



# Geometry through Linear Algebra



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**Abstract**—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.

### 1 PAIR OF STRAIGHT LINES

1.1. Find the value of  $h$  so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.1)$$

may represent two straight lines.

**Solution:**

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \quad (1.1.2)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.3)$$

$$f = 20 \quad (1.1.4)$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \quad (1.1.5)$$

Expanding equation (1.1.5) along row 1 gives

$$\begin{aligned} \Rightarrow 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + \\ 11 \times (\frac{31h}{2} - 132) = 0 \end{aligned}$$

$$\Rightarrow 20h^2 - 341h + \frac{2907}{2} = 0 \quad (1.1.6)$$

$$\Rightarrow \boxed{h = \frac{17}{2}} \quad (1.1.7)$$

$$\Rightarrow \boxed{h = \frac{171}{20}} \quad (1.1.8)$$

If  $h = \frac{17}{2}$  or  $h = \frac{171}{20}$ , the equation given will represent two straight lines.

Sub  $h = \frac{17}{2}$  in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.9)$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (1.1.10)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.11)$$

$$\mathbf{f} = 20 \quad (1.1.12)$$

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The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.13)$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \quad (1.1.14)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.1.15)$$

$$(1.1.16)$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \quad (1.1.17)$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \quad (1.1.18)$$

$$\Rightarrow m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \quad (1.1.19)$$

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.1.20)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.21)$$

we know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.22)$$

Convolution of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  can be done by converting  $\mathbf{n}_1$  into a toeplitz matrix and multiplying with  $\mathbf{n}_2$

From equation (1.1.21)

$$\mathbf{n}_1 = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.23)$$

$$\Rightarrow \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.24)$$

$\Rightarrow$  Equation (1.1.21) satisfies (1.1.22)

$c_1$  and  $c_2$  can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.1.25)$$

Substituting (1.1.21) in (1.1.25), the augmented

matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \quad (1.1.26)$$

$$\Rightarrow c_1 = 4, c_2 = 5 \quad (1.1.27)$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\begin{aligned} \Rightarrow (-2x - 3y - 4)(3x - 4y - 5) &= 0 \\ \Rightarrow (2x + 3y + 4)(3x + 4y + 5) &= 0 \end{aligned} \quad (1.1.28)$$

Equation (1.1.28) represents equations of two straight lines.

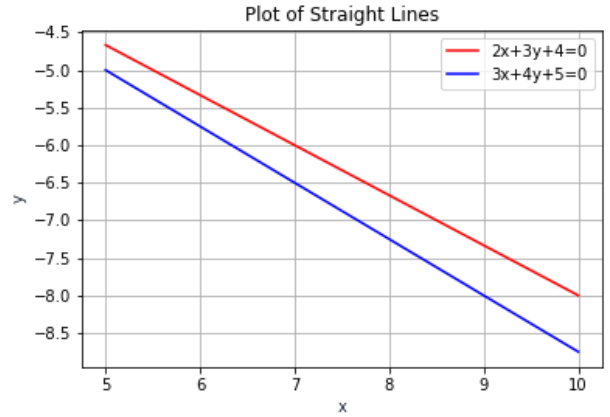


Fig. 1.1.1: Plot of Straight lines when  $h = \frac{17}{2}$

Similarly, Sub  $h = \frac{171}{20}$  in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \quad (1.1.29)$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \quad (1.1.30)$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{3} \\ \frac{310}{3} \end{pmatrix} \quad (1.1.31)$$

$$\mathbf{f} = \frac{200}{3} \quad (1.1.32)$$

The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.33)$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \quad (1.1.34)$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \quad (1.1.35)$$

$$\Rightarrow m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \quad (1.1.36)$$

$$\mathbf{m}_1 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (1.1.37)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.38)$$

Convolution of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  can be done by converting  $\mathbf{n}_1$  into a toeplitz matrix and multiplying with  $\mathbf{n}_2$

From equation (1.1.38)

$$\mathbf{n}_1 = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.39)$$

$$\Rightarrow \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.40)$$

$\Rightarrow$  Equation (1.1.38) satisfies (1.1.22)

$c_1$  and  $c_2$  can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -2\mathbf{u} \quad (1.1.41)$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 4R_2}{5}]{R_2 \leftarrow \frac{5R_2 - 8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \quad (1.1.42)$$

$$\Rightarrow c_1 = 10, c_2 = \frac{20}{3} \quad (1.1.43)$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\Rightarrow \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \quad (1.1.44)$$

Equation (1.1.44) represents equations of two straight lines.

1.2. Prove that the following equations represent two straight lines. Also find their point of in-

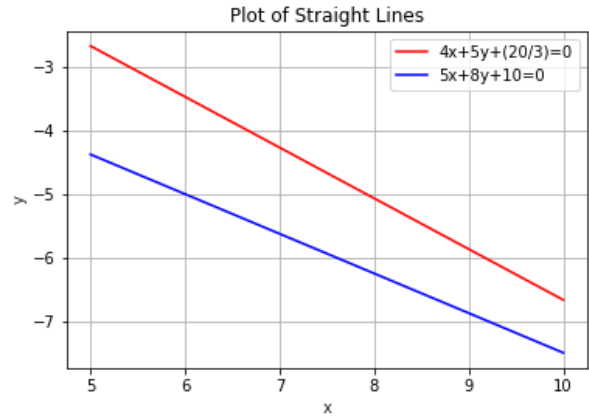


Fig. 1.1.2: Plot of Straight lines when  $h = \frac{171}{20}$

tersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \quad (1.2.1)$$

**Solution:**  $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$  of (1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \quad (1.2.2)$$

Expanding equation (1.2.2), we get zero.

Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.2.3)$$

$$\Rightarrow 3m^2 - 8m - 3 = 0 \quad (1.2.4)$$

$$\text{Solving, } m = 3, -\frac{1}{3} \quad (1.2.5)$$

The normal vectors of the lines then become

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.2.6)$$

$$\mathbf{n}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.7)$$

Equations of the lines can therefore be written as

$$\left(\frac{1}{3} \ 1\right) \mathbf{x} = c \quad (1.2.8)$$

$$\Rightarrow (1 \ 3) \mathbf{x} = c_1, \quad (1.2.9)$$

$$(-3 \ 1) \mathbf{x} = c_2 \quad (1.2.10)$$

$$\Rightarrow [(1 \ 3) \mathbf{x} - c_1][(-3 \ 1) \mathbf{x} - c_2] \quad (1.2.11)$$

represents the equation specified in (1.2.1)

Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \quad (1.2.12)$$

$$(1.2.13)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3 \times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \quad (1.2.14)$$

$$\xrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.15)$$

$$\xrightarrow{R_1 \leftarrow R_1 + 3 \times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.16)$$

$$\Rightarrow c_2 = 2 \text{ and } c_1 = -9 \quad (1.2.17)$$

The individual line equations therefore become

$$(1 \ 3)\mathbf{x} = -9, \quad (1.2.18)$$

$$(-3 \ 1)\mathbf{x} = 2 \quad (1.2.19)$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.20)$$

The LHS part of (1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.21)$$

The augmented matrix for the set of equations represented in (1.2.18), (1.2.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \quad (1.2.22)$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.23)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.24)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \quad (1.2.25)$$

$$\text{Hence, the intersection point is } \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.2.26)$$

Angle between two lines  $\theta$  can be given by

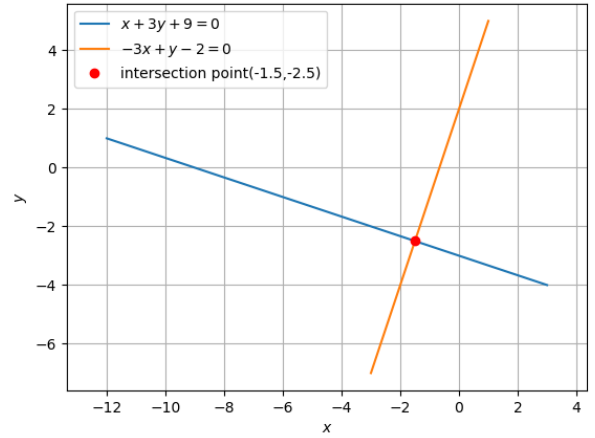


Fig. 1.2.1: plot showing intersection of lines

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.2.27)$$

$$\cos \theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \quad (1.2.28)$$

$$\Rightarrow \theta = 90^\circ \quad (1.2.29)$$

1.3. Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \quad (1.3.1)$$

**Solution:**

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (1.3.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (1.3.3)$$

$$f = -2 \quad (1.3.4)$$

$$\left| \begin{array}{ccc} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{array} \right| \xrightarrow[R_1 \rightarrow R_1 + R_3]{R_1 \rightarrow R_1 - R_2} \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{array} \right| = 0 \quad (1.3.5)$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$|V| < 0 \implies \left| \begin{array}{cc} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array} \right| = \frac{-9}{4} < 0 \quad (1.3.6)$$

Let the pair of straight of lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.3.7)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.3.8)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.3.9)$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \quad (1.3.10)$$

$$m_1 = 1, m_2 = -2 \quad (1.3.11)$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.3.12)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \quad (1.3.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.3.14)$$

$$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (1.3.15)$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.3.16)$$

Using row reduction we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \end{pmatrix} \quad (1.3.17)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_2 \leftarrow R_2 / -3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (1.3.18)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad (1.3.19)$$

$$C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.3.20)$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.21)$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.22)$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (1.3.23)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \quad (1.3.24)$$

consider the augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \quad (1.3.25)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.3.26)$$

$$\xrightarrow[R_1 \leftarrow R_1 + R_2]{R_1 \leftarrow R_1 / 3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.3.27)$$

Therefore point of intersection is  $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

Angle between two lines  $\theta$  can be given by

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.3.28)$$

$$\cos \theta = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} \quad (1.3.29)$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right) \Rightarrow \theta = \tan^{-1}3 \quad (1.3.30)$$

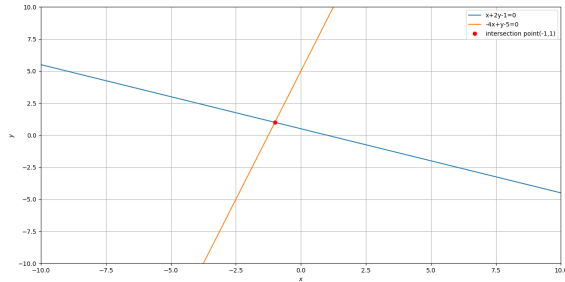


Fig. 1.3.1: plot showing intersection of lines

1.4. Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \quad (1.4.1)$$

represents two parallel lines.

**Solution:** The given equation (1.4.1) can be written as

$$\mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (1.4.2)$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \quad (1.4.3)$$

Equation (1.4.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \quad (1.4.4)$$

Vector form of straight lines,

$$\mathbf{n}_1^T \mathbf{x} = \mathbf{c}_1 \quad (1.4.5)$$

$$\mathbf{n}_2^T \mathbf{x} = \mathbf{c}_2 \quad (1.4.6)$$

Equating their product with (1.4.2)

$$(\mathbf{n}_1^T \mathbf{x} - \mathbf{c}_1)(\mathbf{n}_2^T \mathbf{x} - \mathbf{c}_2) = \mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 \quad (1.4.7)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.8)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix} \quad (1.4.9)$$

$$c_1 c_2 = -5 \quad (1.4.10)$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \quad (1.4.11)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.4.12)$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.4.13)$$

From (1.4.2) equation (1.4.11) becomes

$$9m^2 + 6m + 1 = 0 \quad (1.4.14)$$

Using (1.4.3),

$$|\mathbf{V}| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \quad (1.4.15)$$

Substituting the values in (1.4.12),

$$m_i = \frac{-3 \pm 0}{9} \quad (1.4.16)$$

$$m_1 = m_2 = \frac{-1}{3} \quad (1.4.17)$$

Substituting values in (1.4.13)

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.18)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.19)$$

Using the above values in (1.4.8),

$$k_1 k_2 = 9 \quad (1.4.20)$$

Taking  $k_1 = 3$  and  $k_2 = 3$  we get

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.21)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.22)$$

Verifying  $\mathbf{n}_1$  and  $\mathbf{n}_2$  by computing the convolution by representing  $\mathbf{n}_1$  as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.23)$$

Finding the Angle between the lines,

$$\theta = \cos^{-1} \left( \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) \quad (1.4.24)$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \quad (1.4.25)$$

$$\|\mathbf{n}_1\| = \sqrt{10} \quad \|\mathbf{n}_2\| = \sqrt{10} \quad (1.4.26)$$

Substituting (1.4.25) and (1.4.26) in (1.4.24) we get,

$$\theta = \cos^{-1}(1) \quad (1.4.27)$$

$$\theta = 0^\circ \quad (1.4.28)$$

From (1.4.17) and (1.4.28) shows the given equation (1.4.1) represents two parallel lines. Hence proved.

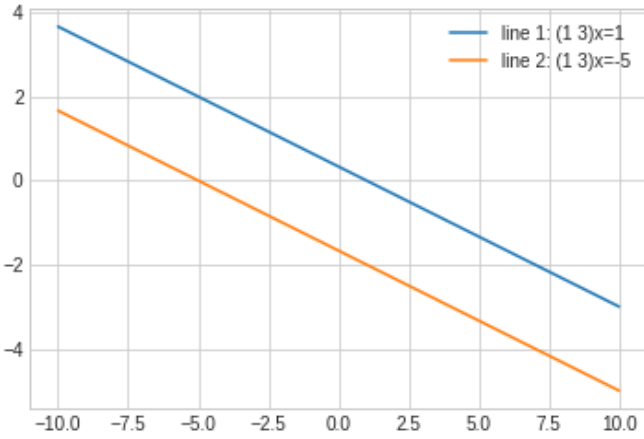


Fig. 1.4.1: Pair of straight lines plot generated using python

1.5. **Solution:** Find the value of  $k$  such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \quad (1.5.1)$$

represent pairs of straight lines.

From (1.5.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \quad (1.5.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.5.3)$$

$$f = k \quad (1.5.4)$$

Compute the slopes of lines given by the roots

of the polynomial  $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.5.5)$$

$$\Rightarrow m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \quad (1.5.6)$$

$$\Rightarrow m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \quad (1.5.7)$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.5.8)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.5.9)$$

Here,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \quad (1.5.10)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -m_2 \\ 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (1.5.11)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.12)$$

Substituting (1.5.10) and (1.5.11) in the above equation, we get

$$k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} \quad (1.5.13)$$

$$\Rightarrow k_1 k_2 = -10 \quad (1.5.14)$$

By inspection, we get the values,  $k_1 = 5, k_2 = -2$ . Substituting the values of  $k_1$  and  $k_2$  in (1.5.10) and (1.5.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (1.5.15)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.5.16)$$

Using Teoplitz matrix representation, the convolution of  $\mathbf{n}_1$  with  $\mathbf{n}_2$ , is as follows:

$$\begin{pmatrix} 2 & 0 & 5 \\ 5 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.17)$$

Hence,  $\mathbf{n}_1$  and  $\mathbf{n}_2$  satisfies (1.5.12). We have,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.5.18)$$

Substituting (1.5.15), (1.5.16) in (1.5.18), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.5.19)$$

Solving for  $c_1$  and  $c_2$ , the augmented matrix is,

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & -2 & -31 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 5R_1]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & \frac{-19}{2} & \frac{-37}{2} \end{pmatrix} \quad (1.5.20)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{3}{2}R_2]{R_2 \leftarrow \frac{R_2}{-19/2}} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.5.21)$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \quad (1.5.22)$$

We know that,

$$f = k = c_1 c_2 \quad (1.5.23)$$

$$\Rightarrow \boxed{k = -15} \quad (1.5.24)$$

Hence the solution. Using (1.5.8) and (1.5.9), the equation of pair of straight lines is given by,

$$(2 \ 5)\mathbf{x} = 3 \quad (1.5.25)$$

$$(3 \ -2)\mathbf{x} = -5 \quad (1.5.26)$$

See Fig. 1.5.1

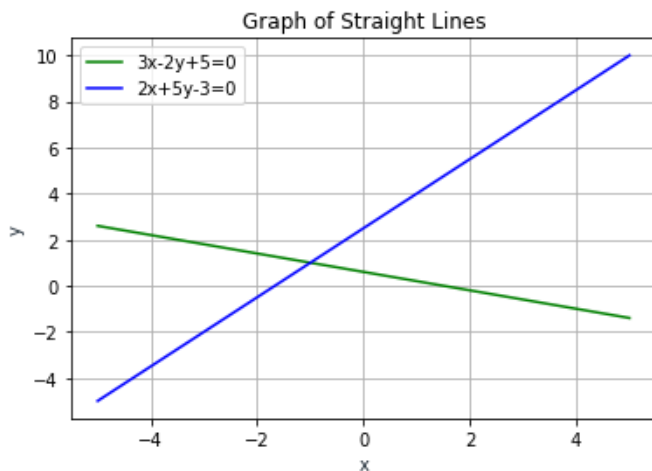


Fig. 1.5.1: Plot of two straight lines.

1.6. Find the value of  $k$  so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.1)$$

**Solution:**

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \quad (1.6.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.3)$$

The equation (1.6.1) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.6.4)$$

$$\Rightarrow \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \quad (1.6.5)$$

$$\Rightarrow \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \quad (1.6.6)$$

$$\Rightarrow 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11 \begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \quad (1.6.7)$$

$$\Rightarrow 2k^2 + 55k + 350 = 0 \quad (1.6.8)$$

$$\Rightarrow (10 + k)(2k + 35) = 0 \quad (1.6.9)$$

$$\Rightarrow k = -10$$

$$k = -\frac{35}{2} \quad (1.6.10)$$

Therefore, for  $k = -10$  and  $k = -\frac{35}{2}$  the given equation represents pair of straight lines.

Now Lets find equation of lines for  $k = -10$ . Substitute  $k = -10$  in (1.6.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.11)$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (1.6.12)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.13)$$

If  $|\mathbf{V}| < 0$  then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \quad (1.6.14)$$

$$\Rightarrow |\mathbf{V}| = -1 \quad (1.6.15)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.6.16)$$



Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.6.17)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.6.18)$$

Equating their product with (1.6.1)

$$\begin{aligned} (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (1.6.19)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.6.20)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} = -2 \begin{pmatrix} \frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \quad (1.6.21)$$

$$c_1 c_2 = f = 2 \quad (1.6.22)$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.6.23)$$

$$\Rightarrow 2m^2 - 10m + 12 = 0 \quad (1.6.24)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.6.25)$$

$$\Rightarrow m_i = \frac{5 \pm \sqrt{1}}{2} \quad (1.6.26)$$

$$\Rightarrow m_1 = 3 \quad (1.6.27)$$

$$m_2 = 2 \quad (1.6.28)$$

The normal vector for two lines is given by

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.6.29)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.6.30)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.6.31)$$

Substituting (1.6.30),(1.6.31) in (1.6.20). we get

$$k_1 k_2 = 2 \quad (1.6.32)$$

The possible combinations of  $(k_1, k_2)$  are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume  $k_1 = 1, k_2 = 2$  we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.6.33)$$

$$\mathbf{n}_2 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (1.6.34)$$

We verify obtained  $\mathbf{n}_1, \mathbf{n}_2$  using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.6.35)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.6.36)$$

Therefore the obtained  $\mathbf{n}_1, \mathbf{n}_2$  are correct.

Substitute (1.6.33), (1.6.34) in (1.6.21) and calculate for  $c_1$  and  $c_2$

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.6.37)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.38)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.6.39)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.6.40)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \quad (1.6.41)$$

$$\Rightarrow c_1 = -13 \quad (1.6.42)$$

$$c_2 = 21 \quad (1.6.43)$$

Substituting (1.6.33),(1.6.34),(1.6.42),(1.6.43) in (1.6.17) and (1.6.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = -13 \quad (1.6.44)$$

$$\begin{pmatrix} -4 & 2 \end{pmatrix} \mathbf{x} = 21 \quad (1.6.45)$$

The plot of these two lines is shown in Fig. 1.6.1.

Now Lets find equation of lines for  $k = -\frac{35}{2}$ . Substitute  $k = -\frac{35}{2}$  in (1.6.1). We get equation

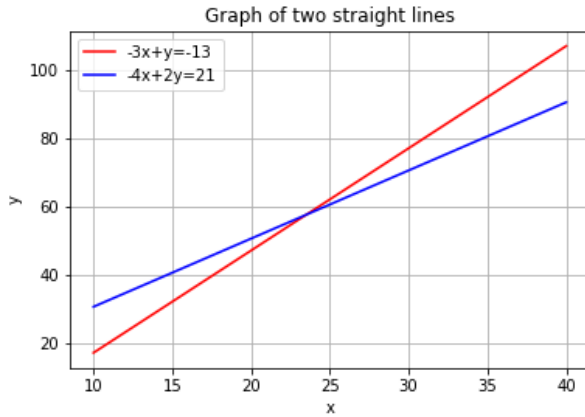


Fig. 1.6.1: Pair of straight lines for  $k = -10$

of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.46)$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \quad (1.6.47)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.48)$$

If  $|\mathbf{V}| < 0$  then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \quad (1.6.49)$$

$$\Rightarrow |\mathbf{V}| = -\frac{841}{16} \quad (1.6.50)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.6.51)$$

Therefore the lines will intersect.

Now from (1.6.20),

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (1.6.52)$$

The slopes of the lines are given by roots of

equation (1.6.23)

$$\Rightarrow 2m^2 - \frac{35}{2}m + 12 = 0 \quad (1.6.53)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.6.54)$$

$$\Rightarrow m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \quad (1.6.55)$$

$$\Rightarrow m_1 = 8 \quad (1.6.56)$$

$$m_2 = \frac{3}{4} \quad (1.6.57)$$

The normal vector for two lines is given by (1.6.29)

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.6.58)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.6.59)$$

Substituting (1.6.58), (1.6.59) in (1.6.52). we get

$$k_1 k_2 = 2 \quad (1.6.60)$$

The possible combinations of  $(k_1, k_2)$  are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume  $k_1 = 1, k_2 = 2$  we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.6.61)$$

$$\mathbf{n}_2 = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \quad (1.6.62)$$

We verify obtained  $\mathbf{n}_1, \mathbf{n}_2$  using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -8 & 0 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \end{pmatrix} \quad (1.6.63)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -\frac{35}{2} \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.6.64)$$

Therefore the obtained  $\mathbf{n}_1, \mathbf{n}_2$  are correct.

Substitute (1.6.61), (1.6.62) in (1.6.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.6.65)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.66)$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.67)$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.6.68)$$

$$\xleftrightarrow{R_1 \leftarrow -29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.6.69)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \quad (1.6.70)$$

$$\Rightarrow c_1 = -\frac{102}{29} \quad (1.6.71)$$

$$c_2 = \frac{59}{29} \quad (1.6.72)$$

Substituting (1.6.61), (1.6.62), (1.6.71), (1.6.72) in (1.6.17) and (1.6.18). we get equation of two straight lines.

$$(-8 \ 1)\mathbf{x} = -\frac{102}{29} \quad (1.6.73)$$

$$\left(-\frac{3}{2} \ 2\right)\mathbf{x} = \frac{59}{29} \quad (1.6.74)$$

1.7. Find the value of  $k$  so that the following equation may represent a pair of straight lines

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \quad (1.7.1)$$

**Solution:** The given second degree equation is, Comparing coefficients of (1.7.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \quad (1.7.2)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.3)$$

$$f = -35 \quad (1.7.4)$$

The given second degree equation (1.7.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \quad (1.7.5)$$

Expanding the determinant,

$$k + 12 = 0 \quad (1.7.6)$$

$$\Rightarrow k = -12 \quad (1.7.7)$$

Hence, from (1.7.7) we find that for  $k = -12$ , the given second degree equation (1.7.1) represents pair of straight lines. For the appropriate value of  $k$ , (1.7.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \quad (1.7.8)$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.7.9)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.7.10)$$

The pair of straight lines is given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.7.11)$$

Putting the values of  $\mathbf{V}$  and  $\mathbf{u}$  we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \quad (1.7.12)$$

Hence, from (1.7.12) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.14)$$

$$c_1 c_2 = -35 \quad (1.7.15)$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \quad (1.7.16)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.7.17)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.7.18)$$

Substituting the values in above equations (1.7.16) we get,

$$-12m^2 + m + 6 = 0 \quad (1.7.19)$$

$$\Rightarrow m_i = \frac{-\frac{1}{2} \pm \sqrt{-\left(-\frac{289}{4}\right)}}{-12} \quad (1.7.20)$$

Solving equation (1.7.20) we get ,

$$m_1 = -\frac{2}{3} \quad (1.7.21)$$

$$m_2 = \frac{3}{4} \quad (1.7.22)$$

Hence putting the values of  $m_1$  and  $m_2$  in (1.7.18) we get

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (1.7.23)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.7.24)$$

Putting values of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  in (1.7.13) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.25)$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.26)$$

Thus, from (1.7.26),  $k_1k_2 = -12$ . Possible combinations of  $(k_1, k_2)$  are  $(6, -2)$ ,  $(-6, 2)$ ,  $(3, -4)$ ,  $(-3, 4)$  Lets assume  $k_1 = 3$ ,  $k_2 = -4$ , then we get,

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.7.27)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (1.7.28)$$

From equation (1.7.14) we get

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.7.29)$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.30)$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \quad (1.7.31)$$

$$3c_2 - 4c_1 = -43 \quad (1.7.32)$$

The augmented matrix of (1.7.31) ,(1.7.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix} \quad (1.7.33)$$

$$\xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \quad (1.7.34)$$

$$\xrightarrow{R_2 = -\frac{2}{17}R_2} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & \frac{11}{7} \end{pmatrix} \quad (1.7.35)$$

$$\xrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & \frac{11}{7} \end{pmatrix} \quad (1.7.36)$$

$$(1.7.37)$$

Hence we get,

$$c_1 = -5 \quad (1.7.38)$$

$$c_2 = 7 \quad (1.7.39)$$

Hence (1.7.9), (1.7.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \quad (1.7.40)$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \quad (1.7.41)$$

The figure below corresponds to the pair of straight lines represented by (1.7.40) and (1.7.41).

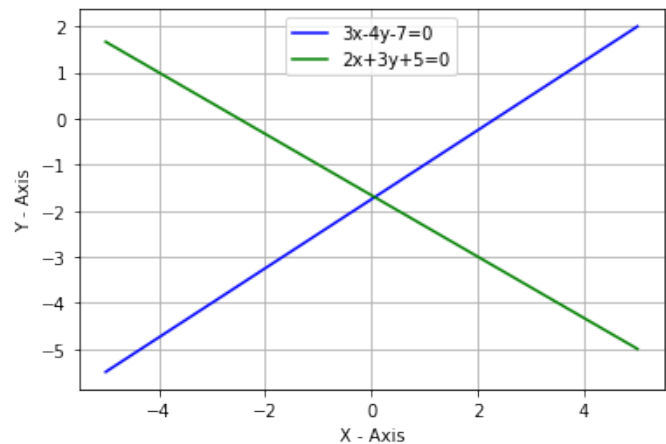


Fig. 1.7.1: Pair of Straight Lines

1.8. Find the value of  $k$  so that following equation may represent pairs of straight lines,

$$kxy - 8x + 9y - 12 = 0 \quad (1.8.1)$$

**Solution:** The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.8.2)$$

In vector form the equation (1.8.2) can be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.8.3)$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (1.8.4)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (1.8.5)$$

Now, comparing equation (1.8.2) to (1.8.1) we get,  $a = c = 0$ ,  $b = \left(\frac{k}{2}\right)$ ,  $d = -4$ ,  $e = \left(\frac{9}{2}\right)$ ,  $f = -12$ . Hence, substituting these values in equation (1.8.4) and (1.8.5) we get,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} \quad (1.8.6)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix} \quad (1.8.7)$$

Now equation (1.8.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.8.8)$$

$$\begin{vmatrix} 0 & \frac{k}{2} & -4 \\ \frac{k}{2} & 0 & \frac{9}{2} \\ -4 & \frac{9}{2} & -12 \end{vmatrix} = 0 \quad (1.8.9)$$

$$\Rightarrow k = 0, k = 6 \quad (1.8.10)$$

Substituting (1.8.10) in (1.8.1) we get,

$$6xy - 8x + 9y - 12 = 0 \quad (1.8.11)$$

$$-8x + 9y - 12 = 0 \quad (1.8.12)$$

Hence value of  $k = 6$  represents pair of straight lines. Also it can be verified that the pair of lines intersect as,

$$|\mathbf{V}| = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} < 0 \quad (1.8.13)$$

Let the pair of straight lines is given by,

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.8.14)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.8.15)$$

Now equating the product of equation (1.8.14) and (1.8.15) with (1.8.3) we get,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \quad (1.8.16)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & \frac{9}{2} \end{pmatrix} \mathbf{x} - 12 \quad (1.8.17)$$

$$\Rightarrow n_1 * n_2 = \{0, 6, 0\} \quad (1.8.18)$$

$$c_1 n_1 + c_2 n_2 = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \quad (1.8.19)$$

$$c_1 c_2 = -12. \quad (1.8.20)$$

Now the slopes of line is given by roots of polynomial,

$$cm^2 + 2bm + a = 0 \quad (1.8.21)$$

$$\Rightarrow 2bm = 0 \quad (1.8.22)$$

$$\Rightarrow m = 0 \quad (1.8.23)$$

Also

$$m_i = \frac{-b \pm \sqrt{-|V|}}{c} \quad (1.8.24)$$

$$\Rightarrow m_i = \frac{-0 \pm \sqrt{9}}{0} \quad (1.8.25)$$

$$\therefore m_1 = 0 \quad (1.8.26)$$

$$m_2 = \infty \quad (1.8.27)$$

The normal vector to the two lines is given by,

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.8.28)$$

$$\Rightarrow n_1 = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.8.29)$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.8.30)$$

Also,

$$k_1 k_2 = 6 \quad (1.8.31)$$

Let  $k_1 = 2$  and  $k_2 = 3$

$$\Rightarrow n_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (1.8.32)$$

$$n_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.8.33)$$

We verify obtained  $n_1$  and  $n_2$  using Toeplitz matrix,

$$n_1 * n_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \quad (1.8.34)$$

Hence (1.8.18) and (1.8.34) are same. Hence verified.

Now substituting it in (1.8.19) we get,

$$c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \quad (1.8.35)$$

Solve using Row reduction Technique we get,

$$\Rightarrow \begin{pmatrix} 3 & 0 & 8 \\ 0 & 2 & -9 \end{pmatrix} \quad (1.8.36)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 2 & -9 \end{pmatrix} \quad (1.8.37)$$

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & -9/2 \end{pmatrix} \quad (1.8.38)$$

$$\Rightarrow c_1 = \frac{8}{3} \quad (1.8.39)$$

$$c_2 = \frac{-9}{2} \quad (1.8.40)$$

substituting the values of  $c_1$ ,  $c_2$  and equation (1.8.32) and (1.8.33) to equation (1.8.14) and (1.8.15) we get equation of two straight lines.

$$\Rightarrow \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} = \frac{8}{3} \quad (1.8.41)$$

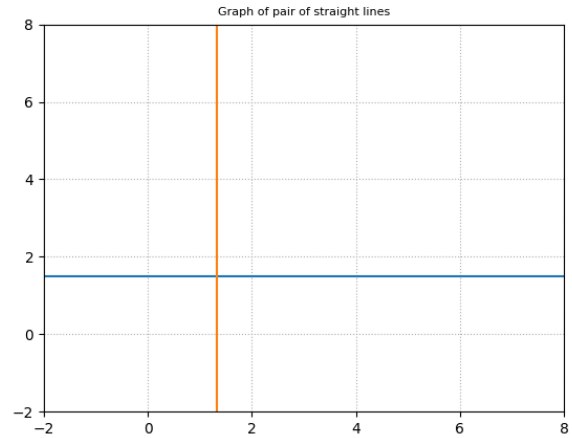


Fig. 1.8.1: Intersection of 2 lines

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} = \frac{-9}{2} \quad (1.8.42)$$

Hence the equation of pair of straight lines are,

$$\left( \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} - \frac{8}{3} \right) \left( \begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - \frac{-9}{2} \right) = 0 \quad (1.8.43)$$

Hence, Plot of the equation (1.8.43) is shown in Figure.1.8.1 Now for value of  $k = 0$  does not represent pair of straight lines.as,

$$|\mathbf{V}| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \neq 0 \quad (1.8.44)$$

Hence, Plot of the equation  $\begin{pmatrix} -8 & 9 \end{pmatrix} \mathbf{x} = 12$  is shown in figure 1.8.2,

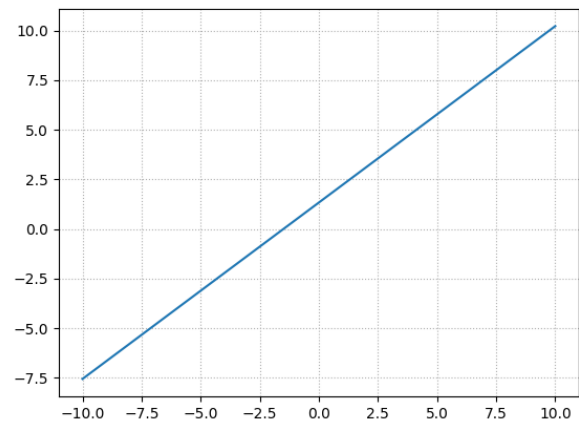


Fig. 1.8.2: Intersection of 2 lines

1.9. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \quad (1.9.1)$$

represent pairs of straight lines.

**Solution:** From (1.9.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \quad (1.9.2)$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{5}{2} & -\frac{7}{2} \end{pmatrix} \quad (1.9.3)$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & k \end{vmatrix} = 0 \quad (1.9.4)$$

$$\begin{aligned} \Rightarrow \left( k - \left( \frac{49}{4} \right) \right) - \frac{5}{3} \left( \frac{5}{3}k - \frac{35}{4} \right) \\ - \frac{5}{2} \left( \frac{-35}{6} + \frac{5}{2} \right) = 0 \end{aligned} \quad (1.9.5)$$

$$\Rightarrow \frac{64}{k} 36 - \frac{128}{12} = 0 \quad (1.9.6)$$

$$\Rightarrow \boxed{k = 6} \quad (1.9.7)$$

Substituting (1.9.7) in (1.9.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \quad (1.9.8)$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (1.9.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & 6 \end{vmatrix} \quad (1.9.9)$$

Simplify the above determinant, we get

$$\delta = 0 \quad (1.9.10)$$

(1.9.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \quad (1.9.11)$$

Since  $\det(V) < 0$  lines would intersect each other

$$\mathbf{n}_1 * \mathbf{n}_2 = \left\{ 1, \frac{10}{3}, 1 \right\} \quad (1.9.12)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{5}{2} \\ -\frac{7}{2} \end{pmatrix} \quad (1.9.13)$$

$$c_1 c_2 = 6 \quad (1.9.14)$$

The slopes of the lines are given by the roots

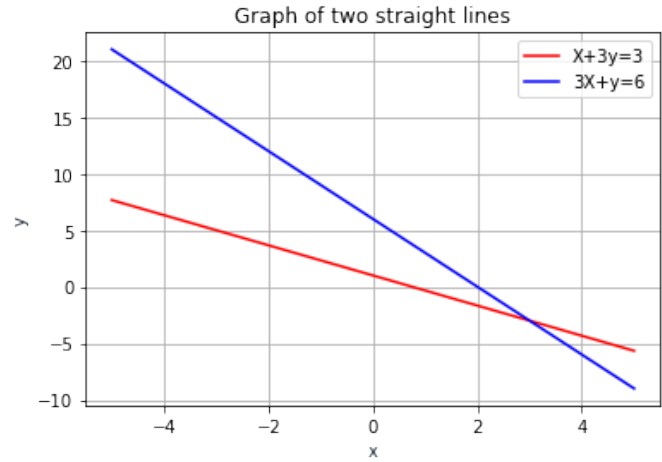


Fig. 1.9.1: Pair of straight lines

of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.9.15)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.9.16)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.9.17)$$

Substituting in above equations (1.9.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \quad (1.9.18)$$

$$\Rightarrow m_i = \frac{\frac{-10}{3} \pm \sqrt{-\left(\frac{-16}{9}\right)}}{1} \quad (1.9.19)$$

Solving equation (1.9.19) we have,

$$m_1 = \frac{-1}{3} \quad (1.9.20)$$

$$m_2 = -3 \quad (1.9.21)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.9.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.9.23)$$

Substituting equations (1.9.22), (1.9.23) in equation (1.9.12) we get

$$k_1 k_2 = 1 \quad (1.9.24)$$

Possible combination of  $(k_1, k_2)$  is (1,1) Lets

assume  $k_1 = 1$ ,  $k_2 = 1$ , we get

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.9.25)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.9.26)$$

we have:

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.9.27)$$

Convolution of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  can be done by converting  $\mathbf{n}_1$  into a teoplitz matrix and multiplying with  $\mathbf{n}_2$

From equation (1.9.25) and (1.9.26)

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.9.28)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.9.29)$$

$c_1$  and  $c_2$  can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.9.30)$$

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -5 \\ \frac{7}{2} \\ -7 \end{pmatrix} \quad (1.9.31)$$

Substituting (1.9.25) and (1.9.26) in (1.9.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \quad (1.9.32)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \quad (1.9.33)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.9.34)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.9.35)$$

From above we get

$$c_1 = 1 \quad (1.9.36)$$

$$c_2 = 6 \quad (1.9.37)$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \quad (1.9.38)$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (1.9.39)$$

1.10. Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (1.10.1)$$

represents two straight lines and find the angle between the lines.

**Solution:** The above equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.10.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (1.10.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (1.10.4)$$

$$f = -35 \quad (1.10.5)$$

(1.10.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.10.6)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (1.10.7)$$

$$\Rightarrow 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \quad (1.10.8)$$

$$(1.10.9)$$

The lines intercept if

$$|\mathbf{V}| < 0 \quad (1.10.10)$$

$$|\mathbf{V}| = -\frac{529}{4} < 0 \quad (1.10.11)$$

From (1.10.8) and (1.10.11) it can be concluded that the given equation represents a pair of intersecting lines. Let the equations of lines be

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.10.12)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.10.13)$$



Since (1.10.2) represents a pair of straight lines it must satisfy

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_1^T \mathbf{x} - c_1) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.10.14)$$

where

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (1.10.15)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.10.16)$$

$$c_1 c_2 = f \quad (1.10.17)$$

Slopes of the lines can be obtained by solving

$$cm^2 + 2bm + a = 0 \quad (1.10.18)$$

$$-10m^2 + 7m + 12 = 0 \quad (1.10.19)$$

$$\Rightarrow m_1 = \frac{-4}{5}, m_2 = \frac{3}{2} \quad (1.10.20)$$

The normal vectors can be expressed in terms of corresponding slopes of lines as

$$\mathbf{n} = k \begin{pmatrix} -m \\ 1 \end{pmatrix} \quad (1.10.21)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} \frac{4}{5} \\ 1 \end{pmatrix} \quad (1.10.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \quad (1.10.23)$$

Substituting (1.10.22) and (1.10.23) in (1.10.15) we get

$$k_1 k_2 = -10 \quad (1.10.24)$$

Assuming  $k_1 = 5$  and  $k_2 = -2$

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.10.25)$$

Verification using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 4 & 0 \\ 5 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (1.10.26)$$

From (1.10.16) we have

$$c_2 \begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ -45 \end{pmatrix} \quad (1.10.27)$$

Solving the augmented matrix

$$\begin{pmatrix} 4 & 3 & -13 \\ 5 & -2 & -45 \end{pmatrix} \xrightarrow{R_2 \leftarrow -4R_2 - 5R_1} \begin{pmatrix} 4 & 3 & -13 \\ 0 & -23 & -115 \end{pmatrix} \quad (1.10.28)$$

$$\xrightarrow{R_2 \leftarrow -\frac{R_2}{23}} \begin{pmatrix} 4 & 3 & -13 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 4 & 0 & -28 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.10.29)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.10.30)$$

$$\Rightarrow c_1 = -7, c_2 = 5 \quad (1.10.31)$$

Thus the equation of lines are

$$(4 \ 5) \mathbf{x} = 5 \quad (1.10.32)$$

$$(3 \ -2) \mathbf{x} = -7 \quad (1.10.33)$$

The angle between the lines can be expressed in terms of normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.10.34)$$

as

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.10.35)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \quad (1.10.36)$$

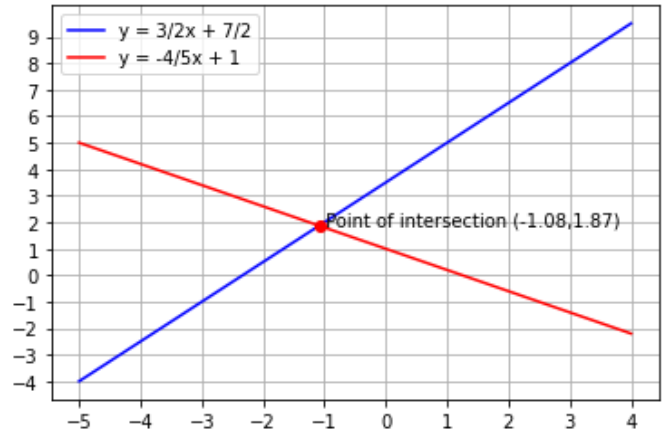


Fig. 1.10.1

1.11. Find the value of  $k$  so that the following equation may represent the pair of straight lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \quad (1.11.1)$$

**Solution:** We need to find the value of  $k$  for which (1.11.1) represents a pair of straight lines.

Converting (1.11.1) into vector form, we get

$$\mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 = 0 \quad (1.11.2)$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \quad (1.11.3)$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \quad (1.11.4)$$

$$f = -9 \quad (1.11.5)$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.11.6)$$

Since (1.11.1) represents a pair of straight lines, then by (1.11.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \quad (1.11.7)$$

By solving, above determinant we get

$$2(9 - 9) + \frac{-1}{2} \left( \frac{-9}{2} + \frac{-3k}{2} \right) + \frac{k}{2} \left( \frac{3}{2} + \frac{k}{2} \right) = 0 \quad (1.11.8)$$

$$\frac{(9 + 3k)}{4} + \frac{k(3 + k)}{4} = 0 \quad (1.11.9)$$

$$k^2 + 6k + 9 = 0 \quad (1.11.10)$$

$$(k + 3)^2 = 0 \quad (1.11.11)$$

$$k = -3 \quad (1.11.12)$$

Hence by (1.11.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \quad (1.11.13)$$

represents family of straight lines for  $k = -3$ . To find the straight lines, we write each of them in their vector form as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.11.14)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.11.15)$$

Equating the product of above with (1.11.2),

we have

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 \quad (1.11.16)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad (1.11.17)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \quad (1.11.18)$$

$$c_1 c_2 = -9 \quad (1.11.19)$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \quad (1.11.20)$$

$$m^2 - m - 2 = 0 \quad (1.11.21)$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} \quad (1.11.22)$$

$$m_1 = \frac{1+3}{2} = 2 \quad (1.11.23)$$

$$m_2 = \frac{1-3}{2} = -1 \quad (1.11.24)$$

$$n_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.11.25)$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.11.26)$$

Substituting (1.11.25) and (1.11.26) in (1.11.17), we get

$$k_1 k_2 = -1 \quad (1.11.27)$$

Taking  $k_1 = -1$  and  $k_2 = 1$ , we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.11.28)$$

$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.11.29)$$

Substituting in (1.11.18) for above values of  $n_1$  and  $n_2$

$$(n_1 n_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (1.11.30)$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (1.11.31)$$

Solving (1.11.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xleftrightarrow{r_2=r_2+2r_1} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \quad (1.11.32)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xleftrightarrow{r_2=r_2/3} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad (1.11.33)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \quad (1.11.34)$$

Hence, we found out

$$c_1 = -3 \quad (1.11.35)$$

$$c_2 = 3 \quad (1.11.36)$$

Thus, pair of straight lines are

$$(2 \ -1)\mathbf{x} = -3 \quad (1.11.37)$$

$$(1 \ 1)\mathbf{x} = 3 \quad (1.11.38)$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.11.39)$$

The plot of above is shown below

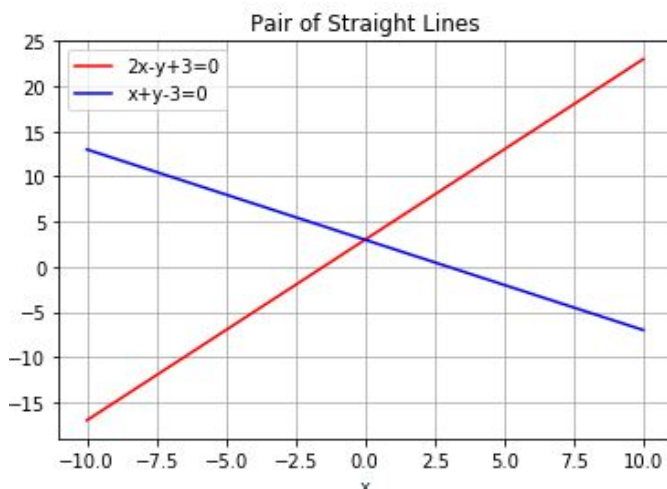


Fig. 1.11.1: Pair of Straight Lines

## 2 GENERAL EQUATION. TRACING OF CURVES

2.1. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre

$$12x^2 - 23xy + 10y^2 - 25x + 26y = 14 \quad (2.1.1)$$

**Solution:** The given equation (2.1.1) can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{-25}{2} & 13 \end{pmatrix} \mathbf{x} - 14 = 0 \quad (2.1.2)$$

where

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \quad (2.1.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \quad (2.1.4)$$

$$f = -14 \quad (2.1.5)$$

$$\det(\mathbf{V}) = \begin{vmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{vmatrix} \quad (2.1.6)$$

$$\Rightarrow \det(\mathbf{V}) = \frac{-49}{4} \quad (2.1.7)$$

$$\Rightarrow \det(\mathbf{V}) < 0 \quad (2.1.8)$$

Since  $\det(\mathbf{V}) < 0$  the given equation (2.1.2) represents the hyperbola. The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.1.9)$$

$$\begin{vmatrix} 12 - \lambda & \frac{-23}{2} \\ \frac{-23}{2} & 10 - \lambda \end{vmatrix} = 0 \quad (2.1.10)$$

$$\Rightarrow 4\lambda^2 - 88\lambda - 49 = 0 \quad (2.1.11)$$

The eigenvalues are the roots of equation 2.1.11 is given by

$$\lambda_1 = \frac{22 + \sqrt{533}}{2} \quad (2.1.12)$$

$$\lambda_2 = \frac{22 - \sqrt{533}}{2} \quad (2.1.13)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.1.14)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.1.15)$$

For  $\lambda_1 = \frac{22-\sqrt{533}}{2}$ ,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (2.1.16)$$

By row reduction ,

$$\begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (2.1.17)$$

$$\xleftarrow{R_1 = \frac{R_1}{\frac{\sqrt{533}+2}{2}}} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (2.1.18)$$

$$\xleftarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \quad (2.1.19)$$

Substituting equation 2.1.19 in equation 2.1.15 we get

$$\begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.20)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let  $v_2 = t$

$$v_1 = \frac{-t(2 - \sqrt{533})}{23} \quad (2.1.21)$$

Eigen vector  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} \frac{-t(2-\sqrt{533})}{23} \\ t \end{pmatrix} \quad (2.1.22)$$

Let  $t = 1$ , we get

$$\mathbf{p}_1 = \begin{pmatrix} \frac{\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \quad (2.1.23)$$

For  $\lambda_2 = \frac{22+\sqrt{533}}{2}$ ,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (2.1.24)$$

By row reduction ,

$$\begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (2.1.25)$$

$$\xleftarrow{R_1 = \frac{R_1}{\frac{2-\sqrt{533}}{2}}} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (2.1.26)$$

$$\xleftarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \quad (2.1.27)$$

Substituting equation 2.1.27 in equation 2.1.15 we get

$$\begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.28)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let  $v_2 = t$

$$v_1 = \frac{-t(2 + \sqrt{533})}{23} \quad (2.1.29)$$

Eigen vector  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-t(2+\sqrt{533})}{23} \\ t \end{pmatrix} \quad (2.1.30)$$

Let  $t = 1$ , we get

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \quad (2.1.31)$$

By eigen decomposition  $\mathbf{V}$  can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (2.1.32)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.1.33)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.1.34)$$

Substituting equations 2.1.23, 2.1.31 in equation 2.1.33 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & \frac{-\sqrt{533}-2}{23} \\ 1 & 1 \end{pmatrix} \quad (2.1.35)$$

Substituting equations 2.1.12, 2.1.13 in 2.1.34 we get

$$\mathbf{D} = \begin{pmatrix} \frac{22-\sqrt{533}}{2} & 0 \\ 0 & \frac{22+\sqrt{533}}{2} \end{pmatrix} \quad (2.1.36)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (2.1.37)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{-40}{49} & \frac{-46}{49} \\ \frac{-46}{49} & \frac{-48}{49} \end{pmatrix} \begin{pmatrix} -25 \\ 13 \end{pmatrix} \quad (2.1.38)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{40}{49} & \frac{46}{49} \\ \frac{46}{49} & \frac{48}{49} \end{pmatrix} \begin{pmatrix} -25 \\ 13 \end{pmatrix} \quad (2.1.39)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.1.40)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 26 > 0 \quad (2.1.41)$$

there isn't a need to swap axes  
In hyperbola,

$$\text{axes} = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (2.1.42)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{2\sqrt{13}}{\sqrt{22 + \sqrt{533}}} \quad (2.1.43)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \frac{2\sqrt{13}}{\sqrt{\sqrt{533} - 22}} \quad (2.1.44)$$

Now (2.1.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.1.45)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.1.46)$$

To get  $\mathbf{y}$ ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (2.1.47)$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.1.48)$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{2(\sqrt{533}-2)}{23} + 1 \\ \frac{2(-\sqrt{533}-2)}{23} + 1 \end{pmatrix} \quad (2.1.49)$$

Substituting the equations (2.1.41), (2.1.36) in equation (2.1.45)

$$\mathbf{y}^T \begin{pmatrix} \frac{22+\sqrt{533}}{2} & 0 \\ 0 & \frac{22-\sqrt{533}}{2} \end{pmatrix} \mathbf{y} - 26 = 0 \quad (2.1.50)$$

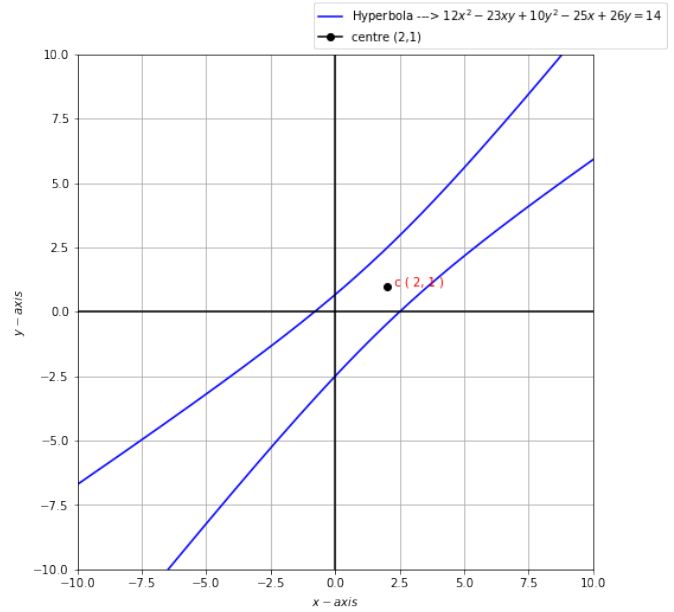


Fig. 2.1.1: Hyperbola when origin is shifted

The figure 2.1.1 verifies the given equation (2.1.2) as hyperbola with centre  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2.2. What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0 \quad (2.2.1)$$

Find the center.

**Solution:** The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.2.2)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \quad (2.2.3)$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (2.2.4)$$

$$f = -2 \quad (2.2.5)$$

Expanding the determinant of  $\mathbf{V}$  we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \quad (2.2.6)$$

Hence from (2.2.6) we conclude that given equation is an ellipse. The characteristic equa-

tion of  $\mathbf{V}$  is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \quad (2.2.7)$$

$$\Rightarrow \lambda^2 - 50\lambda + 400 = 0 \quad (2.2.8)$$

Hence the characteristic equation of  $\mathbf{V}$  is given by (2.2.8). The roots of (2.2.8) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \quad (2.2.9)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.2.10)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (2.2.11)$$

for  $\lambda_1 = 10$ ,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \quad (2.2.12)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.2.13)$$

Again, for  $\lambda_2 = 40$ ,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \quad (2.2.14)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (2.2.15)$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \quad (2.2.16)$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \quad (2.2.17)$$

Now (2.2.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (2.2.18)$$

And,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (2.2.19)$$

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.2.20)$$

The centre/vertex of the conic section in (2.2.2) is given by  $\mathbf{c}$  in (2.2.19). We compute  $\mathbf{V}^{-1}$  as

follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.21)$$

$$\xrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.22)$$

Hence  $\mathbf{V}^{-1}$  is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.23)$$

Now  $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}$  is given by,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \quad (2.2.24)$$

And,  $\mathbf{V}^{-1} \mathbf{u}$  is given by,

$$\mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.2.25)$$

By putting the value of (2.2.25), the center of the ellipse is given by (2.2.19) as follows,

$$\mathbf{c} = -\frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \quad (2.2.26)$$

Also the semi-major axis ( $a$ ) and semi-minor axis ( $b$ ) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \quad (2.2.27)$$

$$b = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \quad (2.2.28)$$

Finally from (2.2.18), the equation of ellipse is given by,

$$\mathbf{y}^T \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \mathbf{y} = 4 \quad (2.2.29)$$

The following figure 2.2.1 is the graphical representation of the ellipse in (2.2.29),

2.3. What conic does the given equations represent?

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \quad (2.3.1)$$

**Solution:** The above equation can be expressed

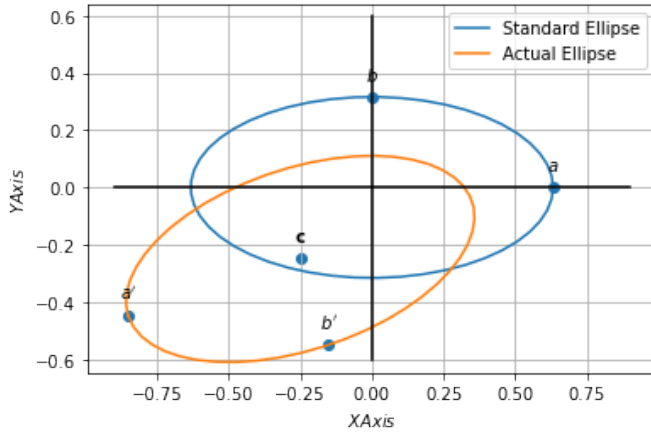


Fig. 2.2.1: Graphical representation of the ellipse

in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.3.2)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{pmatrix} \quad (2.3.3)$$

$$\mathbf{u} = \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (2.3.4)$$

$$f = 4 \quad (2.3.5)$$

The above equation (2.3.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.3.6)$$

Verify the given equation as if it is pair of straight lines

$$\Delta = \begin{vmatrix} 6 & \frac{-5}{2} & 7 \\ \frac{-5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix} \quad (2.3.7)$$

$$\Rightarrow 6 \begin{vmatrix} -6 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{vmatrix} - \frac{-5}{2} \begin{vmatrix} -\frac{5}{2} & \frac{5}{2} \\ 7 & 4 \end{vmatrix} + 7 \begin{vmatrix} -\frac{5}{2} & -6 \\ 7 & \frac{5}{2} \end{vmatrix} = 0 \quad (2.3.8)$$

$$\Rightarrow \Delta = 0 \quad (2.3.9)$$

Since equation (2.3.6) is satisfied, we could say that the given equation represents two straight lines

$$\Delta_V = \begin{vmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{vmatrix} < 0 \quad (2.3.10)$$

Let the equations of lines be,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_1^T \mathbf{x} - c_1) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.3.11)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 7 & \frac{5}{2} \end{pmatrix} \mathbf{x} + 4 \quad (2.3.12)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} \quad (2.3.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (2.3.14)$$

$$c_1 c_2 = 4 \quad (2.3.15)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.3.16)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (2.3.17)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.3.18)$$

Substituting the given data in above equations (2.3.16) we get,

$$-6m^2 - 5m + 6 = 0 \quad (2.3.19)$$

$$\Rightarrow m_i = \frac{\frac{-5}{2} \pm \sqrt{-\left(\frac{-169}{4}\right)}}{-6} \quad (2.3.20)$$

Solving equation (2.3.20) we get,

$$m_1 = -\frac{3}{2}, m_2 = \frac{2}{3} \quad (2.3.21)$$

$$= \mathbf{n}_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.3.22)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.3.23)$$

Verification using Toeplitz matrix, From equa-

tion (2.3.22)

$$\mathbf{n}_1 = \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.3.24)$$

$$\Rightarrow \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.3.25)$$

$\Rightarrow$  Equation (2.3.22) satisfies (2.3.23)  
 $c_1$  and  $c_2$  can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.3.26)$$

Substituting (2.3.22) in (2.3.26), the augmented matrix is,

$$\begin{pmatrix} -3 & -2 & 14 \\ -2 & 3 & 5 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + 2R_1]{R_1 \leftarrow -R_1/3} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{14}{3} \\ 0 & \frac{13}{3} & -\frac{13}{3} \end{pmatrix} \quad (2.3.27)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{2}{3}R_2]{R_2 \leftarrow \frac{3}{13}R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \end{pmatrix} \quad (2.3.28)$$

$$\Rightarrow c_1 = -4, c_2 = -1 \quad (2.3.29)$$

Equations (2.3.11), can be modified as, from (2.3.22) and (2.3.29) in we get,

$$\begin{pmatrix} -3 & -2 \end{pmatrix} \mathbf{x} = -4 \quad (2.3.30)$$

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = -1 \quad (2.3.31)$$

$$\Rightarrow (-3x - 2y + 4)(-2x + 3y + 1) = 0$$

$$\Rightarrow \boxed{(3x + 2y - 4)(2x - 3y - 1) = 0} \quad (2.3.32)$$

The angle between the lines can be expressed as,

$$\mathbf{n}_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.3.33)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (2.3.34)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{0}{\sqrt{169}}\right) = 90^\circ. \quad (2.3.35)$$

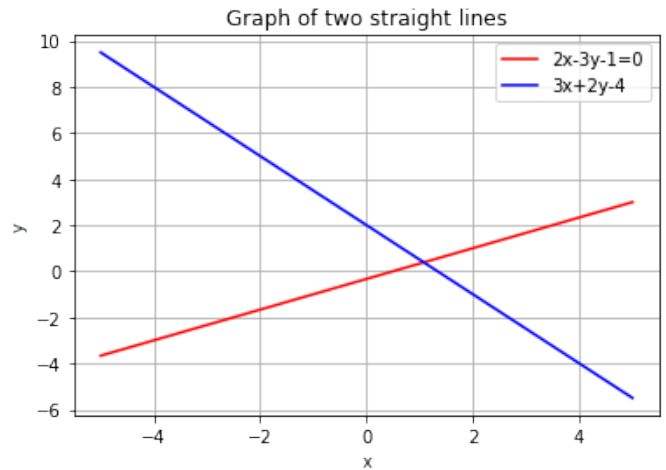


Fig. 2.3.1: Pair of straight lines