 \tag{1.1.12.9}$$

$$k^2 + 6k + 9 = 0 \tag{1.1.12.10}$$

$$(k+3)^2 = 0 \tag{1.1.12.11}$$

$$k = -3 \tag{1.1.12.12}$$

Hence by (1.1.12.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \tag{1.1.12.13}$$

represents family of straight lines for $k = -3$. To find the staright lines, we write each of thrm in their vector form as

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{1.1.12.14}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{1.1.12.15}$$

Equating the product of above with (1.1.12.2), we have

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} k/2 \\ 3 \end{pmatrix}\mathbf{x} - 9 \tag{1.1.12.16}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \tag{1.1.12.17}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_1} = -2\begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \tag{1.1.12.18}$$

$$c_1c_2 = -9 \tag{1.1.12.19}$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \tag{1.1.12.20}$$

$$m^2 - m - 2 = 0 \tag{1.1.12.21}$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} \tag{1.1.12.22}$$

$$m_1 = \frac{1+3}{2} = 2 \tag{1.1.12.23}$$

$$m_2 = \frac{1-3}{2} = -1 \tag{1.1.12.24}$$

$$n_1 = k_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.12.25}$$

$$n_2 = k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.12.26}$$

Substituing (1.1.12.25) and (1.1.12.26) in (1.1.12.17), we get

$$k_1k_2 = -1 \tag{1.1.12.27}$$

Taking $k_1 = -1$ and $k_2 = 1$, we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.1.12.28}$$

$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.12.29}$$

Substituting in (1.1.12.18) for above values ofn_1 and n_2

$$\begin{pmatrix} n_1 n_2 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{1.1.12.30}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{1.1.12.31}$$

Solving (1.1.12.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xleftrightarrow{r_2 = r_2 + 2r_1} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \tag{1.1.12.32}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xleftrightarrow{r_2 = r_2/3} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.12.33}$$

$$\begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = \begin{pmatrix}3\\-3\end{pmatrix} \xleftrightarrow{r_1=r_1-r_2}$$

$$\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = \begin{pmatrix}6\\-3\end{pmatrix} \quad (1.1.12.34)$$

Hence, we found out

$$c_1 = -3 \quad (1.1.12.35)$$

$$c_2 = 3 \quad (1.1.12.36)$$

Thus, pair of staright lines are

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -3 \quad (1.1.12.37)$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \quad (1.1.12.38)$$

where

$$\mathbf{x} = \begin{pmatrix}x\\y\end{pmatrix} \quad (1.1.12.39)$$

The plot of above is shown below

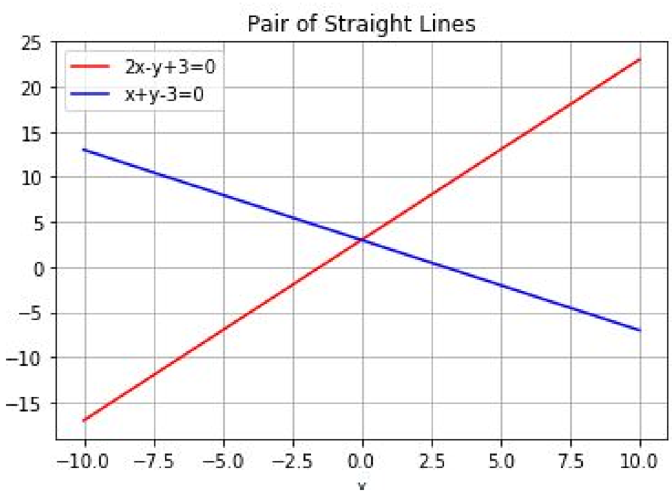

Fig. 1.1.12.1: Pair of Straight Lines

1.1.13. Prove that the equation $12x^2+7xy-10y^2+13x+45y-35=0$ represents two straight lines and find the angle between them.
Solution: The general second order equation is given by ,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.1.13.1)$$

Given,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (1.1.13.2)$$

The above equation can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (1.1.13.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix} \quad (1.1.13.4)$$

$$\mathbf{u} = \begin{pmatrix}\frac{13}{2}\\ \frac{45}{2}\end{pmatrix} \quad (1.1.13.5)$$

$$f = -35 \quad (1.1.13.6)$$

(1.1.13.3) represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} = 0 \quad (1.1.13.7)$$

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} = \begin{vmatrix}12 & \frac{7}{2} & \frac{13}{2}\\ \frac{7}{2} & -10 & \frac{45}{2}\\ \frac{13}{2} & \frac{45}{2} & -35\end{vmatrix} \quad (1.1.13.8)$$

$$\implies 12\begin{vmatrix}-10 & \frac{45}{2}\\ \frac{45}{2} & -35\end{vmatrix} - \frac{7}{2}\begin{vmatrix}\frac{7}{2} & \frac{45}{2}\\ \frac{13}{2} & -35\end{vmatrix} + \frac{13}{2}\begin{vmatrix}\frac{7}{2} & -10\\ \frac{13}{2} & \frac{45}{2}\end{vmatrix} = 0 \quad (1.1.13.9)$$

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \quad (1.1.13.10)$$

$$\left|\mathbf{V}\right| = -\frac{529}{4} < 0 \quad (1.1.13.11)$$

From (1.1.13.9) and (1.1.13.11) it can be concluded that the given equation represents a pair of intersecting lines.
Let (α, β) be their point of intersection, then

$$\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\frac{-13}{2}\\ -\frac{45}{2}\end{pmatrix} \quad (1.1.13.12)$$

$$\implies \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-1\\ 2\end{pmatrix} \quad (1.1.13.13)$$

$$\text{From } \textit{Spectral theorem, } \mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (1.1.13.14)$$

$$\mathbf{V} = \begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix} \quad (1.1.13.15)$$

$$\mathbf{P} = \begin{pmatrix}\frac{-\sqrt{533}-22}{2} & \frac{-22+\sqrt{533}}{2}\\ 1 & 1\end{pmatrix} \quad (1.1.13.16)$$

$$\mathbf{D} = \begin{pmatrix}1 + \frac{\sqrt{533}}{2} & 0\\ 0 & 1 - \frac{\sqrt{533}}{2}\end{pmatrix} \quad (1.1.13.17)$$

Using *Spectral decomposition* of matrix we can express equation as

$$u_1(x-\alpha)+u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha)+v_2(y-\beta)) \tag{1.1.13.18}$$

Substituting values in above equation we get;

$$\frac{\sqrt{533}-22}{2}(x+1)+(y-2) = \pm\sqrt{-\frac{1-\frac{\sqrt{533}}{2}}{1+\frac{\sqrt{533}}{2}}}\left(\frac{-22-\sqrt{533}}{2}(x+1)+(y-2)\right) \tag{1.1.13.19}$$

Simplifying (1.1.13.19),

$$3x-2y+7=0 \text{ and } 4x+5y-5=0 \tag{1.1.13.20}$$

$$\implies (3x-2y+7)(4x+5y-5)=0 \tag{1.1.13.21}$$

Thus the equation of lines are

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 5 \tag{1.1.13.22}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{1.1.13.23}$$

Angle between the straight lines: The angle

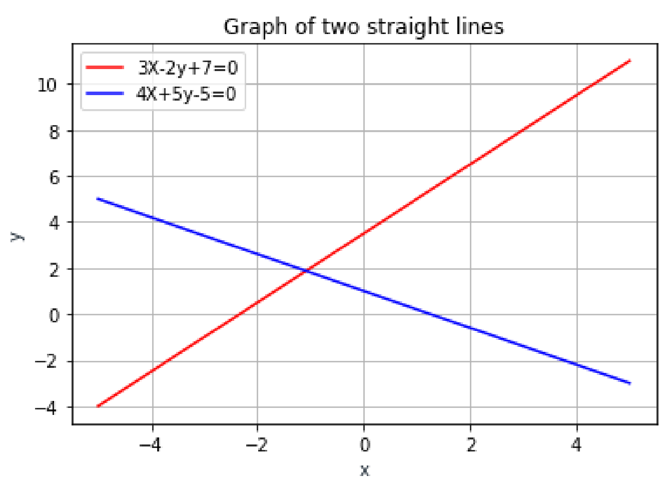

Fig. 1: Pair of straight lines

between the lines can be expressed in terms of normal vectors

$$\mathbf{n_1} = \begin{pmatrix}4\\5\end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{1.1.13.24}$$

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{1.1.13.25}$$

$$\implies \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \tag{1.1.13.26}$$

1.1.14. Find the value of h so that the equation

$$6x^2+2hxy+12y^2+22x+31y+20=0 \tag{1.1.14.1}$$

may represent two straight lines.
Solution: The general equation second degree is given by

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{1.1.14.2}$$

(1.1.14.2) represents pair of straight lines if

$$\begin{vmatrix}a & h & d\\ h & c & e\\ d & e & f\end{vmatrix} = 0 \tag{1.1.14.3}$$

From (1.1.14.3), given equation represents pair of straight lines if

$$\begin{vmatrix}6 & h & 11\\ h & 12 & \frac{31}{2}\\ 11 & \frac{31}{2} & 20\end{vmatrix} = 0 \tag{1.1.14.4}$$

$$\implies h = \frac{17}{2} \text{ or } h = \frac{171}{20} \tag{1.1.14.5}$$

Verify (1.1.14.5) using python code from

https://github.com/shreeprasadbhat/matrix−theory/tree/master/assignment5/codes/solve_determinant.py

The general equation second degree is given by

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{1.1.14.6}$$

Let (α,β) be their point of intersection, then

$$\begin{pmatrix}a & h\\ h & b\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-d\\ -e\end{pmatrix} \tag{1.1.14.7}$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + c \tag{1.1.14.8}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{1.1.14.9}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \tag{1.1.14.10}$$

(1.1.14.6) under transformation (1.1.14.10) will become,

$$aX^2 + 2bXY + cY^2 = 0 \tag{1.1.14.11}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{1.1.14.12}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{1.1.14.13}$$

$$\begin{pmatrix} X' & Y' \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \tag{1.1.14.14}$$

where $X' = Xu_1 + Yu_2$ and $Y' = Xv_1 + Yv_2$

$$\implies \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \tag{1.1.14.15}$$

This is called *Spectral decomposition* of matrix

$$X' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}Y' \tag{1.1.14.16}$$

$$u_1X + u_2Y = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1X + v_2Y) \tag{1.1.14.17}$$

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \tag{1.1.14.18}$$

Given equation is

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \tag{1.1.14.19}$$

Substituting in (1.1.14.7)

$$\begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -11 \\ -\frac{31}{2} \end{pmatrix} \tag{1.1.14.20}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.1.14.21}$$

Verify (1.1.14.21) using python code from

https://github.com/shreeprasadbhat/matrix−theory/tree/master/assignment5/codes/find_intersection.py

Taking $h = \frac{17}{2}$

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{1.1.14.22}$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{1.1.14.23}$$

$$\mathbf{P} = \begin{pmatrix} \frac{-5\sqrt{13}-6}{17} & \frac{-6+5\sqrt{13}}{17} \\ 1 & 1 \end{pmatrix} \tag{1.1.14.24}$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{5\sqrt{13}}{2} & 0 \\ 0 & 9 + \frac{5\sqrt{13}}{2} \end{pmatrix} \tag{1.1.14.25}$$

Verify (1.1.14.24) and (1.1.14.25) using python code from

https://github.com/shreeprasadbhat/matrix−theory/tree/master/assignment5/codes/diagonalize1.py

Substituting (1.1.14.21), (1.1.14.24) and (1.1.14.25) in (1.1.14.18),

$$\frac{-5\sqrt{13}-6}{17}(x+1) + (y-2) = \pm\sqrt{-\frac{9+\frac{5\sqrt{13}}{2}}{9-\frac{5\sqrt{13}}{2}}}\left(\frac{-6+5\sqrt{13}}{17}(x+1) + (y+2)\right) \tag{1.1.14.26}$$

Simplifying (1.1.14.26),

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0 \tag{1.1.14.27}$$

$$\implies (2x+3y+4)(3x+4y+5) = 0 \tag{1.1.14.28}$$

Verify (1.1.14.27) using python code from

https://github.com/shreeprasadbhat/matrix−theory/tree/master/assignment5/codes/calculate1.py

Taking $h = \frac{171}{20}$

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{1.1.14.29}$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{171}{2} \\ \frac{171}{2} & 12 \end{pmatrix} \tag{1.1.14.30}$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{3649}-20}{57} & \frac{-20+\sqrt{3649}}{57} \end{pmatrix} \tag{1.1.14.31}$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{3\sqrt{3649}}{20} & 0 \\ 0 & 9 + \frac{3\sqrt{3649}}{20} \end{pmatrix} \tag{1.1.14.32}$$

Verify (1.1.14.31) and (1.1.14.32) using python code from

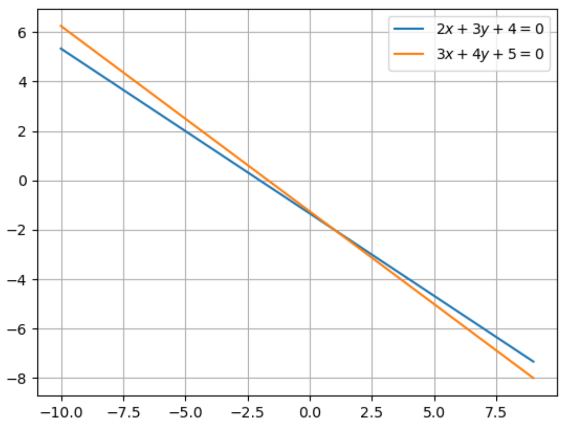

Fig. 1: Pair of straight lines $3x + 4y + 5 = 0$ and $2x + 3y + 4 = 0$

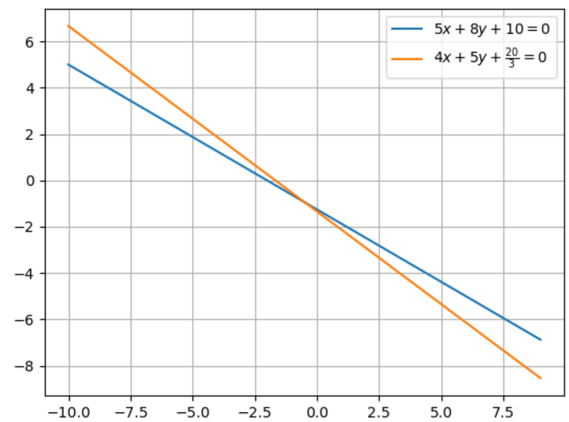

Fig. 1: Pair of straight lines $4x + 5y + \frac{20}{3} = 0$ and $5x + 8y + 10 = 0$

https://github.com/shreeprasadbhat/matrix−theory/tree/master/assignment5/codes/diagonalize2.py

Substituting (1.1.14.21),(1.1.14.31) and (1.1.14.32) in (1.1.14.18),

$$\frac{-\sqrt{3649}-20}{57}(x+1)+(y-2) = \pm\sqrt{-\frac{9+\frac{3\sqrt{3649}}{20}}{9-\frac{3\sqrt{3649}}{20}}}\left(\frac{-20+\sqrt{3649}}{57}(x+1)+(y+2)\right) \tag{1.1.14.33}$$

Simplifying (1.1.14.32),

$$2x+3y+4=0 \text{ and } 3x+4y+5=0 \tag{1.1.14.34}$$

$$\implies (2x+3y+4)(3x+4y+5)=0 \tag{1.1.14.35}$$

Verify (1.1.14.33) using python code from

https://github.com/shreeprasadbhat/matrix−theory/tree/master/assignment5/codes/calculate2.py

2 General Equation. Tracing of Curves

2.1 40

2.1.1. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre

$$12x^2-23xy+10y^2-25x+26y=14 \tag{2.1.1.1}$$

Solution: The given equation (2.1.1.1) can be expressed as

$$\mathbf{x}^T\begin{pmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{pmatrix}\mathbf{x}+2\begin{pmatrix}\frac{-25}{2} & 13\end{pmatrix}\mathbf{x}-14=0 \tag{2.1.1.2}$$

where

$$\mathbf{V}=\begin{pmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{pmatrix} \tag{2.1.1.3}$$

$$\mathbf{u}=\begin{pmatrix}\frac{-25}{2}\\ 13\end{pmatrix} \tag{2.1.1.4}$$

$$f=-14 \tag{2.1.1.5}$$

$$\det(\mathbf{V})=\begin{vmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{vmatrix} \tag{2.1.1.6}$$

$$\implies \det(\mathbf{V})=\frac{-49}{4} \tag{2.1.1.7}$$

$$\implies \det(\mathbf{V})<0 \tag{2.1.1.8}$$

Since $\det(\mathbf{V})<0$ the given equation (2.1.1.2) represents the hyperbola The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda \mathbf{I} \right| = 0 \tag{2.1.1.9}$$

$$\begin{vmatrix} 12-\lambda & \frac{-23}{2} \\ \frac{-23}{2} & 10-\lambda \end{vmatrix} = 0 \tag{2.1.1.10}$$

$$\implies 4\lambda^2 - 88\lambda - 49 = 0 \tag{2.1.1.11}$$

The eigenvalues are the roots of equation 2.1.1.11 is given by

$$\lambda_1 = \frac{22 + \sqrt{533}}{2} \tag{2.1.1.12}$$

$$\lambda_2 = \frac{22 - \sqrt{533}}{2} \tag{2.1.1.13}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \tag{2.1.1.14}$$

$$\implies (\mathbf{V} - \lambda \mathbf{I})\mathbf{p} = 0 \tag{2.1.1.15}$$

For $\lambda_1 = \frac{22-\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{\sqrt{553}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{2.1.1.16}$$

By row reduction ,

$$\begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{2.1.1.17}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\frac{\sqrt{533}+2}{2}}} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{2.1.1.18}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{23}{2} R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \tag{2.1.1.19}$$

Subsituting equation 2.1.1.19 in equation 2.1.1.15 we get

$$\begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.1.20}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
Let $v_2 = t$

$$v_1 = \frac{-t(2 - \sqrt{533})}{23} \tag{2.1.1.21}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{-t(2-\sqrt{533})}{23} \\ t \end{pmatrix} \tag{2.1.1.22}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \tag{2.1.1.23}$$

For $\lambda_2 = \frac{22+\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{2-\sqrt{553}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.1.1.24}$$

By row reduction ,

$$\begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.1.1.25}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\frac{2-\sqrt{533}}{2}}} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.1.1.26}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{23}{2} R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \tag{2.1.1.27}$$

Subsituting equation 2.1.1.27 in equation 2.1.1.15 we get

$$\begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.1.28}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
Let $v_2 = t$

$$v_1 = \frac{-t(2 + \sqrt{533})}{23} \tag{2.1.1.29}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(2+\sqrt{533})}{23} \\ t \end{pmatrix} \tag{2.1.1.30}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{-\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \tag{2.1.1.31}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.1.1.32}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.1.1.33}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.1.1.34}$$

Substituting equations 2.1.1.23, 2.1.1.31 in equation 2.1.1.33 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & \frac{-\sqrt{533}-2}{23} \\ 1 & 1 \end{pmatrix} \tag{2.1.1.35}$$

Substituting equations 2.1.1.12, 2.1.1.13 in 2.1.1.34 we get

$$\mathbf{D} = \begin{pmatrix} \frac{22-\sqrt{533}}{2} & 0 \\ 0 & \frac{22+\sqrt{533}}{2} \end{pmatrix} \tag{2.1.1.36}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.1.1.37}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-40}{49} & \frac{-46}{49} \\ \frac{-46}{49} & \frac{-48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{2.1.1.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{40}{49} & \frac{46}{49} \\ \frac{46}{49} & \frac{48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{2.1.1.39}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.1.40}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 26 > 0 \tag{2.1.1.41}$$

there isn't a need to swap axes
In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{2.1.1.42}$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{2\sqrt{13}}{\sqrt{22 + \sqrt{533}}} \tag{2.1.1.43}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \frac{2\sqrt{13}}{\sqrt{\sqrt{533} - 22}} \tag{2.1.1.44}$$

Now (2.1.1.2) can be written as,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.1.1.45}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.1.1.46}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.1.1.47}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.1.48}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{2(\sqrt{533}-2)}{23} + 1 \\ \frac{2(-\sqrt{533}-2)}{23} + 1 \end{pmatrix} \tag{2.1.1.49}$$

Subsituting the eqauations (2.1.1.41), (2.1.1.36) in equation (2.1.1.45)

$$\mathbf{y}^T \begin{pmatrix} \frac{22+\sqrt{533}}{2} & 0 \\ 0 & \frac{22-\sqrt{533}}{2} \end{pmatrix} \mathbf{y} - 26 = 0 \tag{2.1.1.50}$$

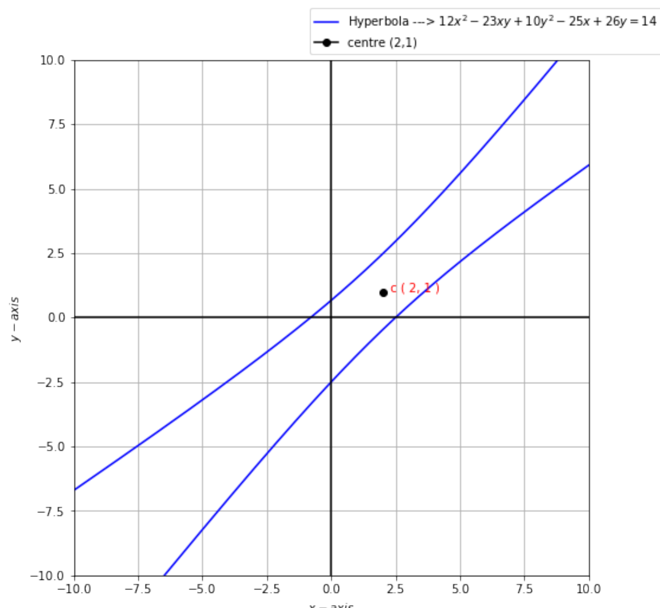

Fig. 2.1.1.1: Hyperbola when origin is shifted

The figure 2.1.1.1 verifies the given equation (2.1.1.2) as hyperbola with centre $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2.1.2. What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0 \tag{2.1.2.1}$$

Find the center.

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.1.2.2}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \tag{2.1.2.3}$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \tag{2.1.2.4}$$

$$f = -2 \tag{2.1.2.5}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \tag{2.1.2.6}$$

Hence from (2.1.2.6) we conclude that given equation is an ellipse. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \tag{2.1.2.7}$$

$$\implies \lambda^2 - 50\lambda + 400 = 0 \tag{2.1.2.8}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.1.2.8). The roots of (2.1.2.8) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \tag{2.1.2.9}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.1.2.10}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.1.2.11}$$

for $\lambda_1 = 10$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \tag{2.1.2.12}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.1.2.13}$$

Again, for $\lambda_2 = 40$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \tag{2.1.2.14}$$

$$\implies \mathbf{p_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.1.2.15}$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{PDP^{-1}P} \qquad = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \tag{2.1.2.16}$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \tag{2.1.2.17}$$

Now (2.1.2.2) can be written as,

$$\mathbf{y^T D y} = \mathbf{u^T V^{-1} u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.1.2.18}$$

And,

$$\mathbf{c} = -\mathbf{V^{-1}u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.1.2.19}$$

$$\mathbf{y} = \mathbf{P^T}\,(\mathbf{x} - \mathbf{c}) \tag{2.1.2.20}$$

The centre/vertex of the conic section in (2.1.2.2) is given by $\mathbf{c}$ in (2.1.2.19). We compute $\mathbf{V^{-1}}$ as follows,

$$\left(\begin{array}{cc|cc} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{array}\right) \xleftrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.1.2.21}$$

$$\xleftrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.1.2.22}$$

Hence $\mathbf{V^{-1}}$ is given by,

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.1.2.23}$$

Now $\mathbf{u^T V^{-1} u}$ is given by,

$$\mathbf{u^T V^{-1} u} = \frac{1}{400}\begin{pmatrix} 1 & 7 \end{pmatrix}\begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \tag{2.1.2.24}$$

And, $\mathbf{V^{-1}u}$ is given by,

$$\mathbf{V^{-1}u} = \frac{1}{400}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.1.2.25}$$

By putting the value of (2.1.2.25), the center of the ellipse is given by (2.1.2.19) as follows,

$$\mathbf{c} = -\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \tag{2.1.2.26}$$

Also the semi-major axis (a) and semi-minor

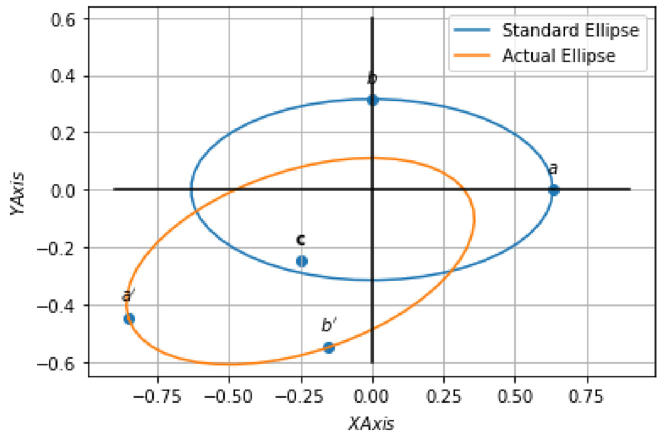

Fig. 2.1.2.1: Graphical representation of the ellipse

axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \tag{2.1.2.27}$$

$$b = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \tag{2.1.2.28}$$

Finally from (2.1.2.18), the equation of ellipse is given by,

$$\mathbf{y}^T \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \mathbf{y} = 4 \tag{2.1.2.29}$$

The following figure 2.1.2.1 is the graphical representation of the ellipse in (2.1.2.29),

2.1.3. What conic does the following equation represent?

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 \tag{2.1.3.1}$$

Find the center.

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.3.2}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{2.1.3.3}$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.1.3.4}$$

$$f = 5 \tag{2.1.3.5}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \tag{2.1.3.6}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \neq 0 \tag{2.1.3.7}$$

Hence from (2.1.3.6) and (2.1.3.7) we conclude that given equation is a parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 3-\lambda & -\sqrt{3} \\ -\sqrt{3} & 1-\lambda \end{vmatrix} = 0 \tag{2.1.3.8}$$

$$\implies \lambda^2 - 4\lambda = 0 \tag{2.1.3.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.1.3.9). The roots of (2.1.3.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 4 \tag{2.1.3.10}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.1.3.11}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\,\mathbf{p} = 0 \tag{2.1.3.12}$$

for $\lambda_1 = 0$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{\sqrt{3}}R_1]{R_2 = R_1 + R_2} \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix} \tag{2.1.3.13}$$

Substituting equation 2.1.3.13 in equation 2.1.3.12 and upon normalizing we get we get

$$\implies \mathbf{p_1} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \tag{2.1.3.14}$$

Again, for $\lambda_2 = 4$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix} \xleftrightarrow[R_1 = -\sqrt{3}R_1]{R_2 = -\sqrt{3}R_1 + R_2} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \tag{2.1.3.15}$$

Substituting equation 2.1.3.15 in equation 2.1.3.12 and upon normalizing we get

$$\mathbf{p_2} = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \tag{2.1.3.16}$$

The matrix $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \quad (2.1.3.17)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \quad (2.1.3.18)$$

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = 3 - 2\sqrt{3} \quad (2.1.3.19)$$

The focal length of the parabola is given by:

$$\left|\frac{\eta}{\lambda_2}\right| = \left|\frac{3-2\sqrt{3}}{4}\right| = 0.116 \quad (2.1.3.20)$$

When $\left|\mathbf{V}\right| = 0$, (2.1.3.2) can be written as

$$\mathbf{y^T D y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \quad (2.1.3.21)$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \quad (2.1.3.22)$$

Substituting the found values

$$\mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix}3 & -2\end{pmatrix} + \frac{3-2\sqrt{3}}{2}\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix} \quad (2.1.3.23)$$

$$\implies \mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix}\frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4}\end{pmatrix} \quad (2.1.3.24)$$

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \begin{pmatrix}\frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4}\end{pmatrix} \quad (2.1.3.25)$$

using equations (2.1.3.4),(2.1.3.5),(2.1.3.14),(2.1.3.24),(2.1.3.25) and (2.1.3.14) in (2.1.3.22)

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -5 \\ \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (2.1.3.26)$$

By performing row reductions on augmented matrix

$$\left(\begin{array}{cc|c} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{array}\right) R_2 \leftrightarrow R_1$$

$$\left(\begin{array}{cc|c} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{array}\right) \quad (2.1.3.27)$$

$$\left(\begin{array}{cc|c} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - \frac{15-2\sqrt{3}}{12}R_1}$$

$$\left(\begin{array}{cc|c} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ \sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{array}\right) \quad (2.1.3.28)$$

Therefore,

$$\left(\begin{array}{cc|c} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ -\sqrt{3} & 1 & \frac{(2+3\sqrt{3})}{4} \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 + \frac{1}{\sqrt{3}}R_1}$$

$$\begin{pmatrix} 3 & -\frac{433}{250} & -\frac{311}{100} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.1.3.29)$$

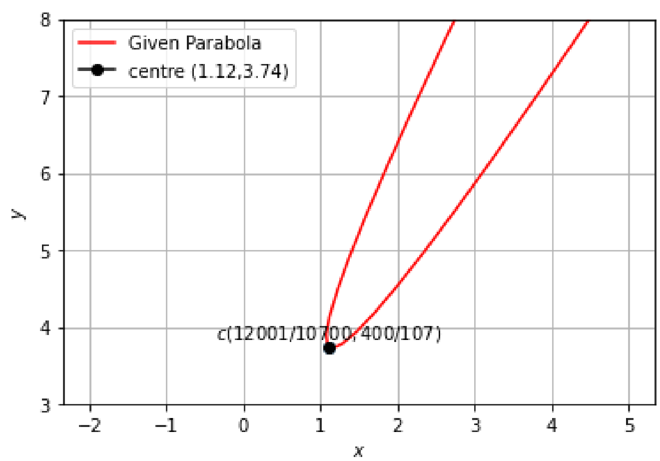

Fig. 2.1.3.1: Parabola with the center c

$$\begin{pmatrix} 3 & -\frac{433}{250} & -\frac{311}{100} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}}$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.1.3.30)$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{-200}{107} R_2}$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.1.3.31)$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + \frac{433}{750} R_2}$$

$$\begin{pmatrix} 1 & 0 & \frac{12001}{10700} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.1.3.32)$$

On solving for values of $\mathbf{c}$ from (2.1.3.32) The vertex of parabola is $\mathbf{c}=\begin{pmatrix} \frac{12001}{10700} \\ \frac{400}{107} \end{pmatrix}$.

2.1.4. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre.

$$2x^2 - 72xy + 23y^2 - 4x - 2y - 48 = 0 \quad (2.1.4.1)$$

Solution:

2.1.5. What conic does the given equations represent?

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \quad (2.1.5.1)$$

Solution: The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.1.5.2)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{pmatrix} \quad (2.1.5.3)$$

$$\mathbf{u} = \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (2.1.5.4)$$

$$f = 4 \quad (2.1.5.5)$$

The above equation (2.1.5.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.1.5.6)$$

Verify the given equation as if it is pair of straight lines

$$\Delta = \begin{vmatrix} 6 & \frac{-5}{2} & 7 \\ \frac{-5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix} \quad (2.1.5.7)$$

$$\implies 6 \begin{vmatrix} -6 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{vmatrix} - \frac{-5}{2} \begin{vmatrix} -\frac{5}{2} & \frac{5}{2} \\ 7 & 4 \end{vmatrix} + 7 \begin{vmatrix} -\frac{5}{2} & -6 \\ 7 & \frac{5}{2} \end{vmatrix} = 0 \quad (2.1.5.8)$$

$$\implies \Delta = 0 \quad (2.1.5.9)$$

Since equation (2.1.5.6) is satisfied, we could say that the given equation represents two straight lines

$$\Delta_V = \begin{vmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{vmatrix} < 0 \quad (2.1.5.10)$$

Let the equations of lines be,

$$\left(\mathbf{n_1}^T \mathbf{x} - c_1\right)\left(\mathbf{n_1}^T \mathbf{x} - c_1\right) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.1.5.11)$$

$$\left(\mathbf{n_1}^T \mathbf{x} - c_1\right)\left(\mathbf{n_2}^T \mathbf{x} - c_2\right) = \mathbf{x}^T \begin{pmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 7 & \frac{5}{2} \end{pmatrix} \mathbf{x} + 4 \quad (2.1.5.12)$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} \quad (2.1.5.13)$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2 \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (2.1.5.14)$$

$$c_1 c_2 = 4 \quad (2.1.5.15)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.1.5.16)$$

$$\implies m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (2.1.5.17)$$

$$\mathbf{n_i} = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.1.5.18)$$

Substituting the given data in above equations

(2.1.5.16) we get,

$$-6m^2 - 5m + 6 = 0 \quad (2.1.5.19)$$

$$\implies m_i = \frac{\frac{-5}{2} \pm \sqrt{-(\frac{-169}{4})}}{-6} \quad (2.1.5.20)$$

Solving equation (2.1.5.20) we get,

$$m_1 = -\frac{3}{2}, m_2 = \frac{2}{3} \quad (2.1.5.21)$$

$$= \mathbf{n_1} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.1.5.22)$$

We know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.1.5.23)$$

Verification using Toeplitz matrix, From equation (2.1.5.22)

$$\mathbf{n_1} = \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.1.5.24)$$

$$\implies \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.1.5.25)$$

$\implies$ Equation (2.1.5.22) satisfies (2.1.5.23)
c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.1.5.26)$$

Substituting (2.1.5.22) in (2.1.5.26), the augmented matrix is,

$$\begin{pmatrix} -3 & -2 & 14 \\ -2 & 3 & 5 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 2R_1]{R_1 \leftarrow -R_1/3} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{14}{3} \\ 0 & \frac{13}{3} & -\frac{13}{3} \end{pmatrix} \quad (2.1.5.27)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{2}{3}R_2]{R_2 \leftarrow \frac{3}{13}R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \end{pmatrix} \quad (2.1.5.28)$$

$$\implies c_1 = -4, c_2 = -1 \quad (2.1.5.29)$$

Equations (2.1.5.11), can be modified as,from (2.1.5.22) and (2.1.5.29) in we get,

$$\begin{pmatrix} -3 & -2 \end{pmatrix} \mathbf{x} = -4 \quad (2.1.5.30)$$

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = -1 \quad (2.1.5.31)$$

$$\implies (-3x - 2y + 4)(-2x + 3y + 1) = 0$$

$$\implies \boxed{(3x + 2y - 4)(2x - 3y - 1) = 0} \quad (2.1.5.32)$$

The angle between the lines can be expressed as,

$$\mathbf{n_1} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.1.5.33)$$

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \quad (2.1.5.34)$$

$$\implies \quad \theta = \cos^{-1}(\frac{0}{\sqrt{169}}) = 90^\circ. \quad (2.1.5.35)$$

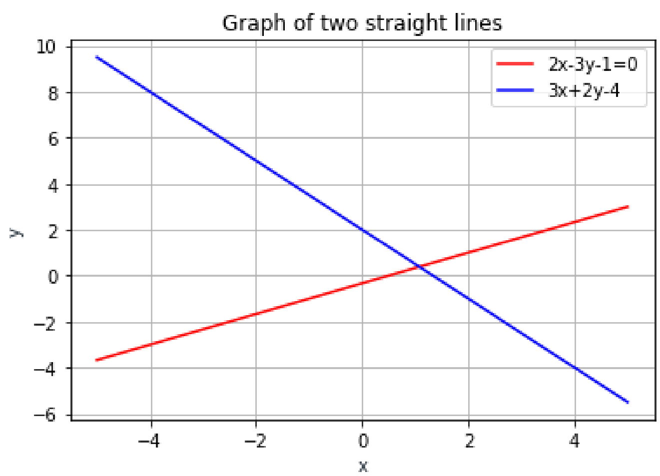

Fig. 2.1.5.1: Pair of straight lines

2.1.6. What conic does the following equation represent? Find its equation and centre.

$$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$$

Solution: The general equation of second degree can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.1.6.1)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.1.6.2)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (2.1.6.3)$$

From (2.1.6.2) and (2.1.6.3)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \tag{2.1.6.4}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -\frac{13}{2} \end{pmatrix} \tag{2.1.6.5}$$

$$\left|\mathbf{V}\right| = \begin{vmatrix} 3 & -4 \\ -4 & 3 \end{vmatrix} = -25 \tag{2.1.6.6}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{2.1.6.7}$$

Since $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = diag\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix} \tag{2.1.6.8}$$

or equivalently

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.1.6.9}$$

Eigen vectors of real symmetric matrix $\mathbf{V}$ are orthogonal. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 3 & 4 \\ 4 & \lambda + 3 \end{vmatrix} = 0 \tag{2.1.6.10}$$

$$\implies \lambda^2 - 25 = 0 \tag{2.1.6.11}$$

$$\implies \lambda_1 = -5, \lambda_2 = 5 \tag{2.1.6.12}$$

From (2.1.6.7) and (2.1.6.12) the equation represents a hyperbola. The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.1.6.13}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{2.1.6.14}$$

For $\lambda_1 = -5$:

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow -\frac{R_1}{4}} \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \tag{2.1.6.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.1.6.16}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.6.17}$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.6.18}$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{2.1.6.19}$$

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} \tag{2.1.6.20}$$

Let $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$ with $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Substituting in (2.1.6.1)

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.1.6.21}$$

with centre

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} -\frac{41}{25} \\ \frac{1}{50} \end{pmatrix} \tag{2.1.6.22}$$

and minor and major axes parameters as

$$\sqrt{\frac{\lambda_1}{f - \mathbf{u^T}\mathbf{V^{-1}}\mathbf{u}}} = \sqrt{\frac{500}{33}}, \quad \sqrt{\frac{\lambda_2}{\mathbf{u^T}\mathbf{V^{-1}}\mathbf{u} - f}} = \sqrt{\frac{500}{33}} \tag{2.1.6.23}$$

The equation of hyperbola is

$$\frac{y_2^2}{\frac{33}{500}} - \frac{y_1^2}{\frac{33}{500}} = 1 \tag{2.1.6.24}$$

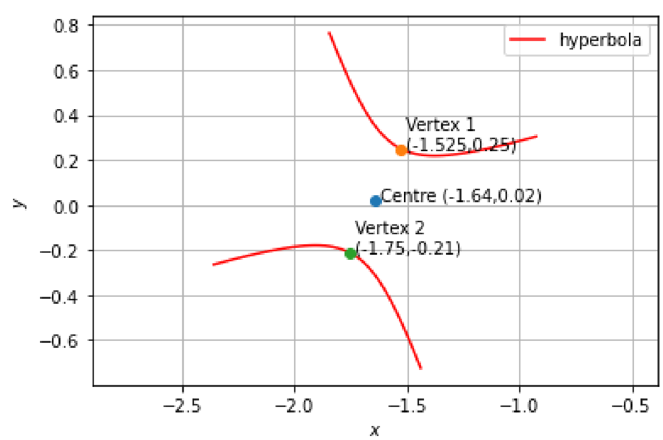

Fig. 2.1.6.1

2.1.7. Find the asymptotes of the hyperbola given below and also the equations to their conjugate hyperbolas.
$8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$ **Solution:** The

above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.7.1}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \tag{2.1.7.2}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.7.3}$$

$$f = -2 \tag{2.1.7.4}$$

Expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \tag{2.1.7.5}$$

Hence from (2.1.7.5) given equation represents the hyperbola The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.1.7.6}$$

$$\begin{vmatrix} 8-\lambda & 5 \\ 5 & -3-\lambda \end{vmatrix} = 0 \tag{2.1.7.7}$$

$$(8-\lambda)(-3-\lambda) - 25 = 0 \tag{2.1.7.8}$$

$$\lambda_1 = \frac{5+\sqrt{221}}{2} \tag{2.1.7.9}$$

$$\lambda_2 = \frac{5-\sqrt{221}}{2} \tag{2.1.7.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.1.7.11}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.1.7.12}$$

For $\lambda_1 = \frac{5+\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{2.1.7.13}$$

By row reduction ,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{2.1.7.14}$$

$$\xleftrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} \frac{-11-\sqrt{221}}{2} & 5 \\ \frac{11-\sqrt{221}}{2} & 5 \end{pmatrix} \tag{2.1.7.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11-\sqrt{221}}{10}R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \tag{2.1.7.16}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \tag{2.1.7.17}$$

Subsituting equation 2.1.7.17 in equation 2.1.7.12 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.7.18}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11+\sqrt{221})}{10} \tag{2.1.7.19}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{t(11+\sqrt{221})}{10} \\ t \end{pmatrix} \tag{2.1.7.20}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \tag{2.1.7.21}$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \tag{2.1.7.22}$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10}R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \tag{2.1.7.23}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \tag{2.1.7.24}$$

Substiuting equation 2.1.7.24 in equation 2.1.7.12 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.7.25}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t(10)}{11+\sqrt{221}} \tag{2.1.7.26}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \tag{2.1.7.27}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \tag{2.1.7.28}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.1.7.29}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.1.7.30}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.1.7.31}$$

Substituting equations 2.1.7.21, 2.1.7.28 in equation 2.1.7.30 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \tag{2.1.7.32}$$

Substituting equations 2.1.7.9, 2.1.7.10 in 2.1.7.31 we get

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \tag{2.1.7.33}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.1.7.34}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.7.35}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.7.36}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.1.7.37}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0 \tag{2.1.7.38}$$

there isn't a need to swap axes In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{2.1.7.39}$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{2}{5+\sqrt{221}}} \tag{2.1.7.40}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{\frac{2}{5-\sqrt{221}}} \tag{2.1.7.41}$$

Now we have,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.1.7.42}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.1.7.43}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.1.7.44}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.1.7.45}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{70} + \frac{3}{7} \\ \frac{10}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \tag{2.1.7.46}$$

Subsituting the eqauations (2.1.7.38), (2.1.7.33) in equation (2.1.7.42)

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} + 2 = 0 \tag{2.1.7.47}$$

Asymptotes of hyperbola Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$8x^2 + 10xy - 3y^2 - 2x + 4y + K = 0 \tag{2.1.7.48}$$

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.7.49}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \tag{2.1.7.50}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.7.51}$$

$$f = K \tag{2.1.7.52}$$

$$\Delta = \begin{vmatrix} 8 & 5 & -1 \\ 5 & -3 & 2 \\ -1 & 2 & K \end{vmatrix} \quad (2.1.7.53)$$

$$\implies K = -1 \quad (2.1.7.54)$$

Similar way expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \quad (2.1.7.55)$$

From (2.1.7.55) we could say that the given equation represents two straight lines Let the equations of lines be,

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_1}^T\mathbf{x} - c_1\right) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.1.7.56)$$

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -1 & 2 \end{pmatrix}\mathbf{x} - 1 \quad (2.1.7.57)$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} \quad (2.1.7.58)$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.1.7.59)$$

$$c_1c_2 = -1 \quad (2.1.7.60)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.1.7.61)$$

$$\implies m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (2.1.7.62)$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.1.7.63)$$

Substituting the given data in above equations (2.1.7.61) we get,

$$-3m^2 + 10m + 8 = 0 \quad (2.1.7.64)$$

$$m_1 = 4, m_2 = \frac{-2}{3} \quad (2.1.7.65)$$

$$= \mathbf{n_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (2.1.7.66)$$

We know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.1.7.67)$$

Verification using Toeplitz matrix, From equation (2.1.7.66)

$$\mathbf{n_1} = \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & -1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (2.1.7.68)$$

$$\implies \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.1.7.69)$$

$\implies$ Equation (2.1.7.66) satisfies (2.1.7.67) c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.1.7.70)$$

Substituting (2.1.7.66) in (2.1.7.70), the augmented matrix is,

$$\begin{pmatrix} -4 & -2 & -2 \\ 1 & -3 & 4 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_1 \leftarrow -R_1/4} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} \end{pmatrix} \quad (2.1.7.71)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{1}{2}R_2]{R_2 \leftarrow -\frac{2}{7}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (2.1.7.72)$$

$$\implies c_1 = 1, c_2 = -1 \quad (2.1.7.73)$$

Equations (2.1.7.56), can be modified as,from (2.1.7.66) and (2.1.7.73) in we get,

$$\begin{pmatrix} -4 & 1 \end{pmatrix}\mathbf{x} = 1 \quad (2.1.7.74)$$

$$\begin{pmatrix} -2 & -3 \end{pmatrix}\mathbf{x} = -1 \quad (2.1.7.75)$$

$$\implies (-4x + y - 1)(-2x - 3y + 1) = 0$$
$$\implies \boxed{(4x - y + 1)(2x + 3y - 1) = 0} \quad (2.1.7.76)$$

The angle between the lines can be expressed as,

$$\mathbf{n_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (2.1.7.77)$$

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \quad (2.1.7.78)$$

$$\implies \theta = \cos^{-1}\left(\frac{0}{\sqrt{221}}\right) = 90°. \quad (2.1.7.79)$$

Equation of Asymptotes: The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant (2.1.7.50)

$$\left| V - \lambda\mathbf{I} \right| = 0 \quad (2.1.7.80)$$

$$\begin{vmatrix} 8-\lambda & 5 \\ 5 & -3-\lambda \end{vmatrix} = 0 \quad (2.1.7.81)$$

$$(8-\lambda)(-3-\lambda) - 25 = 0 \quad (2.1.7.82)$$

$$\lambda_1 = \frac{5+\sqrt{221}}{2} \quad (2.1.7.83)$$

$$\lambda_2 = \frac{5-\sqrt{221}}{2} \quad (2.1.7.84)$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.1.7.85)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.1.7.86)$$

For $\lambda_1 = \frac{5+\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \quad (2.1.7.87)$$

By row reduction ,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \quad (2.1.7.88)$$

$$\xleftrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} \frac{-11-\sqrt{221}}{2} & 5 \\ \frac{11-\sqrt{221}}{2} & 5 \end{pmatrix} \quad (2.1.7.89)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \quad (2.1.7.90)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \quad (2.1.7.91)$$

Subsituting equation 2.1.7.91 in equation 2.1.7.86 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.7.92)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11+\sqrt{221})}{10} \quad (2.1.7.93)$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{t(11+\sqrt{221})}{10} \\ t \end{pmatrix} \quad (2.1.7.94)$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \quad (2.1.7.95)$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \quad (2.1.7.96)$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \quad (2.1.7.97)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \quad (2.1.7.98)$$

Subsituting equation 2.1.7.98 in equation 2.1.7.86 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.7.99)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t(10)}{11+\sqrt{221}} \quad (2.1.7.100)$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \quad (2.1.7.101)$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \quad (2.1.7.102)$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (2.1.7.103)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \quad (2.1.7.104)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.1.7.105)$$

Substituting equations 2.1.7.95, 2.1.7.102 in equation 2.1.7.104 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \tag{2.1.7.106}$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \tag{2.1.7.107}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.1.7.108}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.7.109}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.7.110}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.1.7.111}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{2.1.7.112}$$

Now,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.1.7.113}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.1.7.114}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.1.7.115}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.1.7.116}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{70} + \frac{3}{7} \\ \frac{10}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \tag{2.1.7.117}$$

Subsituting the eqauations (2.1.7.112), (2.1.7.107) in equation (2.1.7.113) Equation of asymptotes is

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} + 1 = 0 \tag{2.1.7.118}$$

And the Equations of Conjugate hyperbola is 2(Equation of Asymptotes)- Equation of hyperbola.

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} = 0 \tag{2.1.7.119}$$

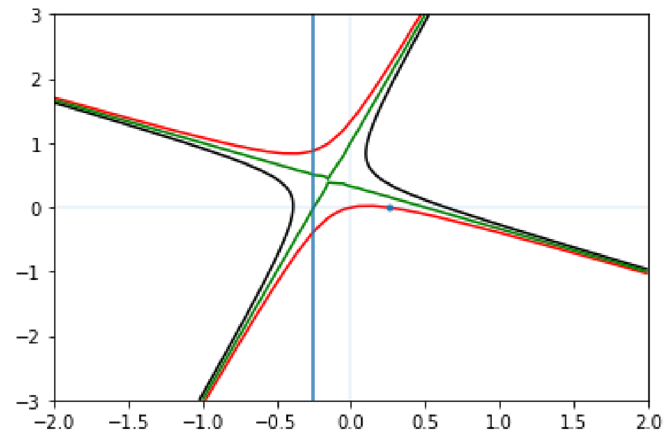

Fig. 2.1.7.1: Hyperbola with assymptotes and its conjugate

2.1.8. What conics do the following equation represents? When possible, find the center and the equation reffered to the center.

$$55x^2 - 120xy + 20y^2 + 64x - 48y = 0 \tag{2.1.8.1}$$

Solution: The general equation of second degree can be represented as:

$$\mathbf{X}^T\mathbf{V}\mathbf{X} + 2\mathbf{u}^T\mathbf{X} + f = 0 \tag{2.1.8.2}$$

The above 2.1.8.1 can also be written as:

$$\mathbf{X}^T\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix}\mathbf{X} + 2\begin{pmatrix} 32 & -24 \end{pmatrix}\mathbf{X} + 0 = 0 \tag{2.1.8.3}$$

So,

$$\mathbf{V} = \begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix} \tag{2.1.8.4}$$

and

$$\mathbf{u} = \begin{pmatrix} 32 \\ -24 \end{pmatrix} \tag{2.1.8.5}$$

$$f = 0 \tag{2.1.8.6}$$

Now,

$$\det \mathbf{V} = \begin{vmatrix} 55 & -60 \\ -60 & 20 \end{vmatrix} \tag{2.1.8.7}$$

$$\implies \det \mathbf{V} = -2500 < 0 \tag{2.1.8.8}$$

As $\det \mathbf{V} < 0$, so we can say that the above conic section 2.1.8.1 is hyperbola.Now,

$$\mathbf{V}^{-1} = \frac{1}{-2500}\begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix} \tag{2.1.8.9}$$

The center of this hyperbola will be:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.1.8.10}$$

$$\implies \mathbf{c} = \frac{1}{2500}\begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix}\begin{pmatrix} 32 \\ -24 \end{pmatrix} \tag{2.1.8.11}$$

$$\implies \mathbf{c} = \begin{pmatrix} -\frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \tag{2.1.8.12}$$

$$\tag{2.1.8.13}$$

Now the characteristic equation of $\mathbf{V}$ is obtained as:

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{2.1.8.14}$$

$$\implies \begin{vmatrix} 55-\lambda & -60 \\ -60 & 20-\lambda \end{vmatrix} = 0 \tag{2.1.8.15}$$

$$\implies \lambda^2 - 75\lambda - 2500 = 0 \tag{2.1.8.16}$$

The eigen values are given by:

$$\lambda_1 = 100 \tag{2.1.8.17}$$

$$\lambda_2 = -25 \tag{2.1.8.18}$$

The eigen vector $\mathbf{P}$ is defined as:

$$\mathbf{VP} = \lambda\mathbf{P} \tag{2.1.8.19}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{P} = \mathbf{0} \tag{2.1.8.20}$$

For λ_1=100,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \tag{2.1.8.21}$$

By row reduction,

$$\begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/(-5)]{R_2 \leftarrow R_2/(-5)} \tag{2.1.8.22}$$

$$\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/3]{R_2 \leftarrow R_2/4} \tag{2.1.8.23}$$

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \tag{2.1.8.24}$$

So,

$$(\mathbf{V} - \lambda_1\mathbf{I})\mathbf{P_1} = \mathbf{0} \tag{2.1.8.25}$$

$$\implies \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.8.26}$$

$$\implies \mathbf{P_1} = \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix} \tag{2.1.8.27}$$

Similarly, For λ_2=100,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} 80 & -60 \\ -60 & 45 \end{pmatrix} \tag{2.1.8.28}$$

By row reduction,

$$\begin{pmatrix} 80 & -60 \\ -60 & 45 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/5]{R_2 \leftarrow R_2/5} \tag{2.1.8.29}$$

$$\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/4]{R_2 \leftarrow R_2/(-3)} \tag{2.1.8.30}$$

$$\begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} \tag{2.1.8.31}$$

So,

$$(\mathbf{V} - \lambda_2\mathbf{I})\mathbf{P_2} = \mathbf{0} \tag{2.1.8.32}$$

$$\implies \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.8.33}$$

$$\implies \mathbf{P_2} = \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix} \tag{2.1.8.34}$$

By eigen decomposition $\mathbf{V}$ can also be written as:

$$\mathbf{V} = \mathbf{PDP}^T \tag{2.1.8.35}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} \end{pmatrix} \tag{2.1.8.36}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.1.8.37}$$

So,

$$\mathbf{P} = \begin{pmatrix} -\frac{4}{3} & 1 \\ 1 & \frac{4}{3} \end{pmatrix} \tag{2.1.8.38}$$

$$\mathbf{D} = \begin{pmatrix} 100 & 0 \\ 0 & -25 \end{pmatrix} \tag{2.1.8.39}$$

and

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 16 > 0 \tag{2.1.8.40}$$

So, the axes are:

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{2}{5} \tag{2.1.8.41}$$

$$b = \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \frac{4}{5} \tag{2.1.8.42}$$

Now, the equation 2.1.8.1 can be written as:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.1.8.43}$$

where,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.1.8.44}$$

So,

$$\mathbf{y}^T\begin{pmatrix}100 & 0\\ 0 & -25\end{pmatrix}\mathbf{y} = 16 \tag{2.1.8.45}$$

$$\implies \mathbf{y}^T\begin{pmatrix}100 & 0\\ 0 & -25\end{pmatrix}\mathbf{y} - 16 = 0 \tag{2.1.8.46}$$

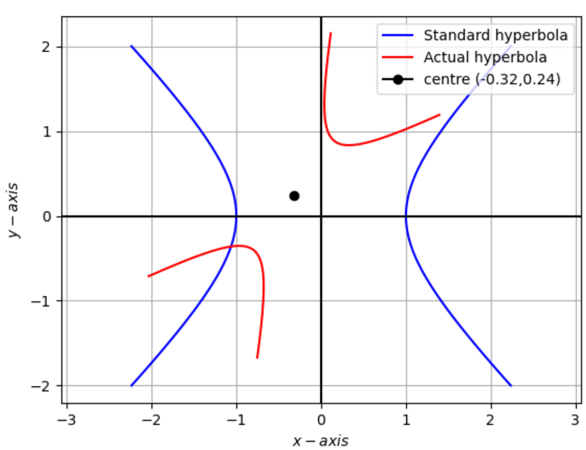

Fig. 2.1.8.1: Comparison of the Standerd and Actual Hyperbola

2.1.9. Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola

$$19x^2 + 24xy + y^2 - 22x - 6y = 0 \tag{2.1.9.1}$$

Solution: The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.1.9.2}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.9.3}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b\\ b & c\end{pmatrix} \tag{2.1.9.4}$$

$$\mathbf{u} = \begin{pmatrix}d\\ e\end{pmatrix} \tag{2.1.9.5}$$

Comparing equations 2.1.9.1 and 2.1.9.3 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}19 & 12\\ 12 & 1\end{pmatrix} \tag{2.1.9.6}$$

$$\mathbf{u} = \begin{pmatrix}-11\\ -3\end{pmatrix} \tag{2.1.9.7}$$

$$f = 0 \tag{2.1.9.8}$$

Expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix}19 & 12\\ 12 & 1\end{vmatrix} < 0 \tag{2.1.9.9}$$

Hence from 2.1.9.9 given equation represents the hyperbola.
The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.1.9.10}$$

$$\begin{vmatrix}19-\lambda & 12\\ 12 & 1-\lambda\end{vmatrix} = 0 \tag{2.1.9.11}$$

$$(19-\lambda)(1-\lambda) - 144 = 0 \tag{2.1.9.12}$$

$$\lambda_1 = -5, \lambda_2 = 25 \tag{2.1.9.13}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.1.9.14}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.1.9.15}$$

For $\lambda_1 = -5$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix}19+5 & 12\\ 12 & 1+5\end{pmatrix} \tag{2.1.9.16}$$

By row reduction ,

$$\begin{pmatrix}24 & 12\\ 12 & 6\end{pmatrix} \tag{2.1.9.17}$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix}24 & 12\\ 0 & 0\end{pmatrix} \tag{2.1.9.18}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{12}} \begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix} \tag{2.1.9.19}$$

Subsituting equation 2.1.9.19 in equation

2.1.9.15 we get

$$\begin{pmatrix}2 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1.9.20}$$

Where, $\mathbf{p} = \begin{pmatrix}u_1\\u_2\end{pmatrix}$ Let $u_1 = t$

$$u_2 = -2t \tag{2.1.9.21}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix}t\\-2t\end{pmatrix} \tag{2.1.9.22}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix}1\\-2\end{pmatrix} \tag{2.1.9.23}$$

For $\lambda_2 = 25$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix}19-25 & 12\\12 & 1-25\end{pmatrix} \tag{2.1.9.24}$$

By row reduction ,

$$\begin{pmatrix}-6 & 12\\12 & -24\end{pmatrix} \tag{2.1.9.25}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix}-6 & 12\\0 & 0\end{pmatrix} \tag{2.1.9.26}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix} \tag{2.1.9.27}$$

Subsituting equation 2.1.9.27 in equation 2.1.9.15 we get

$$\begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1.9.28}$$

Where, $\mathbf{p} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ Let $v_1 = t$

$$v_2 = \frac{t}{2} \tag{2.1.9.29}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix}t\\\frac{t}{2}\end{pmatrix} \tag{2.1.9.30}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix} \tag{2.1.9.31}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{PDP}^T \tag{2.1.9.32}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} \tag{2.1.9.33}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} \tag{2.1.9.34}$$

Substituting equations 2.1.9.23, 2.1.9.31 in equation 2.1.9.33 we get

$$\mathbf{P} = \begin{pmatrix}1 & 1\\-2 & \frac{1}{2}\end{pmatrix} \tag{2.1.9.35}$$

Substituting equation 2.1.9.13 in 2.1.9.34 we get

$$\mathbf{D} = \begin{pmatrix}-5 & 0\\0 & 25\end{pmatrix} \tag{2.1.9.36}$$

Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$19x^2 + 24xy + y^2 - 22x - 6y + K = 0 \tag{2.1.9.37}$$

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.9.38}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}19 & 12\\12 & 1\end{pmatrix} \tag{2.1.9.39}$$

$$\mathbf{u} = \begin{pmatrix}-11\\-3\end{pmatrix} \tag{2.1.9.40}$$

$$f = K \tag{2.1.9.41}$$

$$\Delta = \begin{vmatrix}19 & 12 & -11\\12 & 1 & -3\\-11 & -3 & K\end{vmatrix} \tag{2.1.9.42}$$

Since the equations represent pair of straight lines, equating the determinant to zero, we can get the value of K

$$\implies K = 4 \tag{2.1.9.43}$$

Let (α,β) be their point of intersection, then

$$\begin{pmatrix}a & b\\b & c\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}-d\\-e\end{pmatrix} \tag{2.1.9.44}$$

Substituting the values, we obtain,

$$\begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix} \tag{2.1.9.45}$$

$$\text{We get, } \alpha = \frac{1}{5}, \beta = \frac{3}{5} \tag{2.1.9.46}$$

Using Affine transformation and Spectral decomposition, we get

$$X' = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} Y' \tag{2.1.9.47}$$

$$\text{where } X' = Xu_1 + Yu_2 \tag{2.1.9.48}$$

$$Y' = Xv_1 + Yv_2 \tag{2.1.9.49}$$

$$X = x - \alpha \text{ and } Y = y - \beta \tag{2.1.9.50}$$

Therefore,

$$u_1(x-\alpha) + u_2(y-\beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x-\alpha) + v_2(y-\beta)) \tag{2.1.9.51}$$

Substituting values, we get

$$(x - \frac{1}{5}) - 2(y - \frac{3}{5}) = \pm \sqrt{\frac{25}{5}} (x - \frac{1}{5}) + \frac{1}{2}(y - \frac{3}{5}) \tag{2.1.9.52}$$

Simplifying above equation

$$8x + 9y - 7 = 0 \tag{2.1.9.53}$$

$$12x + y + 7 = 0 \tag{2.1.9.54}$$

$$\implies (8x + 9y - 7)(12x + y + 7) = 0 \tag{2.1.9.55}$$

Thus the equation of lines are

$$\begin{pmatrix} 8 & 9 \end{pmatrix} \mathbf{x} = 7 \tag{2.1.9.56}$$

$$\begin{pmatrix} 12 & 1 \end{pmatrix} \mathbf{x} = -7 \tag{2.1.9.57}$$

The Equation of Conjugate hyperbola is given by:

2(Equation of Asymptotes)- Equation of hyperbola.

From Eq 2.1.9.1 and 2.1.9.37, we obtain equation of Conjugate hyperbola as:-

$$19x^2 + 24xy + y^2 - 22x - 6y + 8 = 0 \tag{2.1.9.58}$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.1.9.59}$$

comparing equation 2.1.9.58 with the general equation of second degree given at 2.1.9.59, it can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.1.9.60}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.1.9.61}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{2.1.9.62}$$

Comparing equations 2.1.9.58 and 2.1.9.60 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \tag{2.1.9.63}$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \tag{2.1.9.64}$$

$$f = 8 \tag{2.1.9.65}$$

Therefore, the equation of the conjugate hyperbola is as given below:-

$$\mathbf{x}^T \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -11 & -3 \end{pmatrix} \mathbf{x} + 8 = 0 \tag{2.1.9.66}$$

2.2 41

2.2.1. Trace the curve

$$(x-y)^2 = x + y + 1 \tag{2.2.1.1}$$

Solution:
We have given equation as :

$$(x-y)^2 = x + y + 1 \tag{2.2.1.2}$$

$$\implies x^2 - 2xy + y^2 - x - y - 1 = 0 \tag{2.2.1.3}$$

The general equation of second degree is given

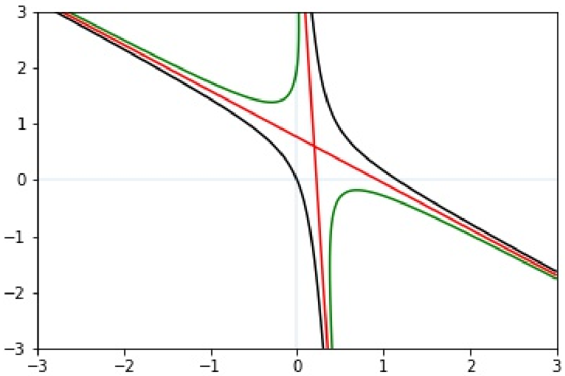

Fig. 2.1.9.1: Hyperbola, Conjugate Hyperbola and Asymptotes

by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.2.1.4}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.2.1.5}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.2.1.6}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.2.1.7}$$

Comparing (2.2.1.3) with (2.2.1.4), we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{2.2.1.8}$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{2.2.1.9}$$

$$f = -1 \tag{2.2.1.10}$$

Expanding the determinant of V we observe,

$$|\mathbf{V}| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0 \tag{2.2.1.11}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 1 & -1 & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{vmatrix} \neq 0 \tag{2.2.1.12}$$

Hence from (2.2.1.11) and (2.2.1.12) we conclude that given equation is an parabola.The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = 0 \tag{2.2.1.13}$$

$$\implies (\lambda - 1)^2 - 1 = 0 \tag{2.2.1.14}$$

The eigenvalues are the roots of (2.2.1.14) given by

$$\lambda_1 = 0, \lambda_2 = 2 \tag{2.2.1.15}$$

The eigenvector $\mathbf{p}$ is defined as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.2.1.16}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.2.1.17}$$

where λ is the eigenvalue. For $\lambda_1 = 0$,

$$\mathbf{V}\mathbf{p} = 0 \tag{2.2.1.18}$$

Row reducing $\mathbf{V}$ yields,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.2.1.19}$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.2.1.20}$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{2.2.1.21}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{2.2.1.22}$$

From equation (2.2.1.19) and (2.2.1.20), we have

$$\mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{and, } \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.2.1.23}$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.2.1.24}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \tag{2.2.1.25}$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{\sqrt{2}}{2} = \sqrt{2} \tag{2.2.1.26}$$

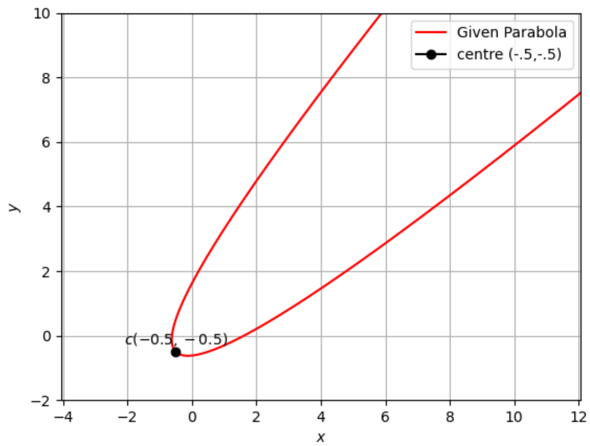

Fig. 2.2.1.1: Parabola with the center c

and its equation is

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \tag{2.2.1.27}$$

where,

$$\eta = \mathbf{u}^T\mathbf{p_1} = -\frac{1}{\sqrt{2}} \tag{2.2.1.28}$$

$$\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.2.1.29}$$

$$\implies \begin{pmatrix}-1 & -1 \\ 1 & -1 \\ -1 & 1\end{pmatrix}\mathbf{c} = \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} \tag{2.2.1.30}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix}-1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix}-1 & -1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0\end{pmatrix} \xleftrightarrow[R_1 \leftarrow -1R_1]{R_3 \leftarrow R_3 - R_1} \begin{pmatrix}1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix}1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0\end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix}1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0\end{pmatrix} \tag{2.2.1.31}$$

So,

$$\mathbf{c} = \begin{pmatrix}-\frac{1}{2} \\ -\frac{1}{2}\end{pmatrix} \tag{2.2.1.32}$$

2.2.2. Trace the curve

$$35x^2 + 30y^2 + 32x - 108y - 12xy + 59 = 0 \tag{2.2.2.1}$$

Solution: The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.2.2.2}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.2.2.3}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b \\ b & c\end{pmatrix} \tag{2.2.2.4}$$

$$\mathbf{u}^T = \begin{pmatrix}d & e\end{pmatrix} \tag{2.2.2.5}$$

Comparing (2.2.2.1) with (2.2.2.2), we get

$$\mathbf{V} = \begin{pmatrix}35 & -6 \\ -6 & 30\end{pmatrix} \tag{2.2.2.6}$$

$$\mathbf{u}^T = \begin{pmatrix}16 & -54\end{pmatrix} \tag{2.2.2.7}$$

If $|\mathbf{V}| > 0$, then (2.2.2.3) is an ellipse.

$$|V| = \begin{vmatrix}35 & -6 \\ -6 & 30\end{vmatrix} = 1014 > 0 \tag{2.2.2.8}$$

(2.2.2.3) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \tag{2.2.2.9}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \qquad |V| = 0 \tag{2.2.2.10}$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{2.2.2.11}$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|35 * 30 - 6 * 6|}\begin{pmatrix}30 & 6 \\ 6 & 35\end{pmatrix}\begin{pmatrix}16 \\ -54\end{pmatrix} \tag{2.2.2.12}$$

$$= \frac{1}{1014}\begin{pmatrix}156 \\ -1794\end{pmatrix} \tag{2.2.2.13}$$

$$= \begin{pmatrix}\frac{2}{13} \\ \frac{-23}{13}\end{pmatrix} \tag{2.2.2.14}$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{2.2.2.15}$$

(2.2.2.9) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad (2.2.2.16)$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}} \qquad (2.2.2.17)$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda - 35 & 6 \\ 6 & \lambda - 30\end{vmatrix} = 0 \qquad (2.2.2.18)$$

$$\implies \lambda^2 - 65\lambda + 1014 = 0 \qquad (2.2.2.19)$$

The eigenvalues are the roots of (2.2.2.19) given by

$$\lambda_1 = 39, \lambda_2 = 26 \qquad (2.2.2.20)$$

Calculating the major and minor axes lengths using (2.2.2.17), we get

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} =$$

$$= \begin{pmatrix}16 - 54\end{pmatrix}\frac{1}{1014}\begin{pmatrix}30 & 6 \\ 6 & 35\end{pmatrix}\begin{pmatrix}16 \\ -54\end{pmatrix}$$

$$= \frac{1}{1014}\begin{pmatrix}16 & -54\end{pmatrix}\begin{pmatrix}156 \\ -1794\end{pmatrix}$$

$$= 98$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 98 - 59 = 39 \qquad (2.2.2.21)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{39}{39}} = 1 \qquad (2.2.2.22)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{39}{26}} = \frac{\sqrt{6}}{2} \qquad (2.2.2.23)$$

2.2.3. Trace the curve

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \qquad (2.2.3.1)$$

Solution: The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \qquad (2.2.3.2)$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \qquad (2.2.3.3)$$

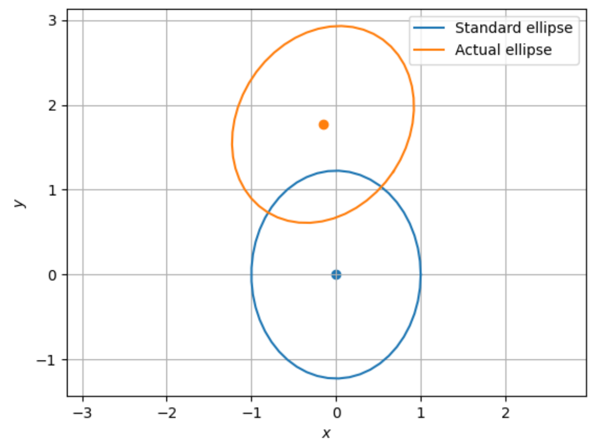

Fig. 2.2.2.1: Ellipse with center $\begin{pmatrix}\frac{2}{13} & \frac{-23}{13}\end{pmatrix}$ and having the axes lengths as 1 and $\frac{\sqrt{6}}{2}$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b \\ b & c\end{pmatrix} \qquad (2.2.3.4)$$

$$\mathbf{u}^T = \begin{pmatrix}d & e\end{pmatrix} \qquad (2.2.3.5)$$

Comparing (2.2.3.1) with (2.2.3.2), we get

$$\mathbf{V} = \begin{pmatrix}14 & -2 \\ -2 & 11\end{pmatrix} \qquad (2.2.3.6)$$

$$\mathbf{u}^T = \begin{pmatrix}-22 & -29\end{pmatrix} \qquad (2.2.3.7)$$

If $|\mathbf{V}| > 0$, then (2.2.3.3) is an ellipse.

$$|V| = \begin{vmatrix}14 & -2 \\ -2 & 11\end{vmatrix} = 150 > 0 \qquad (2.2.3.8)$$

(2.2.3.3) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \qquad (2.2.3.9)$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \qquad |V| = 0 \qquad (2.2.3.10)$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \qquad (2.2.3.11)$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|14 \times 11 - (-2 \times -2)|}\begin{pmatrix}11 & 2\\ 2 & 14\end{pmatrix}\begin{pmatrix}-22\\ -29\end{pmatrix} \tag{2.2.3.12}$$

$$= \frac{1}{150}\begin{pmatrix}300\\ 450\end{pmatrix} \tag{2.2.3.13}$$

$$= \begin{pmatrix}2\\ 3\end{pmatrix} \tag{2.2.3.14}$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{2.2.3.15}$$

(2.2.3.9) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.2.3.16}$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}} \tag{2.2.3.17}$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda - 14 & 2\\ 2 & \lambda - 11\end{vmatrix} = 0 \tag{2.2.3.18}$$

$$\implies \lambda^2 - 25\lambda + 150 = 0 \tag{2.2.3.19}$$

The eigenvalues are the roots of (2.2.3.19) given by

$$\lambda_1 = 15, \lambda_2 = 10 \tag{2.2.3.20}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.2.3.21}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.2.3.22}$$

where λ is the eigenvalue. For $\lambda_1 = 15$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix} \tag{2.2.3.23}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\ -1\end{pmatrix} \tag{2.2.3.24}$$

such that $\|\mathbf{p}_1\| = 1$. Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\ 2\end{pmatrix} \tag{2.2.3.25}$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^{T} \quad \because \mathbf{P}^{-1} = \mathbf{P}^{T} \tag{2.2.3.26}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{2.2.3.27}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix} \tag{2.2.3.28}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \begin{pmatrix}15 & 0\\ 0 & 10\end{pmatrix} \tag{2.2.3.29}$$

Calculating the ellipse parameters using (2.2.3.17), we get

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} =$$

$$= \begin{pmatrix}-22 & -29\end{pmatrix}\frac{1}{150}\begin{pmatrix}11 & 2\\ 2 & 14\end{pmatrix}\begin{pmatrix}-22\\ -29\end{pmatrix}$$

$$= \frac{1}{150}\begin{pmatrix}300 & 450\end{pmatrix}\begin{pmatrix}22\\ 29\end{pmatrix}$$

$$= 131$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 131 - 71 = 60 \tag{2.2.3.30}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{60}{15}} = 2 \tag{2.2.3.31}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{60}{10}} = \sqrt{6} \tag{2.2.3.32}$$

Thus, the given curve is found to be an ellipse from (2.2.3.8) with center at $\begin{pmatrix}2 & 3\end{pmatrix}$ and the major and minor axes lengths are calculated as $\sqrt{6}$, 2. An ellipse with these parameters along with one having center as origin are plotted as shown.

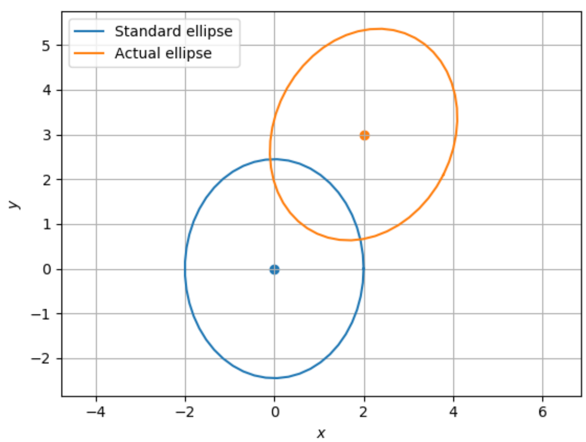

Fig. 2.2.3.1: Ellipse with center (2 3) and having the axes lengths as $\sqrt{6}$ and 2 along with an ellipse with center as origin