



Geometry through Linear Algebra



G V V Sharma*

CONTENTS

- 1 **Pair of Straight Lines** 1
- 2 **General Equation. Tracing of Curves** 15

Abstract—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.

1 PAIR OF STRAIGHT LINES

1.1. Find the value of h so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.1)$$

may represent two straight lines.

Solution:

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \quad (1.1.2)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.3)$$

$$f = 20 \quad (1.1.4)$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \quad (1.1.5)$$

Expanding equation (1.1.5) along row 1 gives

$$\begin{aligned} \Rightarrow 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + \\ 11 \times (\frac{31h}{2} - 132) = 0 \end{aligned}$$

$$\Rightarrow 20h^2 - 341h + \frac{2907}{2} = 0 \quad (1.1.6)$$

$$\Rightarrow h = \frac{17}{2} \quad (1.1.7)$$

$$\Rightarrow h = \frac{171}{20} \quad (1.1.8)$$

If $h = \frac{17}{2}$ or $h = \frac{171}{20}$, the equation given will represent two straight lines.

Sub $h = \frac{17}{2}$ in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.9)$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (1.1.10)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.11)$$

$$\mathbf{f} = 20 \quad (1.1.12)$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.13)$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \quad (1.1.14)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.1.15)$$

$$(1.1.16)$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \quad (1.1.17)$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \quad (1.1.18)$$

$$\Rightarrow m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \quad (1.1.19)$$

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.1.20)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.21)$$

we know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.22)$$

Convolution of \mathbf{n}_1 and \mathbf{n}_2 can be done by converting \mathbf{n}_1 into a toeplitz matrix and multiplying with \mathbf{n}_2

From equation (1.1.21)

$$\mathbf{n}_1 = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.23)$$

$$\Rightarrow \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.24)$$

\Rightarrow Equation (1.1.21) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.1.25)$$

Substituting (1.1.21) in (1.1.25), the augmented

matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \quad (1.1.26)$$

$$\Rightarrow c_1 = 4, c_2 = 5 \quad (1.1.27)$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\begin{aligned} \Rightarrow (-2x - 3y - 4)(3x - 4y - 5) &= 0 \\ \Rightarrow (2x + 3y + 4)(3x + 4y + 5) &= 0 \end{aligned} \quad (1.1.28)$$

Equation (1.1.28) represents equations of two straight lines.

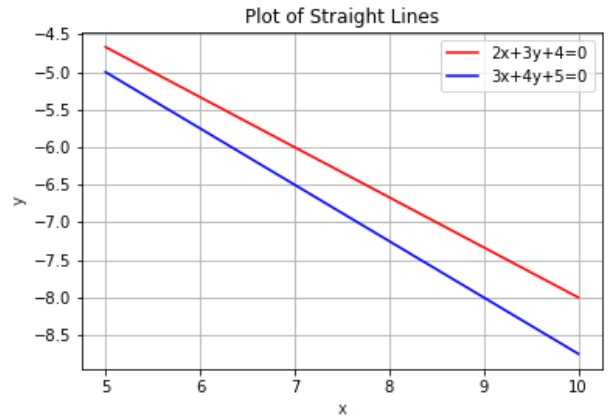


Fig. 1.1.1: Plot of Straight lines when $h = \frac{17}{2}$

Similarly, Sub $h = \frac{171}{20}$ in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \quad (1.1.29)$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \quad (1.1.30)$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{3} \\ \frac{310}{3} \end{pmatrix} \quad (1.1.31)$$

$$\mathbf{f} = \frac{200}{3} \quad (1.1.32)$$

The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.33)$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \quad (1.1.34)$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \quad (1.1.35)$$

$$\Rightarrow m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \quad (1.1.36)$$

$$\mathbf{m}_1 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (1.1.37)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.38)$$

Convolution of \mathbf{n}_1 and \mathbf{n}_2 can be done by converting \mathbf{n}_1 into a toeplitz matrix and multiplying with \mathbf{n}_2

From equation (1.1.38)

$$\mathbf{n}_1 = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.39)$$

$$\Rightarrow \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.40)$$

\Rightarrow Equation (1.1.38) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -2\mathbf{u} \quad (1.1.41)$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 4R_2}{5}]{R_2 \leftarrow \frac{5R_2 - 8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \quad (1.1.42)$$

$$\Rightarrow c_1 = 10, c_2 = \frac{20}{3} \quad (1.1.43)$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\Rightarrow \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \quad (1.1.44)$$

Equation (1.1.44) represents equations of two straight lines.

1.2. Prove that the following equations represent two straight lines. Also find their point of in-

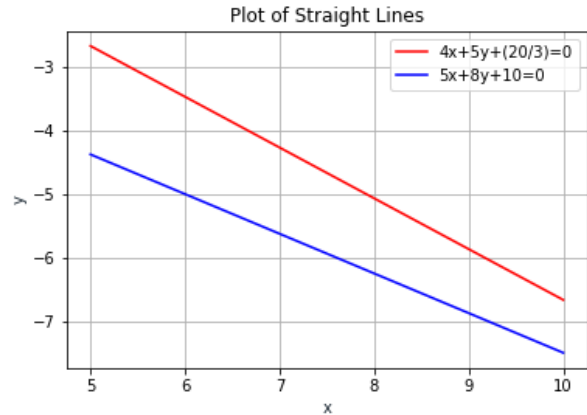


Fig. 1.1.2: Plot of Straight lines when $h = \frac{171}{20}$

tersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \quad (1.2.1)$$

Solution: $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \quad (1.2.2)$$

Expanding equation (1.2.2), we get zero.

Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.2.3)$$

$$\Rightarrow 3m^2 - 8m - 3 = 0 \quad (1.2.4)$$

$$\text{Solving, } m = 3, -\frac{1}{3} \quad (1.2.5)$$

The normal vectors of the lines then become

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.2.6)$$

$$\mathbf{n}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.7)$$

Equations of the lines can therefore be written as

$$\left(\frac{1}{3} \ 1\right) \mathbf{x} = c \quad (1.2.8)$$

$$\Rightarrow (1 \ 3) \mathbf{x} = c_1, \quad (1.2.9)$$

$$(-3 \ 1) \mathbf{x} = c_2 \quad (1.2.10)$$

$$\Rightarrow [(1 \ 3) \mathbf{x} - c_1][(-3 \ 1) \mathbf{x} - c_2] \quad (1.2.11)$$

represents the equation specified in (1.2.1)

Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \quad (1.2.12)$$

$$(1.2.13)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3 \times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \quad (1.2.14)$$

$$\xrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.15)$$

$$\xrightarrow{R_1 \leftarrow R_1 + 3 \times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.16)$$

$$\Rightarrow c_2 = 2 \text{ and } c_1 = -9 \quad (1.2.17)$$

The individual line equations therefore become

$$(1 \ 3)\mathbf{x} = -9, \quad (1.2.18)$$

$$(-3 \ 1)\mathbf{x} = 2 \quad (1.2.19)$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.20)$$

The LHS part of (1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.21)$$

The augmented matrix for the set of equations represented in (1.2.18), (1.2.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \quad (1.2.22)$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.23)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.24)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \quad (1.2.25)$$

$$\text{Hence, the intersection point is } \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.2.26)$$

Angle between two lines θ can be given by

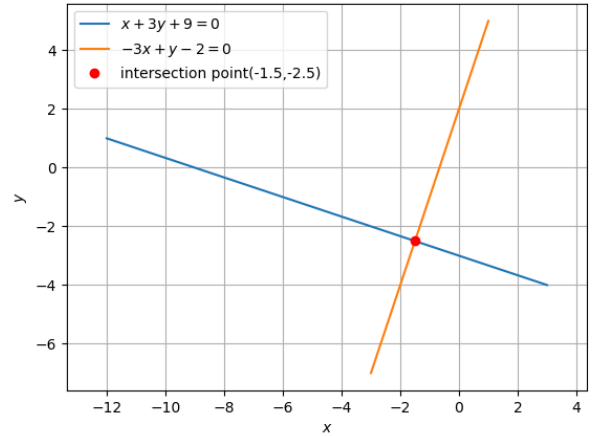


Fig. 1.2.1: plot showing intersection of lines

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.2.27)$$

$$\cos \theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \quad (1.2.28)$$

$$\Rightarrow \theta = 90^\circ \quad (1.2.29)$$

1.3. Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \quad (1.3.1)$$

Solution:

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (1.3.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (1.3.3)$$

$$f = -2 \quad (1.3.4)$$

$$\left| \begin{array}{ccc} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{array} \right| \xrightarrow[R_1 \rightarrow R_1 + R_3]{R_1 \rightarrow R_1 - R_2} \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{array} \right| = 0 \quad (1.3.5)$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$|V| < 0 \implies \left| \begin{array}{cc} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array} \right| = \frac{-9}{4} < 0 \quad (1.3.6)$$

Let the pair of straight of lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.3.7)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.3.8)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.3.9)$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \quad (1.3.10)$$

$$m_1 = 1, m_2 = -2 \quad (1.3.11)$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.3.12)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \quad (1.3.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.3.14)$$

$$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (1.3.15)$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.3.16)$$

Using row reduction we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \end{pmatrix} \quad (1.3.17)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_2 \leftarrow R_2 / -3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (1.3.18)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad (1.3.19)$$

$$C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.3.20)$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.21)$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.22)$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (1.3.23)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \quad (1.3.24)$$

consider the augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \quad (1.3.25)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.3.26)$$

$$\xrightarrow[R_1 \leftarrow R_1 + R_2]{R_1 \leftarrow R_1 / 3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.3.27)$$

Therefore point of intersection is $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Angle between two lines θ can be given by

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.3.28)$$

$$\cos \theta = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} \quad (1.3.29)$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right) \Rightarrow \theta = \tan^{-1}3 \quad (1.3.30)$$

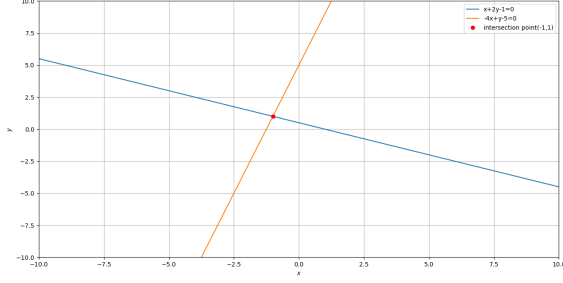


Fig. 1.3.1: plot showing intersection of lines

1.4. Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \quad (1.4.1)$$

represents two parallel lines.

Solution: The given equation (1.4.1) can be written as

$$\mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (1.4.2)$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \quad (1.4.3)$$

Equation (1.4.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \quad (1.4.4)$$

Vector form of straight lines,

$$\mathbf{n}_1^T \mathbf{x} = \mathbf{c}_1 \quad (1.4.5)$$

$$\mathbf{n}_2^T \mathbf{x} = \mathbf{c}_2 \quad (1.4.6)$$

Equating their product with (1.4.2)

$$(\mathbf{n}_1^T \mathbf{x} - \mathbf{c}_1)(\mathbf{n}_2^T \mathbf{x} - \mathbf{c}_2) = \mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 \quad (1.4.7)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.8)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix} \quad (1.4.9)$$

$$c_1 c_2 = -5 \quad (1.4.10)$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \quad (1.4.11)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.4.12)$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.4.13)$$

From (1.4.2) equation (1.4.11) becomes

$$9m^2 + 6m + 1 = 0 \quad (1.4.14)$$

Using (1.4.3),

$$|\mathbf{V}| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \quad (1.4.15)$$

Substituting the values in (1.4.12),

$$m_i = \frac{-3 \pm 0}{9} \quad (1.4.16)$$

$$m_1 = m_2 = \frac{-1}{3} \quad (1.4.17)$$

Substituting values in (1.4.13)

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.18)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.19)$$

Using the above values in (1.4.8),

$$k_1 k_2 = 9 \quad (1.4.20)$$

Taking $k_1 = 3$ and $k_2 = 3$ we get

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.21)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.22)$$

Verifying \mathbf{n}_1 and \mathbf{n}_2 by computing the convolution by representing \mathbf{n}_1 as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.23)$$

Finding the Angle between the lines,

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) \quad (1.4.24)$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \quad (1.4.25)$$

$$\|\mathbf{n}_1\| = \sqrt{10} \quad \|\mathbf{n}_2\| = \sqrt{10} \quad (1.4.26)$$

Substituting (1.4.25) and (1.4.26) in (1.4.24) we get,

$$\theta = \cos^{-1}(1) \quad (1.4.27)$$

$$\theta = 0^\circ \quad (1.4.28)$$

From (1.4.17) and (1.4.28) shows the given equation (1.4.1) represents two parallel lines. Hence proved.

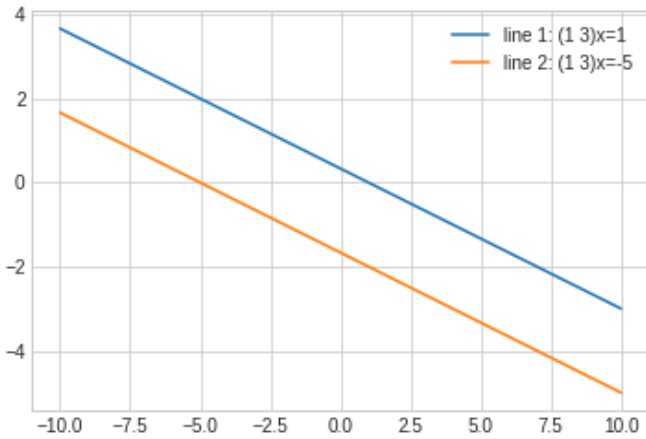


Fig. 1.4.1: Pair of straight lines plot generated using python

1.5. **Solution:** Find the value of k such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \quad (1.5.1)$$

represent pairs of straight lines.

From (1.5.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \quad (1.5.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.5.3)$$

$$f = k \quad (1.5.4)$$

Compute the slopes of lines given by the roots

of the polynomial $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.5.5)$$

$$\Rightarrow m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \quad (1.5.6)$$

$$\Rightarrow m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \quad (1.5.7)$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.5.8)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.5.9)$$

Here,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \quad (1.5.10)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -m_2 \\ 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (1.5.11)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.12)$$

Substituting (1.5.10) and (1.5.11) in the above equation, we get

$$k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} \quad (1.5.13)$$

$$\Rightarrow k_1 k_2 = -10 \quad (1.5.14)$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of k_1 and k_2 in (1.5.10) and (1.5.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (1.5.15)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.5.16)$$

Using Teoplitz matrix representation, the convolution of \mathbf{n}_1 with \mathbf{n}_2 , is as follows:

$$\begin{pmatrix} 2 & 0 & 5 \\ 5 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.17)$$

Hence, \mathbf{n}_1 and \mathbf{n}_2 satisfies (1.5.12). We have,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.5.18)$$

Substituting (1.5.15), (1.5.16) in (1.5.18), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.5.19)$$

Solving for c_1 and c_2 , the augmented matrix is,

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & -2 & -31 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 5R_1]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & \frac{-19}{2} & \frac{-37}{2} \end{pmatrix} \quad (1.5.20)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{3}{2}R_2]{R_2 \leftarrow \frac{R_2}{-19/2}} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.5.21)$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \quad (1.5.22)$$

We know that,

$$f = k = c_1 c_2 \quad (1.5.23)$$

$$\Rightarrow \boxed{k = -15} \quad (1.5.24)$$

Hence the solution. Using (1.5.8) and (1.5.9), the equation of pair of straight lines is given by,

$$(2 \ 5)\mathbf{x} = 3 \quad (1.5.25)$$

$$(3 \ -2)\mathbf{x} = -5 \quad (1.5.26)$$

See Fig. 1.5.1

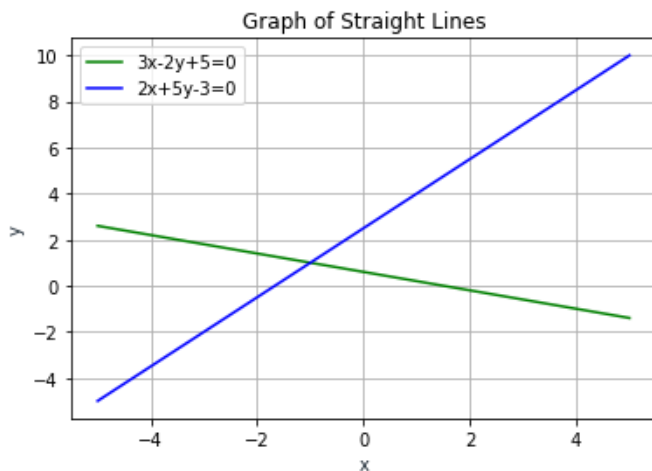


Fig. 1.5.1: Plot of two straight lines.

1.6. Find the value of k so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.1)$$

Solution:

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \quad (1.6.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.3)$$

The equation (1.6.1) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.6.4)$$

$$\Rightarrow \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \quad (1.6.5)$$

$$\Rightarrow \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \quad (1.6.6)$$

$$\Rightarrow 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11 \begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \quad (1.6.7)$$

$$\Rightarrow 2k^2 + 55k + 350 = 0 \quad (1.6.8)$$

$$\Rightarrow (10 + k)(2k + 35) = 0 \quad (1.6.9)$$

$$\Rightarrow k = -10$$

$$k = -\frac{35}{2} \quad (1.6.10)$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.

Now Lets find equation of lines for $k = -10$. Substitute $k = -10$ in (1.6.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.11)$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (1.6.12)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.13)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \quad (1.6.14)$$

$$\Rightarrow |\mathbf{V}| = -1 \quad (1.6.15)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.6.16)$$

Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.6.17)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.6.18)$$

Equating their product with (1.6.1)

$$\begin{aligned} (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (1.6.19)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.6.20)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} = -2 \begin{pmatrix} \frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \quad (1.6.21)$$

$$c_1 c_2 = f = 2 \quad (1.6.22)$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.6.23)$$

$$\Rightarrow 2m^2 - 10m + 12 = 0 \quad (1.6.24)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.6.25)$$

$$\Rightarrow m_i = \frac{5 \pm \sqrt{1}}{2} \quad (1.6.26)$$

$$\Rightarrow m_1 = 3 \quad (1.6.27)$$

$$m_2 = 2 \quad (1.6.28)$$

The normal vector for two lines is given by

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.6.29)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.6.30)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.6.31)$$

Substituting (1.6.30),(1.6.31) in (1.6.20). we get

$$k_1 k_2 = 2 \quad (1.6.32)$$

The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1, k_2 = 2$ we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.6.33)$$

$$\mathbf{n}_2 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (1.6.34)$$

We verify obtained $\mathbf{n}_1, \mathbf{n}_2$ using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.6.35)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.6.36)$$

Therefore the obtained $\mathbf{n}_1, \mathbf{n}_2$ are correct.

Substitute (1.6.33), (1.6.34) in (1.6.21) and calculate for c_1 and c_2

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.6.37)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.38)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.6.39)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.6.40)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \quad (1.6.41)$$

$$\Rightarrow c_1 = -13 \quad (1.6.42)$$

$$c_2 = 21 \quad (1.6.43)$$

Substituting (1.6.33),(1.6.34),(1.6.42),(1.6.43) in (1.6.17) and (1.6.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = -13 \quad (1.6.44)$$

$$\begin{pmatrix} -4 & 2 \end{pmatrix} \mathbf{x} = 21 \quad (1.6.45)$$

The plot of these two lines is shown in Fig. 1.6.1.

Now Lets find equation of lines for $k = -\frac{35}{2}$. Substitute $k = -\frac{35}{2}$ in (1.6.1). We get equation

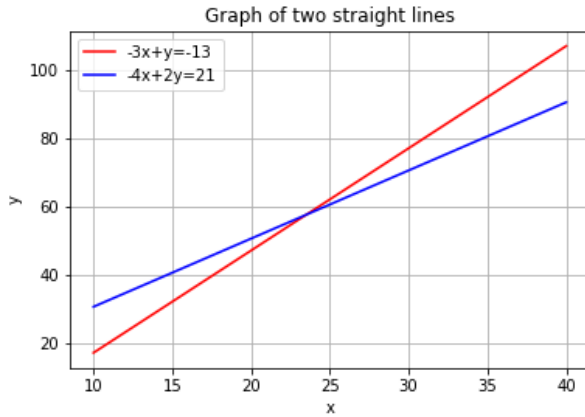


Fig. 1.6.1: Pair of straight lines for $k = -10$

of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.46)$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \quad (1.6.47)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.48)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \quad (1.6.49)$$

$$\Rightarrow |\mathbf{V}| = -\frac{841}{16} \quad (1.6.50)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.6.51)$$

Therefore the lines will intersect.

Now from (1.6.20),

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (1.6.52)$$

The slopes of the lines are given by roots of

equation (1.6.23)

$$\Rightarrow 2m^2 - \frac{35}{2}m + 12 = 0 \quad (1.6.53)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.6.54)$$

$$\Rightarrow m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \quad (1.6.55)$$

$$\Rightarrow m_1 = 8 \quad (1.6.56)$$

$$m_2 = \frac{3}{4} \quad (1.6.57)$$

The normal vector for two lines is given by (1.6.29)

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.6.58)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.6.59)$$

Substituting (1.6.58), (1.6.59) in (1.6.52). we get

$$k_1 k_2 = 2 \quad (1.6.60)$$

The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1, k_2 = 2$ we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.6.61)$$

$$\mathbf{n}_2 = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \quad (1.6.62)$$

We verify obtained $\mathbf{n}_1, \mathbf{n}_2$ using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -8 & 0 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \end{pmatrix} \quad (1.6.63)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -\frac{35}{2} \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.6.64)$$

Therefore the obtained $\mathbf{n}_1, \mathbf{n}_2$ are correct.

Substitute (1.6.61), (1.6.62) in (1.6.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.6.65)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.66)$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.67)$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.6.68)$$

$$\xleftrightarrow{R_1 \leftarrow -29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.6.69)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \quad (1.6.70)$$

$$\Rightarrow c_1 = -\frac{102}{29} \quad (1.6.71)$$

$$c_2 = \frac{59}{29} \quad (1.6.72)$$

Substituting (1.6.61), (1.6.62), (1.6.71), (1.6.72) in (1.6.17) and (1.6.18). we get equation of two straight lines.

$$(-8 \ 1)\mathbf{x} = -\frac{102}{29} \quad (1.6.73)$$

$$\left(-\frac{3}{2} \ 2\right)\mathbf{x} = \frac{59}{29} \quad (1.6.74)$$

1.7. Find the value of k so that the following equation may represent a pair of straight lines

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \quad (1.7.1)$$

Solution: The given second degree equation is, Comparing coefficients of (1.7.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \quad (1.7.2)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.3)$$

$$f = -35 \quad (1.7.4)$$

The given second degree equation (1.7.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \quad (1.7.5)$$

Expanding the determinant,

$$k + 12 = 0 \quad (1.7.6)$$

$$\Rightarrow k = -12 \quad (1.7.7)$$

Hence, from (1.7.7) we find that for $k = -12$, the given second degree equation (1.7.1) represents pair of straight lines. For the appropriate value of k , (1.7.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \quad (1.7.8)$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.7.9)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.7.10)$$

The pair of straight lines is given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.7.11)$$

Putting the values of \mathbf{V} and \mathbf{u} we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \quad (1.7.12)$$

Hence, from (1.7.12) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.14)$$

$$c_1 c_2 = -35 \quad (1.7.15)$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \quad (1.7.16)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.7.17)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.7.18)$$

Substituting the values in above equations (1.7.16) we get,

$$-12m^2 + m + 6 = 0 \quad (1.7.19)$$

$$\Rightarrow m_i = \frac{-\frac{1}{2} \pm \sqrt{-\left(-\frac{289}{4}\right)}}{-12} \quad (1.7.20)$$

Solving equation (1.7.20) we get ,

$$m_1 = -\frac{2}{3} \quad (1.7.21)$$

$$m_2 = \frac{3}{4} \quad (1.7.22)$$

Hence putting the values of m_1 and m_2 in (1.7.18) we get

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (1.7.23)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.7.24)$$

Putting values of \mathbf{n}_1 and \mathbf{n}_2 in (1.7.13) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.25)$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.26)$$

Thus, from (1.7.26), $k_1k_2 = -12$. Possible combinations of (k_1, k_2) are $(6, -2)$, $(-6, 2)$, $(3, -4)$, $(-3, 4)$ Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.7.27)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (1.7.28)$$

From equation (1.7.14) we get

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.7.29)$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.30)$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \quad (1.7.31)$$

$$3c_2 - 4c_1 = -43 \quad (1.7.32)$$

The augmented matrix of (1.7.31) ,(1.7.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} R_1 = \frac{1}{2} R_1 \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix} \quad (1.7.33)$$

$$R_2 = R_2 - 3R_1 \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \quad (1.7.34)$$

$$R_2 = -\frac{2}{17} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \quad (1.7.35)$$

$$R_1 = R_1 - \frac{3}{2} R_2 \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix} \quad (1.7.36)$$

$$(1.7.37)$$

Hence we get,

$$c_1 = -5 \quad (1.7.38)$$

$$c_2 = 7 \quad (1.7.39)$$

Hence (1.7.9), (1.7.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \quad (1.7.40)$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \quad (1.7.41)$$

The figure below corresponds to the pair of straight lines represented by (1.7.40) and (1.7.41).

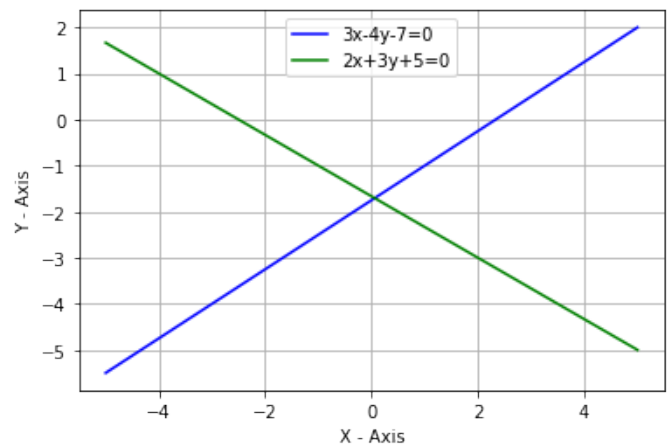


Fig. 1.7.1: Pair of Straight Lines

1.8. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \quad (1.8.1)$$

represent pairs of straight lines. **Solution:**

From (1.8.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \quad (1.8.2)$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \quad (1.8.3)$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & k \end{vmatrix} = 0 \quad (1.8.4)$$

$$\Rightarrow \left(k - \left(\frac{49}{4} \right) \right) - \frac{5}{3} \left(\frac{5}{3}k - \frac{35}{4} \right) - \frac{5}{2} \left(\frac{-35}{6} + \frac{5}{2} \right) = 0 \quad (1.8.5)$$

$$\Rightarrow \frac{64}{k} 36 - \frac{128}{12} = 0 \quad (1.8.6)$$

$$\Rightarrow \boxed{k = 6} \quad (1.8.7)$$

Substituting (1.8.7) in (1.8.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \quad (1.8.8)$$

Hence value of $k=6$ represents pair of straight lines. Substituting value of $k=6$ in (1.8.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & 6 \end{vmatrix} \quad (1.8.9)$$

Simplify the above determinant, we get

$$\delta = 0 \quad (1.8.10)$$

(1.8.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \quad (1.8.11)$$

Since $\det(V) < 0$ lines would intersect each other

$$\mathbf{n}_1 * \mathbf{n}_2 = \left\{ 1, \frac{10}{3}, 1 \right\} \quad (1.8.12)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \quad (1.8.13)$$

$$c_1 c_2 = 6 \quad (1.8.14)$$

The slopes of the lines are given by the roots

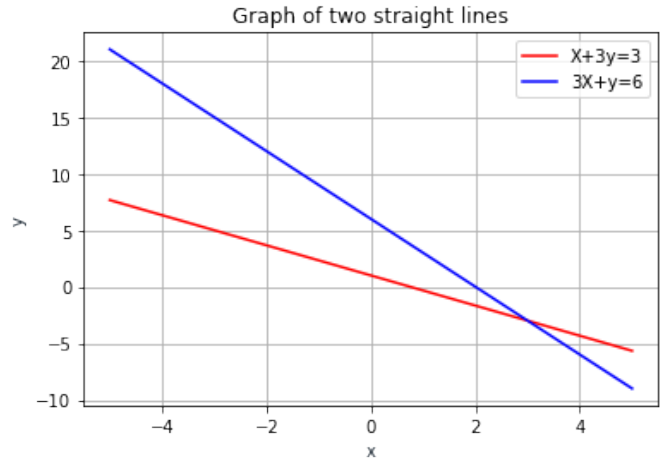


Fig. 1.8.1: Pair of straight lines

of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.8.15)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.8.16)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.8.17)$$

Substituting in above equations (1.8.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \quad (1.8.18)$$

$$\Rightarrow m_i = \frac{\frac{-10}{3} \pm \sqrt{-\left(\frac{-16}{9}\right)}}{1} \quad (1.8.19)$$

Solving equation (1.8.19) we have,

$$m_1 = \frac{-1}{3} \quad (1.8.20)$$

$$m_2 = -3 \quad (1.8.21)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.8.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.8.23)$$

Substituting equations (1.8.22), (1.8.23) in equation (1.8.12) we get

$$k_1 k_2 = 1 \quad (1.8.24)$$

Possible combination of (k_1, k_2) is (1,1) Lets

assume $k_1 = 1, k_2 = 1$, we get

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.8.25)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.8.26)$$

we have:

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.8.27)$$

Convolution of \mathbf{n}_1 and \mathbf{n}_2 can be done by converting \mathbf{n}_1 into a teoplitz matrix and multiplying with \mathbf{n}_2

From equation (1.8.25) and (1.8.26)

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.8.28)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.8.29)$$

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.8.30)$$

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -5 \\ 2 \\ -7 \end{pmatrix} \quad (1.8.31)$$

Substituting (1.8.25) and (1.8.26) in (1.8.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \quad (1.8.32)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \quad (1.8.33)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.8.34)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.8.35)$$

From above we get

$$c_1 = 1 \quad (1.8.36)$$

$$c_2 = 6 \quad (1.8.37)$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \quad (1.8.38)$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (1.8.39)$$

1.9. Find the value of k so that the following equation may represent the pair of straight lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \quad (1.9.1)$$

Solution: We need to find the value of k for which (1.9.1) represents a pair of straight lines. Converting (1.9.1) into vector form, we get

$$\mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 = 0 \quad (1.9.2)$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \quad (1.9.3)$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \quad (1.9.4)$$

$$f = -9 \quad (1.9.5)$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.9.6)$$

Since (1.9.1) represents a pair of straight lines, then by (1.9.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \quad (1.9.7)$$

By solving, above determinant we get

$$2(9 - 9) + \frac{-1}{2} \left(\frac{-9}{2} + \frac{-3k}{2} \right) + \frac{k}{2} \left(\frac{3}{2} + \frac{k}{2} \right) = 0 \quad (1.9.8)$$

$$\frac{(9 + 3k)}{4} + \frac{k(3 + k)}{4} = 0 \quad (1.9.9)$$

$$k^2 + 6k + 9 = 0 \quad (1.9.10)$$

$$(k + 3)^2 = 0 \quad (1.9.11)$$

$$k = -3 \quad (1.9.12)$$

Hence by (1.9.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \quad (1.9.13)$$

represents family of straight lines for $k = -3$. To find the straight lines, we write each of them

in their vector form as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.9.14)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.9.15)$$

Equating the product of above with (1.9.2), we have

$$\begin{aligned} (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \\ \mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 \end{aligned} \quad (1.9.16)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad (1.9.17)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \quad (1.9.18)$$

$$c_1 c_2 = -9 \quad (1.9.19)$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \quad (1.9.20)$$

$$m^2 - m - 2 = 0 \quad (1.9.21)$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} \quad (1.9.22)$$

$$m_1 = \frac{1+3}{2} = 2 \quad (1.9.23)$$

$$m_2 = \frac{1-3}{2} = -1 \quad (1.9.24)$$

$$n_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.9.25)$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.9.26)$$

Substituting (1.9.25) and (1.9.26) in (1.9.17), we get

$$k_1 k_2 = -1 \quad (1.9.27)$$

Taking $k_1 = -1$ and $k_2 = 1$, we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.9.28)$$

$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.9.29)$$

Substituting in (1.9.18) for above values of n_1

and n_2

$$(n_1 n_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (1.9.30)$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (1.9.31)$$

Solving (1.9.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xrightarrow{r_2 = r_2 + 2r_1} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \quad (1.9.32)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xrightarrow{r_2 = r_2/3} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad (1.9.33)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \xrightarrow{r_1 = r_1 - r_2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \quad (1.9.34)$$

Hence, we found out

$$c_1 = -3 \quad (1.9.35)$$

$$c_2 = 3 \quad (1.9.36)$$

Thus, pair of staright lines are

$$(2 \ -1) \mathbf{x} = -3 \quad (1.9.37)$$

$$(1 \ 1) \mathbf{x} = 3 \quad (1.9.38)$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.9.39)$$

The plot of above is shown below

2 GENERAL EQUATION. TRACING OF CURVES

- 2.1. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre

$$12x^2 - 23xy + 10y^2 - 25x + 26y = 14 \quad (2.1.1)$$

Solution:

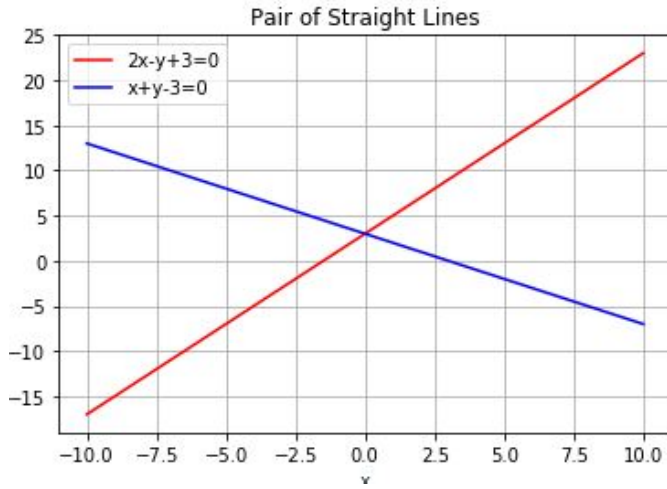


Fig. 1.9.1: Pair of Straight Lines

2.2. What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0 \quad (2.2.1)$$

Find the center.

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.2.2)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \quad (2.2.3)$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (2.2.4)$$

$$f = -2 \quad (2.2.5)$$

Expanding the determinant of \mathbf{V} we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \quad (2.2.6)$$

Hence from (2.2.6) we conclude that given equation is an ellipse. The characteristic equation of \mathbf{V} is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \quad (2.2.7)$$

$$\Rightarrow \lambda^2 - 50\lambda + 400 = 0 \quad (2.2.8)$$

Hence the characteristic equation of \mathbf{V} is given by (2.2.8). The roots of (2.2.8) i.e the eigen-

values are given by

$$\lambda_1 = 10, \lambda_2 = 40 \quad (2.2.9)$$

The eigen vector \mathbf{p} is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.2.10)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (2.2.11)$$

for $\lambda_1 = 10$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \quad (2.2.12)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.2.13)$$

Again, for $\lambda_2 = 40$,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \quad (2.2.14)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (2.2.15)$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad \mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \quad (2.2.16)$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \quad (2.2.17)$$

Now (2.2.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (2.2.18)$$

And,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (2.2.19)$$

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.2.20)$$

The centre/vertex of the conic section in (2.2.2) is given by \mathbf{c} in (2.2.19). We compute \mathbf{V}^{-1} as follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.21)$$

$$\xrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.22)$$

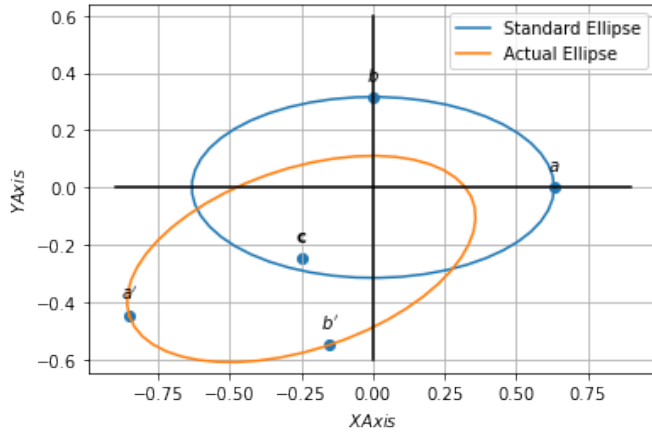


Fig. 2.2.1: Graphical representation of the ellipse

Hence \mathbf{V}^{-1} is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.23)$$

Now $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}$ is given by,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \quad (2.2.24)$$

And, $\mathbf{V}^{-1} \mathbf{u}$ is given by,

$$\mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.2.25)$$

By putting the value of (2.2.25), the center of the ellipse is given by (2.2.19) as follows,

$$\mathbf{c} = -\frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \quad (2.2.26)$$

Also the semi-major axis (a) and semi-minor axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \quad (2.2.27)$$

$$b = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \quad (2.2.28)$$

Finally from (2.2.18), the equation of ellipse is given by,

$$\mathbf{y}^T \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \mathbf{y} = 4 \quad (2.2.29)$$

The following figure 2.2.1 is the graphical representation of the ellipse in (2.2.29),