Coordinate Geometry Exercises

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Contents

Abstract—**This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.**

1 Conics

1.1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix}2\\0\end{pmatrix}\right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.1.1}$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.2}$$
$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.1.3}$$
$$\mathbf{u_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.1.4}$$
$$f_1 = -4 \tag{1.1.5}$$
$$\mathbf{O_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.1.6}$$

Taking equation of the second circle to be,

$$\left\|\mathbf{x} - \begin{pmatrix}2\\0\end{pmatrix}\right\|^2 = 2^2 \tag{1.1.7}$$
$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{1.1.8}$$
$$\mathbf{u_2} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.1.9}$$
$$f_2 = 0 \tag{1.1.10}$$
$$\mathbf{O_2} = \begin{pmatrix}2\\0\end{pmatrix} \tag{1.1.11}$$

Now,Subtracting equation (1.1.8) from (1.1.3) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{1.1.12}$$
$$2\mathbf{u}^T\mathbf{x} = -4 \tag{1.1.13}$$
$$\begin{pmatrix}-4 & 0\end{pmatrix}\mathbf{x} = -4 \tag{1.1.14}$$

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Which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1 \tag{1.1.15}$$

$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{1.1.16}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.17}$$

$$\mathbf{q} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.1.18}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.1.19}$$

Substituting (1.1.17) in (1.1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.20}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.21}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.22}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.23}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.24}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.25}$$

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.26}$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.27}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.1.28}$$

$$\lambda^2 = 3 \tag{1.1.29}$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{1.1.30}$$

Substituting the value of λ in(1.1.17)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.31}$$

$$\mathbf{A} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{1.1.32}$$

$$\mathbf{B} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{1.1.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix} \tag{1.1.34}$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}-1\\\sqrt{3}\end{pmatrix} \tag{1.1.35}$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{1.1.36}$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{1.1.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \tag{1.1.38}$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.1.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.1.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.1.41}$$

$$\theta_1 = 120° \tag{1.1.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \tag{1.1.43}$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.1.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.1.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.1.46}$$

$$\theta_2 = 120° \tag{1.1.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.48}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.50}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.52}$$
$$= \frac{8\pi}{3} - 2\sqrt{3} \tag{1.1.53}$$

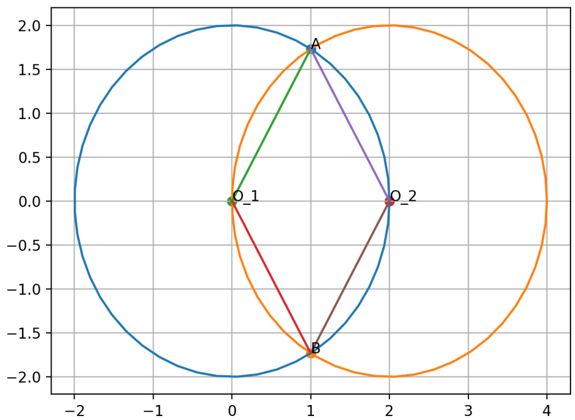

Fig. 1.1: Figure depicting intersection points of circle

1.2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre(h,k) is given by,

$$x^T x + 2u^T x + f = 0 \tag{1.2.1}$$

where,

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.2.2}$$

The radius and centre are respectively given by,

$$r = 5 \tag{1.2.3}$$
$$\mathbf{c} = -u = k\mathbf{e} \tag{1.2.4}$$

Where ,

$$\mathbf{e} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.2.5}$$
$$\mathbf{x_1} = \begin{pmatrix}2\\3\end{pmatrix} \tag{1.2.6}$$

From the given data , we modify equation 1.2.1 as,

$$\mathbf{x_1}^T\mathbf{x_1} + 2\begin{pmatrix}-k & 0\end{pmatrix}\begin{pmatrix}-k\\0\end{pmatrix} + f = 0 \tag{1.2.7}$$
$$\|\mathbf{x_1}\|^2 + 2\left(k^2\right) + f = 0 \tag{1.2.8}$$
$$2k^2 + f = -\|\mathbf{x_1}\|^2 \tag{1.2.9}$$

Subsituting $\mathbf{u}$ in equation 1.2.2 , we get ,

$$f = \begin{pmatrix}-k & 0\end{pmatrix}\begin{pmatrix}-k\\0\end{pmatrix} - r^2 \tag{1.2.10}$$
$$f = \left(k^2\right) - r^2 \tag{1.2.11}$$
$$k^2 - f = r^2 \tag{1.2.12}$$

From equations 1.2.9 and 1.2.12,

$$\begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}k^2\\f\end{pmatrix} = \begin{pmatrix}-\|\mathbf{x_1}\|^2\\r^2\end{pmatrix} \tag{1.2.13}$$

Here ,$\|\mathbf{x_1}\|$ is given by ,

$$\|\mathbf{x_1}\| = \sqrt{2^2 + 3^2} \tag{1.2.14}$$
$$\|\mathbf{x_1}\| = \sqrt{13} \tag{1.2.15}$$

Subsituting equation 1.2.6,1.2.3 in equation 1.2.13 we get ,

$$\begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}k^2\\f\end{pmatrix} = \begin{pmatrix}-13\\25\end{pmatrix} \tag{1.2.16}$$

The augumented matrix of 1.2.16 is given by ,

$$\left(\begin{array}{cc|c}2 & 1 & -13\\1 & -1 & 25\end{array}\right) \tag{1.2.17}$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c}2 & 1 & -13\\1 & -1 & 25\end{array}\right) \xleftrightarrow{R_2\leftrightarrow R_1} \left(\begin{array}{cc|c}1 & -1 & 25\\2 & 1 & -13\end{array}\right) \tag{1.2.18}$$
$$\left(\begin{array}{cc|c}1 & -1 & 25\\2 & 1 & -13\end{array}\right) \xleftrightarrow{R_2=R_2-2R_1} \left(\begin{array}{cc|c}1 & -1 & 25\\0 & 3 & -63\end{array}\right) \tag{1.2.19}$$
$$\left(\begin{array}{cc|c}1 & -1 & 25\\0 & 3 & -63\end{array}\right) \xleftrightarrow{R_2=\frac{R_2}{3}} \left(\begin{array}{cc|c}1 & -1 & 25\\0 & 1 & -21\end{array}\right) \tag{1.2.20}$$
$$\left(\begin{array}{cc|c}1 & -1 & 25\\0 & 1 & -21\end{array}\right) \xleftrightarrow{R_1=R_1+R_2} \left(\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & -21\end{array}\right) \tag{1.2.21}$$

Equation 1.2.16 can we rewritten as ,

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}k^2\\f\end{pmatrix} = \begin{pmatrix}4\\-21\end{pmatrix} \tag{1.2.22}$$

Expanding the above equation 1.2.22 we get ,

$$k^2 = 4 \tag{1.2.23}$$
$$k = \pm 2 \tag{1.2.24}$$
$$f = -21 \tag{1.2.25}$$

To get the centre substitute equation 1.2.24 in equation 1.2.4 To verify the above results we plot the circle with centre $\mathbf{c}$ as $\begin{pmatrix}2\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\0\end{pmatrix}$,

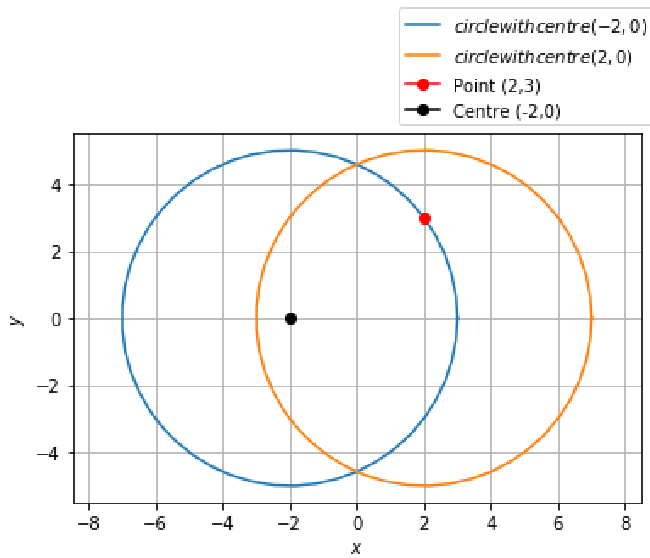

Fig. 1.2: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

qFrom the above figure 1.2 it is clear that circle with centre $\mathbf{c}$=$\begin{pmatrix}-2\\0\end{pmatrix}$ passes through the point $\mathbf{x_1}$ Desired equation of circle is given by ,

$$c = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.2.26}$$
$$f = -21 \tag{1.2.27}$$

1.3. Find the equation of the circle passing through $\begin{pmatrix}0\\0\end{pmatrix}$ and making intercepts a and b on the coordinate axes.
1.4. Find the equation of a circle with centre $\begin{pmatrix}2\\2\end{pmatrix}$ and passes through the point $\begin{pmatrix}4\\5\end{pmatrix}$.

Solution: he general equation of a circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.4.1}$$
$$\text{If } r \text{ is radius, } f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.4.2}$$
$$\text{center } \mathbf{c} = -\mathbf{u} \tag{1.4.3}$$

Given centre is $\begin{pmatrix}2\\2\end{pmatrix}$

$$\implies \mathbf{c} = \begin{pmatrix}2\\2\end{pmatrix} \tag{1.4.4}$$
$$\implies \mathbf{u} = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{1.4.5}$$

Equation (1.4.1) becomes

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} + f = 0 \tag{1.4.6}$$

This passes through point $\begin{pmatrix}4\\5\end{pmatrix}$

Substituting $\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix}$ in (1.4.6)

$$\begin{pmatrix}4 & 5\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + f = 0 \tag{1.4.7}$$
$$\implies f = -5 \tag{1.4.8}$$

Also, radius can be determined as follows

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.4.9}$$
$$\implies -5 = \begin{pmatrix}-2 & -2\end{pmatrix}\begin{pmatrix}-2\\-2\end{pmatrix} - r^2 \tag{1.4.10}$$
$$\implies -5 = 8 - r^2 \tag{1.4.11}$$
$$\implies r = \sqrt{13} \tag{1.4.12}$$

The equation of required circle is

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} - 5 = 0 \tag{1.4.13}$$

See Fig. 1.4

1.5. Find the locus of all the unit vectors in the xy-plane.
1.6. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.6.1}$$

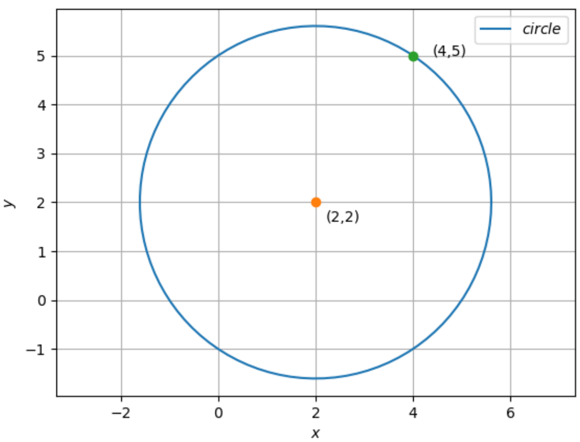

Fig. 1.4: plot showing the circle

The centre and the radius can be obtained as,

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.2}$$

$$f = -3 \tag{1.6.3}$$

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.6.4}$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \tag{1.6.5}$$

$\because$ The tangents are parallel to the x-axis, their direction and normal vectors, $\mathbf{m}$ and $\mathbf{n}$ are respectively,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.6.6}$$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.6.7}$$

For a circle, given the normal vector $\mathbf{n}$, the tangent points of contact to circle given by equation (1.6.1) are given by

$$\mathbf{q_i} = (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \tag{1.6.8}$$

where

$$\kappa_i = \pm\sqrt{\frac{\mathbf{u^T u} - f}{\mathbf{n^T n}}} \tag{1.6.9}$$

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} - (-3)}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{1.6.10}$$

$$\implies \kappa = \pm\sqrt{\frac{4}{1}} \tag{1.6.11}$$

$$\implies \kappa = \pm 2 \tag{1.6.12}$$

and from (1.6.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_1} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.13}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.6.14}$$

$$\mathbf{q_2} = -2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.15}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.6.16}$$

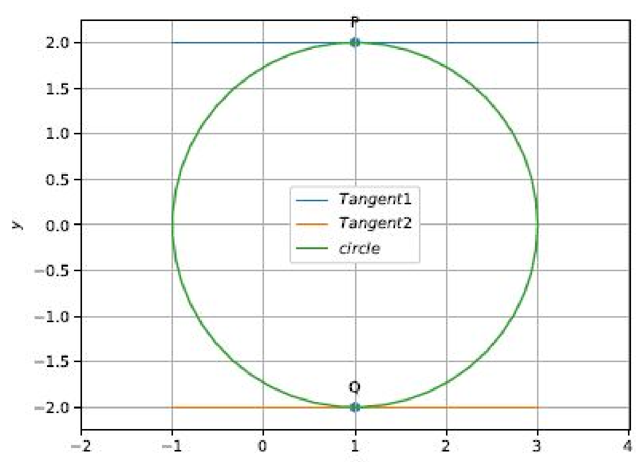

Fig. 1.6: Figure depicting tangents of circle parallel to x-axis

1.7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^T\mathbf{x} = 4$.
Solution: The equation of a circle can be expressed as,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{1.7.1}$$

where $\mathbf{c}$ is the center.
Comparing equation (1.7.1) with the circle equation given,

$$\mathbf{x}^T\mathbf{x} = 4 \tag{1.7.2}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad f = -4 \tag{1.7.3}$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \tag{1.7.4}$$

$$\implies \boxed{r = 2} \tag{1.7.5}$$

From equation (1.7.5), the point at which circle touches x-axis is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The direction vector of x-axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
The direction vector of the given line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ is $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$.
The angle that the line makes with the x-axis is given by,

$$\cos\theta = \frac{\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\|\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 & 0 \end{pmatrix}\right\|} = \frac{\sqrt{3}}{2} \tag{1.7.6}$$

$$\implies \boxed{\theta = 30^\circ} \tag{1.7.7}$$

Using equation (1.7.5) and (1.7.7),the area of the sector is obtained as,

$$\implies \boxed{\frac{\theta}{360^\circ}\pi r^2 = \frac{30^\circ}{360^\circ}\pi(2)^2 = \frac{\pi}{3}} \tag{1.7.8}$$

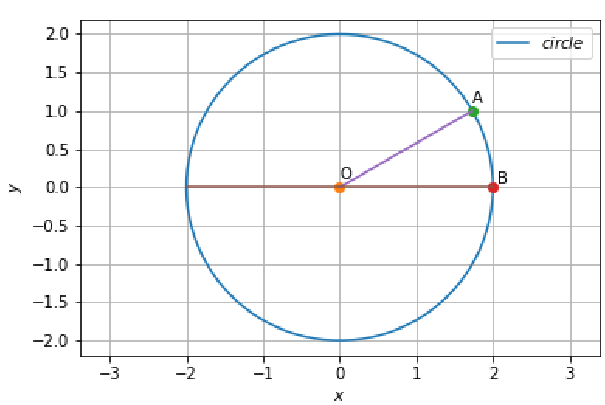

Fig. 1.7: Region enclosed by x-axis, line and circle

To find points $\mathbf{A}$ and $\mathbf{B}$,

The parametric form of x-axis is,

$$\mathbf{B} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.9}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.7.10}$$

From the intersection of circle and line, the value of λ can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.11}$$

$$= \frac{4-0}{1} = 4 \tag{1.7.12}$$

$$\implies \lambda = \pm 2 \tag{1.7.13}$$

Sub equation (1.7.13) in (1.7.10),

$$\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.7.14}$$

As given in question as first quadrant,

$$\implies \boxed{\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}} \tag{1.7.15}$$

Similarly, to find point $\mathbf{A}$, The parametric form of line is,

$$\mathbf{A} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.16}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{1.7.17}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.18}$$

$$= \frac{4-0}{4} = 1 \tag{1.7.19}$$

$$\implies \lambda = \pm 1 \tag{1.7.20}$$

$$\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \tag{1.7.21}$$

$$\implies \boxed{\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \tag{1.7.22}$$

1.8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
1.9. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.
1.10. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
1.11. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2$.
1.12. Find the slope of the tangent to the curve $y =$

$\frac{x-1}{x-2}, x \neq 2$ at $x = 10$.
Solution:

$$y = \frac{x-1}{x-2} \tag{1.12.1}$$

Equation (1.12.1) can be expressed as

$$y(x-2) = x - 1 \tag{1.12.2}$$
$$yx - 2y - x + 1 = 0 \tag{1.12.3}$$

From above we can say,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.12.4}$$
$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} & -1 \end{pmatrix} \tag{1.12.5}$$
$$f = 1 \tag{1.12.6}$$

Now,

$$\because |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0, \tag{1.12.7}$$

(1.12.1) is the equation of a hyperbola. To verify that this we will find the the characteristic equation of $\mathbf{V}$.

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} = 0 \tag{1.12.8}$$
$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{1.12.9}$$

The eigenvalues are the roots of (1.12.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \tag{1.12.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{1.12.11}$$
$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{1.12.12}$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.12.13}$$
$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.12.14}$$

Now,λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.12.15}$$
$$\implies \mathbf{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.12.16}$$

From Equations,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^{T} \quad \because \mathbf{P}^{-1} = \mathbf{P}^{T} \tag{1.12.17}$$
$$\text{or, } \mathbf{D} = \mathbf{P}^{T}\mathbf{V}\mathbf{P} \tag{1.12.18}$$

We can say that

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.12.19}$$
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{1.12.20}$$

$\because \mathbf{u}^{T}\mathbf{V}^{-1}\mathbf{u} - f > 0$,there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-}1\mathbf{u} \tag{1.12.21}$$
$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^{T}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^{T}\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{1.12.22}$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{1.12.23}$$
$$\sqrt{\frac{\mathbf{u}^{T}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.12.24}$$
$$\sqrt{\frac{f - \mathbf{u}^{T}\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{1.12.25}$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \tag{1.12.26}$$

Let us assume slope to be l,now finding the direction vector and normal vector of the tangent with slope l.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \tag{1.12.27}$$
$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \tag{1.12.28}$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{1.12.29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.12.30}$$

By using (1.12.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \tag{1.12.31}$$

Now substituting this κ in (1.12.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{\frac{-l}{4}} + 1 \end{pmatrix} \tag{1.12.32}$$

We know that x=10.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \tag{1.12.33}$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \tag{1.12.34}$$

$$\sqrt{-\frac{1}{4l}} = 4 \tag{1.12.35}$$

$$-\frac{1}{4l} = 16 \tag{1.12.36}$$

$$l = -\frac{1}{64} \tag{1.12.37}$$

The slope of the tangent to the curve y=$\frac{x-1}{x-2}$, x≠2 at x=10 is $\frac{1}{64}$. So,from the above we can say that κ=4,-4 and from equation (1.12.27) and (1.12.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \tag{1.12.38}$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \tag{1.12.39}$$

Now using equation (1.12.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1}(\kappa_1\mathbf{n} - \mathbf{u}) \tag{1.12.40}$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(-4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.12.41}$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{1.12.42}$$

$$\mathbf{q}_2 = \mathbf{V}^{-1}(\kappa_2\mathbf{n} - \mathbf{u}) \tag{1.12.43}$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.12.44}$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \tag{1.12.45}$$

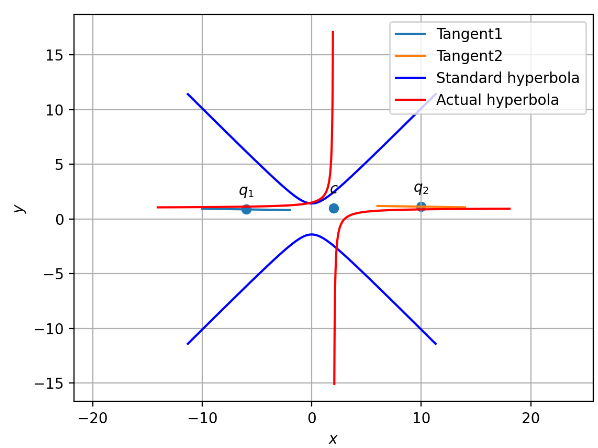

Fig. 1.12: Tangent 2 shows the tangent

1.13. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Solution: $y = (x-2)^2$ can be written as,

$$x^2 - 4x - y + 4 = 0 \tag{1.13.1}$$

From (1.13.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \mathbf{u} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}; f = 4 \tag{1.13.2}$$

$$\left|V\right| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{1.13.3}$$

(1.13.3) implies that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left|V - \lambda I\right| = 0$$
$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$
$$\implies \lambda = 0, 1 \tag{1.13.4}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{Vx} = \lambda\mathbf{x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.13.5}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.13.6}$$

By Eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{PDP}^T$$
$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.13.7}$$
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.13.8}$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{1.13.9}$$
$$where, \eta = \mathbf{u}^T\mathbf{p}_1 = -\frac{1}{2} \tag{1.13.10}$$

Substituting values from (1.13.2), (1.13.5) and (1.13.10) in (1.13.9),

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \tag{1.13.11}$$

Removing last row and representing (1.13.11) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -2 & -1 & -4 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-2R_2)} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}}$$
$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.13.12}$$

From (1.13.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.13.13}$$

Direction vector of the chord joining A(4,4) and B(2,0) can be calculated as,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.13.14}$$

We know that,

$$\mathbf{m}^T\mathbf{n} = 0; \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.13.15}$$

To find the point of contact $\mathbf{q}$, which is intersection point for normal of the chord AB and also tangent of the curve,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.13.16}$$
$$where, \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = \frac{1}{2} \tag{1.13.17}$$

Substituting the values from (1.13.2),(1.13.15) and (1.13.17) in (1.13.16),

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{1.13.18}$$

Removing last row and representing (1.13.18) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -1 & -1 & -4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow (-R_1)} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$
$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-R_2)} \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.13.19}$$

From (1.13.19), it can be observed,

$$\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.13.20}$$

which is the required point of contact

1.14. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

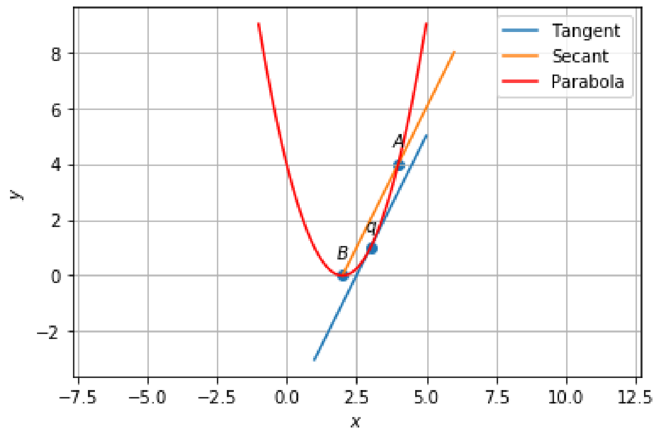

Fig. 1.13: Parabola with AB as chord, a tangent parallel to the chord

Solution: The given curve

$$y = \frac{1}{x-1} \tag{1.14.1}$$

can be expressed as

$$xy - y - 1 = 0 \tag{1.14.2}$$

Hence, we have

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix}, f = -1 \tag{1.14.3}$$

Since $\left|\mathbf{V}\right| < 0$, the equation (1.14.2) represents hyperbola. To find the values of λ_1 and λ_2, consider the characteristic equation,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{1.14.4}$$

$$\implies \left|\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right| = 0 \tag{1.14.5}$$

$$\implies \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \tag{1.14.6}$$

$$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.14.7}$$

In addition, given the slope -1, the direction and normal vectors are given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.14.8}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.14.9}$$

The parameters of hyperbola are as follows:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{1.14.10}$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \tag{1.14.11}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.14.12}$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} = \sqrt{2} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \end{cases} \tag{1.14.13}$$

which represents the standard hyperbola equation,

$$\frac{x^2}{2} - \frac{x^2}{2} = 1 \tag{1.14.14}$$

The points of contact are given by

$$K = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} = \pm\frac{1}{2} \tag{1.14.15}$$

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{1.14.16}$$

$$\mathbf{q_1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \tag{1.14.17}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.14.18}$$

$$\mathbf{q_2} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{-1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \tag{1.14.19}$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1.14.20}$$

$\therefore$ The tangents are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 \tag{1.14.21}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = 0 \tag{1.14.22}$$

The desired equations of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$ are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{1.14.23}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = -1 \tag{1.14.24}$$

The above results are verified in the following figure.

1.15. Find the equation of all lines having slope -2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.

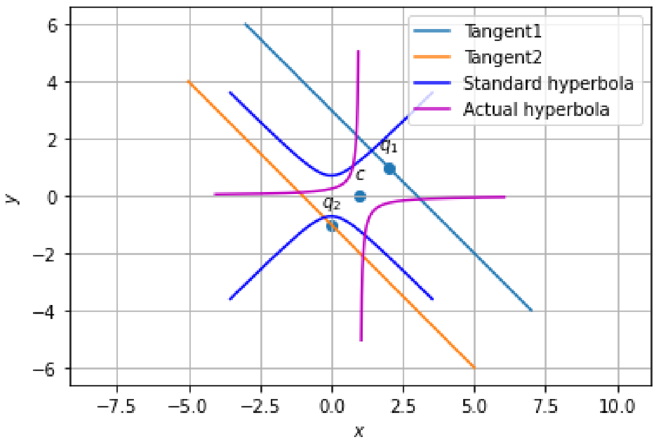

Fig. 1.14: The standard and actual hyperbola.

Solution: Given the curve,

$$y = \frac{1}{x-3} \tag{1.15.1}$$

$$\implies xy - 3y - 1 = 0 \tag{1.15.2}$$

From (1.15.2) we get,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{-3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = -1 \tag{1.15.3}$$

Now,

$$\because \left|V\right| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = \frac{-1}{2} < 0 \tag{1.15.4}$$

(1.15.1) is equation of hyperbola. Now,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \tag{1.15.5}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{1.15.6}$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.15.7}$$

The eigen vector $\mathbf{p}$ is given by,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{1.15.8}$$

For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.15.9}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.15.10}$$

Similarly for λ_2,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_{-}R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.15.11}$$

$$\implies \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.15.12}$$

Now,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.15.13}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{1.15.14}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{1.15.15}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0$, there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.15.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.15.17}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = \sqrt{2} \tag{1.15.18}$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \tag{1.15.19}$$

The direction and normal vectors of the tangent with slope -2 are given as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.15.20}$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{1.15.21}$$

$$k = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.15.22}$$

Thus,

$$\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n} = 8 \tag{1.15.23}$$

$$k = \pm\frac{1}{2\sqrt{2}} \tag{1.15.24}$$

$$\mathbf{q_1} = \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \tag{1.15.25}$$

$$\mathbf{q_2} = \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \tag{1.15.26}$$

The desired tangents are,

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix}\right\} = 0 \tag{1.15.27}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 + 2\sqrt{2} \tag{1.15.28}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix}\right\} = 0 \tag{1.15.29}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 - 2\sqrt{2} \tag{1.15.30}$$

Below figure corresponds to the tangents on the hyperbola, represented by (1.15.28) and (1.15.30) each having slope of -2.

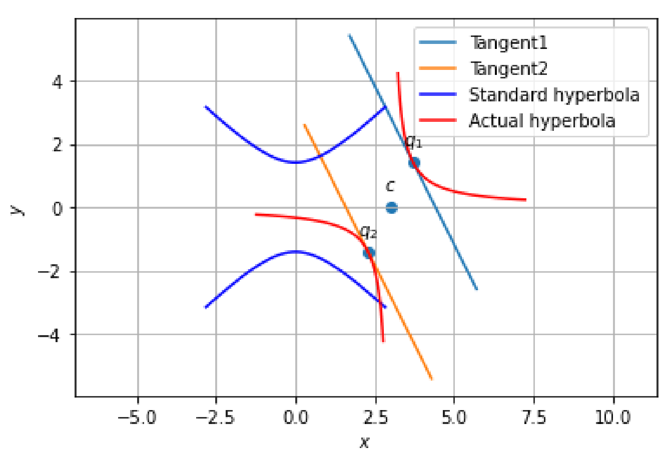

Fig. 1.15: Tangents to the hyperbola

1.16. Find points on the curve $\mathbf{x}^T\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\mathbf{x} = 1$ at which tangents are
  a) parallel to x-axis
  b) parallel to y-axis.
1.17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
1.18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$
  a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = -9$
  b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix}\mathbf{x} = 13$.
1.19. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix}\mathbf{x} + 5 = 0$.
1.20. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.
1.21. The line $\begin{pmatrix} -m & 1 \end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.
1.22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$
1.23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
1.24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.
1.25. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.
1.26. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.
1.27. Find the area of the region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}\mathbf{x} = 1$
1.28. Find the area of the region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}\mathbf{x} = 1$
1.29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
1.30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
1.31. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = -2$.
1.32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
1.33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
1.34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.

1.35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x}-8\begin{pmatrix}1 & 0\end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.

1.36. AOBA is the part of the ellipse $\mathbf{x}^T\begin{pmatrix}9 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.

1.37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.

1.38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.

1.39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.

1.40. Find the area between $y = x^2$ and $y = x$.

1.41. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

1.42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix}-3 & 2\end{pmatrix}\mathbf{x} = 12$.

1.43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & \frac{1}{4}\end{pmatrix}\mathbf{x} = 1$ and the line $\begin{pmatrix}\frac{1}{a} & \frac{1}{b}\end{pmatrix}\mathbf{x} = 1$

1.44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 2$ and the x-axis.

1.45. Find the area bounded by the curves

$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{1.45.1}$$

1.46. Find the area of the region

$$\left\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{1.46.1}$$

1.47. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.

2 QR Decomposition

2.1. $\begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}$

Solution: Let

$$\alpha = \begin{pmatrix}1\\2\end{pmatrix} \tag{2.1.1}$$

$$\beta = \begin{pmatrix}-1\\3\end{pmatrix} \tag{2.1.2}$$

We can express these as

$$\alpha = k_1\mathbf{u}_1 \tag{2.1.3}$$

$$\beta = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.1.4}$$

where

$$k_1 = \|\alpha\| \tag{2.1.5}$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\beta}{\|\mathbf{u}_1\|^2} \tag{2.1.7}$$

$$\mathbf{u}_2 = \frac{\beta - r_1\mathbf{u}_1}{\|\beta - r_1\mathbf{u}_1\|} \tag{2.1.8}$$

$$k_2 = \mathbf{u}_2^T\beta \tag{2.1.9}$$

From (2.1.3) and (2.1.4),

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{2.1.10}$$

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \mathbf{QR} \tag{2.1.11}$$

From above we can see that $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.1.12}$$

Now by using equations (2.1.5) to (2.1.9)

$$k_1 = \sqrt{5} \tag{2.1.13}$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}}\begin{pmatrix}1\\2\end{pmatrix}, \tag{2.1.14}$$

$$r_1 = \sqrt{5} \tag{2.1.15}$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}}\begin{pmatrix}-2\\1\end{pmatrix} \tag{2.1.16}$$

$$k_2 = \sqrt{5} \tag{2.1.17}$$

Thus obtained QR decomposition is

$$\begin{pmatrix}1 & -1\\2 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & \sqrt{5}\\0 & \sqrt{5}\end{pmatrix} \tag{2.1.18}$$

2.2. $\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$

2.3. $\begin{pmatrix}1 & 3\\2 & 7\end{pmatrix}$

2.4. $\begin{pmatrix}2 & 3\\5 & 7\end{pmatrix}$

2.5. $\begin{pmatrix}2 & 1\\7 & 4\end{pmatrix}$

2.6. $\begin{pmatrix}2 & 5\\1 & 3\end{pmatrix}$

2.7. $\begin{pmatrix}3 & 1\\5 & 2\end{pmatrix}$

2.8. $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

2.9. $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

2.10. $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2.11. $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$

2.12. $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

2.13. $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

2.14. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

2.15. Find QR decomposition of $\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$

Solution: Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.15.1}$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{2.15.2}$$

The column vectors can be expressed as follows,

$$\mathbf{a} = k_1 \mathbf{u}_1 \tag{2.15.3}$$

$$\mathbf{b} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{2.15.4}$$

Here,

$$k_1 = \|\mathbf{a}\| \tag{2.15.5}$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \tag{2.15.6}$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \tag{2.15.7}$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \tag{2.15.8}$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \tag{2.15.9}$$

The (2.15.3) and (2.15.4) can be written as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.15.10}$$

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \mathbf{QR} \tag{2.15.11}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \tag{2.15.12}$$

Now using equations (2.15.5) to (2.15.9) we get,

$$k_1 = \sqrt{2^2 + 3^2} = \sqrt{13} \tag{2.15.13}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.15.14}$$

$$r_1 = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = -\frac{6}{\sqrt{13}} \tag{2.15.15}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.15.16}$$

$$k_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{17}{\sqrt{13}} \tag{2.15.17}$$

Thus putting the values from (2.15.13) to (2.15.17) in (2.15.11) we obtain QR decomposition,

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \sqrt{13} & -\frac{6}{\sqrt{13}} \\ 0 & \frac{17}{\sqrt{13}} \end{pmatrix} \tag{2.15.18}$$

2.16. Find the QR decomposition of $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

Solution:
Let $\mathbf{c_1}$ and $\mathbf{c_2}$ be the column vectors of the given matrix.

$$\mathbf{c_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.16.1}$$

$$\mathbf{c_2} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{2.16.2}$$

The column vectors can be represented as,

$$\mathbf{c_1} = k_1 \mathbf{u}_1 \tag{2.16.3}$$

$$\mathbf{c_2} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{2.16.4}$$

where,

$$k_1 = \|\mathbf{c_1}\| \tag{2.16.5}$$

$$\mathbf{u_1} = \frac{\mathbf{c_1}}{k_1} \tag{2.16.6}$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{c_2}}{\|\mathbf{u}_1\|^2} \tag{2.16.7}$$

$$\mathbf{u_2} = \frac{\mathbf{c_2} - r_1 \mathbf{u_1}}{\|\mathbf{c_2} - r_1 \mathbf{u_1}\|} \tag{2.16.8}$$

$$k_2 = \mathbf{u_2}^T \mathbf{c_2} \tag{2.16.9}$$

From (2.16.3) and (2.16.4),

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\ 0 & k_2\end{pmatrix} \tag{2.16.10}$$

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \mathbf{QR} \tag{2.16.11}$$

Where $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.16.12}$$

Using equations (2.16.5) to (2.16.9) we get,

$$k_1 = \sqrt{3^2 + 1^2} = \sqrt{10} \tag{2.16.13}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{pmatrix} \tag{2.16.14}$$

$$r_1 = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \sqrt{10} \tag{2.16.15}$$

$$\mathbf{u_2} = \begin{pmatrix}\frac{-1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{pmatrix} \tag{2.16.16}$$

$$k_2 = \begin{pmatrix}\frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \sqrt{10} \tag{2.16.17}$$

Now putting the values from (2.16.13) to (2.16.17), we obtain the QR decomposition of given matrix,

$$\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & \sqrt{10}\\ 0 & \sqrt{10}\end{pmatrix} \tag{2.16.18}$$

2.17. Find QR decomposition of $\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$\mathbf{A} = \mathbf{QR} \tag{2.17.1}$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix Given

$$\mathbf{A} = \begin{pmatrix}4 & 3\\5 & -2\end{pmatrix} \tag{2.17.2}$$

Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix

$$\mathbf{a} = \begin{pmatrix}4\\5\end{pmatrix} \tag{2.17.3}$$

$$\mathbf{b} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{2.17.4}$$

The above column vectors (2.17.3) ,(2.17.4) can be expressed as ,

$$\mathbf{a} = t_1\mathbf{u}_1 \tag{2.17.5}$$

$$\mathbf{b} = s_1\mathbf{u}_1 + t_2\mathbf{u}_2 \tag{2.17.6}$$

Where,

$$t_1 = \|\mathbf{a}\| \tag{2.17.7}$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{t_1} \tag{2.17.8}$$

$$s_1 = \frac{\mathbf{u}_1^T\mathbf{b}}{\|\mathbf{u}_1\|^2} \tag{2.17.9}$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - s_1\mathbf{u}_1}{\|\mathbf{b} - s_1\mathbf{u}_1\|} \tag{2.17.10}$$

$$t_2 = \mathbf{u}_2^T\mathbf{b} \tag{2.17.11}$$

The (2.17.5) and (2.17.6) can be written as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}t_1 & s_1\\ 0 & t_2\end{pmatrix} \tag{2.17.12}$$

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \mathbf{QR} \tag{2.17.13}$$

Here, $\mathbf{R}$ is an upper triangular matrix and $\mathbf{Q}$ is an orthogonal matrix such that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.17.14}$$

Now using equations from (2.17.7) to (2.17.11) we get,

$$t_1 = \sqrt{4^2 + 5^2} = \sqrt{41} \tag{2.17.15}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{41}}\begin{pmatrix}4\\5\end{pmatrix} \tag{2.17.16}$$

$$s_1 = \begin{pmatrix}\frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \frac{2}{\sqrt{41}} \tag{2.17.17}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{41}}\begin{pmatrix}5\\-4\end{pmatrix} \tag{2.17.18}$$

$$t_2 = \begin{pmatrix}\frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \frac{23}{\sqrt{41}} \tag{2.17.19}$$

Substituting the values from (2.17.15) to (2.17.19) in (2.17.13) we obtain QR decomposition as,

$$\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix} = \begin{pmatrix}\frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}}\\ \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}\sqrt{41} & \frac{2}{\sqrt{41}}\\ 0 & \frac{23}{\sqrt{41}}\end{pmatrix} \tag{2.17.20}$$

2.18. Perform the QR decomposition of matrix

$$\mathbf{A} = \begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} \tag{2.18.1}$$

Solution:
If α and β are the columns of a (2×2) matrix **A**,
then **A** can be decomposed as

$$\mathbf{A} = \mathbf{QR} \quad (2.18.2)$$

$$\text{where, } \mathbf{U} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}, \quad (2.18.3)$$

$$\text{uppertriangular matrix } \mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.18.4)$$

$$k_1 = \|\alpha\|, \mathbf{u_1} = \frac{\alpha}{k_1} \quad (2.18.5)$$

$$r_1 = \frac{\mathbf{u_1}^T \beta}{\|\mathbf{u_1}\|^2} \quad (2.18.6)$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|}, k_2 = \mathbf{u_2}^T\beta \quad (2.18.7)$$

$$\alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.18.8)$$

$$\text{From, (2.18.5), } k_1 = \|\alpha\| = \sqrt{10} \quad (2.18.9)$$

$$\text{and } \mathbf{u_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.18.10)$$

$$\text{From (2.18.6), } r_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.11)$$

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{5}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.18.12)$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (2.18.13)$$

$$\text{From (2.18.7), } \mathbf{u_2} = \frac{\begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{pmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4}}} \quad (2.18.14)$$

$$\implies \mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}, \quad (2.18.15)$$

$$k_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.16)$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (2.18.17)$$

The matrix **A** can now be rewritten using (2.18.2) as

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{5}{\sqrt{10}} \\ 0 & \frac{5}{\sqrt{10}} \end{pmatrix} \quad (2.18.18)$$

2.19. Find the QR decomposition of the given matrix.

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \quad (2.19.1)$$

Solution: QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \quad (2.19.2)$$

where **Q** is an orthogonal matrix and **R** is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \quad (2.19.3)$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (2.19.4)$$

Equation (2.19.3) can be written in form of (2.19.4) as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix} \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} = \mathbf{QR} \quad (2.19.5)$$

where,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (2.19.6)$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.19.7)$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{-2}{\sqrt{5}} \quad (2.19.8)$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (2.19.9)$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{6}{\sqrt{5}} \quad (2.19.10)$$

Substituting equation (2.19.6) to (2.19.10) in (2.19.5),

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{5}} \end{pmatrix} \quad (2.19.11)$$

The QR decomposition is,

$$\begin{pmatrix}1 & 2\\ 2 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & -\frac{2}{\sqrt{5}}\\ 0 & \frac{6}{\sqrt{5}}\end{pmatrix} \quad (2.19.12)$$

2.20. Find the QR decomposition on a given 2×2 matrix.

$$\begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix} \quad (2.20.1)$$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \quad (2.20.2)$$

Here $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix} \quad (2.20.3)$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix}2\\1\end{pmatrix} \quad \mathbf{b} = \begin{pmatrix}1\\-2\end{pmatrix} \quad (2.20.4)$$

Equation (2.20.3) can be written in **QR** form as:

$$\mathbf{QR} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}u_1 & u_3\\ 0 & u_2\end{pmatrix} \quad (2.20.5)$$

Now,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2+2^2} = \sqrt{5} \quad (2.20.6)$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix} \quad (2.20.7)$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = 0 \quad (2.20.8)$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}}\end{pmatrix} \quad (2.20.9)$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = \sqrt{5} \quad (2.20.10)$$

Substituting equation (2.20.6) to (2.20.10) in (2.20.5),to obtain the QR Decomposition of the given matrix as:

$$\begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & 0\\ 0 & \sqrt{5}\end{pmatrix} \quad (2.20.11)$$

In equation (2.20.11) **R** is diagonal because the columns and rows are orthogonal to each other.

2.21. Perform QR decomposition on matrix **A**

$$\mathbf{A} = \begin{pmatrix}1 & 4\\ 3 & -5\end{pmatrix} \quad (2.21.1)$$

Solution:
The columns of matrix **A** can be represented in α and β as

$$\implies \alpha = \begin{pmatrix}1\\3\end{pmatrix} \quad (2.21.2)$$

$$\beta = \begin{pmatrix}4\\-5\end{pmatrix} \quad (2.21.3)$$

For QR decomposition, matrix **A** can be expressed as

$$\mathbf{A} = \mathbf{QR} \quad (2.21.4)$$

where, **Q** and **R** are expressed as

$$\mathbf{Q} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \quad (2.21.5)$$

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1\\ 0 & k_2\end{pmatrix} \quad (2.21.6)$$

Note that **R** is an upper triangular matrix.
Now,we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \quad (2.21.7)$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \quad (2.21.8)$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}4\\-5\end{pmatrix} \quad (2.21.9)$$

$$\implies r_1 = -\frac{11}{\sqrt{10}} \quad (2.21.10)$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \quad (2.21.11)$$

Consider

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix}4\\-5\end{pmatrix} + \frac{11}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \quad (2.21.12)$$

$$\implies \beta - r_1\mathbf{u_1} = \begin{pmatrix}\frac{51}{10}\\ -\frac{17}{10}\end{pmatrix} \quad (2.21.13)$$

$$\|\beta - r_1\mathbf{u_1}\| = \frac{17}{\sqrt{10}} \quad (2.21.14)$$

Substitute (2.21.13),(2.21.14) in (2.21.11), we get

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix} \tag{2.21.15}$$

$$k_2 = \mathbf{u_2}^T \beta = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{2.21.16}$$

$$\implies k_2 = \frac{17}{\sqrt{10}} \tag{2.21.17}$$

Therefore, from (2.21.5) and (2.21.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \tag{2.21.18}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \tag{2.21.19}$$

Note that,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{2.21.20}$$

Now matrix $\mathbf{A}$ can be written as (2.21.4)

$$\begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \tag{2.21.21}$$

3 Singular Value Decomposition

3.1. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{3.1.1}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.2}$$

Solution:
The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.3}$$

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.1.4}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.1.5}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{3.1.6}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}^T\mathbf{A}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{3.1.7}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{3.1.8}$$

Calculating eigen value of $\mathbf{M}^T\mathbf{M}$.

$$\begin{vmatrix} 6-\lambda & 13 \\ 13 & 38-\lambda \end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \tag{3.1.9}$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22 \tag{3.1.10}$$

Eigen vectors of $\mathbf{M}\mathbf{M}^T$.

$$\begin{vmatrix} 13-\lambda & -17 & 8 \\ 17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \tag{3.1.11}$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \tag{3.1.12}$$

Hence,The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{5} \\ \frac{-3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \tag{3.1.13}$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12-\sqrt{17}}{5}\right)^2 + \left(\frac{1-3\sqrt{17}}{5}\right)^2 + 1^2} \tag{3.1.14}$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} \end{pmatrix} \tag{3.1.15}$$

$$\tag{3.1.16}$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \tag{3.1.17}$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+30\sqrt{7}}}\begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \tag{3.1.18}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} \\ \frac{5}{\sqrt{340+30\sqrt{17}}} \end{pmatrix} \tag{3.1.19}$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \tag{3.1.20}$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}}\begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \end{pmatrix} \tag{3.1.21}$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.22}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.23}$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{3.1.24}$$

Now, $\mathbf{V} = \mathbf{M}^T \frac{\mathbf{u_i}}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.25}$$

So,from equation (3.1.6)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} = \tag{3.1.26}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.27}$$

$$\begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{3.1.28}$$

$$\begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}^T \tag{3.1.29}$$

Now, Finding Moore-Penrose Pseudo inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \tag{3.1.30}$$

We,know that, $\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+0\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix} \tag{3.1.31}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.32}$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.33}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.34}$$

$$\mathbf{x} = \begin{pmatrix} \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \\ \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \end{pmatrix} \tag{3.1.35}$$

Simplifying the values of x_1 and x_2

$$x_2 = \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \tag{3.1.36}$$

$$= \frac{-702100}{591700} \tag{3.1.37}$$

$$= -\frac{7}{59} \tag{3.1.38}$$

$$x_1 = \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \tag{3.1.39}$$

$$= \frac{2507500}{591700} \tag{3.1.40}$$

$$= \frac{25}{59} \tag{3.1.41}$$

Now, Verifying the values using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.1.42}$$

Solving R.H.S

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.1.43}$$

Now using equation (3.1.7) in (3.1.43)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.1.44}$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xleftrightarrow{R_2-\frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \tag{3.1.45}$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \tag{3.1.46}$$

$$6x_1 + 13x_2 = 1 \tag{3.1.47}$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \tag{3.1.48}$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{3.1.49}$$

3.2. Find the distance of the point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane $\begin{pmatrix} 6 & -3 & 2 \end{pmatrix}\mathbf{x} = 4$

Solution:

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{3.2.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \tag{3.2.2}$$

$$\implies 6a - 3b + 2c = 0 \tag{3.2.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \tag{3.2.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{3.2.5}$$

Now we solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.2.6}$$

Putting values in (3.2.6),

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 3 & \frac{3}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\ 5\\ -3\end{pmatrix} \tag{3.2.7}$$

Now, to solve (3.2.7), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{3.2.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}10 & \frac{9}{2}\\ \frac{9}{2} & \frac{13}{4}\end{pmatrix} \tag{3.2.9}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & 3\\ 0 & 1 & \frac{3}{2}\\ 3 & \frac{3}{2} & \frac{45}{4}\end{pmatrix} \tag{3.2.10}$$

From (3.2.6) putting (3.2.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.2.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{3.2.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{3.2.13}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & 3\\ 0 & 1-\lambda & \frac{3}{2}\\ 3 & \frac{3}{2} & \frac{45}{4}-\lambda\end{pmatrix} = 0 \tag{3.2.14}$$

$$\implies \lambda^3 - \frac{53}{4}\lambda^2 + \frac{49}{4}\lambda = 0 \tag{3.2.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{4} \tag{3.2.16}$$

$$\lambda_2 = 1 \tag{3.2.17}$$

$$\lambda_3 = 0 \tag{3.2.18}$$

Hence the eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{4}{15}\\ \frac{2}{15}\\ 1\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{2}\\ 1\\ 0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-3\\ -\frac{3}{2}\\ 1\end{pmatrix} \tag{3.2.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{4}{7\sqrt{5}}\\ \frac{2}{7\sqrt{5}}\\ \frac{3\sqrt{5}}{7}\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\\ 0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-\frac{6}{7}\\ -\frac{3}{7}\\ \frac{2}{7}\end{pmatrix} \tag{3.2.20}$$

Hence we obtain $\mathbf{U}$ of (3.2.8) as follows,

$$\mathbf{U} = \begin{pmatrix}\frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7}\\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7}\\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7}\end{pmatrix} \tag{3.2.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (3.2.8) as follows,

$$\mathbf{S} = \begin{pmatrix}\frac{7}{2} & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} \tag{3.2.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{3.2.23}$$

$$\implies \begin{pmatrix}10-\lambda & \frac{9}{2}\\ \frac{9}{2} & \frac{13}{4}-\lambda\end{pmatrix} = 0 \tag{3.2.24}$$

$$\implies \lambda^2 - \frac{53}{4}\lambda + \frac{49}{4} = 0 \tag{3.2.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{49}{4} \tag{3.2.26}$$

$$\lambda_5 = 1 \tag{3.2.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix}2\\ 1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-\frac{1}{2}\\ 1\end{pmatrix} \tag{3.2.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix} \tag{3.2.29}$$

Hence we obtain $\mathbf{V}$ of (3.2.8) as follows,

$$\mathbf{V} = \begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix} \tag{3.2.30}$$

Finally from (3.2.8) we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix}\frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7}\\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7}\\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7}\end{pmatrix}\begin{pmatrix}\frac{7}{2} & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}^T \tag{3.2.31}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.2.32}$$

From (3.2.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{27}{7\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \\ -\frac{33}{7} \end{pmatrix} \tag{3.2.33}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{54}{49\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \end{pmatrix} \tag{3.2.34}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{3.2.35}$$

Verifying the solution of (3.2.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.2.36}$$

Evaluating the R.H.S in (3.2.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{3.2.37}$$

$$\implies \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{3.2.38}$$

Solving the augmented matrix of (3.2.38) we get,

$$\begin{pmatrix} 10 & \frac{9}{2} & -7 \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{10}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \tag{3.2.39}$$

$$\xleftrightarrow{R_2=R_2-\frac{9}{2}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & \frac{49}{40} & \frac{73}{20} \end{pmatrix} \tag{3.2.40}$$

$$\xleftrightarrow{R_2=\frac{40}{49}R_2} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{3.2.41}$$

$$\xleftrightarrow{R_1=R_1-\frac{9}{20}R_1} \begin{pmatrix} 1 & 0 & -\frac{100}{49} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{3.2.42}$$

Hence, Solution of (3.2.36) is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{3.2.43}$$

Comparing results of $\mathbf{x}$ from (3.2.35) and (3.2.43) we conclude that the solution is verified.