



# Geometry through Linear Algebra



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## CONTENTS

<b>1</b>	<b>Planes and Lines</b>	<b>1</b>
1.1	Distance from a plane to a point	1

**Abstract**—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.

## 1 PLANES AND LINES

### 1.1 Distance from a plane to a point

1.1.1. Find the foot of perpendicular from the point

$\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$  on the plane  $(3 \ 2 \ -6)\mathbf{x} = 2$ .

**Solution:** Consider orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ . Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.1.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \quad (1.1.1.2)$$

$$\Rightarrow 3a + 2b - 6c = 0 \quad (1.1.1.3)$$

Let  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.4)$$

Let  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \quad (1.1.1.5)$$

Solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.1.6)$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.1.7)$$

Solving (1.1.1.7) using Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (1.1.1.8)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T \mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T \mathbf{M}$ . We

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have,

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{5}{6} & \frac{1}{9} \\ \frac{4}{6} & \frac{10}{9} \end{pmatrix} \quad (1.1.1.9)$$

$$\mathbf{M} \mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \quad (1.1.1.10)$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.1.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V} \mathbf{\Sigma}^{-1} \mathbf{U}^T \mathbf{b} \quad (1.1.1.12)$$

Where  $\mathbf{\Sigma}^{-1}$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{\Sigma}$  and is obtained by inverting only non-zero elements in  $\mathbf{\Sigma}$

Calculating eigen values of  $\mathbf{M} \mathbf{M}^T$ ,

$$|\mathbf{M} \mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (1.1.1.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda \end{vmatrix} = 0 \quad (1.1.1.14)$$

$$\Rightarrow \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \quad (1.1.1.15)$$

From the characteristic equation (1.1.1.15), the eigen values of  $\mathbf{M} \mathbf{M}^T$  are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.1.16)$$

The eigen vectors of  $\mathbf{M} \mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.1.1.17)$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{1}{7} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-7}{12} \\ \frac{18}{7} \\ \frac{1}{6} \end{pmatrix} \quad (1.1.1.18)$$

Hence we obtain  $\mathbf{U}$  as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{18}{7} \\ \frac{1}{7} & 0 & \frac{1}{6} \end{pmatrix} \quad (1.1.1.19)$$

By computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{\Sigma}$  as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.1.20)$$

Calculating eigen values of  $\mathbf{M}^T \mathbf{M}$ ,

$$|\mathbf{M}^T \mathbf{M} - \lambda \mathbf{I}| = 0 \quad (1.1.1.21)$$

$$\Rightarrow \begin{vmatrix} \frac{5}{4} - \lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} - \lambda \end{vmatrix} = 0 \quad (1.1.1.22)$$

$$\Rightarrow \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \quad (1.1.1.23)$$

From the characteristic equation, the eigen values of  $\mathbf{M}^T \mathbf{M}$  are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad (1.1.1.24)$$

Hence the eigen vectors of  $\mathbf{M}^T \mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \quad (1.1.1.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.26)$$

Hence we obtain  $\mathbf{V}$  as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.27)$$

From (1.1.1.6), the Singular Value Decomposition of  $\mathbf{M}$  is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{1}{7} & 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \quad (1.1.1.28)$$

And, the Moore-Penrose Pseudo inverse of  $\mathbf{\Sigma}$  is given by,

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.1.29)$$

From (1.1.1.12) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \quad (1.1.1.30)$$

$$\Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.31)$$

$$\mathbf{x} = \mathbf{V} \Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.32)$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{M} \mathbf{x} = \mathbf{b} \implies \mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.1.33)$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.34)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.35)$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xrightarrow{R_1 = \frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \quad (1.1.1.36)$$

$$\xrightarrow{R_2 = R_2 - \frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{15}{45} & \frac{8}{3} \end{pmatrix} \quad (1.1.1.37)$$

$$\xrightarrow{R_2 = \frac{45}{45} R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.38)$$

$$\xrightarrow{R_1 = R_1 - \frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.39)$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.40)$$

From the equations (1.1.1.32) and (1.1.1.40), the solution  $\mathbf{x}$  is verified.

#### 1.1.2. Solve the following

a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.1)$$

to the plane,

$$(2 \quad -3 \quad 1) \mathbf{x} = 0 \quad (1.1.2.2)$$

**Solution:** The equation of plane is given as,

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.3)$$

Hence the normal vector  $\mathbf{n}$  is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.4)$$

Let, the normal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the normal vector  $\mathbf{n}$  be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.5)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.6)$$

$$\implies (a \quad b \quad c) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.7)$$

Let,  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.8)$$

Let,  $a=1$  and  $b=0$ ,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.9)$$

Now solving the equation,

$$\mathbf{M} \mathbf{x} = \mathbf{b} \quad (1.1.2.10)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.11)$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.12)$$

To solve (1.1.2.10) we perform singular value decomposition on  $\mathbf{M}$  given by,

$$\mathbf{M} = \mathbf{U} \mathbf{S} \mathbf{V}^T \quad (1.1.2.13)$$

substituting the value of  $\mathbf{M}$  from equation (1.1.2.13) to (1.1.2.10),

$$\implies \mathbf{U} \mathbf{S} \mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.14)$$

$$\implies \mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (1.1.2.15)$$

where,  $\mathbf{S}_+$  is Moore-Pen-rose Pseudo-Inverse of  $\mathbf{S}$ . Columns of  $\mathbf{U}$  are eigenvectors of  $\mathbf{MM}^T$ , columns of  $\mathbf{V}$  are eigenvectors of  $\mathbf{M}^T\mathbf{M}$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ . First calculating the eigenvectors corresponding to  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.16)$$

Eigenvalues corresponding to  $\mathbf{M}^T\mathbf{M}$  is,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.2.17)$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \quad (1.1.2.18)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (1.1.2.19)$$

$$\therefore \lambda_1 = 14 \quad (1.1.2.20)$$

$$\lambda_2 = 1 \quad (1.1.2.21)$$

Hence the eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$  respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.22)$$

$$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.23)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} \quad (1.1.2.24)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \quad (1.1.2.25)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (1.1.2.26)$$

Now calculating the eigenvectors corresponding to  $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.1.2.27)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.28)$$

Eigenvalues corresponding to  $\mathbf{MM}^T$  is,

$$|\mathbf{MM}^T - \lambda\mathbf{I}| = 0 \quad (1.1.2.29)$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\Rightarrow -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to  $\lambda_3$ ,  $\lambda_4$  and  $\lambda_5$  respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v}_4 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now  $\mathbf{S}$  corresponding to eigenvalues  $\lambda_3, \lambda_4$

and  $\lambda_5$  is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of  $\mathbf{M}$  as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & -\sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & -\sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\Rightarrow \mathbf{V} \mathbf{S}_+ = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

$\therefore$  from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.51)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.52)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.53)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \quad (1.1.2.54)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \quad (1.1.2.55)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{7}{5} \end{pmatrix} \quad (1.1.2.56)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14} R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{7}{14} \end{pmatrix} \quad (1.1.2.57)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5} R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{7}{14} \end{pmatrix} \quad (1.1.2.58)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.59)$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.

b) Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.60)$$

to the plane,

$$(1.1.2.61)$$

$$(2 \ -3 \ 1) \mathbf{x} = 0 \quad (1.1.2.62)$$

**Solution:** The equation of plane is give

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.63)$$

Hence the normal vector  $\mathbf{n}$  is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.64)$$

Let, the normal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the normal vector  $\mathbf{n}$  be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.65)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.66)$$

$$\Rightarrow \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.67)$$

Let,  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.68)$$

Let,  $a=1$  and  $b=0$ ,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.69)$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.70)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.71)$$

To solve (1.1.2.70) we perform singular value decomposition on  $\mathbf{M}$  given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.72)$$

substituting the value of  $\mathbf{M}$  from equation (1.1.2.72) to (1.1.2.70),

$$\Rightarrow \mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.73)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (1.1.2.74)$$

where,  $\mathbf{S}_+$  is Moore-Pen-rose Pseudo-Inverse of  $\mathbf{S}$ . Columns of  $\mathbf{U}$  are eigenvectors of  $\mathbf{M}\mathbf{M}^T$ , columns of  $\mathbf{V}$  are eigenvectors of  $\mathbf{M}^T\mathbf{M}$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ . First calculat-

ing the eigenvectors corresponding to  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.75)$$

Eigenvalues corresponding to  $\mathbf{M}^T\mathbf{M}$  is,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.2.76)$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \quad (1.1.2.77)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (1.1.2.78)$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \quad (1.1.2.79)$$

Hence the eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$  respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad (1.1.2.80)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (1.1.2.81)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (1.1.2.82)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (1.1.2.83)$$

Now calculating the eigenvectors corresponding to  $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.1.2.84)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.85)$$

Eigenvalues corresponding to  $\mathbf{M}\mathbf{M}^T$  is,

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (1.1.2.86)$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.87)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.88)$$

$$\Rightarrow -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (1.1.2.89)$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1 \quad (1.1.2.90)$$

$$\lambda_5 = 0 \quad (1.1.2.91)$$

Hence the eigenvectors corresponding to  $\lambda_3$ ,

$\lambda_4$  and  $\lambda_5$  respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -\frac{2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.92)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.93)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.94)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.95)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.96)$$

Now  $\mathbf{S}$  corresponding to eigenvalues  $\lambda_3$ ,  $\lambda_4$  and  $\lambda_5$  is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.97)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.98)$$

Hence we get singular value decomposition of  $\mathbf{M}$  as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.99)$$

Now substituting the values of (1.1.2.83), (1.1.2.98), (1.1.2.96) and (1.1.2.71) in

(1.1.2.74),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.100)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \quad (1.1.2.101)$$

$$\mathbf{v} \mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.102)$$

$$\Rightarrow \mathbf{v} \mathbf{S}_+ = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.103)$$

$\therefore$  from equation (1.1.2.74),

$$\mathbf{x} = \frac{1}{\sqrt{13} \sqrt{14}} \begin{pmatrix} -2 & 3 \sqrt{14} & 0 \\ 3 & 2 \sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \quad (1.1.2.104)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.2.105)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.106)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.107)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (1.1.2.108)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix} \quad (1.1.2.109)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix} \quad (1.1.2.110)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.2.111)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.2.112)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.2.113)$$

Hence from equations (1.1.2.105) and (1.1.2.113) we conclude that the solution is verified.