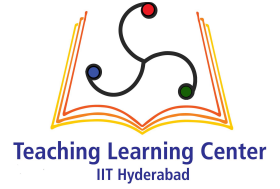




Coordinate Geometry Exercises



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CONTENTS

Abstract—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.

1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T \mathbf{x} = 4$ and $\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.1)$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (1.2)$$

$$\mathbf{x}^T \mathbf{x} - 4 = 0 \quad (1.3)$$

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.4)$$

$$f_1 = -4 \quad (1.5)$$

$$\mathbf{O}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.6)$$

Taking equation of the second circle to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\|^2 = 2^2 \quad (1.7)$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}_2^T \mathbf{x} = 0 \quad (1.8)$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.9)$$

$$f_2 = 0 \quad (1.10)$$

$$\mathbf{O}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (1.11)$$

Now, Subtracting equation (1.8) from (1.3) We get,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{u}_2^T \mathbf{x} + f_1 - \mathbf{x}^T \mathbf{x} = 0 \quad (1.12)$$

$$2\mathbf{u}_2^T \mathbf{x} = -4 \quad (1.13)$$

$$\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} = -4 \quad (1.14)$$

Which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \quad (1.15)$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.16)$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (1.17)$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.18)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.19)$$

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Substituting (1.17) in (1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (1.20)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 + f_1 = 0 \quad (1.21)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (1.22)$$

$$\mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (1.23)$$

$$\|\mathbf{q}\|^2 + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (1.24)$$

$$\|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (1.25)$$

$$\lambda(\lambda \|\mathbf{m}\|^2 + 2\mathbf{q}^T \mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \quad (1.26)$$

$$\lambda^2 \|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \quad (1.27)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.28)$$

$$\lambda^2 = 3 \quad (1.29)$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \quad (1.30)$$

Substituting the value of λ in (1.17)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (1.31)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (1.32)$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (1.33)$$

Now finding the direction vector \mathbf{m}_{O_1A} , \mathbf{m}_{O_1B} , \mathbf{m}_{O_2A} and \mathbf{m}_{O_2B} .

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \quad (1.34)$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad (1.35)$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (1.36)$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (1.37)$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B} = \|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\| \cos \theta_1 \quad (1.38)$$

$$\frac{\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B}}{\|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\|} = \cos \theta_1 \quad (1.39)$$

$$\frac{-2}{4} = \cos \theta_1 \quad (1.40)$$

$$\frac{-1}{2} = \cos \theta_1 \quad (1.41)$$

$$\theta_1 = 120^\circ \quad (1.42)$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B} = \|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\| \cos \theta_2 \quad (1.43)$$

$$\frac{\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B}}{\|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\|} = \cos \theta_2 \quad (1.44)$$

$$\frac{-2}{4} = \cos \theta_2 \quad (1.45)$$

$$\frac{-1}{2} = \cos \theta_2 \quad (1.46)$$

$$\theta_2 = 120^\circ \quad (1.47)$$

Finding area of $\mathbf{O}_1\mathbf{AB}$ and $\mathbf{O}_2\mathbf{AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2\sqrt{3} \quad (1.48)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \quad (1.49)$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360} r^2 - \frac{1}{2} 2\sqrt{3} \quad (1.50)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \quad (1.51)$$

Area of $\mathbf{O}_1\mathbf{AO}_2\mathbf{B}$

$$A_{O_1AO_2B} = \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} + \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \quad (1.52)$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \quad (1.53)$$

2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre(h,k) is given by,

$$x^T x + 2u^T x + f = 0 \quad (2.1)$$

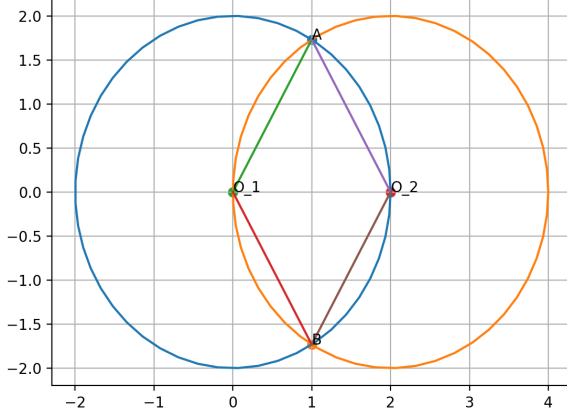


Fig. 1: Figure depicting intersection points of circle

where,

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (2.2)$$

The radius and centre are respectively given by,

$$r = 5 \quad (2.3)$$

$$\mathbf{c} = -\mathbf{u} = k\mathbf{e} \quad (2.4)$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.5)$$

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (2.6)$$

From the given data , we modify equation 2.1 as,

$$\mathbf{x}_1^T \mathbf{x}_1 + 2 \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \quad (2.7)$$

$$\|\mathbf{x}_1\|^2 + 2(k^2) + f = 0 \quad (2.8)$$

$$2k^2 + f = -\|\mathbf{x}_1\|^2 \quad (2.9)$$

Substituting \mathbf{u} in equation 2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \quad (2.10)$$

$$f = (k^2) - r^2 \quad (2.11)$$

$$k^2 - f = r^2 \quad (2.12)$$

From equations 2.9 and 2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x}_1\|^2 \\ r^2 \end{pmatrix} \quad (2.13)$$

Here $\|\mathbf{x}_1\|$ is given by ,

$$\|\mathbf{x}_1\| = \sqrt{2^2 + 3^2} \quad (2.14)$$

$$\|\mathbf{x}_1\| = \sqrt{13} \quad (2.15)$$

Substituting equation 2.6, 2.3 in equation 2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \quad (2.16)$$

The augmented matrix of 2.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \quad (2.17)$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \quad (2.18)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \quad (2.19)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \xrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \quad (2.20)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \xrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array} \right) \quad (2.21)$$

Equation 2.16 can be rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \quad (2.22)$$

Expanding the above equation 2.22 we get ,

$$k^2 = 4 \quad (2.23)$$

$$k = \pm 2 \quad (2.24)$$

$$f = -21 \quad (2.25)$$

To get the centre substitute equation 2.24 in equation 2.4 To verify the above results we plot the circle with centre \mathbf{c} as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$,

From the above figure 1 it is clear that circle with centre $\mathbf{c} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ passes through the point \mathbf{x}_1

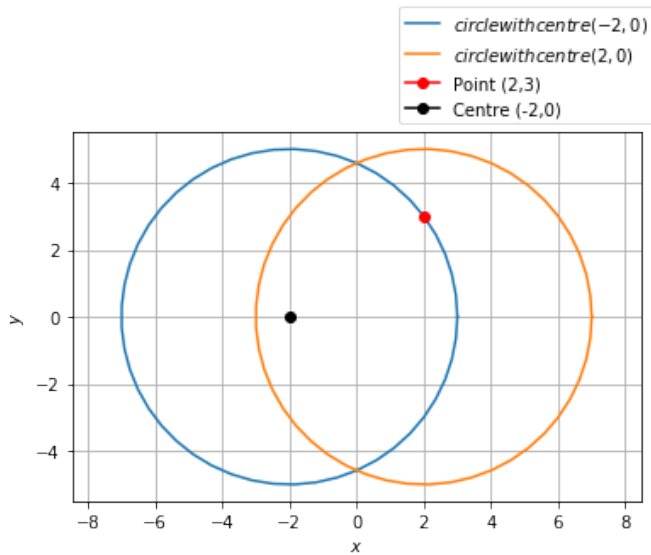


Fig. 1: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

Desired equation of circle is given by ,

$$c = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (2.26)$$

$$f = -21 \quad (2.27)$$

3. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.
4. Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.
5. Find the locus of all the unit vectors in the xy-plane.
6. Find the points on the curve $\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 0$ and the circle $\mathbf{x}^T \mathbf{x} = 4$.
8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
9. Find the area of the circle $4\mathbf{x}^T \mathbf{x} = 9$.
10. Find the area bounded by curves $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$ and $\|\mathbf{x}\| = 1$
11. Find the smaller area enclosed by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$.
12. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
13. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.
14. Find the equation of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}$, $x \neq 1$
15. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}$, $x \neq 3$.
16. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at which tangents are
 - a) parallel to x-axis
 - b) parallel to y-axis.
17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$
 - a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$
 - b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.
19. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0$.
20. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.
21. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m .
22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$
23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis in the first quadrant.
25. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.
26. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.
27. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
28. Find the area of the region bounded by the

$$\text{ellipse } \mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$$

the parabola $y^2 = 6$.

29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a .
30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
31. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.
32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T \mathbf{x} - 8 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0$ and inside of the parabola $y^2 = 4x$.
36. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB .
37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
40. Find the area between $y = x^2$ and $y = x$.
41. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.
43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$
44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.
45. Find the area bounded by the curves

$$\{(x, y) : y > x^2, y = |x|\} \quad (45.1)$$
46. Find the area of the region

$$\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T \mathbf{x} = 9\} \quad (46.1)$$
47. Find the area of the circle $\mathbf{x}^T \mathbf{x} = 16$ exterior to