Geometry through Linear Algebra

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Abstract—**This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.**

1 Planes and Lines

1.1 Distance from a plane to a point

1.1.1. Solve the following

   a) Find the foot of perpendicular from the point $\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ on the plane $\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\mathbf{x} = 2$.

   Solution: Consider orthogonal vectors $\mathbf{m_1}$

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and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \qquad (1.1.1.1)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \qquad (1.1.1.2)$$

$$\implies 3a + 2b - 6c = 0 \qquad (1.1.1.3)$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad (1.1.1.4)$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \qquad (1.1.1.5)$$

Solving the equation,

$$\mathbf{Mx} = \mathbf{b} \qquad (1.1.1.6)$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \qquad (1.1.1.7)$$

Solving (1.1.1.7) using Singular Value De-

composition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U\Sigma V}^T \qquad (1.1.1.8)$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. We have,

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \qquad (1.1.1.9)$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \qquad (1.1.1.10)$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U\Sigma V}^T\mathbf{x} = \mathbf{b} \qquad (1.1.1.11)$$

$$\implies \mathbf{x} = \mathbf{V\Sigma^{-1}U^T b} \qquad (1.1.1.12)$$

Where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$ and is obtained by inversing only non-zero elements in $\mathbf{\Sigma}$
Calculating eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.1.13)$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda \end{vmatrix} = 0 \quad (1.1.1.14)$$

$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \quad (1.1.1.15)$$

From the characteristic equation (1.1.1.15), the eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \qquad (1.1.1.16)$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{3} \\ 1 \end{pmatrix} \qquad (1.1.1.17)$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-7}{12} \\ \frac{-7}{18} \\ \frac{7}{6} \end{pmatrix} \qquad (1.1.1.18)$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \qquad (1.1.1.19)$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (1.1.1.20)$$

Calculating eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \qquad (1.1.1.21)$$

$$\implies \begin{vmatrix} \frac{5}{4} - \lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} - \lambda \end{vmatrix} = 0 \qquad (1.1.1.22)$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \qquad (1.1.1.23)$$

From the characteristic equation, the eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \qquad (1.1.1.24)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \qquad (1.1.1.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \qquad (1.1.1.26)$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \qquad (1.1.1.27)$$

From (1.1.1.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \qquad (1.1.1.28)$$

And, the Moore-Penrose Pseudo inverse of $\mathbf{\Sigma}$ is given by,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.1.1.29)$$

From (1.1.1.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \quad (1.1.1.30)$$

$$\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.31)$$

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.32)$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{Mx} = \mathbf{b} \implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \quad (1.1.1.33)$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T\mathbf{Mx} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.34)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.35)$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xrightarrow{R_1 = \frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \quad (1.1.1.36)$$

$$\xrightarrow{R_2 = R_2 - \frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{49}{45} & \frac{8}{3} \end{pmatrix} \quad (1.1.1.37)$$

$$\xrightarrow{R_2 = \frac{45}{49}R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.38)$$

$$\xrightarrow{R_1 = R_1 - \frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.39)$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.40)$$

From the equations (1.1.1.32) and (1.1.1.40), the solution $\mathbf{x}$ is verified.

b) Find the foot of perpendicular from point $B = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ to the plane $\begin{pmatrix} 2 & 3 & -4 \end{pmatrix}\mathbf{x} = -5$.

Solution: Let us consider orthogonal vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ to the given normal vector $\mathbf{n}$. Let

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Then,

$$\mathbf{m}^T\mathbf{n} = 0 \quad (1.1.1.41)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \quad (1.1.1.42)$$

$$\implies 2a + 3b - 4c = 0 \quad (1.1.1.43)$$

Let $a = 1$, $b = 0$, so that

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.44)$$

and $a = 0$, $b = 1$, so that

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \quad (1.1.1.45)$$

We, now, solve the equation

$$\mathbf{Mx} = \mathbf{b} \quad (1.1.1.46)$$

which, upon substitution, becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad (1.1.1.47)$$

Any $m \times n$ matrix $\mathbf{M}$ can be factorized in SVD form as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.1.48)$$

where $\mathbf{U}$ and $\mathbf{V}$ are matrices of eigen vectors which are orthogonal. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is the diagonal matrix of singular values of $\mathbf{M}$ of the eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \quad (1.1.1.49)$$

Putting (1.1.1.48) into (1.1.1.46), we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (1.1.1.50)$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (1.1.1.51)$$

where $\mathbf{S}_+$ is the Moore-Penrose Pseudoinverse of $\mathbf{S}$.

The eigenvalues of $\mathbf{M}^T\mathbf{M}$:

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (1.1.1.52)$$

$$\implies \begin{vmatrix} \frac{10}{8} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{vmatrix} = 0 \quad (1.1.1.53)$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{116}{64} = 0 \quad (1.1.1.54)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (1.1.1.55)$$

$$\lambda_2 = 1 \quad (1.1.1.56)$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector $\mathbf{v_1}$ can be calculated using row reduction as :

$$\begin{pmatrix} -\frac{9}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \xleftarrow{R_1 \leftarrow -\frac{16}{9}R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \quad (1.1.1.57)$$

$$\xleftarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad (1.1.1.58)$$

Hence,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.59)$$

Similarly, for $\lambda_2 = 1$,

$$\mathbf{v_2} = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.60)$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.61)$$

Now,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \quad (1.1.1.62)$$

Now, calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$

$$\begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \lambda \end{vmatrix} = 0 \quad (1.1.1.63)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (1.1.1.64)$$

$$\lambda_2 = 1 \quad (1.1.1.65)$$

$$\lambda_3 = 0 \quad (1.1.1.66)$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector can be computed as:

$$\begin{pmatrix} 1 - \frac{29}{16} & 0 & \frac{1}{2} \\ 0 & 1 - \frac{29}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \frac{29}{16} \end{pmatrix} \quad (1.1.1.67)$$

$$\leftrightarrow \begin{pmatrix} -\frac{13}{16} & 0 & \frac{1}{2} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (1.1.1.68)$$

$$\xleftarrow{R_1 \leftarrow -\frac{16}{13}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (1.1.1.69)$$

$$\xleftarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \quad (1.1.1.70)$$

$$\xleftarrow{R_2 \leftarrow -\frac{16}{13}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \quad (1.1.1.71)$$

$$\xleftarrow{R_2 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.1.1.72)$$

Hence, the eigen vector $\mathbf{u}_1$:

$$\mathbf{u_1} = \begin{pmatrix} \frac{8}{\sqrt{377}} \\ \frac{12}{\sqrt{377}} \\ \frac{13}{\sqrt{377}} \end{pmatrix} \quad (1.1.1.73)$$

For $\lambda_2 = 1$, the eigen vector is:

$$\begin{pmatrix} 1-1 & 0 & \frac{1}{2} \\ 0 & 1-1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - 1 \end{pmatrix} \quad (1.1.1.74)$$

$$\leftrightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{3}{16} \end{pmatrix} \quad (1.1.1.75)$$

Hence, the eigen vector $\mathbf{u}_2$:

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.1.76)$$

Similarly, for $\lambda_3 = 0$, the eigen vector is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \quad (1.1.1.77)$$

$$\xleftarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1 - \frac{3}{4}R_2} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.1.1.78)$$

Hence, the eigen vector $\mathbf{u}_3$:

$$\mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ -\frac{4}{\sqrt{29}} \end{pmatrix} \quad (1.1.1.79)$$

So, the orthonormal matrix $\mathbf{U}$ of eigen vectors is:

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & -\frac{4}{\sqrt{29}} \end{pmatrix} \quad (1.1.1.80)$$

The matrix of singular values of $\mathbf{M}$ is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.1.1.81)$$

The Moore-Penrose pseudoinverse of $\mathbf{S}$ is computed as

$$\mathbf{S}_+ = (\mathbf{SS}^T)^{-1}\mathbf{S}^T \quad (1.1.1.82)$$

$$= \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.1.83)$$

To solve for $\mathbf{x}$ in (1.1.1.51), noting that $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \\ 0 \end{pmatrix} \quad (1.1.1.84)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \quad (1.1.1.85)$$

Thus, the foot of perpendicular is:

$$\mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \quad (1.1.1.86)$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.87)$$

The solution can be verified using

$$\mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \quad (1.1.1.88)$$

The LHS gives

$$\mathbf{M}^T\mathbf{Mx} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.89)$$

$$\implies \mathbf{M}^T\mathbf{Mx} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.90)$$

Now, finding $\mathbf{x}$ from

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.91)$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} & -3 \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \xleftarrow{R_1 \leftarrow \frac{3}{10}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix}$$
$$(1.1.1.92)$$

$$\xleftarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & \frac{29}{20} & \frac{58}{20} \end{pmatrix} \xleftarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & 1 & 2 \end{pmatrix}$$
$$(1.1.1.93)$$

$$\xleftarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix}$$
$$(1.1.1.94)$$

Hence, the solution is given by

$$\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.95)$$

Comparing the results in Eq.(1.1.1.87) and (1.1.1.95), it is concluded that the solution is verified.

1.1.2. Solve the following
   a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.1)$$

to the plane,

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}\mathbf{x} = 0 \quad (1.1.2.2)$$

Solution: The equation of plane is given as,

$$\mathbf{n}^T\mathbf{x} = c \quad (1.1.2.3)$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.4)$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (1.1.2.5)$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \qquad (1.1.2.6)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \qquad (1.1.2.7)$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad (1.1.2.8)$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \qquad (1.1.2.9)$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \qquad (1.1.2.10)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \qquad (1.1.2.11)$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \qquad (1.1.2.12)$$

To solve (1.1.2.10) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \qquad (1.1.2.13)$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.13) to (1.1.2.10),

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \qquad (1.1.2.14)$$
$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} \qquad (1.1.2.15)$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculat-

ing the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}$$
$$(1.1.2.16)$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \qquad (1.1.2.17)$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \qquad (1.1.2.18)$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \qquad (1.1.2.19)$$

$$\therefore \lambda_1 = 14 \qquad (1.1.2.20)$$

$$\lambda_2 = 1 \qquad (1.1.2.21)$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \qquad (1.1.2.22)$$

$$\mathbf{v_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \qquad (1.1.2.23)$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad (1.1.2.24)$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad (1.1.2.25)$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \qquad (1.1.2.26)$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \qquad (1.1.2.27)$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \qquad (1.1.2.28)$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.2.29)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\implies -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v_5} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4

and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S_+} = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{V}\mathbf{S_+} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\implies \mathbf{V}\mathbf{S_+} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

$\therefore$ from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.51)$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.2.52)$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$
$$(1.1.2.53)$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$$
$$(1.1.2.54)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix}$$
$$(1.1.2.55)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{17}{5} \end{pmatrix}$$
$$(1.1.2.56)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{17}{14} \end{pmatrix}$$
$$(1.1.2.57)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{17}{14} \end{pmatrix}$$
$$(1.1.2.58)$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix}$$
$$(1.1.2.59)$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.

b) Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.60)$$

to the plane,

$$(1.1.2.61)$$

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (1.1.2.62)$$

Solution: The equation of plane is give

$$\mathbf{n}^T\mathbf{x} = c \quad (1.1.2.63)$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.64)$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.65)$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \quad (1.1.2.66)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.67)$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.68)$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.69)$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.70)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.71)$$

To solve (1.1.2.70) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.72)$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.72) to (1.1.2.70),

$$\implies \mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (1.1.2.73)$$
$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (1.1.2.74)$$

where, $\mathbf{S}_+$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{M}\mathbf{M}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculat-

ing the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}$$
(1.1.2.75)

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \qquad (1.1.2.76)$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \qquad (1.1.2.77)$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \qquad (1.1.2.78)$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \qquad (1.1.2.79)$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \qquad (1.1.2.80)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad (1.1.2.81)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad (1.1.2.82)$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \qquad (1.1.2.83)$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \qquad (1.1.2.84)$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \qquad (1.1.2.85)$$

Eigenvalues corresponding to $\mathbf{MM}^T$ is,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \qquad (1.1.2.86)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \qquad (1.1.2.87)$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \qquad (1.1.2.88)$$

$$\implies -\lambda(\lambda - 1)(\lambda - 14) = 0 \qquad (1.1.2.89)$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1 \qquad (1.1.2.90)$$

$$\lambda_5 = 0 \qquad (1.1.2.91)$$

Hence the eigenvectors corresponding to λ_3,

λ_4 and λ_5 respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.92)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.93)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.94)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.95)$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.96)$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (1.1.2.97)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.1.2.98)$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}}\begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}\begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T$$
(1.1.2.99)

Now substituting the values of (1.1.2.83), (1.1.2.98), (1.1.2.96) and (1.1.2.71) in

(1.1.2.74),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

(1.1.2.100)

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix}$$

(1.1.2.101)

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(1.1.2.102)

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix}$$

(1.1.2.103)

∴ from equation (1.1.2.74),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix}$$

(1.1.2.104)

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(1.1.2.105)

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b}$$

(1.1.2.106)

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

(1.1.2.107)

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

(1.1.2.108)

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xleftarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix}$$

(1.1.2.109)

$$\xleftarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix}$$

(1.1.2.110)

$$\xleftarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix}$$

(1.1.2.111)

$$\xleftarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(1.1.2.112)

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(1.1.2.113)

Hence from equations (1.1.2.105) and (1.1.2.113) we conclude that the solution is verified.

c) Find the foot of the perpendicular from $\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$ on the plane $\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}\mathbf{x} = 0$

Solution: Let orthogonal vectors be $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0$$

(1.1.2.114)

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

(1.1.2.115)

$$\implies -5a + b + 3c = 0$$

(1.1.2.116)

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

(1.1.2.117)

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

(1.1.2.118)

From (1.1.2.117) and (1.1.2.118),

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \qquad (1.1.2.119)$$

Now solving the equation

$$\mathbf{Mx} = \mathbf{b} \qquad (1.1.2.120)$$

Substituting the given point and (1.1.2.119) in (1.1.2.120)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \qquad (1.1.2.121)$$

Using the Singular value decomposition to solve (1.1.2.121) as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \qquad (1.1.2.122)$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{\Sigma}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \qquad (1.1.2.123)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \qquad (1.1.2.124)$$

Substituting (1.1.2.122) in (1.1.2.120)

$$\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{x} = \mathbf{b} \qquad (1.1.2.125)$$
$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b} \qquad (1.1.2.126)$$

where $\mathbf{\Sigma}^{-1}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$.

Now finding the eigen values of $\mathbf{M}\mathbf{M}^T$

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \qquad (1.1.2.127)$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0 \qquad (1.1.2.128)$$

$$\implies \lambda^3 - 15\lambda^2 + 14\lambda = 0 \qquad (1.1.2.129)$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad \lambda_3 = 0 \qquad (1.1.2.130)$$

Therefore eigen vectors of $\mathbf{M}\mathbf{M}^T$,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$
$$(1.1.2.131)$$

Normalizing the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}$$
$$(1.1.2.132)$$

Hence from the above we get,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{-3}{\sqrt{14}} \\ 0 & \frac{13}{\sqrt{182}} & \frac{1}{\sqrt{14}} \end{pmatrix} \qquad (1.1.2.133)$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 14 \\ 0 & 0 \end{pmatrix} \qquad (1.1.2.134)$$

Now calculating eigen values of $\mathbf{M}^T\mathbf{M}$

$$\left|\mathbf{M}^T\mathbf{M} - \lambda I\right| = 0 \qquad (1.1.2.135)$$

$$\begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0 \qquad (1.1.2.136)$$

$$\implies \lambda^2 - 15\lambda + 14 = 0 \qquad (1.1.2.137)$$

hence the eigen values of $\mathbf{M}^T\mathbf{M}$

$$\lambda_1 = 1 \quad \lambda_2 = 14 \qquad (1.1.2.138)$$

Therefore eigen vectors $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \qquad (1.1.2.139)$$

Normalizing the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \qquad (1.1.2.140)$$

Hence $\mathbf{V}$ is given as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \qquad (1.1.2.141)$$

Moore Pseudo inverse of Σ is,

$$\Sigma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \qquad (1.1.2.142)$$

Substituting (1.1.2.133), (1.1.2.141) and (1.1.2.142) in (1.1.2.126),

$$U^T b = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{182}} & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{1}} \end{pmatrix}$$

$$(1.1.2.143)$$

$$\Sigma^{-1} U^T b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix}$$

$$(1.1.2.144)$$

$$V\Sigma^{-1} U^T b = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix}$$

$$(1.1.2.145)$$

$$\implies x = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix}$$

$$(1.1.2.146)$$

Now verifying (1.1.2.146) using (1.1.2.120)

$$Mx = b \implies M^T M x = M^T b \qquad (1.1.2.147)$$

Substituting (1.1.2.119), (1.1.2.123) and given point in (1.1.2.147)

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} x = \begin{pmatrix} -11 \\ 10 \end{pmatrix} \qquad (1.1.2.148)$$

$$(1.1.2.149)$$

Solving the augmented matrix.

$$\begin{pmatrix} 5 & -6 & -11 \\ -6 & 10 & 10 \end{pmatrix} \xleftarrow{R_1 = \frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ -6 & 10 & 10 \end{pmatrix}$$

$$(1.1.2.150)$$

$$\xleftarrow{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & \frac{14}{5} & \frac{-16}{5} \end{pmatrix}$$

$$(1.1.2.151)$$

$$\xleftarrow{R_2 = \frac{5R_2}{14}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix}$$

$$(1.1.2.152)$$

$$\xleftarrow{R_1 = R_1 + \frac{6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{-25}{7} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix}$$

$$(1.1.2.153)$$

From (1.1.2.153) we get,

$$x = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \qquad (1.1.2.154)$$

Hence from (1.1.2.146) and (1.1.2.154) the x is verified

d) Find the coordinates of foot of perpendicular from $D = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ to the plane

$$2x - 3y + z = 0 \qquad (1.1.2.155)$$

using SVD

Solution: First we find orthogonal vectors m_1 and m_2 to the given plane n. Let, $m = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$m^T n = 0$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$\implies 2a - 3b + c = 0 \qquad (1.1.2.156)$$

By substituting $a = 1; b = 0$ in (1.1.2.156),

$$m_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad (1.1.2.157)$$

By substituting $a = 0; b = 1$ in (1.1.2.156),

$$m_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \qquad (1.1.2.158)$$

Now M can be written as,

$$M = \begin{pmatrix} m_1 & m_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \qquad (1.1.2.159)$$

such that solving $Mx = b$ gives the required solution.

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad (1.1.2.160)$$

Applying Singular Value Decomposition on M,

$$M = USV^T \qquad (1.1.2.161)$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.162}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.163}$$

From (1.1.2.160) and (1.1.2.161),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$
$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{1.1.2.164}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^3 + 15\lambda^2 - 14\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = 14; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{1.1.2.165}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$
$$\tag{1.1.2.166}$$

Normalizing the above eigenvectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}$$
$$\tag{1.1.2.167}$$

From (1.1.2.167) we obtain $\mathbf{U}$ as,

$$\mathbf{U} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.168}$$

Using values from (1.1.2.165),

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.169}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - 15\lambda + 14 = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 14; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Normalizing the above eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.170}$$

From (1.1.2.170) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.171}$$

From (1.1.2.161) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T$$
$$\tag{1.1.2.172}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.173}$$

From (1.1.2.164),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{2}}{\sqrt{91}} \\ \frac{3}{\sqrt{13}} \\ \frac{2\sqrt{2}}{7} \end{pmatrix}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.1.2.174}$$

To verify the solution obtained from (1.1.2.174),

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.175}$$

Substituting the values from (1.1.2.162) in (1.1.2.175),

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

Converting the above equation into augmented form and solving for $\mathbf{x}$,

$$\begin{pmatrix} 5 & -6 & -3 \\ -6 & 10 & 6 \end{pmatrix} \xleftarrow{R_2 \leftarrow \frac{5R_2+6R_1}{14}} \begin{pmatrix} 5 & -6 & -3 \\ 0 & 1 & \frac{6}{7} \end{pmatrix}$$

$$\xleftarrow{R_1 \leftarrow \frac{R_1+6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix}$$

$$(1.1.2.176)$$

From (1.1.2.176) it can be observed that,

$$\mathbf{x} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \qquad (1.1.2.177)$$

1.1.3. a) Perform SVD and find the foot of the perpendicular from any point outside the plane. $\begin{pmatrix} 14 & -3 & -18 \end{pmatrix}\mathbf{x} = -1$ **Solution:** Let the point be $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$, First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \qquad (1.1.3.1)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 14 \\ -3 \\ 18 \end{pmatrix} = 0 \qquad (1.1.3.2)$$

$$\implies 14a - 3b + 18c = 0 \qquad (1.1.3.3)$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{-7}{9} \end{pmatrix} \qquad (1.1.3.4)$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{6} \end{pmatrix} \qquad (1.1.3.5)$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \qquad (1.1.3.6)$$

Putting values in (1.1.3.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-7}{9} & \frac{1}{6} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \qquad (1.1.3.7)$$

In order to solve (1.1.3.7), perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \qquad (1.1.3.8)$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{130}{81} & \frac{-7}{54} \\ \frac{-7}{54} & \frac{37}{36} \end{pmatrix} \qquad (1.1.3.9)$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{-7}{9} \\ 0 & 1 & \frac{1}{6} \\ \frac{-7}{9} & \frac{1}{6} & \frac{205}{324} \end{pmatrix} \qquad (1.1.3.10)$$

From (1.1.3.6) putting (1.1.3.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \qquad (1.1.3.11)$$

$$\implies \mathbf{x} = \mathbf{VS_+U^T b} \qquad (1.1.3.12)$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left| \mathbf{MM}^T - \lambda\mathbf{I} \right| = 0$$

$$(1.1.3.13)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 1 \\ \frac{1}{2} & 1 & \frac{205}{324}-\lambda \end{pmatrix} = 0$$

$$(1.1.3.14)$$

$$\implies -\lambda^3 + \frac{853}{324}\lambda^2 - \frac{529}{324}\lambda = 0$$

$$(1.1.3.15)$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{529}{324} \qquad (1.1.3.16)$$

$$\lambda_2 = 1 \qquad (1.1.3.17)$$

$$\lambda_3 = 0 \qquad (1.1.3.18)$$

Hence the eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-252}{205} \\ \frac{54}{205} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{7}{9} \\ \frac{-1}{6} \\ 1 \end{pmatrix} \qquad (1.1.3.19)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-252}{23\sqrt{205}} \\ \frac{54}{23\sqrt{205}} \\ \frac{\sqrt{205}}{23} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{205}} \\ \frac{14}{\sqrt{205}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{14}{23} \\ \frac{-3}{23} \\ \frac{18}{23} \end{pmatrix}$$
$$(1.1.3.20)$$

Hence we obtain $\mathbf{U}$ of (1.1.3.8) as follows, $\mathbf{U}$

$$\begin{pmatrix} \frac{-252}{23\sqrt{205}} & \frac{3}{\sqrt{205}} & \frac{14}{23} \\ \frac{54}{23\sqrt{205}} & \frac{14}{\sqrt{205}} & -\frac{3}{23} \\ \frac{\sqrt{205}}{23} & 0 & \frac{18}{23} \end{pmatrix} \quad (1.1.3.21)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (1.1.3.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{23}{18} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.3.22)$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left| \mathbf{M}^T\mathbf{M} - \lambda\mathbf{I} \right| = 0 \quad (1.1.3.23)$$

$$\implies \begin{pmatrix} \frac{130}{81} - \lambda & \frac{-7}{54} \\ \frac{-7}{54} & \frac{37}{36} - \lambda \end{pmatrix} = 0 \quad (1.1.3.24)$$

$$\implies \lambda^2 - \frac{853}{324}\lambda + \frac{529}{324} = 0 \quad (1.1.3.25)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{529}{324} \quad (1.1.3.26)$$
$$\lambda_5 = 1 \quad (1.1.3.27)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-14}{3} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{14} \\ 1 \end{pmatrix} \quad (1.1.3.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-14}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{205}} \\ \frac{14}{\sqrt{205}} \end{pmatrix} \quad (1.1.3.29)$$

Hence we obtain $\mathbf{V}$ of (1.1.3.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{-14}{\sqrt{205}} & \frac{3}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} & \frac{14}{\sqrt{205}} \end{pmatrix} \quad (1.1.3.30)$$

From (1.1.3.8) we get the Singular Value

Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{-252}{23\sqrt{205}} & \frac{3}{\sqrt{205}} & \frac{14}{23} \\ \frac{54}{23\sqrt{205}} & \frac{14}{\sqrt{205}} & -\frac{3}{23} \\ \frac{\sqrt{205}}{23} & 0 & \frac{18}{23} \end{pmatrix} \begin{pmatrix} \frac{23}{18} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-14}{\sqrt{205}} & \frac{3}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} & \frac{14}{\sqrt{205}} \end{pmatrix}^T$$
$$(1.1.3.31)$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{18}{23} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.3.32)$$

From (1.1.3.11) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{1367}{23\sqrt{205}} \\ \frac{48}{\sqrt{205}} \\ -\frac{71}{23} \end{pmatrix} \quad (1.1.3.33)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{24606}{529\sqrt{205}} \\ \frac{48}{\sqrt{205}} \end{pmatrix} \quad (1.1.3.34)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2052}{529} \\ \frac{1374}{529} \end{pmatrix} \quad (1.1.3.35)$$

Verifying the solution of (1.1.3.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.3.36)$$

Evaluating the R.H.S in (1.1.3.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{53}{9} \\ \frac{13}{6} \end{pmatrix} \quad (1.1.3.37)$$

$$\implies \begin{pmatrix} \frac{130}{81} & \frac{-7}{54} \\ \frac{-7}{54} & \frac{37}{36} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{53}{9} \\ \frac{13}{6} \end{pmatrix} \quad (1.1.3.38)$$

Solving the augmented matrix of (1.1.3.38) we get,

$$\begin{pmatrix} \frac{130}{81} & \frac{-7}{54} & \frac{53}{9} \\ \frac{-7}{54} & \frac{37}{36} & \frac{13}{6} \end{pmatrix} \xleftrightarrow{R_1 = \frac{81}{130}R_1} \begin{pmatrix} 1 & \frac{-21}{260} & \frac{477}{130} \\ \frac{-7}{54} & \frac{37}{36} & \frac{13}{6} \end{pmatrix}$$
$$(1.1.3.39)$$

$$\xleftrightarrow{R_2 = R_2 + \frac{7}{54}R_1} \begin{pmatrix} 1 & \frac{-21}{260} & \frac{477}{130} \\ 0 & \frac{529}{520} & \frac{687}{260} \end{pmatrix}$$
$$(1.1.3.40)$$

$$\xleftrightarrow{R_2 = \frac{520}{529}R_2} \begin{pmatrix} 1 & \frac{-21}{260} & \frac{477}{130} \\ 0 & 1 & \frac{1374}{529} \end{pmatrix}$$
$$(1.1.3.41)$$

$$\xleftrightarrow{R_1 = R_1 + \frac{21}{260}R_2} \begin{pmatrix} 1 & 0 & \frac{2052}{529} \\ 0 & 1 & \frac{1374}{529} \end{pmatrix}$$
$$(1.1.3.42)$$

From equation (1.1.3.42), solution is given

by,

$$\mathbf{x} = \begin{pmatrix} \frac{2052}{529} \\ \frac{1374}{529} \end{pmatrix} \quad (1.1.3.43)$$

Comparing results of $\mathbf{x}$ from (1.1.3.35) and (1.1.3.43), we can say that the solution is verified.

b) Find the foot of the perpendicular to the plane

$$2x + 3y - 2z + 4 = 0 \quad (1.1.3.44)$$

from the point $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ using SVD. **Solution:**

The given plane equation is

$$\begin{pmatrix} 2 & 3 & -2 \end{pmatrix} \mathbf{x} = 0 \quad (1.1.3.45)$$

$$(1.1.3.46)$$

The equation of plane is

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.3.47)$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \quad (1.1.3.48)$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.3.49)$$

then, $\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.3.50)$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 0 \quad (1.1.3.51)$$

Let, a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (1.1.3.52)$$

Let, a=0 and b=1,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \quad (1.1.3.53)$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \quad (1.1.3.54)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad (1.1.3.55)$$

To solve (1.1.3.54) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \quad (1.1.3.56)$$

substituting the value of $\mathbf{M}$ from equation (1.1.3.56) to (1.1.3.54),

$$\implies \mathbf{USV}^T \mathbf{x} = \mathbf{b} \quad (1.1.3.57)$$
$$\implies \mathbf{x} = \mathbf{VS_+U}^T \mathbf{b} \quad (1.1.3.58)$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$.

Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} \end{pmatrix}$$

$$(1.1.3.59)$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left| \mathbf{M}^T\mathbf{M} - \lambda \mathbf{I} \right| = 0 \quad (1.1.3.60)$$

$$\implies \begin{pmatrix} 2 - \lambda & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} - \lambda \end{pmatrix} \quad (1.1.3.61)$$

$$\implies (\lambda - \frac{17}{4})(\lambda - 1) = 0 \quad (1.1.3.62)$$

$$\therefore \lambda_1 = \frac{17}{4}, \lambda_2 = 1, \quad (1.1.3.63)$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (1.1.3.64)$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad (1.1.3.65)$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \qquad (1.1.3.66)$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \qquad (1.1.3.67)$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \qquad (1.1.3.68)$$

$$\implies \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{13}{4} \end{pmatrix} \qquad (1.1.3.69)$$

Eigenvalues corresponding to $\mathbf{MM}^T$ is,

$$\left| \mathbf{MM}^T - \lambda \mathbf{I} \right| = 0 \qquad (1.1.3.70)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{13}{4}-\lambda \end{pmatrix} \qquad (1.1.3.71)$$

$$\implies \lambda(\lambda-1)(\lambda - \frac{17}{4}) = 0 \qquad (1.1.3.72)$$

$$\therefore \lambda_3 = \frac{17}{4}, \lambda_4 = 1, \lambda_5 = 0 \qquad (1.1.3.73)$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{4}{13} \\ \frac{6}{13} \\ 1 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} \frac{-3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_5} = \begin{pmatrix} -1 \\ \frac{-3}{2} \\ 1 \end{pmatrix} \qquad (1.1.3.74)$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \begin{pmatrix} \frac{4}{\sqrt{221}} \\ \frac{6}{\sqrt{221}} \\ \frac{13}{\sqrt{221}} \end{pmatrix} \qquad (1.1.3.75)$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \qquad (1.1.3.76)$$

$$\mathbf{v_5} = \begin{pmatrix} \frac{-2}{\sqrt{17}} \\ \frac{-3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix} \qquad (1.1.3.77)$$

$$\implies \mathbf{U} = \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix} \qquad (1.1.3.78)$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\frac{17}{4}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (1.1.3.79)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S_+} = \begin{pmatrix} \frac{2}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.1.3.80)$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{17}{4}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}^T$$

$$(1.1.3.81)$$

Now substituting the values of (1.1.3.67), (1.1.3.80), (1.1.3.78) and (1.1.3.55) in (1.1.3.58),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{4}{\sqrt{221}} & \frac{-3}{\sqrt{13}} & \frac{-2}{\sqrt{17}} \\ \frac{6}{\sqrt{221}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{17}} \\ \frac{13}{\sqrt{221}} & 0 & \frac{2}{\sqrt{17}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \qquad (1.1.3.82)$$

$$\implies \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \qquad (1.1.3.83)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \quad (1.1.3.84)$$

$$\implies \mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \quad (1.1.3.85)$$

$$\mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix}$$
$$(1.1.3.86)$$

$$\implies \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
$$(1.1.3.87)$$

$\therefore$ from equation (1.1.3.58),

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.1.3.88)$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.3.89)$$

$$\implies \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$
$$(1.1.3.90)$$

$$\implies \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
$$(1.1.3.91)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 2 & \frac{3}{2} & 3 \\ \frac{3}{2} & \frac{13}{4} & -2 \end{pmatrix} \xleftarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{4} & -2 \end{pmatrix}$$
$$(1.1.3.92)$$

$$\xleftarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ 0 & \frac{17}{8} & -\frac{17}{4} \end{pmatrix}$$
$$(1.1.3.93)$$

$$\xleftarrow{R_2 \leftarrow \frac{8}{17}R_2} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{2} \\ 0 & 1 & -2 \end{pmatrix}$$
$$(1.1.3.94)$$

$$\xleftarrow{R_1 \leftarrow R_1 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$$
$$(1.1.3.95)$$

$$\implies \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
$$(1.1.3.96)$$

Hence from equations (1.1.3.88) and (1.1.3.96) we conclude that the solution is verified.

a) Determine the distance from the Y-axis to the plane $5x - 2z - 3 = 0$
Solution: Equation of plane can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \quad (1.1.3.97)$$

Rewriting given equation of plane in (1.1.3.97) form

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \quad (1.1.3.98)$$

where : $\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 3$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \quad (1.1.3.99)$$

Here p is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Find two vectors that are $\perp$ to $\mathbf{n}$

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (1.1.3.100)$$

Put $a = 0$ and $b = 1$ in (1.1.3.99), $\implies c = 0$
Put $a = 1$ and $b = 0$ in (1.1.3.99), $\implies c = \frac{5}{2}$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, y = 0$ in (1.1.3.98), we get $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Since given plane is parallel to y-axis, we can use any point P on y-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.1.3.101)$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (1.1.3.100) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (1.1.3.102)$$

Equation $\mathbf{P}$ and $\mathbf{Q}$, and computing pseudo inverse using SVD should give the value of λ_1 and λ_2 (since plane and y-axis never intersect pseudo inverse should give the points which are closest)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (1.1.3.103)$$

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \qquad (1.1.3.104)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \qquad (1.1.3.105)$$

$$\mathbf{Mx} = \mathbf{b} \qquad (1.1.3.106)$$

$$\mathbf{x} = \mathbf{M^+ b} \qquad (1.1.3.107)$$

where $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

Diagonalize $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \\ \frac{5}{2} & 0 & \frac{25}{4} \end{pmatrix}$$
$$(1.1.3.108)$$

$$= \begin{pmatrix} 0 & \frac{2}{5} & -\frac{5}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{29}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{5} & 0 & 1 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix}$$
$$(1.1.3.109)$$

$$= \mathbf{U}\Sigma^T\Sigma\mathbf{U}^T \qquad (1.1.3.110)$$

Verify (1.1.3.109) from,

```
codes/diagonalize1.py
```

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{29}{4} & 0 \\ 0 & 1 \end{pmatrix}$$
$$(1.1.3.111)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{29}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (1.1.3.112)$$

$$= \mathbf{V}\Sigma^T\Sigma\mathbf{V}^T \qquad (1.1.3.113)$$

Verify (1.1.3.112) from,

```
codes/diagonalize2.py
```

Compute SVD of $\mathbf{M}$ from (1.1.3.109) and (1.1.3.114),

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T \qquad (1.1.3.114)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{5} & -\frac{5}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$(1.1.3.115)$$

$$\mathbf{M}^+ = \mathbf{V}\Sigma^T\mathbf{U}^T \qquad (1.1.3.116)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{5} & 0 & 1 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix}$$
$$(1.1.3.117)$$

$$= \begin{pmatrix} \frac{4}{29} & 0 & \frac{10}{29} \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.1.3.118)$$

Verify (1.1.3.118) from,

```
codes/pseudo_inverse.py
```

Substitute (1.1.3.118) in (1.1.3.107),

$$\mathbf{x} = \begin{pmatrix} \frac{4}{29} & 0 & \frac{10}{29} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{14}{29} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$
$$(1.1.3.119)$$

Substituting λ_1, λ_2 in (1.1.3.102)

$$\mathbf{Q} = \begin{pmatrix} \frac{15}{29} \\ 0 \\ -\frac{6}{29} \end{pmatrix} \qquad (1.1.3.120)$$

Distance between point $\mathbf{P}$ and $\mathbf{Q}$ is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(\frac{15}{29}\right)^2 + 0 + \left(-\frac{6}{29}\right)^2} = \frac{3}{\sqrt{29}}$$
$$(1.1.3.121)$$

Hence, distance from y-axis to $5x - 2z - 3 = 0$ is $\frac{3}{\sqrt{29}}$.

Verifying solution to (1.1.3.106) by least squares method

$$\mathbf{M}^T(\mathbf{b} - \mathbf{Mx}) = 0 \qquad (1.1.3.122)$$

$$\implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \qquad (1.1.3.123)$$

Substituting $\mathbf{M}, \mathbf{b}$ from (1.1.3.105) in

(1.1.3.123)

$$\begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

(1.1.3.124)

$$\begin{pmatrix} \frac{29}{4} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ 0 \end{pmatrix} \qquad (1.1.3.125)$$

$$\implies \frac{29}{4}\lambda_1 = -\frac{7}{2} \qquad (1.1.3.126)$$

$$\lambda_1 = -\frac{7}{2} \times \frac{4}{29} = -\frac{14}{29}$$

(1.1.3.127)

$$\text{and } \lambda_2 = 0 \qquad (1.1.3.128)$$

$$\mathbf{x} = \begin{pmatrix} -\frac{14}{29} \\ 0 \end{pmatrix} \qquad (1.1.3.129)$$

Comparing (1.1.3.119) and (1.1.3.129) solution is verified.

b) Determine the distance from the Z-axis to the plane 5x - 12y - 8 = 0

Solution: Equation of plane can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \qquad (1.1.3.130)$$

Rewriting given equation of plane in (1.1.3.130) form

$$\begin{pmatrix} 5 & -12 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8 \qquad (1.1.3.131)$$

where the value of

$$\mathbf{n} = \begin{pmatrix} 5 \\ -12 \\ 0 \end{pmatrix} \qquad (1.1.3.132)$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (1.1.3.133)$$

$$c = 8 \qquad (1.1.3.134)$$

We need to represent the equation of plane in parametric form,

$$\mathbf{Q} = \mathbf{p} + \lambda_1 \mathbf{q} + \lambda_2 \mathbf{r} \qquad (1.1.3.135)$$

Here p is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Now, we need to find these two vectors $\mathbf{q}$

and $\mathbf{r}$ which are $\perp$ to $\mathbf{n}$

$$\begin{pmatrix} 5 & -12 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \implies 5a - 12b = 0$$

(1.1.3.136)

Put $a = 0$ and $c = 1$ in (1.1.3.136), $\implies b = 0$

Put $a = 1$ and $c = 0$ in (1.1.3.136), $\implies b = \frac{5}{12}$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, z = 0$ in (1.1.3.131), we get $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Since given plane is parallel to Z-axis, we can use any point P on Z-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (1.1.3.137)$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (1.1.3.136) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (1.1.3.138)$$

Computation of Pseudo Inverse using SVD in order to determine the value of λ_1 and λ_2 :

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (1.1.3.139)$$

$$\lambda_1 \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \qquad (1.1.3.140)$$

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \qquad (1.1.3.141)$$

$$\mathbf{Mx} = \mathbf{b} \qquad (1.1.3.142)$$

$$\implies \mathbf{x} = \mathbf{M}^+\mathbf{b} \qquad (1.1.3.143)$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.3.144)$$

$$\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (1.1.3.145)$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad (1.1.3.146)$$

Applying Singular Value Decomposition on **M**,

$$\mathbf{M} = \mathbf{USV}^T \quad (1.1.3.147)$$

Where the columns of **V** are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of **U** are the eigenvectors of $\mathbf{MM}^T$ and **S** is diagonal matrix of Singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.3.148)$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.1.3.149)$$

As we know that,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b}$$
$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \quad (1.1.3.150)$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of **S**. Calculating eigenvalues of $\mathbf{MM}^T$,

$$\left| \mathbf{MM}^T - \lambda\mathbf{I} \right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144}-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^3 - \frac{313}{144}\lambda^2 + \frac{169}{144}\lambda = 0$$

Hence eigenvalues of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{169}{144}; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \quad (1.1.3.151)$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} -\frac{5}{12} \\ 1 \\ 0 \end{pmatrix}$$
$$(1.1.3.152)$$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.3.153)$$

Using values from (1.1.3.151),

$$\mathbf{S} = \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.3.154)$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left| \mathbf{M}^T\mathbf{M} - \lambda\mathbf{I} \right| = 0$$
$$\implies \begin{vmatrix} \frac{169}{144}-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - \frac{313}{144}\lambda + \frac{169}{144} = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{169}{144}; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.3.155)$$

From (1.1.3.155) we obtain **V** as,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.3.156)$$

Now, we can compute *SVD* of **M** :

$$\mathbf{M} = \mathbf{USV}^T \quad (1.1.3.157)$$

$$= \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$(1.1.3.158)$$

$$\mathbf{M}^+ = \mathbf{VS}^T\mathbf{U}^T \quad (1.1.3.159)$$

$$= \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.1.3.160)$$

Substitute (1.1.3.160) in (1.1.3.143),

$$\mathbf{x} = \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$
$$(1.1.3.161)$$

$$\mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \quad (1.1.3.162)$$

$$\implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \quad (1.1.3.163)$$

Substituting λ_1, λ_2 in (1.1.3.138)

$$\mathbf{Q} = \begin{pmatrix} -\frac{204}{169} \\ -\frac{85}{169} \\ 0 \end{pmatrix} \qquad (1.1.3.164)$$

Distance between point $\mathbf{P}$ and $\mathbf{Q}$ is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(-\frac{204}{169}\right)^2 + \left(-\frac{85}{169}\right)^2 + 0} \qquad (1.1.3.165)$$

$$\|\mathbf{P} - \mathbf{Q}\| = \frac{17}{13} \qquad (1.1.3.166)$$

Hence, the distance from the Z-axis to the plane $5x - 12y - 8 = 0$ is $\frac{17}{13}$. Now, we can verify the solution using Least Squares Method,

$$\mathbf{M}^T(\mathbf{b} - \mathbf{Mx}) = 0 \qquad (1.1.3.167)$$

$$\implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \qquad (1.1.3.168)$$

Substituting $\mathbf{M}, \mathbf{b}$ from (1.1.3.141) in (1.1.3.168)

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \qquad (1.1.3.169)$$

$$\begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{17}{12} \\ 0 \end{pmatrix} \qquad (1.1.3.170)$$

$$\implies \frac{169}{144}\lambda_1 = -\frac{17}{12} \qquad (1.1.3.171)$$

$$\lambda_1 = -\frac{17}{12} \times \frac{144}{169} = -\frac{204}{169} \qquad (1.1.3.172)$$

$$\text{and } \lambda_2 = 0 \qquad (1.1.3.173)$$

$$\implies \mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \qquad (1.1.3.174)$$

Comparing (1.1.3.161) and (1.1.3.174) solution is verified.

1.1.4. Find the foot of the perpendicular using svd drawn from $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ to the plane

$$\begin{pmatrix} 2 & -1 & -2 \end{pmatrix}\mathbf{x} + 4 = 0 \qquad (1.1.4.1)$$

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let,

$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0 \qquad (1.1.4.2)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0 \qquad (1.1.4.3)$$

$$\implies 2a - b - 2c = 0 \qquad (1.1.4.4)$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad (1.1.4.5)$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \qquad (1.1.4.6)$$

Let us solve the equation,

$$\mathbf{Mx} = \mathbf{b} \qquad (1.1.4.7)$$

Substituting (1.1.4.5) and (1.1.4.6) in (1.1.4.7),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \qquad (1.1.4.8)$$

To solve (1.1.4.8), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \qquad (1.1.4.9)$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \qquad (1.1.4.10)$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \qquad (1.1.4.11)$$

Substituting (1.1.4.9) in (1.1.4.7),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \qquad (1.1.4.12)$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} \qquad (1.1.4.13)$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.

Let us calculate eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.4.14)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \quad (1.1.4.15)$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \quad (1.1.4.16)$$

From equation (1.1.4.16) eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.4.17)$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ -1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$$
$$(1.1.4.18)$$

Normalizing the eigen vectors in equation (1.1.4.18)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$
$$(1.1.4.19)$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \quad (1.1.4.20)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.4.21)$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (1.1.4.22)$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \quad (1.1.4.23)$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (1.1.4.24)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad (1.1.4.25)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \quad (1.1.4.26)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.1.4.27)$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.1.4.28)$$

From (1.1.4.7), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}^T$$
$$(1.1.4.29)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.4.30)$$

From (1.1.4.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{11}{3} \end{pmatrix} \quad (1.1.4.31)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{8}{9\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (1.1.4.32)$$

$$\mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \quad (1.1.4.33)$$

Verifying the solution of (1.1.4.33) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.4.34)$$

Evaluating the R.H.S in (1.1.4.34) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (1.1.4.35)$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (1.1.4.36)$$

Solving the augmented matrix of (1.1.4.36) we

get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix}$$
(1.1.4.37)

$$\xleftrightarrow{R_2 = R_2 + \frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{9}{8} & -\frac{1}{4} \end{pmatrix}$$
(1.1.4.38)

$$\xleftrightarrow{R_2 = \frac{8}{9} R_2} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix}$$
(1.1.4.39)

$$\xleftrightarrow{R_1 = R_1 + \frac{R_2}{4}} \begin{pmatrix} 1 & 0 & -\frac{5}{9} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix}$$
(1.1.4.40)

From equation (1.1.4.40), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix}$$
(1.1.4.41)

Comparing results of $\mathbf{x}$ from (1.1.4.33) and (1.1.4.41), we can say that the solution is verified.

1.1.5. Find the foot of the perpendicular to the given plane

$$2x + 3y - 4z + 5 = 0$$
(1.1.5.1)

from
a)

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$
(1.1.5.2)

Solution: The given equation of plane can be represented as

$$\begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \mathbf{x} = -5$$
(1.1.5.3)

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$
(1.1.5.4)

We need to find two vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$
(1.1.5.5)

Put $a = 1$ and $b = 0$ in (1.1.5.5),we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$
(1.1.5.6)

Put $a = 0$ and $b = 1$ in (1.1.5.5),we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix}$$
(1.1.5.7)

Now, solving the equation

$$\mathbf{Mx} = \mathbf{b}$$
(1.1.5.8)

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$
(1.1.5.9)

$$\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$
(1.1.5.10)

Now, to solve equation (1.1.5.8), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T$$
(1.1.5.11)

Substituting the value of $\mathbf{M}$ from equation (1.1.5.11) to(1.1.5.8),

$$\mathbf{USV}^T \mathbf{x} = \mathbf{b}$$
(1.1.5.12)

$$\implies \mathbf{x} = \mathbf{VS_+ U}^T \mathbf{b}$$
(1.1.5.13)

Where, $\mathbf{S_+}$ is the Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T \mathbf{M}$, columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T \mathbf{M}$.

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix}$$
(1.1.5.14)

Eigen values corresponding to $\mathbf{M}^T \mathbf{M}$ are given by,

$$\left| \mathbf{M}^T \mathbf{M} - \lambda \mathbf{I} \right| = 0$$
(1.1.5.15)

$$\implies \left| \begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{pmatrix} \right| = 0$$
(1.1.5.16)

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{29}{16} = 0$$
(1.1.5.17)

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{29}{16} \qquad (1.1.5.18)$$

$$\lambda_2 = 1 \qquad (1.1.5.19)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \qquad (1.1.5.20)$$

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \qquad (1.1.5.21)$$

Normalizing the eigen vectors, we obtain $\mathbf{V}$ of (1.1.5.11) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \qquad (1.1.5.22)$$

$\mathbf{S}$ of the diagonal matrix of (1.1.5.11) is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (1.1.5.23)$$

Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \qquad (1.1.5.24)$$

Eigen values corresponding to $\mathbf{M}\mathbf{M}^T$ are given by

$$\left| \mathbf{M}\mathbf{M}^T - \lambda\mathbf{I} \right| = 0$$

$$(1.1.5.25)$$

$$\implies \left\| \begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{pmatrix} \right\| = 0$$

$$(1.1.5.26)$$

$$\implies \lambda^3 - \frac{45}{16}\lambda^2 + \frac{29}{16}\lambda = 0$$

$$(1.1.5.27)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_3 = \frac{29}{16} \qquad (1.1.5.28)$$

$$\lambda_4 = 1 \qquad (1.1.5.29)$$

$$\lambda_5 = 0 \qquad (1.1.5.30)$$

Hence we obtain $\mathbf{U}$ of (1.1.5.11) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{29} \\ \sqrt{\frac{13}{29}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \qquad (1.1.5.31)$$

Finally from (1.1.5.11) we get the Singular Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{29} \\ \sqrt{\frac{13}{29}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T$$

$$(1.1.5.32)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.1.5.33)$$

Substituting the values of (1.1.5.31),(1.1.5.22),(1.1.5.33) in (1.1.5.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \qquad (1.1.5.34)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \qquad (1.1.5.35)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad (1.1.5.36)$$

Verifying the solution of (1.1.5.36) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \qquad (1.1.5.37)$$

Evaluating the R.H.S in (1.1.5.37) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad (1.1.5.38)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad (1.1.5.39)$$

The augmented matrix of (1.1.5.39) is,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & 3 \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \qquad (1.1.5.40)$$

Solving the augmented matrix into Row re-

duced echelon form of (1.1.5.40) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & 3 \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix}$$

(1.1.5.41)

$$\xrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ 0 & \frac{29}{20} & -\frac{29}{10} \end{pmatrix}$$

(1.1.5.42)

$$\xrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ 0 & 1 & -2 \end{pmatrix}$$

(1.1.5.43)

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$$

(1.1.5.44)

Therefore,

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.5.45}$$

Comparing results of $\mathbf{x}$ from (1.1.5.36) and (1.1.5.45) we conclude that the solution is verified.

b)

$$\mathbf{c} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \tag{1.1.5.46}$$

Solution:
The given equation of plane can be represented as

$$\begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \mathbf{x} = -5 \tag{1.1.5.47}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \tag{1.1.5.48}$$

We need to find two vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{1.1.5.49}$$

Put $a = 1$ and $b = 0$ in (1.1.5.49),we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.5.50}$$

Put $a = 0$ and $b = 1$ in (1.1.5.49),we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \tag{1.1.5.51}$$

Now, solving the equation

$$\mathbf{Mx} = \mathbf{c} \tag{1.1.5.52}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{1.1.5.53}$$

$$\mathbf{c} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \tag{1.1.5.54}$$

Now, to solve equation (1.1.5.52), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.5.55}$$

Substituting the value of $\mathbf{M}$ from equation (1.1.5.55) to(1.1.5.52),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{c} \tag{1.1.5.56}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{c} \tag{1.1.5.57}$$

Where, $\mathbf{S_+}$ is the Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \tag{1.1.5.58}$$

Eigen values corresponding to $\mathbf{M}^T\mathbf{M}$ are given by,

$$\left| \mathbf{M}^T\mathbf{M} - \lambda\mathbf{I} \right| = 0 \tag{1.1.5.59}$$

$$\implies \left| \begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{pmatrix} \right| = 0 \tag{1.1.5.60}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{29}{16} = 0 \tag{1.1.5.61}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{29}{16} \tag{1.1.5.62}$$

$$\lambda_2 = 1 \tag{1.1.5.63}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \tag{1.1.5.64}$$

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.5.65}$$

Normalizing the eigen vectors, we obtain $\mathbf{V}$ of (1.1.5.55) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.5.66}$$

$\mathbf{S}$ of the diagonal matrix of (1.1.5.55) is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.5.67}$$

Now, calculating eigen value of $\mathbf{MM}^T$,

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.5.68}$$

Eigen values corresponding to $\mathbf{MM}^T$ are given by

$$\left| \mathbf{MM}^T - \lambda\mathbf{I} \right| = 0$$
$$\tag{1.1.5.69}$$

$$\implies \left\| \begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{pmatrix} \right\| = 0$$
$$\tag{1.1.5.70}$$

$$\implies \lambda^3 - \frac{45}{16}\lambda^2 + \frac{29}{16}\lambda = 0$$
$$\tag{1.1.5.71}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_3 = \frac{29}{16} \tag{1.1.5.72}$$

$$\lambda_4 = 1 \tag{1.1.5.73}$$

$$\lambda_5 = 0 \tag{1.1.5.74}$$

Hence we obtain $\mathbf{U}$ of (1.1.5.55) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.5.75}$$

Finally from (1.1.5.55) we get the Singular Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T$$
$$\tag{1.1.5.76}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.5.77}$$

Substituting the values of (1.1.5.75),(1.1.5.66),(1.1.5.77) in (1.1.5.57) we get,

$$\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{125}{\sqrt{377}} \\ \frac{-3}{\sqrt{13}} \\ \frac{5}{\sqrt{29}} \end{pmatrix} \tag{1.1.5.78}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{500}{29\sqrt{13}} \\ -\frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.5.79}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{97}{29} \\ \frac{102}{29} \end{pmatrix} \tag{1.1.5.80}$$

Verifying the solution of (1.1.5.80) using,

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{c} \tag{1.1.5.81}$$

Evaluating the R.H.S in (1.1.5.81) we get,

$$\mathbf{M}^T\mathbf{c} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ \frac{27}{4} \end{pmatrix} \tag{1.1.5.82}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{11}{2} \\ \frac{27}{4} \end{pmatrix} \tag{1.1.5.83}$$

The augmented matrix of (1.1.5.83) is,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & \frac{11}{2} \\ \frac{3}{8} & \frac{25}{16} & \frac{27}{4} \end{pmatrix} \tag{1.1.5.84}$$

Solving the augmented matrix into Row re-

duced echelon form of (1.1.5.84) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & \frac{11}{2} \\ \frac{4}{3} & \frac{25}{8} & \frac{27}{16} \\ \frac{3}{8} & \frac{16}{16} & \frac{4}{4} \end{pmatrix} \xleftarrow{R_1 \leftarrow \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{27} \\ \frac{3}{3} & \frac{25}{16} & \frac{27}{4} \\ \frac{3}{8} & 16 & 4 \end{pmatrix} \quad (1.1.5.85)$$

$$\xleftarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ 0 & \frac{29}{20} & \frac{51}{10} \end{pmatrix} \quad (1.1.5.86)$$

$$\xleftarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ 0 & 1 & \frac{102}{29} \end{pmatrix} \quad (1.1.5.87)$$

$$\xleftarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & \frac{97}{29} \\ 0 & 1 & \frac{102}{29} \end{pmatrix} \quad (1.1.5.88)$$

Therefore,

$$\mathbf{x} = \begin{pmatrix} \frac{97}{29} \\ \frac{102}{29} \end{pmatrix} \quad (1.1.5.89)$$

Comparing results of **x** from (1.1.5.80) and (1.1.5.89), Hence, the solution is verified.

1.2 Two planes

1.2.1. Write the equation of line through $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and

perpendicular to the plane

$$2x - y + 2z - 5 = 0 \quad (1.2.1.1)$$

Determine the co-ordinates of the point in which the plane is met by this line.

Solution:
The general equation of a plane is given by

$$px + qy + rz = c \quad (1.2.1.2)$$

and can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \quad (1.2.1.3)$$

where

$$\mathbf{n} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad (1.2.1.4)$$

The equation of a line passing through the point a and having direction vector **b** is given by

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} \quad (1.2.1.5)$$

Rewriting given equation of plane in (1.2.1.3)

form, we get

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \quad (1.2.1.6)$$

where : $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 5$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \quad (1.2.1.7)$$

Here **p** is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to **n**. Find two vectors that are $\perp$ to **n**

$$\implies \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (1.2.1.8)$$

Put $a = 1$ and $b = 2$ in (1.2.1.8), $\implies c = 0$
Put $a = 0$ and $b = 2$ in (1.2.1.8), $\implies c = 1$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

Let us find point **p** on the plane. Put $x = 1, y = -1$ in (1.2.1.6), we get $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Since given plane is perpendicular to given line, if we take $\mathbf{P} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ it will give us the shortest distance to the plane.

Let **Q** be the point on plane with shortest distance to **P**. **Q** can be expressed in (1.2.1.7) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad (1.2.1.9)$$

Equate **P** and **Q**, and compute λ_1, λ_2 using

pseudoinverse from *SVD*

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \quad (1.2.1.10)$$

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \quad (1.2.1.11)$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \quad (1.2.1.12)$$

$$\mathbf{Mx} = \mathbf{b} \quad (1.2.1.13)$$

$$\implies \mathbf{x} = \mathbf{M^+ b} \quad (1.2.1.14)$$

where $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$

Diagonalize $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$
$$(1.2.1.15)$$

We get three eigen values as 9, 1 and 0. Normalizing the eigen vectors, $\mathbf{U}$ is calculated as :

$$\mathbf{U} = \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{2}{3} \\ \frac{4}{3\sqrt{2}} & 0 & \frac{-1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{pmatrix} \quad (1.2.1.16)$$

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$
$$(1.2.1.17)$$

We get two eigen values as 9, 1. Normalizing the eigen vectors, $\mathbf{V}$ is calculated as :

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (1.2.1.18)$$

Taking square root of eigen values, We get $\mathbf{\Sigma}$ as :

$$\mathbf{\Sigma} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.2.1.19)$$

Thus, we performed Singular Value Decomposition for $\mathbf{M}$. It is easy to check that $\mathbf{U\Sigma V}^T =$

$\mathbf{M}$.

For calculating the psuedoinverse, we know

$$\mathbf{M^+} = \mathbf{V\Sigma}^{-1}\mathbf{U}^T \quad (1.2.1.20)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}$$
$$(1.2.1.21)$$

$$= \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{-4}{9} \\ \frac{-4}{9} & \frac{2}{9} & \frac{5}{9} \end{pmatrix} \quad (1.2.1.22)$$

Substitute (1.2.1.22) in (1.2.1.14),

$$\mathbf{x} = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{-4}{9} \\ \frac{-4}{9} & \frac{2}{9} & \frac{5}{9} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{28}{9} \\ \frac{-8}{9} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$
$$(1.2.1.23)$$

Substituting λ_1, λ_2 in (1.2.1.9)

$$\mathbf{Q} = \frac{1}{9}\begin{pmatrix} 37 \\ 31 \\ 1 \end{pmatrix} \quad (1.2.1.24)$$

Hence the equation of required line using (1.2.1.5) is:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad (1.2.1.25)$$
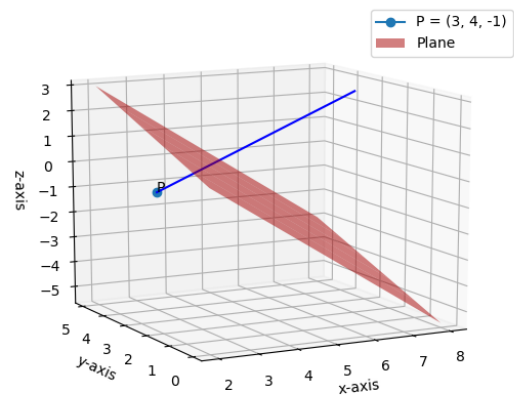

Fig. 1.2.1.1: Line passing through $\begin{pmatrix} 3 & 4 & -1 \end{pmatrix}$ and perpendicular to the plane $2x - y + 2z = 5$.

Verifying solution to (1.2.1.13) with *least squares* method

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \tag{1.2.1.26}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.2.1.27}$$

Substituting $\mathbf{M}, \mathbf{b}$ from (1.2.1.12) in (1.2.1.27)

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$$
$$\tag{1.2.1.28}$$

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix} \tag{1.2.1.29}$$

$$\implies 5\lambda_1 + 4\lambda_2 = 12 \tag{1.2.1.30}$$

$$4\lambda_1 + 5\lambda_2 = 8 \tag{1.2.1.31}$$

$$\lambda_1 = \frac{28}{9} \tag{1.2.1.32}$$

$$\text{and } \lambda_2 = \frac{-8}{9} \tag{1.2.1.33}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{28}{9} \\ \frac{-8}{9} \end{pmatrix} \tag{1.2.1.34}$$

Comparing (1.2.1.23) and (1.2.1.34) solution is verified.

1.2.2. Set up the equation of a plane through the point A $(-2, -3, 4)$ and perpendicular to the line

$$\frac{x}{4} = \frac{y - 3}{6} = \frac{z + 2}{-12} \tag{1.2.2.1}$$

Solution: Let the equation of plane is

$$ax + by + cz + d = 0 \tag{1.2.2.2}$$

Direction ratio of the line (1.2.2.1) is given as

$$\mathbf{D} = \begin{pmatrix} 4 \\ 6 \\ -12 \end{pmatrix} \tag{1.2.2.3}$$

Now let consider

$$\mathbf{A} = \begin{pmatrix} -2 & -3 & 4 \end{pmatrix} \mathbf{A}\mathbf{D} + d = 0 \tag{1.2.2.4}$$

$$\implies d = 37 \tag{1.2.2.5}$$

Since plane is passing through the point A (-2, -3, 4) and perpendicular to the line (1.2.2.1). Hence equation of the plane is

$$2x + 3y - 6z + 37 = 0 \tag{1.2.2.6}$$

$$\implies 2x + 3y - 6z = -37 \tag{1.2.2.7}$$

equation (1.2.2.7) can written as :

$$\begin{pmatrix} 2 & 3 & -6 \end{pmatrix}\mathbf{x} = -37 \tag{1.2.2.8}$$

$$\tag{1.2.2.9}$$

For foot perpendicular we need to find the distance between the plane and point P $(0, 3, -2)$.

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0 \tag{1.2.2.10}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = 0 \tag{1.2.2.11}$$

$$\implies 2a + 3b - 6c = 0 \tag{1.2.2.12}$$

$$\tag{1.2.2.13}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{3} \end{pmatrix} \tag{1.2.2.14}$$

$$\tag{1.2.2.15}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \tag{1.2.2.16}$$

Now we solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.2.2.17}$$

$$\tag{1.2.2.18}$$

Putting values in (1.2.2.17),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \tag{1.2.2.19}$$

Now, to solve (1.2.2.19), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.2.2.20}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of

singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{9} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{4} \end{pmatrix} \qquad (1.2.2.21)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{36} \end{pmatrix} \qquad (1.2.2.22)$$

From (1.2.2.17) putting (1.2.2.20) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \qquad (1.2.2.23)$$
$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \qquad (1.2.2.24)$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$
$$(1.2.2.25)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{1}{3} \\ 0 & 1-\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{36}-\lambda \end{pmatrix} = 0$$
$$(1.2.2.26)$$

$$\implies \lambda(\lambda-1)(\lambda-\frac{49}{36}) = 0$$
$$(1.2.2.27)$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \qquad (1.2.2.28)$$
$$\lambda_2 = 1 \qquad (1.2.2.29)$$
$$\lambda_3 = 0 \qquad (1.2.2.30)$$
$$(1.2.2.31)$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{12}{13} \\ \frac{18}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{2} \\ 1 \end{pmatrix}$$
$$(1.2.2.32)$$

Normalizing the eigen vectors we get,

$$\mathbf{u_1} = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{18}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-3}{2} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-2}{7} \\ \frac{-3}{7} \\ \frac{6}{7} \end{pmatrix}$$
$$(1.2.2.33)$$

Hence we obtain $\mathbf{U}$ of (1.2.2.20) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{12}{7\sqrt{13}} & \frac{-3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{\sqrt{13}}{7} & 0 & \frac{6}{7} \end{pmatrix} \qquad (1.2.2.34)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (1.2.2.20) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (1.2.2.35)$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \qquad (1.2.2.36)$$
$$\implies \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9}-\lambda \end{vmatrix} = 0 \qquad (1.2.2.37)$$
$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \qquad (1.2.2.38)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \qquad (1.2.2.39)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \qquad (1.2.2.40)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \qquad (1.2.2.41)$$

Hence we obtain $\mathbf{V}$ of (1.2.2.20) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \qquad (1.2.2.42)$$

Finally from (1.2.2.20) we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{12}{7\sqrt{13}} & \frac{-3}{\sqrt{13}} & \frac{-2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & \frac{-3}{7} \\ \frac{\sqrt{13}}{7} & 0 & \frac{6}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T$$
$$(1.2.2.43)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.2.2.44)$$

From (1.2.2.24) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{\sqrt{13}} \\ \frac{6}{\sqrt{13}} \\ -3 \end{pmatrix} \qquad (1.2.2.45)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{24}{7\sqrt{13}} \\ \frac{6}{\sqrt{13}} \end{pmatrix} \qquad (1.2.2.46)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad = \begin{pmatrix} \frac{-6}{7} \\ \frac{12}{7} \end{pmatrix} \qquad (1.2.2.47)$$

Verifying the solution of (1.2.2.47) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \qquad (1.2.2.48)$$

Evaluating the R.H.S in (1.2.2.48) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{-2}{3} \\ 2 \end{pmatrix} \qquad (1.2.2.49)$$

$$\implies \begin{pmatrix} \frac{10}{9} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-2}{3} \\ 2 \end{pmatrix} \qquad (1.2.2.50)$$

Solving the augmented matrix of (1.2.2.50) we get,

$$\begin{pmatrix} \frac{10}{9} & \frac{1}{6} & \frac{-2}{3} \\ \frac{1}{6} & \frac{5}{4} & 2 \end{pmatrix} \xrightarrow{R_1 = \frac{9R_1}{10}} \begin{pmatrix} 1 & \frac{3}{20} & \frac{-3}{5} \\ \frac{1}{6} & \frac{5}{4} & 2 \end{pmatrix} \qquad (1.2.2.51)$$

$$\xrightarrow{R_2 = R_2 - \frac{R_1}{6}} \begin{pmatrix} 1 & \frac{3}{20} & \frac{-3}{5} \\ 0 & \frac{49}{40} & \frac{21}{10} \end{pmatrix} \qquad (1.2.2.52)$$

$$\xrightarrow{R_2 = \frac{40}{49}R_2} \begin{pmatrix} 1 & \frac{3}{20} & \frac{-3}{5} \\ 0 & 1 & \frac{12}{7} \end{pmatrix} \qquad (1.2.2.53)$$

$$\xrightarrow{R_1 = R_1 - \frac{3R_2}{20}} \begin{pmatrix} 1 & 0 & \frac{-6}{7} \\ 0 & 1 & \frac{12}{7} \end{pmatrix} \qquad (1.2.2.54)$$

Hence, Solution of (1.2.2.48) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-6}{7} \\ \frac{12}{7} \end{pmatrix} \qquad (1.2.2.55)$$

Comparing results of $\mathbf{x}$ from (1.2.2.47) and (1.2.2.55) we conclude that the solution is verified.

1.3 The Pencil of Planes. The Bundle of Planes

1.3.1. Write the equation of a plane through the point A $(-3, 4, -1)$ and perpendicular to the line

$$\frac{x+2}{-3} = \frac{y-2}{1} = \frac{z-4}{2} \qquad (1.3.1.1)$$

Solution:
Let the equation of plane is

$$ax + by + cz + d = 0 \qquad (1.3.1.2)$$

Direction ratio of the line (1.3.1.1) is given as

$$\mathbf{D} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \qquad (1.3.1.3)$$

Now let consider

$$\mathbf{A} = \begin{pmatrix} -3 & 4 & -1 \end{pmatrix} \qquad (1.3.1.4)$$

Since plane is passing through the point A $(-3, 4, -1)$ and perpendicular to the line (1.3.1.1), hence

$$\mathbf{A}\mathbf{D} + d = 0 \qquad (1.3.1.5)$$

$$\implies d = -11 \qquad (1.3.1.6)$$

Hence equation of the plane is

$$-3x + y + 2z - 11 = 0 \qquad (1.3.1.7)$$

$$\implies -3x + y + 2z = 11 \qquad (1.3.1.8)$$

equation (1.3.1.8) can written as :

$$\begin{pmatrix} -3 & 1 & 2 \end{pmatrix}\mathbf{x} = 11 \qquad (1.3.1.9)$$

For foot perpendicular we need to find the distance between the plane and point P $(-2, 2, 4)$. First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0 \qquad (1.3.1.10)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 0 \qquad (1.3.1.11)$$

$$\implies -3a + b + 2c = 0 \qquad (1.3.1.12)$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} \qquad (1.3.1.13)$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \qquad (1.3.1.14)$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \qquad (1.3.1.15)$$

Putting values in (1.3.1.15),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \qquad (1.3.1.16)$$

Now, to solve (1.3.1.16), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \qquad (1.3.1.17)$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{13}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \qquad (1.3.1.18)$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \qquad (1.3.1.19)$$

From (1.3.1.15) putting (1.3.1.17) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \qquad (1.3.1.20)$$
$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \qquad (1.3.1.21)$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \quad (1.3.1.22)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{3}{2} \\ 0 & 1-\lambda & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{5}{2}-\lambda \end{pmatrix} = 0 \quad (1.3.1.23)$$

$$\implies \lambda(\lambda - 1)(\lambda - \frac{7}{2}) = 0 \quad (1.3.1.24)$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{7}{2} \qquad (1.3.1.25)$$
$$\lambda_2 = 1 \qquad (1.3.1.26)$$
$$\lambda_3 = 0 \qquad (1.3.1.27)$$

Hence the eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

$$(1.3.1.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{\sqrt{35}} \\ -\frac{1}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix}$$
$$(1.3.1.29)$$

Hence we obtain $\mathbf{U}$ of (1.3.1.17) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \end{pmatrix} \qquad (1.3.1.30)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (1.3.1.17) as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\frac{7}{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (1.3.1.31)$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \qquad (1.3.1.32)$$

$$\implies \begin{pmatrix} \frac{13}{4}-\lambda & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \qquad (1.3.1.33)$$

$$\implies \lambda^2 - \frac{9}{2}\lambda + \frac{7}{2} = 0 \qquad (1.3.1.34)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{7}{2} \qquad (1.3.1.35)$$
$$\lambda_5 = 1 \qquad (1.3.1.36)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \qquad (1.3.1.37)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \qquad (1.3.1.38)$$

Hence we obtain $\mathbf{V}$ of (1.3.1.17) as follows,

$$\mathbf{V} = \begin{pmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \qquad (1.3.1.39)$$

Finally from (1.3.1.17) we get the Singualr

Value Decomposition of **M** as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{7}{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}^T$$

$$(1.3.1.40)$$

Now, Moore-Penrose Pseudo inverse of **S** is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \sqrt{\frac{2}{7}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.3.1.41)$$

From (1.3.1.21) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{\sqrt{35}} \\ \frac{2\sqrt{2}}{\sqrt{5}} \\ \frac{8\sqrt{2}}{\sqrt{7}} \end{pmatrix} \qquad (1.3.1.42)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{10}}{35} \\ \frac{2\sqrt{10}}{5} \end{pmatrix} \qquad (1.3.1.43)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{10}{7} \\ \frac{6}{7} \end{pmatrix} \qquad (1.3.1.44)$$

Verifying the solution of (1.3.1.44) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \qquad (1.3.1.45)$$

Evaluating the R.H.S in (1.3.1.45) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \qquad (1.3.1.46)$$

$$\implies \begin{pmatrix} \frac{13}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad (1.3.1.47)$$

Solving the augmented matrix of (1.3.1.47) we get,

$$\begin{pmatrix} \frac{13}{4} & -\frac{3}{4} & 4 \\ -\frac{3}{4} & \frac{5}{4} & 0 \end{pmatrix} \xleftarrow{R_1 = \frac{4}{13}R_1} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ -\frac{3}{4} & \frac{5}{4} & 0 \end{pmatrix}$$

$$(1.3.1.48)$$

$$\xleftarrow{R_2 = R_2 + \frac{3}{4}R_1} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ 0 & \frac{14}{13} & \frac{12}{13} \end{pmatrix}$$

$$(1.3.1.49)$$

$$\xleftarrow{R_2 = \frac{13}{14}R_2} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ 0 & 1 & \frac{6}{7} \end{pmatrix}$$

$$(1.3.1.50)$$

$$\xleftarrow{R_1 = R_1 + \frac{3}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix}$$

$$(1.3.1.51)$$

Hence, Solution of (1.3.1.45) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{10}{7} \\ \frac{6}{7} \end{pmatrix} \qquad (1.3.1.52)$$

Comparing results of **x** from (1.3.1.44) and (1.3.1.52) we conclude that the solution is verified.

1.3.2. Write the equation of the line through $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and perpendicular to the plane $2x - y + 2z - 5 = 0$. Determine the coordinates of the point in which the plane is met by this line.

Solution: Given a point $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and a plane $\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\mathbf{x} = 5$. We know that the equation of a plane is given by

$$\mathbf{n}^T\mathbf{x} = c \qquad (1.3.2.1)$$

Hence, normal vector **n** is given by

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \qquad (1.3.2.2)$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be two vectors that are normal to normal vector **n**. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then if

$$\mathbf{n}^T\mathbf{m} = 0 \qquad (1.3.2.3)$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \qquad (1.3.2.4)$$

Taking $a = 1$, $b = 0$, we get $c = -1$, and hence

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad (1.3.2.5)$$

Take $a = 0$ and $b = 1$, we get $c = \frac{1}{2}$, and hence

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \qquad (1.3.2.6)$$

Since foot of perpendicular is the point where the plane is met by a line perpendicular to the same plane. So, to get foot of perpendicular,

we solve

$$\mathbf{Mx} = \mathbf{b} \qquad (1.3.2.7)$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}, b = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \qquad (1.3.2.8)$$

To solve (1.3.2.7), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \qquad (1.3.2.9)$$

Substituting the value of $\mathbf{M}$ from (1.3.2.9) in (1.3.2.7), we get

$$\mathbf{USV^Tx} = \mathbf{b} \qquad (1.3.2.10)$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \qquad (1.3.2.11)$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigen-values of $\mathbf{M^TM}$. First calculating the eigenvectors corresponding to $\mathbf{M^TM}$.

$$\mathbf{M^TM} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix}$$
$$(1.3.2.12)$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left| \mathbf{M^TM} - \lambda\mathbf{I} \right| = 0 \qquad (1.3.2.13)$$

$$\left| \begin{pmatrix} 2 - \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} - \lambda \end{pmatrix} \right| = 0 \qquad (1.3.2.14)$$

$$\left(\frac{5}{4} - \lambda \right)(2 - \lambda) - \frac{1}{4} = 0 \qquad (1.3.2.15)$$

$$\left(\lambda - \frac{9}{4} \right)(\lambda - 1) = 0 \qquad (1.3.2.16)$$

Hence,

$$\lambda_1 = \frac{9}{4}, \lambda_2 = 1 \qquad (1.3.2.17)$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad (1.3.2.18)$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (1.3.2.19)$$

Normalizing, the eigen vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, we get

$$\mathbf{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad (1.3.2.20)$$

$$\mathbf{v_2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (1.3.2.21)$$

Hence,

$$\mathbf{V} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \qquad (1.3.2.22)$$

Now calculating the eigenvectors corresponding to $\mathbf{MM^T}$

$$\mathbf{MM^T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \qquad (1.3.2.23)$$

Eigen values of $\mathbf{MM^T}$ can be found out as

$$\left| \mathbf{MM^T} - \lambda\mathbf{I} \right| = 0 \qquad (1.3.2.24)$$

$$\left| \begin{pmatrix} 1 - \lambda & 0 & -1 \\ 0 & 1 - \lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} - \lambda \end{pmatrix} \right| = 0 \qquad (1.3.2.25)$$

$$(1 - \lambda)\left((1 - \lambda)\left(\frac{5}{4} - \lambda \right) - \frac{1}{4} \right) - 1 + \lambda = 0$$
$$(1.3.2.26)$$

$$\lambda\left(\lambda - \frac{9}{4} \right)(\lambda - 1) = 0 \qquad (1.3.2.27)$$

Hence,

$$\lambda_3 = 0, \lambda_4 = 1, \lambda_5 = \frac{9}{4} \qquad (1.3.2.28)$$

Eigen-vector corresponding to $\lambda = 0$,

$$\mathbf{v_3} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \qquad (1.3.2.29)$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_4} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad (1.3.2.30)$$

Eigen-vector corresponding to $\lambda = \frac{9}{4}$,

$$\mathbf{v_5} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} \qquad (1.3.2.31)$$

Normalizing, the eigen vectors $\mathbf{v_3}$, $\mathbf{v_4}$ and $\mathbf{v_5}$, we get

$$\mathbf{v_3} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{pmatrix} \qquad (1.3.2.32)$$

$$\mathbf{v_4} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \qquad (1.3.2.33)$$

$$\mathbf{v_5} = \frac{1}{3\sqrt{5}}\begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \\ \frac{-5}{3\sqrt{5}} \end{pmatrix} \qquad (1.3.2.34)$$

Hence,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \qquad (1.3.2.35)$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_5, λ_4 and λ_3 is as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (1.3.2.36)$$

Now, Moore-Pen-Rose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S_+} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.3.2.37)$$

Hence, we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{-1}{3} \\ \frac{-5}{3\sqrt{5}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \qquad (1.3.2.38)$$

Substituting values of (1.3.2.8), (1.3.2.22),

(1.3.2.35) and (1.3.2.36) into (1.3.2.11), we get

$$\mathbf{U^T b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{-5}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \qquad (1.3.2.39)$$

$$\implies \mathbf{U^T b} = \begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \qquad (1.3.2.40)$$

Now,

$$\mathbf{VS_+} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.3.2.41)$$

$$\implies \mathbf{VS_+} = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{-2}{3} & 2 & 0 \end{pmatrix} \qquad (1.3.2.42)$$

Now, by (1.3.2.11), we have

$$\mathbf{x} = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{4}{3} & 1 & 0 \\ \frac{-2}{3} & 2 & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} \\ 0 \end{pmatrix} \qquad (1.3.2.43)$$

$$\implies \mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad (1.3.2.44)$$

Now, we verify our solution using

$$\mathbf{M^T M x} = \mathbf{M^T b} \qquad (1.3.2.45)$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \qquad (1.3.2.46)$$

$$\implies \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 4 \\ \frac{7}{2} \end{pmatrix} \qquad (1.3.2.47)$$

Solving the augumented matrix, we get

$$\begin{pmatrix} 2 & \frac{-1}{2} & 4 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \xleftarrow{r_1 = (1/2)*(r_1)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \qquad (1.3.2.48)$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ \frac{-1}{2} & \frac{5}{4} & \frac{7}{2} \end{pmatrix} \xleftarrow{r_2 = r_2 + (1/2)*(r_1)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & \frac{9}{8} & \frac{9}{2} \end{pmatrix} \qquad (1.3.2.49)$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & \frac{9}{8} & \frac{9}{2} \end{pmatrix} \xleftarrow{r_2 = (8/9)*(r_2)} \begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & 1 & 4 \end{pmatrix} \qquad (1.3.2.50)$$

$$\begin{pmatrix} 1 & \frac{-1}{4} & 2 \\ 0 & 1 & 4 \end{pmatrix} \xleftarrow{r_1 = r_1 + (-1/4)*(r_2)} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix} \qquad (1.3.2.51)$$

Thus,

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad (1.3.2.52)$$

verifying the result from SVD.

Now, we solve for third coordinate of foot of perpendicular by,

$$\mathbf{n}^T\mathbf{x} = 5 \qquad (1.3.2.53)$$

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ z \end{pmatrix} = 5 \qquad (1.3.2.54)$$

$$z = \frac{-5}{2} \qquad (1.3.2.55)$$

Normalizing z, we get

$$z = \frac{\left(\frac{-5}{2}\right)}{3} \implies z = \frac{-5}{6} \qquad (1.3.2.56)$$

Hence, coordinate of foot of perpendicular is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ \frac{-5}{6} \end{pmatrix} \qquad (1.3.2.57)$$

Now, we try to find equation of straight line through $\mathbf{P} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and having direction cosines as $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \qquad (1.3.2.58)$$