



Geometry through Linear Algebra



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Abstract—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.

1 PLANES AND LINES

1.1 Distance from a plane to a point

1.1.1. Solve the following

a) Find the foot of perpendicular from the point

$$\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \text{ on the plane } (3 \ 2 \ -6)\mathbf{x} = 2.$$

Solution: Consider orthogonal vectors \mathbf{m}_1 and \mathbf{m}_2 to the given normal vector \mathbf{n} . Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.1.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \quad (1.1.1.2)$$

$$\Rightarrow 3a + 2b - 6c = 0 \quad (1.1.1.3)$$

Let $a=1$ and $b=0$ we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.4)$$

Let $a=0$ and $b=1$ we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \quad (1.1.1.5)$$

Solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.1.6)$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.1.7)$$

Solving (1.1.1.7) using Singular Value Decomposition on \mathbf{M} as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (1.1.1.8)$$

Where the columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T \mathbf{M}$, the columns of \mathbf{U} are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$. We

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have,

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \quad (1.1.1.9)$$

$$\mathbf{M} \mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \quad (1.1.1.10)$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.1.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V} \mathbf{\Sigma}^{-1} \mathbf{U}^T \mathbf{b} \quad (1.1.1.12)$$

Where $\mathbf{\Sigma}^{-1}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$ and is obtained by inverting only non-zero elements in $\mathbf{\Sigma}$

Calculating eigen values of $\mathbf{M} \mathbf{M}^T$,

$$|\mathbf{M} \mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (1.1.1.13)$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 0 & \frac{1}{2} \\ 0 & 1 - \lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} - \lambda \end{vmatrix} = 0 \quad (1.1.1.14)$$

$$\Rightarrow \lambda^3 - \frac{85}{36} \lambda^2 + \frac{49}{36} \lambda = 0 \quad (1.1.1.15)$$

From the characteristic equation (1.1.1.15), the eigen values of $\mathbf{M} \mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.1.16)$$

The eigen vectors of $\mathbf{M} \mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ \frac{1}{13} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{3} \\ 1 \end{pmatrix} \quad (1.1.1.17)$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-7}{12} \\ \frac{-7}{18} \\ \frac{7}{6} \end{pmatrix} \quad (1.1.1.18)$$

Hence we obtain \mathbf{U} as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \quad (1.1.1.19)$$

By computing the singular values from eigen

values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.1.20)$$

Calculating eigen values of $\mathbf{M}^T \mathbf{M}$,

$$|\mathbf{M}^T \mathbf{M} - \lambda \mathbf{I}| = 0 \quad (1.1.1.21)$$

$$\Rightarrow \begin{vmatrix} \frac{5}{4} - \lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} - \lambda \end{vmatrix} = 0 \quad (1.1.1.22)$$

$$\Rightarrow \lambda^2 - \frac{85}{36} \lambda + \frac{49}{36} = 0 \quad (1.1.1.23)$$

From the characteristic equation, the eigen values of $\mathbf{M}^T \mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad (1.1.1.24)$$

Hence the eigen vectors of $\mathbf{M}^T \mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \quad (1.1.1.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.26)$$

Hence we obtain \mathbf{V} as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.27)$$

From (1.1.1.6), the Singular Value Decomposition of \mathbf{M} is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \quad (1.1.1.28)$$

And, the Moore-Penrose Pseudo inverse of $\mathbf{\Sigma}$ is given by,

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.1.29)$$

From (1.1.1.12) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \quad (1.1.1.30)$$

$$\Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.31)$$

$$\mathbf{x} = \mathbf{V} \Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.32)$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{M} \mathbf{x} = \mathbf{b} \implies \mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.1.33)$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.34)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.35)$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xrightarrow{R_1 = \frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \quad (1.1.1.36)$$

$$\xrightarrow{R_2 = R_2 - \frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{15}{45} & \frac{8}{3} \end{pmatrix} \quad (1.1.1.37)$$

$$\xrightarrow{R_2 = \frac{45}{15} R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.38)$$

$$\xrightarrow{R_1 = R_1 - \frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.39)$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.40)$$

From the equations (1.1.1.32) and (1.1.1.40), the solution \mathbf{x} is verified.

1.1.2. Solve the following

a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.1)$$

to the plane,

$$(2 \ -3 \ 1) \mathbf{x} = 0 \quad (1.1.2.2)$$

Solution: The equation of plane is given as,

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.3)$$

Hence the normal vector \mathbf{n} is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.4)$$

Let, the normal vectors \mathbf{m}_1 and \mathbf{m}_2 to the normal vector \mathbf{n} be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.5)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.6)$$

$$\implies (a \ b \ c) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.7)$$

Let, $a=0$ and $b=1$ we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.8)$$

Let, $a=1$ and $b=0$,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.9)$$

Now solving the equation,

$$\mathbf{M} \mathbf{x} = \mathbf{b} \quad (1.1.2.10)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.11)$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.12)$$

To solve (1.1.2.10) we perform singular value decomposition on \mathbf{M} given by,

$$\mathbf{M} = \mathbf{U} \mathbf{S} \mathbf{V}^T \quad (1.1.2.13)$$

substituting the value of \mathbf{M} from equation

(1.1.2.13) to (1.1.2.10),

$$\Rightarrow \mathbf{USV}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.14)$$

$$\Rightarrow \mathbf{x} = \mathbf{VS}_+ \mathbf{U}^T \mathbf{b} \quad (1.1.2.15)$$

where, \mathbf{S}_+ is Moore-Pen-rose Pseudo-Inverse of \mathbf{S} . Columns of \mathbf{U} are eigenvectors of \mathbf{MM}^T , columns of \mathbf{V} are eigenvectors of $\mathbf{M}^T \mathbf{M}$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T \mathbf{M}$.

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.16)$$

Eigenvalues corresponding to $\mathbf{M}^T \mathbf{M}$ is,

$$|\mathbf{M}^T \mathbf{M} - \lambda \mathbf{I}| = 0 \quad (1.1.2.17)$$

$$\Rightarrow \begin{pmatrix} 5 - \lambda & -6 \\ -6 & 10 - \lambda \end{pmatrix} \quad (1.1.2.18)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (1.1.2.19)$$

$$\therefore \lambda_1 = 14 \quad (1.1.2.20)$$

$$\lambda_2 = 1 \quad (1.1.2.21)$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.22)$$

$$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.23)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.24)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.25)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (1.1.2.26)$$

Now calculating the eigenvectors corresponding to \mathbf{MM}^T

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.1.2.27)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.28)$$

Eigenvalues corresponding to \mathbf{MM}^T is,

$$|\mathbf{MM}^T - \lambda \mathbf{I}| = 0 \quad (1.1.2.29)$$

$$\Rightarrow \begin{pmatrix} 1 - \lambda & 0 & -2 \\ 0 & 1 - \lambda & 3 \\ -2 & 3 & 13 - \lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\Rightarrow -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to λ_3 , λ_4 and λ_5 respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v}_4 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now \mathbf{S} corresponding to eigenvalues λ_3, λ_4

and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of \mathbf{M} as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & -\sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & -\sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\Rightarrow \mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

\therefore from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.51)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.52)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.53)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \quad (1.1.2.54)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \quad (1.1.2.55)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{7}{5} \end{pmatrix} \quad (1.1.2.56)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14} R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{7}{14} \end{pmatrix} \quad (1.1.2.57)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5} R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{7}{14} \end{pmatrix} \quad (1.1.2.58)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.59)$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.

b) Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.60)$$

to the plane,

$$(1.1.2.61)$$

$$(2 \ -3 \ 1) \mathbf{x} = 0 \quad (1.1.2.62)$$

Solution: The equation of plane is give

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.63)$$

Hence the normal vector \mathbf{n} is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.64)$$

Let, the normal vectors \mathbf{m}_1 and \mathbf{m}_2 to the normal vector \mathbf{n} be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.65)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.66)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.67)$$

Let, $a=0$ and $b=1$ we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.68)$$

Let, $a=1$ and $b=0$,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.69)$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.70)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.71)$$

To solve (1.1.2.70) we perform singular value decomposition on \mathbf{M} given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.72)$$

substituting the value of \mathbf{M} from equation (1.1.2.72) to (1.1.2.70),

$$\Rightarrow \mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.73)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (1.1.2.74)$$

where, \mathbf{S}_+ is Moore-Pen-rose Pseudo-Inverse of \mathbf{S} . Columns of \mathbf{U} are eigenvectors of $\mathbf{M}\mathbf{M}^T$, columns of \mathbf{V} are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculat-

ing the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.75)$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.2.76)$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \quad (1.1.2.77)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (1.1.2.78)$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \quad (1.1.2.79)$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad (1.1.2.80)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (1.1.2.81)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (1.1.2.82)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (1.1.2.83)$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.1.2.84)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.85)$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (1.1.2.86)$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.87)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.88)$$

$$\Rightarrow -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (1.1.2.89)$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1 \quad (1.1.2.90)$$

$$\lambda_5 = 0 \quad (1.1.2.91)$$

Hence the eigenvectors corresponding to λ_3 ,

λ_4 and λ_5 respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -\frac{2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.92)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.93)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.94)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.95)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.96)$$

Now \mathbf{S} corresponding to eigenvalues λ_3 , λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.97)$$

Now, Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.98)$$

Hence we get singular value decomposition of \mathbf{M} as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.99)$$

Now substituting the values of (1.1.2.83), (1.1.2.98), (1.1.2.96) and (1.1.2.71) in

(1.1.2.74),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.100)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \quad (1.1.2.101)$$

$$\mathbf{v} \mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.102)$$

$$\Rightarrow \mathbf{v} \mathbf{S}_+ = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.103)$$

\therefore from equation (1.1.2.74),

$$\mathbf{x} = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \quad (1.1.2.104)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.2.105)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.106)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.107)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (1.1.2.108)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix} \quad (1.1.2.109)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix} \quad (1.1.2.110)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.2.111)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.2.112)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.2.113)$$

Hence from equations (1.1.2.105) and (1.1.2.113) we conclude that the solution is verified.

c) Find the foot of the perpendicular from $\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$

on the plane $(2 \ -3 \ 1)\mathbf{x} = 0$

Solution: Let orthogonal vectors be \mathbf{m}_1 and \mathbf{m}_2 to the given normal vector \mathbf{n} . Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.114)$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.115)$$

$$\Rightarrow -5a + b + 3c = 0 \quad (1.1.2.116)$$

Let $a=1$ and $b=0$ we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.117)$$

Let $a=0$ and $b=1$ we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.118)$$

From (1.1.2.117) and (1.1.2.118),

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.119)$$

Now solving the equation

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.120)$$

Substituting the given point and (1.1.2.119) in (1.1.2.120)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.121)$$

Using the Singular value decomposition to solve (1.1.2.121) as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (1.1.2.122)$$

Where the columns of \mathbf{V} are the eigen vectors of $\mathbf{M}^T \mathbf{M}$, the columns of \mathbf{U} are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{\Sigma}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$.

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.123)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.124)$$

Substituting (1.1.2.122) in (1.1.2.120)

$$\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.125)$$

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T \mathbf{b} \quad (1.1.2.126)$$

where $\mathbf{\Sigma}^{-1}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$.

Now finding the eigen values of $\mathbf{M}\mathbf{M}^T$

$$|\mathbf{M}\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (1.1.2.127)$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0 \quad (1.1.2.128)$$

$$\Rightarrow \lambda^3 - 15\lambda^2 + 14\lambda = 0 \quad (1.1.2.129)$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad \lambda_3 = 0 \quad (1.1.2.130)$$

Therefore eigen vectors of $\mathbf{M}\mathbf{M}^T$,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.131)$$

Normalizing the eigen vectors,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{1}{\sqrt{182}} \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.132)$$

Hence from the above we get,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{-3}{\sqrt{14}} \\ 0 & \frac{1}{\sqrt{182}} & \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.133)$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get Σ as,

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 14 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.134)$$

Now calculating eigen values of $\mathbf{M}^T\mathbf{M}$

$$|\mathbf{M}^T\mathbf{M} - \lambda I| = 0 \quad (1.1.2.135)$$

$$\begin{vmatrix} 5 - \lambda & -6 \\ -6 & 10 - \lambda \end{vmatrix} = 0 \quad (1.1.2.136)$$

$$\Rightarrow \lambda^2 - 15\lambda + 14 = 0 \quad (1.1.2.137)$$

hence the eigen values of $\mathbf{M}^T\mathbf{M}$

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad (1.1.2.138)$$

Therefore eigen vectors $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \quad (1.1.2.139)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.140)$$

Hence \mathbf{V} is given as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.141)$$

Moore Pseudo inverse of Σ is,

$$\Sigma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \quad (1.1.2.142)$$

Substituting (1.1.2.133), (1.1.2.141) and (1.1.2.142) in (1.1.2.126),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{182}} & \frac{1}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.143)$$

$$\Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{-13}{7\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \\ \frac{-10}{7\sqrt{13}} \end{pmatrix} \quad (1.1.2.144)$$

$$\mathbf{V} \Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{-13}{7\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \\ \frac{-10}{7\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.145)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.146)$$

Now verifying (1.1.2.146) using (1.1.2.120)

$$\mathbf{M}\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.2.147)$$

Substituting (1.1.2.119), (1.1.2.123) and given point in (1.1.2.147)

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 \\ 10 \end{pmatrix} \quad (1.1.2.148)$$

$$(1.1.2.149)$$

Solving the augmented matrix.

$$\begin{pmatrix} 5 & -6 & -11 \\ -6 & 10 & 10 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ -6 & 10 & 10 \end{pmatrix} \quad (1.1.2.150)$$

$$\xrightarrow{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & \frac{14}{5} & \frac{-16}{5} \end{pmatrix} \quad (1.1.2.151)$$

$$\xrightarrow{R_2 = \frac{5R_2}{14}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \quad (1.1.2.152)$$

$$\xrightarrow{R_1 = R_1 + \frac{6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{-25}{7} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \quad (1.1.2.153)$$

From (1.1.2.153) we get,

$$\mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.154)$$

Hence from (1.1.2.146) and (1.1.2.154) the \mathbf{x} is verified

- d) Find the coordinates of foot of perpendicular from $\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ to the plane

$$2x - 3y + z = 0 \quad (1.1.2.155)$$

using SVD **Solution:** First we find orthogonal vectors \mathbf{m}_1 and \mathbf{m}_2 to the given plane \mathbf{n} .

Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\begin{aligned} \mathbf{m}^T \mathbf{n} &= 0 \\ \Rightarrow (a \ b \ c) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} &= 0 \\ \Rightarrow 2a - 3b + c &= 0 \end{aligned} \quad (1.1.2.156)$$

By substituting $a = 1; b = 0$ in (1.1.2.156),

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.157)$$

By substituting $a = 0; b = 1$ in (1.1.2.156),

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.158)$$

Now \mathbf{M} can be written as,

$$\mathbf{M} = (\mathbf{m}_1 \ \mathbf{m}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.159)$$

such that solving $\mathbf{M}\mathbf{x} = \mathbf{b}$ gives the required solution.

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad (1.1.2.160)$$

Applying Singular Value Decomposition on \mathbf{M} ,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.161)$$

Where the columns of \mathbf{V} are the eigenvectors

of $\mathbf{M}^T\mathbf{M}$, the columns of \mathbf{U} are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and \mathbf{S} is diagonal matrix of singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.162)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.163)$$

From (1.1.2.160) and (1.1.2.161),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (1.1.2.164)$$

Where \mathbf{S}_+ is Moore-Penrose Pseudo-Inverse of \mathbf{S} . Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\begin{aligned} |\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| &= 0 \\ \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda^3 + 15\lambda^2 - 14\lambda &= 0 \end{aligned}$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = 14; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \quad (1.1.2.165)$$

And the corresponding eigenvectors are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}; \quad \mathbf{u}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u}_3 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.166)$$

Normalizing the above eigenvectors,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{1}{\sqrt{182}} \end{pmatrix}; \quad \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.167)$$

From (1.1.2.167) we obtain \mathbf{U} as,

$$\mathbf{U} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.168)$$

Using values from (1.1.2.165),

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.169)$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\begin{aligned} |\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| &= 0 \\ \Rightarrow \begin{vmatrix} 5 - \lambda & -6 \\ -6 & 10 - \lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda^2 - 15\lambda + 14 &= 0 \end{aligned}$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 14; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Normalizing the above eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.170)$$

From (1.1.2.170) we obtain \mathbf{V} as,

$$\mathbf{V} = \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{2\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.171)$$

From (1.1.2.161) we get the Singular Value Decomposition of \mathbf{M} ,

$$\mathbf{M} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \quad (1.1.2.172)$$

Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.173)$$

From (1.1.2.164),

$$\begin{aligned} \mathbf{U}^T \mathbf{b} &= \begin{pmatrix} \frac{12\sqrt{2}}{\sqrt{91}} \\ \frac{3}{\sqrt{13}} \\ \frac{2\sqrt{2}}{7} \end{pmatrix} \\ \mathbf{S}_+ \mathbf{U}^T \mathbf{b} &= \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \\ \mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} &= \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \end{aligned} \quad (1.1.2.174)$$

To verify the solution obtained from (1.1.2.174),

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.175)$$

Substituting the values from (1.1.2.162) in (1.1.2.175),

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

Converting the above equation into augmented form and solving for \mathbf{x} ,

$$\begin{aligned} \begin{pmatrix} 5 & -6 & -3 \\ -6 & 10 & 6 \end{pmatrix} &\xrightarrow{R_2 \leftarrow \frac{5R_2 + 6R_1}{14}} \begin{pmatrix} 5 & -6 & -3 \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \\ &\xrightarrow{R_1 \leftarrow \frac{R_1 + 6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \end{aligned} \quad (1.1.2.176)$$

From (1.1.2.176) it can be observed that,

$$\mathbf{x} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \quad (1.1.2.177)$$