



# Geometry through Linear Algebra



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**Abstract**—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.

## 1 PLANES AND LINES

### 1.1 Distance from a plane to a point

#### 1.1.1. Solve the following

a) Find the foot of perpendicular from the point

$$\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \text{ on the plane } (3 \ 2 \ -6)\mathbf{x} = 2.$$

**Solution:** Consider orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ . Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.1.1)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \quad (1.1.1.2)$$

$$\Rightarrow 3a + 2b - 6c = 0 \quad (1.1.1.3)$$

Let  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.4)$$

Let  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \quad (1.1.1.5)$$

Solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.1.6)$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.1.7)$$

Solving (1.1.1.7) using Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (1.1.1.8)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T \mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T \mathbf{M}$ . We

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have,

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \quad (1.1.1.9)$$

$$\mathbf{M} \mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \quad (1.1.1.10)$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.1.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V} \mathbf{\Sigma}^{-1} \mathbf{U}^T \mathbf{b} \quad (1.1.1.12)$$

Where  $\mathbf{\Sigma}^{-1}$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{\Sigma}$  and is obtained by inverting only non-zero elements in  $\mathbf{\Sigma}$

Calculating eigen values of  $\mathbf{M} \mathbf{M}^T$ ,

$$|\mathbf{M} \mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (1.1.1.13)$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 0 & \frac{1}{2} \\ 0 & 1 - \lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} - \lambda \end{vmatrix} = 0 \quad (1.1.1.14)$$

$$\Rightarrow \lambda^3 - \frac{85}{36} \lambda^2 + \frac{49}{36} \lambda = 0 \quad (1.1.1.15)$$

From the characteristic equation (1.1.1.15), the eigen values of  $\mathbf{M} \mathbf{M}^T$  are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.1.16)$$

The eigen vectors of  $\mathbf{M} \mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ \frac{1}{13} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{3} \\ 1 \end{pmatrix} \quad (1.1.1.17)$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u}_1 = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{-7}{12} \\ \frac{-7}{18} \\ \frac{7}{6} \end{pmatrix} \quad (1.1.1.18)$$

Hence we obtain  $\mathbf{U}$  as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \quad (1.1.1.19)$$

By computing the singular values from eigen

values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{\Sigma}$  as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.1.20)$$

Calculating eigen values of  $\mathbf{M}^T \mathbf{M}$ ,

$$|\mathbf{M}^T \mathbf{M} - \lambda \mathbf{I}| = 0 \quad (1.1.1.21)$$

$$\Rightarrow \begin{vmatrix} \frac{5}{4} - \lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} - \lambda \end{vmatrix} = 0 \quad (1.1.1.22)$$

$$\Rightarrow \lambda^2 - \frac{85}{36} \lambda + \frac{49}{36} = 0 \quad (1.1.1.23)$$

From the characteristic equation, the eigen values of  $\mathbf{M}^T \mathbf{M}$  are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad (1.1.1.24)$$

Hence the eigen vectors of  $\mathbf{M}^T \mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \quad (1.1.1.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.26)$$

Hence we obtain  $\mathbf{V}$  as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.27)$$

From (1.1.1.6), the Singular Value Decomposition of  $\mathbf{M}$  is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \quad (1.1.1.28)$$

And, the Moore-Penrose Pseudo inverse of  $\mathbf{\Sigma}$  is given by,

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.1.29)$$

From (1.1.1.12) we get,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \quad (1.1.1.30)$$

$$\Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.31)$$

$$\mathbf{x} = \mathbf{V} \Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.32)$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \implies \mathbf{M}^T \mathbf{M}\mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.1.33)$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T \mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.34)$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \quad (1.1.1.35)$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xrightarrow{R_1 = \frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \quad (1.1.1.36)$$

$$\xrightarrow{R_2 = R_2 - \frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{15}{45} & \frac{8}{3} \end{pmatrix} \quad (1.1.1.37)$$

$$\xrightarrow{R_2 = \frac{45}{45} R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.38)$$

$$\xrightarrow{R_1 = R_1 - \frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \quad (1.1.1.39)$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \quad (1.1.1.40)$$

From the equations (1.1.1.32) and (1.1.1.40), the solution  $\mathbf{x}$  is verified.

- b) Find the foot of perpendicular from point  $B = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$  to the plane  $(2 \ 3 \ -4)\mathbf{x} = -5$ .

**Solution:** Let us consider orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ . Let

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Then,

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.1.41)$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \quad (1.1.1.42)$$

$$\implies 2a + 3b - 4c = 0 \quad (1.1.1.43)$$

Let  $a = 1$ ,  $b = 0$ , so that

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (1.1.1.44)$$

and  $a = 0$ ,  $b = 1$ , so that

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \quad (1.1.1.45)$$

We, now, solve the equation

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.1.46)$$

which, upon substitution, becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad (1.1.1.47)$$

Any  $m \times n$  matrix  $\mathbf{M}$  can be factorized in SVD form as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.1.48)$$

where  $\mathbf{U}$  and  $\mathbf{V}$  are matrices of eigen vectors which are orthogonal. Columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T \mathbf{M}$ , columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is the diagonal matrix of singular values of  $\mathbf{M}$  of the eigenvalues of  $\mathbf{M}^T \mathbf{M}$ .

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \quad (1.1.1.49)$$

Putting (1.1.1.48) into (1.1.1.46), we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.1.50)$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (1.1.1.51)$$

where  $\mathbf{S}_+$  is the Moore-Penrose Pseudoinverse of  $\mathbf{S}$ .

The eigenvalues of  $\mathbf{M}^T\mathbf{M}$ :

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.1.52)$$

$$\Rightarrow \begin{vmatrix} \frac{10}{8} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{vmatrix} = 0 \quad (1.1.1.53)$$

$$\Rightarrow \lambda^2 - \frac{45}{16}\lambda + \frac{116}{64} = 0 \quad (1.1.1.54)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (1.1.1.55)$$

$$\lambda_2 = 1 \quad (1.1.1.56)$$

For  $\lambda_1 = \frac{29}{16}$ , the eigen vector  $\mathbf{v}_1$  can be calculated using row reduction as :

$$\begin{pmatrix} -\frac{9}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \xrightarrow{R_1 \leftarrow -\frac{16}{9}R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \quad (1.1.1.57)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad (1.1.1.58)$$

Hence,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.59)$$

Similarly, for  $\lambda_2 = 1$ ,

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.60)$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.1.61)$$

Now,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \quad (1.1.1.62)$$

Now, calculating eigenvalues of  $\mathbf{M}\mathbf{M}^T$

$$\begin{vmatrix} 1 - \lambda & 0 & \frac{1}{2} \\ 0 & 1 - \lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \lambda \end{vmatrix} = 0 \quad (1.1.1.63)$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \quad (1.1.1.64)$$

$$\lambda_2 = 1 \quad (1.1.1.65)$$

$$\lambda_3 = 0 \quad (1.1.1.66)$$

For  $\lambda_1 = \frac{29}{16}$ , the eigen vector can be computed as:

$$\begin{pmatrix} 1 - \frac{29}{16} & 0 & \frac{1}{2} \\ 0 & 1 - \frac{29}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \frac{29}{16} \end{pmatrix} \quad (1.1.1.67)$$

$$\leftrightarrow \begin{pmatrix} -\frac{13}{16} & 0 & \frac{1}{2} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (1.1.1.68)$$

$$\xrightarrow{R_1 \leftarrow -\frac{16}{13}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \quad (1.1.1.69)$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \quad (1.1.1.70)$$

$$\xrightarrow{R_2 \leftarrow -\frac{16}{13}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \quad (1.1.1.71)$$

$$\xrightarrow{R_2 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.1.1.72)$$

Hence, the eigen vector  $\mathbf{u}_1$ :

$$\mathbf{u}_1 = \begin{pmatrix} \frac{8}{\sqrt{377}} \\ \frac{12}{\sqrt{377}} \\ \frac{13}{\sqrt{377}} \end{pmatrix} \quad (1.1.1.73)$$

For  $\lambda_2 = 1$ , the eigen vector is:

$$\begin{pmatrix} 1 - 1 & 0 & \frac{1}{2} \\ 0 & 1 - 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - 1 \end{pmatrix} \quad (1.1.1.74)$$

$$\leftrightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{3}{16} \end{pmatrix} \quad (1.1.1.75)$$

Hence, the eigen vector  $\mathbf{u}_2$ :

$$\mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.1.76)$$

Similarly, for  $\lambda_3 = 0$ , the eigen vector is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \quad (1.1.1.77)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1 - \frac{3}{4}R_2} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.1.1.78)$$

Hence, the eigen vector  $\mathbf{u}_3$ :

$$\mathbf{u}_3 = \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ -\frac{4}{\sqrt{29}} \end{pmatrix} \quad (1.1.1.79)$$

So, the orthonormal matrix  $\mathbf{U}$  of eigen vectors is:

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & -\frac{4}{\sqrt{29}} \end{pmatrix} \quad (1.1.1.80)$$

The matrix of singular values of  $\mathbf{M}$  is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.1.1.81)$$

The Moore-Penrose pseudoinverse of  $\mathbf{S}$  is computed as

$$\mathbf{S}_+ = (\mathbf{S}\mathbf{S}^T)^{-1}\mathbf{S}^T \quad (1.1.1.82)$$

$$= \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.1.83)$$

To solve for  $\mathbf{x}$  in (1.1.1.51), noting that  $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ ,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \\ 0 \end{pmatrix} \quad (1.1.1.84)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \quad (1.1.1.85)$$

Thus, the foot of perpendicular is:

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \quad (1.1.1.86)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.87)$$

The solution can be verified using

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.1.88)$$

The LHS gives

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.89)$$

$$\Rightarrow \mathbf{M}^T \mathbf{M} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.90)$$

Now, finding  $\mathbf{x}$  from

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.91)$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} & -3 \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{3}{10}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \quad (1.1.1.92)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & \frac{29}{20} & \frac{58}{20} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & 1 & 2 \end{pmatrix} \quad (1.1.1.93)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \quad (1.1.1.94)$$

Hence, the solution is given by

$$\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.1.1.95)$$

Comparing the results in Eq.(1.1.1.87) and (1.1.1.95), it is concluded that the solution is verified.

## 1.1.2. Solve the following

a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.1)$$

to the plane,

$$(2 \quad -3 \quad 1)\mathbf{x} = 0 \quad (1.1.2.2)$$

**Solution:** The equation of plane is given as,

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.3)$$

Hence the normal vector  $\mathbf{n}$  is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.4)$$

Let, the normal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the normal vector  $\mathbf{n}$  be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.5)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.6)$$

$$\Rightarrow \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.7)$$

Let,  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.8)$$

Let,  $a=1$  and  $b=0$ ,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.9)$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.10)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.11)$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.12)$$

To solve (1.1.2.10) we perform singular value decomposition on  $\mathbf{M}$  given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.13)$$

substituting the value of  $\mathbf{M}$  from equation (1.1.2.13) to (1.1.2.10),

$$\Rightarrow \mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.14)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+^T \mathbf{U}^T \mathbf{b} \quad (1.1.2.15)$$

where,  $\mathbf{S}_+$  is Moore-Pen-rose Pseudo-Inverse of  $\mathbf{S}$ . Columns of  $\mathbf{U}$  are eigenvectors of  $\mathbf{M}\mathbf{M}^T$ , columns of  $\mathbf{V}$  are eigenvectors of  $\mathbf{M}^T\mathbf{M}$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ . First calculat-

ing the eigenvectors corresponding to  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.16)$$

Eigenvalues corresponding to  $\mathbf{M}^T\mathbf{M}$  is,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.2.17)$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0 \quad (1.1.2.18)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (1.1.2.19)$$

$$\therefore \lambda_1 = 14 \quad (1.1.2.20)$$

$$\lambda_2 = 1 \quad (1.1.2.21)$$

Hence the eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$  respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.22)$$

$$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.23)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.24)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.25)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (1.1.2.26)$$

Now calculating the eigenvectors corresponding to  $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.1.2.27)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.28)$$

Eigenvalues corresponding to  $\mathbf{M}\mathbf{M}^T$  is,

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (1.1.2.29)$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\Rightarrow -\lambda(\lambda-1)(\lambda-14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to  $\lambda_3$ ,  $\lambda_4$  and  $\lambda_5$  respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -2 \\ \frac{3}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v}_4 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now  $\mathbf{S}$  corresponding to eigenvalues  $\lambda_3$ ,  $\lambda_4$

and  $\lambda_5$  is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of  $\mathbf{M}$  as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{V}\mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\Rightarrow \mathbf{V}\mathbf{S}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

$\therefore$  from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.51)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.52)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.53)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \quad (1.1.2.54)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \quad (1.1.2.55)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{17}{5} \end{pmatrix} \quad (1.1.2.56)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14} R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \quad (1.1.2.57)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5} R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \quad (1.1.2.58)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \quad (1.1.2.59)$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.

b) Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.60)$$

to the plane,

$$(1.1.2.61)$$

$$(2 \ -3 \ 1) \mathbf{x} = 0 \quad (1.1.2.62)$$

**Solution:** The equation of plane is give

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.63)$$

Hence the normal vector  $\mathbf{n}$  is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.64)$$

Let, the normal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the normal vector  $\mathbf{n}$  be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.2.65)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.66)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.67)$$

Let,  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.68)$$

Let,  $a=1$  and  $b=0$ ,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.69)$$

Now solving the equation,

$$\mathbf{M} \mathbf{x} = \mathbf{b} \quad (1.1.2.70)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.71)$$

To solve (1.1.2.70) we perform singular value decomposition on  $\mathbf{M}$  given by,

$$\mathbf{M} = \mathbf{U} \mathbf{S} \mathbf{V}^T \quad (1.1.2.72)$$

substituting the value of  $\mathbf{M}$  from equation (1.1.2.72) to (1.1.2.70),

$$\Rightarrow \mathbf{U} \mathbf{S} \mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.2.73)$$

$$\Rightarrow \mathbf{x} = \mathbf{V} \mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (1.1.2.74)$$

where,  $\mathbf{S}_+$  is Moore-Pen-rose Pseudo-Inverse of  $\mathbf{S}$ . Columns of  $\mathbf{U}$  are eigenvectors of  $\mathbf{M} \mathbf{M}^T$ , columns of  $\mathbf{V}$  are eigenvectors of  $\mathbf{M}^T \mathbf{M}$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T \mathbf{M}$ . First calculat-



ing the eigenvectors corresponding to  $\mathbf{M}^T \mathbf{M}$ .

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.75)$$

Eigenvalues corresponding to  $\mathbf{M}^T \mathbf{M}$  is,

$$|\mathbf{M}^T \mathbf{M} - \lambda \mathbf{I}| = 0 \quad (1.1.2.76)$$

$$\Rightarrow \begin{pmatrix} 5 - \lambda & -6 \\ -6 & 10 - \lambda \end{pmatrix} \quad (1.1.2.77)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (1.1.2.78)$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \quad (1.1.2.79)$$

Hence the eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$  respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.80)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad (1.1.2.81)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (1.1.2.82)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (1.1.2.83)$$

Now calculating the eigenvectors corresponding to  $\mathbf{M} \mathbf{M}^T$

$$\mathbf{M} \mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.1.2.84)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.85)$$

Eigenvalues corresponding to  $\mathbf{M} \mathbf{M}^T$  is,

$$|\mathbf{M} \mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (1.1.2.86)$$

$$\Rightarrow \begin{pmatrix} 1 - \lambda & 0 & -2 \\ 0 & 1 - \lambda & 3 \\ -2 & 3 & 13 - \lambda \end{pmatrix} \quad (1.1.2.87)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.88)$$

$$\Rightarrow -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (1.1.2.89)$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1 \quad (1.1.2.90)$$

$$\lambda_5 = 0 \quad (1.1.2.91)$$

Hence the eigenvectors corresponding to  $\lambda_3$ ,

$\lambda_4$  and  $\lambda_5$  respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.92)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.93)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.94)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.95)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.96)$$

Now  $\mathbf{S}$  corresponding to eigenvalues  $\lambda_3, \lambda_4$  and  $\lambda_5$  is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.97)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.98)$$

Hence we get singular value decomposition of  $\mathbf{M}$  as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.99)$$

Now substituting the values of (1.1.2.83), (1.1.2.98), (1.1.2.96) and (1.1.2.71) in

(1.1.2.74),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.100)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \quad (1.1.2.101)$$

$$\mathbf{v}\mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.102)$$

$$\Rightarrow \mathbf{v}\mathbf{S}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.103)$$

$\therefore$  from equation (1.1.2.74),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \quad (1.1.2.104)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.2.105)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.106)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.1.2.107)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (1.1.2.108)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix} \quad (1.1.2.109)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix} \quad (1.1.2.110)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.2.111)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.1.2.112)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.2.113)$$

Hence from equations (1.1.2.105) and (1.1.2.113) we conclude that the solution is verified.

c) Find the foot of the perpendicular from  $\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$

on the plane  $(2 \ -3 \ 1)\mathbf{x} = 0$

**Solution:** Let orthogonal vectors be  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ . Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.2.114)$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (1.1.2.115)$$

$$\Rightarrow -5a + b + 3c = 0 \quad (1.1.2.116)$$

Let  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.117)$$

Let  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.118)$$

From (1.1.2.117) and (1.1.2.118),

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.119)$$

Now solving the equation

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.120)$$

Substituting the given point and (1.1.2.119) in (1.1.2.120)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.121)$$

Using the Singular value decomposition to solve (1.1.2.121) as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (1.1.2.122)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{\Sigma}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.123)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.124)$$

Substituting (1.1.2.122) in (1.1.2.120)

$$\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{x} = \mathbf{b} \quad (1.1.2.125)$$

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b} \quad (1.1.2.126)$$

where  $\mathbf{\Sigma}^{-1}$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{\Sigma}$ .

Now finding the eigen values of  $\mathbf{M}\mathbf{M}^T$

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (1.1.2.127)$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0 \quad (1.1.2.128)$$

$$\Rightarrow \lambda^3 - 15\lambda^2 + 14\lambda = 0 \quad (1.1.2.129)$$

Hence eigen values of  $\mathbf{M}\mathbf{M}^T$ ,

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad \lambda_3 = 0 \quad (1.1.2.130)$$

Therefore eigen vectors of  $\mathbf{M}\mathbf{M}^T$ ,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.131)$$

Normalizing the eigen vectors,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{1}{\sqrt{182}} \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.132)$$

Hence from the above we get,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{-3}{\sqrt{14}} \\ 0 & \frac{1}{\sqrt{182}} & \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.133)$$

By computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{\Sigma}$  as,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 14 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.134)$$

Now calculating eigen values of  $\mathbf{M}^T\mathbf{M}$

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.2.135)$$

$$\begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0 \quad (1.1.2.136)$$

$$\Rightarrow \lambda^2 - 15\lambda + 14 = 0 \quad (1.1.2.137)$$

hence the eigen values of  $\mathbf{M}^T\mathbf{M}$

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad (1.1.2.138)$$

Therefore eigen vectors  $\mathbf{M}^T\mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \quad (1.1.2.139)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.140)$$

Hence  $\mathbf{V}$  is given as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.141)$$

Moore Pseudo inverse of  $\Sigma$  is,

$$\Sigma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \quad (1.1.2.142)$$

Substituting (1.1.2.133), (1.1.2.141) and (1.1.2.142) in (1.1.2.126),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{182}} & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.143)$$

$$\Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} \quad (1.1.2.144)$$

$$\mathbf{V} \Sigma^{-1} \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.145)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.146)$$

Now verifying (1.1.2.146) using (1.1.2.120)

$$\mathbf{M}\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{M}^T \mathbf{M}\mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (1.1.2.147)$$

Substituting (1.1.2.119), (1.1.2.123) and given point in (1.1.2.147)

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 \\ 10 \end{pmatrix} \quad (1.1.2.148)$$

$$(1.1.2.149)$$

Solving the augmented matrix.

$$\begin{pmatrix} 5 & -6 & -11 \\ -6 & 10 & 10 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ -6 & 10 & 10 \end{pmatrix} \quad (1.1.2.150)$$

$$\xrightarrow{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & \frac{5}{5} & \frac{-16}{5} \end{pmatrix} \quad (1.1.2.151)$$

$$\xrightarrow{R_2 = \frac{5R_2}{14}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \quad (1.1.2.152)$$

$$\xrightarrow{R_1 = R_1 + \frac{6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{-25}{7} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \quad (1.1.2.153)$$

From (1.1.2.153) we get,

$$\mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \quad (1.1.2.154)$$

Hence from (1.1.2.146) and (1.1.2.154) the  $\mathbf{x}$  is verified

d) Find the coordinates of foot of perpendicular

from  $\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$  to the plane

$$2x - 3y + z = 0 \quad (1.1.2.155)$$

using SVD

**Solution:** First we find orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given plane  $\mathbf{n}$ . Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0$$

$$\Rightarrow \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow 2a - 3b + c = 0 \quad (1.1.2.156)$$

By substituting  $a = 1; b = 0$  in (1.1.2.156),

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (1.1.2.157)$$

By substituting  $a = 0; b = 1$  in (1.1.2.156),

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (1.1.2.158)$$

Now  $\mathbf{M}$  can be written as,

$$\mathbf{M} = (\mathbf{m}_1 \quad \mathbf{m}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \quad (1.1.2.159)$$

such that solving  $\mathbf{M}\mathbf{x} = \mathbf{b}$  gives the required solution.

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad (1.1.2.160)$$

Applying Singular Value Decomposition on  $\mathbf{M}$ ,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.2.161)$$

Where the columns of  $\mathbf{V}$  are the eigenvectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigenvectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular values of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (1.1.2.162)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (1.1.2.163)$$

From (1.1.2.160) and (1.1.2.161),

$$\begin{aligned} \mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} &= \mathbf{b} \\ \Rightarrow \mathbf{x} &= \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \end{aligned} \quad (1.1.2.164)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ . Calculating eigenvalues of  $\mathbf{M}\mathbf{M}^T$ ,

$$\begin{aligned} |\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| &= 0 \\ \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda^3 + 15\lambda^2 - 14\lambda &= 0 \end{aligned}$$

Hence eigenvalues of  $\mathbf{M}\mathbf{M}^T$  are,

$$\lambda_1 = 14; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \quad (1.1.2.165)$$

And the corresponding eigenvectors are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}; \quad \mathbf{u}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u}_3 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.166)$$

Normalizing the above eigenvectors,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{1}{\sqrt{182}} \end{pmatrix}; \quad \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.167)$$

From (1.1.2.167) we obtain  $\mathbf{U}$  as,

$$\mathbf{U} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.168)$$

Using values from (1.1.2.165),

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.169)$$

Calculating the eigenvalues of  $\mathbf{M}^T\mathbf{M}$ ,

$$\begin{aligned} |\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| &= 0 \\ \Rightarrow \begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda^2 - 15\lambda + 14 &= 0 \end{aligned}$$

Hence, eigenvalues of  $\mathbf{M}^T\mathbf{M}$  are,

$$\lambda_4 = 14; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Normalizing the above eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.170)$$

From (1.1.2.170) we obtain  $\mathbf{V}$  as,

$$\mathbf{V} = \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{2\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \quad (1.1.2.171)$$

From (1.1.2.161) we get the Singular Value Decomposition of  $\mathbf{M}$ ,

$$\mathbf{M} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{2\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \quad (1.1.2.172)$$

Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.173)$$

From (1.1.2.164),

$$\begin{aligned} \mathbf{U}^T\mathbf{b} &= \begin{pmatrix} \frac{12\sqrt{2}}{\sqrt{91}} \\ \frac{3}{\sqrt{13}} \\ \frac{2\sqrt{2}}{7} \end{pmatrix} \\ \mathbf{S}_+\mathbf{U}^T\mathbf{b} &= \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \\ \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} &= \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \end{aligned} \quad (1.1.2.174)$$

To verify the solution obtained from (1.1.2.174),

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.2.175)$$

Substituting the values from (1.1.2.162) in (1.1.2.175),

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

Converting the above equation into augmented form and solving for  $\mathbf{x}$ ,

$$\begin{pmatrix} 5 & -6 & -3 \\ -6 & 10 & 6 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{5R_2 + 6R_1}{14}} \begin{pmatrix} 5 & -6 & -3 \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1 + 6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \quad (1.1.2.176)$$

From (1.1.2.176) it can be observed that,

$$\mathbf{x} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \quad (1.1.2.177)$$

- a) Determine the distance from the Y-axis to the plane  $5x - 2z - 3 = 0$

**Solution:** Equation of plane can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \quad (1.1.2.178)$$

Rewriting given equation of plane in (1.1.2.178) form

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \quad (1.1.2.179)$$

where :  $\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ ,  $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  and  $c = 3$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1 \mathbf{q} + \lambda_2 \mathbf{r} \quad (1.1.2.180)$$

Here  $p$  is any point on plane and  $\mathbf{q}, \mathbf{r}$  are two vectors parallel to plane and hence  $\perp$  to  $\mathbf{n}$ . Find two vectors that are  $\perp$  to  $\mathbf{n}$

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (1.1.2.181)$$

Put  $a = 0$  and  $b = 1$  in (1.1.2.180),  $\implies c = 0$

Put  $a = 1$  and  $b = 0$  in (1.1.2.180),  $\implies c = \frac{5}{2}$

$$\text{Hence } \mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Let us find point  $\mathbf{p}$  on the plane. Put  $x = 1, y = 0$  in (1.1.2.179), we get  $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Since given plane is parallel to y-axis, we can use any point  $P$  on y-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.1.2.182)$$

Let  $\mathbf{Q}$  be the point on plane with shortest distance to  $\mathbf{P}$ .  $\mathbf{Q}$  can be expressed in (1.1.2.181) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (1.1.2.183)$$

Equation  $\mathbf{P}$  and  $\mathbf{Q}$ , and computing pseudo inverse using SVD should give the value of  $\lambda_1$  and  $\lambda_2$  (since plane and y-axis never intersect pseudo inverse should give the points which are closest)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.1.2.184)$$

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \quad (1.1.2.185)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \quad (1.1.2.186)$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.2.187)$$

$$\mathbf{x} = \mathbf{M}^+ \mathbf{b} \quad (1.1.2.188)$$

$$\text{where } \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

Diagonalize  $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \\ \frac{5}{2} & 0 & \frac{25}{4} \end{pmatrix} \quad (1.1.2.189)$$

$$= \begin{pmatrix} 0 & \frac{2}{5} & -\frac{5}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{29}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{5} & 0 & 1 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix} \quad (1.1.2.190)$$

$$= \mathbf{U}\Sigma^T\Sigma\mathbf{U}^T \quad (1.1.2.191)$$

Verify (1.1.2.190) from,

codes/diagonalize1.py

Diagonalize  $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{29}{4} & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.2.192)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{29}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.1.2.193)$$

$$= \mathbf{V}\Sigma^T\Sigma\mathbf{V}^T \quad (1.1.2.194)$$

Verify (1.1.2.193) from,

codes/diagonalize2.py

Compute SVD of  $\mathbf{M}$  from (1.1.2.190) and (1.1.2.195),

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T \quad (1.1.2.195)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{5} & -\frac{5}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.2.196)$$

$$\mathbf{M}^+ = \mathbf{V}\Sigma^T\mathbf{U}^T \quad (1.1.2.197)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{5} & 0 & 1 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix} \quad (1.1.2.198)$$

$$= \begin{pmatrix} \frac{4}{29} & 0 & \frac{10}{29} \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.199)$$

Verify (1.1.2.199) from,

codes/pseudo\_inverse.py

Substitute (1.1.2.199) in (1.1.2.188),

$$\mathbf{x} = \begin{pmatrix} \frac{4}{29} & 0 & \frac{10}{29} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{14}{29} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (1.1.2.200)$$

Substituting  $\lambda_1, \lambda_2$  in (1.1.2.183)

$$\mathbf{Q} = \begin{pmatrix} \frac{15}{29} \\ 0 \\ -\frac{6}{29} \end{pmatrix} \quad (1.1.2.201)$$

Distance between point  $\mathbf{P}$  and  $\mathbf{Q}$  is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(\frac{15}{29}\right)^2 + 0 + \left(-\frac{6}{29}\right)^2} = \frac{3}{\sqrt{29}} \quad (1.1.2.202)$$

Hence, distance from y-axis to  $5x-2z-3=0$  is  $\frac{3}{\sqrt{29}}$ .

Verifying solution to (1.1.2.187) by least squares method

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \quad (1.1.2.203)$$

$$\Rightarrow \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.2.204)$$

Substituting  $\mathbf{M}, \mathbf{b}$  from (1.1.2.186) in (1.1.2.204)

$$\begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \quad (1.1.2.205)$$

$$\begin{pmatrix} \frac{29}{4} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ 0 \end{pmatrix} \quad (1.1.2.206)$$

$$\Rightarrow \frac{29}{4}\lambda_1 = -\frac{7}{2} \quad (1.1.2.207)$$

$$\lambda_1 = -\frac{7}{2} \times \frac{4}{29} = -\frac{14}{29} \quad (1.1.2.208)$$

$$\text{and } \lambda_2 = 0 \quad (1.1.2.209)$$

$$\mathbf{x} = \begin{pmatrix} -\frac{14}{29} \\ 0 \end{pmatrix} \quad (1.1.2.210)$$

Comparing (1.1.2.200) and (1.1.2.210) solution is verified.

1.1.3. Find the foot of the perpendicular using svd

drawn from  $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$  to the plane

$$(2 \ -1 \ -2)\mathbf{x} + 4 = 0 \quad (1.1.3.1)$$

**Solution:** Let us consider orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ . Let,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \text{ then}$$

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.1.3.2)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0 \quad (1.1.3.3)$$

$$\Rightarrow 2a - b - 2c = 0 \quad (1.1.3.4)$$

Let  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (1.1.3.5)$$

Let  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \quad (1.1.3.6)$$

Let us solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (1.1.3.7)$$

Substituting (1.1.3.5) and (1.1.3.6) in (1.1.3.7),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad (1.1.3.8)$$

To solve (1.1.3.8), we will perform Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (1.1.3.9)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T \mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T \mathbf{M}$ .

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \quad (1.1.3.10)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \quad (1.1.3.11)$$

Substituting (1.1.3.9) in (1.1.3.7),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (1.1.3.12)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (1.1.3.13)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ .

Let us calculate eigen values of  $\mathbf{M}\mathbf{M}^T$ ,

$$|\mathbf{M}\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (1.1.3.14)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{5}{4}-\lambda \end{vmatrix} = 0 \quad (1.1.3.15)$$

$$\Rightarrow \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \quad (1.1.3.16)$$

From equation (1.1.3.16) eigen values of  $\mathbf{M}\mathbf{M}^T$  are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad (1.1.3.17)$$

The eigen vectors of  $\mathbf{M}\mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ -1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 1 \end{pmatrix} \quad (1.1.3.18)$$

Normalizing the eigen vectors in equation (1.1.3.18)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \quad (1.1.3.19)$$

Hence we obtain  $\mathbf{U}$  as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \quad (1.1.3.20)$$

After computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{S}$  as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.3.21)$$



Now, lets calculate eigen values of  $\mathbf{M}^T\mathbf{M}$ ,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (1.1.3.22)$$

$$\Rightarrow \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \quad (1.1.3.23)$$

$$\Rightarrow \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (1.1.3.24)$$

Hence eigen values of  $\mathbf{M}^T\mathbf{M}$  are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad (1.1.3.25)$$

Hence the eigen vectors of  $\mathbf{M}^T\mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \quad (1.1.3.26)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.1.3.27)$$

Hence we obtain  $\mathbf{V}$  as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.1.3.28)$$

From (1.1.3.7), the Singular Value Decomposition of  $\mathbf{M}$  is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (1.1.3.29)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.3.30)$$

From (1.1.3.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{11}{3} \end{pmatrix} \quad (1.1.3.31)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{8}{9\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (1.1.3.32)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \quad (1.1.3.33)$$

Verifying the solution of (1.1.3.33) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.3.34)$$

Evaluating the R.H.S in (1.1.3.34) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (1.1.3.35)$$

$$\Rightarrow \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (1.1.3.36)$$

Solving the augmented matrix of (1.1.3.36) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \xrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \quad (1.1.3.37)$$

$$\xrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{9}{8} & -\frac{1}{4} \end{pmatrix} \quad (1.1.3.38)$$

$$\xrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \quad (1.1.3.39)$$

$$\xrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & -\frac{5}{9} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \quad (1.1.3.40)$$

From equation (1.1.3.40), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \quad (1.1.3.41)$$

Comparing results of  $\mathbf{x}$  from (1.1.3.33) and (1.1.3.41), we can say that the solution is verified.