Geometry through Linear Algebra

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Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1 Pair of Straight Lines

1.1. Find the value of h so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \tag{1.1.1}$$

may represent two straight lines.
Solution:

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \tag{1.1.2}$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{1.1.3}$$

$$f = 20 \tag{1.1.4}$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \tag{1.1.5}$$

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Expanding equation (1.1.5) along row 1 gives

$$\implies 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + 11 \times (\frac{31h}{2} - 132) = 0$$

$$\implies 20h^2 - 341h + \frac{2907}{2} = 0 \tag{1.1.6}$$

$$\implies \boxed{h = \frac{17}{2}} \tag{1.1.7}$$

$$\implies \boxed{h = \frac{171}{20}} \tag{1.1.8}$$

If $h = \frac{17}{2}$ or $h = \frac{171}{20}$, the equation given will represent two straight lines.

Sub $h = \frac{17}{2}$ in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \tag{1.1.9}$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{1.1.10}$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{1.1.11}$$

$$\mathbf{f} = 20 \tag{1.1.12}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \tag{1.1.13}$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \tag{1.1.14}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.1.15}$$

$$\tag{1.1.16}$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \tag{1.1.17}$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \tag{1.1.18}$$

$$\implies m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \tag{1.1.19}$$

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{1.1.20}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.21}$$

we know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.22}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$

From equation (1.1.21)

$$\mathbf{n_1} = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.23}$$

$$\implies \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.24}$$

$\implies$ Equation (1.1.21) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.25}$$

Substituting (1.1.21) in (1.1.25), the augmented matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.1.26}$$

$$\implies c_1 = 4, c_2 = 5 \tag{1.1.27}$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\implies (-2x - 3y - 4)(3x - 4y - 5) = 0$$

$$\implies \boxed{(2x + 3y + 4)(3x + 4y + 5) = 0} \tag{1.1.28}$$

Equation (1.1.28) represents equations of two straight lines.

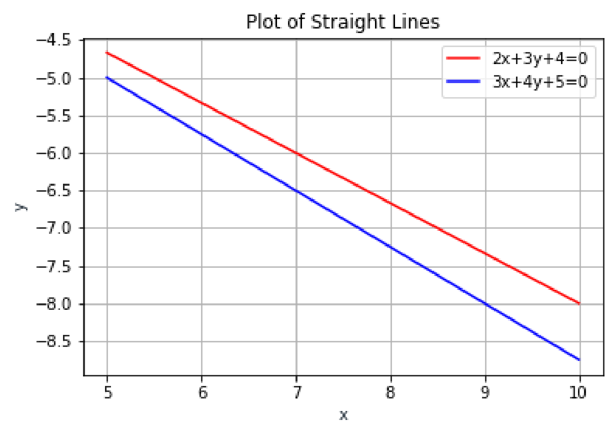

Fig. 1.1.1: Plot of Straight lines when $h = \frac{17}{2}$

Similarly, Sub $h = \frac{171}{20}$ in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \tag{1.1.29}$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \tag{1.1.30}$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{6} \\ \frac{310}{6} \end{pmatrix} \tag{1.1.31}$$

$$\mathbf{f} = \frac{200}{3} \tag{1.1.32}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \tag{1.1.33}$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \tag{1.1.34}$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \tag{1.1.35}$$

$$\implies m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \tag{1.1.36}$$

$$\mathbf{m_1} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \tag{1.1.37}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{1.1.38}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$

From equation (1.1.38)

$$\mathbf{n_1} = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{1.1.39}$$

$$\implies \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.1.40}$$

$\implies$ Equation (1.1.38) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.1.41}$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1-4R_2}{5}]{R_2 \leftarrow \frac{5R_2-8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \tag{1.1.42}$$

$$\implies c_1 = 10, c_2 = \frac{20}{3} \tag{1.1.43}$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\implies \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \tag{1.1.44}$$

Equation (1.1.44) represents equations of two straight lines.

1.2. Prove that the following equations represent two straight lines.Also find their point of intersection and the angle between them

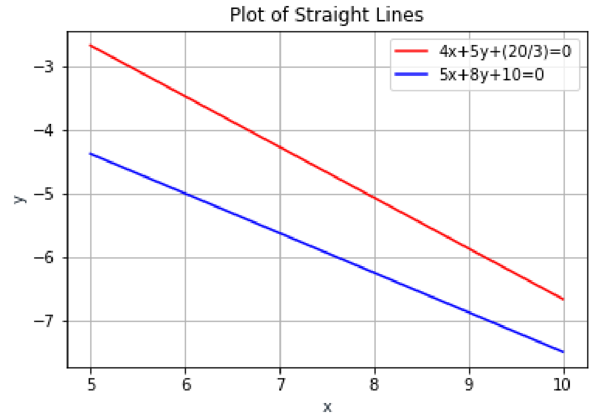

Fig. 1.1.2: Plot of Straight lines when $h = \frac{171}{20}$

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \tag{1.2.1}$$

Solution: $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \tag{1.2.2}$$

Expanding equation (1.2.2), we get zero.

Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \tag{1.2.3}$$

$$\implies 3m^2 - 8m - 3 = 0 \tag{1.2.4}$$

$$\text{Solving, } m = 3, -\frac{1}{3} \tag{1.2.5}$$

The normal vectors of the lines then become

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.2.6}$$

$$\mathbf{n_2} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.2.7}$$

Equations of the lines can therefore be written as

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = c \tag{1.2.8}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = c_1, \tag{1.2.9}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{1.2.10}$$

$$\implies \left[\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - c_1 \right] \left[\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} - c_2 \right] \tag{1.2.11}$$

represents the equation specified in (1.2.1)

Comparing the equations, we have

$$\begin{pmatrix}1 & -3\\3 & 1\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix}=\begin{pmatrix}29\\-3\end{pmatrix} \tag{1.2.12}$$

$$\tag{1.2.13}$$

Row reducing the augmented matrix

$$\begin{pmatrix}1 & -3 & 29\\3 & 1 & -3\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2-3\times R_1}\begin{pmatrix}1 & -3 & 29\\0 & 10 & -90\end{pmatrix} \tag{1.2.14}$$

$$\xleftrightarrow{R_2\leftarrow R_2\times\frac{1}{10}}\begin{pmatrix}1 & -3 & 29\\0 & 1 & -9\end{pmatrix} \tag{1.2.15}$$

$$\xleftrightarrow{R_1\leftarrow R_1+3\times R_2}\begin{pmatrix}1 & 0 & 2\\0 & 1 & -9\end{pmatrix} \tag{1.2.16}$$

$$\implies c_2 = 2 \text{ and } c_1 = -9 \tag{1.2.17}$$

The individual line equations therefore become

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = -9, \tag{1.2.18}$$

$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 2 \tag{1.2.19}$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix}1\\3\end{pmatrix}*\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}a\\2b\\c\end{pmatrix} \tag{1.2.20}$$

The LHS part of (1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix}1 & 0\\3 & 1\\0 & 3\end{pmatrix}\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}-3\\-8\\3\end{pmatrix}=\begin{pmatrix}a\\2b\\c\end{pmatrix} \tag{1.2.21}$$

The augmented matrix for the set of equations represented in (1.2.18), (1.2.19) is

$$\begin{pmatrix}1 & 3 & -9\\-3 & 1 & 2\end{pmatrix} \tag{1.2.22}$$

Row reducing the matrix

$$\begin{pmatrix}1 & 3 & -9\\-3 & 1 & 2\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2+3\times R_1}\begin{pmatrix}1 & 3 & -9\\0 & 10 & -25\end{pmatrix} \tag{1.2.23}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{3}{10}\times R_2}\begin{pmatrix}1 & 0 & -\frac{3}{2}\\0 & 10 & -25\end{pmatrix} \tag{1.2.24}$$

$$\xleftrightarrow{R_2\leftarrow\frac{R_2}{10}}\begin{pmatrix}1 & 0 & -\frac{3}{2}\\0 & 1 & -\frac{5}{2}\end{pmatrix} \tag{1.2.25}$$

$$\text{Hence, the intersection point is}\begin{pmatrix}-\frac{3}{2}\\-\frac{5}{2}\end{pmatrix} \tag{1.2.26}$$

Angle between two lines θ can be given by

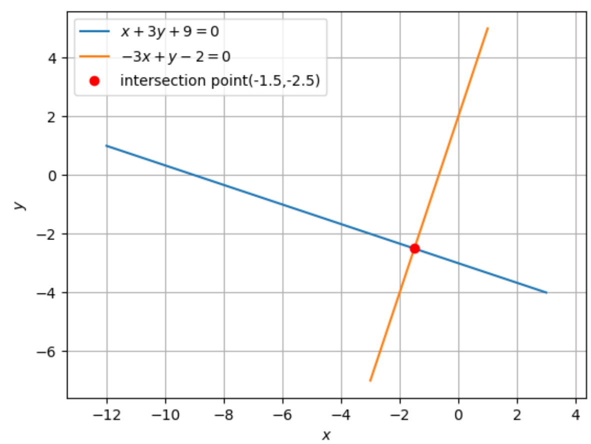

Fig. 1.2.1: plot showing intersection of lines

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{1.2.27}$$

$$\cos\theta = \frac{\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}-3\\1\end{pmatrix}}{\sqrt{(3)^2+1}\times\sqrt{(-3)^2+1}} = 0 \tag{1.2.28}$$

$$\implies \theta = 90^\circ \tag{1.2.29}$$

1.3. Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \tag{1.3.1}$$

Solution:

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{1.3.2}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{1.3.3}$$

$$f = -2 \tag{1.3.4}$$

$$\begin{vmatrix} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} \xleftrightarrow[R_1 \to R_1 + R_3]{R_1 \to R_1 - R_2} \begin{vmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} = 0 \tag{1.3.5}$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$\left|V\right| < 0 \implies \begin{vmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{-9}{4} < 0 \tag{1.3.6}$$

Let the pair of straight of lines be given by

$$\mathbf{n_1}^T \mathbf{x} = c_1 \tag{1.3.7}$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \tag{1.3.8}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{1.3.9}$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \tag{1.3.10}$$

$$m_1 = 1, m_2 = -2 \tag{1.3.11}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} and \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.3.12}$$

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \tag{1.3.13}$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2\mathbf{u} \tag{1.3.14}$$

$$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} & \frac{-1}{2} \end{pmatrix} \tag{1.3.15}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{1.3.16}$$

Using row reduction we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \end{pmatrix} \tag{1.3.17}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 / -3]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{1.3.18}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{1.3.19}$$

$$C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.3.20}$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.3.21}$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.3.22}$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1.3.23}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{1.3.24}$$

consider the augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \tag{1.3.25}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.3.26}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + R_2]{R_1 \leftarrow R_1 / 3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.3.27}$$

Therefore point of intersection is $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{1.3.28}$$

$$\cos\theta = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} \tag{1.3.29}$$

$$\theta = \cos^{-1}(\frac{-1}{\sqrt{10}}) \implies \theta = tan^{-1}3 \quad (1.3.30)$$

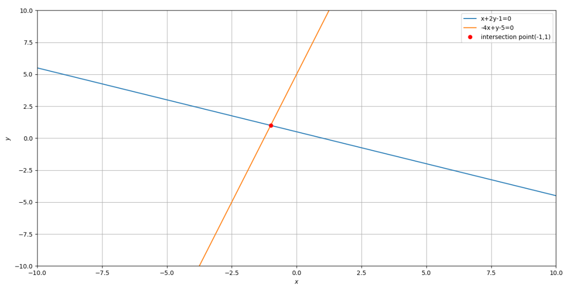

Fig. 1.3.1: plot showing intersection of lines

1.4. Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \quad (1.4.1)$$

represents two parallel lines.
Solution: The given equation (1.4.1) can be written as

$$\mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (1.4.2)$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \quad (1.4.3)$$

Equation (1.4.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \quad (1.4.4)$$

Vector form of straight lines,

$$\mathbf{n_1}^T \mathbf{x} = \mathbf{c_1} \quad (1.4.5)$$

$$\mathbf{n_2}^T \mathbf{x} = \mathbf{c_2} \quad (1.4.6)$$

Equating their product with (1.4.2)

$$(\mathbf{n_1}^T \mathbf{x} - \mathbf{c_1})(\mathbf{n_2}^T \mathbf{x} - \mathbf{c_2}) = \mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 \quad (1.4.7)$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.8)$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2\begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad (1.4.9)$$

$$c_1 c_2 = -5 \quad (1.4.10)$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \quad (1.4.11)$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \quad (1.4.12)$$

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.4.13)$$

From (1.4.2) equation (1.4.11) becomes

$$9m^2 + 6m + 1 = 0 \quad (1.4.14)$$

Using (1.4.3),

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \quad (1.4.15)$$

Substituting the values in (1.4.12),

$$m_i = \frac{-3 \pm 0}{9} \quad (1.4.16)$$

$$m_1 = m_2 = \frac{-1}{3} \quad (1.4.17)$$

Substituting values in (1.4.13)

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.18)$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.19)$$

Using the above values in (1.4.8),

$$k_1 k_2 = 9 \quad (1.4.20)$$

Taking $k_1 = 3$ and $k_2 = 3$ we get

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.21)$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.22)$$

Verifying $\mathbf{n_1}$ and $\mathbf{n_2}$ by computing the convolution by representing $\mathbf{n_1}$ as Toeplitz matrix,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.23)$$

Finding the Angle between the lines,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|}\right) \tag{1.4.24}$$

$$\mathbf{n_1}^T\mathbf{n_2} = \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = 10 \tag{1.4.25}$$

$$\|\mathbf{n_1}\| = \sqrt{10} \quad \|\mathbf{n_2}\| = \sqrt{10} \tag{1.4.26}$$

Substituting (1.4.25) and (1.4.26) in (1.4.24) we get,

$$\theta = \cos^{-1}(1) \tag{1.4.27}$$

$$\theta = 0^\circ \tag{1.4.28}$$

From (1.4.17) and (1.4.28) shows the given equation (1.4.1) represents two parallel lines. Hence proved.

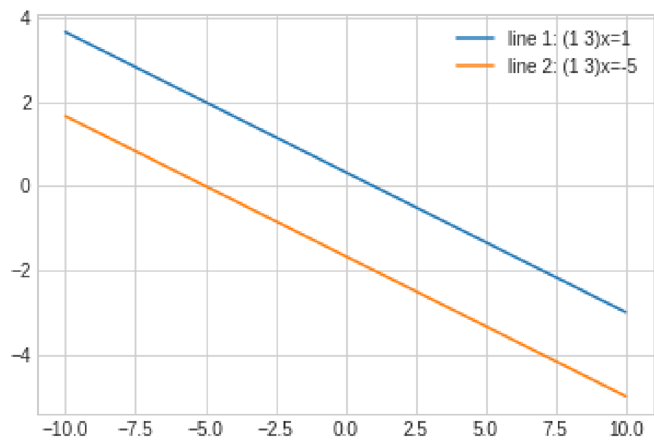

Fig. 1.4.1: Pair of straight lines plot generated using python

1.5. **Solution:** Find the value of k such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \tag{1.5.1}$$

represent pairs of straight lines.
From (1.5.1) we get,

$$\mathbf{V} = \begin{pmatrix}6 & \frac{11}{2}\\ \frac{11}{2} & -10\end{pmatrix} \tag{1.5.2}$$

$$\mathbf{u} = \begin{pmatrix}\frac{1}{2}\\ \frac{31}{2}\end{pmatrix} \tag{1.5.3}$$

$$f = k \tag{1.5.4}$$

Compute the slopes of lines given by the roots of the polynomial $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{1.5.5}$$

$$\implies m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \tag{1.5.6}$$

$$\implies m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \tag{1.5.7}$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.5.8}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.5.9}$$

Here,

$$\mathbf{n}_1 = k_1\begin{pmatrix}-m_1\\1\end{pmatrix} = k_1\begin{pmatrix}\frac{2}{5}\\1\end{pmatrix} \tag{1.5.10}$$

$$\mathbf{n}_2 = k_2\begin{pmatrix}-m_2\\1\end{pmatrix} = k_2\begin{pmatrix}\frac{-3}{2}\\1\end{pmatrix} \tag{1.5.11}$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix}a\\2b\\c\end{pmatrix} \tag{1.5.12}$$

Substituting (1.5.10) and (1.5.11) in the above equation, we get

$$k_1\begin{pmatrix}\frac{2}{5}\\1\end{pmatrix} * k_2\begin{pmatrix}\frac{-3}{2}\\1\end{pmatrix} = \begin{pmatrix}6\\11\\-10\end{pmatrix} \tag{1.5.13}$$

$$\implies k_1k_2 = -10 \tag{1.5.14}$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of k_1 and k_2 in (1.5.10) and (1.5.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix}2\\5\end{pmatrix} \tag{1.5.15}$$

$$\mathbf{n}_2 = \begin{pmatrix}3\\-2\end{pmatrix} \tag{1.5.16}$$

Using Teoplitz matrix representation, the convolution of $\mathbf{n}_1$ with $\mathbf{n}_2$, is as follows:

$$\begin{pmatrix}2 & 0 & 5\\5 & 2 & 0\\0 & 5 & 2\end{pmatrix}\begin{pmatrix}3\\-2\\0\end{pmatrix} = \begin{pmatrix}6\\11\\-10\end{pmatrix} = \begin{pmatrix}a\\2b\\c\end{pmatrix} \tag{1.5.17}$$

Hence, $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfies (1.5.12). We have,

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\mathbf{u} \tag{1.5.18}$$

Substituting (1.5.15), (1.5.16) in (1.5.18), we get

$$\begin{pmatrix}2 & 3\\5 & -2\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = -2\begin{pmatrix}\frac{1}{2}\\\frac{31}{2}\end{pmatrix} \tag{1.5.19}$$

Solving for c_1 and c_2, the augmented matrix is,

$$\begin{pmatrix}2 & 3 & -1\\5 & -2 & -31\end{pmatrix} \xleftrightarrow[R_2\leftarrow R_2-5R_1]{R_1\leftarrow\frac{R_1}{2}} \begin{pmatrix}1 & \frac{3}{2} & \frac{-1}{2}\\0 & \frac{-19}{2} & \frac{-57}{2}\end{pmatrix} \tag{1.5.20}$$

$$\xleftrightarrow[R_1\leftarrow R_1-\frac{3}{2}R_2]{R_2\leftarrow\frac{R_2}{-19/2}} \begin{pmatrix}1 & 0 & -5\\0 & 1 & 3\end{pmatrix} \tag{1.5.21}$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \tag{1.5.22}$$

We know that,

$$f = k = c_1c_2 \tag{1.5.23}$$

$$\implies \boxed{k = -15} \tag{1.5.24}$$

Hence the solution. Using (1.5.8) and (1.5.9), the equation of pair of straight lines is given by,

$$\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 3 \tag{1.5.25}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -5 \tag{1.5.26}$$

See Fig. 1.5.1

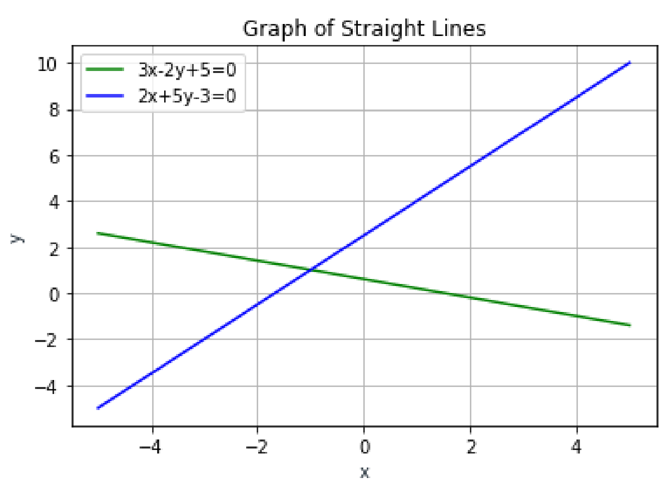

Fig. 1.5.1: Plot of two straight lines.

1.6. Find the value of k so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \tag{1.6.1}$$

Solution:

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b\\b & c\end{pmatrix} = \begin{pmatrix}12 & \frac{k}{2}\\\frac{k}{2} & 2\end{pmatrix} \tag{1.6.2}$$

$$\mathbf{u} = \begin{pmatrix}d\\e\end{pmatrix} = \begin{pmatrix}\frac{11}{2}\\-\frac{5}{2}\end{pmatrix} \tag{1.6.3}$$

The equation (1.6.1) represents pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\\mathbf{u}^T & f\end{vmatrix} = 0 \tag{1.6.4}$$

$$\implies \begin{vmatrix}12 & \frac{k}{2} & \frac{11}{2}\\\frac{k}{2} & 2 & -\frac{5}{2}\\\frac{11}{2} & -\frac{5}{2} & 2\end{vmatrix} = 0 \tag{1.6.5}$$

$$\implies \begin{vmatrix}24 & k & 11\\k & 4 & -5\\11 & -5 & 4\end{vmatrix} = 0 \tag{1.6.6}$$

$$\implies 24\begin{vmatrix}4 & -5\\-5 & 4\end{vmatrix} - k\begin{vmatrix}k & -5\\11 & 4\end{vmatrix} + 11\begin{vmatrix}k & 4\\11 & -5\end{vmatrix} = 0 \tag{1.6.7}$$

$$\implies 2k^2 + 55k + 350 = 0 \tag{1.6.8}$$

$$\implies (10 + k)(2k + 35) = 0 \tag{1.6.9}$$

$$\implies k = -10$$

$$k = -\frac{35}{2} \tag{1.6.10}$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.
Now Lets find equation of lines for $k = -10$. Substitute $k = -10$ in (1.6.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \tag{1.6.11}$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}12 & -5\\-5 & 2\end{pmatrix} \tag{1.6.12}$$

$$\mathbf{u} = \begin{pmatrix}\frac{11}{2}\\-\frac{5}{2}\end{pmatrix} \tag{1.6.13}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix}12 & -5\\-5 & 2\end{vmatrix} \tag{1.6.14}$$

$$\implies \left|\mathbf{V}\right| = -1 \tag{1.6.15}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{1.6.16}$$

Therefore the lines will intersect.
The equation of two lines is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{1.6.17}$$
$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{1.6.18}$$

Equating their product with (1.6.1)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.6.19}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{1.6.20}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = -2\begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.6.21}$$
$$c_1c_2 = f = 2 \tag{1.6.22}$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \tag{1.6.23}$$
$$\implies 2m^2 - 10m + 12 = 0 \tag{1.6.24}$$
$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{1.6.25}$$
$$\implies m_i = \frac{5 \pm \sqrt{1}}{2} \tag{1.6.26}$$
$$\implies m_1 = 3 \tag{1.6.27}$$
$$m_2 = 2 \tag{1.6.28}$$

The normal vector for two lines is given by

$$\mathbf{n_i} = k_i\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.6.29}$$
$$\implies \mathbf{n_1} = k_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.6.30}$$
$$\mathbf{n_2} = k_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.6.31}$$

Substituting (1.6.30),(1.6.31) in (1.6.20). we get

$$k_1k_2 = 2 \tag{1.6.32}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.6.33}$$
$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{1.6.34}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{1.6.35}$$
$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.6.36}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (1.6.33), (1.6.34) in (1.6.21) and calculate for c_1 and c_2

$$c_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{1.6.37}$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \tag{1.6.38}$$
$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \tag{1.6.39}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \tag{1.6.40}$$
$$\implies \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \tag{1.6.41}$$
$$\implies c_1 = -13 \tag{1.6.42}$$
$$c_2 = 21 \tag{1.6.43}$$

Substituting (1.6.33),(1.6.34),(1.6.42),(1.6.43) in (1.6.17) and (1.6.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} = -13 \tag{1.6.44}$$
$$\begin{pmatrix} -4 & 2 \end{pmatrix}\mathbf{x} = 21 \tag{1.6.45}$$

The plot of these two lines is shown in Fig. 1.6.1.
Now Lets find equation of lines for $k = -\frac{35}{2}$.
Substitute $k = -\frac{35}{2}$ in (1.6.1). We get equation

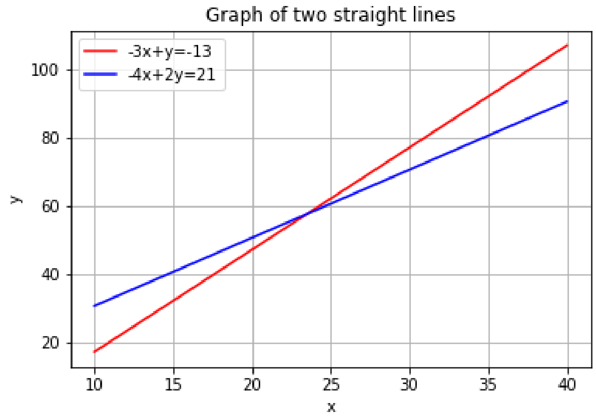

Fig. 1.6.1: Pair of straight lines for $k = -10$

of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \tag{1.6.46}$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \tag{1.6.47}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{1.6.48}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \tag{1.6.49}$$

$$\implies \left|\mathbf{V}\right| = -\frac{841}{16} \tag{1.6.50}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{1.6.51}$$

Therefore the lines will intersect.
Now from (1.6.20),

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{1.6.52}$$

The slopes of the lines are given by roots of equation (1.6.23)

$$\implies 2m^2 - \frac{35}{2}m + 12 = 0 \tag{1.6.53}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{1.6.54}$$

$$\implies m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \tag{1.6.55}$$

$$\implies m_1 = 8 \tag{1.6.56}$$

$$m_2 = \frac{3}{4} \tag{1.6.57}$$

The normal vector for two lines is given by (1.6.29)

$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{1.6.58}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \tag{1.6.59}$$

Substituting (1.6.58),(1.6.59) in (1.6.52). we get

$$k_1 k_2 = 2 \tag{1.6.60}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{1.6.61}$$

$$\mathbf{n_2} = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \tag{1.6.62}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{1.6.63}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.6.64}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (1.6.61), (1.6.62) in (1.6.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{1.6.65}$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \tag{1.6.66}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \tag{1.6.67}$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \tag{1.6.68}$$

$$\xleftrightarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \tag{1.6.69}$$

$$\implies \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \tag{1.6.70}$$

$$\implies c_1 = -\frac{102}{29} \tag{1.6.71}$$

$$c_2 = \frac{59}{29} \tag{1.6.72}$$

Substituting (1.6.61),(1.6.62),(1.6.71),(1.6.72) in (1.6.17) and (1.6.18). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \tag{1.6.73}$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \tag{1.6.74}$$

1.7. Find the value of k so that the following equation may represent a pair of straight lines -

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \tag{1.7.1}$$

Solution: The given second degree equation is, Comparing coefficients of (1.7.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \tag{1.7.2}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{1.7.3}$$

$$f = -35 \tag{1.7.4}$$

The given second degree equation (1.7.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \tag{1.7.5}$$

Expanding the determinant,

$$k + 12 = 0 \tag{1.7.6}$$

$$\implies k = -12 \tag{1.7.7}$$

Hence, from (1.7.7) we find that for $k = -12$, the given second degree equation (1.7.1) represents pair of straight lines. For the appropriate value of k, (1.7.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \tag{1.7.8}$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n_1}^T \mathbf{x} = c_1 \tag{1.7.9}$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \tag{1.7.10}$$

The pair of straight lines is given by,

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^\mathbf{T} \mathbf{x} + f = 0 \tag{1.7.11}$$

Putting the values of $\mathbf{V}$ and $\mathbf{u}$ we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \tag{1.7.12}$$

Hence, from (1.7.12) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \tag{1.7.13}$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{1.7.14}$$

$$c_1 c_2 = -35 \tag{1.7.15}$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \tag{1.7.16}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.7.17}$$

$$\mathbf{n_i} = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.7.18}$$

Substituting the values in above equations (1.7.16) we get,

$$-12m^2 + m + 6 = 0 \tag{1.7.19}$$

$$\implies m_i = \frac{-\frac{1}{2} \pm \sqrt{-(-\frac{289}{4})}}{-12} \tag{1.7.20}$$

Solving equation (1.7.20) we get ,

$$m_1 = -\frac{2}{3} \quad (1.7.21)$$

$$m_2 = \frac{3}{4} \quad (1.7.22)$$

Hence putting the values of m_1 and m_2 in (1.7.18) we get

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (1.7.23)$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.7.24)$$

Putting values of $\mathbf{n_1}$ and $\mathbf{n_2}$ in (1.7.13) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.25)$$

$$\implies \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.26)$$

Thus, from (1.7.26), $k_1k_2 = -12$. Possible combinations of (k_1, k_2) are (6,-2), (-6,2), (3,-4), (-3,4) Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.7.27)$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (1.7.28)$$

From equation (1.7.14) we get

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.7.29)$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.30)$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \quad (1.7.31)$$

$$3c_2 - 4c_1 = -43 \quad (1.7.32)$$

The augmented matrix of (1.7.31) ,(1.7.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix} \quad (1.7.33)$$

$$\xleftrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \quad (1.7.34)$$

$$\xleftrightarrow{R_2 = -\frac{2}{17}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \quad (1.7.35)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix} \quad (1.7.36)$$

$$(1.7.37)$$

Hence we get,

$$c_1 = -5 \quad (1.7.38)$$

$$c_2 = 7 \quad (1.7.39)$$

Hence (1.7.9), (1.7.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \quad (1.7.40)$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \quad (1.7.41)$$

The figure below corresponds to the pair of straight lines represented by (1.7.40) and (1.7.41).

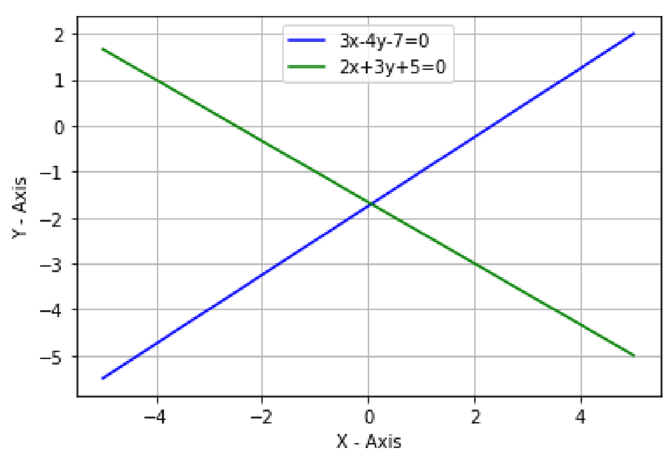

Fig. 1.7.1: Pair of Straight Lines

1.8. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \quad (1.8.1)$$

represent pairs of straight lines. **Solution:**

From (1.8.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \tag{1.8.2}$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \tag{1.8.3}$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & k \end{vmatrix} = 0 \tag{1.8.4}$$

$$\implies \left(k - \left(\frac{49}{4}\right)\right) - \frac{5}{3}\left(\frac{5}{3}k - \frac{35}{4}\right)$$

$$-\frac{5}{2}\left(\frac{-35}{6} + \frac{5}{2}\right) = 0 \tag{1.8.5}$$

$$\implies \frac{64}{k}36 - \frac{128}{12} = 0 \tag{1.8.6}$$

$$\implies \boxed{k = 6} \tag{1.8.7}$$

Substituting (1.8.7) in (1.8.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \tag{1.8.8}$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (1.8.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & 6 \end{vmatrix} \tag{1.8.9}$$

Simplyfying the above determinant , we get

$$\delta = 0 \tag{1.8.10}$$

(1.8.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \tag{1.8.11}$$

Since $\det(V) < 0$ lines would intersect each other

$$\mathbf{n_1} * \mathbf{n_2} = \{1, \frac{10}{3}, 1\} \tag{1.8.12}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.8.13}$$

$$c_1c_2 = 6 \tag{1.8.14}$$

The slopes of the lines are given by the roots

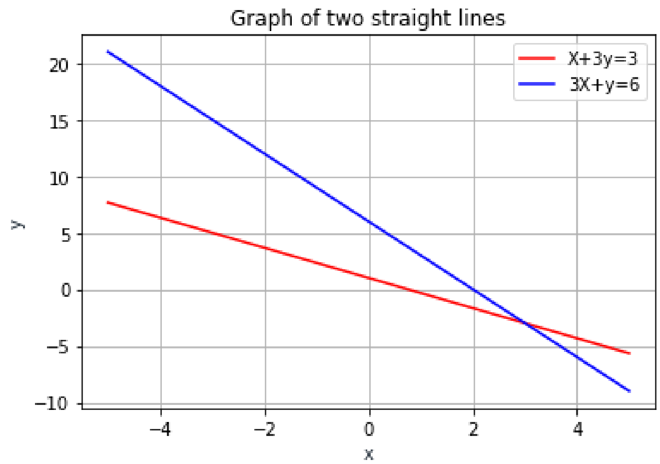

Fig. 1.8.1: Pair of straight lines

of the polynomial

$$cm^2 + 2bm + a = 0 \tag{1.8.15}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{1.8.16}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{1.8.17}$$

Substituting in above equations (1.8.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \tag{1.8.18}$$

$$\implies m_i = \frac{\frac{-10}{3} \pm \sqrt{-(\frac{-16}{9})}}{1} \tag{1.8.19}$$

Solving equation (1.8.19) we have ,

$$m_1 = \frac{-1}{3} \tag{1.8.20}$$

$$m_2 = -3 \tag{1.8.21}$$

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.8.22}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.8.23}$$

Substituting equations (1.8.22), (1.8.23) in equation (1.8.12) we get

$$k_1k_2 = 1 \tag{1.8.24}$$

Possible combination of (k_1, k_2) is (1,1) Lets assume $k_1 = 1$, $k_2 = 1$, we get

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{1.8.25}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.8.26}$$

we have:

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.8.27}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a teoplitz matrix and multiplying with $\mathbf{n_2}$

From equation (1.8.25) and (1.8.26)

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.8.28}$$

$$\implies \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{1.8.29}$$

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{1.8.30}$$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.8.31}$$

Substituting (1.8.25) and (1.8.26) in (1.8.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \tag{1.8.32}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \tag{1.8.33}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.8.34}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.8.35}$$

From above we get

$$c_1 = 1 \tag{1.8.36}$$

$$c_2 = 6 \tag{1.8.37}$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \tag{1.8.38}$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{1.8.39}$$

1.9. Find the value of k so that the following equation may represent the pair of staright lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \tag{1.9.1}$$

Solution: We need to find the value of k for which (1.9.1) represents a pair of straight lines. Converting (1.9.1) into vector form, we get

$$\mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 = 0 \tag{1.9.2}$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \tag{1.9.3}$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \tag{1.9.4}$$

$$f = -9 \tag{1.9.5}$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.9.6}$$

Since (1.9.1) represents a pair of straight lines, then by (1.9.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \tag{1.9.7}$$

By solving, above determinant we get

$$2(9-9) + \frac{-1}{2}(\frac{-9}{2} + \frac{-3k}{2}) + \frac{k}{2}(\frac{3}{2} + \frac{k}{2}) = 0 \tag{1.9.8}$$

$$\frac{(9+3k)}{4} + \frac{k(3+k)}{4} = 0 \tag{1.9.9}$$

$$k^2 + 6k + 9 = 0 \tag{1.9.10}$$

$$(k+3)^2 = 0 \tag{1.9.11}$$

$$k = -3 \tag{1.9.12}$$

Hence by (1.9.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \tag{1.9.13}$$

represents family of straight lines for $k = -3$. To find the staright lines, we write each of thrm in their vector form as

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{1.9.14}$$
$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{1.9.15}$$

Equating the product of above with (1.9.2), we have

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} k/2 \\ 3 \end{pmatrix}\mathbf{x} - 9 \tag{1.9.16}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \tag{1.9.17}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_1} = -2\begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \tag{1.9.18}$$

$$c_1c_2 = -9 \tag{1.9.19}$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \tag{1.9.20}$$
$$m^2 - m - 2 = 0 \tag{1.9.21}$$
$$m = \frac{1 \pm \sqrt{1+8}}{2} \tag{1.9.22}$$
$$m_1 = \frac{1+3}{2} = 2 \tag{1.9.23}$$
$$m_2 = \frac{1-3}{2} = -1 \tag{1.9.24}$$
$$n_1 = k_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.9.25}$$
$$n_2 = k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.9.26}$$

Substituing (1.9.25) and (1.9.26) in (1.9.17), we get

$$k_1k_2 = -1 \tag{1.9.27}$$

Taking $k_1 = -1$ and $k_2 = 1$, we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.9.28}$$
$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.9.29}$$

Substituting in (1.9.18) for above values ofn_1 and n_2

$$\begin{pmatrix} n_1 n_2 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{1.9.30}$$
$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{1.9.31}$$

Solving (1.9.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xleftrightarrow{r_2=r_2+2r_1} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \tag{1.9.32}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xleftrightarrow{r_2=r_2/3} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.9.33}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \tag{1.9.34}$$

Hence, we found out

$$c_1 = -3 \tag{1.9.35}$$
$$c_2 = 3 \tag{1.9.36}$$

Thus, pair of staright lines are

$$\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = -3 \tag{1.9.37}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{1.9.38}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.9.39}$$

The plot of above is shown below

2 General Equation. Tracing of Curves

2.1. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre

$$12x^2 - 23xy + 10y^2 - 25x + 26y = 14 \tag{2.1.1}$$

Solution: The given equation (2.1.1) can be

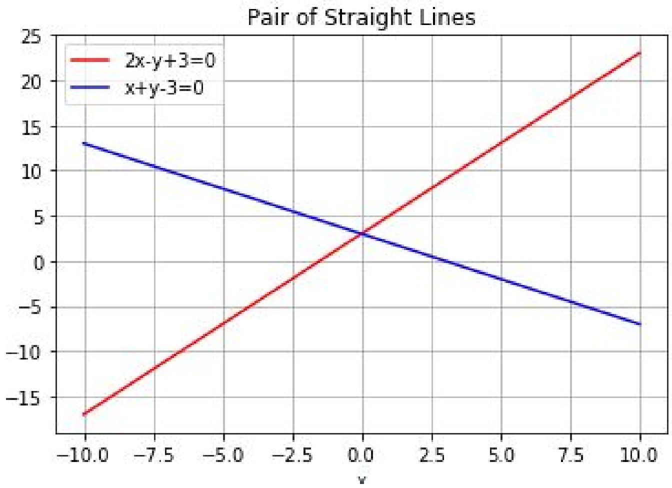

Fig. 1.9.1: Pair of Straight Lines

expressed as

$$\mathbf{x}^T\begin{pmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{pmatrix}\mathbf{x}+2\begin{pmatrix}\frac{-25}{2} & 13\end{pmatrix}\mathbf{x}-14=0 \tag{2.1.2}$$

where

$$\mathbf{V}=\begin{pmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{pmatrix} \tag{2.1.3}$$

$$\mathbf{u}=\begin{pmatrix}\frac{-25}{2}\\ 13\end{pmatrix} \tag{2.1.4}$$

$$f=-14 \tag{2.1.5}$$

$$\det(\mathbf{V})=\begin{vmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{vmatrix} \tag{2.1.6}$$

$$\implies \det(\mathbf{V})=\frac{-49}{4} \tag{2.1.7}$$

$$\implies \det(\mathbf{V})<0 \tag{2.1.8}$$

Since $\det(\mathbf{V})<0$ the given equation (2.1.2) represents the hyperbola The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V-\lambda\mathbf{I}\right|=0 \tag{2.1.9}$$

$$\begin{vmatrix}12-\lambda & \frac{-23}{2}\\ \frac{-23}{2} & 10-\lambda\end{vmatrix}=0 \tag{2.1.10}$$

$$\implies 4\lambda^2-88\lambda-49=0 \tag{2.1.11}$$

The eigenvalues are the roots of equation 2.1.11 is given by

$$\lambda_1=\frac{22+\sqrt{533}}{2} \tag{2.1.12}$$

$$\lambda_2=\frac{22-\sqrt{533}}{2} \tag{2.1.13}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p}=\lambda\mathbf{p} \tag{2.1.14}$$

$$\implies (\mathbf{V}-\lambda\mathbf{I})\mathbf{p}=0 \tag{2.1.15}$$

For $\lambda_1=\frac{22-\sqrt{533}}{2}$,

$$(\mathbf{V}-\lambda_1\mathbf{I})=\begin{pmatrix}\frac{\sqrt{553}+2}{2} & \frac{-23}{2}\\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2}\end{pmatrix} \tag{2.1.16}$$

By row reduction ,

$$\begin{pmatrix}\frac{\sqrt{533}+2}{2} & \frac{-23}{2}\\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2}\end{pmatrix} \tag{2.1.17}$$

$$\xleftrightarrow{R_1=\frac{R_1}{\frac{\sqrt{533}+2}{2}}}\begin{pmatrix}1 & \frac{2-\sqrt{533}}{23}\\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2}\end{pmatrix} \tag{2.1.18}$$

$$\xleftrightarrow{R_2=R_2+\frac{23}{2}R_1}\begin{pmatrix}1 & \frac{2-\sqrt{533}}{23}\\ 0 & 0\end{pmatrix} \tag{2.1.19}$$

Subsituting equation 2.1.19 in equation 2.1.15 we get

$$\begin{pmatrix}1 & \frac{2-\sqrt{533}}{23}\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{2.1.20}$$

Where, $\mathbf{p}=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$

Let $v_2=t$

$$v_1=\frac{-t(2-\sqrt{533})}{23} \tag{2.1.21}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1}=\begin{pmatrix}\frac{-t(2-\sqrt{533})}{23}\\ t\end{pmatrix} \tag{2.1.22}$$

Let $t=1$, we get

$$\mathbf{p_1}=\begin{pmatrix}\frac{\sqrt{533}-2}{23}\\ 1\end{pmatrix} \tag{2.1.23}$$

For $\lambda_2=\frac{22+\sqrt{533}}{2}$,

$$(\mathbf{V}-\lambda_2\mathbf{I})=\begin{pmatrix}\frac{2-\sqrt{553}}{2} & \frac{-23}{2}\\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2}\end{pmatrix} \tag{2.1.24}$$

By row reduction ,

$$\begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.1.25}$$

$$\xleftrightarrow{R_1=\frac{R_1}{\frac{2-\sqrt{533}}{2}}} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.1.26}$$

$$\xleftrightarrow{R_2=R_2+\frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \tag{2.1.27}$$

Subsituting equation 2.1.27 in equation 2.1.15 we get

$$\begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.28}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
Let $v_2 = t$

$$v_1 = \frac{-t(2+\sqrt{533})}{23} \tag{2.1.29}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(2+\sqrt{533})}{23} \\ t \end{pmatrix} \tag{2.1.30}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{-\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \tag{2.1.31}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.1.32}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.1.33}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.1.34}$$

Substituting equations 2.1.23, 2.1.31 in equation 2.1.33 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & \frac{-\sqrt{533}-2}{23} \\ 1 & 1 \end{pmatrix} \tag{2.1.35}$$

Substituting equations 2.1.12, 2.1.13 in 2.1.34 we get

$$\mathbf{D} = \begin{pmatrix} \frac{22-\sqrt{533}}{2} & 0 \\ 0 & \frac{22+\sqrt{533}}{2} \end{pmatrix} \tag{2.1.36}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.1.37}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-40}{49} & \frac{-46}{49} \\ \frac{-46}{49} & \frac{-48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{2.1.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{40}{49} & \frac{46}{49} \\ \frac{46}{49} & \frac{48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{2.1.39}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.40}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 26 > 0 \tag{2.1.41}$$

there isn't a need to swap axes
In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{2.1.42}$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} = \frac{2\sqrt{13}}{\sqrt{22+\sqrt{533}}} \tag{2.1.43}$$

$$\sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \frac{2\sqrt{13}}{\sqrt{\sqrt{533}-22}} \tag{2.1.44}$$

Now (2.1.2) can be written as,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.1.45}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x}-\mathbf{c}) \tag{2.1.46}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.1.47}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.48}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{2(\sqrt{533}-2)}{23}+1 \\ \frac{2(-\sqrt{533}-2)}{23}+1 \end{pmatrix} \tag{2.1.49}$$

Subsituting the eqauations (2.1.41), (2.1.36) in equation (2.1.45)

$$\mathbf{y}^T\begin{pmatrix} \frac{22+\sqrt{533}}{2} & 0 \\ 0 & \frac{22-\sqrt{533}}{2} \end{pmatrix}\mathbf{y} - 26 = 0 \tag{2.1.50}$$

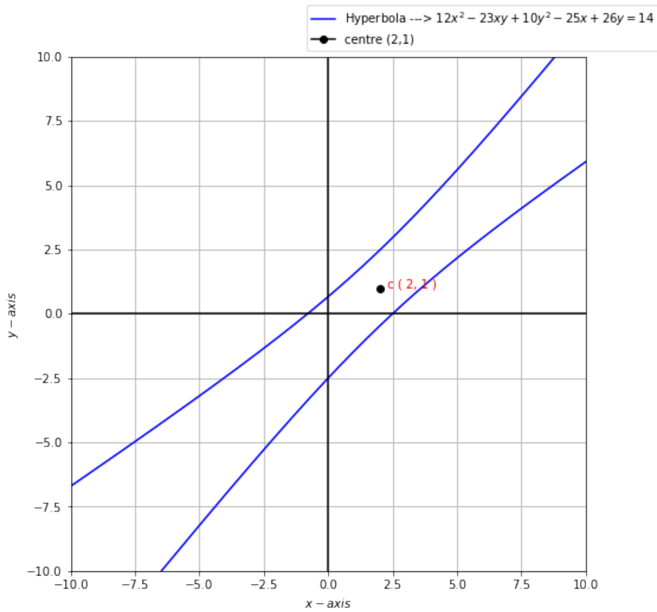

Fig. 2.1.1: Hyperbola when origin is shifted

The figure 2.1.1 verifies the given equation (2.1.2) as hyperbola with centre $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2.2. What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0 \tag{2.2.1}$$

Find the center.

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.2.2}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \tag{2.2.3}$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \tag{2.2.4}$$

$$f = -2 \tag{2.2.5}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \tag{2.2.6}$$

Hence from (2.2.6) we conclude that given equation is an ellipse. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \tag{2.2.7}$$

$$\implies \lambda^2 - 50\lambda + 400 = 0 \tag{2.2.8}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.2.8). The roots of (2.2.8) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \tag{2.2.9}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.2.10}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.2.11}$$

for $\lambda_1 = 10$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \tag{2.2.12}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.2.13}$$

Again, for $\lambda_2 = 40$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \tag{2.2.14}$$

$$\implies \mathbf{p_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.2.15}$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P} \qquad = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \tag{2.2.16}$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \tag{2.2.17}$$

Now (2.2.2) can be written as,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.2.18}$$

And,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.2.19}$$

$$\mathbf{y} = \mathbf{P}^T\left(\mathbf{x} - \mathbf{c}\right) \tag{2.2.20}$$

The centre/vertex of the conic section in (2.2.2) is given by $\mathbf{c}$ in (2.2.19). We compute $\mathbf{V}^{-1}$ as follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xleftrightarrow[R_2=\frac{13}{400}R_2]{R_2=R_2+\frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.2.21}$$

$$\xleftrightarrow[R_1=R_1+\frac{9}{13}R_2]{R_1=\frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.2.22}$$

Hence $\mathbf{V^{-1}}$ is given by,

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.2.23}$$

Now $\mathbf{u^T V^{-1} u}$ is given by,

$$\mathbf{u^T V^{-1} u} = \frac{1}{400}\begin{pmatrix} 1 & 7 \end{pmatrix}\begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \tag{2.2.24}$$

And, $\mathbf{V^{-1}u}$ is given by,

$$\mathbf{V^{-1}u} = \frac{1}{400}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.2.25}$$

By putting the value of (2.2.25), the center of the ellipse is given by (2.2.19) as follows,

$$\mathbf{c} = -\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \tag{2.2.26}$$

Also the semi-major axis (a) and semi-minor axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u^T V^{-1} u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \tag{2.2.27}$$

$$b = \sqrt{\frac{\mathbf{u^T V^{-1} u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \tag{2.2.28}$$

Finally from (2.2.18), the equation of ellipse is given by,

$$\mathbf{y^T}\begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix}\mathbf{y} = 4 \tag{2.2.29}$$

The following figure 2.2.1 is the graphical representation of the ellipse in (2.2.29),

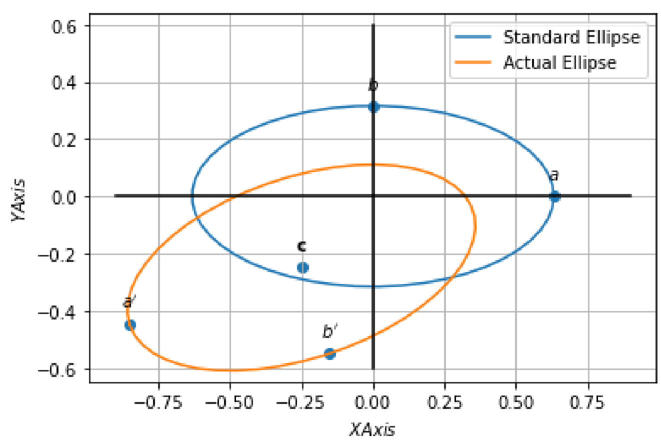

Fig. 2.2.1: Graphical representation of the ellipse