



## Coordinate Geometry Exercises



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**Abstract**—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.

### 1 POINTS AND VECTORS

#### 1.1 Triangle

1.1.1. Construct a triangle of sides  $a = 4$ ,  $b = 5$  and  $c = 6$ .

- 1.1.2. Construct an isosceles triangle whose base is  $a = 8\text{cm}$  and altitude  $AD = h = 4\text{cm}$
- 1.1.3. In  $\triangle ABC$ , given that  $a+b+c = 11$ ,  $\angle B = 45^\circ$  and  $\angle C = 45^\circ$ , find  $a, b, c$  and sketch the triangle.
- 1.1.4. Draw  $\triangle ABC$  with  $a = 6$ ,  $c = 5$  and  $\angle B = 60^\circ$ .
- 1.1.5. Draw  $\triangle ABC$  with  $a = 7$ ,  $\angle B = 45^\circ$  and  $\angle A = 105^\circ$ .
- 1.1.6.  $\triangle ABC$  is right angled at **B**. If  $a = 12$  and  $b+c = 18$ , find  $b, c$  and draw the triangle.
- 1.1.7. In  $\triangle ABC$ ,  $a = 8$ ,  $\angle B = 45^\circ$  and  $c - b = 3.5$ . Sketch  $\triangle ABC$ .
- 1.1.8. In  $\triangle ABC$ ,  $a = 6$ ,  $\angle B = 60^\circ$  and  $b - c = 2$ . Sketch  $\triangle ABC$ .
- 1.1.9. Draw  $\triangle ABC$ , given that  $a+b+c = 11$ ,  $\angle B = 30^\circ$  and  $\angle C = 90^\circ$ .
- 1.1.10. Construct  $\triangle xyz$  where  $xy = 4.5$ ,  $yz = 5$  and  $zx = 6$ .
- 1.1.11. Draw an equilateral triangle of side 5.5.
- 1.1.12. Draw  $\triangle PQR$  with  $PQ = 4$ ,  $QR = 3.5$  and  $PR = 4$ . What type of triangle is this?
- 1.1.13. Construct  $\triangle ABC$  such that  $AB = 2.5$ ,  $BC = 6$  and  $AC = 6.5$ . Find  $\angle B$ .
- 1.1.14. Construct  $\triangle PQR$ , given that  $PQ = 3$ ,  $QR = 5.5$  and  $\angle PQR = 60^\circ$ .
- 1.1.15. Construct  $\triangle DEF$  such that  $DE = 5$ ,  $DF = 3$  and  $\angle D = 90^\circ$ .
- 1.1.16. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is  $110^\circ$ .
- 1.1.17. Construct  $\triangle ABC$  with  $BC = 7.5$ ,  $AC = 5$  and  $\angle C = 60^\circ$ .

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- 1.1.18. Construct  $\triangle XYZ$  if  $XY = 6$ ,  $\angle X = 30^\circ$  and  $\angle Y = 100^\circ$ .
- 1.1.19. If  $AC = 7$ ,  $\angle A = 60^\circ$  and  $\angle B = 50^\circ$ , can you draw the triangle?
- 1.1.20. Construct  $\triangle ABC$  given that  $\angle A = 60^\circ$ ,  $\angle B = 30^\circ$  and  $AB = 5.8$ .
- 1.1.21. Construct  $\triangle PQR$  if  $PQ = 5$ ,  $\angle Q = 105^\circ$  and  $\angle R = 40^\circ$ .
- 1.1.22. Can you construct  $\triangle DEF$  such that  $EF = 7.2$ ,  $\angle E = 110^\circ$  and  $\angle F = 180^\circ$ ?
- 1.1.23. Construct  $\triangle LMN$  right angled at  $M$  such that  $LN = 5$  and  $MN = 3$ .
- 1.1.24. Construct  $\triangle PQR$  right angled at  $Q$  such that  $QR = 8$  and  $PR = 10$ .
- 1.1.25. Construct right angled  $\triangle$  whose hypotenuse is 6 and one of the legs is 4.
- 1.1.26. Construct an isosceles right angled  $\triangle ABC$  right angled at  $C$  such  $AC = 6$ .
- 1.1.27. Construct the triangles in Table 1.1.27.
- 1.2.7. Draw  $GOLD$  such that  $OL = 7.5$ ,  $GL = 6$ ,  $GD = 6$ ,  $LD = 5$ ,  $OD = 10$ .
- 1.2.8. DRAW rhombus  $BEND$  such that  $BN = 5.6$ ,  $DE = 6.5$ .
- 1.2.9. construct a quadrilateral  $MIST$  where  $MI = 3.5$ ,  $IS = 6.5$ ,  $\angle M = 75^\circ$ ,  $\angle I = 105^\circ$  and  $\angle S = 120^\circ$ .
- 1.2.10. Can you construct the above quadrilateral  $MIST$  if  $\angle M = 100^\circ$  instead of  $75^\circ$ ?
- 1.2.11. Can you construct the quadrilateral  $PLAN$  if  $PL = 6$ ,  $LA = 9.5$ ,  $\angle P = 75^\circ$ ,  $\angle L = 150^\circ$  and  $\angle A = 140^\circ$ ?
- 1.2.12. Construct  $MORE$  where  $MO = 6$ ,  $OR = 4.5$ ,  $\angle M = 60^\circ$ ,  $\angle O = 105^\circ$ ,  $\angle R = 105^\circ$ .
- 1.2.13. Construct  $PLAN$  where  $PL = 4$ ,  $LA = 6.5$ ,  $\angle P = 90^\circ$ ,  $\angle A = 110^\circ$  and  $\angle N = 85^\circ$ .
- 1.2.14. Draw rectangle  $OKAY$  with  $OK = 7$  and  $KA = 5$ .
- 1.2.15. Construct  $ABCD$ , where  $AB = 4$ ,  $BC = 5$ ,  $CD = 6.5$ ,  $\angle B = 105^\circ$  and  $\angle C = 80^\circ$ .
- 1.2.16. Construct  $DEAR$  with  $DE = 4$ ,  $EA = 5$ ,  $AR = 4.5$ ,  $\angle E = 60^\circ$  and  $\angle A = 90^\circ$ .
- 1.2.17. Construct  $TRUE$  with  $TR = 3.5$ ,  $RU = 3$ ,  $UE = 4$ ,  $\angle R = 75^\circ$  and  $\angle U = 120^\circ$ .
- 1.2.18. Can you construct a rhombus  $ABCD$  with  $AC = 6$  and  $BD = 7$ ?
- 1.2.19. Draw a square  $READ$  with  $RE = 5.1$ .
- 1.2.20. Draw a rhombus whose diagonals are 5.2 and 6.4.
- 1.2.21. Draw a rectangle with adjacent sides 5 and 4.
- 1.2.22. Draw a parallelogram  $OKAY$  with  $OK = 5.5$  and  $KA = 4.2$ .
- 1.2.23. Construct a kite  $EASY$  if  $AY = 8$ ,  $EY = 4$  and  $SY = 6$ .

S.No	Triangle	Given Measurements		
1	$\triangle ABC$	$\angle A = 85^\circ$	$\angle B = 115^\circ$	$AB = 5$
2	$\triangle PQR$	$\angle Q = 30^\circ$	$\angle R = 60^\circ$	$QR = 4.7$
3	$\triangle ABC$	$\angle A = 70^\circ$	$\angle B = 50^\circ$	$AC = 3$
4	$\triangle LMN$	$\angle L = 60^\circ$	$\angle N = 120^\circ$	$LM = 5$
5	$\triangle ABC$	$BC = 2$	$AB = 4$	$AC = 2$
6	$\triangle PQR$	$PQ = 2.5$	$QR = 4$	$PR = 3.5$
7	$\triangle XYZ$	$XY = 3$	$YZ = 4$	$XZ = 5$
8	$\triangle DEF$	$DE = 4.5$	$EF = 5.5$	$DF = 4$

TABLE 1.1.27

## 1.2 Quadrilateral

- 1.2.1. Construct a quadrilateral  $ABCD$  such that  $AB = 5$ ,  $\angle A = 50^\circ$ ,  $AC = 4$ ,  $BD = 5$  and  $AD = 6$ .
- 1.2.2. Construct  $PQRS$  where  $PQ = 4$ ,  $QR = 6$ ,  $RS = 5$ ,  $PS = 5.5$  and  $PR = 7$ .
- 1.2.3. Draw  $JUMP$  with  $JU = 3.5$ ,  $UM = 4$ ,  $MP = 5$ ,  $PJ = 4.5$  and  $PU = 6.5$ .
- 1.2.4. Construct a quadrilateral  $ABCD$  such that  $BC = 4.5$ ,  $AC = 5.5$ ,  $CD = 5$ ,  $BD = 7$  and  $AD = 5.5$ .
- 1.2.5. Can you construct a quadrilateral  $PQRS$  with  $PQ = 3$ ,  $RS = 3$ ,  $PS = 7.5$ ,  $PR = 8$  and  $SQ = 4$ ?
- 1.2.6. Construct  $LIFT$  such that  $LI = 4$ ,  $IF = 3$ ,  $TL = 2.5$ ,  $LF = 4.5$ ,  $IT = 4$ .

## 2 LINEAR FORMS

## 3 QUADRATIC FORMS

### 3.1 Circle

- 3.1.1. Draw a circle of diameter 6.1
- 3.1.2. With the same centre **O**, draw two circles of radii 4 and 2.5
- 3.1.3. Draw a circle with centre **B** and radius 6. If **C** be a point 10 units away from its centre, construct the pair of tangents  $AC$  and  $CD$  to the circle.
- 3.1.4. Draw a circle of radius 3 and any two of its diameters. Draw the ends of these diameters. What figure do you get?

3.1.5. Let **A** and **B** be the centres of two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is  $AB \perp CD$ ?

3.1.6. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.

**Solution:** Take the centre of both circles to be at the origin.

3.1.7. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.

**Solution:** Take the diameter to be on the  $x$ -axis.

3.1.8. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of  $60^\circ$ .

**Solution:** The tangent is perpendicular to the radius.

3.1.9. Draw a line segment  $AB$  of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (3.1.9.1)$$

3.1.10. Let  $ABC$  be a right triangle in which  $a = 8, c = 6$  and  $\angle B = 90^\circ$ .  $BD$  is the perpendicular from **B** on  $AC$  (altitude). The circle through **B, C, D** (circumcircle of  $\triangle BCD$ ) is drawn. Construct the tangents from **A** to this circle.

3.1.11. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (3.2.1.2)$$

$$\mathbf{x}^T \mathbf{x} - 4 = 0 \quad (3.2.1.3)$$

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.1.4)$$

$$f_1 = -4 \quad (3.2.1.5)$$

$$\mathbf{O}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.1.6)$$

Taking equation of the second circle to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\|^2 = 2^2 \quad (3.2.1.7)$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}_2^T \mathbf{x} = 0 \quad (3.2.1.8)$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (3.2.1.9)$$

$$f_2 = 0 \quad (3.2.1.10)$$

$$\mathbf{O}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.2.1.11)$$

Now, Subtracting equation (3.2.1.8) from (3.2.1.3) We get,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{u}_2^T \mathbf{x} + f_1 - \mathbf{x}^T \mathbf{x} = 0 \quad (3.2.1.12)$$

$$2\mathbf{u}_2^T \mathbf{x} = -4 \quad (3.2.1.13)$$

$$\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} = -4 \quad (3.2.1.14)$$

Which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \quad (3.2.1.15)$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.1.16)$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (3.2.1.17)$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.1.18)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.1.19)$$

## 3.2 Conics

3.2.1. Find the area of the region enclosed between the two circles:  $\mathbf{x}^T \mathbf{x} = 4$  and  $\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\| = 2$ .

**Solution:** General equation of circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.2.1.1)$$

Substituting (3.2.1.17) in (3.2.1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (3.2.1.20)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 + f_1 = 0 \quad (3.2.1.21)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (3.2.1.22)$$

$$\mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (3.2.1.23)$$

$$\|\mathbf{q}\|^2 + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (3.2.1.24)$$

$$\|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (3.2.1.25)$$

$$\lambda(\lambda \|\mathbf{m}\|^2 + 2\mathbf{q}^T \mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \quad (3.2.1.26)$$

$$\lambda^2 \|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \quad (3.2.1.27)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (3.2.1.28)$$

$$\lambda^2 = 3 \quad (3.2.1.29)$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \quad (3.2.1.30)$$

Substituting the value of  $\lambda$  in (3.2.1.17)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (3.2.1.31)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (3.2.1.32)$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (3.2.1.33)$$

Now finding the direction vector  $\mathbf{m}_{O_1A}$ ,  $\mathbf{m}_{O_1B}$ ,  $\mathbf{m}_{O_2A}$  and  $\mathbf{m}_{O_2B}$ .

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \quad (3.2.1.34)$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad (3.2.1.35)$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (3.2.1.36)$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (3.2.1.37)$$

Now finding the angle  $\angle O_1AB$ .

$$\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B} = \|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\| \cos \theta_1 \quad (3.2.1.38)$$

$$\frac{\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B}}{\|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\|} = \cos \theta_1 \quad (3.2.1.39)$$

$$\frac{-2}{4} = \cos \theta_1 \quad (3.2.1.40)$$

$$\frac{-1}{2} = \cos \theta_1 \quad (3.2.1.41)$$

$$\theta_1 = 120^\circ \quad (3.2.1.42)$$

Now finding the angle  $\angle O_2AB$ .

$$\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B} = \|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\| \cos \theta_2 \quad (3.2.1.43)$$

$$\frac{\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B}}{\|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\|} = \cos \theta_2 \quad (3.2.1.44)$$

$$\frac{-2}{4} = \cos \theta_2 \quad (3.2.1.45)$$

$$\frac{-1}{2} = \cos \theta_2 \quad (3.2.1.46)$$

$$\theta_2 = 120^\circ \quad (3.2.1.47)$$

Finding area of  $\mathbf{O}_1\mathbf{AB}$  and  $\mathbf{O}_2\mathbf{AB}$ .

$$A_{O_1AB} = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2 \sqrt{3} \quad (3.2.1.48)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} \quad (3.2.1.49)$$

$$A_{O_2AB} = \frac{\pi \theta_2}{360} r^2 - \frac{1}{2} 2 \sqrt{3} \quad (3.2.1.50)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} \quad (3.2.1.51)$$

Area of  $\mathbf{O}_1\mathbf{AO}_2\mathbf{B}$

$$A_{O_1AO_2B} = \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} + \frac{120}{360} 4\pi - \frac{1}{2} 2 \sqrt{3} \quad (3.2.1.52)$$

$$= \frac{8\pi}{3} - 2 \sqrt{3} \quad (3.2.1.53)$$

3.2.2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .

**Solution:**

Equation of the circle with radius  $r$  and centre  $(h, k)$  is given by,

$$x^2 + y^2 + 2ux + 2vy + f = 0 \quad (3.2.2.1)$$

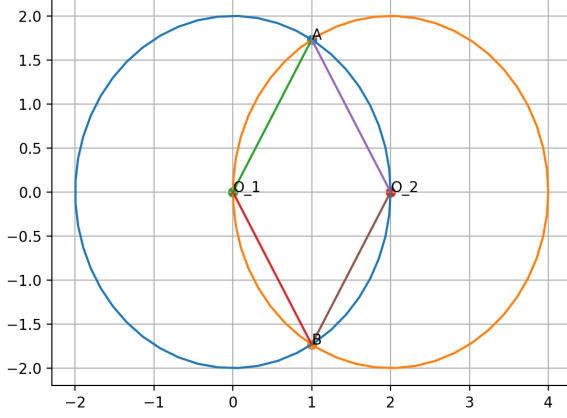


Fig. 3.2.1: Figure depicting intersection points of circle

where,

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (3.2.2.2)$$

The radius and centre are respectively given by,

$$r = 5 \quad (3.2.2.3)$$

$$\mathbf{c} = -\mathbf{u} = k\mathbf{e} \quad (3.2.2.4)$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.2.5)$$

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (3.2.2.6)$$

From the given data , we modify equation 3.2.2.1 as,

$$\mathbf{x}_1^T \mathbf{x}_1 + 2 \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \quad (3.2.2.7)$$

$$\|\mathbf{x}_1\|^2 + 2(k^2) + f = 0 \quad (3.2.2.8)$$

$$2k^2 + f = -\|\mathbf{x}_1\|^2 \quad (3.2.2.9)$$

Substituting  $\mathbf{u}$  in equation 3.2.2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \quad (3.2.2.10)$$

$$f = (k^2) - r^2 \quad (3.2.2.11)$$

$$k^2 - f = r^2 \quad (3.2.2.12)$$

From equations 3.2.2.9 and 3.2.2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x}_1\|^2 \\ r^2 \end{pmatrix} \quad (3.2.2.13)$$

Here  $\|\mathbf{x}_1\|$  is given by ,

$$\|\mathbf{x}_1\| = \sqrt{2^2 + 3^2} \quad (3.2.2.14)$$

$$\|\mathbf{x}_1\| = \sqrt{13} \quad (3.2.2.15)$$

Substituting equation 3.2.2.6, 3.2.2.3 in equation 3.2.2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \quad (3.2.2.16)$$

The augmented matrix of 3.2.2.16 is given by ,

$$\left( \begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \quad (3.2.2.17)$$

By using row reduction technique, we get ,

$$\left( \begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left( \begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \quad (3.2.2.18)$$

$$\left( \begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left( \begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \quad (3.2.2.19)$$

$$\left( \begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left( \begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \quad (3.2.2.20)$$

$$\left( \begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \xleftrightarrow{R_1 = R_1 + R_2} \left( \begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array} \right) \quad (3.2.2.21)$$

Equation 3.2.2.16 can be rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \quad (3.2.2.22)$$

Expanding the above equation 3.2.2.22 we get ,

$$k^2 = 4 \quad (3.2.2.23)$$

$$k = \pm 2 \quad (3.2.2.24)$$

$$f = -21 \quad (3.2.2.25)$$

To get the centre substitute equation 3.2.2.24 in equation 3.2.2.4 To verify the above results we plot the circle with centre  $\mathbf{c}$  as  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ , From the above figure 3.2.2 it is clear that

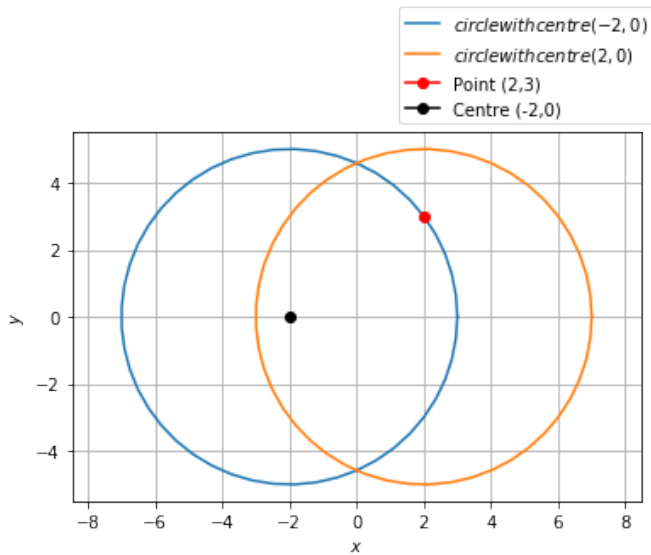


Fig. 3.2.2: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

circle with centre  $\mathbf{c} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$  passes through the point  $\mathbf{x}_1$

Desired equation of circle is given by ,

$$\mathbf{c} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (3.2.2.26)$$

$$f = -21 \quad (3.2.2.27)$$

3.2.3. Find the equation of the circle passing through  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and making intercepts a and b on the coordinate axes.

3.2.4. Find the equation of a circle with centre  $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

and passes through the point  $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ .

**Solution:** the general equation of a circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.2.4.1)$$

$$\text{If } r \text{ is radius, } f = \mathbf{u}^T \mathbf{u} - r^2 \quad (3.2.4.2)$$

$$\text{center } \mathbf{c} = -\mathbf{u} \quad (3.2.4.3)$$

Given centre is  $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (3.2.4.4)$$

$$\Rightarrow \mathbf{u} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (3.2.4.5)$$

Equation (3.2.4.1) becomes

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -4 & -4 \end{pmatrix} \mathbf{x} + f = 0 \quad (3.2.4.6)$$

This passes through point  $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Substituting  $\mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$  in (3.2.4.6)

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 & -4 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} + f = 0 \quad (3.2.4.7)$$

$$\Rightarrow f = -5 \quad (3.2.4.8)$$

Also, radius can be determined as follows

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (3.2.4.9)$$

$$\Rightarrow -5 = \begin{pmatrix} -2 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} - r^2 \quad (3.2.4.10)$$

$$\Rightarrow -5 = 8 - r^2 \quad (3.2.4.11)$$

$$\Rightarrow r = \sqrt{13} \quad (3.2.4.12)$$

The equation of required circle is

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -4 & -4 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (3.2.4.13)$$

See Fig. 3.2.4

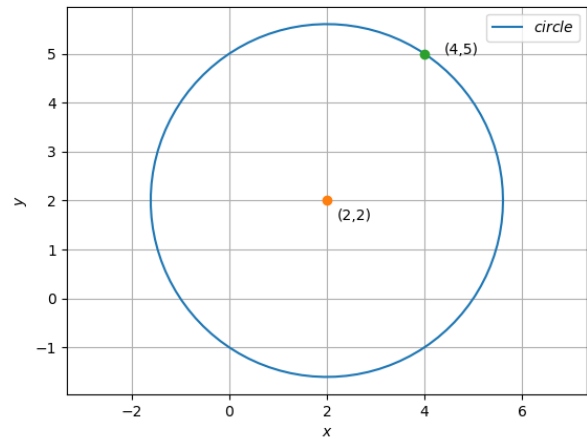


Fig. 3.2.4: plot showing the circle

3.2.5. Find the locus of all the unit vectors in the xy-plane.

3.2.6. Find the points on the curve  $\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 3 = 0$  at which the tangents are parallel to the x-axis.

**Solution:** General equation of circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.2.6.1)$$

The centre and the radius can be obtained as,

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (3.2.6.2)$$

$$f = -3 \quad (3.2.6.3)$$

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.6.4)$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \quad (3.2.6.5)$$

$\therefore$  The tangents are parallel to the x-axis, their direction and normal vectors,  $\mathbf{m}$  and  $\mathbf{n}$  are respectively,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.6.6)$$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.6.7)$$

For a circle, given the normal vector  $\mathbf{n}$ , the tangent points of contact to circle given by equation (3.2.6.1) are given by

$$\mathbf{q}_i = (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \quad (3.2.6.8)$$

where

$$\kappa_i = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\mathbf{n}^T \mathbf{n}}} \quad (3.2.6.9)$$

$$\kappa = \pm \sqrt{\frac{\begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} - (-3)}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \quad (3.2.6.10)$$

$$\Rightarrow \kappa = \pm \sqrt{\frac{4}{1}} \quad (3.2.6.11)$$

$$\Rightarrow \kappa = \pm 2 \quad (3.2.6.12)$$

and from (3.2.6.8), the point of contact  $\mathbf{q}_i$  are,

$$\mathbf{q}_1 = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (3.2.6.13)$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.2.6.14)$$

$$\mathbf{q}_2 = -2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (3.2.6.15)$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (3.2.6.16)$$

3.2.7. Find the area of the region in the first quadrant enclosed by x-axis, line  $(1 - \sqrt{3})\mathbf{x} = 0$  and

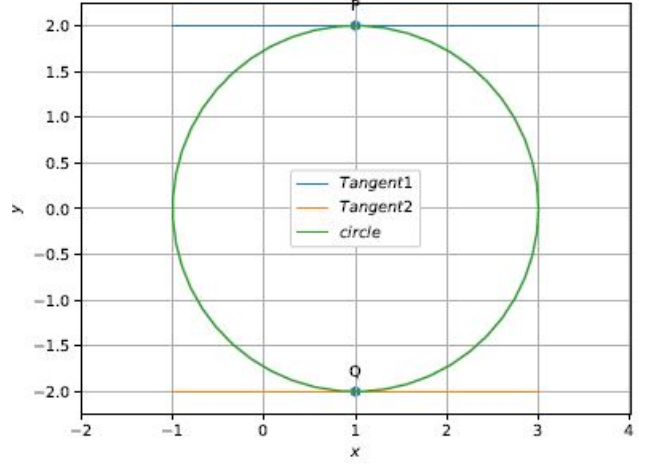


Fig. 3.2.6: Figure depicting tangents of circle parallel to x-axis

the circle  $\mathbf{x}^T \mathbf{x} = 4$ .

**Solution:** The equation of a circle can be expressed as,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (3.2.7.1)$$

where  $\mathbf{c}$  is the center.

Comparing equation (3.2.7.1) with the circle equation given,

$$\mathbf{x}^T \mathbf{x} = 4 \quad (3.2.7.2)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad f = -4 \quad (3.2.7.3)$$

$$r = \sqrt{\mathbf{c}^T \mathbf{c} - f} = \sqrt{4} \quad (3.2.7.4)$$

$$\Rightarrow \boxed{r = 2} \quad (3.2.7.5)$$

From equation (3.2.7.5), the point at which circle touches x-axis is  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ .

The direction vector of x-axis is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

The direction vector of the given line  $(1 - \sqrt{3})\mathbf{x} = 0$  is  $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ .

The angle that the line makes with the x-axis is given by,

$$\cos \theta = \frac{\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\| \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \| \| \begin{pmatrix} 1 & 0 \end{pmatrix} \|} = \frac{\sqrt{3}}{2} \quad (3.2.7.6)$$

$$\Rightarrow \boxed{\theta = 30^\circ} \quad (3.2.7.7)$$

Using equation (3.2.7.5) and (3.2.7.7), the area of the sector is obtained as,

$$\Rightarrow \frac{\theta}{360^\circ} \pi r^2 = \frac{30^\circ}{360^\circ} \pi (2)^2 = \frac{\pi}{3} \quad (3.2.7.8)$$

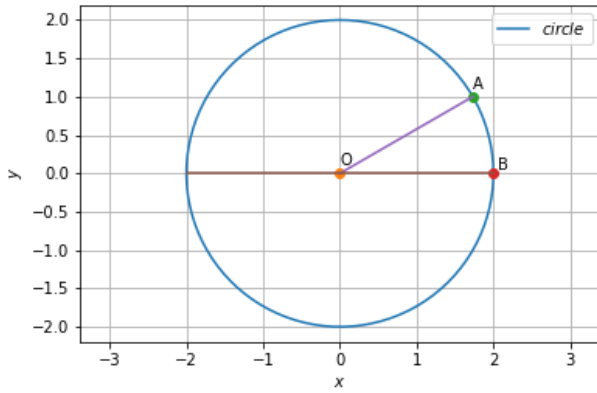


Fig. 3.2.7: Region enclosed by  $x$ -axis, line and circle

To find points **A** and **B**,  
The parametric form of  $x$ -axis is,

$$\mathbf{B} = \mathbf{q} + \lambda \mathbf{m} \quad (3.2.7.9)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.7.10)$$

From the intersection of circle and line, the value of  $\lambda$  can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (3.2.7.11)$$

$$= \frac{4 - 0}{1} = 4 \quad (3.2.7.12)$$

$$\Rightarrow \lambda = \pm 2 \quad (3.2.7.13)$$

Sub equation (3.2.7.13) in (3.2.7.10),

$$\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (3.2.7.14)$$

As given in question as first quadrant,

$$\Rightarrow \boxed{\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}} \quad (3.2.7.15)$$

Similarly, to find point **A**, The parametric form

of line is,

$$\mathbf{A} = \mathbf{q} + \lambda \mathbf{m} \quad (3.2.7.16)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad (3.2.7.17)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (3.2.7.18)$$

$$= \frac{4 - 0}{4} = 1 \quad (3.2.7.19)$$

$$\Rightarrow \lambda = \pm 1 \quad (3.2.7.20)$$

$$\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \quad (3.2.7.21)$$

$$\Rightarrow \boxed{\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \quad (3.2.7.22)$$

3.2.8. Find the area lying in the first quadrant and bounded by the circle  $\mathbf{x}^T \mathbf{x} = 4$  and the lines  $x = 0$  and  $x = 2$ .

3.2.9. Find the area of the circle  $4\mathbf{x}^T \mathbf{x} = 9$ .

3.2.10. Find the area bounded by curves  $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$  and  $\|\mathbf{x}\| = 1$

**Solution:**

General equation of circle is  $\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0$

Taking equation of the first curve to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = 1^2 \quad (3.2.10.1)$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}_1^T \mathbf{x} = 0 \quad (3.2.10.2)$$

$$\mathbf{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (3.2.10.3)$$

$$f_1 = 0 \quad (3.2.10.4)$$

$$\mathbf{O}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.10.5)$$

Taking equation of the second curve to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_2^T \mathbf{x} + f_2 = 0 \quad (3.2.10.6)$$

$$\mathbf{x}^T \mathbf{x} - 1 = 0 \quad (3.2.10.7)$$

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.10.8)$$

$$f_2 = -1 \quad (3.2.10.9)$$

$$\mathbf{O}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.10.10)$$

Now, subtracting equation (3.2.10.2) from



(3.2.10.7) We get,

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}_1^T \mathbf{x} - \mathbf{x}^T \mathbf{x} - f_2 = 0 \quad (3.2.10.11)$$

$$2\mathbf{u}_1^T \mathbf{x} = -1 \quad (3.2.10.12)$$

$$\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = -1 \quad (3.2.10.13)$$

which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1/2 \quad (3.2.10.14)$$

$$\mathbf{x} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.10.15)$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (3.2.10.16)$$

$$\mathbf{q} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \quad (3.2.10.17)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.10.18)$$

Substituting (3.2.10.16) in (3.2.10.6)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_2^T \mathbf{x} + f_2 = 0 \quad (3.2.10.19)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 + f_2 = 0 \quad (3.2.10.20)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) + f_2 = 0 \quad (3.2.10.21)$$

$$\mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) + f_2 = 0 \quad (3.2.10.22)$$

$$\|\mathbf{q}\|^2 + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \|\mathbf{m}\|^2 + f_2 = 0 \quad (3.2.10.23)$$

$$\|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 + f_2 = 0 \quad (3.2.10.24)$$

Taking  $\lambda$  as common :

$$\lambda(\lambda \|\mathbf{m}\|^2 + 2\mathbf{q}^T \mathbf{m}) = -f_2 - \|\mathbf{q}\|^2 \quad (3.2.10.25)$$

$$\lambda^2 \|\mathbf{m}\|^2 = -f_2 - \|\mathbf{q}\|^2 \quad (3.2.10.26)$$

$$\lambda^2 = \frac{-f_2 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (3.2.10.27)$$

$$\lambda^2 = \frac{3}{4} \quad (3.2.10.28)$$

$$\lambda = +\sqrt{\frac{3}{4}}, -\sqrt{\frac{3}{4}} \quad (3.2.10.29)$$

$$\lambda = +\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \quad (3.2.10.30)$$

Substituting the value of  $\lambda$  in (3.2.10.16)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (3.2.10.31)$$

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.2.10.32)$$

$$\mathbf{B} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.2.10.33)$$

Now finding the direction vector  $\mathbf{m}_{O_1A}$ ,  $\mathbf{m}_{O_1B}$ ,  $\mathbf{m}_{O_2A}$  and  $\mathbf{m}_{O_2B}$ .

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.2.10.34)$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.2.10.35)$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.2.10.36)$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.2.10.37)$$

Now finding the angle  $\angle O_1AB$ .

$$\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B} = \|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\| \cos \theta_1 \quad (3.2.10.38)$$

$$\frac{\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B}}{\|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\|} = \cos \theta_1 \quad (3.2.10.39)$$

$$\frac{-2}{4} = \cos \theta_1 \quad (3.2.10.40)$$

$$\frac{-1}{2} = \cos \theta_1 \quad (3.2.10.41)$$

$$\theta_1 = 120^\circ \quad (3.2.10.42)$$

Now finding the angle  $\angle O_2AB$ .

$$\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B} = \|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\| \cos \theta_2 \quad (3.2.10.43)$$

$$\frac{\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B}}{\|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\|} = \cos \theta_2 \quad (3.2.10.44)$$

$$\frac{-2}{4} = \cos \theta_2 \quad (3.2.10.45)$$

$$\frac{-1}{2} = \cos \theta_2 \quad (3.2.10.46)$$

$$\theta_2 = 120^\circ \quad (3.2.10.47)$$

Finding area of  $\mathbf{O}_1\mathbf{AB}$  and  $\mathbf{O}_2\mathbf{AB}$ .

$$A_{O_1AB} = \frac{\pi\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \quad (3.2.10.48)$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \quad (3.2.10.49)$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \quad (3.2.10.50)$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \quad (3.2.10.51)$$

Area of  $\mathbf{O}_1\mathbf{AO}_2\mathbf{B}$

$$A_{O_1AO_2B} = \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \quad (3.2.10.52)$$

$$= \frac{2\pi}{3} - 2\sqrt{3} \quad (3.2.10.53)$$

equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (3.2.11.1)$$

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \quad (3.2.11.2)$$

$$\mathbf{x}^T\mathbf{x} - 4 = 0 \quad (3.2.11.3)$$

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.11.4)$$

$$f_1 = -4 \quad (3.2.11.5)$$

$$\mathbf{O}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.11.6)$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \quad (3.2.11.7)$$

$$\Rightarrow \boxed{r = 2} \quad (3.2.11.8)$$

From equation (3.2.11.8), the point at which circle touches  $x$ -axis is  $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ .

The direction vector of the given line  $(1 \ 1)\mathbf{x} = 2$  is  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

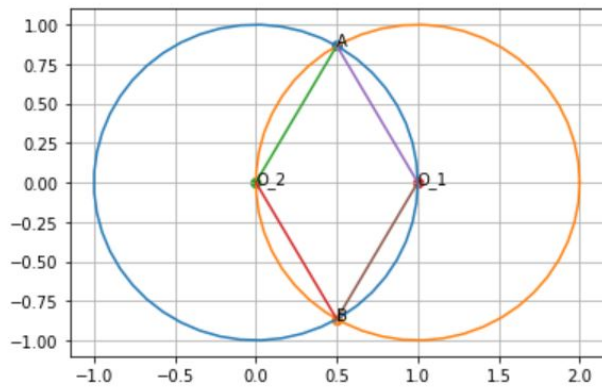


Fig. 3.2.10: Figure depicting intersection points of circle

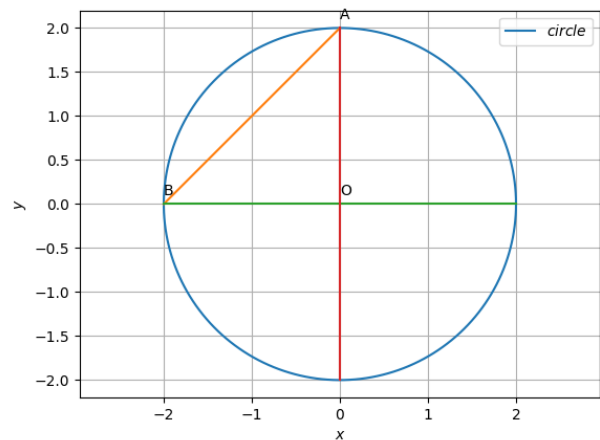


Fig. 3.2.11: Smaller area enclosed by line and circle

3.2.11. Find the smaller area enclosed by the circle  $\mathbf{x}^T\mathbf{x} = 4$  and the line  $(1 \ 1)\mathbf{x} = 2$ .

**Solution:**

Find the smaller area enclosed by the circle  $\mathbf{xx}^T = 4$  and the line  $(1 \ 1)\mathbf{x} = 2$ . General

To find point **A** and **B**, The parametric form of

line is,

$$\mathbf{A} = \mathbf{q} + \lambda \mathbf{m} \quad (3.2.11.9)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.2.11.10)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (3.2.11.11)$$

$$= \frac{4-2}{2} = 1 \quad (3.2.11.12)$$

$$\Rightarrow \lambda = \pm 1 \quad (3.2.11.13)$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (3.2.11.14)$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.11.15)$$

$$(\mathbf{A} - \mathbf{O}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (3.2.11.16)$$

$$(\mathbf{B} - \mathbf{O}) = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (3.2.11.17)$$

Inner product of  $(\mathbf{A} - \mathbf{O})$  and  $(\mathbf{B} - \mathbf{O})$  is given as:

$$(\mathbf{A} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = 0 \quad (3.2.11.18)$$

Therefore,  $(\mathbf{A} - \mathbf{O}) \perp (\mathbf{B} - \mathbf{O})$

Smaller area enclosed by circle and line  $\mathbf{AB}$  is:  
Area = (Area of circle in 2nd Quadrant) - (Area of right triangle formed by line AB, X and Y axis)

$$Area = \frac{\pi \theta_1}{360} r^2 - \frac{1}{2} \times 2 \times 2 \quad (3.2.11.19)$$

$$= \frac{90}{360} \pi \times 2^2 - 2 \quad (3.2.11.20)$$

$$= \pi - 2 \quad (3.2.11.21)$$

Hence, the smaller area enclosed by the circle  $\mathbf{xx}^T = 4$  and the line  $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$  is  $(\pi - 2)$

3.2.12. Find the slope of the tangent to the curve  $y = \frac{x-1}{x-2}$ ,  $x \neq 2$  at  $x = 10$ .

**Solution:**

$$y = \frac{x-1}{x-2} \quad (3.2.12.1)$$

Equation (3.2.12.1) can be expressed as

$$y(x-2) = x-1 \quad (3.2.12.2)$$

$$yx - 2y - x + 1 = 0 \quad (3.2.12.3)$$

From above we can say,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.12.4)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} & -1 \end{pmatrix} \quad (3.2.12.5)$$

$$f = 1 \quad (3.2.12.6)$$

Now,

$$\because |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0, \quad (3.2.12.7)$$

(3.2.12.1) is the equation of a hyperbola. To verify that this we will find the characteristic equation of  $\mathbf{V}$ .

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} = 0 \quad (3.2.12.8)$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (3.2.12.9)$$

The eigenvalues are the roots of (3.2.12.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (3.2.12.10)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{Vp} = \lambda \mathbf{p} \quad (3.2.12.11)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V}) \mathbf{p} = 0 \quad (3.2.12.12)$$

where  $\lambda$  is the eigenvalue. For  $\lambda_1 = \frac{1}{2}$ ,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.2.12.13)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3.2.12.14)$$

Now,  $\lambda$  is the eigenvalue. For  $\lambda_2 = -\frac{1}{2}$ ,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.2.12.15)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.2.12.16)$$

From Equations,

$$\mathbf{V} = \mathbf{PDP}^{-1} = \mathbf{PDP}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (3.2.12.17)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{VP} \quad (3.2.12.18)$$

We can say that

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (3.2.12.19)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (3.2.12.20)$$

$\because \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f > 0$ , there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (3.2.12.21)$$

$$\text{axes} = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (3.2.12.22)$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad (3.2.12.23)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (3.2.12.24)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \quad (3.2.12.25)$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \quad (3.2.12.26)$$

Let us assume slope to be 1, now finding the direction vector and normal vector of the tangent with slope 1.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \quad (3.2.12.27)$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \quad (3.2.12.28)$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \quad (3.2.12.29)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (3.2.12.30)$$

By using (3.2.12.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \quad (3.2.12.31)$$

Now substituting this  $\kappa$  in (3.2.12.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{-\frac{l}{4}} + 1 \end{pmatrix} \quad (3.2.12.32)$$

We know that  $x=10$ .

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \quad (3.2.12.33)$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \quad (3.2.12.34)$$

$$\sqrt{-\frac{1}{4l}} = 4 \quad (3.2.12.35)$$

$$-\frac{1}{4l} = 16 \quad (3.2.12.36)$$

$$l = -\frac{1}{64} \quad (3.2.12.37)$$

The slope of the tangent to the curve  $y = \frac{x-1}{x-2}$ ,  $x \neq 2$  at  $x=10$  is  $\frac{1}{64}$ . So, from the above we can say that  $\kappa=4, -4$  and from equation (3.2.12.27) and (3.2.12.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \quad (3.2.12.38)$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \quad (3.2.12.39)$$

Now using equation (3.2.12.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1} (\kappa_1 \mathbf{n} - \mathbf{u}) \quad (3.2.12.40)$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left( -4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (3.2.12.41)$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \quad (3.2.12.42)$$

$$\mathbf{q}_2 = \mathbf{V}^{-1} (\kappa_2 \mathbf{n} - \mathbf{u}) \quad (3.2.12.43)$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left( 4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (3.2.12.44)$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \quad (3.2.12.45)$$

3.2.13. Find a point on the curve  $y = (x-2)^2$  at which the tangent is parallel to the chord joining the

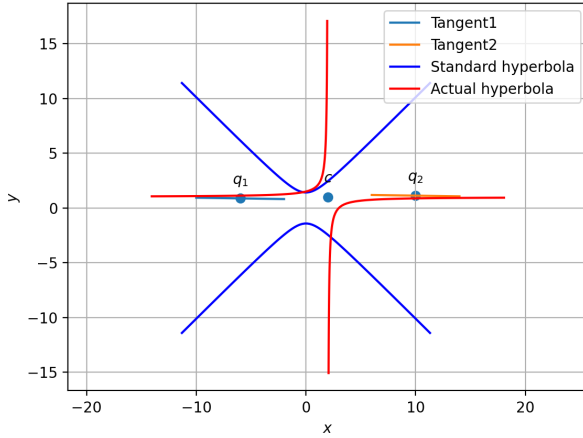


Fig. 3.2.12: Tangent 2 shows the tangent

points  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ .

**Solution:**  $y = (x - 2)^2$  can be written as,

$$x^2 - 4x - y + 4 = 0 \quad (3.2.13.1)$$

From (3.2.13.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \mathbf{u} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}; f = 4 \quad (3.2.13.2)$$

$$|V| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \quad (3.2.13.3)$$

(3.2.13.3) implies that the curve is a parabola. Now, finding the eigen values corresponding to the  $\mathbf{V}$ ,

$$\begin{aligned} |V - \lambda I| &= 0 \\ \begin{vmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} &= 0 \\ \implies \lambda &= 0, 1 \end{aligned} \quad (3.2.13.4)$$

Calculating the eigenvectors corresponding to  $\lambda = 0, 1$  respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0; \implies \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.13.5)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0; \implies \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.13.6)$$

By Eigen decomposition on  $\mathbf{V}$ ,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$$

$$\text{where, } \mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.13.7)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.13.8)$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (3.2.13.9)$$

$$\text{where, } \eta = \mathbf{u}^T \mathbf{p}_1 = -\frac{1}{2} \quad (3.2.13.10)$$

Substituting values from (3.2.13.2), (3.2.13.5) and (3.2.13.10) in (3.2.13.9),

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \quad (3.2.13.11)$$

Removing last row and representing (3.2.13.11) as augmented matrix and then converting the matrix to echelon form,

$$\begin{aligned} \begin{pmatrix} -2 & -1 & -4 \\ 1 & 0 & 2 \end{pmatrix} &\xrightarrow{R_1 \leftrightarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \\ \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} &\xrightarrow{R_2 \leftarrow (-2R_2)} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}} \\ &\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \quad (3.2.13.12)$$

From (3.2.13.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.2.13.13)$$

Direction vector of the chord joining A(4,4) and B(2,0) can be calculated as,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.2.13.14)$$

We know that,

$$\mathbf{m}^T \mathbf{n} = 0; \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (3.2.13.15)$$

To find the point of contact  $\mathbf{q}$ , which is intersection point for normal of the chord AB and

also tangent of the curve,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (3.2.13.16)$$

$$\text{where, } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}} = \frac{1}{2} \quad (3.2.13.17)$$

Substituting the values from (3.2.13.2), (3.2.13.15) and (3.2.13.17) in (3.2.13.16),

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \quad (3.2.13.18)$$

Removing last row and representing (3.2.13.18) as augmented matrix and then converting the matrix to echelon form,

$$\begin{aligned} \begin{pmatrix} -1 & -1 & -4 \\ 1 & 0 & 3 \end{pmatrix} &\xrightarrow{R_1 \leftarrow (-R_1)} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \\ \begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} &\xrightarrow{R_2 \leftarrow (-R_2)} \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} & \end{aligned} \quad (3.2.13.19)$$

From (3.2.13.19), it can be observed,

$$\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (3.2.13.20)$$

which is the required point of contact

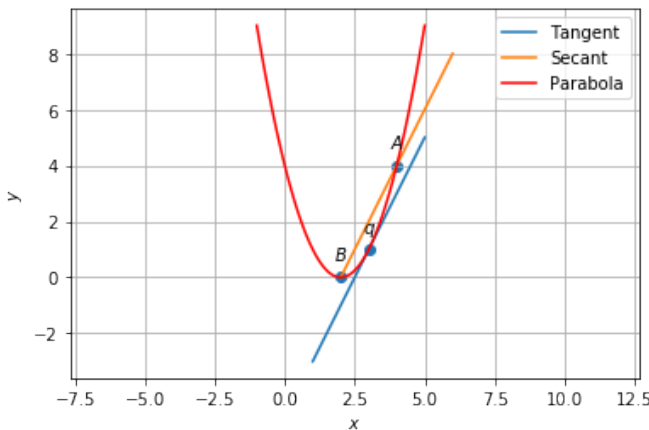


Fig. 3.2.13: Parabola with AB as chord, a tangent parallel to the chord

**Solution:** The given curve

$$y = \frac{1}{x-1} \quad (3.2.14.1)$$

can be expressed as

$$xy - y - 1 = 0 \quad (3.2.14.2)$$

Hence, we have

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix}, f = -1 \quad (3.2.14.3)$$

Since  $|\mathbf{V}| < 0$ , the equation (3.2.14.2) represents hyperbola. To find the values of  $\lambda_1$  and  $\lambda_2$ , consider the characteristic equation,

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (3.2.14.4)$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \right| = 0 \quad (3.2.14.5)$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \lambda \end{pmatrix} \right| = 0 \quad (3.2.14.6)$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \quad (3.2.14.7)$$

In addition, given the slope -1, the direction and normal vectors are given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3.2.14.8)$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.2.14.9)$$

The parameters of hyperbola are as follows:

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (3.2.14.10)$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \quad (3.2.14.11)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.14.12)$$

$$\text{axes} = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \end{cases} \quad (3.2.14.13)$$

which represents the standard hyperbola equation,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (3.2.14.14)$$

3.2.14. Find the equation of all lines having slope -1 that are tangents to the curve  $\frac{1}{x-1}, x \neq 1$

The points of contact are given by

$$K = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} = \pm \frac{1}{2} \quad (3.2.14.15)$$

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \quad (3.2.14.16)$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left[ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right] \quad (3.2.14.17)$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (3.2.14.18)$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left[ \frac{-1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right] \quad (3.2.14.19)$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (3.2.14.20)$$

$\therefore$  The tangents are given by

$$(1 \ 1) \left( \mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (3.2.14.21)$$

$$(1 \ 1) \left( \mathbf{x} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) = 0 \quad (3.2.14.22)$$

The desired equations of all lines having slope -1 that are tangents to the curve  $\frac{1}{x-1}, x \neq 1$  are given by

$$(1 \ 1)\mathbf{x} = 3 \quad (3.2.14.23)$$

$$(1 \ 1)\mathbf{x} = -1 \quad (3.2.14.24)$$

The above results are verified in the following figure.

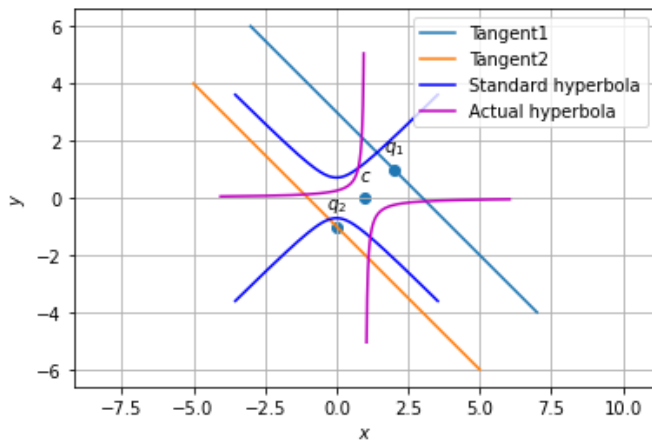


Fig. 3.2.14: The standard and actual hyperbola.

**Solution:** Given the curve,

$$y = \frac{1}{x-3} \quad (3.2.15.1)$$

$$\Rightarrow xy - 3y - 1 = 0 \quad (3.2.15.2)$$

From (3.2.15.2) we get,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = -1 \quad (3.2.15.3)$$

Now,

$$\therefore |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = \frac{-1}{2} < 0 \quad (3.2.15.4)$$

(3.2.15.1) is equation of hyperbola. Now,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \lambda \end{vmatrix} = 0 \quad (3.2.15.5)$$

$$\Rightarrow \lambda^2 - \frac{1}{4} = 0 \quad (3.2.15.6)$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (3.2.15.7)$$

The eigen vector  $\mathbf{p}$  is given by,

$$(\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (3.2.15.8)$$

For  $\lambda_1 = \frac{1}{2}$ ,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (3.2.15.9)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.2.15.10)$$

Similarly for  $\lambda_2$ ,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.2.15.11)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.2.15.12)$$

3.2.15. Find the equation of all lines having slope -2 which are tangents to the curve  $\frac{1}{x-3}, x \neq 3$ .

Now,

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (3.2.15.13)$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \quad (3.2.15.14)$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 \quad (3.2.15.15)$$

$\because \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 > 0$ , there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.15.16)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (3.2.15.17)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = \sqrt{2} \quad (3.2.15.18)$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (3.2.15.19)$$

The direction and normal vectors of the tangent with slope  $-2$  are given as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (3.2.15.20)$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \quad (3.2.15.21)$$

$$k = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (3.2.15.22)$$

Thus,

$$\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n} = 8 \quad (3.2.15.23)$$

$$k = \pm \frac{1}{2\sqrt{2}} \quad (3.2.15.24)$$

$$\mathbf{q}_1 = \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.2.15.25)$$

$$\mathbf{q}_2 = \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.2.15.26)$$

The desired tangents are,

$$(2 \ -1) \left\{ \mathbf{x} - \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\} = 0 \quad (3.2.15.27)$$

$$\Rightarrow (2 \ -1) \mathbf{x} = 6 + 2\sqrt{2} \quad (3.2.15.28)$$

$$(2 \ -1) \left\{ \mathbf{x} - \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\} = 0 \quad (3.2.15.29)$$

$$\Rightarrow (2 \ -1) \mathbf{x} = 6 - 2\sqrt{2} \quad (3.2.15.30)$$

Below figure corresponds to the tangents on the hyperbola, represented by (3.2.15.28) and (3.2.15.30) each having slope of  $-2$ .

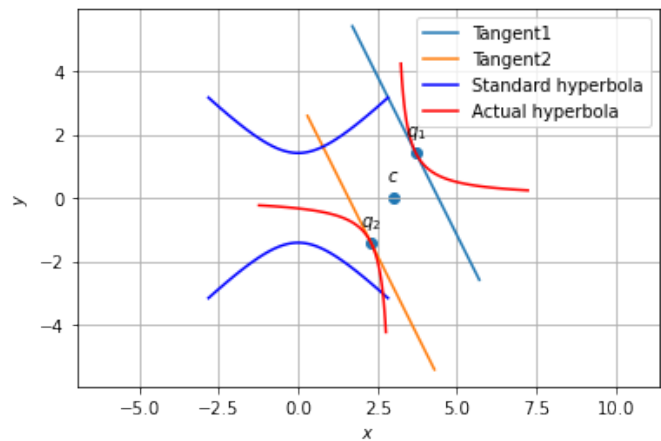


Fig. 3.2.15: Tangents to the hyperbola

3.2.16. Find points on the curve  $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$  at

which tangents are

a) parallel to x-axis

b) parallel to y-axis.

**Solution:**

General equation of conics is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.2.16.1)$$

Comparing with the equation given,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \quad (3.2.16.2)$$

$$\mathbf{u} = \mathbf{0} \quad (3.2.16.3)$$

$$f = -1 \quad (3.2.16.4)$$

$$|\mathbf{V}| = \left| \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \right| > 0 \quad (3.2.16.5)$$

$\because |\mathbf{V}| > 0$ , the given equation is of ellipse.

a) The tangents are parallel to the x-axis, hence,



their direction and normal vectors,  $\mathbf{m}_1$  and  $\mathbf{n}_1$  are respectively,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.16.6)$$

$$\mathbf{n}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.16.7)$$

For an ellipse, given the normal vector  $\mathbf{n}$ , the tangent points of contact to the ellipse are given by

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) = \mathbf{V}^{-1} \kappa \mathbf{n} \quad (3.2.16.8)$$

where

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (3.2.16.9)$$

$$= \pm \sqrt{\frac{-f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (3.2.16.10)$$

$$\mathbf{V}^{-1} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \quad (3.2.16.11)$$

$$\kappa_1 = \pm \sqrt{\frac{-(-1)}{(0 \ 1) \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \quad (3.2.16.12)$$

$$\Rightarrow \kappa_1 = \pm \sqrt{\frac{1}{16}} \quad (3.2.16.13)$$

$$\Rightarrow \kappa_1 = \pm \frac{1}{4} \quad (3.2.16.14)$$

From (3.2.16.8), the point of contact  $\mathbf{q}_i$  are,

$$\mathbf{q}_1 = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.16.15)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} \quad (3.2.16.16)$$

$$= \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (3.2.16.17)$$

$$\mathbf{q}_2 = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \left(-\frac{1}{4}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.16.18)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix} \quad (3.2.16.19)$$

$$= \begin{pmatrix} 0 \\ -4 \end{pmatrix} \quad (3.2.16.20)$$

b) The tangents are parallel to the y-axis, hence, their direction and normal vectors,  $\mathbf{m}_2$

and  $\mathbf{n}_2$  are respectively,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.16.21)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.16.22)$$

Using equation (3.2.16.9), the values of  $\kappa$  for this case are

$$\kappa_2 = \pm \sqrt{\frac{-(-1)}{(1 \ 0) \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}} \quad (3.2.16.23)$$

$$\Rightarrow \kappa_2 = \pm \sqrt{\frac{1}{9}} \quad (3.2.16.24)$$

$$\Rightarrow \kappa_2 = \pm \frac{1}{3} \quad (3.2.16.25)$$

and from (3.2.16.8), the point of contact  $\mathbf{q}_i$  are,

$$\mathbf{q}_3 = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.16.26)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \quad (3.2.16.27)$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.2.16.28)$$

$$\mathbf{q}_4 = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \left(-\frac{1}{3}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.16.29)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} \quad (3.2.16.30)$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad (3.2.16.31)$$

3.2.17. Find the equations of the tangent and normal to the given curves at the indicated points:  $y = x^2$  at  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

3.2.18. Find the equation of the tangent line to the curve  $y = x^2 - 2x + 7$

a) parallel to the line  $(2 \ -1)\mathbf{x} = -9$

b) perpendicular to the line  $(-15 \ 5)\mathbf{x} = 13$ .

**Solution:**

Given equation

$$y = x^2 - 2x + 7 \quad (3.2.18.1)$$

The equation (3.2.18.1) can be written as,

$$x^2 - 2x - y + 7 = 0 \quad (3.2.18.2)$$

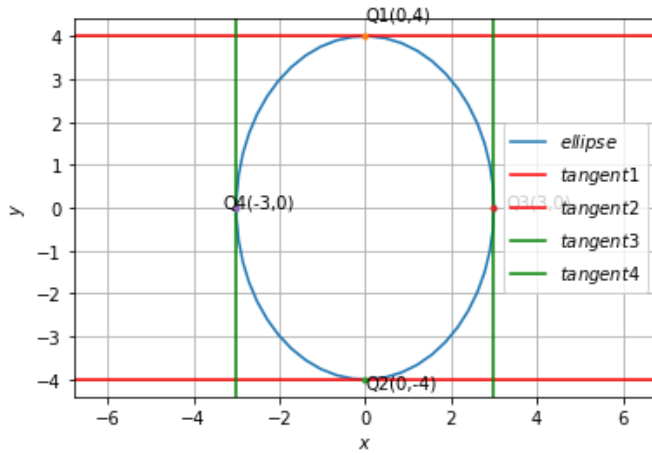


Fig. 3.2.16: Figure depicting point of contact of tangents of ellipse parallel to x-axis and y-axis

Comparing it with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (3.2.18.3)$$

$$\therefore a = 1, b = 0, c = 0, d = -1, e = \frac{-1}{2}, f = 7.$$

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.18.4)$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \quad (3.2.18.5)$$

$$\text{Now, } |V| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \quad (3.2.18.6)$$

$\Rightarrow$  that the curve is a parabola. Now, finding the eigen values corresponding to the  $\mathbf{V}$ ,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.2.18.7)$$

$$\begin{vmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \quad (3.2.18.8)$$

$$\Rightarrow \lambda = 0, 1. \quad (3.2.18.9)$$

Calculating the eigenvectors corresponding to  $\lambda = 0, 1$  respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \quad (3.2.18.10)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.18.11)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = \mathbf{x} \Rightarrow \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.18.12)$$

Now by eigen decomposition on  $\mathbf{V}$ ,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (3.2.18.13)$$

$$\text{where, } \mathbf{P} = (\mathbf{p}_1 \mathbf{p}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.18.14)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.18.15)$$

Hence equation (3.2.18.5) becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.18.16)$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.18.17)$$

a) The given parallel line equation is

$$(2 \quad -1) \mathbf{x} = -9 \quad (3.2.18.18)$$

$$\Rightarrow 2x - y + 9 = 0 \quad (3.2.18.19)$$

Now the tangent to parabola is parallel to the line equation (3.2.18.19), the general straight line equation is of the form

$$ax + by + c = 0 \quad (3.2.18.20)$$

The normal vector ( $\mathbf{n}$ ) and direction ( $\mathbf{m}$ ) are given by,

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.2.18.21)$$

$$\mathbf{m} = \begin{pmatrix} b \\ -a \end{pmatrix} \quad (3.2.18.22)$$

Comparing (3.2.18.19), (3.2.18.13), (3.2.18.21), the direction vectors ( $\mathbf{m}$ ) and normal ( $\mathbf{n}$ ) vectors are,

$$\mathbf{m} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad (3.2.18.23)$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (3.2.18.24)$$

Now, the equation for the point of contact

for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}^T \\ \mathbf{v} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (3.2.18.25)$$

$$\text{where, } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}} = \frac{1}{2} \quad (3.2.18.26)$$

Hence substituting the values of (3.2.18.5), (3.2.18.24), (3.2.18.13), (3.2.18.26) in equation (3.2.18.25) we get,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -7 \\ 2 \\ 0 \end{pmatrix} \quad (3.2.18.27)$$

Solving for  $\mathbf{q}$  by removing the zero row and representing (3.2.18.27) as augmented matrix and then converting the matrix to echelon form,

$$\Rightarrow \begin{pmatrix} 0 & -1 & -7 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -7 \end{pmatrix} \quad (3.2.18.28)$$

$$\xrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \end{pmatrix} \quad (3.2.18.29)$$

Hence from equation (3.2.18.29) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \quad (3.2.18.30)$$

Now  $\mathbf{q}$  is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \quad (3.2.18.31)$$

where  $c$  is,

$$c = \mathbf{n}^T \mathbf{q} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = -3 \quad (3.2.18.32)$$

Hence equation of tangent to the curve (3.2.18.1) parallel to (3.2.18.19) is given by substituting the value of  $c$  and  $\mathbf{n}$  from equation (3.2.18.32) and (3.2.18.24) respectively to the equation (3.2.18.31),

$$\Rightarrow \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -3 \quad (3.2.18.33)$$

Figure 3.2.18 verifies that the  $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -3$  is a tangent to parabola  $y = x^2 - 2x + 7$

b) The given perpendicular line equation is

$$\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13 \quad (3.2.18.34)$$

$$\Rightarrow -15x + 5y - 13 = 0 \quad (3.2.18.35)$$

Now the tangent to parabola is perpendicular to the line equation (3.2.18.35), the general straight line equation is of the form

$$ax + by + c = 0 \quad (3.2.18.36)$$

Therefore, if we find the line that is parallel to the line (3.2.18.35), it will be parallel to the tangent itself. For the given line the normal vector ( $\mathbf{n}$ ) and direction ( $\mathbf{m}$ ) are given by,

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.2.18.37)$$

$$\mathbf{m} = \begin{pmatrix} b \\ -a \end{pmatrix} \quad (3.2.18.38)$$

Comparing (3.2.18.35), (3.2.18.37), (3.2.18.38), the direction vectors ( $\mathbf{m}$ ) and normal ( $\mathbf{n}$ ) vectors are,

$$\mathbf{m} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \quad (3.2.18.39)$$

$$\mathbf{n} = \begin{pmatrix} -15 \\ 5 \end{pmatrix} \quad (3.2.18.40)$$

The parallel line for this vector will have the normal vector ( $\mathbf{n}_1$ ) and direction ( $\mathbf{m}_1$ ) are given by

$$\mathbf{m}_1 = \begin{pmatrix} 15 \\ -5 \end{pmatrix} \quad (3.2.18.41)$$

$$\mathbf{n}_1 = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \quad (3.2.18.42)$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}_1^T \\ \mathbf{v} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n}_1 - \mathbf{u} \end{pmatrix} \quad (3.2.18.43)$$

$$\text{where, } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}_1} = \frac{-1}{30} \quad (3.2.18.44)$$

Hence substituting the values of (3.2.18.5), (3.2.18.42), (3.2.18.13), (3.2.18.44) in equation

tion (3.2.18.43) we get,

$$\begin{pmatrix} \frac{-7}{6} & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} \frac{-7}{6} \\ \frac{5}{6} \\ 0 \end{pmatrix} \quad (3.2.18.45)$$

Solving for  $\mathbf{q}$  by removing the zero row and representing (3.2.18.45) as augmented matrix and then converting the matrix to echelon form,

$$\Rightarrow \begin{pmatrix} \frac{-7}{6} & -1 & \frac{-7}{6} \\ 1 & 0 & \frac{5}{6} \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ \frac{-7}{6} & -1 & \frac{-7}{6} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2.18.46)$$

$$\xrightarrow{R_2 \leftarrow R_2 - (\frac{7}{6})R_1} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ 0 & -1 & \frac{-217}{36} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2.18.47)$$

$$\xrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ 0 & 1 & \frac{217}{36} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2.18.48)$$

Hence from equation (3.2.18.48) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} \quad (3.2.18.49)$$

Now  $\mathbf{q}$  is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}_1^T \mathbf{x} = c \quad (3.2.18.50)$$

where  $c$  is,

$$c = \mathbf{n}_1^T \mathbf{q} = \begin{pmatrix} 5 & 15 \end{pmatrix} \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} = \frac{3405}{36} \quad (3.2.18.51)$$

Hence equation of tangent to the curve (3.2.18.1) parallel to (3.2.18.35) is given by substituting the value of  $c$  and  $\mathbf{n}_1$  from equation (3.2.18.51) and (3.2.18.42) respectively to the equation (3.2.18.50),

$$\Rightarrow \begin{pmatrix} 5 & 15 \end{pmatrix} \mathbf{x} = \frac{3405}{36} \quad (3.2.18.52)$$

Figure 3.2.18 verifies that the  $\begin{pmatrix} 5 & 15 \end{pmatrix} \mathbf{x} = \frac{3405}{36}$  is a tangent to parabola  $y = x^2 - 2x + 7$

3.2.19. Find the equation of the tangent to the curve,

$$y = \sqrt{3x-2} \quad (3.2.19.1)$$

which is parallel to the line,

$$\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0 \quad (3.2.19.2)$$

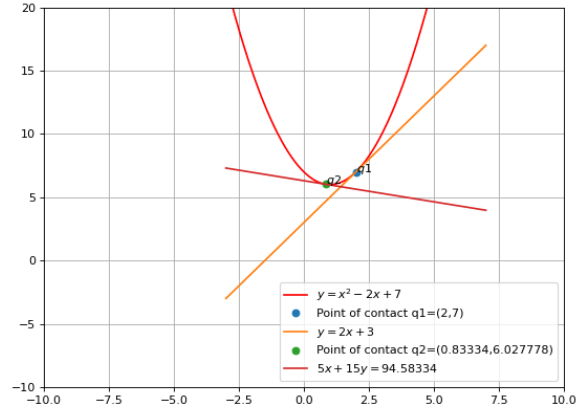


Fig. 3.2.18: Tangents to parabola  $y = x^2 - 2x + 7$

**Solution:** The equation (3.2.19.1) can be written as,

$$y^2 - 3x + 2 = 0 \quad (3.2.19.3)$$

Comparing it with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (3.2.19.4)$$

$$\therefore a = b = e = 0, d = \frac{-3}{2}, c = 1, f = 2.$$

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.19.5)$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix} \quad (3.2.19.6)$$

$$\text{Now, } |V| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \quad (3.2.19.7)$$

$\Rightarrow$  that the curve is a parabola. Now, finding the eigen values corresponding to the  $\mathbf{V}$ ,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.2.19.8)$$

$$\begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \quad (3.2.19.9)$$

$$\Rightarrow \lambda = 0, 1. \quad (3.2.19.10)$$

Calculating the eigenvectors corresponding to  $\lambda = 0, 1$  respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \quad (3.2.19.11)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.19.12)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{x} \implies \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.19.13)$$

Now by eigen decomposition on  $\mathbf{V}$ ,

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (3.2.19.14)$$

$$\text{where, } \mathbf{P} = (\mathbf{p}_1 \mathbf{p}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.19.15)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.19.16)$$

Hence equation (3.2.19.14) becomes,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.19.17)$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.19.18)$$

Now the tangent to parabola is parallel to the line equation (3.2.19.2), Hence the direction vectors ( $\mathbf{m}$ ) and normal ( $\mathbf{n}$ ) vectors are,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (3.2.19.19)$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (3.2.19.20)$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}^T \\ \mathbf{v} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (3.2.19.21)$$

$$\text{where, } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}} = \frac{-3}{4} \quad (3.2.19.22)$$

Hence substituting the values of (3.2.19.22), (3.2.19.20), (3.2.19.14) and (3.2.19.6) in equation (3.2.19.21) we get,

$$\begin{pmatrix} -3 & \frac{-3}{4} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -2 \\ 0 \\ \frac{-3}{4} \end{pmatrix} \quad (3.2.19.23)$$

Solving for  $\mathbf{q}$  by removing the zero row and representing (3.2.19.23) as augmented matrix

and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} -3 & \frac{-3}{4} & -2 \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow (-\frac{R_1}{3})} \begin{pmatrix} 1 & \frac{1}{4} & \frac{2}{3} \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \quad (3.2.19.24)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{1}{4} R_2} \begin{pmatrix} 1 & 0 & \frac{41}{48} \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \quad (3.2.19.25)$$

Hence from equation (3.2.19.25) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{-3}{4} \end{pmatrix} \quad (3.2.19.26)$$

Now  $\mathbf{q}$  is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \quad (3.2.19.27)$$

where  $c$  is,

$$c = \mathbf{n}^T \mathbf{q} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{41}{48} \\ \frac{-3}{4} \end{pmatrix} = \frac{23}{24} \quad (3.2.19.28)$$

Hence equation of tangent to the curve (3.2.19.1) parallel to (3.2.19.2) is given by substituting the value of  $c$  and  $\mathbf{n}$  from equation (3.2.19.28) and (3.2.19.20) respectively to the equation (3.2.19.27),

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{23}{24} \quad (3.2.19.29)$$

Figure 3.2.19 verifies that the  $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{23}{24}$  is a tangent to parabola  $y = \sqrt{3x - 2}$

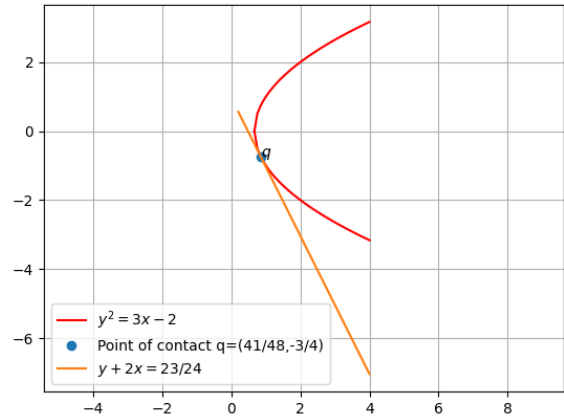


Fig. 3.2.19: Tangent to parabola  $y = \sqrt{3x - 2}$

3.2.20. Find the point at which the line  $(-1 \ 1)\mathbf{x} = 1$  is a tangent to the curve  $y^2 = 4x$ .

**Solution:** Comparing  $y^2 = 4x$  to standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (3.2.20.1)$$

$\therefore a = b = e = 0, d = -2, c = 1, f = 0.$

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.20.2)$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (3.2.20.3)$$

$$\text{Now, } |V| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \quad (3.2.20.4)$$

$\Rightarrow$  That the curve is a parabola.

$$\text{Since } \mathbf{V}\mathbf{p}_1 = 0 \quad (3.2.20.5)$$

$$\therefore \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.20.6)$$

Since the slope of the line is 1 The direction vector  $\mathbf{m}$  is as follows:

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.2.20.7)$$

$$\text{Since } \mathbf{m}^T \mathbf{n} = 0 \quad (3.2.20.8)$$

$$\therefore \mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3.2.20.9)$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (3.2.20.10)$$

$$\text{where, } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}} = -2 \quad (3.2.20.11)$$

By substituting the values ,we get:

$$\begin{pmatrix} -4 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \quad (3.2.20.12)$$

Solving for  $\mathbf{q}$  by removing the zero row and representing (3.2.20.12) as augmented matrix

and then converting the matrix to echelon form,

$$\Rightarrow \begin{pmatrix} -4 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftarrow (-\frac{R_1}{4})} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad (3.2.20.13)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad (3.2.20.14)$$

Threorefore the point at which the line  $(-1 \ 1)\mathbf{x} = 1$  is a tangent to the curve  $y^2 = 4x$  is  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

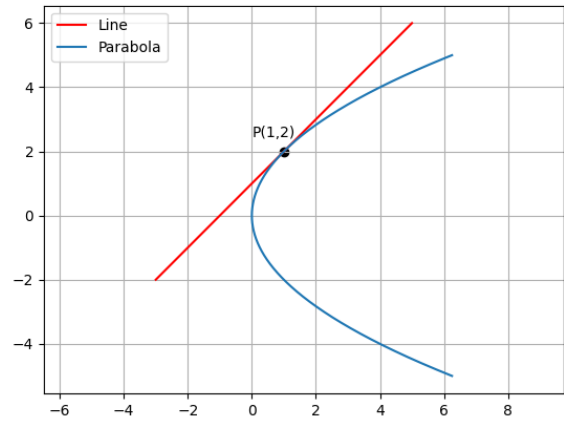


Fig. 3.2.20: Figure depicting the point at which the line is tangent to the parabola

3.2.21. The line  $(-m \ 1)\mathbf{x} = 1$  is a tangent to the curve  $y^2 = 4x$ . Find the value of  $m$ .

3.2.22. Find the normal at the point  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  on the curve  $2y + x^2 = 3$

**Solution:** Given,

$$x^2 + 2y - 3 = 0 \quad (3.2.22.1)$$

From (3.2.22.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.22.2)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.22.3)$$

$$f = -3 \quad (3.2.22.4)$$

From (3.2.22.2),

$$|V| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \quad (3.2.22.5)$$

Now (3.2.22.5) implies that the curve is a parabola. We can find the Eigen values corresponding to the  $V$ ,

$$\begin{aligned} |V - \lambda I| &= 0 \\ \begin{vmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda &= 0, 1 \end{aligned} \quad (3.2.22.6)$$

Calculating the Eigen Vectors corresponding to  $\lambda = 0, 1$  respectively,

$$Vx = \lambda x$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x = 0; \Rightarrow p_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.2.22.7)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x = 0; \Rightarrow p_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.22.8)$$

By Eigen decomposition on  $V$ ,

$$V = PDP^T$$

$$\text{where, } P = (p_1 \ p_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.22.9)$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.22.10)$$

To find the vertex of the parabola,

$$\begin{pmatrix} u^T + \eta p_1^T \\ V \end{pmatrix} c = \begin{pmatrix} -f \\ \eta p_1 - u \end{pmatrix} \quad (3.2.22.11)$$

$$\text{where, } \eta = u^T p_1 = 1 \quad (3.2.22.12)$$

Substituting values from (3.2.22.2), (3.2.22.7) and (3.2.22.12) in (3.2.22.11),

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} c = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \quad (3.2.22.13)$$

Removing last row and representing (3.2.22.13) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix} \quad (3.2.22.14)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix} \xrightarrow{R_2 \leftarrow -\frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \quad (3.2.22.15)$$

From (3.2.22.15) it can be observed that,

$$c = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \quad (3.2.22.16)$$

Now to evaluate the direction vector  $m$ ,

$$m^T(Vq + u) = 0 \quad (3.2.22.17)$$

$$m^T \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 0 \quad (3.2.22.18)$$

$$m^T \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 0 \quad (3.2.22.19)$$

$$m^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \quad (3.2.22.20)$$

$$\Rightarrow m = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3.2.22.21)$$

Now to obtain the equation of normal using,

$$m^T(x - q) = 0 \quad (3.2.22.22)$$

$$(1 \ -1) \left( x - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = 0 \quad (3.2.22.23)$$

$$(1 \ -1)x = 0 \quad (3.2.22.24)$$

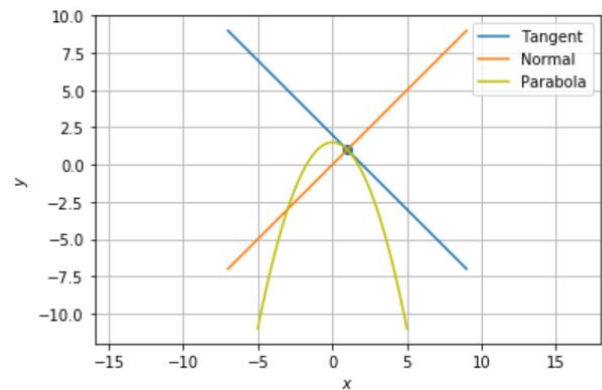


Fig. 3.2.22: Parabola showing tangent perpendicular to the normal

3.2.23. Find the normal to the curve  $x^2 = 4y$  passing through  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

3.2.24. Find the area of the region bounded by the curve  $y^2 = x$  and the lines  $x = 1$ ,  $x = 4$  and the x-axis in the first quadrant.

3.2.25. Find the area of the region bounded by  $y^2 = 9x$ ,  $x = 2$ ,  $x = 4$  and the x-axis in the first quadrant.

3.2.26. Find the area of the region bounded by  $x^2 =$

4y, y = 2, y = 4 and the y-axis in the first quadrant.

3.2.27. Find the area of the region bounded by the ellipse  $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

3.2.28. Find the area of the region bounded by the ellipse  $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

3.2.29. The area between  $x = y^2$  and  $x = 4$  is divided into two equal parts by the line  $x = a$ , find the value of  $a$ .

3.2.30. Find the area of the region bounded by the parabola  $y = x^2$  and  $y = |x|$ .

3.2.31. Find the area bounded by the curve  $x^2 = 4y$  and the line  $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$ .

3.2.32. Find the area of the region bounded by the curve  $y^2 = 4x$  and the line  $x = 3$ .

3.2.33. Find the area of the region bounded by the curve  $y^2 = x$ , y-axis and the line  $y = 3$ .

3.2.34. Find the area of the region bounded by the two parabolas  $y = x^2, y^2 = x$ .

3.2.35. Find the area lying above x-axis and included between the circle  $\mathbf{x}^T \mathbf{x} - 8 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0$  and inside of the parabola  $y^2 = 4x$ .

3.2.36. AOBA is the part of the ellipse  $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$  in the first quadrant such that  $OA = 2$  and  $OB = 6$ . Find the area between the arc  $AB$  and the chord  $AB$ .

3.2.37. Find the area lying between the curves  $y^2 = 4x$  and  $y = 2x$ .

3.2.38. Find the area of the region bounded by the curves  $y = x^2 + 2, y = x, x = 0$  and  $x = 3$ .

3.2.39. Find the area under  $y = x^2, x = 1, x = 2$  and x-axis.

3.2.40. Find the area between  $y = x^2$  and  $y = x$ .

3.2.41. Find the area of the region lying in the first quadrant and bounded by  $y = 4x^2, x = 0, y = 1$  and  $y = 4$ .

3.2.42. Find the area enclosed by the parabola  $4y = 3x^2$  and the line  $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$ .

3.2.43. Find the area of the smaller region bounded by the ellipse  $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$  and the line  $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$

3.2.44. Find the area of the region enclosed by the parabola  $x^2 = y$ , the line  $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$  and the x-axis.

$$\{(x, y) : y > x^2, y = |x|\} \quad (3.2.45.1)$$

3.2.46. Find the area of the region

$$\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T \mathbf{x} = 9\} \quad (3.2.46.1)$$

3.2.47. Find the area of the circle  $\mathbf{x}^T \mathbf{x} = 16$  exterior to the parabola  $y^2 = 6x$ .

## 4 MATRIX DECOMPOSITIONS

### 4.1 QR Decomposition

$$4.1.1. \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

**Solution:** Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (4.1.1.1)$$

$$\beta = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (4.1.1.2)$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \quad (4.1.1.3)$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (4.1.1.4)$$

where

$$k_1 = \|\alpha\| \quad (4.1.1.5)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \quad (4.1.1.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (4.1.1.7)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (4.1.1.8)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (4.1.1.9)$$

From (4.1.1.3) and (4.1.1.4),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.1.10)$$

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (4.1.1.11)$$

From above we can see that  $\mathbf{R}$  is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.1.12)$$



Now by using equations (4.1.1.5) to (4.1.1.9)

$$k_1 = \sqrt{5} \quad (4.1.1.13)$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (4.1.1.14)$$

$$r_1 = \sqrt{5} \quad (4.1.1.15)$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (4.1.1.16)$$

$$k_2 = \sqrt{5} \quad (4.1.1.17)$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \quad (4.1.1.18)$$

$$4.1.2. \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$4.1.3. \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$$

$$4.1.4. \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$$

$$4.1.5. \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$$

$$4.1.6. \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

$$4.1.7. \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

$$4.1.8. \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$$

$$4.1.9. \begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$$

$$4.1.10. \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$$

$$4.1.11. \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$$

$$4.1.12. \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$$

$$4.1.13. \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$4.1.14. \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

$$4.1.15. \text{ Find QR decomposition of } \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$$

**Solution:** Let  $\mathbf{a}$  and  $\mathbf{b}$  be the column vectors

of the given matrix.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.1.15.1)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.15.2)$$

The column vectors can be expressed as follows,

$$\mathbf{a} = k_1 \mathbf{u}_1 \quad (4.1.15.3)$$

$$\mathbf{b} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (4.1.15.4)$$

Here,

$$k_1 = \|\mathbf{a}\| \quad (4.1.15.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (4.1.15.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (4.1.15.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (4.1.15.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (4.1.15.9)$$

The (4.1.15.3) and (4.1.15.4) can be written as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.15.10)$$

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (4.1.15.11)$$

Now,  $\mathbf{R}$  is an upper triangular matrix and also,

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.15.12)$$

Now using equations (4.1.15.5) to (4.1.15.9) we get,

$$k_1 = \sqrt{2^2 + 3^2} = \sqrt{13} \quad (4.1.15.13)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.1.15.14)$$

$$r_1 = \left( \frac{2}{\sqrt{13}} \quad \frac{3}{\sqrt{13}} \right) \begin{pmatrix} 3 \\ -4 \end{pmatrix} = -\frac{6}{\sqrt{13}} \quad (4.1.15.15)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (4.1.15.16)$$

$$k_2 = \left( \frac{3}{\sqrt{13}} \quad -\frac{2}{\sqrt{13}} \right) \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{17}{\sqrt{13}} \quad (4.1.15.17)$$

Thus putting the values from (4.1.15.13) to (4.1.15.17) in (4.1.15.11) we obtain QR de-

composition,

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \sqrt{13} & -\frac{6}{\sqrt{13}} \\ 0 & \frac{17}{\sqrt{13}} \end{pmatrix} \quad (4.1.15.18)$$

4.1.16. Find the QR decomposition of  $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

**Solution:**

Let  $\mathbf{c}_1$  and  $\mathbf{c}_2$  be the column vectors of the given matrix.

$$\mathbf{c}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (4.1.16.1)$$

$$\mathbf{c}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad (4.1.16.2)$$

The column vectors can be represented as,

$$\mathbf{c}_1 = k_1 \mathbf{u}_1 \quad (4.1.16.3)$$

$$\mathbf{c}_2 = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (4.1.16.4)$$

where,

$$k_1 = \|\mathbf{c}_1\| \quad (4.1.16.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{c}_1}{k_1} \quad (4.1.16.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{c}_2}{\|\mathbf{u}_1\|^2} \quad (4.1.16.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{c}_2 - r_1 \mathbf{u}_1}{\|\mathbf{c}_2 - r_1 \mathbf{u}_1\|} \quad (4.1.16.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{c}_2 \quad (4.1.16.9)$$

From (4.1.16.3) and (4.1.16.4),

$$(\mathbf{c}_1 \ \mathbf{c}_2) = (\mathbf{u}_1 \ \mathbf{u}_2) \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.16.10)$$

$$(\mathbf{c}_1 \ \mathbf{c}_2) = \mathbf{Q} \mathbf{R} \quad (4.1.16.11)$$

Where  $\mathbf{R}$  is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.16.12)$$

Using equations (4.1.16.5) to (4.1.16.9) we get,

$$k_1 = \sqrt{3^2 + 1^2} = \sqrt{10} \quad (4.1.16.13)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \quad (4.1.16.14)$$

$$r_1 = \left( \frac{3}{\sqrt{10}} \ \frac{1}{\sqrt{10}} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \sqrt{10} \quad (4.1.16.15)$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{-1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \quad (4.1.16.16)$$

$$k_2 = \left( \frac{-1}{\sqrt{10}} \ \frac{3}{\sqrt{10}} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \sqrt{10} \quad (4.1.16.17)$$

Now putting the values from (4.1.16.13) to (4.1.16.17), we obtain the QR decomposition of given matrix,

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \sqrt{10} \\ 0 & \sqrt{10} \end{pmatrix} \quad (4.1.16.18)$$

4.1.17. Find QR decomposition of  $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$

**Solution:** The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$\mathbf{A} = \mathbf{Q} \mathbf{R} \quad (4.1.17.1)$$

where  $\mathbf{Q}$  is an orthogonal matrix and  $\mathbf{R}$  is an upper triangular matrix Given

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} \quad (4.1.17.2)$$

Let  $\mathbf{a}$  and  $\mathbf{b}$  be the column vectors of the given matrix

$$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (4.1.17.3)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (4.1.17.4)$$

The above column vectors (4.1.17.3), (4.1.17.4) can be expressed as ,

$$\mathbf{a} = t_1 \mathbf{u}_1 \quad (4.1.17.5)$$

$$\mathbf{b} = s_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 \quad (4.1.17.6)$$

Where,

$$t_1 = \|\mathbf{a}\| \quad (4.1.17.7)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{t_1} \quad (4.1.17.8)$$

$$s_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (4.1.17.9)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - s_1 \mathbf{u}_1}{\|\mathbf{b} - s_1 \mathbf{u}_1\|} \quad (4.1.17.10)$$

$$t_2 = \mathbf{u}_2^T \mathbf{b} \quad (4.1.17.11)$$

The (4.1.17.5) and (4.1.17.6) can be written as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} t_1 & s_1 \\ 0 & t_2 \end{pmatrix} \quad (4.1.17.12)$$

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (4.1.17.13)$$

Here,  $\mathbf{R}$  is an upper triangular matrix and  $\mathbf{Q}$  is an orthogonal matrix such that

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.17.14)$$

Now using equations from (4.1.17.7) to (4.1.17.11) we get,

$$t_1 = \sqrt{4^2 + 5^2} = \sqrt{41} \quad (4.1.17.15)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{41}} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (4.1.17.16)$$

$$s_1 = \left( \frac{4}{\sqrt{41}} \quad \frac{5}{\sqrt{41}} \right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{2}{\sqrt{41}} \quad (4.1.17.17)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{41}} \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (4.1.17.18)$$

$$t_2 = \left( \frac{5}{\sqrt{41}} \quad \frac{-4}{\sqrt{41}} \right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{23}{\sqrt{41}} \quad (4.1.17.19)$$

Substituting the values from (4.1.17.15) to (4.1.17.19) in (4.1.17.13) we obtain QR decomposition as,

$$\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \\ \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}} \end{pmatrix} \begin{pmatrix} \sqrt{41} & \frac{2}{\sqrt{41}} \\ 0 & \frac{23}{\sqrt{41}} \end{pmatrix} \quad (4.1.17.20)$$

4.1.18. Perform the QR decomposition of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \quad (4.1.18.1)$$

**Solution:**

If  $\alpha$  and  $\beta$  are the columns of a (2×2) matrix  $\mathbf{A}$ ,

then  $\mathbf{A}$  can be decomposed as

$$\mathbf{A} = \mathbf{Q} \mathbf{R} \quad (4.1.18.2)$$

$$\text{where, } \mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}, \quad (4.1.18.3)$$

$$\text{uppertriangular matrix } \mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.18.4)$$

$$k_1 = \|\alpha\|, \mathbf{u}_1 = \frac{\alpha}{k_1} \quad (4.1.18.5)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (4.1.18.6)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|}, k_2 = \mathbf{u}_2^T \beta \quad (4.1.18.7)$$

$$\alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.1.18.8)$$

$$\text{From, (4.1.18.5), } k_1 = \|\alpha\| = \sqrt{10} \quad (4.1.18.9)$$

$$\text{and } \mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.18.10)$$

$$\text{From (4.1.18.6), } r_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{10}} \quad (4.1.18.11)$$

$$\beta - r_1 \mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{5}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.18.12)$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (4.1.18.13)$$

$$\text{From (4.1.18.7), } \mathbf{u}_2 = \frac{\begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{pmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4}}} \quad (4.1.18.14)$$

$$\Rightarrow \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}, \quad (4.1.18.15)$$

$$k_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{5}{\sqrt{10}} \quad (4.1.18.16)$$

Note that,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (4.1.18.17)$$

The matrix  $\mathbf{A}$  can now be rewritten using (4.1.18.2) as

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{5}{\sqrt{10}} \\ 0 & \frac{5}{\sqrt{10}} \end{pmatrix} \quad (4.1.18.18)$$

4.1.19. Find the QR decomposition of the given matrix.

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \quad (4.1.19.1)$$

**Solution:** QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \quad (4.1.19.2)$$

where  $\mathbf{Q}$  is an orthogonal matrix and  $\mathbf{R}$  is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \quad (4.1.19.3)$$

The column vectors of the matrix is given by

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (4.1.19.4)$$

Equation (4.1.19.3) can be written in form of (4.1.19.4) as,

$$(\mathbf{a} \quad \mathbf{b}) = (\mathbf{q}_1 \quad \mathbf{q}_2) \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} = \mathbf{QR} \quad (4.1.19.5)$$

where,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (4.1.19.6)$$

$$\mathbf{q}_1 = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.1.19.7)$$

$$u_3 = \frac{\mathbf{q}_1^T \mathbf{b}}{\|\mathbf{q}_1\|^2} = \left( \frac{1}{\sqrt{5}} \quad \frac{2}{\sqrt{5}} \right) \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{-2}{\sqrt{5}} \quad (4.1.19.8)$$

$$\mathbf{q}_2 = \frac{\mathbf{b} - u_3 \mathbf{q}_1}{\|\mathbf{b} - u_3 \mathbf{q}_1\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.1.19.9)$$

$$u_2 = \mathbf{q}_2^T \mathbf{b} = \left( \frac{2}{\sqrt{5}} \quad -\frac{1}{\sqrt{5}} \right) \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{6}{\sqrt{5}} \quad (4.1.19.10)$$

Substituting equation (4.1.19.6) to (4.1.19.10) in (4.1.19.5),

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{5}} \end{pmatrix} \quad (4.1.19.11)$$

The QR decomposition is,

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{5}} \end{pmatrix} \quad (4.1.19.12)$$

4.1.20. Find the QR decomposition on a given 2×2 matrix.

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad (4.1.20.1)$$

**Solution:** The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \quad (4.1.20.2)$$

Here  $\mathbf{Q}$  is an orthogonal matrix and  $\mathbf{R}$  is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad (4.1.20.3)$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (4.1.20.4)$$

Equation (4.1.20.3) can be written in  $\mathbf{QR}$  form

as:

$$\mathbf{QR} = (\mathbf{q}_1 \quad \mathbf{q}_2) \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \quad (4.1.20.5)$$

Now,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (4.1.20.6)$$

$$\mathbf{q}_1 = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.1.20.7)$$

$$u_3 = \frac{\mathbf{q}_1^T \mathbf{b}}{\|\mathbf{q}_1\|^2} = \left( \frac{2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0 \quad (4.1.20.8)$$

$$\mathbf{q}_2 = \frac{\mathbf{b} - u_3 \mathbf{q}_1}{\|\mathbf{b} - u_3 \mathbf{q}_1\|} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.1.20.9)$$

$$u_2 = \mathbf{q}_2^T \mathbf{b} = \left( \frac{1}{\sqrt{5}} \quad -\frac{2}{\sqrt{5}} \right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \sqrt{5} \quad (4.1.20.10)$$

Substituting equation (4.1.20.6) to (4.1.20.10) in (4.1.20.5), to obtain the QR Decomposition of the given matrix as:

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \quad (4.1.20.11)$$

In equation (4.1.20.11)  $\mathbf{R}$  is diagonal because the columns and rows are orthogonal to each other.

4.1.21. Perform QR decomposition on matrix  $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} \quad (4.1.21.1)$$

**Solution:**

The columns of matrix  $\mathbf{A}$  can be represented in  $\alpha$  and  $\beta$  as

$$\Rightarrow \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.21.2)$$

$$\beta = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad (4.1.21.3)$$

For QR decomposition, matrix  $\mathbf{A}$  can be expressed as

$$\mathbf{A} = \mathbf{QR} \quad (4.1.21.4)$$

where,  $\mathbf{Q}$  and  $\mathbf{R}$  are expressed as

$$\mathbf{Q} = (\mathbf{u}_1 \quad \mathbf{u}_2) \quad (4.1.21.5)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.21.6)$$

Note that  $\mathbf{R}$  is an upper triangular matrix.

Now, we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \quad (4.1.21.7)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.21.8)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad (4.1.21.9)$$

$$\Rightarrow r_1 = -\frac{11}{\sqrt{10}} \quad (4.1.21.10)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (4.1.21.11)$$

Consider

$$\beta - r_1 \mathbf{u}_1 = \begin{pmatrix} 4 \\ -5 \end{pmatrix} + \frac{11}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.21.12)$$

$$\Rightarrow \beta - r_1 \mathbf{u}_1 = \begin{pmatrix} \frac{51}{10} \\ -\frac{17}{10} \end{pmatrix} \quad (4.1.21.13)$$

$$\|\beta - r_1 \mathbf{u}_1\| = \frac{17}{\sqrt{10}} \quad (4.1.21.14)$$

Substitute (4.1.21.13), (4.1.21.14) in (4.1.21.11), we get

$$\mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix} \quad (4.1.21.15)$$

$$k_2 = \mathbf{u}_2^T \beta = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad (4.1.21.16)$$

$$\Rightarrow k_2 = \frac{17}{\sqrt{10}} \quad (4.1.21.17)$$

Therefore, from (4.1.21.5) and (4.1.21.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \quad (4.1.21.18)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \quad (4.1.21.19)$$

Note that,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (4.1.21.20)$$

Now matrix  $\mathbf{A}$  can be written as (4.1.21.4)

$$\begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \quad (4.1.21.21)$$

#### 4.1.22. Perform QR decomposition on matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} \quad (4.1.22.1)$$

**Solution:** The columns of matrix  $\mathbf{A}$  can be represented in  $\alpha$  and  $\beta$  as

$$\Rightarrow \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.22.2)$$

$$\beta = \begin{pmatrix} -7 \\ 1 \end{pmatrix} \quad (4.1.22.3)$$

For QR decomposition, matrix  $\mathbf{A}$  can be expressed as

$$\mathbf{A} = \mathbf{QR} \quad (4.1.22.4)$$

where,  $\mathbf{Q}$  and  $\mathbf{R}$  are expressed as

$$\mathbf{Q} = (\mathbf{u}_1 \quad \mathbf{u}_2) \quad (4.1.22.5)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.22.6)$$

Note that  $\mathbf{R}$  is an upper triangular matrix. Now, we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \quad (4.1.22.7)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.22.8)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} = \frac{1}{\sqrt{10}} (1 \quad 3) \begin{pmatrix} -7 \\ 1 \end{pmatrix} \quad (4.1.22.9)$$

$$\Rightarrow r_1 = -\frac{4}{\sqrt{10}} \quad (4.1.22.10)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (4.1.22.11)$$

Consider

$$\beta - r_1 \mathbf{u}_1 = \begin{pmatrix} -7 \\ 1 \end{pmatrix} + \frac{4}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.22.12)$$

$$\Rightarrow \beta - r_1 \mathbf{u}_1 = \begin{pmatrix} -66 \\ 10 \end{pmatrix} \quad (4.1.22.13)$$

$$\|\beta - r_1 \mathbf{u}_1\| = \frac{22}{\sqrt{10}} \quad (4.1.22.14)$$

Substitute (4.1.22.13), (4.1.22.14) in

(4.1.22.11), we get

$$\mathbf{u}_2 = \begin{pmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \quad (4.1.22.15)$$

$$k_2 = \mathbf{u}_2^T \beta = \begin{pmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} -7 \\ 1 \end{pmatrix} \quad (4.1.22.16)$$

$$\Rightarrow k_2 = \frac{22}{\sqrt{10}} \quad (4.1.22.17)$$

Therefore, from (4.1.22.5) and (4.1.22.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \quad (4.1.22.18)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & -\frac{4}{\sqrt{10}} \\ 0 & \frac{22}{\sqrt{10}} \end{pmatrix} \quad (4.1.22.19)$$

Note that,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (4.1.22.20)$$

Now matrix  $\mathbf{A}$  can be written as (4.1.22.4)

$$\begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{4}{\sqrt{10}} \\ 0 & \frac{22}{\sqrt{10}} \end{pmatrix} \quad (4.1.22.21)$$

4.1.23. Given a matrix  $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ , find its **QR** decomposition **Solution:**  
Given

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} \quad (4.1.23.1)$$

Let us use the Gram-Schmidt approach to obtain QR decomposition of  $\mathbf{A}$ . Consider column vectors say  $\mathbf{a}_1$  and  $\mathbf{a}_2$  of  $\mathbf{A}$  which is given by

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (4.1.23.2)$$

$$\mathbf{a}_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (4.1.23.3)$$

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (4.1.23.4)$$

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \quad (4.1.23.5)$$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2^T \cdot \mathbf{e}_1) \mathbf{e}_1 \quad (4.1.23.6)$$

$$= \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \left(-\frac{14}{5}\right) \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \quad (4.1.23.7)$$

$$= \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{42}{25} \\ -\frac{56}{25} \end{pmatrix} = \begin{pmatrix} -\frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \quad (4.1.23.8)$$

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \quad (4.1.23.9)$$

The matrix  $\mathbf{Q}$  and  $\mathbf{R}$  is given by,

$$\mathbf{Q} = (\mathbf{e}_1 \ \mathbf{e}_2) = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \quad (4.1.23.10)$$

$$\mathbf{R} = \begin{pmatrix} \mathbf{a}_1^T \cdot \mathbf{e}_1 & \mathbf{a}_2^T \cdot \mathbf{e}_1 \\ 0 & \mathbf{a}_2^T \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 5 & -\frac{14}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \quad (4.1.23.11)$$

Hence, the  $\mathbf{QR}$  decomposition of matrix  $\mathbf{A}$  is as follows:

$$\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 5 & -\frac{14}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \quad (4.1.23.12)$$

4.1.24. Perform the QR decomposition of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}. \text{ Solution:}$$

Let  $\mathbf{a}$  and  $\mathbf{b}$  are the columns of matrix  $\mathbf{A}$ . The matrix  $\mathbf{A}$  can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \quad (4.1.24.1)$$

$$\mathbf{U} = (\mathbf{u}_1 \ \mathbf{u}_2) \quad (4.1.24.2)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.24.3)$$

where

$$k_1 = \|\mathbf{a}\| \quad (4.1.24.4)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (4.1.24.5)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (4.1.24.6)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (4.1.24.7)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (4.1.24.8)$$

Then the given matrix can be represented as,

$$(\mathbf{a} \ \mathbf{b}) = (\mathbf{u}_1 \ \mathbf{u}_2) \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.24.9)$$

The the columns of matrix  $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$  are  $\mathbf{a}$  and  $\mathbf{b}$  where

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.24.10)$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (4.1.24.11)$$

Now for the given matrix, From (4.1.24.4) and (4.1.24.5)

$$k_1 = \|\mathbf{a}\| = 5 \quad (4.1.24.12)$$

$$\mathbf{u}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.24.13)$$

From (4.1.24.6)

$$r_1 = \frac{1}{5} \begin{pmatrix} 3 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{-11}{5} \quad (4.1.24.14)$$

From (4.1.24.7)

$$\mathbf{b} - r_1 \mathbf{u}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{-11}{5} \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix} \quad (4.1.24.15)$$

$$\|\mathbf{b} - r_1 \mathbf{u}_1\| = \frac{2}{5} \quad (4.1.24.16)$$

$$\Rightarrow \mathbf{u}_2 = \frac{5}{2} \begin{pmatrix} \frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \quad (4.1.24.17)$$

From (4.1.24.8)

$$k_2 = \mathbf{u}_2^T \mathbf{b} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{2}{5} \quad (4.1.24.18)$$

Now we can observe that  $\mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = \mathbf{I}$

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.1.24.19)$$

From (4.1.24.9), The matrix  $\mathbf{A}$  can now be written as,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -5 & \frac{-11}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \quad (4.1.24.20)$$

4.1.25. Perform QR decomposition on matrix A given by

$$\mathbf{A} = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$$

**Solution:** Representing matrix  $A$  in terms of its column vectors as

$$A = (\mathbf{a} \ \mathbf{b}) \quad (4.1.25.1)$$

Let

$$\mathbf{q}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (4.1.25.2)$$

An orthonormal vector to  $\mathbf{q}_1$  can be obtained by subtracting the projection of  $\mathbf{b}$  on  $\mathbf{q}_1$  from  $\mathbf{b}$ . Thus

$$\mathbf{q}_2 = \frac{\mathbf{b} - k\mathbf{q}_1}{\|\mathbf{b} - k\mathbf{q}_1\|} \quad (4.1.25.3)$$

where

$$k = \frac{\mathbf{b}^T \mathbf{q}_1}{\|\mathbf{q}_1\|^2} \quad (4.1.25.4)$$

From (4.1.25.2) and (4.1.25.3)

$$\mathbf{a} = \|\mathbf{a}\| \mathbf{q}_1 \quad (4.1.25.5)$$

$$\mathbf{b} = k\mathbf{q}_1 + \|\mathbf{b} - k\mathbf{q}_1\| \mathbf{q}_2 \quad (4.1.25.6)$$

$$\Rightarrow (\mathbf{a} \ \mathbf{b}) = (\mathbf{q}_1 \ \mathbf{q}_2) \begin{pmatrix} \|\mathbf{a}\| & k \\ 0 & \|\mathbf{b} - k\mathbf{q}_1\| \end{pmatrix} \quad (4.1.25.7)$$

$$\Rightarrow \mathbf{A} = \mathbf{QR} \quad (4.1.25.8)$$

QR decomposition of a matrix  $A$  is essentially representation of column vectors of matrix  $A$  in terms of linear combination of orthonormal basis of column space of  $A$ . For matrix  $A$

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (4.1.25.9)$$

$$(4.1.25.10)$$

Let

$$\mathbf{q}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.25.11)$$

$$(4.1.25.12)$$

From (4.1.25.3) and (4.1.25.4)

$$\mathbf{q}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (4.1.25.13)$$

$$\Rightarrow \mathbf{Q} = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix} \quad (4.1.25.14)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \quad (4.1.25.15)$$

Therefore the matrix  $A$  can be decomposed as

$$\mathbf{A} = \begin{pmatrix} \frac{3}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \\ -\frac{4}{\sqrt{5}} & \frac{3}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \quad (4.1.25.16)$$

4.1.26. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix} \quad (4.1.26.1)$$

**Solution:** Let  $c_1$  and  $c_2$  be the column vectors of given matrix  $A$

$$c_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.1.26.2)$$

$$c_2 = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \quad (4.1.26.3)$$

We can express the matrix  $A$  as,

$$\mathbf{A} = \mathbf{QR} \quad (4.1.26.4)$$

Where,  $\mathbf{Q}$  is an orthogonal matrix given as,

$$\mathbf{Q} = (\mathbf{u}_1 \ \mathbf{u}_2) \quad (4.1.26.5)$$

and  $\mathbf{R}$  is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.26.6)$$

Now, we can express  $\alpha$  and  $\beta$  as,

$$c_1 = k_1 \mathbf{u}_1 \quad (4.1.26.7)$$

$$c_2 = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (4.1.26.8)$$

$$\text{where, } k_1 = \|c_1\| = \sqrt{2^2 + 1^2} = \sqrt{5} \quad (4.1.26.9)$$

Solving equation (4.1.26.7) for  $\mathbf{u}_1$ ,

$$\mathbf{u}_1 = \frac{c_1}{k_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.1.26.10)$$

$$\text{Now, } r_1 = \frac{\mathbf{u}_1^T c_2}{\|\mathbf{u}_1\|^2} \quad (4.1.26.11)$$

$$\Rightarrow \frac{\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} -6 \\ -2 \end{pmatrix}}{1} \quad (4.1.26.12)$$



$$\text{Hence, } r_1 = -\frac{-14}{\sqrt{5}} \quad (4.1.26.13)$$

$$\mathbf{u}_2 = \frac{c_2 - r_1 \mathbf{u}_1}{\|c_2 - r_1 \mathbf{u}_1\|} \quad (4.1.26.14)$$

$$\Rightarrow \frac{\begin{pmatrix} -6 \\ -2 \end{pmatrix} - \left(-\frac{14}{\sqrt{5}}\right) \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} -6 \\ -2 \end{pmatrix} - \left(-\frac{14}{\sqrt{5}}\right) \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 1 \end{pmatrix} \right\|}} \quad (4.1.26.15)$$

$$\Rightarrow \mathbf{u}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (4.1.26.16)$$

$$\text{Now, } k_2 = u_2^T c_2 \quad (4.1.26.17)$$

$$\Rightarrow \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} -6 \\ -2 \end{pmatrix} \quad (4.1.26.18)$$

$$\Rightarrow k_2 = \frac{2}{\sqrt{5}} \quad (4.1.26.19)$$

Hence substituting the values of unknown parameter from equations (4.1.26.9), (4.1.26.19), (4.1.26.10), (4.1.26.16) and (4.1.26.13) to equation (4.1.26.5) and (4.1.26.6) we get,

$$\mathbf{Q} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad (4.1.26.20)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & \frac{-14}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.1.26.21)$$

4.1.27. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \quad (4.1.27.1)$$

**Solution:** If  $\mathbf{A} \in \mathbf{R}^{m \times n}$  has linearly independent columns then it can be factored as

$$\mathbf{A} = \mathbf{Q}\mathbf{R}$$

where  $\mathbf{Q}$  is a orthogonal matrix and  $\mathbf{R}$  is a upper triangular matrix with non zero diagonal elements

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \quad (4.1.27.2)$$

The column vectors of  $\mathbf{A}$  are,

$$\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \quad (4.1.27.3)$$

(4.1.27.2) can be written as,

$$\mathbf{Q}\mathbf{R} = (\mathbf{p}_1 \quad \mathbf{p}_2) \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \quad (4.1.27.4)$$

Now,

$$u_1 = \|a\| = \sqrt{4^2 + 7^2} = \sqrt{65} \quad (4.1.27.5)$$

$$\mathbf{q}_1 = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{4}{\sqrt{65}} \\ \frac{7}{\sqrt{65}} \end{pmatrix} \quad (4.1.27.6)$$

$$u_3 = \frac{\mathbf{q}_1^T \mathbf{b}}{\|\mathbf{q}_1\|^2} = \left( \frac{4}{\sqrt{65}} \quad \frac{7}{\sqrt{65}} \right) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{47}{\sqrt{65}} \quad (4.1.27.7)$$

$$\mathbf{q}_2 = \frac{\mathbf{b} - u_3 \mathbf{q}_1}{\|\mathbf{b} - u_3 \mathbf{q}_1\|} = \begin{pmatrix} \frac{7}{\sqrt{65}} \\ -\frac{4}{\sqrt{65}} \end{pmatrix} \quad (4.1.27.8)$$

$$u_2 = \mathbf{q}_2^T \mathbf{b} = \left( \frac{7}{\sqrt{65}} \quad -\frac{4}{\sqrt{65}} \right) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{65}} \quad (4.1.27.9)$$

Substituting (4.1.27.5) to (4.1.27.9) in (4.1.27.4),

$$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{65}} & \frac{7}{\sqrt{65}} \\ \frac{7}{\sqrt{65}} & -\frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} \sqrt{65} & \frac{47}{\sqrt{65}} \\ 0 & \frac{1}{\sqrt{65}} \end{pmatrix} \quad (4.1.27.10)$$

Which can also be written as,

$$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{4}{\sqrt{65}} & -\frac{7}{\sqrt{65}} \\ -\frac{7}{\sqrt{65}} & \frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} -\sqrt{65} & -\frac{47}{\sqrt{65}} \\ 0 & -\frac{1}{\sqrt{65}} \end{pmatrix} \quad (4.1.27.11)$$

4.1.28. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 6 & -2 \end{pmatrix} \quad (4.1.28.1)$$

**Solution:** Let  $c_1$  and  $c_2$  be the column vectors of given matrix  $\mathbf{A}$

$$c_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad (4.1.28.2)$$

$$c_2 = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad (4.1.28.3)$$

We can express the matrix  $\mathbf{A}$  as,

$$\mathbf{A} = \mathbf{Q}\mathbf{R} \quad (4.1.28.4)$$

Where,  $\mathbf{Q}$  is an orthogonal matrix given as,

$$\mathbf{Q} = (\mathbf{u}_1 \quad \mathbf{u}_2) \quad (4.1.28.5)$$

and  $\mathbf{R}$  is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.28.6)$$

Now, we can express  $\alpha$  and  $\beta$  as,

$$c_1 = k_1 \mathbf{u}_1 \quad (4.1.28.7)$$

$$c_2 = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad 4.1.29. \text{ Find the QR decomposition of} \quad (4.1.28.8)$$

$$\text{where, } k_1 = \|\mathbf{c}_1\| = \sqrt{4^2 + (6^2)} = \sqrt{52} \quad (4.1.28.9)$$

Solving equation (4.1.28.7) for  $\mathbf{u}_1$ ,

$$\mathbf{u}_1 = \frac{c_1}{k_1} = \frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad (4.1.28.10)$$

$$\text{Now, } r_1 = \frac{\mathbf{u}_1^T c_2}{\|\mathbf{u}_1\|^2} \quad (4.1.28.11)$$

$$\Rightarrow \frac{\frac{1}{\sqrt{52}} \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix}}{1} \quad (4.1.28.12)$$

$$\text{Hence, } r_1 = -\frac{-24}{\sqrt{52}} \quad (4.1.28.13)$$

$$\mathbf{u}_2 = \frac{c_2 - r_1 \mathbf{u}_1}{\|c_2 - r_1 \mathbf{u}_1\|} \quad (4.1.28.14)$$

$$\Rightarrow \frac{\begin{pmatrix} -3 \\ -2 \end{pmatrix} - \left( \frac{-24}{\sqrt{52}} \right) \left( \frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right)}{\left\| \begin{pmatrix} -3 \\ -2 \end{pmatrix} - \left( -\frac{24}{\sqrt{52}} \frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right) \right\|} \quad (4.1.28.15)$$

$$\Rightarrow \mathbf{u}_2 = \frac{1}{\sqrt{335}} \begin{pmatrix} -15 \\ 10 \end{pmatrix} \quad (4.1.28.16)$$

$$\text{Now, } k_2 = \mathbf{u}_2^T c_2 \quad (4.1.28.17)$$

$$\Rightarrow \frac{1}{\sqrt{335}} \begin{pmatrix} -15 & 10 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad (4.1.28.18)$$

$$\Rightarrow k_2 = \frac{25}{\sqrt{335}} \quad (4.1.28.19)$$

Hence substituting the values of unknown parameter from equations (4.1.28.9), (4.1.28.19), (4.1.28.10), (4.1.28.16) and (4.1.28.13) to

equation (4.1.28.5) and (4.1.28.6) we get,

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{\sqrt{52}} & \frac{-15}{\sqrt{335}} \\ \frac{6}{\sqrt{52}} & \frac{10}{\sqrt{335}} \end{pmatrix} \quad (4.1.28.20)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{52} & \frac{-24}{\sqrt{335}} \\ 0 & \frac{25}{\sqrt{335}} \end{pmatrix} \quad (4.1.28.21)$$

$$\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} \quad (4.1.29.1)$$

4.1.30. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \quad (4.1.30.1)$$

**Solution:**

The matrix  $\mathbf{A}$  can be written as,

$$\mathbf{A} = (\mathbf{a} \quad \mathbf{b}) \quad (4.1.30.2)$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are column vectors. From (4.1.29.1)

$$\mathbf{a} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \quad (4.1.30.3)$$

$$\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (4.1.30.4)$$

The QR decomposition of the given matrix is given by

$$\mathbf{A} = \mathbf{QR} \quad (4.1.30.5)$$

here  $\mathbf{R}$  is a upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.30.6)$$

where

$$\mathbf{Q} = (\mathbf{q}_1 \quad \mathbf{q}_2) \quad \mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \quad (4.1.30.7)$$

The above values are given by,

$$r_1 = \|\mathbf{a}\| \quad (4.1.30.8)$$

$$\mathbf{q}_1 = \frac{\mathbf{a}}{r_1} \quad (4.1.30.9)$$

$$r_2 = \frac{\mathbf{q}_1^T \mathbf{b}}{\|\mathbf{q}_1\|^2} \quad (4.1.30.10)$$

$$\mathbf{q}_2 = \frac{\mathbf{b} - r_2 \mathbf{q}_1}{\|\mathbf{b} - r_2 \mathbf{q}_1\|} \quad (4.1.30.11)$$

$$r_3 = \mathbf{q}_2^T \mathbf{b} \quad (4.1.30.12)$$

Substituting (4.1.30.3) and (4.1.30.4) we get

$$r_1 = \sqrt{8^2 + 3^2} = \sqrt{73} \quad (4.1.30.13)$$

$$\mathbf{q}_1 = \frac{1}{\sqrt{73}} \begin{pmatrix} 8 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{73}} \\ \frac{3}{\sqrt{73}} \end{pmatrix} \quad (4.1.30.14)$$

$$r_2 = \frac{1}{\left(\sqrt{\frac{64}{73} + \frac{9}{73}}\right)^2} \begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{3}{\sqrt{73}} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{46}{\sqrt{73}} \quad (4.1.30.15)$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{73}} \left( \begin{pmatrix} 5 \\ 2 \end{pmatrix} - \frac{46}{\sqrt{73}} \begin{pmatrix} \frac{8}{\sqrt{73}} \\ \frac{3}{\sqrt{73}} \end{pmatrix} \right) = \begin{pmatrix} \frac{-3}{73\sqrt{73}} \\ \frac{8}{73\sqrt{73}} \end{pmatrix} \quad (4.1.30.16)$$

$$r_3 = \begin{pmatrix} \frac{-3}{73\sqrt{73}} & \frac{8}{73\sqrt{73}} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{73\sqrt{73}} \quad (4.1.30.17)$$

Hence substituting these values in (4.1.30.7) and then back in (4.1.30.5) we get,

$$\mathbf{A} = \begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{-3}{73\sqrt{73}} \\ \frac{3}{\sqrt{73}} & \frac{8}{73\sqrt{73}} \end{pmatrix} \begin{pmatrix} \sqrt{73} & \frac{46}{\sqrt{73}} \\ 0 & \frac{1}{73\sqrt{73}} \end{pmatrix} \quad (4.1.30.18)$$

Hence QR decomposition is,

$$\begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{-3}{73\sqrt{73}} \\ \frac{3}{\sqrt{73}} & \frac{8}{73\sqrt{73}} \end{pmatrix} \begin{pmatrix} \sqrt{73} & \frac{46}{\sqrt{73}} \\ 0 & \frac{1}{73\sqrt{73}} \end{pmatrix} \quad (4.1.30.19)$$

4.1.31. Perform QR decomposition on the matrix  $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad (4.1.31.1)$$

**Solution:** The columns of the matrix  $\mathbf{A}$  can be represented as:

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (4.1.31.2)$$

$$\beta = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (4.1.31.3)$$

For QR decomposition, matrix  $\mathbf{A}$  is represented in the form:

$$\mathbf{A} = \mathbf{Q}\mathbf{R} \quad (4.1.31.4)$$

where  $\mathbf{Q}$  and  $\mathbf{R}$  are:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \quad (4.1.31.5)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.31.6)$$

Here  $\mathbf{R}$  is an upper triangular matrix and  $\mathbf{Q}$  is an orthogonal matrix such that,

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.31.7)$$

Now we calculate the above values,

$$k_1 = \|\alpha\| \quad (4.1.31.8)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \quad (4.1.31.9)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (4.1.31.10)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (4.1.31.11)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (4.1.31.12)$$

Substituting (4.1.31.2) and (4.1.31.3) in the above equations, we get

$$k_1 = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (4.1.31.13)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.1.31.14)$$

$$r_1 = \frac{1}{\left(\sqrt{\frac{1}{5} + \frac{4}{5}}\right)^2} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (4.1.31.15)$$

$$\Rightarrow r_1 = \frac{11}{\sqrt{5}} \quad (4.1.31.16)$$

$$\beta - r_1 \mathbf{u}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \left(\frac{11}{\sqrt{5}}\right) \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-2}{5} \end{pmatrix} \quad (4.1.31.17)$$

$$\mathbf{u}_2 = \frac{\begin{pmatrix} \frac{4}{5} \\ \frac{-2}{5} \end{pmatrix}}{\sqrt{\frac{4^2}{5} + \frac{-2^2}{5}}} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} \quad (4.1.31.18)$$

$$k_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{5}} \quad (4.1.31.19)$$

Therefore, from (4.1.31.5) and (4.1.31.6) we get,

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \quad (4.1.31.20)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.1.31.21)$$

where

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (4.1.31.22)$$

Therefore matrix  $\mathbf{A}$  in QR decomposed form is,

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.1.31.23)$$

4.1.32. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \quad (4.1.32.1)$$

**Solution:** Let  $\alpha$  and  $\beta$  be the column vectors of given matrix  $\mathbf{A}$

$$\alpha = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.32.2)$$

$$\beta = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (4.1.32.3)$$

We can express these as,

$$\alpha = k_1 \mathbf{u}_1 \quad (4.1.32.4)$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (4.1.32.5)$$

Where,

$$k_1 = \|\alpha\| \quad (4.1.32.6)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \quad (4.1.32.7)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (4.1.32.8)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (4.1.32.9)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (4.1.32.10)$$

From (4.1.32.4) and (4.1.32.5)

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.32.11)$$

$$\mathbf{A} = \mathbf{QR} \quad (4.1.32.12)$$

From the above equation we can see that  $\mathbf{R}$  is an upper triangular matrix and  $\mathbf{Q}$  is an orthogonal matrix

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.32.13)$$

Now by using equations (4.1.32.2) to

(4.1.32.10)

$$k_1 = \sqrt{9 + 16} = 5 \quad (4.1.32.14)$$

$$\mathbf{u}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix} \quad (4.1.32.15)$$

$$r_1 = \frac{\begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}{\left(\frac{3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2} = \frac{-11}{5} \quad (4.1.32.16)$$

$$\mathbf{u}_2 = \frac{\begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{-11}{5} \begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix}}{\left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{-11}{5} \begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix} \right\|} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \quad (4.1.32.17)$$

$$k_2 = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{2}{5} \quad (4.1.32.18)$$

From equations (4.1.32.11) and (4.1.32.12) the obtained **QR** decomposition is

$$\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 5 & \frac{-11}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \quad (4.1.32.19)$$

4.1.33. Perform QR decomposition of matrix  $\begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix}$

**Solution:** Let  $\mathbf{a}$  and  $\mathbf{b}$  are the columns of matrix  $\mathbf{A}$ . The matrix  $\mathbf{A}$  can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \quad (4.1.33.1)$$

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \quad (4.1.33.2)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.33.3)$$

where

$$k_1 = \|\mathbf{a}\| \quad (4.1.33.4)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (4.1.33.5)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (4.1.33.6)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (4.1.33.7)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (4.1.33.8)$$

The given matrix can be represented as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.33.9)$$

The columns of matrix  $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix}$  are  $\mathbf{a}$  and  $\mathbf{b}$  where

$$\mathbf{a} = \begin{pmatrix} 6 \\ -8 \end{pmatrix} \quad (4.1.33.10)$$

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (4.1.33.11)$$

Now for the given matrix, From (4.1.33.4) and (4.1.33.5)

$$k_1 = \|\mathbf{a}\| = 10 \quad (4.1.33.12)$$

$$\mathbf{u}_1 = \frac{1}{10} \begin{pmatrix} 6 \\ -8 \end{pmatrix} \quad (4.1.33.13)$$

From (4.1.33.6)

$$r_1 = \frac{1}{10} \begin{pmatrix} 6 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1 \quad (4.1.33.14)$$

From (4.1.33.7)

$$\mathbf{b} - r_1 \mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{6}{10} \\ \frac{-8}{10} \end{pmatrix} = \begin{pmatrix} \frac{16}{10} \\ \frac{12}{10} \end{pmatrix} \quad (4.1.33.15)$$

$$\|\mathbf{b} - r_1 \mathbf{u}_1\| = \frac{20}{10} = 2 \quad (4.1.33.16)$$

$$\Rightarrow \mathbf{u}_2 = \frac{1}{10} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \quad (4.1.33.17)$$

From (4.1.33.8)

$$k_2 = \mathbf{u}_2^T \mathbf{b} = \begin{pmatrix} \frac{8}{10} & \frac{6}{10} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{20}{10} = 2 \quad (4.1.33.18)$$

Now we can observe that  $\mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = \mathbf{I}$

$$\begin{pmatrix} \frac{6}{10} & \frac{8}{10} \\ \frac{-8}{10} & \frac{6}{10} \end{pmatrix} \begin{pmatrix} \frac{6}{10} & \frac{-8}{10} \\ \frac{8}{10} & \frac{6}{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.1.33.19)$$

From (4.1.33.9), The matrix  $\mathbf{A}$  can now be written as,

$$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix} = \begin{pmatrix} \frac{6}{10} & \frac{8}{10} \\ \frac{-8}{10} & \frac{6}{10} \end{pmatrix} \begin{pmatrix} 10 & -1 \\ 0 & 2 \end{pmatrix} \quad (4.1.33.20)$$

4.1.34. Find QR decomposition of  $\begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix}$  **Solution:**

Let  $\alpha$  and  $\beta$  be transpose of column vectors of

the given matrix.

$$\alpha = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.34.1)$$

$$\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.1.34.2)$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \quad (4.1.34.3)$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (4.1.34.4)$$

where

$$k_1 = \|\alpha\| \quad (4.1.34.5)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \quad (4.1.34.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (4.1.34.7)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (4.1.34.8)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (4.1.34.9)$$

From (4.1.34.3) and (4.1.34.4),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.34.10)$$

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (4.1.34.11)$$

From above we can see that  $\mathbf{R}$  is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.34.12)$$

Now by using equations (4.1.34.5) to (4.1.34.9)

$$k_1 = 5 \quad (4.1.34.13)$$

$$\mathbf{u}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad (4.1.34.14)$$

$$r_1 = \frac{-1}{5} \quad (4.1.34.15)$$

$$\mathbf{u}_2 = \frac{5}{7} \begin{pmatrix} \frac{28}{25} \\ \frac{21}{25} \end{pmatrix} \quad (4.1.34.16)$$

$$k_2 = \frac{7}{5} \quad (4.1.34.17)$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 5 & -\frac{1}{5} \\ 0 & \frac{7}{5} \end{pmatrix} \quad (4.1.34.18)$$

4.1.35. Find the QR decomposition of  $\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix}$

**Solution:**

Let  $\mathbf{a}$  and  $\mathbf{b}$  are the column vectors of the given matrix. So,

$$\mathbf{a} = \begin{pmatrix} 55 \\ -60 \end{pmatrix} \quad (4.1.35.1)$$

$$\mathbf{b} = \begin{pmatrix} -60 \\ 20 \end{pmatrix} \quad (4.1.35.2)$$

$\mathbf{a}$  and  $\mathbf{b}$  can be expressed as:

$$\mathbf{a} = K_1 \mathbf{u}_1 \quad (4.1.35.3)$$

$$\mathbf{b} = r_1 \mathbf{u}_1 + K_2 \mathbf{u}_2 \quad (4.1.35.4)$$

Where,

$$K_1 = \|\mathbf{a}\| \quad (4.1.35.5)$$

$$\Rightarrow \mathbf{u}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (4.1.35.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (4.1.35.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (4.1.35.8)$$

$$K_2 = \mathbf{u}_2^T \mathbf{b} \quad (4.1.35.9)$$

Then we can express the given matrix as:

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} K_1 & r_1 \\ 0 & K_2 \end{pmatrix} \quad (4.1.35.10)$$

$$\text{or, } \begin{pmatrix} a & b \end{pmatrix} = \mathbf{QR} \quad (4.1.35.11)$$

Now,

$$K_1 = \|\mathbf{a}\| = \sqrt{55^2 + (-60)^2} = 5\sqrt{265} \quad (4.1.35.12)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{5\sqrt{265}} = \frac{1}{\sqrt{265}} \begin{pmatrix} 11 \\ -12 \end{pmatrix} \quad (4.1.35.13)$$

$$\|\mathbf{u}_1\| = 1 \quad (4.1.35.14)$$

$$r_1 = \mathbf{u}_1^T \mathbf{b} = \frac{1}{\sqrt{265}} \begin{pmatrix} 11 & -12 \end{pmatrix} \begin{pmatrix} -60 \\ 20 \end{pmatrix} \quad (4.1.35.15)$$

$$\Rightarrow r_1 = -\frac{900}{\sqrt{265}} \quad (4.1.35.16)$$

So,

$$r_1 \mathbf{u}_1 = -\frac{900}{\sqrt{265}} \frac{1}{\sqrt{265}} \begin{pmatrix} 11 \\ -12 \end{pmatrix} \quad (4.1.35.17)$$

$$\Rightarrow r_1 \mathbf{u}_1 = \begin{pmatrix} -\frac{1980}{2160} \\ \frac{53}{53} \end{pmatrix} \quad (4.1.35.18)$$

$$\mathbf{b} - r_1 \mathbf{u}_1 = \begin{pmatrix} -60 \\ 20 \end{pmatrix} - \begin{pmatrix} -\frac{1980}{2160} \\ \frac{53}{53} \end{pmatrix} \quad (4.1.35.19)$$

$$\Rightarrow \mathbf{b} - r_1 \mathbf{u}_1 = -\frac{100}{53} \begin{pmatrix} 12 \\ 11 \end{pmatrix} \quad (4.1.35.20)$$

$$\|\mathbf{b} - r_1 \mathbf{u}_1\| = \frac{100\sqrt{5}}{\sqrt{53}} \quad (4.1.35.21)$$

Similarly, we can obtain:

$$\mathbf{u}_2 = -\frac{1}{\sqrt{265}} \begin{pmatrix} 12 \\ 11 \end{pmatrix} = \begin{pmatrix} -\frac{12}{\sqrt{265}} \\ \frac{11}{\sqrt{265}} \end{pmatrix} \quad (4.1.35.22)$$

$$K_2 = \mathbf{u}_2^T \mathbf{b} = -\frac{1}{\sqrt{265}} \begin{pmatrix} 12 & 11 \end{pmatrix} \begin{pmatrix} -60 \\ 20 \end{pmatrix} \quad (4.1.35.23)$$

$$= \frac{500}{\sqrt{265}} \quad (4.1.35.24)$$

Now, after QR decomposition of the given matrix we can get:

$$\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix} = \begin{pmatrix} \frac{11}{\sqrt{265}} & \frac{-12}{\sqrt{265}} \\ \frac{-12}{\sqrt{265}} & \frac{11}{\sqrt{265}} \end{pmatrix} \begin{pmatrix} 5\sqrt{265} & \frac{-900}{\sqrt{265}} \\ 0 & \frac{500}{\sqrt{265}} \end{pmatrix} \quad (4.1.35.25)$$

4.1.36. Find QR decomposition for matrix

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (4.1.36.1)$$

**Solution:**

$$\text{Let } \mathbf{V} = (\mathbf{c}_1 \quad \mathbf{c}_2) = \begin{pmatrix} a & h \\ h & b \end{pmatrix} \quad (4.1.36.2)$$

Let  $\mathbf{Q}$  be an orthogonal and  $\mathbf{R}$  be an upper triangular matrix such that,

$$\mathbf{V} = \mathbf{QR} \quad (4.1.36.3)$$

$$= (\mathbf{q}_1 \quad \mathbf{q}_2) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \quad (4.1.36.4)$$

$$(\mathbf{c}_1 \quad \mathbf{c}_2) = (\mathbf{q}_1 r_{11} \quad \mathbf{q}_1 r_{12} + \mathbf{q}_2 r_{22}) \quad (4.1.36.5)$$

$$\Rightarrow \mathbf{c}_1 = \mathbf{q}_1 r_{11} \quad (4.1.36.6)$$

$$r_{11} = \|\mathbf{c}_1\| = \sqrt{a^2 + h^2} \quad (4.1.36.7)$$

$$\mathbf{q}_1 = \frac{\mathbf{c}_1}{r_{11}} \quad (4.1.36.8)$$

$$\mathbf{c}_2 = \mathbf{q}_1 r_{12} + \mathbf{q}_2 r_{22} \quad (4.1.36.9)$$

$$r_{21} = \mathbf{q}_1^T \mathbf{c}_2 \quad (4.1.36.10)$$

$$r_{22} = \|\mathbf{c}_2 - \mathbf{q}_1 r_{12}\| \quad (4.1.36.11)$$

$$\mathbf{q}_2 = \frac{\mathbf{c}_2 - \mathbf{q}_1 r_{12}}{r_{22}} \quad (4.1.36.12)$$

Given,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (4.1.36.13)$$

From (4.1.36.7),

$$r_{11} = \sqrt{6^2 + \left(\frac{17}{2}\right)^2} = \frac{\sqrt{433}}{2} \quad (4.1.36.14)$$

Substitute (4.1.36.14) in (4.1.36.8)

$$\mathbf{q}_1 = \begin{pmatrix} \frac{12}{\sqrt{433}} \\ \frac{17}{\sqrt{433}} \end{pmatrix} \quad (4.1.36.15)$$

Substitute (4.1.36.15) in (4.1.36.10)

$$r_{21} = \frac{306}{\sqrt{433}} \quad (4.1.36.16)$$

Substitute (4.1.36.11) in (4.1.36.16)

$$r_{22} = \frac{\sqrt{433}}{866} \quad (4.1.36.17)$$

Substitute (4.1.36.17) in (4.1.36.12)

$$\mathbf{q}_2 = \begin{pmatrix} \frac{17}{\sqrt{433}} \\ -\frac{12}{\sqrt{433}} \end{pmatrix} \quad (4.1.36.18)$$

Hence QR decomposition of V is,

$$\mathbf{V} = \begin{pmatrix} \frac{12}{\sqrt{433}} & \frac{17}{\sqrt{433}} \\ \frac{17}{\sqrt{433}} & -\frac{12}{\sqrt{433}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{433}}{2} & \frac{306}{\sqrt{433}} \\ 0 & \frac{\sqrt{433}}{866} \end{pmatrix} \quad (4.1.36.19)$$

4.1.37. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix} \quad (4.1.37.1)$$

**Solution:** Let  $\mathbf{x}$  and  $\mathbf{y}$  be the column vectors

of the given matrix.

$$\mathbf{x} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad (4.1.37.2)$$

$$\mathbf{y} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (4.1.37.3)$$

The column vectors can be expressed as follows,

$$\mathbf{x} = k_1 \mathbf{u}_1 \quad (4.1.37.4)$$

$$\mathbf{y} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (4.1.37.5)$$

Here,

$$k_1 = \|\mathbf{x}\| \quad (4.1.37.6)$$

$$\mathbf{u}_1 = \frac{\mathbf{x}}{k_1} \quad (4.1.37.7)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{y}}{\|\mathbf{u}_1\|^2} \quad (4.1.37.8)$$

$$\mathbf{u}_2 = \frac{\mathbf{y} - r_1 \mathbf{u}_1}{\|\mathbf{y} - r_1 \mathbf{u}_1\|} \quad (4.1.37.9)$$

$$k_2 = \mathbf{u}_2^T \mathbf{y} \quad (4.1.37.10)$$

The (4.1.37.4) and (4.1.37.5) can be written as,

$$\begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (4.1.37.11)$$

$$\begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (4.1.37.12)$$

Now,  $\mathbf{R}$  is an upper triangular matrix and also,

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (4.1.37.13)$$

Now using equations (4.1.37.6) to (4.1.37.10) we get,

$$k_1 = \sqrt{7^2 + 2^2} = \sqrt{53} \quad (4.1.37.14)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{53}} \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad (4.1.37.15)$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{7}{\sqrt{53}} \\ \frac{2}{\sqrt{53}} \end{pmatrix} \quad (4.1.37.16)$$

$$r_1 = \left( \frac{7}{\sqrt{53}} \quad \frac{2}{\sqrt{53}} \right) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{29}{\sqrt{53}} \quad (4.1.37.17)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{53}} \begin{pmatrix} 2 \\ -7 \end{pmatrix} \quad (4.1.37.18)$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{2}{\sqrt{53}} \\ -\frac{7}{\sqrt{53}} \end{pmatrix} \quad (4.1.37.19)$$

$$k_2 = \left( \frac{2}{\sqrt{53}} \quad -\frac{7}{\sqrt{53}} \right) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = -\frac{22}{\sqrt{53}} \quad (4.1.37.20)$$

Thus putting the values from (4.1.37.14) to (4.1.37.20) in (4.1.37.12) we obtain QR decomposition,

$$\begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}} \\ -\frac{2}{\sqrt{53}} & -\frac{7}{\sqrt{53}} \end{pmatrix} \begin{pmatrix} \sqrt{53} & \frac{29}{\sqrt{53}} \\ 0 & -\frac{22}{\sqrt{53}} \end{pmatrix} \quad (4.1.37.21)$$

Which can also be written as,

$$\begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{7}{\sqrt{53}} & -\frac{2}{\sqrt{53}} \\ -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}} \end{pmatrix} \begin{pmatrix} -\sqrt{53} & -\frac{29}{\sqrt{53}} \\ 0 & \frac{22}{\sqrt{53}} \end{pmatrix} \quad (4.1.37.22)$$

4.1.38. Find the QR decomposition of

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \quad (4.1.38.1)$$

**Solution:**

$$\text{If, } \mathbf{V} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \text{ Consider, } \mathbf{V} = (\mathbf{a} \ \mathbf{b})$$

$$\text{Where, } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (4.1.38.2)$$

$$\text{Then, } \mathbf{u}_1 = \mathbf{a}, \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$

$$\mathbf{u}_2 = \mathbf{b} - (\mathbf{b} \cdot \mathbf{e}_1) \mathbf{e}_1, \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \quad (4.1.38.3)$$

$$\text{and, } \mathbf{V} = (\mathbf{e}_1 \ \mathbf{e}_2) \begin{pmatrix} \mathbf{a} \cdot \mathbf{e}_1 & \mathbf{b} \cdot \mathbf{e}_1 \\ 0 & \mathbf{b} \cdot \mathbf{e}_2 \end{pmatrix} = \mathbf{QR} \quad (4.1.38.4)$$

Performing QR decomposition on  $\mathbf{V}$  we get,

$$\mathbf{e}_1 = \frac{1}{10\sqrt{2}} \begin{pmatrix} 14 \\ -2 \end{pmatrix} \quad (4.1.38.5)$$

$$\mathbf{e}_2 = \frac{1}{\sqrt{50}} \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (4.1.38.6)$$

$$\mathbf{Q} = (\mathbf{e}_1 \ \mathbf{e}_2) \quad (4.1.38.7)$$

$$\mathbf{R} = \begin{pmatrix} 10\sqrt{2} & -5\sqrt{2} \\ 0 & \frac{75}{\sqrt{50}} \end{pmatrix} \quad (4.1.38.8)$$

It is easy to verify that  $\mathbf{QR} = \mathbf{V}$  and  $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$ . Thus,  $\mathbf{V}$  is decomposed into an orthogonal matrix and an upper triangular matrix.

## 4.2 Singular Value Decomposition

4.2.1. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (4.2.1.1)$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (4.2.1.2)$$

**Solution:**

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (4.2.1.3)$$

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (4.2.1.4)$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.1.5)$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{USV}^T \quad (4.2.1.6)$$

Where the columns of  $\mathbf{V}$  are the eigenvectors of  $\mathbf{A}^T \mathbf{A}$ , the columns of  $\mathbf{U}$  are the eigenvectors of  $\mathbf{AA}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{A}^T \mathbf{A}$ .

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \quad (4.2.1.7)$$

$$\mathbf{MM}^T = \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \quad (4.2.1.8)$$

Calculating eigen value of  $\mathbf{M}^T \mathbf{M}$ .

$$\begin{vmatrix} 6-\lambda & 13 \\ 13 & 38-\lambda \end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \quad (4.2.1.9)$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22 \quad (4.2.1.10)$$

Eigen vectors of  $\mathbf{MM}^T$ .

$$\begin{vmatrix} 13-\lambda & -17 & 8 \\ 17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \quad (4.2.1.11)$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \quad (4.2.1.12)$$



Hence, The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{5} \\ \frac{-3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \quad (4.2.1.13)$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12 - \sqrt{17}}{5}\right)^2 + \left(\frac{1 - 3\sqrt{17}}{5}\right)^2 + 1^2} \quad (4.2.1.14)$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} \end{pmatrix} \quad (4.2.1.15)$$

$$(4.2.1.16)$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \quad (4.2.1.17)$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+30\sqrt{17}}} \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \quad (4.2.1.18)$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} \\ \frac{5}{\sqrt{340+30\sqrt{17}}} \end{pmatrix} \quad (4.2.1.19)$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \quad (4.2.1.20)$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}} \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \quad (4.2.1.21)$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \quad (4.2.1.22)$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \quad (4.2.1.23)$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \quad (4.2.1.24)$$

Now,  $\mathbf{V} = \mathbf{M}^T \frac{\mathbf{u}_i}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (4.2.1.25)$$

So, from equation (4.2.1.6)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} = \quad (4.2.1.26)$$

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \quad (4.2.1.27)$$

$$\begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \quad (4.2.1.28)$$

$$\begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}^T \quad (4.2.1.29)$$

Now, Finding Moore-Penrose Pseudo inverse of  $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \quad (4.2.1.30)$$

We know that,  $\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T \mathbf{b}))$

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+0\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix} \quad (4.2.1.31)$$

$$\mathbf{S}_+(\mathbf{U}^T \mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}} \sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}} \sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (4.2.1.32)$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}} \sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}} \sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}} \sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}} \sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (4.2.1.33)$$

$$\begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}} \sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}} \sqrt{-5\sqrt{17}+22}} \end{pmatrix} \quad (4.2.1.34)$$

$$\mathbf{x} = \begin{pmatrix} \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \\ \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \end{pmatrix} \quad (4.2.1.35)$$

Simplifying the values of  $x_1$  and  $x_2$

$$x_2 = \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \quad (4.2.1.36)$$

$$= \frac{-702100}{591700} \quad (4.2.1.37)$$

$$= -\frac{7}{59} \quad (4.2.1.38)$$

$$x_1 = \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \quad (4.2.1.39)$$

$$= \frac{2507500}{591700} \quad (4.2.1.40)$$

$$= \frac{25}{59} \quad (4.2.1.41)$$

Now, Verifying the values using

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (4.2.1.42)$$

Solving R.H.S

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.2.1.43)$$

Now using equation (4.2.1.7) in (4.2.1.43)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.2.1.44)$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xleftarrow{R_2 - \frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \quad (4.2.1.45)$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \quad (4.2.1.46)$$

$$6x_1 + 13x_2 = 1 \quad (4.2.1.47)$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \quad (4.2.1.48)$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \quad (4.2.1.49)$$

4.2.2. Find the distance of the point  $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$  from the

plane  $\begin{pmatrix} 6 & -3 & 2 \end{pmatrix} \mathbf{x} = 4$

**Solution:**

First we find orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to

the given normal vector  $\mathbf{n}$ . Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (4.2.2.1)$$

$$\Rightarrow \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \quad (4.2.2.2)$$

$$\Rightarrow 6a - 3b + 2c = 0 \quad (4.2.2.3)$$

Putting  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad (4.2.2.4)$$

Putting  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \quad (4.2.2.5)$$

Now we solve the equation,

$$\mathbf{M} \mathbf{x} = \mathbf{b} \quad (4.2.2.6)$$

Putting values in (4.2.2.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & \frac{3}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \quad (4.2.2.7)$$

Now, to solve (4.2.2.7), we perform Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (4.2.2.8)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix} \quad (4.2.2.9)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4} \end{pmatrix} \quad (4.2.2.10)$$

From (4.2.2.6) putting (4.2.2.8) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (4.2.2.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \quad (4.2.2.12)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ . Now, calculating eigen value of  $\mathbf{M}\mathbf{M}^T$ ,

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (4.2.2.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4}-\lambda \end{vmatrix} = 0 \quad (4.2.2.14)$$

$$\Rightarrow \lambda^3 - \frac{53}{4}\lambda^2 + \frac{49}{4}\lambda = 0 \quad (4.2.2.15)$$

Hence eigen values of  $\mathbf{M}\mathbf{M}^T$  are,

$$\lambda_1 = \frac{49}{4} \quad (4.2.2.16)$$

$$\lambda_2 = 1 \quad (4.2.2.17)$$

$$\lambda_3 = 0 \quad (4.2.2.18)$$

Hence the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{15} \\ \frac{2}{15} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix} \quad (4.2.2.19)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{7\sqrt{5}} \\ \frac{2}{7\sqrt{5}} \\ \frac{3\sqrt{5}}{7} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{6}{7} \\ -\frac{3}{7} \\ \frac{2}{7} \end{pmatrix} \quad (4.2.2.20)$$

Hence we obtain  $\mathbf{U}$  of (4.2.2.8) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \quad (4.2.2.21)$$

After computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{S}$  of (4.2.2.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (4.2.2.22)$$

Now, calculating eigen value of  $\mathbf{M}^T\mathbf{M}$ ,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (4.2.2.23)$$

$$\Rightarrow \begin{vmatrix} 10-\lambda & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4}-\lambda \end{vmatrix} = 0 \quad (4.2.2.24)$$

$$\Rightarrow \lambda^2 - \frac{53}{4}\lambda + \frac{49}{4} = 0 \quad (4.2.2.25)$$

Hence eigen values of  $\mathbf{M}^T\mathbf{M}$  are,

$$\lambda_4 = \frac{49}{4} \quad (4.2.2.26)$$

$$\lambda_5 = 1 \quad (4.2.2.27)$$

Hence the eigen vectors of  $\mathbf{M}^T\mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \quad (4.2.2.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.2.2.29)$$

Hence we obtain  $\mathbf{V}$  of (4.2.2.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.2.2.30)$$

Finally from (4.2.2.8) we get the Singular

Value Decomposition of  $\mathbf{M}$  as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (4.2.2.31)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.2.2.32)$$

From (4.2.2.12) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{27}{7\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \\ -\frac{33}{7} \end{pmatrix} \quad (4.2.2.33)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{54}{49\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \end{pmatrix} \quad (4.2.2.34)$$

$$\mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{100}{146} \\ \frac{49}{49} \end{pmatrix} \quad (4.2.2.35)$$

Verifying the solution of (4.2.2.35) using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (4.2.2.36)$$

Evaluating the R.H.S in (4.2.2.36) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \quad (4.2.2.37)$$

$$\Rightarrow \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \quad (4.2.2.38)$$

Solving the augmented matrix of (4.2.2.38) we get,

$$\begin{pmatrix} 10 & \frac{9}{2} & -7 \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \xrightarrow{R_1 = \frac{1}{10} R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \quad (4.2.2.39)$$

$$\xrightarrow{R_2 = R_2 - \frac{9}{2} R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & \frac{49}{40} & \frac{73}{20} \end{pmatrix} \quad (4.2.2.40)$$

$$\xrightarrow{R_2 = \frac{40}{49} R_2} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \quad (4.2.2.41)$$

$$\xrightarrow{R_1 = R_1 - \frac{9}{20} R_2} \begin{pmatrix} 1 & 0 & -\frac{100}{49} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \quad (4.2.2.42)$$

Hence, Solution of (4.2.2.36) is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \quad (4.2.2.43)$$

Comparing results of  $\mathbf{x}$  from (4.2.2.35) and (4.2.2.43) we conclude that the solution is verified.

4.2.3. Check whether the given line equations intersect. If they do not intersect find the closest points on the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (4.2.3.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (4.2.3.2)$$

**Solution:**

Given

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (4.2.3.3)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (4.2.3.4)$$

The above equations (4.2.3.3), (4.2.3.4) are in the form

$$L_1 : \quad \mathbf{x} = \mathbf{a}_1 + \lambda_1 \mathbf{b}_1 \quad (4.2.3.5)$$

$$L_2 : \quad \mathbf{x} = \mathbf{a}_2 + \lambda_2 \mathbf{b}_2 \quad (4.2.3.6)$$

Here ,

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \quad (4.2.3.7)$$

$$\mathbf{a}_2 = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} \quad (4.2.3.8)$$

$$\mathbf{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (4.2.3.9)$$

$$\mathbf{b}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (4.2.3.10)$$

Now let us assume the lines  $L_1$  and  $L_2$  are

intersecting at a point. Therefore ,

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (4.2.3.11)$$

$$\lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \quad (4.2.3.12)$$

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \quad (4.2.3.13)$$

The augmented matrix of (4.2.3.13) is given by

$$\left( \begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array} \right) \quad (4.2.3.14)$$

$$\left( \begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array} \right) \xrightarrow{R_2=R_2-\frac{2}{3}R_1} \left( \begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array} \right) \quad (4.2.3.15)$$

$$\left( \begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array} \right) \xrightarrow{R_3=R_3-2R_1} \left( \begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{array} \right) \quad (4.2.3.16)$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect. Hence our assumptions is wrong

Equation (4.2.3.13) can be expressed as

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.3.17)$$

By singular value decomposition  $\mathbf{M}$  can be expressed as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (4.2.3.18)$$

Where the columns of  $\mathbf{V}$  are the eigenvectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigenvectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix} \quad (4.2.3.19)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \quad (4.2.3.20)$$

The characteristic equation of  $\mathbf{M}^T\mathbf{M}$  is obtained by evaluating the determinant

$$\begin{vmatrix} 49 - \lambda & -19 \\ -19 & 9 - \lambda \end{vmatrix} = 0 \quad (4.2.3.21)$$

$$\Rightarrow \lambda^2 - 58\lambda + 80 = 0 \quad (4.2.3.22)$$

The eigenvalues are the roots of equation 4.2.3.22 is given by

$$\lambda_{11} = 29 + \sqrt{761} \quad (4.2.3.23)$$

$$\lambda_{12} = 29 - \sqrt{761} \quad (4.2.3.24)$$

The eigen vectors comes out to be ,

$$\mathbf{u}_{11} = \begin{pmatrix} \frac{-20-\sqrt{761}}{19} \\ 1 \end{pmatrix}, \mathbf{u}_{12} = \begin{pmatrix} \frac{-20+\sqrt{761}}{19} \\ 1 \end{pmatrix} \quad (4.2.3.25)$$

Normalising the eigen vectors,

$$l_{11} = \sqrt{\left(\frac{-20-\sqrt{761}}{19}\right)^2 + 1^2} \quad (4.2.3.26)$$

$$\Rightarrow l_{11} = \frac{\sqrt{1522 + 40\sqrt{761}}}{19} \quad (4.2.3.27)$$

$$\mathbf{u}_{11} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} \\ \frac{\sqrt{1522+40\sqrt{761}}}{\sqrt{1522+40\sqrt{761}}} \end{pmatrix} \quad (4.2.3.28)$$

$$l_{12} = \sqrt{\left(\frac{-20+\sqrt{761}}{19}\right)^2 + 1^2} \quad (4.2.3.29)$$

$$\Rightarrow l_{12} = \frac{\sqrt{1522 - 40\sqrt{761}}}{19} \quad (4.2.3.30)$$

$$\mathbf{u}_{12} = \begin{pmatrix} \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{\sqrt{1522-40\sqrt{761}}}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \quad (4.2.3.31)$$

$$\mathbf{V} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} & \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{\sqrt{1522+40\sqrt{761}}}{\sqrt{1522+40\sqrt{761}}} & \frac{\sqrt{1522-40\sqrt{761}}}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \quad (4.2.3.32)$$

$\mathbf{S}$  is given by

$$\mathbf{S} = \begin{pmatrix} \sqrt{29 + \sqrt{761}} & 0 \\ 0 & \sqrt{29 - \sqrt{761}} \\ 0 & 0 \end{pmatrix} \quad (4.2.3.33)$$

The characteristic equation of  $\mathbf{MM}^T$  is obtained by evaluating the determinant

$$\begin{vmatrix} 10 - \lambda & 8 & 20 \\ 8 & 8 - \lambda & 16 \\ 20 & 16 & 40 - \lambda \end{vmatrix} = 0 \quad (4.2.3.34)$$

$$\Rightarrow \lambda^3 - 58\lambda^2 + 80\lambda = 0 \quad (4.2.3.35)$$

The eigenvalues are the roots of equation 4.2.3.35 is given by

$$\lambda_{21} = 29 + \sqrt{761} \quad (4.2.3.36)$$

$$\lambda_{22} = 29 - \sqrt{761} \quad (4.2.3.37)$$

$$\lambda_{23} = 0 \quad (4.2.3.38)$$

The eigen vectors comes out to be ,

$$\mathbf{u}_{21} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-\sqrt{761}+21}{16} \\ -1 \end{pmatrix}, \mathbf{u}_{22} = \begin{pmatrix} \frac{1}{2} \\ \frac{-\sqrt{761}-21}{16} \\ 1 \end{pmatrix}, \mathbf{u}_{23} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad (4.2.3.39)$$

Normalising the eigen vectors,

$$l_{21} = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{21 - \sqrt{761}}{16}\right)^2 + (-1)^2} \quad (4.2.3.40)$$

$$\Rightarrow l_{21} = \frac{\sqrt{1522 - 42\sqrt{761}}}{16} \quad (4.2.3.41)$$

$$\mathbf{u}_{21} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \quad (4.2.3.42)$$

$$l_{22} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-21 - \sqrt{761}}{16}\right)^2 + 1^2} \quad (4.2.3.43)$$

$$\Rightarrow l_{22} = \frac{\sqrt{1522 + 42\sqrt{761}}}{16} \quad (4.2.3.44)$$

$$\mathbf{u}_{22} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} \end{pmatrix} \quad (4.2.3.45)$$

$$l_{23} = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \quad (4.2.3.46)$$

$$\mathbf{u}_{23} = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.3.47)$$

$$\mathbf{U} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.3.48)$$

From equation (4.2.3.18) we rewrite  $\mathbf{M}$  as follows,

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.3.49)$$

$$\begin{pmatrix} \sqrt{29 + \sqrt{761}} \\ 0 \\ 0 \end{pmatrix} \quad (4.2.3.50)$$

$$\begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} \end{pmatrix} \quad (4.2.3.51)$$

By substituting the equation (4.2.3.18) in equation (4.2.3.17) we get

$$\mathbf{USV}^T \mathbf{x} = \mathbf{b} \quad (4.2.3.52)$$

$$\Rightarrow \mathbf{x} = \mathbf{VS}_+ \mathbf{U}^T \mathbf{b} \quad (4.2.3.53)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{29+\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29-\sqrt{761}}} & 0 \end{pmatrix} \quad (4.2.3.54)$$

From (4.2.3.53) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{\sqrt{761}-45}{\sqrt{1522-42\sqrt{761}}} \\ \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ -\frac{11}{\sqrt{5}} \end{pmatrix} \quad (4.2.3.55)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{761\sqrt{15}-761-45\sqrt{11415}+45\sqrt{761}}{10654} \\ \frac{10654}{45\sqrt{11415}+45\sqrt{761}+761\sqrt{15}+761} \\ \frac{10654}{10654} \end{pmatrix} \quad (4.2.3.56)$$

$$\mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \\ \frac{21}{20} \end{pmatrix} \quad (4.2.3.57)$$

Verifying the solution of (4.2.3.57) using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (4.2.3.58)$$

Evaluating the R.H.S in (4.2.3.58) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \quad (4.2.3.59)$$

$$\Rightarrow \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \quad (4.2.3.60)$$

Solving the augmented matrix of (4.2.3.60) we get,

$$\left( \begin{array}{ccc|c} 49 & -19 & 7 & 7 \\ -19 & 9 & -1 & -1 \end{array} \right) \xrightarrow{R_2=R_2+\frac{19}{49}R_1} \left( \begin{array}{ccc|c} 49 & -19 & 7 & 7 \\ 0 & \frac{80}{49} & \frac{12}{7} & \frac{12}{7} \end{array} \right) \quad (4.2.3.61)$$

$$\xrightarrow{R_1=\frac{1}{49}R_1} \left( \begin{array}{ccc|c} 1 & -\frac{19}{49} & \frac{7}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{12}{7} & \frac{12}{7} \end{array} \right) \quad (4.2.3.62)$$

$$\xrightarrow{R_2=\frac{80}{49}R_2} \left( \begin{array}{ccc|c} 1 & -\frac{19}{49} & \frac{7}{49} & \frac{7}{49} \\ 0 & 1 & \frac{21}{20} & \frac{21}{20} \end{array} \right) \quad (4.2.3.63)$$

$$\xrightarrow{R_1=R_1+\frac{19}{49}R_2} \left( \begin{array}{ccc|c} 1 & 0 & \frac{11}{20} & \frac{11}{20} \\ 0 & 1 & \frac{21}{20} & \frac{21}{20} \end{array} \right) \quad (4.2.3.64)$$

Hence, Solution of (4.2.3.58) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \\ \frac{21}{20} \end{pmatrix} \quad (4.2.3.65)$$

Comparing results of  $\mathbf{x}$  from (4.2.3.57) and (4.2.3.65) we conclude that the solution is verified.

4.2.4. Check if the lines  $L_1, L_2$  are skew. If so, find the closest points on those lines using Singular

Value Decomposition(SVD)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (4.2.4.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (4.2.4.2)$$

**Solution:**

The matrix  $\mathbf{M}$  of dimensions  $(m \times n)$  can be decomposed using SVD as

$$\mathbf{M} = \mathbf{U} \mathbf{S} \mathbf{V}^T \quad (4.2.4.3)$$

where, columns of  $\mathbf{U}_{(m \times m)}$  are eigen vectors of  $\mathbf{M} \mathbf{M}^T$

columns of  $\mathbf{V}_{(n \times n)}$  are eigen vectors of  $\mathbf{M}^T \mathbf{M}$   
 $\mathbf{S}$  is a diagonal matrix containing singular values of  $\mathbf{M}$ . Also,  $\mathbf{U}$  and  $\mathbf{V}$  are orthogonal matrices

$$\mathbf{U} \mathbf{U}^T = \mathbf{U}^T \mathbf{U} = \mathbf{I} \quad (4.2.4.4)$$

$$\mathbf{V} \mathbf{V}^T = \mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (4.2.4.5)$$

Given line equations intersect if

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (4.2.4.6)$$

This can be written as

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \quad (4.2.4.7)$$

$$\mathbf{M} \mathbf{x} = \mathbf{b} \quad (4.2.4.8)$$

$$\text{where, } \mathbf{x} = \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (4.2.4.9)$$

The augmented matrix is

$$\left( \begin{array}{ccc|c} 3 & 1 & 5 \\ 2 & 2 & -1 \\ 6 & 2 & -1 \end{array} \right) \xrightarrow{\begin{array}{l} R_3 \leftarrow R_3 - 2 \times R_1 \\ R_2 \leftarrow R_2 - R_1 \times \frac{2}{3} \end{array}} \left( \begin{array}{ccc|c} 3 & 1 & 5 \\ 0 & \frac{5}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{array} \right) \quad (4.2.4.10)$$

So, the given pair of lines do not intersect and also their direction vectors are not parallel. Hence they are skew lines.

To find  $\mathbf{U}$ ,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \quad (4.2.4.11)$$

To calculate its Eigen values,

$$\begin{vmatrix} 10 - \lambda & 8 & 20 \\ 8 & 8 - \lambda & 16 \\ 20 & 16 & 40 - \lambda \end{vmatrix} = 0 \quad (4.2.4.12)$$

$$\Rightarrow \lambda^3 + 58\lambda^2 + 80\lambda = 0 \quad (4.2.4.13)$$

$$\lambda_1 = 29 - \sqrt{761}, \lambda_2 = 29 + \sqrt{761}, \lambda_3 = 0 \quad (4.2.4.14)$$

with corresponding Eigen vectors as

$$\mathbf{u}_1 = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ -\frac{21+\sqrt{761}}{16} \\ 1 \end{pmatrix} \quad (4.2.4.15)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{-21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ -\frac{21+\sqrt{761}}{16} \\ 1 \end{pmatrix} \quad (4.2.4.16)$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{(-2)^2 + 1}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad (4.2.4.17)$$

Solving, the  $\mathbf{U}$  matrix becomes

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.4.18)$$

Also, from the obtained Eigen values, the  $\mathbf{S}$  matrix becomes

$$\mathbf{S} = \begin{pmatrix} \sqrt{29 - \sqrt{761}} & 0 \\ 0 & \sqrt{29 + \sqrt{761}} \\ 0 & 0 \end{pmatrix} \quad (4.2.4.19)$$

The Moore-Penrose pseudo inverse of  $\mathbf{S}$  is

given by

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{29 - \sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29 + \sqrt{761}}} & 0 \end{pmatrix} \quad (4.2.4.20)$$

Now to find  $\mathbf{V}$ ,

Rewriting (4.2.4.3)

$$\mathbf{V} = (\mathbf{M}^T \mathbf{U}) \mathbf{S}_+^T \quad (4.2.4.21)$$

$\mathbf{M}^T \mathbf{U}$  becomes

$$\begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} \quad (4.2.4.22)$$

$$\begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.4.23)$$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \end{pmatrix} \quad (4.2.4.24)$$

Therefore from (4.2.4.20), (4.2.4.21), (4.2.4.24),

$$\mathbf{V} = \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \frac{\sqrt{29-\sqrt{761}}}{\sqrt{1522+42\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \frac{\sqrt{29+\sqrt{761}}}{\sqrt{1522-42\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \frac{\sqrt{29-\sqrt{761}}}{\sqrt{1522+42\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \frac{\sqrt{29+\sqrt{761}}}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \quad (4.2.4.25)$$

Now, to calculate  $\mathbf{x}$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.4.26)$$

$$\Rightarrow \mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (4.2.4.27)$$

$$\Rightarrow \mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{U}^T \mathbf{b} \quad (4.2.4.28)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T \mathbf{b})) \quad (4.2.4.29)$$

Calculating  $\mathbf{U}^T \mathbf{b}$ , we have

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \quad (4.2.4.30)$$

$$\mathbf{S}_+(\mathbf{U}^T \mathbf{b}) = \begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \frac{\sqrt{29-\sqrt{761}}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \frac{\sqrt{29+\sqrt{761}}}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \quad (4.2.4.31)$$



$$\mathbf{V}(\mathbf{S}_+(\mathbf{U}^T \mathbf{b}))$$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \quad (4.2.4.32)$$

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \quad (4.2.4.33)$$

Solving,

$$\mathbf{x} = \begin{pmatrix} \frac{8371}{15220} \\ \frac{-15981}{15220} \end{pmatrix} = \begin{pmatrix} \frac{11}{20} \\ \frac{-21}{20} \end{pmatrix} \quad (4.2.4.34)$$

Verifying the solution,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.4.35)$$

$$\Rightarrow \mathbf{M}^T \mathbf{M}\mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (4.2.4.36)$$

$$\mathbf{M}^T \mathbf{b} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \quad (4.2.4.37)$$

$$= \begin{pmatrix} 7 \\ 1 \end{pmatrix} \quad (4.2.4.38)$$

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \quad (4.2.4.39)$$

$$= \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix} \quad (4.2.4.40)$$

$$\text{From, (4.2.4.36)} \quad \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \quad (4.2.4.41)$$

Solving for  $\mathbf{x}$

$$\begin{pmatrix} 49 & 19 & 7 \\ 19 & 9 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1 \times \frac{19}{49}} \begin{pmatrix} 49 & 19 & 7 \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \quad (4.2.4.42)$$

$$\xrightarrow{R_1 \leftarrow R_1 \times \frac{1}{49}} \begin{pmatrix} 1 & \frac{19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \quad (4.2.4.43)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2 \times \frac{19}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{80} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \quad (4.2.4.44)$$

$$\xrightarrow{R_2 \leftarrow R_2 \times \frac{49}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{80} \\ 0 & 1 & \frac{-21}{20} \end{pmatrix} \quad (4.2.4.45)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{11}{80} \\ \frac{-21}{20} \end{pmatrix} \quad (4.2.4.46)$$

4.2.5. Find the point on the plane closest to the point

$$\begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix} \text{ and the plane is determined by the points}$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$$

**Solution:** The equation of plane is given by,

$$\mathbf{n}^T \mathbf{x} = c \quad (4.2.5.1)$$

$$\mathbf{n}^T \mathbf{A} = \mathbf{n}^T \mathbf{B} = \mathbf{n}^T \mathbf{C} = c \quad (4.2.5.2)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B} \quad \mathbf{B} - \mathbf{C})^T \mathbf{n} = 0 \quad (4.2.5.3)$$

Using row reduction on above matrix,

$$\begin{pmatrix} -2 & -3 & -2 \\ 6 & 3 & -2 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ 6 & 3 & -2 \end{pmatrix} \quad (4.2.5.4)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 6R_1} \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ 0 & -6 & -8 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{-6}} \begin{pmatrix} 1 & \frac{3}{2} & 1 \\ 0 & 1 & \frac{4}{3} \end{pmatrix} \quad (4.2.5.5)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{4}{3} \end{pmatrix} \quad (4.2.5.6)$$

Thus,

$$\mathbf{n} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} \quad (4.2.5.7)$$

$$c = \mathbf{n}^T \mathbf{A} = 19 \quad (4.2.5.8)$$

Thus the equation of the plane is,

$$(3 \ -4 \ 3)\mathbf{x} = 19 \quad (4.2.5.9)$$

Let  $\mathbf{m}_1$  and  $\mathbf{m}_2$  be the two orthogonal vectors

to the given normal. Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (4.2.5.10)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} = 0 \quad (4.2.5.11)$$

$$\Rightarrow 3a - 4b + 3c = 0 \quad (4.2.5.12)$$

Let  $a = 1, b = 0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (4.2.5.13)$$

Let  $a = 0, b = 1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{4}{3} \end{pmatrix} \quad (4.2.5.14)$$

Solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.5.15)$$

Putting the values in (4.2.5.15),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{4}{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix} \quad (4.2.5.16)$$

To solve (4.2.5.16), we perform Singular Value Decomposition on  $\mathbf{M}$ ,

$$\mathbf{M} = \mathbf{USV}^T \quad (4.2.5.17)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T \mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{MM}^T$  and  $\mathbf{S}$  is diagonal matrix of

singular value of eigenvalues of  $\mathbf{M}^T \mathbf{M}$ .

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 2 & -\frac{4}{3} \\ -\frac{4}{3} & \frac{25}{9} \end{pmatrix} \quad (4.2.5.18)$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{4}{3} \\ -1 & \frac{4}{3} & \frac{25}{9} \end{pmatrix} \quad (4.2.5.19)$$

Putting (4.2.5.17) in (4.2.5.15) we get,

$$\mathbf{USV}^T \mathbf{x} = \mathbf{b} \quad (4.2.5.20)$$

$$\Rightarrow \mathbf{x} = \mathbf{VS}_+ \mathbf{U}^T \mathbf{b} \quad (4.2.5.21)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ . Now, calculating eigen values of  $\mathbf{MM}^T$ ,

$$|\mathbf{MM}^T - \lambda \mathbf{I}| = 0 \quad (4.2.5.22)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{4}{3} \\ -1 & \frac{4}{3} & \frac{25}{9}-\lambda \end{vmatrix} = 0 \quad (4.2.5.23)$$

$$\Rightarrow \lambda^3 - \frac{43}{9}\lambda^2 + \frac{34}{9}\lambda = 0 \quad (4.2.5.24)$$

Thus the eigen values of  $\mathbf{MM}^T$  are,

$$\lambda_1 = \frac{34}{9} \quad (4.2.5.25)$$

$$\lambda_2 = 1 \quad (4.2.5.26)$$

$$\lambda_3 = 0 \quad (4.2.5.27)$$

The eigen vectors comes out to be,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{9}{25} \\ \frac{12}{25} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{4}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ -\frac{4}{3} \\ 1 \end{pmatrix} \quad (4.2.5.28)$$

Normalising the eigen vectors,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-9}{5\sqrt{34}} \\ \frac{12}{5\sqrt{34}} \\ \frac{5}{\sqrt{34}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{3}{\sqrt{34}} \\ \frac{-4}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{pmatrix} \quad (4.2.5.29)$$

Hence we obtain  $\mathbf{U}$  matrix as,

$$\mathbf{U} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \quad (4.2.5.30)$$

Now,

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (4.2.5.31)$$

Calculating the eigen values of  $\mathbf{M}^T\mathbf{M}$ ,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (4.2.5.32)$$

$$\Rightarrow \begin{pmatrix} 2 - \lambda & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} - \lambda \end{pmatrix} = 0 \quad (4.2.5.33)$$

$$\Rightarrow \lambda^2 - \frac{43}{9}\lambda + \frac{34}{9} = 0 \quad (4.2.5.34)$$

The eigen values are,

$$\lambda_1 = \frac{34}{9} \quad (4.2.5.35)$$

$$\lambda_2 = 1 \quad (4.2.5.36)$$

The eigen vectors are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \quad (4.2.5.37)$$

Normalising the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \quad (4.2.5.38)$$

Hence we obtain  $\mathbf{V}$  matrix as,

$$\mathbf{V} = \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \quad (4.2.5.39)$$

Thus we get the Singular Value Decomposition of  $\mathbf{M}$  as,

$$\mathbf{M} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}^T \quad (4.2.5.40)$$

The Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.2.5.41)$$

From (4.2.5.21) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{231}{5\sqrt{34}} \\ \frac{39}{5} \\ \frac{25}{\sqrt{34}} \end{pmatrix} \quad (4.2.5.42)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{693}{170} \\ \frac{39}{5} \end{pmatrix} \quad (4.2.5.43)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \quad (4.2.5.44)$$

Verifying the solution of (4.2.5.44) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (4.2.5.45)$$

Evaluating the R.H.S in (4.2.5.45) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \quad (4.2.5.46)$$

$$\Rightarrow \begin{pmatrix} 2 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \quad (4.2.5.47)$$

Solving the augmented matrix of (4.2.5.47) we get,

$$\begin{pmatrix} 2 & \frac{-4}{3} & -3 \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \quad (4.2.5.48)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{4}{3}R_1} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ 0 & \frac{17}{9} & \frac{15}{2} \end{pmatrix} \quad (4.2.5.49)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & \frac{17}{9} & \frac{15}{2} \end{pmatrix} \quad (4.2.5.50)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{9}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & 1 & \frac{135}{17} \end{pmatrix} \quad (4.2.5.51)$$

Hence, solution of (4.2.5.45) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \quad (4.2.5.52)$$

Comparing results of  $\mathbf{x}$  from (4.2.5.44) and (4.2.5.52) we conclude that the solution is verified. Find the foot of the perpendicular

using svd drawn from  $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$  to the plane

$$(2 \quad -1 \quad 2)\mathbf{x} + 3 = 0 \quad (4.2.5.53)$$

**Solution:**

4.2.6. Find the distance of the given point  $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$  from

the plane  $(2 \quad -1 \quad 2)\mathbf{x} = 3$ .

**Solution:** Let us consider orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ . Let,

$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (4.2.6.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0 \quad (4.2.6.2)$$

$$\Rightarrow 2a - b + 2c = 0 \quad (4.2.6.3)$$

Let  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (4.2.6.4)$$

Let  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \quad (4.2.6.5)$$

Let us solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.6.6)$$

Substituting (4.2.6.4) and (4.2.6.5) in (4.2.6.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad (4.2.6.7)$$

To solve (4.2.6.7), we will perform Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (4.2.6.8)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T \mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T \mathbf{M}$ .

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \quad (4.2.6.9)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \quad (4.2.6.10)$$

Substituting (4.2.6.8) in (4.2.6.6),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (4.2.6.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (4.2.6.12)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ .

Let us calculate eigen values of  $\mathbf{M}\mathbf{M}^T$ ,

$$|\mathbf{M}\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (4.2.6.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4}-\lambda \end{vmatrix} = 0 \quad (4.2.6.14)$$

$$\Rightarrow \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \quad (4.2.6.15)$$

From equation (4.2.6.15) eigen values of  $\mathbf{M}\mathbf{M}^T$  are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (4.2.6.16)$$

The eigen vectors of  $\mathbf{M}\mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \quad (4.2.6.17)$$

Normalizing the eigen vectors in equation (4.2.6.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \quad (4.2.6.18)$$

Hence we obtain  $\mathbf{U}$  as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \quad (4.2.6.19)$$

After computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{S}$  as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (4.2.6.20)$$

Now, lets calculate eigen values of  $\mathbf{M}^T \mathbf{M}$ ,

$$|\mathbf{M}^T \mathbf{M} - \lambda \mathbf{I}| = 0 \quad (4.2.6.21)$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{vmatrix} = 0 \quad (4.2.6.22)$$

$$\Rightarrow \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (4.2.6.23)$$

Hence eigen values of  $\mathbf{M}^T \mathbf{M}$  are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad (4.2.6.24)$$

Hence the eigen vectors of  $\mathbf{M}^T\mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (4.2.6.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.2.6.26)$$

Hence we obtain  $\mathbf{V}$  as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (4.2.6.27)$$

From (4.2.6.6), the Singular Value Decomposition of  $\mathbf{M}$  is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (4.2.6.28)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.2.6.29)$$

From (4.2.6.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{11}{3\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ \frac{10}{3} \end{pmatrix} \quad (4.2.6.30)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{22}{9\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.6.31)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \quad (4.2.6.32)$$

Verifying the solution of (4.2.6.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (4.2.6.33)$$

Evaluating the R.H.S in (4.2.6.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \quad (4.2.6.34)$$

$$\Rightarrow \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \quad (4.2.6.35)$$

Solving the augmented matrix of (4.2.6.35) we

get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & 2 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \quad (4.2.6.36)$$

$$\xrightarrow{R_2 = R_2 + \frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{9}{8} & -1 \end{pmatrix} \quad (4.2.6.37)$$

$$\xrightarrow{R_2 = \frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \quad (4.2.6.38)$$

$$\xrightarrow{R_1 = R_1 + \frac{R_2}{4}} \begin{pmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \quad (4.2.6.39)$$

From equation (4.2.6.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \quad (4.2.6.40)$$

Comparing results of  $\mathbf{x}$  from (4.2.6.32) and (4.2.6.40), we can say that the solution is verified.

4.2.7. Find the distance of the point  $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$  from the plane  $(1 \ 2 \ -2)\mathbf{x} = 9$

**Solution:**

Find the distance of the point  $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$  from the

plane  $(1 \ 2 \ -2)\mathbf{x} = 9$  First we find orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector

$\mathbf{n}$ . Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T\mathbf{n} = 0 \quad (4.2.7.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0 \quad (4.2.7.2)$$

$$\Rightarrow a + 2b - 2c = 0 \quad (4.2.7.3)$$

Putting  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (4.2.7.4)$$

Putting  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (4.2.7.5)$$

Now we solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.7.6)$$

Putting values in (4.2.7.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \quad (4.2.7.7)$$

In order to solve (4.2.7.7), perform Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (4.2.7.8)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \quad (4.2.7.9)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{5}{4} \end{pmatrix} \quad (4.2.7.10)$$

From (4.2.7.6) putting (4.2.7.8) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (4.2.7.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+^T\mathbf{U}^T\mathbf{b} \quad (4.2.7.12)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ . Now, calculating eigen value of  $\mathbf{M}\mathbf{M}^T$ ,

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (4.2.7.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 1 \\ \frac{1}{2} & 1 & \frac{5}{4}-\lambda \end{vmatrix} = 0 \quad (4.2.7.14)$$

$$\Rightarrow -4\lambda^3 + 13\lambda^2 - 9\lambda = 0 \quad (4.2.7.15)$$

Hence eigen values of  $\mathbf{M}\mathbf{M}^T$  are,

$$\lambda_1 = \frac{9}{4} \quad (4.2.7.16)$$

$$\lambda_2 = 1 \quad (4.2.7.17)$$

$$\lambda_3 = 0 \quad (4.2.7.18)$$

Hence the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix} \quad (4.2.7.19)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad (4.2.7.20)$$

Hence we obtain  $\mathbf{U}$  of (4.2.7.8) as follows,

$$\begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \quad (4.2.7.21)$$

After computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{S}$  of (4.2.7.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (4.2.7.22)$$

Now, calculating eigen value of  $\mathbf{M}^T\mathbf{M}$ ,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (4.2.7.23)$$

$$\Rightarrow \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{2} \\ \frac{1}{2} & 2-\lambda \end{vmatrix} = 0 \quad (4.2.7.24)$$

$$\Rightarrow \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (4.2.7.25)$$

Hence eigen values of  $\mathbf{M}^T\mathbf{M}$  are,

$$\lambda_4 = \frac{9}{4} \quad (4.2.7.26)$$

$$\lambda_5 = 1 \quad (4.2.7.27)$$

Hence the eigen vectors of  $\mathbf{M}^T\mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (4.2.7.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.7.29)$$

Hence we obtain  $\mathbf{V}$  of (4.2.7.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (4.2.7.30)$$

From (4.2.7.8) we get the Singular Value Decomposition of  $\mathbf{M}$ ,

$$\mathbf{M} = \begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}^T \quad (4.2.7.31)$$

Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.2.7.32)$$

From (4.2.7.11) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{3\sqrt{5}}{5} \\ \frac{5}{5} \\ -\frac{6}{5} \end{pmatrix} \quad (4.2.7.33)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ \frac{5}{5} \\ -\frac{6}{5} \end{pmatrix} \quad (4.2.7.34)$$

$$\mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (4.2.7.35)$$

Verifying the solution of (4.2.7.35) using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (4.2.7.36)$$

Evaluating the R.H.S in (4.2.7.36) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix} \quad (4.2.7.37)$$

$$\Rightarrow \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix} \quad (4.2.7.38)$$

Solving the augmented matrix of (4.2.7.38) we get,

$$\left( \begin{array}{ccc|c} \frac{5}{4} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 2 & -2 & -2 \end{array} \right) \xrightarrow{R_1 = \frac{4}{5} R_1} \left( \begin{array}{ccc|c} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{1}{2} & 2 & -2 & -2 \end{array} \right) \quad (4.2.7.39)$$

$$\xrightarrow{R_2 = R_2 - \frac{1}{2} R_1} \left( \begin{array}{ccc|c} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ 0 & \frac{9}{5} & -\frac{9}{5} & -\frac{9}{5} \end{array} \right) \quad (4.2.7.40)$$

$$\xrightarrow{R_2 = \frac{5}{9} R_2} \left( \begin{array}{ccc|c} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -1 & -1 \end{array} \right) \quad (4.2.7.41)$$

$$\xrightarrow{R_1 = R_1 - \frac{2}{5} R_2} \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{array} \right) \quad (4.2.7.42)$$

From equation (4.2.7.42), solution is given by,

$$\mathbf{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (4.2.7.43)$$

Comparing results of  $\mathbf{x}$  from (4.2.7.35) and (4.2.7.43), we can say that the solution is verified.

4.2.8. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \quad (4.2.8.1)$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \quad (4.2.8.2)$$

are perpendicular, find the value of  $k$ .

**Solution:** In the given problem,

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \quad (4.2.8.3)$$

To find the value of  $k$ , let's assume that the given lines are perpendicular to each other. Then the dot product of their direction vectors should be 0. i.e.,

$$\mathbf{m}_1 \mathbf{m}_2 = 0 \quad (4.2.8.4)$$

$$\Rightarrow \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = 0 \quad (4.2.8.5)$$

$$\Rightarrow k = -\frac{10}{7} \quad (4.2.8.6)$$

The lines will intersect if

$$\mathbf{A}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{A}_2 + \lambda_2 \mathbf{m}_2 \quad (4.2.8.7)$$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \quad (4.2.8.8)$$

$$\Rightarrow \lambda_1 \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad (4.2.8.9)$$

$$\Rightarrow \begin{pmatrix} -3 & 3k \\ 2k & 1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad (4.2.8.10)$$

$$\Rightarrow \begin{pmatrix} -3 & -\frac{30}{7} \\ -\frac{20}{7} & 1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad (4.2.8.11)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -3 & -\frac{30}{7} & 2 \\ -\frac{20}{7} & 1 & -1 \\ 2 & -5 & 3 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + \frac{20}{7}R_1]{R_1 \leftarrow -\frac{R_1}{3}} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & \frac{249}{49} & -\frac{61}{21} \\ 2 & -5 & 3 \end{pmatrix} \quad (4.2.8.12)$$

$$\xrightarrow[R_2 \leftarrow \frac{49}{249}R_2]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & \frac{249}{49} & -\frac{61}{21} \\ 2 & -5 & 3 \end{pmatrix} \quad (4.2.8.13)$$

$$\xrightarrow[R_2 \leftarrow \frac{49}{249}R_2]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & 1 & -\frac{427}{747} \\ 0 & -\frac{55}{7} & \frac{13}{3} \end{pmatrix} \quad (4.2.8.14)$$

$$\xrightarrow[R_3 \leftarrow -\frac{747}{118}R_3]{R_3 \leftarrow R_3 + \frac{55}{7}R_2} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & 1 & -\frac{427}{747} \\ 0 & 0 & 1 \end{pmatrix} \quad (4.2.8.15)$$

$$\xrightarrow[R_1 \leftarrow R_1 + \frac{2}{3}R_3 - \frac{10}{7}R_2]{R_2 \leftarrow R_2 + \frac{427}{47}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.2.8.16)$$

The above matrix has  $rank = 3$ . Hence, the lines do not intersect which implies that the given lines are skew lines. To find the closest points using SVD, consider the equation (4.2.8.11) which can be expressed as

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (4.2.8.17)$$

By singular value decomposition  $\mathbf{M}$  can be expressed as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (4.2.8.18)$$

where the columns of  $\mathbf{V}$  are the eigenvectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigenvectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{1037}{49} & 0 \\ 0 & \frac{2174}{49} \end{pmatrix} \quad (4.2.8.19)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} \frac{1341}{49} & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 \end{pmatrix} \quad (4.2.8.20)$$

To get  $\mathbf{V}$  and  $\mathbf{S}$  The characteristic equation of

$\mathbf{M}^T\mathbf{M}$  is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1037}{49} - \lambda & 0 \\ 0 & \frac{2174}{49} - \lambda \end{vmatrix} = 0 \quad (4.2.8.21)$$

$$\Rightarrow \lambda^2 - \frac{286699}{637}\lambda + \left[ \frac{1037 \times 2174}{49^2} \right] = 0 \quad (4.2.8.22)$$

The eigenvalues are the roots of equation 4.2.8.22 is given by

$$\lambda_{11} = \frac{2174}{49} \quad (4.2.8.23)$$

$$\lambda_{12} = \frac{1037}{49} \quad (4.2.8.24)$$

The corresponding eigen vectors are,

$$\mathbf{u}_{11} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4.2.8.25)$$

$$\mathbf{u}_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (4.2.8.26)$$

$$\therefore \mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.2.8.27)$$

$\mathbf{S}$  is given by

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{2174}}{7} & 0 \\ 0 & \frac{\sqrt{1037}}{7} \\ 0 & 0 \end{pmatrix} \quad (4.2.8.28)$$

To get  $\mathbf{U}$  The characteristic equation of  $\mathbf{M}\mathbf{M}^T$  is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1341}{49} - \lambda & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} - \lambda & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 - \lambda \end{vmatrix} = 0 \quad (4.2.8.29)$$

$$\Rightarrow -\lambda^3 + \frac{3211}{49}\lambda^2 - \frac{2254438}{2401}\lambda = 0 \quad (4.2.8.30)$$

The eigenvalues are the roots of equation 4.2.8.30 is given by

$$\lambda_{21} = \frac{2174}{49} \quad (4.2.8.31)$$

$$\lambda_{22} = \frac{1037}{49} \quad (4.2.8.32)$$

$$\lambda_{23} = 0 \quad (4.2.8.33)$$



The corresponding eigen vectors are ,

$$\mathbf{u}_{21} = \begin{pmatrix} -\frac{6}{7} \\ \frac{1}{5} \\ -1 \end{pmatrix}, \mathbf{u}_{22} = \begin{pmatrix} -\frac{3}{2} \\ \frac{10}{7} \\ 1 \end{pmatrix}, \mathbf{u}_{23} = \begin{pmatrix} -\frac{602}{747} \\ \frac{384}{249} \\ 1 \end{pmatrix} \quad (4.2.8.34)$$

Normalizing the eigen vectors,

$$\|\mathbf{u}_{21}\| = \sqrt{\left(\frac{-6}{7}\right)^2 + \left(\frac{1}{5}\right)^2 + 1} = \frac{\sqrt{2174}}{35} \quad (4.2.8.35)$$

$$\Rightarrow \mathbf{u}_{21} = \begin{pmatrix} -\frac{210}{7\sqrt{2174}} \\ \frac{35}{5\sqrt{2174}} \\ -\frac{35}{\sqrt{2174}} \end{pmatrix} \quad (4.2.8.36)$$

$$\|\mathbf{u}_{22}\| = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{10}{7}\right)^2 + 1} = \frac{\sqrt{1037}}{14} \quad (4.2.8.37)$$

$$\Rightarrow \mathbf{u}_{22} = \begin{pmatrix} -\frac{42}{2\sqrt{1037}} \\ \frac{20}{\sqrt{1037}} \\ \frac{14}{\sqrt{1037}} \end{pmatrix} \quad (4.2.8.38)$$

$$\|\mathbf{u}_{23}\| = \sqrt{\left(\frac{-602}{747}\right)^2 + \left(\frac{384}{249}\right)^2 + 1} = \frac{\sqrt{4027743}}{1000} \quad (4.2.8.39)$$

$$\Rightarrow \mathbf{u}_{23} = \begin{pmatrix} -\frac{602000}{747\sqrt{4027743}} \\ \frac{384000}{249\sqrt{4027743}} \\ \frac{1000}{\sqrt{4027743}} \end{pmatrix} \quad (4.2.8.40)$$

$$\mathbf{U} = \begin{pmatrix} -\frac{210}{7\sqrt{2174}} & -\frac{42}{2\sqrt{1037}} & -\frac{602000}{747\sqrt{4027743}} \\ \frac{35}{5\sqrt{2174}} & \frac{20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}} \\ -\frac{35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}} \end{pmatrix} \quad (4.2.8.41)$$

To get  $\mathbf{x}$  Using (4.2.8.18) we rewrite  $\mathbf{M}$  as

follows,

$$\begin{pmatrix} -3 & -\frac{30}{7} \\ -\frac{20}{7} & 1 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} \frac{-210}{7\sqrt{2174}} & \frac{-42}{2\sqrt{1037}} & \frac{-602000}{747\sqrt{4027743}} \\ \frac{35}{5\sqrt{2174}} & \frac{20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}} \\ \frac{-35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T \quad (4.2.8.42)$$

By substituting the equation (4.2.8.18) in equation (4.2.8.17) we get

$$\mathbf{USV}^T \mathbf{x} = \mathbf{b} \quad (4.2.8.43)$$

$$\Rightarrow \mathbf{x} = \mathbf{VS}_+ \mathbf{U}^T \mathbf{b} \quad (4.2.8.44)$$

where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{7}{\sqrt{2174}} & 0 & 0 \\ 0 & \frac{7}{\sqrt{1037}} & 0 \end{pmatrix} \quad (4.2.8.45)$$

From (4.2.8.44) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-172}{\sqrt{2174}} \\ \frac{20}{\sqrt{1037}} \\ \frac{-115000}{747\sqrt{4027743}} \end{pmatrix} \quad (4.2.8.46)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-602}{1087} \\ \frac{140}{1037} \end{pmatrix} \quad (4.2.8.47)$$

$$\mathbf{x} = \mathbf{VS}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{140}{1037} \\ \frac{-602}{1087} \end{pmatrix} \quad (4.2.8.48)$$

Verification of  $\mathbf{x}$  Verifying the solution of (4.2.8.48) using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (4.2.8.49)$$

Evaluating the R.H.S in (4.2.8.49) we get,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ -\frac{172}{7} \end{pmatrix} \quad (4.2.8.50)$$

$$\Rightarrow \begin{pmatrix} \frac{1037}{49} & 0 \\ 0 & \frac{2174}{49} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ -\frac{172}{7} \end{pmatrix} \quad (4.2.8.51)$$

Solving the augmented matrix of (4.2.8.51) we get,

$$\left( \begin{array}{ccc|ccc} \frac{1037}{49} & 0 & \frac{20}{7} & R_1 \leftarrow \frac{49}{1037} R_1 & & \\ 0 & \frac{2174}{49} & -\frac{172}{7} & & R_2 \leftarrow \frac{49}{2174} R_2 & \end{array} \right) \begin{pmatrix} 1 & 0 & \frac{140}{1037} \\ 0 & 1 & -\frac{602}{1087} \end{pmatrix} \quad (4.2.8.52)$$

Hence, Solution of (4.2.8.49) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{140}{1037} \\ \frac{-602}{1087} \end{pmatrix} \quad (4.2.8.53)$$

Comparing results of  $\mathbf{x}$  from (4.2.8.48) and (4.2.8.53) we conclude that the solution is verified.

## 5 OPTIMIZATION

### 5.1 Linear Programming

#### 5.1.1. Solve

$$\min_{\mathbf{x}} Z = (3 \ 2) \mathbf{x} \quad (5.1.1.1)$$

$$s.t. \quad \begin{pmatrix} -1 & -1 \\ 3 & 5 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} -8 \\ 15 \end{pmatrix} \quad (5.1.1.2)$$

$$\mathbf{x} \geq \mathbf{0} \quad (5.1.1.3)$$

#### 5.1.2. Solve

$$\min_{\mathbf{x}} Z = (200 \ 500) \mathbf{x} \quad (5.1.2.1)$$

$$s.t. \quad \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} -10 \\ 24 \end{pmatrix} \quad (5.1.2.2)$$

$$\mathbf{x} \geq \mathbf{0} \quad (5.1.2.3)$$

#### 5.1.3. Maximise $Z=3x+4y$

subject to the constraints :  $x+y \leq 4$ ,  $x \geq 0$ ,  $y \geq 0$ .

#### 5.1.4. Minimise $Z=-3x+4y$

subject to  $x+2y \leq 8$ ,  $3x+2y \leq 12$ ,  $x \geq 0$ ,  $y \geq 0$ .

#### 5.1.5. Maximise $Z=5x+3y$ subject to $3x+5y \leq 15$ , $5x+2y \leq 10$ , $x \geq 0$ , $y \geq 0$ .

**Solution:**

$$Z - 5x - 3y = 0 \quad (5.1.5.1)$$

$$3x + 5y + s_1 = 15 \quad (5.1.5.2)$$

$$5x + 2y + s_2 = 10 \quad (5.1.5.3)$$

We will write the simplex tableau

$$\begin{pmatrix} x & y & s_1 & s_2 & c \\ 3 & 5 & 1 & 0 & 15 \\ 5 & 2 & 0 & 1 & 10 \\ -5 & -3 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.5.4)$$

Keeping the pivot element as 5, we will use

gauss-jordan elimination.

$$\begin{pmatrix} x & y & s_1 & s_2 & c \\ 0 & \frac{19}{5} & 1 & \frac{-3}{5} & 9 \\ 1 & \frac{2}{5} & 0 & \frac{1}{5} & 2 \\ 0 & -1 & 0 & 1 & 10 \end{pmatrix} \quad (5.1.5.5)$$

Keeping the pivot element as  $\frac{19}{5}$ , we will use gauss-jordan elimination.

$$\begin{pmatrix} x & y & s_1 & s_2 & c \\ 0 & 1 & \frac{5}{19} & \frac{-3}{19} & \frac{45}{19} \\ 1 & 0 & \frac{-2}{19} & \frac{1}{19} & \frac{20}{19} \\ 0 & 0 & \frac{5}{19} & \frac{16}{19} & \frac{235}{19} \end{pmatrix} \quad (5.1.5.6)$$

In this tableau, there are no negative elements in the bottom row. We have therefore determined the optimal solution to be:

$$(x, y, s_1, s_2) = \left( \frac{20}{19}, \frac{45}{19}, 0, 0 \right) \quad (5.1.5.7)$$

$$Z = 5x + 3y \quad (5.1.5.8)$$

$$Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} \quad (5.1.5.9)$$

$$Z = \frac{235}{19} \quad (5.1.5.10)$$

The given problem can be expressed in general

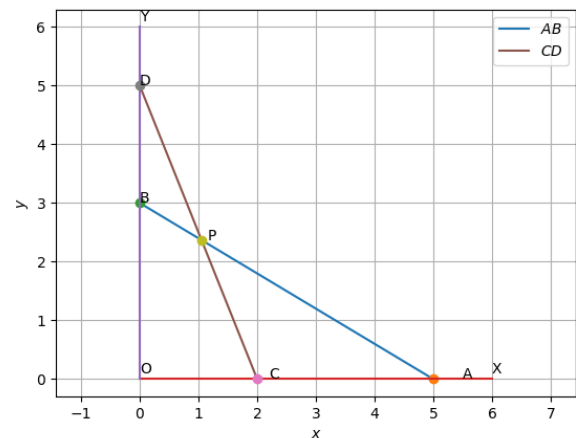


Fig. 5.1.5: optimal point through the intersection of various lines

as matrix inequality as:

$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad (5.1.5.11)$$

$$s.t. \quad \mathbf{Ax} \leq \mathbf{b}, \quad (5.1.5.12)$$

$$\mathbf{x} \geq \mathbf{0} \quad (5.1.5.13)$$

$$\mathbf{y} \geq \mathbf{0} \quad (5.1.5.14)$$

where

$$\mathbf{c} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad (5.1.5.15)$$

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 5 & 2 \end{pmatrix} \quad (5.1.5.16)$$

$$\mathbf{b} = \begin{pmatrix} 15 \\ 10 \end{pmatrix} \quad (5.1.5.17)$$

and can be solved using *cvxpy*. Hence,

$$\mathbf{x} = \begin{pmatrix} 1.05263158 \\ 2.36842105 \end{pmatrix}, Z = 12.36842102 \quad (5.1.5.18)$$

5.1.6. Minimise  $Z=3x+5y$  such that  $x+3y \geq 3$ ,  $x+y \geq 2$ ,  $x, y \geq 0$ .

5.1.7. Maximise  $Z=3x+2y$  subject to  $x+2y \leq 10$ ,  $3x+y \leq 15$ ,  $x, y \geq 0$ .

**Solution:**

$$\text{Maximize : } 3x_1 + 2x_2 \quad (5.1.7.1)$$

$$\text{Subject - to : } x_1 + 2x_2 \leq 10 \quad (5.1.7.2)$$

$$3x_1 + x_2 \leq 15 \quad (5.1.7.3)$$

The Problem is converted into canonical form by adding slack variables. Then Problem becomes,

$$\text{Maximize : } 3x_1 + 2x_2 + 0s_1 + 0s_2 \quad (5.1.7.4)$$

$$\text{Constraints : } x_1 + 2x_2 + s_1 = 10 \quad (5.1.7.5)$$

$$3x_1 + x_2 + s_2 = 15 \quad (5.1.7.6)$$

we write the Simplex tableau ,

$$\begin{pmatrix} x_1 & x_2 & s_1 & s_2 & c \\ 1 & 2 & 1 & 0 & 10 \\ 3 & 1 & 0 & 1 & 15 \\ -3 & -2 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.7.7)$$

Keeping the Pivot element as 3 and by using

gauss-jordan Elimination we get

$$\begin{pmatrix} x_1 & x_2 & s_1 & s_2 & c \\ 0 & \frac{5}{3} & 1 & \frac{-1}{3} & 5 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 5 \\ 0 & -1 & 0 & 1 & 15 \end{pmatrix} \quad (5.1.7.8)$$

Keeping the Pivot element as  $\frac{5}{3}$  and by using gauss-jordan Elimination we get

$$\begin{pmatrix} x_1 & x_2 & s_1 & s_2 & c \\ 0 & 1 & \frac{3}{5} & \frac{-1}{5} & 2 \\ 1 & 0 & \frac{-1}{5} & \frac{2}{5} & 3 \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} & 18 \end{pmatrix} \quad (5.1.7.9)$$

In this tableau Since all indicators in last row are non-negative ,we found optimal solution to given problem. Therefore Optimal Solution will be:

$$(x_1, x_2) = (4, 3) \quad (5.1.7.10)$$

$$Z = 3x_1 + 2x_2 \quad (5.1.7.11)$$

$$Z = 3 \times 4 + 2 \times 3 \quad (5.1.7.12)$$

$$Z = 18 \quad (5.1.7.13)$$

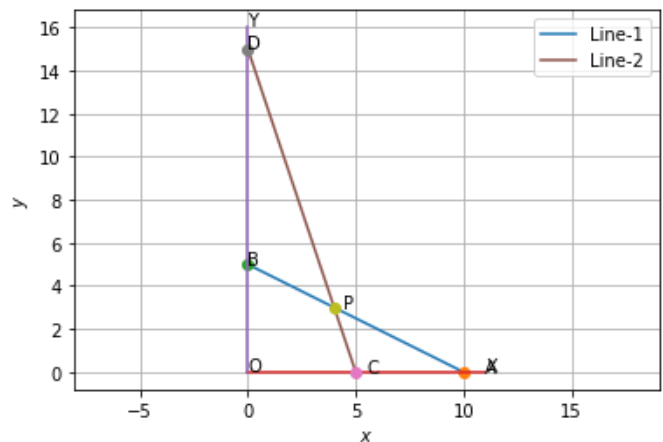


Fig. 5.1.7: Region OBPC is Valid region

This Problem can be represented in matrix form as follows,

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} \quad (5.1.7.14)$$

$$s.t. \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 10 \\ 15 \end{pmatrix} \quad (5.1.7.15)$$

$$\mathbf{x} \geq \mathbf{0} \quad (5.1.7.16)$$

$$\mathbf{y} \geq \mathbf{0} \quad (5.1.7.17)$$

this is solved using cvxpy in python, we get

$$\mathbf{x} = \begin{pmatrix} 3.99999999 \\ 2.99999999 \end{pmatrix}, Z = 17.99999996 \quad (5.1.7.18)$$

5.1.8. Minimise  $Z=x+2y$  subject to  $2x+y \geq 3$ ,  $x+2y \geq 6$ ,  $x, y \geq 0$ .

Show that the minimum of  $Z$  occurs at more than two points.

**Solution:** Here, to minimize the above given objective function, simplex method can be used.

The simplex algorithm for minimization problems works by converting the problem to a maximization problem, by forming an augmented matrix which includes given objective function in last row and constraints in above rows. It is transposed and then maximization is applied for the augmented matrix formed from those equations after transposing. Given objective function

$$f(x, y) = Z = x + 2y \quad (5.1.8.1)$$

subject to

$$2x + y \geq 3 \quad (5.1.8.2)$$

$$x + 2y \geq 6 \quad (5.1.8.3)$$

$$x, y \geq 0 \quad (5.1.8.4)$$

The matrix form of given objective function and constraints is given as below

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 6 \\ 1 & 2 & 0 \end{pmatrix} \quad (5.1.8.5)$$

After transposing the above matrix, we get

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 6 & 0 \end{pmatrix} \quad (5.1.8.6)$$

As per above transposed matrix, the constraints and objective function are as below.

$$f(x, y) = Z = 3x + 6y \quad (5.1.8.7)$$

subject to

$$2x + y \geq 1 \quad (5.1.8.8)$$

$$x + 2y \geq 2 \quad (5.1.8.9)$$

$$x, y \geq 0 \quad (5.1.8.10)$$

Apply Maximization now.

Adding slack variables  $s_1, s_2$ , the inequalities can be re-written as below.

$$f(x, y) = Z = 3x + 6y \Rightarrow Z - 3x - 6y = 0 \quad (5.1.8.11)$$

$$2x + y \geq 1 \Rightarrow 2x + y + s_1 = 1 \quad (5.1.8.12)$$

$$x + 2y \geq 2 \Rightarrow x + 2y + s_2 = 2 \quad (5.1.8.13)$$

The augmented matrix that can be formed using objective function and constraints is given as below.

$$\begin{pmatrix} 1 & -3 & -6 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \quad (5.1.8.14)$$

Here  $x$  and  $y$  are non-basic variables.  $s_1, s_2$  are basic variables. Selecting  $x$  as entering variable.

Entering through  $x$  in row(1).

Reducing augmented matrix for eliminating  $x$  in row(1) through row(2) and row(3).

$$\begin{pmatrix} 1 & -3 & -6 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \xleftarrow{R_1 \leftarrow R_1 + R_2 + R_3} \quad (5.1.8.15)$$

$$\xleftarrow{R_3 \leftarrow R_2 - 2R_3} \quad (5.1.8.16)$$

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 & -2 & -3 \end{pmatrix} \quad (5.1.8.17)$$

eliminating  $x$  in row(1) through row(1).

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \quad (5.1.8.18)$$

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 1 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -3 & 1 & -2 & -3 \end{pmatrix} \quad (5.1.8.19)$$

Here, the feasible solution may not be obtained since row(1) contains negative coefficients of non-basic variables. Hence the entering variable has to be changed to  $y$  (another non-basic variable). Entering through  $y$  in row(2) since ratio  $\frac{1}{1}$  a positive number and minimum.

Reducing augmented matrix for eliminating  $y$  in row(1), row(2) through row(2) and row(3).

$$\begin{pmatrix} 1 & -3 & -6 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \quad (5.1.8.20)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3R_3} \quad (5.1.8.21)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_3 - R_2} \quad (5.1.8.22)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & -1 & 2 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \quad (5.1.8.23)$$

Here, the feasible solutions may be obtained as below and there may be multiple optimal solutions for  $Z$  since the row(1) contain zero coefficients of non-basic variables i.e.  $Z$  can take any any values of  $x$  and  $y$ . Hence, the objective function may have multiple minima points.

$$\Rightarrow (x, y) = (0, 0) \quad (5.1.8.24)$$

$$\Rightarrow Z = x + 2y \quad (5.1.8.25)$$

$$\Rightarrow Z = 0 + 0 = 0 \quad (5.1.8.26)$$

This Problem can be represented in matrix form as follows,

$$\min_x Z = \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} \quad (5.1.8.27)$$

$$s.t. \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} \geq \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad (5.1.8.28)$$

$$\mathbf{x} \geq \mathbf{0} \quad (5.1.8.29)$$

$$\mathbf{y} \geq \mathbf{0} \quad (5.1.8.30)$$

The output as obtained through Python imple-



Fig. 5.1.8: Python plot .

mentation:

$$\mathbf{x} = \begin{pmatrix} 6.89748716e - 11 \\ 5.47282454e - 10 \end{pmatrix}, Z = 1.16353978e - 09 \quad (5.1.8.31)$$

5.1.9. Minimise and Maximise  $Z=5x+10y$  subject to  $x+2y \leq 120$ ,  $x+y \geq 60$ ,  $x-2y \geq 0$ ,  $x, y \geq 0$ .

5.1.10. Minimise and Maximise  $Z=x+2y$  subject to  $x+2y \geq 100$ ,  $2x-y \leq 0$ ,  $2x+y \leq 200$ ;  $x, y \geq 0$ .

**Solution:** In order to obtain the maximum and minimum value we need to solve the system of inequalities by adding slack variables. The equations now become:

$$x + 2y - Z = 0 \quad (5.1.10.1)$$

$$x + 2y - S_1 = 100 \quad (5.1.10.2)$$

$$2x - y + S_2 = 0 \quad (5.1.10.3)$$

$$2x + y + S_3 = 200 \quad (5.1.10.4)$$

The simplex table can be formed as

$$\begin{array}{c|cccccc} (x & y & s_1 & s_2 & s_3 & b \\ \hline 1 & 2 & -1 & 0 & 0 & 50 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 200 \\ \hline 1 & 2 & 0 & 0 & 0 & 0 \end{array} \quad (5.1.10.5)$$

The pivot element is 2 as the minimum ratio 50 occurs for  $y$  as the entering variable. Now

reducing the simplex matrix we get

$$\begin{pmatrix} x & y & s_1 & s_2 & s_3 & b \\ 1 & 1 & -\frac{1}{2} & 0 & 0 & 50 \\ 2 & 0 & -\frac{1}{2} & 1 & 0 & 50 \\ 3 & 0 & \frac{1}{2} & 0 & 1 & 150 \\ 1 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(5.1.10.6)

This can be expressed in the form of matrix inequality for maximization and minimization respectively as:

$$\max_{\{x\}} \mathbf{c}^T \mathbf{x} \quad (5.1.10.7)$$

$$s.t \quad \mathbf{Ax} \leq \mathbf{b}; \mathbf{x} \geq 0 \quad (5.1.10.8)$$

$$\min_{\{x\}} \mathbf{c}^T \mathbf{x} \quad (5.1.10.9)$$

$$s.t \quad \mathbf{Ax} \geq \mathbf{b}; \mathbf{x} \geq 0 \quad (5.1.10.10)$$

where

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5.1.10.11)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 2 & 1 \end{pmatrix} \quad (5.1.10.12)$$

$$\mathbf{b} = \begin{pmatrix} 100 \\ 0 \\ 200 \end{pmatrix} \quad (5.1.10.13)$$

Solving for Z by this reduction method we get

$$\text{Max}Z = 400 \quad (5.1.10.14)$$

$$\text{Min}Z = 100 \quad (5.1.10.15)$$

This can be solved in Python which generates the result as shown in Fig 1.

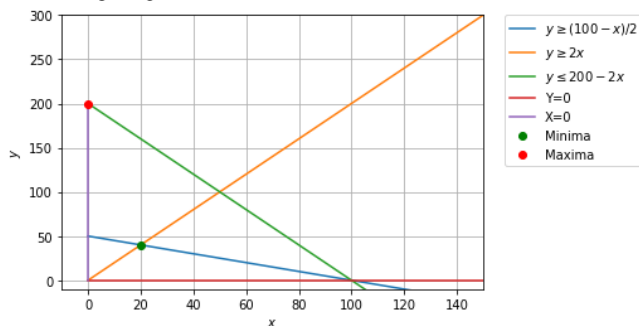


Fig. 1: Plot from python code with Maxima and Minima points

5.1.11. Maximise  $Z = -x + 2y$ , subject to the constraints:  $x \geq 3$ ,  $x + y \geq 5$ ,  $x + 2y \geq 6$ ,  $y \geq 0$ .

5.1.12. Maximise  $Z = x + y$ , subject to  $x - y \leq -1$ ,  $-x + y \leq 0$ ,  $x, y \geq 0$ .

5.1.13. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.

5.1.14. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.

5.1.15. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.

(i) What number of rackets and bats must be made if the factory is to work at full capacity?  
(ii) If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.

5.1.16. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?

5.1.17. A factory manufactures two types of screws,

A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

5.1.18. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

5.1.19. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?

5.1.20. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000

respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

5.1.21. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods  $F_1$  and  $F_2$  are available. Food  $F_1$  costs Rs 4 per unit food and  $F_2$  costs Rs 6 per unit. One unit of food  $F_1$  contains 3 units of vitamin A and 4 units of minerals. One unit of food  $F_2$  contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

5.1.22. There are two types of fertilisers  $F_1$  and  $F_2$ .  $F_1$  consists of 10% nitrogen and 6% phosphoric acid and  $F_2$  consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If  $F_1$  costs Rs 6/kg and  $F_2$  costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

5.1.23. The corner points of the feasible region determined by the following system of linear inequalities:  $2x+y \leq 10$ ,  $x+3y \leq 15$ ,  $x, y \geq 0$  are  $(0,0)$ ,  $(5,0)$ ,  $(3,4)$  and  $(0,5)$ . Let  $Z = px + qy$ , where  $p, q > 0$ . Condition on  $p$  and  $q$  so that the maximum of  $Z$  occurs at both  $(3,4)$  and  $(0,5)$  is

- (A)  $p = q$
- (B)  $p = 2q$
- (C)  $p = 3q$
- (D)  $q = 3p$

5.1.24. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

5.1.25. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

5.1.26. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

5.1.27. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Machines			
Types of toys	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

5.1.28. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy

class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

5.1.29. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per quintal (in Rs)		
From/To	A	B
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

5.1.30. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From/To	A	B
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

5.1.31. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is



the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	Brand Q
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

5.1.32. Refer to Question 29. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

5.1.33. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?

5.1.34. Find the shortest distance of the point  $\begin{pmatrix} 0 \\ c \end{pmatrix}$  from the parabola  $y = x^2$ , where  $\frac{1}{2} \leq c \leq 5$ .

5.1.35. Find the maximum area of an isosceles triangle inscribed in the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (5.1.35.1)$$

with its vertex at one end of the major axis.

5.1.36. **(Manufacturing problem)** A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week

to realise a maximum profit? What is the maximum profit per week?

5.1.37. **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

5.1.38. Solve:

$$\max_{\{x\}} Z = \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} \quad (5.1.38.1)$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \leq \begin{pmatrix} 50 \\ 90 \end{pmatrix} \quad (5.1.38.2)$$

$$\mathbf{x} \geq 0 \quad (5.1.38.3)$$

**Solution:** Adding slack variables to the left side of (5.1.38.2) and (5.1.38.3), we get

$$Z - 4x - y = 0 \quad (5.1.38.4)$$

$$x + y + s_1 = 50 \quad (5.1.38.5)$$

$$3x + y + s_2 = 90 \quad (5.1.38.6)$$

Forming simplex tableau,

$$\begin{pmatrix} x & y & s_1 & s_2 & b \\ 1 & 1 & 1 & 0 & 50 \\ \textcircled{3} & 1 & 0 & 1 & 90 \\ -4 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.38.7)$$

-4 is the smallest entry in the bottom row. Therefore, we determine that x is the starting variable.

Also, the smallest positive ratio is 30, therefore, we chose  $s_2$  as the departing variable.

Hence, keeping the pivot element as 3, we

perform Gauss Jordan elimination,

$$\begin{pmatrix} x & y & s_1 & s_2 & b \\ 1 & 1 & 1 & 0 & 50 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 30 \\ -4 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.38.8)$$

$$\begin{pmatrix} x & y & s_1 & s_2 & b \\ 0 & \frac{2}{3} & 1 & \frac{-1}{3} & 20 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 30 \\ 0 & \frac{1}{3} & 0 & \frac{4}{3} & 120 \end{pmatrix} \quad (5.1.38.9)$$

Note that  $x$  has replaced in the basis column  $s_2$  and the improved solution

$$(x, y, s_1, s_2) = (30, 0, 20, 0) \quad (5.1.38.10)$$

maximizes  $Z$  to value

$$Z = 4(30) + 3(0) \quad (5.1.38.11)$$

$$Z = 120 \quad (5.1.38.12)$$

The given problem can be expressed in the form of matrix inequality as:

$$\max_{\{x\}} \mathbf{c}^T \mathbf{x} \quad (5.1.38.13)$$

$$s.t \quad \mathbf{A} \mathbf{x} \leq \mathbf{b} \quad (5.1.38.14)$$

$$\mathbf{x} \geq 0 \quad (5.1.38.15)$$

$$(5.1.38.16)$$

where

$$\mathbf{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (5.1.38.17)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \quad (5.1.38.18)$$

$$\mathbf{b} = \begin{pmatrix} 50 \\ 90 \end{pmatrix} \quad (5.1.38.19)$$

can be solved using Python. The plot obtained from python is attached below:

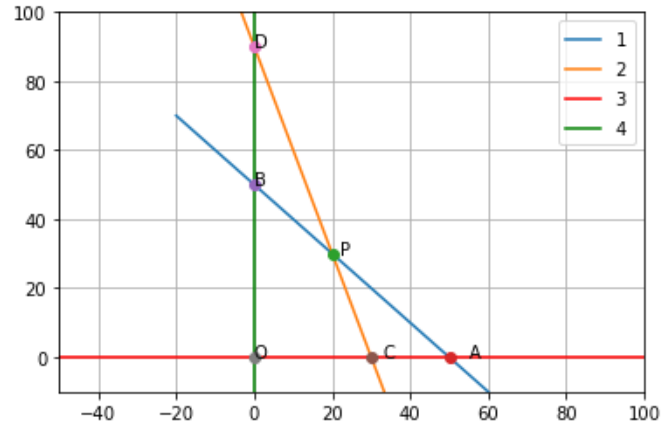


Fig. 5.1.38: Plot obtained from python code

inequality is true:

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2) \quad (5.2.1.2)$$

and for  $\lambda \in [0, 1]$

$$\begin{aligned} & \lambda(4x_1^2 - 4x_1 + 4) + (1 - \lambda)((4x_2^2 - 4x_2 + 4) \geq \\ & 4(\lambda x_1 + (1 - \lambda)x_2)^2 - 4(\lambda x_1 + (1 - \lambda)x_2) + 4 \end{aligned} \quad (5.2.1.3)$$

$$x_1^2(4\lambda - 4\lambda^2) + x_2^2(4\lambda - 4\lambda^2) - 2x_1x_2(4\lambda - 4\lambda^2) \geq 0 \quad (5.2.1.4)$$

$$4\lambda(1 - \lambda)(x_1 - x_2)^2 \geq 0 \quad (5.2.1.5)$$

The above inequality is always true for all values from the domain. Hence the given function  $f(x)$  is convex.

Using gradient descent method,

$$x_n = x_{n-1} - \mu \frac{df(x)}{dx} \quad (5.2.1.6)$$

Here,  $\mu$  is learning rate.

$$\frac{df(x)}{dx} = 8x - 4 \quad (5.2.1.7)$$

After substituting 5.2.1.7 in 5.2.1.6 we get:

$$x_n = x_{n-1} - \mu(8x_{n-1} - 4) \quad (5.2.1.8)$$

In equation (5.2.1.8),  $\mu$  is a variable parameter known as step size.  $x_{n+1}$  is the next position. The minus sign refers to the minimization part of gradient descent. Assume,  $\mu = 0.01$ ,  $x_0 = 2$  and following the above method, we keep doing iterations until  $x_{n+1} - x_n$  becomes less than the value of

## 5.2 Gradient Descent

5.2.1. Find the maximum and minimum values, if any, of the following functions given by

a)  $f(x) = (2x - 1)^2 + 3$

**Solution:** Given Equation can be written as:

$$f(x) = (2x - 1)^2 + 3 = 4x^2 - 4x + 4 \quad (5.2.1.1)$$

A function is said to be convex if following

precision we have chosen.

$$x_n = 0.5 \quad (5.2.1.9)$$

The following python code computes the minimum value as plotted in Fig. 5.2.1. Hence, The minimum value of  $f(x)$  at  $x = 0.5$  is 3.

```
./codes/Assignment_4.py
```

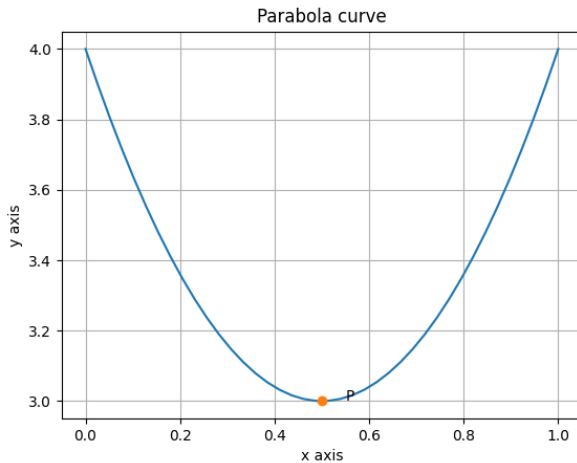


Fig. 5.2.1: Plot of the given polynomial

b)

$$f(x) = 9x^2 + 12x + 2 \quad (5.2.1.10)$$

**Solution:** A function is said to be convex if following inequality is true:

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2) \quad (5.2.1.11)$$

and for  $\lambda \in [0, 1]$

$$\lambda(9x_1^2 + 12x_1 + 2) + (1 - \lambda)(9x_2^2 + 12x_2 + 2) \geq 9(\lambda x_1 + (1 - \lambda)x_2)^2 + 12(\lambda x_1 + (1 - \lambda)x_2) + 2 \quad (5.2.1.12)$$

$$x_1^2(9\lambda - 9\lambda^2) + x_2^2(9\lambda - 9\lambda^2) - 2x_1x_2(9\lambda - 9\lambda^2) \geq 0 \quad (5.2.1.13)$$

$$(9\lambda - 9\lambda^2)(x_1^2 + x_2^2 - 2x_1x_2) \quad (5.2.1.14)$$

$$9\lambda(1 - \lambda)(x_1 - x_2)^2 \geq 0 \quad (5.2.1.15)$$

Equation (5.2.1.15) holds true for all  $\lambda \in (0, 1)$ . Hence the given function  $f(x)$  is convex. For a general quadratic equation

$$f(x) = ax^2 + bx + c \quad (5.2.1.16)$$

The update equation for gradient descent to find minimum of a function is given by:

$$\lambda_{n+1} = \lambda_n - \mu f'(\lambda_n) \quad (5.2.1.17)$$

$$= \lambda_n - \mu(2a\lambda_n + b) \quad (5.2.1.18)$$

In equation (5.2.1.15)  $\lambda_0$  is an initial guess and  $\mu$  is a variable parameter, known as step size  $\lambda_{n+1}$  is the next position. The minus sign refers to the minimization part of gradient descent. Assume,

$$\lambda_0 = 1 \quad (5.2.1.19)$$

$$\mu = 0.001 \quad (5.2.1.20)$$

$$precision = 0.00000001 \quad (5.2.1.21)$$

$$\Rightarrow \lambda_1 = 1 - 0.001(2 \times 9 \times 1 + 12) \quad (5.2.1.22)$$

$$\Rightarrow \lambda_1 = 1 - 0.03 \quad (5.2.1.23)$$

$$= 0.97 \quad (5.2.1.24)$$

following the above method, we keep doing iterations until  $\lambda_{n+1} - \lambda_n$  becomes less than the value of precision we have chosen. Results Using python, the results are:

- The local minimum occurs at - 0.666666130125316.
- The value of  $f(x)$  at minima is - 1.999999999974087

Figure 1 shows plot of parabola obtained from python code:

- $f(x) = -(x - 1)^2 + 10$
- $f(x) = x^2$ .

5.2.2. Find the absolute maximum and absolute minimum value of the following functions in the given intervals

- $f(x) = 4x - \frac{1}{2}x^2, x \in \left(-2, \frac{9}{2}\right)$
- $f(x) = (x - 1)^2 + 3, x \in (-3, 1)$

5.2.3. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2 \quad (5.2.3.1)$$

5.2.4. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

5.2.5. Find two numbers whose sum is 24 and whose

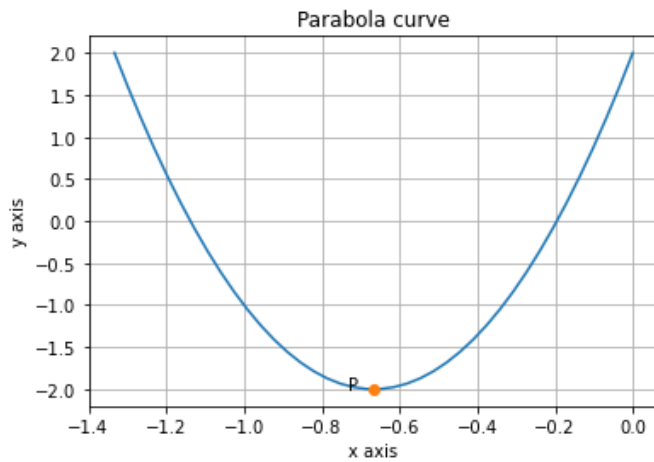


Fig. 1: Plot obtained from python code

product is as large as possible.

- 5.2.6. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.
- 5.2.7. The sum of the perimeter of a circle and square is  $k$ , where  $k$  is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.
- 5.2.8. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.
- 5.2.9. Find the shortest distance of the point  $\begin{pmatrix} 0 \\ c \end{pmatrix}$  from the parabola  $y = x^2$ , where  $\frac{1}{2} \leq c \leq 5$ .
- 5.2.10. Find the maximum area of an isosceles triangle inscribed in the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (5.2.10.1)$$

with its vertex at one end of the major axis.