



Linear Algebra and Matrices



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Without loss of generality, $k\mathbf{m}$, for any real scalar k is also a direction vector. In the rest

of the paper, \mathbf{m} and $k\mathbf{m}$ are interchanged for computational simplicity. Thus, if m be the slope of the line PQ ,

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.2.4)$$

- 1.3. Let \mathbf{P}, \mathbf{Q} be two points on a line. The vector equation of the line is given by

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{m}, \quad \lambda \in \mathbb{R} \quad (1.3.1)$$

$$\mathbf{m} = \mathbf{P} - \mathbf{Q} \quad (1.3.2)$$

(1.3.1) can be used in 3D as well.

- 1.4. The *normal vector* \mathbf{n} to a line is orthogonal to the direction vector \mathbf{m} so that

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.4.1)$$

If \mathbf{P} be a point on the line, the equation of the line can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{P}) = 0 \quad (1.4.2)$$

$$\text{or, } \mathbf{n}^T \mathbf{x} = c, \quad (1.4.3)$$

where

$$c = \mathbf{n}^T \mathbf{P} \quad (1.4.4)$$

which is the desired equation of the straight line. By subsuming the c in (1.4.3) within \mathbf{n} , the equation of a line can also be expressed as

$$\mathbf{n}^T \mathbf{x} = 1 \quad (1.4.5)$$

Note that in 3D, (1.4.2) and (1.4.3) are used to represent the equation of a plane.

- 1.5. *Orthogonality:* Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.5.1)$$

are the vertices of a right angled triangle.

Solution: Let

$$\mathbf{v}_1 = \mathbf{A} - \mathbf{C} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad (1.5.2)$$

$$\mathbf{v}_2 = \mathbf{B} - \mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \quad (1.5.3)$$

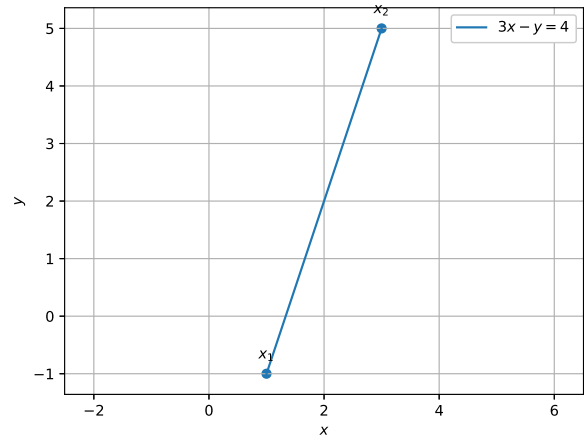


Fig. 1.6: Line obtained in Problem 1.6.

Then

$$\mathbf{v}_1^T \mathbf{v}_2 = \begin{pmatrix} -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 \quad (1.5.4)$$

$$\Rightarrow AC \perp BC \quad (1.5.5)$$

and \mathbf{v}_1 and \mathbf{v}_2 are said to be orthogonal.

- 1.6. Find the equation of the line through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ with slope - 4

Solution: From (1.2.4), the direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad (1.6.1)$$

and from (1.4.1), the normal vector is

$$\mathbf{n} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (1.6.2)$$

Using (1.4.2), the equation of the line is

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} = 0 \quad (1.6.3)$$

$$\Rightarrow \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = -5 \quad (1.6.4)$$

Fig. 1.6 shows the line passing through the given point.

- 1.7. Write the equation of the line through the points $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

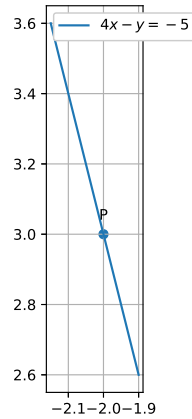


Fig. 1.7: Line obtained in Problem 1.7.

Solution: From (1.4.5),

$$\mathbf{n}^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \quad (1.7.1)$$

$$\mathbf{n}^T \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \quad (1.7.2)$$

resulting in the the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.7.3)$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \quad (1.7.4)$$

Performing row reduction,

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \quad (1.7.5)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 8 & -2 \end{pmatrix} \quad (1.7.6)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{8}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \quad (1.7.7)$$

$$\xleftrightarrow{R_1 \leftarrow 4R_1 + R_2} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \quad (1.7.8)$$

$$\xleftrightarrow{\begin{matrix} R_2 \leftarrow \frac{R_2}{4} \\ R_1 \leftarrow \frac{R_1}{4} \end{matrix}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \quad (1.7.9)$$

From (1.7.9),

$$\mathbf{n} = \frac{1}{4} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad (1.7.10)$$

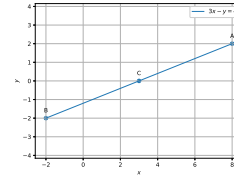


Fig. 1.8: Points on a line and points forming a triangle in Example 1.8.

Thus the equation of the desired line is

$$\frac{1}{4} \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 1 \quad (1.7.11)$$

$$\text{or, } \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 4 \quad (1.7.12)$$

Fig. 1.7 shows the line passing through the given points.

- 1.8. (*Linear Dependence*) Prove that the three points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \end{pmatrix}$ are collinear

Solution: Let

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (1.8.1)$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \end{pmatrix}$$

Then, the given points are collinear if

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = \mathbf{0} \quad (1.8.2)$$

has a nontrivial solution as well, i.e.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \mathbf{0} \quad (1.8.3)$$

Substituting (1.8.1) in (1.8.2) results in the matrix equation

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (1.8.4)$$

Performing row operations on the matrix,

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_1 - 5R_2} \begin{pmatrix} 5 & -10 \\ 0 & 0 \end{pmatrix} \quad (1.8.5)$$

which can be expressed as

$$\begin{pmatrix} 5 & -10 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (1.8.6)$$

$$\text{or, } \mathbf{x} = x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (1.8.7)$$

Thus, there are infinite solutions. The vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent and the given

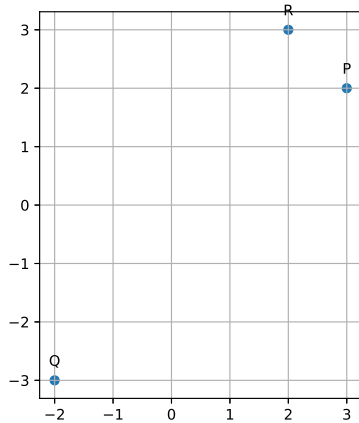


Fig. 1.10: Points on a triangle in Problem 1.10.

points lie on a straight line.

- 1.9. Alternatively, if the given points are collinear, from (1.4.5),

$$\begin{pmatrix} 3 & 0 \\ -2 & -2 \\ 8 & 2 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.9.1)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & -2 & 1 \\ 8 & 2 & 1 \end{pmatrix} \quad (1.9.2)$$

$$\xrightarrow[R_2 \leftarrow -3R_2 + 2R_1]{R_3 \leftarrow -3R_3 - 8R_1} \begin{pmatrix} 3 & 0 & 1 \\ 0 & -6 & 5 \\ 0 & 6 & -5 \end{pmatrix} \quad (1.9.3)$$

$$\xrightarrow{R_3 \leftarrow -R_3 + R_2} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.9.4)$$

The above matrix has a zero row in echelon form, hence (1.9.1) is consistent and the given points are on a straight line. Also,

$$\mathbf{n} = \frac{1}{6} \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix} \quad (1.9.5)$$

- 1.10. (Linear Independence) Do the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle?

Solution: In this case

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad (1.10.1)$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \quad (1.10.2)$$

Thus,

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 = 0 \quad (1.10.3)$$

$$\Rightarrow \begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix} \mathbf{x} = 0 \quad (1.10.4)$$

Using row operations,

$$\begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 5 & -4 \\ 0 & 2 \end{pmatrix} \quad (1.10.5)$$

$$\xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \quad (1.10.6)$$

resulting in a *full rank* matrix. Hence,

$$\mathbf{x} = 0 \quad (1.10.7)$$

and \mathbf{v}_1 and \mathbf{v}_2 are *linearly independent*. The points lie on a triangle.

- 1.11. Alternatively, from (1.4.5), row reducing the augmented matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow -R_3 + R_2} \begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad (1.11.1)$$

The above matrix has a nonzero row in echelon form, hence the given points do not lie on a straight line. So they lie on a triangle.

- 1.12. Find the angle between the lines

$$(1 - \sqrt{3})\mathbf{x} = 5 \quad (1.12.1)$$

$$(\sqrt{3} - 1)\mathbf{x} = -6.$$

Solution: The angle between the lines can be expressed in terms of the normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \quad (1.12.2)$$

as

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.12.3)$$

$$= \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ \quad (1.12.4)$$

1.13. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad (1.13.1)$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \quad (1.13.2)$$

Solution: If the angle between the vectors be θ , the projection is defined as

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\|\mathbf{a}\| \cos \theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{(\mathbf{a}^T \mathbf{b})}{\|\mathbf{b}\|^2} \mathbf{b} \quad (1.13.3)$$

Substituting the values from (1.13.1) and (1.13.2),

$$\text{proj}_{\mathbf{b}} \mathbf{a} = \frac{5}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad (1.13.4)$$

1.14. (*Reflection*) Assuming that straight lines work as a plane mirror for a point, find the image of the point $\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the line

$$L: (1 \ -3)\mathbf{x} = -4. \quad (1.14.1)$$

Solution: From the given equation, the line parameters are

$$\mathbf{n} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, c = -4, \mathbf{m} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad (1.14.2)$$

Let \mathbf{R} be the reflection of \mathbf{P} such that PR bisects the line L at \mathbf{Q} . Then \mathbf{Q} bisects PR . This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (1.14.3)$$

$$\mathbf{n}^T \mathbf{Q} = c \quad \because \mathbf{Q} \text{ lies on the given line} \quad (1.14.4)$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \quad \because \mathbf{m} \perp \mathbf{P} - \mathbf{R} \quad (1.14.5)$$

From (1.14.3) and (1.14.4),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \quad (1.14.6)$$

From (1.14.6) and (1.14.5),

$$(\mathbf{m} \ \mathbf{n})^T \mathbf{R} = (\mathbf{m} \ -\mathbf{n})^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.14.7)$$

Letting

$$\mathbf{V} = (\mathbf{m} \ \mathbf{n}) \quad (1.14.8)$$

with the condition that \mathbf{m}, \mathbf{n} are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (1.14.9)$$

Noting that

$$(\mathbf{m} \ -\mathbf{n}) = (\mathbf{m} \ \mathbf{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.14.10)$$

(1.14.7) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.14.11)$$

$$\Rightarrow \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.14.12)$$

$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c\mathbf{n} \quad (1.14.13)$$

upon substituting from (1.14.8) in (1.14.13). It can be verified that the reflection is also given by

$$\mathbf{R} = (\mathbf{m} \ \mathbf{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\mathbf{m} \ \mathbf{n})^T \mathbf{P} + 2c\mathbf{n} \quad (1.14.14)$$

$$= (\mathbf{m} \ -\mathbf{n}) \begin{pmatrix} \mathbf{m}^T \\ \mathbf{n}^T \end{pmatrix} \mathbf{P} + 2c\mathbf{n} \quad (1.14.15)$$

$$\Rightarrow \mathbf{R} = (\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T) \mathbf{P} + 2c\mathbf{n} \quad (1.14.16)$$

If \mathbf{m}, \mathbf{n} are not orthonormal, (1.14.16) can be expressed as

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T \mathbf{m} + \mathbf{n}^T \mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (1.14.17)$$

1.15. (*Gram-schmidt orthogonalization*) Let

$$\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \quad (1.15.1)$$

$$\beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \quad (1.15.2)$$

Find β_1, β_2 such that

$$\beta = \beta_1 + \beta_2, \quad \beta_1 \parallel \alpha, \beta_2 \perp \alpha \quad (1.15.3)$$

Solution: Let $\beta_1 = k\alpha$. Then, $\beta_1 \parallel \alpha$ and

$$\beta = k\alpha + \beta_2 \quad (1.15.4)$$

$$\Rightarrow \alpha^T \beta = k \|\alpha\|^2 + k\beta_1^T \beta_2 \quad (1.15.5)$$

$$\text{or, } k = \frac{\alpha^T \beta}{\|\alpha\|^2}, \quad \because \beta_1 \perp \beta_2 \quad (1.15.6)$$

Thus,

$$\beta_1 = \frac{\alpha^T \beta}{\|\alpha\|^2} \alpha = \frac{1}{2} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \quad (1.15.7)$$

$$\beta_2 = \beta - \beta_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \quad (1.15.8)$$

Thus, any given set of vectors can be expressed as a linear combination of another set of orthogonal vectors.

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{pmatrix} \quad (2.1.2)$$

$$\xrightarrow[R_3 \leftarrow 2R_3 - 5R_1]{R_2 \leftarrow \frac{R_2 + R_1}{2}} \begin{pmatrix} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -19 & 9 & -3 \end{pmatrix} \quad (2.1.3)$$

$$\xrightarrow[R_3 \leftarrow \frac{R_3 + 19R_2}{4}]{R_1 \leftarrow R_1 - 5R_2} \begin{pmatrix} 2 & 0 & -8 & -4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 7 & 4 \end{pmatrix} \quad (2.1.4)$$

$$\xrightarrow[R_3 \leftarrow 7R_2 - R_3]{R_1 \leftarrow \frac{7R_1 + 8R_3}{2}} \begin{pmatrix} 7 & 0 & 0 & 2 \\ 0 & 7 & 0 & 3 \\ 0 & 0 & 7 & 4 \end{pmatrix} \quad (2.1.5)$$

$$\Rightarrow \mathbf{n} = \frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad (2.1.6)$$

Thus, the equation of the plane passing through the given points is

$$(2 \ 3 \ 4)\mathbf{x} = 7 \quad (2.1.7)$$

2.2. Find the angle between the two planes

$$(2 \ 1 \ -2)\mathbf{x} = 5 \quad (2.2.1)$$

$$(3 \ -6 \ -2)\mathbf{x} = 7 \quad (2.2.2)$$

Solution: The angle between two planes is the same as the angle between their normal vectors. For

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} \quad (2.2.3)$$

using (1.12.3),

$$\cos \theta = \frac{6 - 6 + 4}{\sqrt{9} \sqrt{49}} = \frac{4}{21} \quad (2.2.4)$$

2 PLANE

2.1. Find the equation of a plane passing through

the points $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$

Solution: The equation of plane is also given by (1.4.5) in 3D. Following the approach in the previous example results in the matrix equation,

$$\begin{pmatrix} 2 & 5 & -3 \\ -2 & -3 & 5 \\ 5 & 3 & -3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.1)$$

3 PSEUDO INVERSE

3.1. To find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.1.1)$$

$$L_2: \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (3.1.2)$$

3.2. If the two lines intersect,

$$\mathbf{x}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (3.2.1)$$

$$\Rightarrow (\mathbf{m}_1 \quad \mathbf{m}_2) \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \quad (3.2.2)$$

$$\text{or, } \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \quad (3.2.3)$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}. \quad (3.2.4)$$

$$\mathbf{M} = (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (3.2.5)$$

(3.2.3) can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (3.2.6)$$

From the augmented matrix in (3.2.3),

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (3.2.7)$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \xrightarrow{R_1=R_1-R_2} \begin{pmatrix} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \quad (3.2.8)$$

The above matrix has a *rank* = 3. Hence the lines do not intersect.

3.3. Let

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (3.3.1)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (3.3.2)$$

be the closest points on L_1 and L_2 respectively. Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1, L_2 and passing through A and B. Thus,

$$\mathbf{m}_1^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.3.3)$$

$$\mathbf{m}_2^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.3.4)$$

$$\Rightarrow \mathbf{M}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.3.5)$$

From (3.3.2) and (3.2.5)

$$\mathbf{A} - \mathbf{B} = \mathbf{x}_1 - \mathbf{x}_2 + \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (3.3.6)$$

and using (3.3.5), in the above,

$$\mathbf{M}^T \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M}^T (\mathbf{x}_2 - \mathbf{x}_1) \quad (3.3.7)$$

3.4. Substituting the values from (3.2.4) in (3.3.7) and forming the augmented matrix,

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \quad (3.4.1)$$

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \xrightarrow{R_2=R_2-R_1} \begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \xrightarrow{R_1=2R_1-R_2} \begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \quad (3.4.3)$$

$$\begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \xrightarrow{R_1=\frac{R_1}{6}, R_2=\frac{R_2}{6}} \begin{pmatrix} 1 & 0 & \frac{11}{6} \\ 0 & 1 & -\frac{7}{6} \end{pmatrix} \quad (3.4.4)$$

$$\lambda_1 = \frac{11}{6}, \lambda_2 = \frac{7}{6} \quad (3.4.5)$$

yielding

$$\mathbf{A} = \frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix}, \mathbf{B} = \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}. \quad (3.4.6)$$

3.5. The distance is then obtained as

$$\|\mathbf{B} - \mathbf{A}\| = \frac{3}{\sqrt{2}} \quad (3.5.1)$$

Fig. 3.5 shows the various points and distances between the lines.

4 QUADRATIC FORMS

4.1. The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.1)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (4.1.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (4.1.4)$$

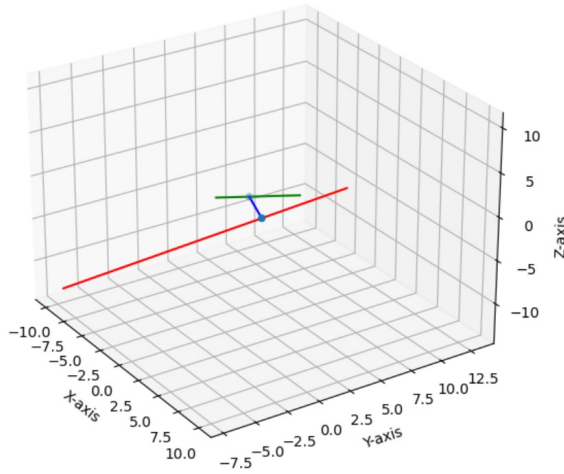


Fig. 3.5: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

4.2. (4.1.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.2.1)$$

otherwise, (4.1.2) represents a conic section.

5 PAIR OF STRAIGHT LINES

5.1. Two intersecting lines are obtained if

$$|V| < 0 \quad (5.1.1)$$

5.2. Let the pair of straight lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (5.2.1)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (5.2.2)$$

Equating their product with (4.1.2),

$$\begin{aligned} & (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ &= \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (5.2.3)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (5.2.4)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (5.2.5)$$

$$c_1 c_2 = f \quad (5.2.6)$$

where * represents convolution.

5.3. The slopes of the lines are given by the roots

of the polynomial

$$cm^2 + 2bm + a = 0 \quad (5.3.1)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-|V|}}{c} \quad (5.3.2)$$

and

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix}, \quad i = 1, 2. \quad (5.3.3)$$

5.4. From (5.2.5),

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (5.4.1)$$

c_1, c_2 can be obtained such that they satisfy (5.2.6).

6 CONIC SECTIONS

6.1. (Affine Transformation and Eigenvalue Decomposition) Using

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (\text{Affine Transformation}) \quad (6.1.1)$$

such that

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D}. \quad (\text{Eigenvalue Decomposition}) \quad (6.1.2)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (6.1.3)$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^T = \mathbf{P}^{-1} \quad (6.1.4)$$

(4.1.2) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |V| \neq 0 \quad (6.1.5)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad |V| = 0 \quad (6.1.6)$$

with

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |V| \neq 0 \quad (6.1.7)$$

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad |V| = 0 \quad (6.1.8)$$

$$\text{where } \eta = \mathbf{n}^T \mathbf{p}_1 \quad (6.1.9)$$

Solution: Proofs for (6.1.5), (6.1.6), (6.1.7) and (6.1.8) are available in Appendix A.

6.2. (Centre/Vertex) The centre/vertex of the conic section in (4.1.2) is given by \mathbf{c} in (6.1.7) or (6.1.8). This is because from (6.1.1),

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (6.2.1)$$

and

$$\mathbf{y} = \mathbf{0} \implies \mathbf{x} = \mathbf{c} \quad (6.2.2)$$

6.3. (Circle) For a circle,

$$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I} \quad (6.3.1)$$

and the centre is obtained from (6.1.7), (6.2.2) as

$$\mathbf{c} = -\mathbf{u} \quad (6.3.2)$$

(6.1.5) becomes

$$\mathbf{y}^T \mathbf{y} = \|\mathbf{y}\|^2 = \left(\sqrt{\mathbf{u}^T \mathbf{u} - f} \right)^2 \quad (6.3.3)$$

and the radius is

$$\sqrt{\mathbf{u}^T \mathbf{u} - f} \quad (6.3.4)$$

6.4. (Ellipse) For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \quad (6.4.1)$$

and (6.1.5) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (6.4.2)$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}. \quad (6.4.3)$$

The centre is obtained from (6.2.2) as (6.1.7).

6.5. (Hyperbola) For

$$|\mathbf{V}| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \quad (6.5.1)$$

and (6.1.5) becomes

$$\lambda_1 y_1^2 - (-\lambda_2) y_2^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (6.5.2)$$

with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}}, \quad (6.5.3)$$

The centre is obtained from (6.2.2) as (6.1.7).

6.6. (Parabola) For

$$|\mathbf{V}| = 0, \quad \text{or, } \lambda_1 = 0. \quad (6.6.1)$$

The vertex of the parabola is obtained using (6.1.8) and the focal length is

$$\left| \frac{2\mathbf{p}_1^T \mathbf{u}}{\lambda_2} \right| \quad (6.6.2)$$

7 TANGENTS AND NORMALS

7.1. *Secant*: The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (7.1.1)$$

with the conic section in (4.1.2) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (7.1.2)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \quad (7.1.3)$$

Solution: Substituting (7.1.1) in (4.1.2),

$$\begin{aligned} (\mathbf{q} + \mu \mathbf{m})^T \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) + 2\mathbf{u}^T (\mathbf{q} + \mu \mathbf{m}) + f &= 0 \\ \implies \mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \\ + \mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f &= 0 \end{aligned} \quad (7.1.4)$$

Solving the above quadratic in (7.1.4) yields (7.1.3).

7.2. *Tangent*: If L in (7.1.1) touches (4.1.2) at exactly one point \mathbf{q} ,

$$\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (7.2.1)$$

Solution: In this case, (7.1.4) has exactly one root. Hence, in (7.1.3)

$$\begin{aligned} [\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 \\ - (\mathbf{m}^T \mathbf{V} \mathbf{m}) (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f) &= 0 \end{aligned} \quad (7.2.2)$$

$\therefore \mathbf{q}$ is the point of contact, \mathbf{q} satisfies (4.1.2) and

$$\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \quad (7.2.3)$$

Substituting (7.2.3) in (7.2.2) and simplifying, we obtain (7.2.1).

7.3. The normal vector is obtained from (7.2.1) and (1.4.1) as

$$\mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u} \quad (7.3.1)$$

7.4. Given the point of contact \mathbf{q} , the equation of a tangent is

$$(\mathbf{V} \mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \quad (7.4.1)$$

Solution: From (7.3.1) and (1.4.2), the equa-

tion of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T (\mathbf{x} - \mathbf{q}) = 0 \quad (7.4.2)$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} - \mathbf{q}^T \mathbf{V}\mathbf{q} - \mathbf{u}^T \mathbf{q} = 0 \quad (7.4.3)$$

which, upon substituting from (7.2.3) and simplifying yields (7.1.1).

7.5. If \mathbf{V}^{-1} exists, given the normal vector \mathbf{n} , the tangent points of contact to (4.1.2) are given by

$$\mathbf{q}_i = \mathbf{V}^{-1} (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \quad (7.5.1)$$

$$\text{where } \kappa_i = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (7.5.2)$$

Solution: From (7.3.1),

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \quad (7.5.3)$$

Substituting (7.5.3) in (7.2.3),

$$\begin{aligned} & (\kappa \mathbf{n} - \mathbf{u})^T \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \\ & + 2\mathbf{u}^T \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + f = 0 \\ \implies & \kappa^2 \mathbf{n}^T \mathbf{V}^{-1} \mathbf{n} - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} + f = 0 \\ \text{or, } \kappa &= \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (7.5.4) \end{aligned}$$

Substituting (7.5.4) in (7.5.3) yields (7.5.2).

7.6. If \mathbf{V} is not invertible, given the normal vector \mathbf{n} , the point of contact to (4.1.2) is given by the matrix equation

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (7.6.1)$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \quad \mathbf{V} \mathbf{p}_1 = 0 \quad (7.6.2)$$

Solution: If \mathbf{V} is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is \mathbf{p}_1 , then,

$$\mathbf{V} \mathbf{p}_1 = 0 \quad (7.6.3)$$

From (7.3.1),

$$\kappa \mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \quad (7.6.4)$$

$$\implies \kappa \mathbf{p}_1^T \mathbf{n} = \mathbf{p}_1^T \mathbf{V} \mathbf{q} + \mathbf{p}_1^T \mathbf{u} \quad (7.6.5)$$

$$\text{or, } \kappa \mathbf{p}_1^T \mathbf{n} = \mathbf{p}_1^T \mathbf{u}, \quad \because \mathbf{p}_1^T \mathbf{V} = 0, \quad (7.6.6)$$

$$\text{from (7.6.3)} \quad (7.6.7)$$

yielding κ in (7.6.2). From (7.6.4),

$$\kappa \mathbf{q}^T \mathbf{n} = \mathbf{q}^T \mathbf{V} \mathbf{q} + \mathbf{q}^T \mathbf{u} \quad (7.6.8)$$

$$\implies \kappa \mathbf{q}^T \mathbf{n} = -f - \mathbf{q}^T \mathbf{u} \quad \text{from (7.2.3),} \quad (7.6.9)$$

$$\text{or, } (\kappa \mathbf{n} + \mathbf{u}) \mathbf{q} = -f \quad (7.6.10)$$

(7.6.4) can be expressed as

$$\mathbf{V} \mathbf{q} = \kappa \mathbf{n} - \mathbf{u}. \quad (7.6.11)$$

(7.6.10) and (7.6.11) clubbed together result in (7.6.1).

7.7. All the results related to conics are summarized in Table 7.7.

8 EXAMPLE: PAIR OF STRAIGHT LINES

8.1. Given,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (8.1.1)$$

it is easy to verify that

$$\begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} = 0 \quad (8.1.2)$$

Hence, (8.1.1) represents a pair of straight lines.

8.2. (8.1.1) can be expressed as (4.1.2) with

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (8.2.1)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (8.2.2)$$

$$f = -35 \quad (8.2.3)$$

From (8.1.1) and (5.3.1),

$$\implies m_i = \frac{-7 \pm \sqrt{49 + 480}}{-20} \quad (8.2.4)$$

$$\implies m_1 = \frac{3}{2}, m_2 = -\frac{4}{5} \quad (8.2.5)$$

Thus,

$$\mathbf{m}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (8.2.6)$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (8.2.7)$$

Conic	Property	Standard Form	Standard Parameters	Point(s) of Contact
Circle	$\mathbf{V} = \mathbf{I}$	$\frac{\mathbf{y}^T \mathbf{D} \mathbf{y}}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f} = 1$	$\mathbf{c} = -\mathbf{u},$ $r = \sqrt{\mathbf{u}^T \mathbf{u} - f}$	$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u})$
Ellipse	$ \mathbf{V} > 0$ $\lambda_1 > 0, \lambda_2 < 0$	$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T$ $\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2)$	$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u},$ $axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} \end{cases}$	$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}}$
Hyperbola	$ \mathbf{V} < 0$ $\lambda_1 > 0, \lambda_2 < 0$		$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u},$ $axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases}$	
Parabola	$ \mathbf{V} = 0$ $\lambda_1 = 0$	$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta (1 \quad 0) \mathbf{y}$	focal length = $\left \frac{\eta}{\lambda_2} \right $ $\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{v} \end{pmatrix} \mathbf{c}$ $= \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix}$ $\eta = 2 \mathbf{p}_1^T \mathbf{u}$	$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{v} \end{pmatrix} \mathbf{q}$ $= \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix}$ $\kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}$

TABLE 7.7: $\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0$ can be expressed in the above standard form for various conics. \mathbf{c} represents the centre/vertex of the conic. \mathbf{q} is/are the point(s) of contact for the tangent(s).

8.3. Using the Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 3 & 0 \\ -2 & 3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (8.3.1)$$

which matches the corresponding coefficients in (8.1.1)

Substituting from (8.2.7) in (5.4.1), the augmented matrix is

$$\begin{pmatrix} 3 & 4 & -13 \\ -2 & 5 & -45 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{R_1 - 4R_2}{3}]{R_2 \leftarrow \frac{2R_1 + 3R_2}{23}} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \end{pmatrix} \quad (8.3.2)$$

$$\Rightarrow c_1 = -7, c_2 = 5 \quad (8.3.3)$$

Fig. 8.3 plots the lines in (8.1.1)

8.4. From (8.2.7) the angle between the two straight lines is given by

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) \quad (8.4.1)$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 2 \quad (8.4.2)$$

$$\|\mathbf{n}_1\| = \sqrt{3^2 + (-2)^2} = \sqrt{13} \quad (8.4.3)$$

$$\|\mathbf{n}_2\| = \sqrt{4^2 + 5^2} = \sqrt{41} \quad (8.4.4)$$

Substituting equations (8.4.2), (8.4.3), (8.4.4)

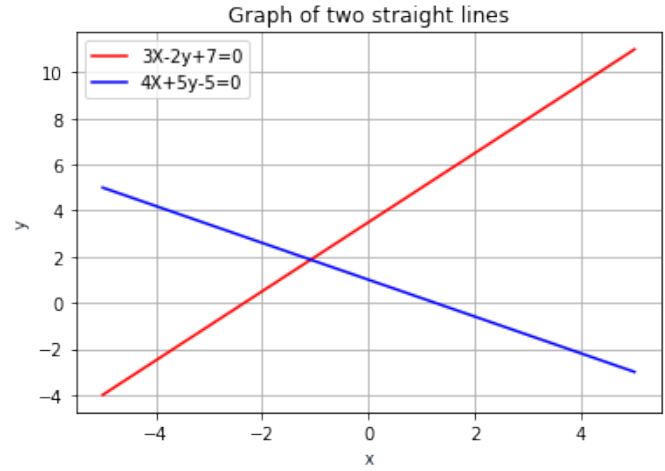


Fig. 8.3: Pair of straight lines

in equation (8.4.1), we get

$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{13} \sqrt{41}} \right) \quad (8.4.5)$$

$$\theta = 85^\circ \quad (8.4.6)$$

9 CIRCLE

9.1. Find the centre and radius of the circle

$$x^2 + y^2 + 8x + 10y - 8 = 0 \quad (9.1.1)$$

Solution: (9.1.1) can be expressed as

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 8 = 0 \quad (9.1.2)$$

which is of the form (4.1.2) with

$$\mathbf{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, f = -8 \quad (9.1.3)$$

From Table 7.7, the center and radius are given by

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}, r = \sqrt{\|\mathbf{u}\|^2 - f} = 7 \quad (9.1.4)$$

9.2. Find the equation of a circle which passes through the points $\mathbf{P} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and whose centre lies on the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (9.2.1)$$

Solution: From (4.1.2) and Table 7.7, the equation of a circle can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (9.2.2)$$

where \mathbf{c} is the centre. Substituting the given points in (9.2.2) and using (9.2.1), the following equations are obtained

$$2 \begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{c} - f = 8 \quad (9.2.3)$$

$$2 \begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{c} - f = 25 \quad (9.2.4)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{c} = 2 \quad (9.2.5)$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 4 & -4 & -1 \\ 6 & 8 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 25 \end{pmatrix} \quad (9.2.6)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 4 & -4 & -1 & 8 \\ 6 & 8 & -1 & 25 \end{pmatrix} \quad (9.2.7)$$

$$\begin{matrix} R_2 \leftarrow -R_2 + 4R_1 \\ R_3 \leftarrow -R_3 + 6R_1 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 8 & 1 & 0 \\ 0 & 2 & -1 & 13 \end{pmatrix} \quad (9.2.8)$$

$$\begin{matrix} R_1 \leftarrow -8R_1 - R_3 \\ R_3 \leftarrow -\frac{4R_3 - R_2}{2} \end{matrix} \begin{pmatrix} 8 & 0 & -1 & 16 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 5 & -52 \end{pmatrix} \quad (9.2.9)$$

$$\begin{matrix} R_1 \leftarrow \frac{5R_1 + R_3}{4} \\ R_2 \leftarrow \frac{5R_2 - R_3}{4} \end{matrix} \begin{pmatrix} 10 & 0 & 0 & 7 \\ 0 & 10 & 0 & 13 \\ 0 & 0 & 5 & -52 \end{pmatrix} \quad (9.2.10)$$

Thus,

$$\mathbf{c} = \frac{1}{10} \begin{pmatrix} 7 \\ 13 \end{pmatrix} \quad (9.2.11)$$

$$f = -\frac{52}{5} \quad (9.2.12)$$

which give the desired equation of the circle. From Table 7.7,

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \frac{1}{10} \sqrt{1258} \quad (9.2.13)$$

Fig. 9.2 verifies the above results.

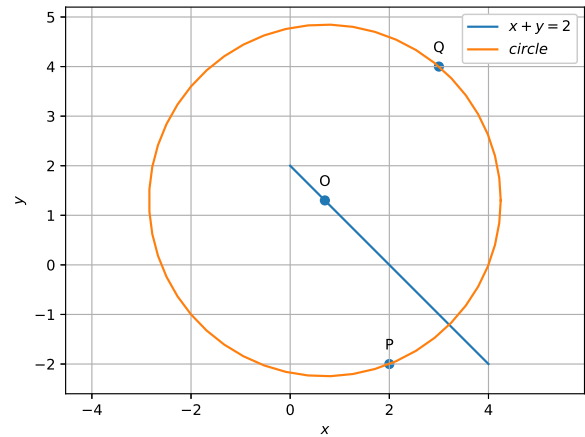


Fig. 9.2: Circle passing through $\begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Center is on line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$.

9.3. Find the points on the curve

$$x^2 + y^2 - 2x - 3 = 0 \quad (9.3.1)$$

at which the tangents are parallel to the x -axis.

Solution: (9.3.1) can be expressed as

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} - 3 = 0 \quad (9.3.2)$$

$$\Rightarrow \mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, f = -3 \quad (9.3.3)$$

From Table 7.7, the centre and radius are

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \quad (9.3.4)$$

\therefore the tangents are parallel to the x -axis, their direction and normal vectors are respectively,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (9.3.5)$$

From Table 7.7,

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\mathbf{n}^T \mathbf{n}}} = \pm \sqrt{\frac{4}{1}} = \pm 2 \quad (9.3.6)$$

and the desired points of contact are

$$\mathbf{q}_1, \mathbf{q}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (9.3.7)$$

Fig. 9.2 verifies the above results.

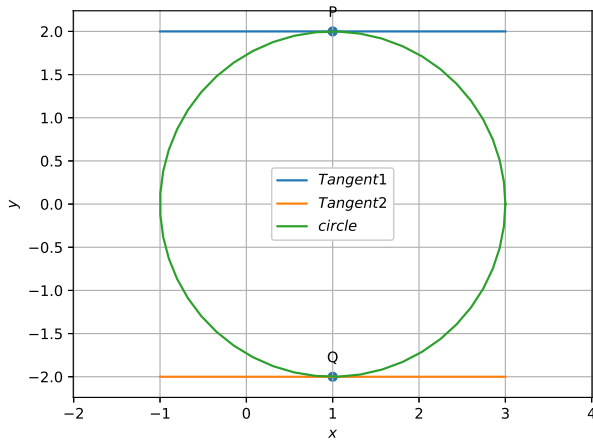


Fig. 9.3: Tangents are parallel to the x -axis.

10 ELLIPSE

10.1. Find $\frac{dy}{dx}$ if

$$E_1 : x^2 + xy + y^2 = 100 \quad (10.1.1)$$

Solution: Expressing (10.1.1) as (4.1.2),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \mathbf{u} = \mathbf{0}, f = -100. \quad (10.1.2)$$

$$\therefore |V| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} > 0, \quad (10.1.3)$$

(10.1.1) is the equation of an ellipse. To verify that this is indeed the case, we do the following exercise. The characteristic equation of \mathbf{V} is obtained by evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 1 & \frac{1}{2} \\ \frac{1}{2} & \lambda - 1 \end{vmatrix} = 0 \quad (10.1.4)$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (10.1.5)$$

The eigenvalues are the roots of (10.1.5) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \quad (10.1.6)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (10.1.7)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = \mathbf{0} \quad (10.1.8)$$

where λ is the eigenvalue. For $\lambda_1 = \frac{3}{2}$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (10.1.9)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (10.1.10)$$

such that $\|\mathbf{p}_1\| = 1$. Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (10.1.11)$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \therefore \mathbf{P}^{-1} = \mathbf{P}^T \quad (10.1.12)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (10.1.13)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (10.1.14)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad (10.1.15)$$

From Table 7.7, ellipse parameters are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \mathbf{0} \quad (10.1.16)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = 10 \sqrt{\frac{2}{3}} \quad (10.1.17)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 10 \sqrt{2} \quad (10.1.18)$$

In Fig. 10.1 the actual ellipse in (10.1.1) is obtained from (6.1.5) using (6.1.1). The anticlockwise 45° rotation is due to the fact that (10.1.14) can be expressed as

$$\mathbf{P} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \quad (10.1.19)$$

Coming back to the original question of finding $\frac{dy}{dx}$, if the point of contact

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad (10.1.20)$$

from (10.1.2), (1.2.4) and (7.2.1),

$$\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \quad (10.1.21)$$

$$\Rightarrow \left(1 + \frac{m}{2} \quad \frac{1}{2} + m\right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \quad (10.1.22)$$

$$\Rightarrow \frac{m}{2} (q_1 + 2q_2) + q_1 + \frac{q_2}{2} = 0 \quad (10.1.23)$$

$$\text{or, } m = \frac{dy}{dx} = -\frac{2q_1 + q_2}{q_1 + 2q_2} \quad (10.1.24)$$

$\therefore \frac{dy}{dx}$ is the slope of the tangent. Note that no results from differential calculus were used to obtain (10.1.24).

10.2. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Solution: This is a standard ellipse given by

$$\mathbf{x}^T \mathbf{D} \mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \quad (10.2.1)$$

$\therefore \mathbf{a}, \mathbf{b}$ satisfy (10.2.1),

$$\mathbf{a}^T \mathbf{D} \mathbf{a} = 1, \quad (10.2.2)$$

$$\mathbf{b}^T \mathbf{D} \mathbf{b} = 1 \quad (10.2.3)$$

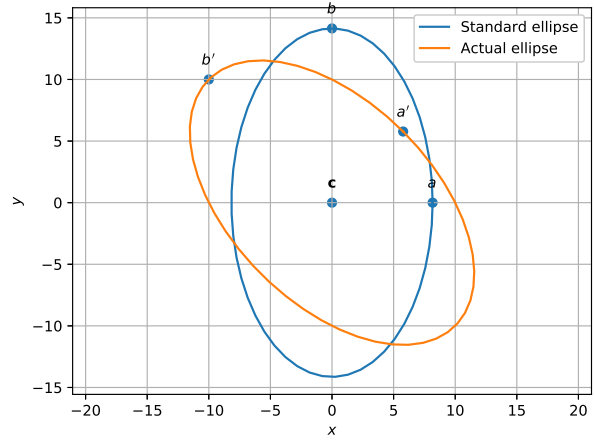


Fig. 10.1: Actual ellipse and transformed ellipse.

which can be expressed as

$$\begin{aligned} \mathbf{a}^T \mathbf{A} \mathbf{d} &= 1, \\ \mathbf{b}^T \mathbf{B} \mathbf{d} &= 1 \end{aligned} \quad (10.2.4)$$

where

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}. \quad (10.2.5)$$

(10.2.4) can then be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{a}^T \mathbf{A} \\ \mathbf{b}^T \mathbf{B} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (10.2.6)$$

which, after substituting the appropriate values can be expressed as

$$\begin{pmatrix} 16 & 9 \\ 1 & 16 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (10.2.7)$$

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix} 16 & 9 & 1 \\ 1 & 16 & 1 \end{pmatrix} \xrightarrow[R_2 \leftarrow -R_2]{R_2 \leftarrow R_1} \begin{pmatrix} 1 & 16 & 1 \\ 0 & 247 & 15 \end{pmatrix} \quad (10.2.8)$$

$$\xrightarrow{R_1 \leftarrow 247R_1 - 16R_2} \begin{pmatrix} 247 & 0 & 7 \\ 0 & 247 & 15 \end{pmatrix} \quad (10.2.9)$$

$$\Rightarrow \mathbf{d} = \frac{1}{247} \begin{pmatrix} 7 \\ 15 \end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247} \begin{pmatrix} 7 & 0 \\ 0 & 15 \end{pmatrix} \quad (10.2.10)$$

The ellipse parameters are obtained from Table

7.7 as

$$\mathbf{c} = \mathbf{0}, \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \quad (10.2.11)$$

Fig. 10.2 verifies the above results.

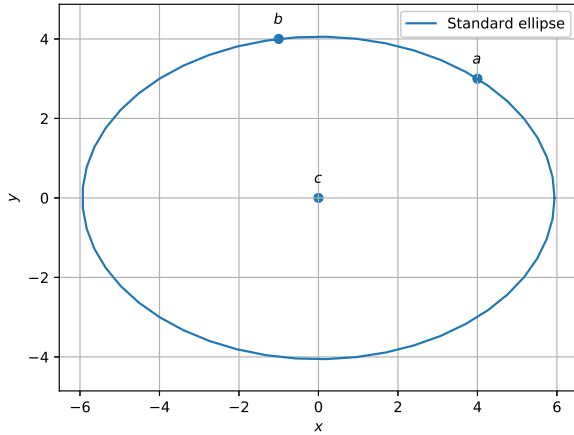


Fig. 10.2: Ellipse through the given points $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

11 HYPERBOLA

11.1. Find the equation of all lines having slope 2 and being tangent to the curve

$$y + \frac{2}{x-3} = 0 \quad (11.1.1)$$

Solution: (11.1.1) can be expressed as

$$xy - 3y + 2 = 0 \quad (11.1.2)$$

which is of the same form as (4.1.2) with

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = -\frac{3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = 2 \quad (11.1.3)$$

Using the approach in Example 10.1,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (11.1.4)$$

$$\therefore \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = -2 < 0, \quad (11.1.5)$$

the major and minor axis are swapped and from Table 7.7 the hyperbola parameters are given

by

$$\mathbf{c} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 2, \quad (11.1.6)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = 2 \quad (11.1.7)$$

with the standard hyperbola equation becoming

$$\frac{y_2^2}{4} - \frac{y_1^2}{4} = 1, \quad (11.1.8)$$

Fig. 11.1 shows the actual hyperbola in (11.1.1) obtained from (11.1.8) using (6.1.1). The direction and normal vectors of the tangent with slope 2 are given by (1.2.4) and (1.4.1) as

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (11.1.9)$$

From (7.5.2) and (9.3.3), using (11.1.3),

$$\kappa = \frac{1}{2}, \mathbf{q}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \quad (11.1.10)$$

The desired tangents are

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} = 0 \implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 2 \quad (11.1.11)$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right\} = 0 \implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 10 \quad (11.1.12)$$

All the above results are verified in Fig. 11.1. As we can see, the hyperbola in (11.1.1) is obtained by rotating the standard hyperbola by \mathbf{P} and then translating it by \mathbf{c} .

12 PARABOLA

12.1. Find the point at which the tangent to the curve

$$y = \sqrt{4x-3} - 1 \quad (12.1.1)$$

has slope $\frac{2}{3}$.

Solution: (12.1.1) can be expressed as

$$(y+1)^2 = 4x-3 \quad (12.1.2)$$

$$\text{or, } y^2 - 4x + 2y + 4 = 0 \quad (12.1.3)$$

which has the form (4.1.2) with parameters

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, f = 4. \quad (12.1.4)$$

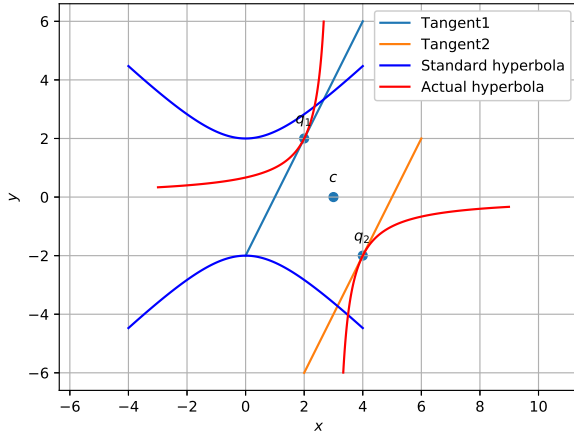


Fig. 11.1: Standard and actual hyperbola.

Thus, the given curve is a parabola. $\therefore \mathbf{V}$ is diagonal and in standard form,

$$\mathbf{P} = \mathbf{I} \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (12.1.5)$$

From Table 7.7, the focus is 4 and the vertex \mathbf{c} is

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \quad (12.1.6)$$

$$\Rightarrow \begin{pmatrix} -4 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \quad (12.1.7)$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \quad (12.1.8)$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}. \quad (12.1.9)$$

Also,

$$\mathbf{V}\mathbf{p} = \mathbf{0} \quad (12.1.10)$$

$$\Rightarrow \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (12.1.11)$$

From (7.6.2), (12.1.9) and (12.1.11),

$$\kappa = -1 \quad (12.1.12)$$

which, upon substitution in (7.6.1) and simplification yields the matrix equation

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \quad (12.1.13)$$

$$\Rightarrow \begin{pmatrix} -4 & 4 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (12.1.14)$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (12.1.15)$$

Fig. 12.1 verifies the above results.

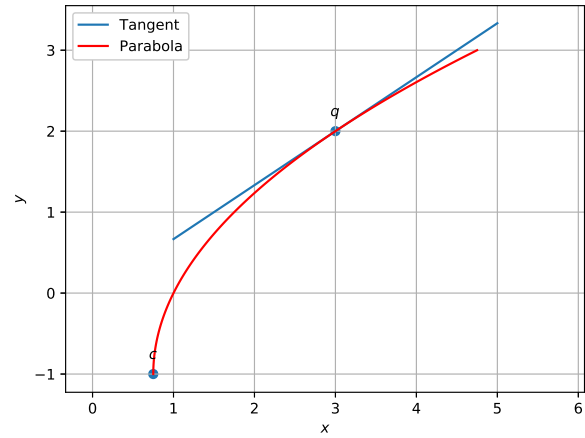


Fig. 12.1: Tangent to parabola in (12.1.1) with slope $\frac{2}{3}$.

12.2. Find a point on the curve

$$y = (x - 2)^2 \quad (12.2.1)$$

at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Solution: (12.2.1) can be expressed as

$$x^2 - 4x - y + 4 = 0 \quad (12.2.2)$$

which has the form (4.1.2) with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = 4. \quad (12.2.3)$$

Using eigenvalue decomposition,

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (12.2.4)$$

Hence, the eigenvector of \mathbf{V} corresponding to

the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (12.2.5)$$

Substituting the above parameters in the equation for the vertex of the parabola in Table 7.7,

$$\begin{pmatrix} -2 & -\frac{5}{2} \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \quad (12.2.6)$$

$$\Rightarrow \begin{pmatrix} -1 & -\frac{5}{2} \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (12.2.7)$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (12.2.8)$$

The direction vector is

$$\mathbf{m} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (12.2.9)$$

and normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (12.2.10)$$

From the equation for the point of contact for the parabola in Table 7.7,

$$\kappa = \frac{1}{2} \quad (12.2.11)$$

resulting in the matrix equation

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \quad (12.2.12)$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (12.2.13)$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (12.2.14)$$

Fig. 12.2 verifies the above results. Note that \mathbf{P} rotates the standard parabola by 90° .

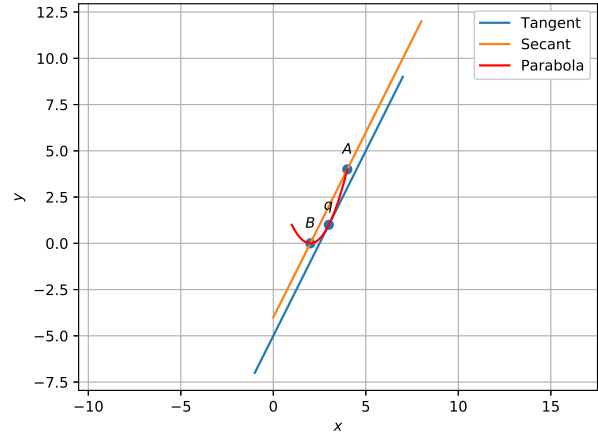


Fig. 12.2: Tangent to parabola in (12.2.1) is parallel to the line joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

13.2.

$$\begin{aligned} \frac{d\mathbf{x}}{dx_1} &= \begin{pmatrix} \frac{dx_1}{dx_1} \\ \frac{dx_2}{dx_1} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ m \end{pmatrix} = \mathbf{m} \end{aligned} \quad (13.2.1)$$

13.3. Show that

$$\begin{aligned} \frac{d(\mathbf{u}^T \mathbf{x})}{d\mathbf{x}} &= \mathbf{u} \\ \frac{d(\mathbf{x}^T \mathbf{V} \mathbf{x})}{d\mathbf{x}} &= 2\mathbf{V}^T \mathbf{x} \end{aligned} \quad (13.3.1)$$

13.4. Differentiating (4.1.2) with respect to x_1 ,

$$\left[\frac{d(\mathbf{x}^T \mathbf{V} \mathbf{x})}{d\mathbf{x}} \right]^T \frac{d\mathbf{x}}{dx_1} + 2 \frac{d(\mathbf{u}^T \mathbf{x})}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} = 0 \quad (13.4.1)$$

$$\Rightarrow 2(\mathbf{V}^T \mathbf{x} + \mathbf{u}) \mathbf{m} = 0 \quad (13.4.2)$$

from (13.2.1) and (13.3.1). Substituting the point of contact $\mathbf{x} = \mathbf{q}$ and simplifying results in

$$(\mathbf{V} \mathbf{q} + \mathbf{u}) \mathbf{m} = 0 \quad (13.4.3)$$

which, upon taking the transpose, yields (7.2.1).

13 VECTOR CALCULUS

13.1. *Definition:* Let $\mathbf{x} \in \mathbb{R}^2$, $f(\mathbf{x}) \in \mathbb{R}$. Then,

$$\frac{df(\mathbf{x})}{d\mathbf{x}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \end{pmatrix} \quad (13.1.1)$$

14 VECTOR INEQUALITIES

14.1. (*Cauchy-Schwarz Inequality*:) Show that

$$|\mathbf{a}^T \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad (14.1.1)$$

Proof. Using the definition of the inner product,

$$\cos \theta = \frac{\mathbf{a}^T \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \quad (14.1.2)$$

$$\therefore |\cos \theta| \leq 1, |\mathbf{a}^T \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad (14.1.3)$$

(*Triangle Inequality*:) Show that

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \quad (14.1.4)$$

Proof. Let \mathbf{O} be the origin. In the triangle formed by \mathbf{O} , \mathbf{a} and $-\mathbf{b}$, the lengths of the sides are

$$\|\mathbf{a}\|, \|\mathbf{b}\|, \|\mathbf{a} + \mathbf{b}\| \quad (14.1.5)$$

\therefore the sum of two sides of a triangle is always greater than the third side,

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \quad (14.1.6)$$

15 QR DECOMPOSITION

15.1. Revisiting Problem (1.15),

$$\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \quad (15.1.1)$$

we can express

$$\begin{aligned} \alpha &= k_1 \mathbf{u}_1 \\ \beta &= r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \end{aligned} \quad (15.1.2)$$

where

$$k_1 = \|\alpha\|, \mathbf{u}_1 = \frac{\alpha}{k_1} \quad (15.1.3)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2}, \mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (15.1.4)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (15.1.5)$$

From (15.1.2),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (15.1.6)$$

This is known as **QR** decomposition, where

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (15.1.7)$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \quad (15.1.8)$$

Note that \mathbf{R} is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I}. \quad (15.1.9)$$

15.2. From (15.1.1),

$$k_1 = \sqrt{10}, \mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \quad (15.2.1)$$

$$r_1 = \frac{1}{2}, \mathbf{u}_2 = \frac{1}{\sqrt{46}} \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \quad (15.2.2)$$

$$k_2 = \sqrt{\frac{23}{2}} \quad (15.2.3)$$

Thus, we obtain the **QR** decomposition

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{46}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{46}} \\ 0 & -\frac{6}{\sqrt{46}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{1}{2} \\ 0 & \sqrt{\frac{23}{2}} \end{pmatrix} \quad (15.2.4)$$

16 SINGULAR VALUE DECOMPOSITION

16.1. We revisit (3.2.6)

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \quad (16.1.1)$$

16.2. Find $\mathbf{M}^T \mathbf{M}$ and $\mathbf{M} \mathbf{M}^T$.

16.3. Obtain the eigen decomposition

$$\mathbf{M}^T \mathbf{M} = \mathbf{P}_1 \mathbf{D}_1 \mathbf{P}_1^T \quad (16.3.1)$$

and

$$\mathbf{M} \mathbf{M}^T = \mathbf{P}_2 \mathbf{D}_2 \mathbf{P}_2^T \quad (16.3.2)$$

16.4. Perform the *QR* decompositions

$$\mathbf{P}_1 = \mathbf{U} \mathbf{R}_1 \mathbf{P}_2 = \mathbf{V} \mathbf{R}_2 \quad (16.4.1)$$

16.5. The singular value decomposition is the given by

$$\mathbf{M} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T, \quad (16.5.1)$$

where Σ has the same shape as \mathbf{M} and

$$\Sigma = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (16.5.2)$$

16.6. (3.2.6) can then be expressed as

$$\mathbf{U}\Sigma\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (16.6.1)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\Sigma^{-1}\mathbf{U}^T\mathbf{b} \quad (16.6.2)$$

where Σ^{-1} is obtained by inverting only the non-zero elements of Σ .

16.7. The relevant codes are available at

codes/line/skew_builtin.py
codes/line/skew_svd.py

APPENDIX A

PROOFS FOR THE CONIC SECTIONS

A.1. Substituting (6.1.1) in (4.1.2)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \quad (A.1.1)$$

which can be expressed as

$$\mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (A.1.2)$$

From (A.1.2) and (6.1.2),

$$\mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.1.3)$$

When \mathbf{V}^{-1} exists,

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or } \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \quad (A.1.4)$$

and substituting (A.1.4) in (A.1.3) yields (6.1.5).

A.2. When $|\mathbf{V}| = 0$, $\lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = \mathbf{0}, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \quad (A.2.1)$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of \mathbf{V} such that (6.1.2)

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad (A.2.2)$$

Substituting (A.2.2) in (A.1.3),

$$\begin{aligned} & \mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T)(\mathbf{p}_1 \quad \mathbf{p}_2) \mathbf{y} \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \Rightarrow \mathbf{y}^T \mathbf{D} \mathbf{y} \\ & \quad + 2((\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_1 \quad (\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \Rightarrow \mathbf{y}^T \mathbf{D} \mathbf{y} \\ & \quad + 2(\mathbf{u}^T \mathbf{p}_1 \quad (\lambda_2 \mathbf{c}^T + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \quad \text{from (A.2.1)} \\ & \Rightarrow \lambda_2 y_2^2 + 2(\mathbf{u}^T \mathbf{p}_1) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^T \mathbf{p}_2 \\ & \quad + \mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.2.3) \end{aligned}$$

which is the equation of a parabola. From (A.2.3), by comparing the coefficients of y_2^2 and y_1 , the focal length of the parabola is obtained as

$$\left| \frac{2\mathbf{u}^T \mathbf{p}_1}{\lambda_2} \right|. \quad (A.2.4)$$

Thus, (A.2.3) can be expressed as (6.1.6) by choosing

$$\eta = 2\mathbf{u}^T \mathbf{p}_1 \quad (A.2.5)$$

and \mathbf{c} in (A.1.3) such that

$$\mathbf{P}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (A.2.6)$$

$$\mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.2.7)$$

Multiplying (A.2.6) by \mathbf{P} yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \mathbf{p}_1, \quad (A.2.8)$$

which, upon substituting in (A.2.7) results in

$$\eta \mathbf{c}^T \mathbf{p}_1 + \mathbf{u}^T \mathbf{c} + f = 0 \quad (A.2.9)$$

(A.2.8) and (A.2.9) can be clubbed together to obtain (6.1.8).