Coordinate Geometry Exercises

G V V Sharma*

Contents

Abstract—**This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.**

1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\lVert \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\rVert = 2$.

   Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.1}$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.2}$$

$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.3}$$

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.4}$$

$$f_1 = -4 \tag{1.5}$$

$$\mathbf{O_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.6}$$

Taking equation of the second circle to be,

$$\left\lVert \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\rVert^2 = 2^2 \tag{1.7}$$

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{1.8}$$

$$\mathbf{u_2} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.9}$$

$$f_2 = 0 \tag{1.10}$$

$$\mathbf{O_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.11}$$

Now,Subtracting equation (1.8) from (1.3) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{1.12}$$

$$2\mathbf{u}^T\mathbf{x} = -4 \tag{1.13}$$

$$\begin{pmatrix} -4 & 0 \end{pmatrix}\mathbf{x} = -4 \tag{1.14}$$

Which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 1 \tag{1.15}$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.17}$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.18}$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.19}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

Substituting (1.17) in (1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.20}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{1.21}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.22}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.23}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.24}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.25}$$

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \tag{1.26}$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \tag{1.27}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.28}$$

$$\lambda^2 = 3 \tag{1.29}$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{1.30}$$

Substituting the value of λ in(1.17)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.31}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.32}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{1.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \tag{1.34}$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \tag{1.35}$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{1.36}$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \tag{1.38}$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.41}$$

$$\theta_1 = 120^\circ \tag{1.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \tag{1.43}$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.46}$$

$$\theta_2 = 120^\circ \tag{1.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.48}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.50}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.52}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \tag{1.53}$$

2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
Solution:
Equation of the circle with radius r and centre(h,k) is given by,

$$x^T x + 2u^T x + f = 0 \tag{2.1}$$

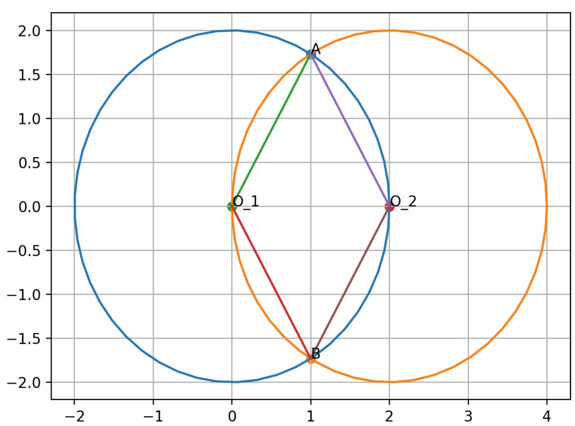

Fig. 1: Figure depicting intersection points of circle

where,

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.2}$$

The radius and centre are respectively given by,

$$r = 5 \tag{2.3}$$

$$\mathbf{c} = -u = k\mathbf{e} \tag{2.4}$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.5}$$

$$\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.6}$$

From the given data , we modify equation 2.1 as,

$$\mathbf{x_1}^T\mathbf{x_1} + 2\begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \tag{2.7}$$

$$\|\mathbf{x_1}\|^2 + 2\left(k^2\right) + f = 0 \tag{2.8}$$

$$2k^2 + f = -\|\mathbf{x_1}\|^2 \tag{2.9}$$

Subsituting $\mathbf{u}$ in equation 2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \tag{2.10}$$

$$f = \left(k^2\right) - r^2 \tag{2.11}$$

$$k^2 - f = r^2 \tag{2.12}$$

From equations 2.9 and 2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x_1}\|^2 \\ r^2 \end{pmatrix} \tag{2.13}$$

Here ,$\|\mathbf{x_1}\|$ is given by ,

$$\|\mathbf{x_1}\| = \sqrt{2^2 + 3^2} \tag{2.14}$$

$$\|\mathbf{x_1}\| = \sqrt{13} \tag{2.15}$$

Subsituting equation 2.6,2.3 in equation 2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \tag{2.16}$$

The augumented matrix of 2.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \tag{2.17}$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \tag{2.18}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \tag{2.19}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \tag{2.20}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array}\right) \tag{2.21}$$

Equation 2.16 can we rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \tag{2.22}$$

Expanding the above equation 2.22 we get ,

$$k^2 = 4 \tag{2.23}$$

$$k = \pm 2 \tag{2.24}$$

$$f = -21 \tag{2.25}$$

To get the centre substitute equation 2.24 in equation 2.4 To verify the above results we plot the circle with centre $\mathbf{c}$ as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, qFrom the above figure 1 it is clear that circle with centre $\mathbf{c}$=$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ passes through the point $\mathbf{x_1}$

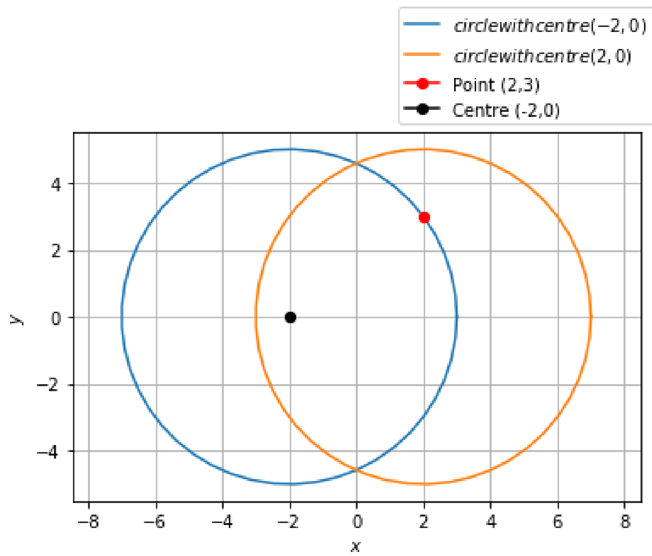

Fig. 1: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

Desired equation of circle is given by ,

$$c = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{2.26}$$

$$f = -21 \tag{2.27}$$

3. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.
4. Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.
5. Find the locus of all the unit vectors in the xy-plane.
6. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^T\mathbf{x} = 4$.
8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
9. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.
10. Find the area bounded by curves $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$ and $\|\mathbf{x}\| = 1$
11. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2$.
12. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.
13. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.
14. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$
15. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.
16. Find points on the curve $\mathbf{x}^T\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\mathbf{x} = 1$ at which tangents are
    a) parallel to x-axis
    b) parallel to y-axis.
17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$
    a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = -9$
    b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix}\mathbf{x} = 13$.
19. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix}\mathbf{x} + 5 = 0$.
20. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.
21. The line $\begin{pmatrix} -m & 1 \end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.
22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$
23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.
25. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.
26. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.
27. Find the area of the region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}\mathbf{x} = 1$
28. Find the area of the region bounded by the

ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
31. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.
32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.
36. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.
37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
40. Find the area between $y = x^2$ and $y = x$.
41. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.
43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$
44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.
45. Find the area bounded by the curves
$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{45.1}$$
46. Find the area of the region
$$\left\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{46.1}$$
47. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.