



Geometry through Linear Algebra



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Abstract—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.

1 PAIR OF STRAIGHT LINES

1.1. Find the value of h so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.1)$$

may represent two straight lines.

Solution:

$$\mathbf{V} = \begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \quad (1.1.2)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.3)$$

$$f = 20 \quad (1.1.4)$$

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \quad (1.1.5)$$

Expanding equation (1.1.5) along row 1 gives

$$\begin{aligned} \Rightarrow 6 \times (240 - \frac{961}{4}) - h \times (20h - \frac{341}{2}) + \\ 11 \times (\frac{31h}{2} - 132) = 0 \end{aligned}$$

$$\Rightarrow 20h^2 - 341h + \frac{2907}{2} = 0 \quad (1.1.6)$$

$$\Rightarrow \boxed{h = \frac{17}{2}} \quad (1.1.7)$$

$$\Rightarrow \boxed{h = \frac{171}{20}} \quad (1.1.8)$$

If $h = \frac{17}{2}$ or $h = \frac{171}{20}$, the equation given will represent two straight lines.

Sub $h = \frac{17}{2}$ in equation (1.1.1) we get,

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \quad (1.1.9)$$

Equation (1.1.9) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (1.1.10)$$

$$\mathbf{u} = \begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \quad (1.1.11)$$

$$\mathbf{f} = 20 \quad (1.1.12)$$

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The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.13)$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \quad (1.1.14)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.1.15)$$

$$(1.1.16)$$

Substituting (1.1.9) in the equation (1.1.14),

$$12m^2 + 17m + 6 = 0 \quad (1.1.17)$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \quad (1.1.18)$$

$$\Rightarrow m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \quad (1.1.19)$$

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.1.20)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.21)$$

we know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.22)$$

Convolution of \mathbf{n}_1 and \mathbf{n}_2 can be done by converting \mathbf{n}_1 into a toeplitz matrix and multiplying with \mathbf{n}_2

From equation (1.1.21)

$$\mathbf{n}_1 = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.23)$$

$$\Rightarrow \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.24)$$

\Rightarrow Equation (1.1.21) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.1.25)$$

Substituting (1.1.21) in (1.1.25), the augmented

matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \quad (1.1.26)$$

$$\Rightarrow c_1 = 4, c_2 = 5 \quad (1.1.27)$$

Substituting (1.1.21) and (1.1.27) in (1.1.13) we get,

$$\begin{aligned} \Rightarrow (-2x - 3y - 4)(3x - 4y - 5) &= 0 \\ \Rightarrow (2x + 3y + 4)(3x + 4y + 5) &= 0 \end{aligned} \quad (1.1.28)$$

Equation (1.1.28) represents equations of two straight lines.

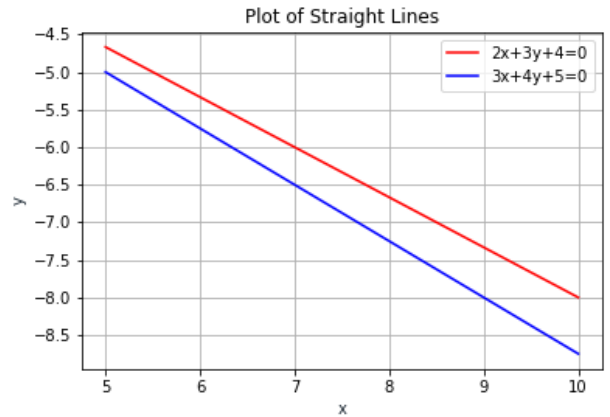


Fig. 1.1.1: Plot of Straight lines when $h = \frac{17}{2}$

Similarly, Sub $h = \frac{171}{20}$ in equation (1.1.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \quad (1.1.29)$$

Equation (1.1.29) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \quad (1.1.30)$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{3} \\ \frac{310}{3} \end{pmatrix} \quad (1.1.31)$$

$$\mathbf{f} = \frac{200}{3} \quad (1.1.32)$$

The pair of straight lines are given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (1.1.33)$$

Substituting (1.1.29) in the equation (1.1.14),

$$40m^2 + 57m + 20 = 0 \quad (1.1.34)$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \quad (1.1.35)$$

$$\Rightarrow m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \quad (1.1.36)$$

$$\mathbf{m}_1 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (1.1.37)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.38)$$

Convolution of \mathbf{n}_1 and \mathbf{n}_2 can be done by converting \mathbf{n}_1 into a toeplitz matrix and multiplying with \mathbf{n}_2

From equation (1.1.38)

$$\mathbf{n}_1 = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (1.1.39)$$

$$\Rightarrow \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.1.40)$$

\Rightarrow Equation (1.1.38) satisfies (1.1.22)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -2\mathbf{u} \quad (1.1.41)$$

Substituting (1.1.38) in (1.1.41), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{-R_1 - 4R_2}{5}]{R_2 \leftarrow \frac{5R_2 - 8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \quad (1.1.42)$$

$$\Rightarrow c_1 = 10, c_2 = \frac{20}{3} \quad (1.1.43)$$

Substituting (1.1.38) and (1.1.43) in (1.1.33) we get,

$$\Rightarrow \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \quad (1.1.44)$$

Equation (1.1.44) represents equations of two straight lines.

1.2. Prove that the following equations represent two straight lines. Also find their point of in-

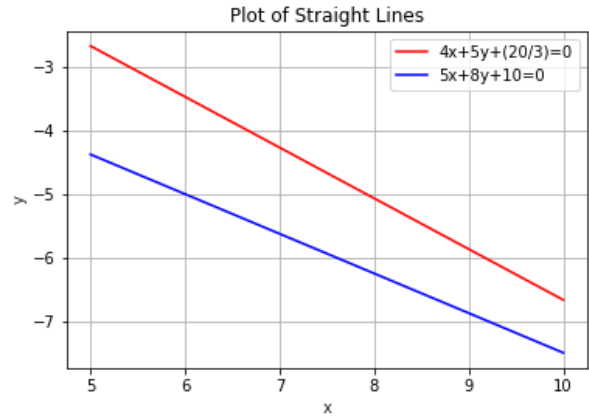


Fig. 1.1.2: Plot of Straight lines when $h = \frac{171}{20}$

tersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \quad (1.2.1)$$

Solution: $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (1.2.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \quad (1.2.2)$$

Expanding equation (1.2.2), we get zero.

Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.2.3)$$

$$\Rightarrow 3m^2 - 8m - 3 = 0 \quad (1.2.4)$$

$$\text{Solving, } m = 3, -\frac{1}{3} \quad (1.2.5)$$

The normal vectors of the lines then become

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.2.6)$$

$$\mathbf{n}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.7)$$

Equations of the lines can therefore be written as

$$\left(\frac{1}{3} \ 1\right) \mathbf{x} = c \quad (1.2.8)$$

$$\Rightarrow (1 \ 3) \mathbf{x} = c_1, \quad (1.2.9)$$

$$(-3 \ 1) \mathbf{x} = c_2 \quad (1.2.10)$$

$$\Rightarrow [(1 \ 3) \mathbf{x} - c_1][(-3 \ 1) \mathbf{x} - c_2] \quad (1.2.11)$$

represents the equation specified in (1.2.1)

Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \quad (1.2.12)$$

$$(1.2.13)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3 \times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \quad (1.2.14)$$

$$\xrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.15)$$

$$\xrightarrow{R_1 \leftarrow R_1 + 3 \times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \quad (1.2.16)$$

$$\Rightarrow c_2 = 2 \text{ and } c_1 = -9 \quad (1.2.17)$$

The individual line equations therefore become

$$(1 \ 3)\mathbf{x} = -9, \quad (1.2.18)$$

$$(-3 \ 1)\mathbf{x} = 2 \quad (1.2.19)$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.20)$$

The LHS part of (1.2.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.2.21)$$

The augmented matrix for the set of equations represented in (1.2.18), (1.2.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \quad (1.2.22)$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.23)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \quad (1.2.24)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \quad (1.2.25)$$

$$\text{Hence, the intersection point is } \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.2.26)$$

Angle between two lines θ can be given by

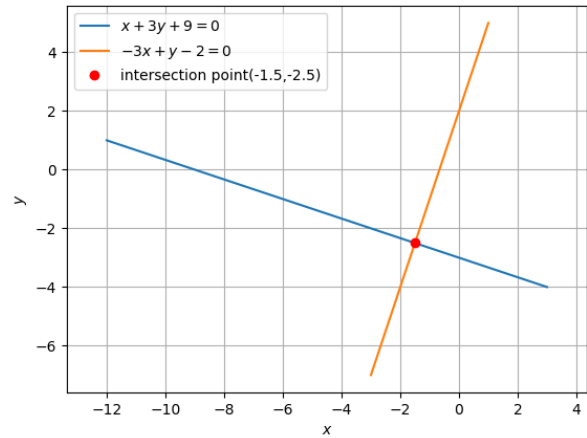


Fig. 1.2.1: plot showing intersection of lines

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.2.27)$$

$$\cos \theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \quad (1.2.28)$$

$$\Rightarrow \theta = 90^\circ \quad (1.2.29)$$

1.3. Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \quad (1.3.1)$$

Solution:

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (1.3.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (1.3.3)$$

$$f = -2 \quad (1.3.4)$$

$$\left| \begin{array}{ccc} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{array} \right| \xrightarrow[R_1 \rightarrow R_1 + R_3]{R_1 \rightarrow R_1 - R_2} \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{array} \right| = 0 \quad (1.3.5)$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$|V| < 0 \implies \left| \begin{array}{cc} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array} \right| = \frac{-9}{4} < 0 \quad (1.3.6)$$

Let the pair of straight of lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.3.7)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.3.8)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.3.9)$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \quad (1.3.10)$$

$$m_1 = 1, m_2 = -2 \quad (1.3.11)$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.3.12)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \quad (1.3.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.3.14)$$

$$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (1.3.15)$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.3.16)$$

Using row reduction we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \end{pmatrix} \quad (1.3.17)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_2 \leftarrow R_2 / -3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (1.3.18)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad (1.3.19)$$

$$C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.3.20)$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.21)$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.3.22)$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (1.3.23)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \quad (1.3.24)$$

consider the augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \quad (1.3.25)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.3.26)$$

$$\xrightarrow[R_1 \leftarrow R_1 + R_2]{R_1 \leftarrow R_1 / 3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.3.27)$$

Therefore point of intersection is $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Angle between two lines θ can be given by

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.3.28)$$

$$\cos \theta = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} \quad (1.3.29)$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right) \Rightarrow \theta = \tan^{-1}3 \quad (1.3.30)$$

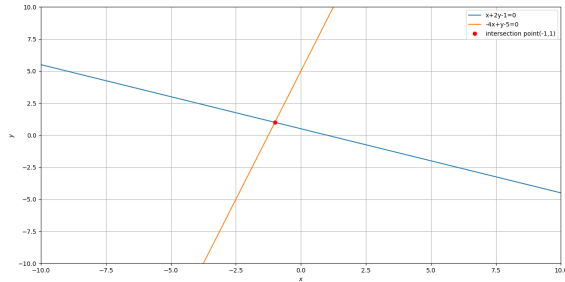


Fig. 1.3.1: plot showing intersection of lines

1.4. Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \quad (1.4.1)$$

represents two parallel lines.

Solution: The given equation (1.4.1) can be written as

$$\mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (1.4.2)$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \quad (1.4.3)$$

Equation (1.4.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \quad (1.4.4)$$

Vector form of straight lines,

$$\mathbf{n}_1^T \mathbf{x} = \mathbf{c}_1 \quad (1.4.5)$$

$$\mathbf{n}_2^T \mathbf{x} = \mathbf{c}_2 \quad (1.4.6)$$

Equating their product with (1.4.2)

$$(\mathbf{n}_1^T \mathbf{x} - \mathbf{c}_1)(\mathbf{n}_2^T \mathbf{x} - \mathbf{c}_2) = \mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 \quad (1.4.7)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.8)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix} \quad (1.4.9)$$

$$c_1 c_2 = -5 \quad (1.4.10)$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \quad (1.4.11)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.4.12)$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.4.13)$$

From (1.4.2) equation (1.4.11) becomes

$$9m^2 + 6m + 1 = 0 \quad (1.4.14)$$

Using (1.4.3),

$$|\mathbf{V}| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \quad (1.4.15)$$

Substituting the values in (1.4.12),

$$m_i = \frac{-3 \pm 0}{9} \quad (1.4.16)$$

$$m_1 = m_2 = \frac{-1}{3} \quad (1.4.17)$$

Substituting values in (1.4.13)

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.18)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.4.19)$$

Using the above values in (1.4.8),

$$k_1 k_2 = 9 \quad (1.4.20)$$

Taking $k_1 = 3$ and $k_2 = 3$ we get

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.21)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.4.22)$$

Verifying \mathbf{n}_1 and \mathbf{n}_2 by computing the convolution by representing \mathbf{n}_1 as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (1.4.23)$$

Finding the Angle between the lines,

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) \quad (1.4.24)$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \quad (1.4.25)$$

$$\|\mathbf{n}_1\| = \sqrt{10} \quad \|\mathbf{n}_2\| = \sqrt{10} \quad (1.4.26)$$

Substituting (1.4.25) and (1.4.26) in (1.4.24) we get,

$$\theta = \cos^{-1}(1) \quad (1.4.27)$$

$$\theta = 0^\circ \quad (1.4.28)$$

From (1.4.17) and (1.4.28) shows the given equation (1.4.1) represents two parallel lines. Hence proved.

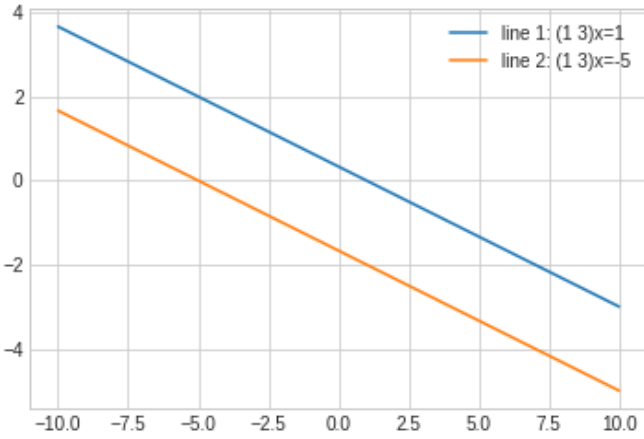


Fig. 1.4.1: Pair of straight lines plot generated using python

1.5. **Solution:** Find the value of k such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \quad (1.5.1)$$

represent pairs of straight lines.

From (1.5.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \quad (1.5.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.5.3)$$

$$f = k \quad (1.5.4)$$

Compute the slopes of lines given by the roots

of the polynomial $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.5.5)$$

$$\Rightarrow m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \quad (1.5.6)$$

$$\Rightarrow m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \quad (1.5.7)$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.5.8)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.5.9)$$

Here,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \quad (1.5.10)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -m_2 \\ 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (1.5.11)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.12)$$

Substituting (1.5.10) and (1.5.11) in the above equation, we get

$$k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} \quad (1.5.13)$$

$$\Rightarrow k_1 k_2 = -10 \quad (1.5.14)$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of k_1 and k_2 in (1.5.10) and (1.5.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (1.5.15)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.5.16)$$

Using Teoplitz matrix representation, the convolution of \mathbf{n}_1 with \mathbf{n}_2 , is as follows:

$$\begin{pmatrix} 2 & 0 & 5 \\ 5 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.5.17)$$

Hence, \mathbf{n}_1 and \mathbf{n}_2 satisfies (1.5.12). We have,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.5.18)$$

Substituting (1.5.15), (1.5.16) in (1.5.18), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (1.5.19)$$

Solving for c_1 and c_2 , the augmented matrix is,

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & -2 & -31 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 5R_1]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & \frac{-19}{2} & \frac{-37}{2} \end{pmatrix} \quad (1.5.20)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{3}{2}R_2]{R_2 \leftarrow \frac{R_2}{-19/2}} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.5.21)$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \quad (1.5.22)$$

We know that,

$$f = k = c_1 c_2 \quad (1.5.23)$$

$$\Rightarrow \boxed{k = -15} \quad (1.5.24)$$

Hence the solution. Using (1.5.8) and (1.5.9), the equation of pair of straight lines is given by,

$$(2 \ 5)\mathbf{x} = 3 \quad (1.5.25)$$

$$(3 \ -2)\mathbf{x} = -5 \quad (1.5.26)$$

See Fig. 1.5.1

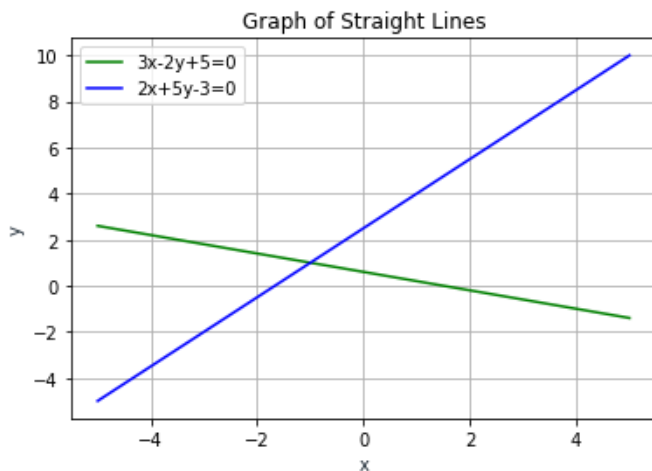


Fig. 1.5.1: Plot of two straight lines.

1.6. Find the value of k so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.1)$$

Solution:

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \quad (1.6.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.3)$$

The equation (1.6.1) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.6.4)$$

$$\Rightarrow \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \quad (1.6.5)$$

$$\Rightarrow \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \quad (1.6.6)$$

$$\Rightarrow 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11 \begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \quad (1.6.7)$$

$$\Rightarrow 2k^2 + 55k + 350 = 0 \quad (1.6.8)$$

$$\Rightarrow (10 + k)(2k + 35) = 0 \quad (1.6.9)$$

$$\Rightarrow k = -10$$

$$k = -\frac{35}{2} \quad (1.6.10)$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.

Now Lets find equation of lines for $k = -10$. Substitute $k = -10$ in (1.6.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.11)$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (1.6.12)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.13)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \quad (1.6.14)$$

$$\Rightarrow |\mathbf{V}| = -1 \quad (1.6.15)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.6.16)$$

Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.6.17)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.6.18)$$

Equating their product with (1.6.1)

$$\begin{aligned} (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (1.6.19)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.6.20)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} = -2 \begin{pmatrix} \frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \quad (1.6.21)$$

$$c_1 c_2 = f = 2 \quad (1.6.22)$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \quad (1.6.23)$$

$$\Rightarrow 2m^2 - 10m + 12 = 0 \quad (1.6.24)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.6.25)$$

$$\Rightarrow m_i = \frac{5 \pm \sqrt{1}}{2} \quad (1.6.26)$$

$$\Rightarrow m_1 = 3 \quad (1.6.27)$$

$$m_2 = 2 \quad (1.6.28)$$

The normal vector for two lines is given by

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.6.29)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.6.30)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.6.31)$$

Substituting (1.6.30),(1.6.31) in (1.6.20). we get

$$k_1 k_2 = 2 \quad (1.6.32)$$

The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1, k_2 = 2$ we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.6.33)$$

$$\mathbf{n}_2 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (1.6.34)$$

We verify obtained $\mathbf{n}_1, \mathbf{n}_2$ using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (1.6.35)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.6.36)$$

Therefore the obtained $\mathbf{n}_1, \mathbf{n}_2$ are correct.

Substitute (1.6.33), (1.6.34) in (1.6.21) and calculate for c_1 and c_2

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.6.37)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.38)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.6.39)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \quad (1.6.40)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \quad (1.6.41)$$

$$\Rightarrow c_1 = -13 \quad (1.6.42)$$

$$c_2 = 21 \quad (1.6.43)$$

Substituting (1.6.33),(1.6.34),(1.6.42),(1.6.43) in (1.6.17) and (1.6.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = -13 \quad (1.6.44)$$

$$\begin{pmatrix} -4 & 2 \end{pmatrix} \mathbf{x} = 21 \quad (1.6.45)$$

The plot of these two lines is shown in Fig. 1.6.1.

Now Lets find equation of lines for $k = -\frac{35}{2}$. Substitute $k = -\frac{35}{2}$ in (1.6.1). We get equation

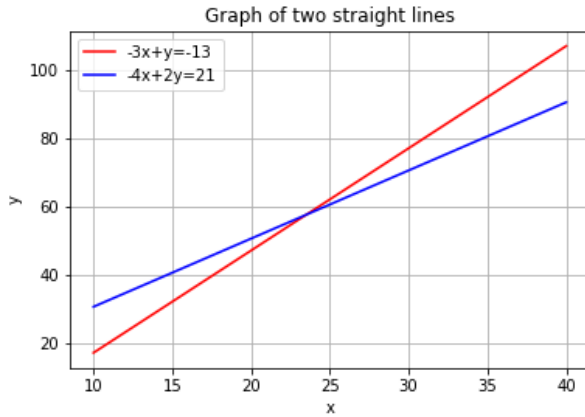


Fig. 1.6.1: Pair of straight lines for $k = -10$

of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.6.46)$$

From (1.6.1), (1.6.2), (1.6.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \quad (1.6.47)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (1.6.48)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \quad (1.6.49)$$

$$\Rightarrow |\mathbf{V}| = -\frac{841}{16} \quad (1.6.50)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (1.6.51)$$

Therefore the lines will intersect.

Now from (1.6.20),

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (1.6.52)$$

The slopes of the lines are given by roots of

equation (1.6.23)

$$\Rightarrow 2m^2 - \frac{35}{2}m + 12 = 0 \quad (1.6.53)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (1.6.54)$$

$$\Rightarrow m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \quad (1.6.55)$$

$$\Rightarrow m_1 = 8 \quad (1.6.56)$$

$$m_2 = \frac{3}{4} \quad (1.6.57)$$

The normal vector for two lines is given by (1.6.29)

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.6.58)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.6.59)$$

Substituting (1.6.58), (1.6.59) in (1.6.52). we get

$$k_1 k_2 = 2 \quad (1.6.60)$$

The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1, k_2 = 2$ we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (1.6.61)$$

$$\mathbf{n}_2 = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \quad (1.6.62)$$

We verify obtained $\mathbf{n}_1, \mathbf{n}_2$ using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -8 & 0 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \end{pmatrix} \quad (1.6.63)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -\frac{35}{2} \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.6.64)$$

Therefore the obtained $\mathbf{n}_1, \mathbf{n}_2$ are correct.

Substitute (1.6.61), (1.6.62) in (1.6.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (1.6.65)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.66)$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \quad (1.6.67)$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.6.68)$$

$$\xleftrightarrow{R_1 \leftarrow -29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \quad (1.6.69)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \quad (1.6.70)$$

$$\Rightarrow c_1 = -\frac{102}{29} \quad (1.6.71)$$

$$c_2 = \frac{59}{29} \quad (1.6.72)$$

Substituting (1.6.61), (1.6.62), (1.6.71), (1.6.72) in (1.6.17) and (1.6.18). we get equation of two straight lines.

$$(-8 \ 1)\mathbf{x} = -\frac{102}{29} \quad (1.6.73)$$

$$\left(-\frac{3}{2} \ 2\right)\mathbf{x} = \frac{59}{29} \quad (1.6.74)$$

1.7. Find the value of k so that the following equation may represent a pair of straight lines

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \quad (1.7.1)$$

Solution: The given second degree equation is, Comparing coefficients of (1.7.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \quad (1.7.2)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.3)$$

$$f = -35 \quad (1.7.4)$$

The given second degree equation (1.7.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \quad (1.7.5)$$

Expanding the determinant,

$$k + 12 = 0 \quad (1.7.6)$$

$$\Rightarrow k = -12 \quad (1.7.7)$$

Hence, from (1.7.7) we find that for $k = -12$, the given second degree equation (1.7.1) represents pair of straight lines. For the appropriate value of k , (1.7.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \quad (1.7.8)$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.7.9)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.7.10)$$

The pair of straight lines is given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.7.11)$$

Putting the values of \mathbf{V} and \mathbf{u} we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \quad (1.7.12)$$

Hence, from (1.7.12) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.14)$$

$$c_1 c_2 = -35 \quad (1.7.15)$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \quad (1.7.16)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.7.17)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.7.18)$$

Substituting the values in above equations (1.7.16) we get,

$$-12m^2 + m + 6 = 0 \quad (1.7.19)$$

$$\Rightarrow m_i = \frac{-\frac{1}{2} \pm \sqrt{-\left(-\frac{289}{4}\right)}}{-12} \quad (1.7.20)$$

Solving equation (1.7.20) we get ,

$$m_1 = -\frac{2}{3} \quad (1.7.21)$$

$$m_2 = \frac{3}{4} \quad (1.7.22)$$

Hence putting the values of m_1 and m_2 in (1.7.18) we get

$$\mathbf{n}_1 = k_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.7.23)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (1.7.24)$$

Putting values of \mathbf{n}_1 and \mathbf{n}_2 in (1.7.13) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.25)$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (1.7.26)$$

Thus, from (1.7.26), $k_1k_2 = -12$. Possible combinations of (k_1, k_2) are $(6, -2)$, $(-6, 2)$, $(3, -4)$, $(-3, 4)$ Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.7.27)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (1.7.28)$$

From equation (1.7.14) we get

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.7.29)$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (1.7.30)$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \quad (1.7.31)$$

$$3c_2 - 4c_1 = -43 \quad (1.7.32)$$

The augmented matrix of (1.7.31) ,(1.7.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} R_1 = \frac{1}{2} R_1 \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix} \quad (1.7.33)$$

$$R_2 = R_2 - 3R_1 \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \quad (1.7.34)$$

$$R_2 = -\frac{2}{17} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \quad (1.7.35)$$

$$R_1 = R_1 - \frac{3}{2} R_2 \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix} \quad (1.7.36)$$

$$(1.7.37)$$

Hence we get,

$$c_1 = -5 \quad (1.7.38)$$

$$c_2 = 7 \quad (1.7.39)$$

Hence (1.7.9), (1.7.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \quad (1.7.40)$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \quad (1.7.41)$$

The figure below corresponds to the pair of straight lines represented by (1.7.40) and (1.7.41).

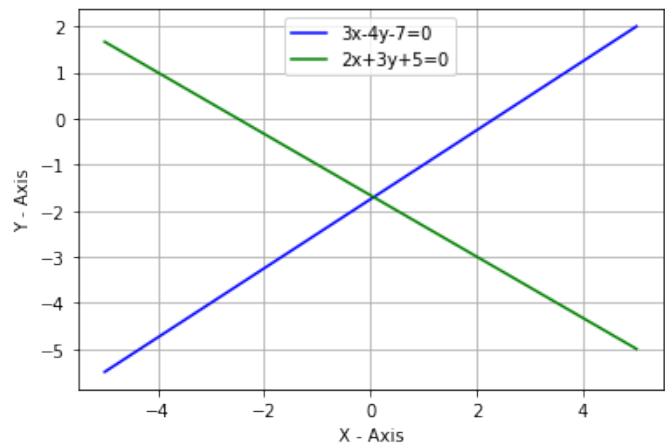


Fig. 1.7.1: Pair of Straight Lines

1.8. Find the value of k so that following equation may represent pairs of straight lines,

$$kxy - 8x + 9y - 12 = 0 \quad (1.8.1)$$

Solution: The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.8.2)$$

In vector form the equation (1.8.2) can be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.8.3)$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (1.8.4)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (1.8.5)$$

Now, comparing equation (1.8.2) to (1.8.1) we get, $a = c = 0$, $b = \left(\frac{k}{2}\right)$, $d = -4$, $e = \left(\frac{9}{2}\right)$, $f = -12$. Hence, substituting these values in equation (1.8.4) and (1.8.5) we get,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} \quad (1.8.6)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix} \quad (1.8.7)$$

Now equation (1.8.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.8.8)$$

$$\begin{vmatrix} 0 & \frac{k}{2} & -4 \\ \frac{k}{2} & 0 & \frac{9}{2} \\ -4 & \frac{9}{2} & -12 \end{vmatrix} = 0 \quad (1.8.9)$$

$$\Rightarrow k = 0, k = 6 \quad (1.8.10)$$

Substituting (1.8.10) in (1.8.1) we get,

$$6xy - 8x + 9y - 12 = 0 \quad (1.8.11)$$

$$-8x + 9y - 12 = 0 \quad (1.8.12)$$

Hence value of $k = 6$ represents pair of straight lines. Also it can be verified that the pair of lines intersect as,

$$|\mathbf{V}| = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} < 0 \quad (1.8.13)$$

Let the pair of straight lines is given by,

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.8.14)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.8.15)$$

Now equating the product of equation (1.8.14) and (1.8.15) with (1.8.3) we get,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \quad (1.8.16)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & \frac{9}{2} \end{pmatrix} \mathbf{x} - 12 \quad (1.8.17)$$

$$\Rightarrow n_1 * n_2 = \{0, 6, 0\} \quad (1.8.18)$$

$$c_1 n_1 + c_2 n_2 = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \quad (1.8.19)$$

$$c_1 c_2 = -12. \quad (1.8.20)$$

Now the slopes of line is given by roots of polynomial,

$$cm^2 + 2bm + a = 0 \quad (1.8.21)$$

$$\Rightarrow 2bm = 0 \quad (1.8.22)$$

$$\Rightarrow m = 0 \quad (1.8.23)$$

Also

$$m_i = \frac{-b \pm \sqrt{-|V|}}{c} \quad (1.8.24)$$

$$\Rightarrow m_i = \frac{-0 \pm \sqrt{9}}{0} \quad (1.8.25)$$

$$\therefore m_1 = 0 \quad (1.8.26)$$

$$m_2 = \infty \quad (1.8.27)$$

The normal vector to the two lines is given by,

$$n_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.8.28)$$

$$\Rightarrow n_1 = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.8.29)$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.8.30)$$

Also,

$$k_1 k_2 = 6 \quad (1.8.31)$$

Let $k_1 = 2$ and $k_2 = 3$

$$\Rightarrow n_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (1.8.32)$$

$$n_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.8.33)$$

We verify obtained n_1 and n_2 using Toeplitz matrix,

$$n_1 * n_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \quad (1.8.34)$$

Hence (1.8.18) and (1.8.34) are same. Hence verified.

Now substituting it in (1.8.19) we get,

$$c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \quad (1.8.35)$$

Solve using Row reduction Technique we get,

$$\Rightarrow \begin{pmatrix} 3 & 0 & 8 \\ 0 & 2 & -9 \end{pmatrix} \quad (1.8.36)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 2 & -9 \end{pmatrix} \quad (1.8.37)$$

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & -9/2 \end{pmatrix} \quad (1.8.38)$$

$$\Rightarrow c_1 = \frac{8}{3} \quad (1.8.39)$$

$$c_2 = \frac{-9}{2} \quad (1.8.40)$$

substituting the values of c_1 , c_2 and equation (1.8.32) and (1.8.33) to equation (1.8.14) and (1.8.15) we get equation of two straight lines.

$$\Rightarrow \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} = \frac{8}{3} \quad (1.8.41)$$

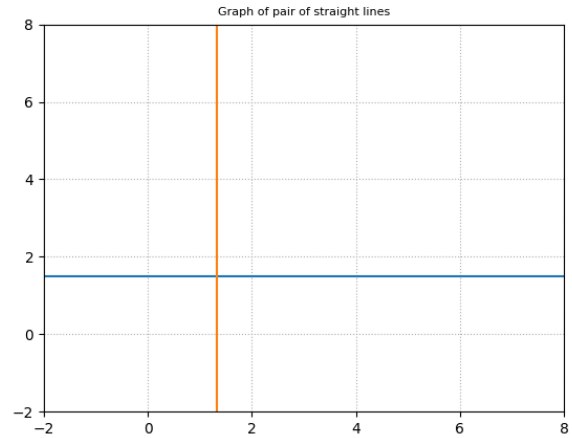


Fig. 1.8.1: Intersection of 2 lines

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} = \frac{-9}{2} \quad (1.8.42)$$

Hence the equation of pair of straight lines are,

$$\left(\begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} - \frac{8}{3} \right) \left(\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - \frac{-9}{2} \right) = 0 \quad (1.8.43)$$

Hence, Plot of the equation (1.8.43) is shown in Figure.1.8.1 Now for value of $k = 0$ does not represent pair of straight lines.as,

$$|\mathbf{V}| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \neq 0 \quad (1.8.44)$$

Hence, Plot of the equation $\begin{pmatrix} -8 & 9 \end{pmatrix} \mathbf{x} = 12$ is shown in figure 1.8.2,

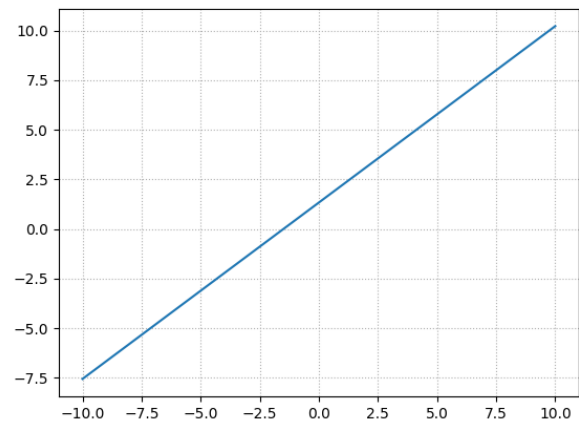


Fig. 1.8.2: Intersection of 2 lines

1.9. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \quad (1.9.1)$$

represent pairs of straight lines.

Solution: From (1.9.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \quad (1.9.2)$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{5}{2} & -\frac{7}{2} \end{pmatrix} \quad (1.9.3)$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & k \end{vmatrix} = 0 \quad (1.9.4)$$

$$\begin{aligned} \Rightarrow \left(k - \left(\frac{49}{4} \right) \right) - \frac{5}{3} \left(\frac{5}{3}k - \frac{35}{4} \right) \\ - \frac{5}{2} \left(\frac{-35}{6} + \frac{5}{2} \right) = 0 \end{aligned} \quad (1.9.5)$$

$$\Rightarrow \frac{64}{k} 36 - \frac{128}{12} = 0 \quad (1.9.6)$$

$$\Rightarrow \boxed{k = 6} \quad (1.9.7)$$

Substituting (1.9.7) in (1.9.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \quad (1.9.8)$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (1.9.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & 6 \end{vmatrix} \quad (1.9.9)$$

Simplify the above determinant, we get

$$\delta = 0 \quad (1.9.10)$$

(1.9.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \quad (1.9.11)$$

Since $\det(V) < 0$ lines would intersect each other

$$\mathbf{n}_1 * \mathbf{n}_2 = \left\{ 1, \frac{10}{3}, 1 \right\} \quad (1.9.12)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{5}{2} \\ -\frac{7}{2} \end{pmatrix} \quad (1.9.13)$$

$$c_1 c_2 = 6 \quad (1.9.14)$$

The slopes of the lines are given by the roots

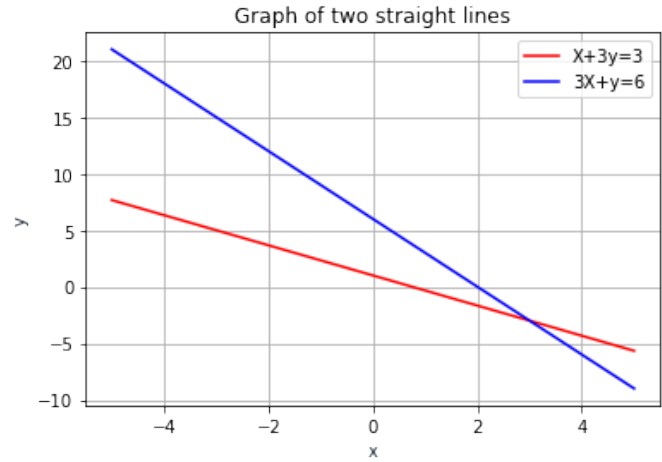


Fig. 1.9.1: Pair of straight lines

of the polynomial

$$cm^2 + 2bm + a = 0 \quad (1.9.15)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (1.9.16)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (1.9.17)$$

Substituting in above equations (1.9.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \quad (1.9.18)$$

$$\Rightarrow m_i = \frac{\frac{-10}{3} \pm \sqrt{-\left(\frac{-16}{9}\right)}}{1} \quad (1.9.19)$$

Solving equation (1.9.19) we have,

$$m_1 = \frac{-1}{3} \quad (1.9.20)$$

$$m_2 = -3 \quad (1.9.21)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.9.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.9.23)$$

Substituting equations (1.9.22), (1.9.23) in equation (1.9.12) we get

$$k_1 k_2 = 1 \quad (1.9.24)$$

Possible combination of (k_1, k_2) is (1,1) Lets

assume $k_1 = 1$, $k_2 = 1$, we get

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (1.9.25)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.9.26)$$

we have:

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.9.27)$$

Convolution of \mathbf{n}_1 and \mathbf{n}_2 can be done by converting \mathbf{n}_1 into a teoplitz matrix and multiplying with \mathbf{n}_2

From equation (1.9.25) and (1.9.26)

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.9.28)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (1.9.29)$$

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (1.9.30)$$

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -5 \\ \frac{7}{2} \\ -7 \end{pmatrix} \quad (1.9.31)$$

Substituting (1.9.25) and (1.9.26) in (1.9.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \quad (1.9.32)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \quad (1.9.33)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.9.34)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.9.35)$$

From above we get

$$c_1 = 1 \quad (1.9.36)$$

$$c_2 = 6 \quad (1.9.37)$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \quad (1.9.38)$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (1.9.39)$$

1.10. Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (1.10.1)$$

represents two straight lines and find the angle between the lines.

Solution: The above equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.10.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (1.10.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (1.10.4)$$

$$f = -35 \quad (1.10.5)$$

(1.10.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.10.6)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (1.10.7)$$

$$\Rightarrow 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \quad (1.10.8)$$

$$(1.10.9)$$

The lines intercept if

$$|\mathbf{V}| < 0 \quad (1.10.10)$$

$$|\mathbf{V}| = -\frac{529}{4} < 0 \quad (1.10.11)$$

From (1.10.8) and (1.10.11) it can be concluded that the given equation represents a pair of intersecting lines. Let the equations of lines be

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.10.12)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.10.13)$$

Since (1.10.2) represents a pair of straight lines it must satisfy

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_1^T \mathbf{x} - c_1) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.10.14)$$

where

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (1.10.15)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (1.10.16)$$

$$c_1 c_2 = f \quad (1.10.17)$$

Slopes of the lines can be obtained by solving

$$cm^2 + 2bm + a = 0 \quad (1.10.18)$$

$$-10m^2 + 7m + 12 = 0 \quad (1.10.19)$$

$$\Rightarrow m_1 = \frac{-4}{5}, m_2 = \frac{3}{2} \quad (1.10.20)$$

The normal vectors can be expressed in terms of corresponding slopes of lines as

$$\mathbf{n} = k \begin{pmatrix} -m \\ 1 \end{pmatrix} \quad (1.10.21)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} \frac{4}{5} \\ 1 \end{pmatrix} \quad (1.10.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \quad (1.10.23)$$

Substituting (1.10.22) and (1.10.23) in (1.10.15) we get

$$k_1 k_2 = -10 \quad (1.10.24)$$

Assuming $k_1 = 5$ and $k_2 = -2$

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.10.25)$$

Verification using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 4 & 0 \\ 5 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (1.10.26)$$

From (1.10.16) we have

$$c_2 \begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ -45 \end{pmatrix} \quad (1.10.27)$$

Solving the augmented matrix

$$\begin{pmatrix} 4 & 3 & -13 \\ 5 & -2 & -45 \end{pmatrix} \xrightarrow{R_2 \leftarrow -4R_2 - 5R_1} \begin{pmatrix} 4 & 3 & -13 \\ 0 & -23 & -115 \end{pmatrix} \quad (1.10.28)$$

$$\xrightarrow{R_2 \leftarrow -\frac{R_2}{23}} \begin{pmatrix} 4 & 3 & -13 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 4 & 0 & -28 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.10.29)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.10.30)$$

$$\Rightarrow c_1 = -7, c_2 = 5 \quad (1.10.31)$$

Thus the equation of lines are

$$(4 \ 5) \mathbf{x} = 5 \quad (1.10.32)$$

$$(3 \ -2) \mathbf{x} = -7 \quad (1.10.33)$$

The angle between the lines can be expressed in terms of normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (1.10.34)$$

as

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.10.35)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \quad (1.10.36)$$

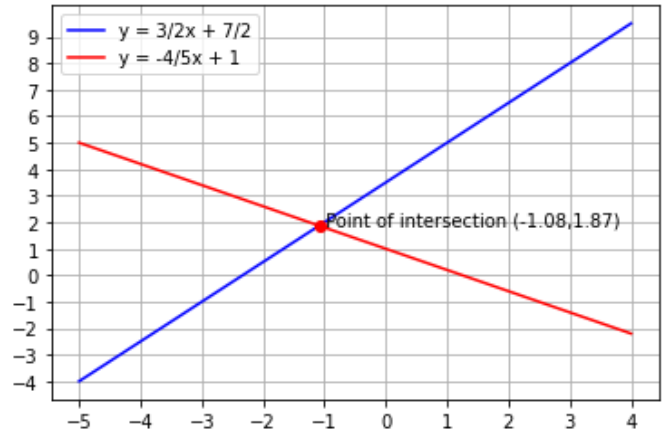


Fig. 1.10.1

1.11. Find the value of k so that the following equation may represent the pair of straight lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \quad (1.11.1)$$

Solution: We need to find the value of k for which (1.11.1) represents a pair of straight lines.

Converting (1.11.1) into vector form, we get

$$\mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 = 0 \quad (1.11.2)$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \quad (1.11.3)$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \quad (1.11.4)$$

$$f = -9 \quad (1.11.5)$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.11.6)$$

Since (1.11.1) represents a pair of straight lines, then by (1.11.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \quad (1.11.7)$$

By solving, above determinant we get

$$2(9 - 9) + \frac{-1}{2} \left(\frac{-9}{2} + \frac{-3k}{2} \right) + \frac{k}{2} \left(\frac{3}{2} + \frac{k}{2} \right) = 0 \quad (1.11.8)$$

$$\frac{(9 + 3k)}{4} + \frac{k(3 + k)}{4} = 0 \quad (1.11.9)$$

$$k^2 + 6k + 9 = 0 \quad (1.11.10)$$

$$(k + 3)^2 = 0 \quad (1.11.11)$$

$$k = -3 \quad (1.11.12)$$

Hence by (1.11.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \quad (1.11.13)$$

represents family of straight lines for $k = -3$. To find the straight lines, we write each of them in their vector form as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.11.14)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.11.15)$$

Equating the product of above with (1.11.2),

we have

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 \quad (1.11.16)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad (1.11.17)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \quad (1.11.18)$$

$$c_1 c_2 = -9 \quad (1.11.19)$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \quad (1.11.20)$$

$$m^2 - m - 2 = 0 \quad (1.11.21)$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} \quad (1.11.22)$$

$$m_1 = \frac{1+3}{2} = 2 \quad (1.11.23)$$

$$m_2 = \frac{1-3}{2} = -1 \quad (1.11.24)$$

$$n_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.11.25)$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.11.26)$$

Substituting (1.11.25) and (1.11.26) in (1.11.17), we get

$$k_1 k_2 = -1 \quad (1.11.27)$$

Taking $k_1 = -1$ and $k_2 = 1$, we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.11.28)$$

$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.11.29)$$

Substituting in (1.11.18) for above values of n_1 and n_2

$$(n_1 n_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (1.11.30)$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (1.11.31)$$

Solving (1.11.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xleftrightarrow{r_2=r_2+2r_1} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \quad (1.11.32)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xleftrightarrow{r_2=r_2/3} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad (1.11.33)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \quad (1.11.34)$$

Hence, we found out

$$c_1 = -3 \quad (1.11.35)$$

$$c_2 = 3 \quad (1.11.36)$$

Thus, pair of straight lines are

$$(2 \ -1)\mathbf{x} = -3 \quad (1.11.37)$$

$$(1 \ 1)\mathbf{x} = 3 \quad (1.11.38)$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.11.39)$$

The plot of above is shown below

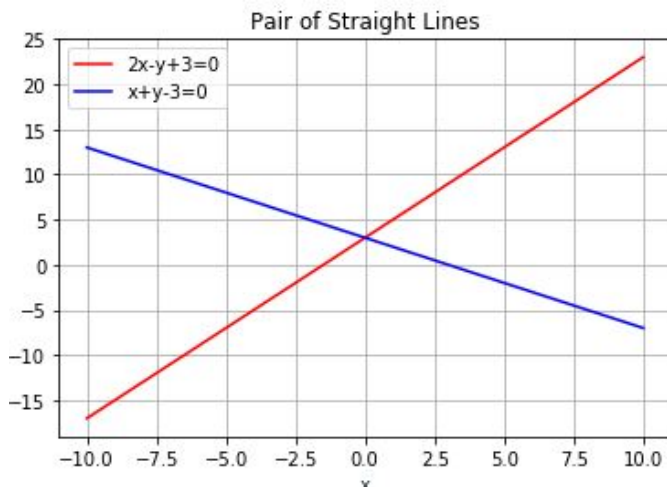


Fig. 1.11.1: Pair of Straight Lines

2 GENERAL EQUATION. TRACING OF CURVES

2.1. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre

$$12x^2 - 23xy + 10y^2 - 25x + 26y = 14 \quad (2.1.1)$$

Solution: The given equation (2.1.1) can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{-25}{2} & 13 \end{pmatrix} \mathbf{x} - 14 = 0 \quad (2.1.2)$$

where

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \quad (2.1.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \quad (2.1.4)$$

$$f = -14 \quad (2.1.5)$$

$$\det(\mathbf{V}) = \begin{vmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{vmatrix} \quad (2.1.6)$$

$$\Rightarrow \det(\mathbf{V}) = \frac{-49}{4} \quad (2.1.7)$$

$$\Rightarrow \det(\mathbf{V}) < 0 \quad (2.1.8)$$

Since $\det(\mathbf{V}) < 0$ the given equation (2.1.2) represents the hyperbola. The characteristic equation of \mathbf{V} is obtained by evaluating the determinant

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (2.1.9)$$

$$\begin{vmatrix} 12 - \lambda & \frac{-23}{2} \\ \frac{-23}{2} & 10 - \lambda \end{vmatrix} = 0 \quad (2.1.10)$$

$$\Rightarrow 4\lambda^2 - 88\lambda - 49 = 0 \quad (2.1.11)$$

The eigenvalues are the roots of equation 2.1.11 is given by

$$\lambda_1 = \frac{22 + \sqrt{533}}{2} \quad (2.1.12)$$

$$\lambda_2 = \frac{22 - \sqrt{533}}{2} \quad (2.1.13)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.1.14)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.1.15)$$

For $\lambda_1 = \frac{22-\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (2.1.16)$$

By row reduction ,

$$\begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (2.1.17)$$

$$\xrightarrow{R_1 = \frac{R_1}{\frac{\sqrt{533}+2}{2}}} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (2.1.18)$$

$$\xrightarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \quad (2.1.19)$$

Substituting equation 2.1.19 in equation 2.1.15 we get

$$\begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.20)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let $v_2 = t$

$$v_1 = \frac{-t(2 - \sqrt{533})}{23} \quad (2.1.21)$$

Eigen vector \mathbf{p}_1 is given by

$$\mathbf{p}_1 = \begin{pmatrix} \frac{-t(2-\sqrt{533})}{23} \\ t \end{pmatrix} \quad (2.1.22)$$

Let $t = 1$, we get

$$\mathbf{p}_1 = \begin{pmatrix} \frac{\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \quad (2.1.23)$$

For $\lambda_2 = \frac{22+\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (2.1.24)$$

By row reduction ,

$$\begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (2.1.25)$$

$$\xrightarrow{R_1 = \frac{R_1}{\frac{2-\sqrt{533}}{2}}} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (2.1.26)$$

$$\xrightarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \quad (2.1.27)$$

Substituting equation 2.1.27 in equation 2.1.15 we get

$$\begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.28)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let $v_2 = t$

$$v_1 = \frac{-t(2 + \sqrt{533})}{23} \quad (2.1.29)$$

Eigen vector \mathbf{p}_2 is given by

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-t(2+\sqrt{533})}{23} \\ t \end{pmatrix} \quad (2.1.30)$$

Let $t = 1$, we get

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \quad (2.1.31)$$

By eigen decomposition \mathbf{V} can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (2.1.32)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.1.33)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.1.34)$$

Substituting equations 2.1.23, 2.1.31 in equation 2.1.33 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & \frac{-\sqrt{533}-2}{23} \\ 1 & 1 \end{pmatrix} \quad (2.1.35)$$

Substituting equations 2.1.12, 2.1.13 in 2.1.34 we get

$$\mathbf{D} = \begin{pmatrix} \frac{22-\sqrt{533}}{2} & 0 \\ 0 & \frac{22+\sqrt{533}}{2} \end{pmatrix} \quad (2.1.36)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (2.1.37)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{-40}{49} & \frac{-46}{49} \\ \frac{-46}{49} & \frac{-48}{49} \end{pmatrix} \begin{pmatrix} -25 \\ 13 \end{pmatrix} \quad (2.1.38)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{40}{49} & \frac{46}{49} \\ \frac{46}{49} & \frac{48}{49} \end{pmatrix} \begin{pmatrix} -25 \\ 13 \end{pmatrix} \quad (2.1.39)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.1.40)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 26 > 0 \quad (2.1.41)$$

there isn't a need to swap axes
In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (2.1.42)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{2\sqrt{13}}{\sqrt{22 + \sqrt{533}}} \quad (2.1.43)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \frac{2\sqrt{13}}{\sqrt{\sqrt{533} - 22}} \quad (2.1.44)$$

Now (2.1.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.1.45)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.1.46)$$

To get \mathbf{y} ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (2.1.47)$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.1.48)$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{2(\sqrt{533}-2)}{23} + 1 \\ \frac{2(-\sqrt{533}-2)}{23} + 1 \end{pmatrix} \quad (2.1.49)$$

Substituting the equations (2.1.41), (2.1.36) in equation (2.1.45)

$$\mathbf{y}^T \begin{pmatrix} \frac{22+\sqrt{533}}{2} & 0 \\ 0 & \frac{22-\sqrt{533}}{2} \end{pmatrix} \mathbf{y} - 26 = 0 \quad (2.1.50)$$

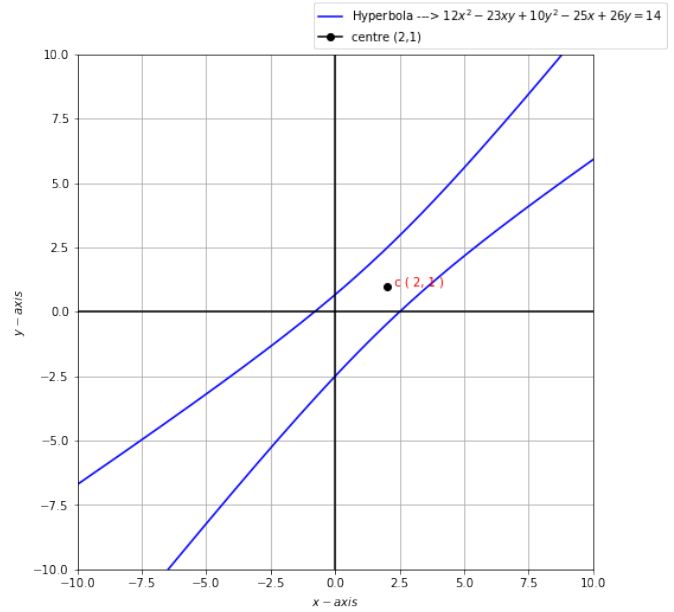


Fig. 2.1.1: Hyperbola when origin is shifted

The figure 2.1.1 verifies the given equation (2.1.2) as hyperbola with centre $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2.2. What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0 \quad (2.2.1)$$

Find the center.

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.2.2)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \quad (2.2.3)$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (2.2.4)$$

$$f = -2 \quad (2.2.5)$$

Expanding the determinant of \mathbf{V} we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \quad (2.2.6)$$

Hence from (2.2.6) we conclude that given equation is an ellipse. The characteristic equa-

tion of \mathbf{V} is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \quad (2.2.7)$$

$$\Rightarrow \lambda^2 - 50\lambda + 400 = 0 \quad (2.2.8)$$

Hence the characteristic equation of \mathbf{V} is given by (2.2.8). The roots of (2.2.8) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \quad (2.2.9)$$

The eigen vector \mathbf{p} is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.2.10)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (2.2.11)$$

for $\lambda_1 = 10$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \quad (2.2.12)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.2.13)$$

Again, for $\lambda_2 = 40$,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \quad (2.2.14)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (2.2.15)$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \quad (2.2.16)$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \quad (2.2.17)$$

Now (2.2.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (2.2.18)$$

And,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (2.2.19)$$

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.2.20)$$

The centre/vertex of the conic section in (2.2.2) is given by \mathbf{c} in (2.2.19). We compute \mathbf{V}^{-1} as

follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.21)$$

$$\xrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.22)$$

Hence \mathbf{V}^{-1} is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.2.23)$$

Now $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}$ is given by,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \quad (2.2.24)$$

And, $\mathbf{V}^{-1} \mathbf{u}$ is given by,

$$\mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.2.25)$$

By putting the value of (2.2.25), the center of the ellipse is given by (2.2.19) as follows,

$$\mathbf{c} = -\frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \quad (2.2.26)$$

Also the semi-major axis (a) and semi-minor axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \quad (2.2.27)$$

$$b = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \quad (2.2.28)$$

Finally from (2.2.18), the equation of ellipse is given by,

$$\mathbf{y}^T \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \mathbf{y} = 4 \quad (2.2.29)$$

The following figure 2.2.1 is the graphical representation of the ellipse in (2.2.29),

2.3. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre.

$$2x^2 - 72xy + 23y^2 - 4x - 2y - 48 = 0 \quad (2.3.1)$$

Solution:

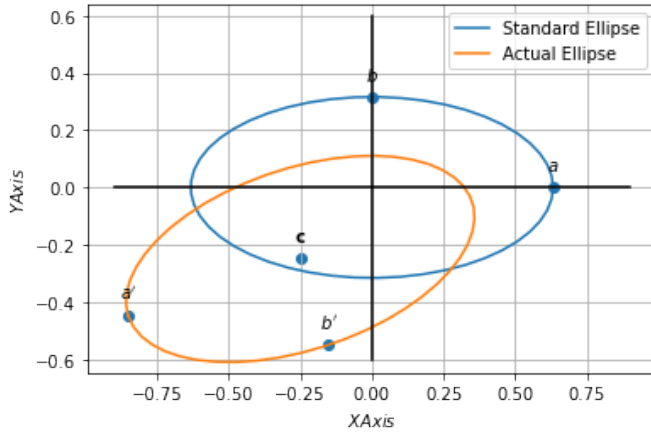


Fig. 2.2.1: Graphical representation of the ellipse

2.4. What conic does the given equations represent?

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \quad (2.4.1)$$

Solution: The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.4.2)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & -\frac{5}{2} \\ -\frac{5}{2} & -6 \end{pmatrix} \quad (2.4.3)$$

$$\mathbf{u} = \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (2.4.4)$$

$$f = 4 \quad (2.4.5)$$

The above equation (2.4.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.4.6)$$

Verify the given equation as if it is pair of straight lines

$$\Delta = \begin{vmatrix} 6 & -\frac{5}{2} & 7 \\ -\frac{5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix} \quad (2.4.7)$$

$$\Rightarrow 6 \begin{vmatrix} -\frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & 4 \end{vmatrix} - \frac{5}{2} \begin{vmatrix} -\frac{5}{2} & \frac{5}{2} \\ 7 & 4 \end{vmatrix} + 7 \begin{vmatrix} -\frac{5}{2} & -6 \\ 7 & \frac{5}{2} \end{vmatrix} = 0 \quad (2.4.8)$$

$$\Rightarrow \Delta = 0 \quad (2.4.9)$$

Since equation (2.4.6) is satisfied, we could say that the given equation represents two straight

lines

$$\Delta_V = \begin{vmatrix} 6 & -\frac{5}{2} \\ -\frac{5}{2} & -6 \end{vmatrix} < 0 \quad (2.4.10)$$

Let the equations of lines be,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.4.11)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 6 & -\frac{5}{2} \\ -\frac{5}{2} & -6 \end{pmatrix} \mathbf{x} + 2 \left(7 \frac{5}{2} \right) \mathbf{x} + 4 \quad (2.4.12)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} \quad (2.4.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (2.4.14)$$

$$c_1 c_2 = 4 \quad (2.4.15)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.4.16)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (2.4.17)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.4.18)$$

Substituting the given data in above equations (2.4.16) we get,

$$-6m^2 - 5m + 6 = 0 \quad (2.4.19)$$

$$\Rightarrow m_i = \frac{-\frac{5}{2} \pm \sqrt{-\frac{169}{4}}}{-6} \quad (2.4.20)$$

Solving equation (2.4.20) we get,

$$m_1 = -\frac{3}{2}, m_2 = \frac{2}{3} \quad (2.4.21)$$

$$= \mathbf{n}_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.4.22)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.4.23)$$

Verification using Toeplitz matrix, From equa-

tion (2.4.22)

$$\mathbf{n}_1 = \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} \quad (2.4.24)$$

$$\Rightarrow \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.4.25)$$

\Rightarrow Equation (2.4.22) satisfies (2.4.23)
 c_1 and c_2 can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.4.26)$$

Substituting (2.4.22) in (2.4.26), the augmented matrix is,

$$\begin{pmatrix} -3 & -2 & 14 \\ -2 & 3 & 5 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + 2R_1]{R_1 \leftarrow -R_1/3} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{14}{3} \\ 0 & \frac{13}{3} & -\frac{13}{3} \end{pmatrix} \quad (2.4.27)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{2}{3}R_2]{R_2 \leftarrow \frac{3}{13}R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \end{pmatrix} \quad (2.4.28)$$

$$\Rightarrow c_1 = -4, c_2 = -1 \quad (2.4.29)$$

Equations (2.4.11), can be modified as, from (2.4.22) and (2.4.29) in we get,

$$(-3 \quad -2)\mathbf{x} = -4 \quad (2.4.30)$$

$$(-2 \quad 3)\mathbf{x} = -1 \quad (2.4.31)$$

$$\begin{aligned} \Rightarrow (-3x - 2y + 4)(-2x + 3y + 1) &= 0 \\ \Rightarrow (3x + 2y - 4)(2x - 3y - 1) &= 0 \end{aligned} \quad (2.4.32)$$

The angle between the lines can be expressed as,

$$\mathbf{n}_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.4.33)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (2.4.34)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{0}{\sqrt{169}}\right) = 90^\circ. \quad (2.4.35)$$

2.5. What conic does the following equation represent? Find its equation and centre.

$$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$$

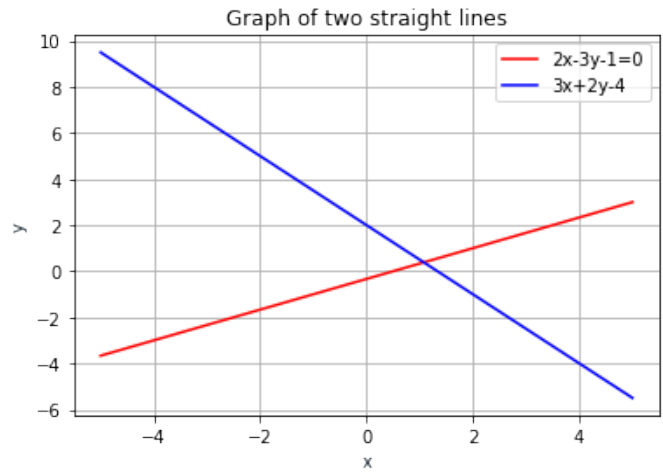


Fig. 2.4.1: Pair of straight lines

Solution: The general equation of second degree can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.5.1)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.5.2)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (2.5.3)$$

From (2.5.2) and (2.5.3)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \quad (2.5.4)$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -\frac{13}{2} \end{pmatrix} \quad (2.5.5)$$

$$|\mathbf{V}| = \begin{vmatrix} 3 & -4 \\ -4 & -3 \end{vmatrix} = -25 \quad (2.5.6)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (2.5.7)$$

Since $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix \mathbf{P} such that

$$\mathbf{PVP}^T = \mathbf{D} = \text{diag}(\lambda_1 \quad \lambda_2) \quad (2.5.8)$$

or equivalently

$$\mathbf{V} = \mathbf{PDP}^T \quad (2.5.9)$$

Eigen vectors of real symmetric matrix \mathbf{V} are orthogonal. The characteristic equation of \mathbf{V} is

obtained by evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 3 & 4 \\ 4 & \lambda + 3 \end{vmatrix} = 0 \quad (2.5.10)$$

$$\Rightarrow \lambda^2 - 25 = 0 \quad (2.5.11)$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 5 \quad (2.5.12)$$

From (2.5.7) and (2.5.12) the equation represents a hyperbola. The eigen vector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.5.13)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (2.5.14)$$

For $\lambda_1 = -5$:

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \xrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow -\frac{R_1}{4}} \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \quad (2.5.15)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad (2.5.16)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.5.17)$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.5.18)$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad (2.5.19)$$

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} \quad (2.5.20)$$

Let $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$ with $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Substituting in (2.5.1)

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.5.21)$$

with centre

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} -\frac{41}{25} \\ \frac{1}{50} \end{pmatrix} \quad (2.5.22)$$

and minor and major axes parameters as

$$\sqrt{\frac{\lambda_1}{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}} = \sqrt{\frac{500}{33}}, \quad \sqrt{\frac{\lambda_2}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}} = \sqrt{\frac{500}{33}} \quad (2.5.23)$$

The equation of hyperbola is

$$\frac{y_2^2}{\frac{33}{500}} - \frac{y_1^2}{\frac{33}{500}} = 1 \quad (2.5.24)$$

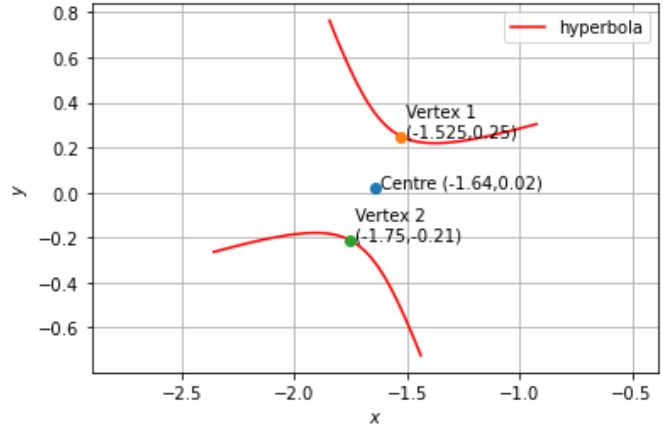


Fig. 2.5.1