Coordinate Geometry Exercises

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Contents

***Abstract*—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.**

1 Conics

1.1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix}2\\0\end{pmatrix}\right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.1.1}$$

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Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.2}$$
$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.1.3}$$
$$\mathbf{u_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.1.4}$$
$$f_1 = -4 \tag{1.1.5}$$
$$\mathbf{O_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.1.6}$$

Taking equation of the second circle to be,

$$\left\|\mathbf{x} - \begin{pmatrix}2\\0\end{pmatrix}\right\|^2 = 2^2 \tag{1.1.7}$$
$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{1.1.8}$$
$$\mathbf{u_2} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.1.9}$$
$$f_2 = 0 \tag{1.1.10}$$
$$\mathbf{O_2} = \begin{pmatrix}2\\0\end{pmatrix} \tag{1.1.11}$$

Now,Subtracting equation (1.1.8) from (1.1.3) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{1.1.12}$$
$$2\mathbf{u}^T\mathbf{x} = -4 \tag{1.1.13}$$
$$\begin{pmatrix}-4 & 0\end{pmatrix}\mathbf{x} = -4 \tag{1.1.14}$$

Which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1 \tag{1.1.15}$$

$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{1.1.16}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.17}$$

$$\mathbf{q} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.1.18}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.1.19}$$

Substituting (1.1.17) in (1.1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.1.20}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.21}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.22}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.1.23}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.24}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.1.25}$$

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.26}$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \tag{1.1.27}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.1.28}$$

$$\lambda^2 = 3 \tag{1.1.29}$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{1.1.30}$$

Substituting the value of λ in(1.1.17)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.1.31}$$

$$\mathbf{A} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{1.1.32}$$

$$\mathbf{B} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{1.1.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix} \tag{1.1.34}$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}-1\\\sqrt{3}\end{pmatrix} \tag{1.1.35}$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{1.1.36}$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{1.1.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \tag{1.1.38}$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.1.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.1.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.1.41}$$

$$\theta_1 = 120° \tag{1.1.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \tag{1.1.43}$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.1.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.1.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.1.46}$$

$$\theta_2 = 120° \tag{1.1.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.48}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.1.50}$$

$$= \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}4\pi - \frac{1}{2}2\sqrt{3} \tag{1.1.52}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \tag{1.1.53}$$

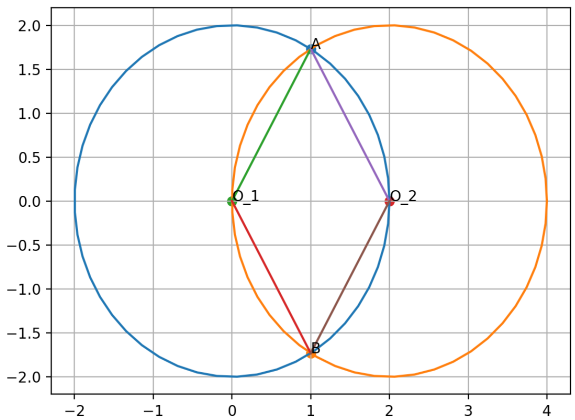

Fig. 1.1: Figure depicting intersection points of circle

1.2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre(h,k) is given by,

$$x^T x + 2u^T x + f = 0 \tag{1.2.1}$$

where,

$$f = \mathbf{u}^T \mathbf{u} - r^2 \tag{1.2.2}$$

The radius and centre are respectively given by,

$$r = 5 \tag{1.2.3}$$

$$\mathbf{c} = -u = k\mathbf{e} \tag{1.2.4}$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.5}$$

$$\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.2.6}$$

From the given data , we modify equation 1.2.1 as,

$$\mathbf{x_1}^T \mathbf{x_1} + 2\begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \tag{1.2.7}$$

$$\|\mathbf{x_1}\|^2 + 2\left(k^2\right) + f = 0 \tag{1.2.8}$$

$$2k^2 + f = -\|\mathbf{x_1}\|^2 \tag{1.2.9}$$

Subsituting $\mathbf{u}$ in equation 1.2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \tag{1.2.10}$$

$$f = \left(k^2\right) - r^2 \tag{1.2.11}$$

$$k^2 - f = r^2 \tag{1.2.12}$$

From equations 1.2.9 and 1.2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x_1}\|^2 \\ r^2 \end{pmatrix} \tag{1.2.13}$$

Here ,$\|\mathbf{x_1}\|$ is given by ,

$$\|\mathbf{x_1}\| = \sqrt{2^2 + 3^2} \tag{1.2.14}$$

$$\|\mathbf{x_1}\| = \sqrt{13} \tag{1.2.15}$$

Subsituting equation 1.2.6,1.2.3 in equation 1.2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \tag{1.2.16}$$

The augumented matrix of 1.2.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \tag{1.2.17}$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \tag{1.2.18}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \tag{1.2.19}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \tag{1.2.20}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array}\right) \tag{1.2.21}$$

Equation 1.2.16 can we rewritten as ,

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}k^2\\f\end{pmatrix}=\begin{pmatrix}4\\-21\end{pmatrix} \tag{1.2.22}$$

Expanding the above equation 1.2.22 we get ,

$$k^2 = 4 \tag{1.2.23}$$
$$k = \pm 2 \tag{1.2.24}$$
$$f = -21 \tag{1.2.25}$$

To get the centre substitute equation 1.2.24 in equation 1.2.4 To verify the above results we plot the circle with centre $\mathbf{c}$ as $\begin{pmatrix}2\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\0\end{pmatrix}$,

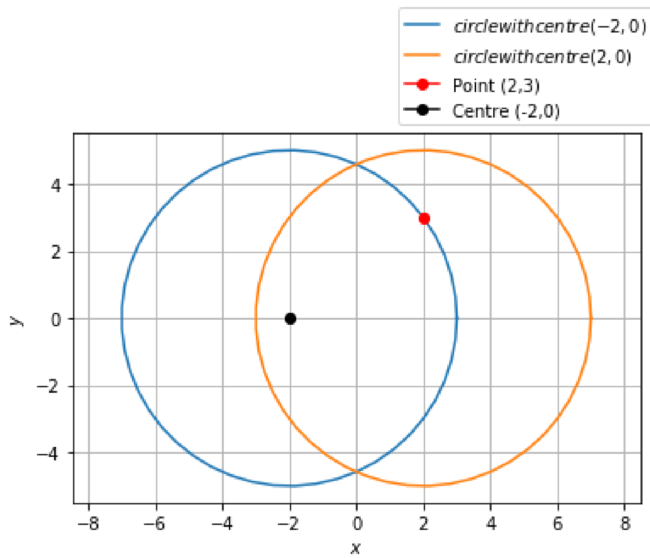

Fig. 1.2: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

qFrom the above figure 1.2 it is clear that circle with centre $\mathbf{c}$=$\begin{pmatrix}-2\\0\end{pmatrix}$ passes through the point $\mathbf{x_1}$ Desired equation of circle is given by ,

$$c = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.2.26}$$
$$f = -21 \tag{1.2.27}$$

1.3. Find the equation of the circle passing through $\begin{pmatrix}0\\0\end{pmatrix}$ and making intercepts a and b on the coordinate axes.

1.4. Find the equation of a circle with centre $\begin{pmatrix}2\\2\end{pmatrix}$ and passes through the point $\begin{pmatrix}4\\5\end{pmatrix}$.

Solution: he general equation of a circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.4.1}$$
$$\text{If } r \text{ is radius, } f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.4.2}$$
$$\text{center } \mathbf{c} = -\mathbf{u} \tag{1.4.3}$$

Given centre is $\begin{pmatrix}2\\2\end{pmatrix}$

$$\implies \mathbf{c} = \begin{pmatrix}2\\2\end{pmatrix} \tag{1.4.4}$$
$$\implies \mathbf{u} = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{1.4.5}$$

Equation (1.4.1) becomes

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} + f = 0 \tag{1.4.6}$$

This passes through point $\begin{pmatrix}4\\5\end{pmatrix}$

Substituting $\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix}$ in (1.4.6)

$$\begin{pmatrix}4 & 5\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + f = 0 \tag{1.4.7}$$
$$\implies f = -5 \tag{1.4.8}$$

Also, radius can be determined as follows

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.4.9}$$
$$\implies -5 = \begin{pmatrix}-2 & -2\end{pmatrix}\begin{pmatrix}-2\\-2\end{pmatrix} - r^2 \tag{1.4.10}$$
$$\implies -5 = 8 - r^2 \tag{1.4.11}$$
$$\implies r = \sqrt{13} \tag{1.4.12}$$

The equation of required circle is

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} - 5 = 0 \tag{1.4.13}$$

See Fig. 1.4

1.5. Find the locus of all the unit vectors in the xy-plane.

1.6. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.

Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.6.1}$$

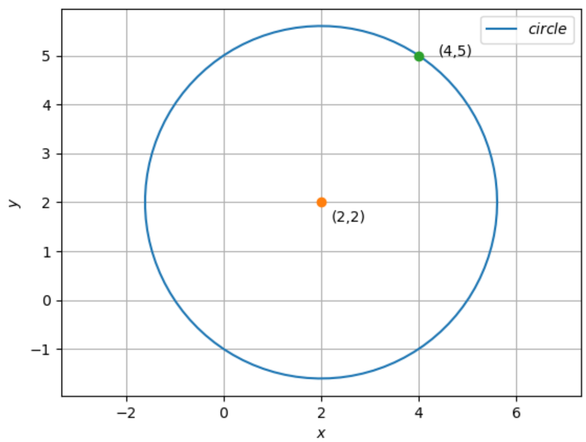

Fig. 1.4: plot showing the circle

The centre and the radius can be obtained as,

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.2}$$

$$f = -3 \tag{1.6.3}$$

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.6.4}$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \tag{1.6.5}$$

$\because$ The tangents are parallel to the x-axis, their direction and normal vectors, $\mathbf{m}$ and $\mathbf{n}$ are respectively,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.6.6}$$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.6.7}$$

For a circle, given the normal vector $\mathbf{n}$, the tangent points of contact to circle given by equation (1.6.1) are given by

$$\mathbf{q_i} = (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \tag{1.6.8}$$

where

$$\kappa_i = \pm\sqrt{\frac{\mathbf{u^T u} - f}{\mathbf{n^T n}}} \tag{1.6.9}$$

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} - (-3)}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{1.6.10}$$

$$\implies \kappa = \pm\sqrt{\frac{4}{1}} \tag{1.6.11}$$

$$\implies \kappa = \pm 2 \tag{1.6.12}$$

and from (1.6.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_1} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.13}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.6.14}$$

$$\mathbf{q_2} = -2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.6.15}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.6.16}$$

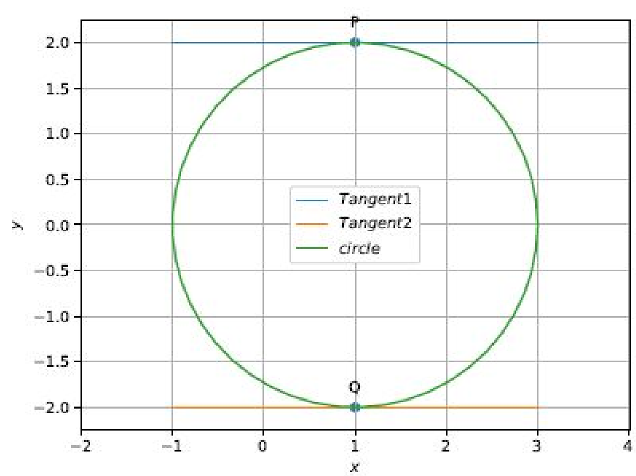

Fig. 1.6: Figure depicting tangents of circle parallel to x-axis

1.7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^T\mathbf{x} = 4$.
Solution: The equation of a circle can be expressed as,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{1.7.1}$$

where $\mathbf{c}$ is the center.
Comparing equation (1.7.1) with the circle equation given,

$$\mathbf{x}^T\mathbf{x} = 4 \tag{1.7.2}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad f = -4 \tag{1.7.3}$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \tag{1.7.4}$$

$$\implies \boxed{r = 2} \tag{1.7.5}$$

From equation (1.7.5), the point at which circle touches x-axis is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The direction vector of x-axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
The direction vector of the given line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} = 0$ is $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$.
The angle that the line makes with the x-axis is given by,

$$\cos\theta = \frac{\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\|\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 & 0 \end{pmatrix}\right\|} = \frac{\sqrt{3}}{2} \tag{1.7.6}$$

$$\implies \boxed{\theta = 30^\circ} \tag{1.7.7}$$

Using equation (1.7.5) and (1.7.7),the area of the sector is obtained as,

$$\implies \boxed{\frac{\theta}{360^\circ}\pi r^2 = \frac{30^\circ}{360^\circ}\pi(2)^2 = \frac{\pi}{3}} \tag{1.7.8}$$

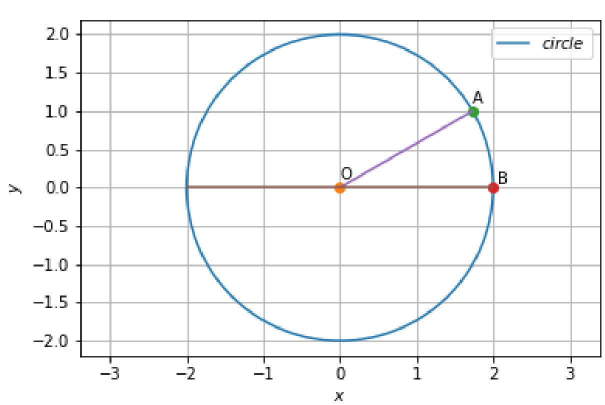

Fig. 1.7: Region enclosed by x-axis, line and circle

To find points $\mathbf{A}$ and $\mathbf{B}$,

The parametric form of x-axis is,

$$\mathbf{B} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.9}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.7.10}$$

From the intersection of circle and line, the value of λ can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.11}$$

$$= \frac{4-0}{1} = 4 \tag{1.7.12}$$

$$\implies \lambda = \pm 2 \tag{1.7.13}$$

Sub equation (1.7.13) in (1.7.10),

$$\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.7.14}$$

As given in question as first quadrant,

$$\implies \boxed{\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}} \tag{1.7.15}$$

Similarly, to find point $\mathbf{A}$, The parametric form of line is,

$$\mathbf{A} = \mathbf{q} + \lambda\mathbf{m} \tag{1.7.16}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{1.7.17}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.7.18}$$

$$= \frac{4-0}{4} = 1 \tag{1.7.19}$$

$$\implies \lambda = \pm 1 \tag{1.7.20}$$

$$\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \tag{1.7.21}$$

$$\implies \boxed{\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \tag{1.7.22}$$

1.8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
1.9. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.
1.10. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
Solution:
General equation of circle is $\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + \mathrm{f} = 0$

Taking equation of the first curve to be,

$$\left\|\mathbf{x}-\begin{pmatrix}1\\0\end{pmatrix}\right\|^2 = 1^2 \tag{1.10.1}$$

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_1}^T\mathbf{x} = 0 \tag{1.10.2}$$

$$\mathbf{u_1} = \begin{pmatrix}-1\\0\end{pmatrix} \tag{1.10.3}$$

$$f_1 = 0 \tag{1.10.4}$$

$$\mathbf{O_1} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.10.5}$$

Taking equation of the second curve to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_2^T\mathbf{x} + f_2 = 0 \tag{1.10.6}$$

$$\mathbf{x}^T\mathbf{x} - 1 = 0 \tag{1.10.7}$$

$$\mathbf{u_2} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.10.8}$$

$$f_2 = -1 \tag{1.10.9}$$

$$\mathbf{O_2} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.10.10}$$

Now, subtracting equation (1.10.2) from (1.10.7) We get,

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_1}^T\mathbf{x} - \mathbf{x}^T\mathbf{x} - f_2 = 0 \tag{1.10.11}$$

$$2\mathbf{u}^T\mathbf{x} = -1 \tag{1.10.12}$$

$$\begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x} = -1 \tag{1.10.13}$$

which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1/2 \tag{1.10.14}$$

$$\mathbf{x} = \begin{pmatrix}1/2\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{1.10.15}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.10.16}$$

$$\mathbf{q} = \begin{pmatrix}1/2\\0\end{pmatrix} \tag{1.10.17}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.10.18}$$

Substituting (1.10.16) in (1.10.6)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_2^T\mathbf{x} + f_2 = 0 \tag{1.10.19}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_2 = 0 \tag{1.10.20}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_2 = 0 \tag{1.10.21}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_2 = 0 \tag{1.10.22}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_2 = 0 \tag{1.10.23}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_2 = 0 \tag{1.10.24}$$

Taking λ as common :

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_2 - \|\mathbf{q}\|^2 \tag{1.10.25}$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_2 - \|\mathbf{q}\|^2 \tag{1.10.26}$$

$$\lambda^2 = \frac{-f_2 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.10.27}$$

$$\lambda^2 = \frac{3}{4} \tag{1.10.28}$$

$$\lambda = +\sqrt{\frac{3}{4}}, -\sqrt{\frac{3}{4}} \tag{1.10.29}$$

$$\lambda = +\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \tag{1.10.30}$$

Substituting the value of λ in (1.10.16)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.10.31}$$

$$\mathbf{A} = \begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.32}$$

$$\mathbf{B} = \begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.34}$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.35}$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.36}$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \tag{1.10.38}$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.10.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.10.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.10.41}$$

$$\theta_1 = 120^\circ \tag{1.10.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \tag{1.10.43}$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.10.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.10.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.10.46}$$

$$\theta_2 = 120^\circ \tag{1.10.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\pi\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.10.48}$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \tag{1.10.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \tag{1.10.50}$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \tag{1.10.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \tag{1.10.52}$$

$$= \frac{2\pi}{3} - 2\sqrt{3} \tag{1.10.53}$$

1.11. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.

Solution:

Find the smaller area enclosed by the circle $\mathbf{x}\mathbf{x}^T = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$. General

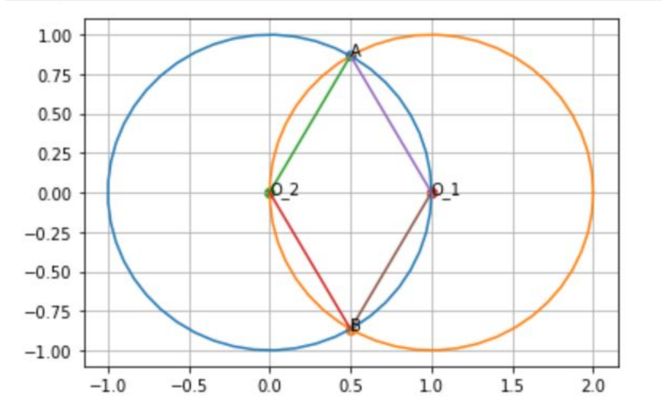

Fig. 1.10: Figure depicting intersection points of circle

equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.11.1}$$

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.11.2}$$

$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.11.3}$$

$$\mathbf{u_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.11.4}$$

$$f_1 = -4 \tag{1.11.5}$$

$$\mathbf{O_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.11.6}$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \tag{1.11.7}$$

$$\implies \boxed{r = 2} \tag{1.11.8}$$

From equation (1.11.8), the point at which circle touches x-axis is $\begin{pmatrix}-2\\0\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$.

The direction vector of the given line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$ is $\begin{pmatrix}-1\\1\end{pmatrix}$.

To find point $\mathbf{A}$ and $\mathbf{B}$, The parametric form of

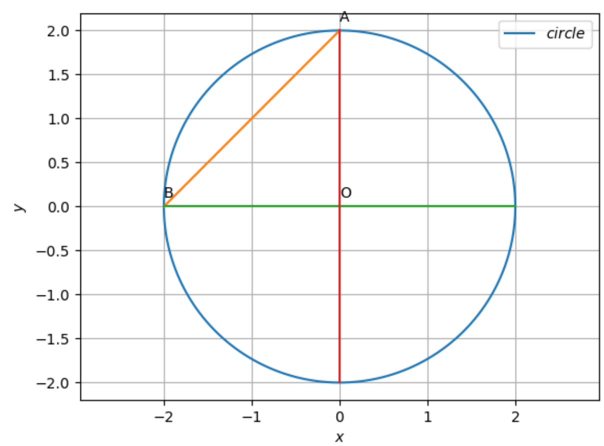

Fig. 1.11: Smaller area enclosed by line and circle

line is,

$$\mathbf{A} = \mathbf{q} + \lambda\mathbf{m} \tag{1.11.9}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.11.10}$$

$$\lambda^2 = \frac{-f_1 - ||\mathbf{q}||^2}{||\mathbf{m}||^2} \tag{1.11.11}$$

$$= \frac{4-2}{2} = 1 \tag{1.11.12}$$

$$\implies \lambda = \pm 1 \tag{1.11.13}$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.11.14}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.11.15}$$

$$(\mathbf{A} - \mathbf{O}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{1.11.16}$$

$$(\mathbf{B} - \mathbf{O}) = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.11.17}$$

Inner product of $(\mathbf{A} - \mathbf{O})$ and $(\mathbf{B} - \mathbf{O})$ is given as:

$$(\mathbf{A} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = 0 \tag{1.11.18}$$

Therefore, $(\mathbf{A} - \mathbf{O}) \perp (\mathbf{B} - \mathbf{O})$
Smaller area enclosed by circle and line **AB** is:
Area = (Area of circle in 2nd Quadrant) - (Area of right triangle formed by line AB, X and Y axis)

$$Area = \frac{\pi\theta_1}{360} r^2 - \frac{1}{2} \times 2 \times 2 \tag{1.11.19}$$

$$= \frac{90}{360} \pi \times 2^2 - 2 \tag{1.11.20}$$

$$= \pi - 2 \tag{1.11.21}$$

Hence, the smaller area enclosed by the circle $\mathbf{x}\mathbf{x}^T = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$ is $(\pi - 2)$

1.12. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.
Solution:

$$y = \frac{x-1}{x-2} \tag{1.12.1}$$

Equation (1.12.1) can be expressed as

$$y(x-2) = x - 1 \tag{1.12.2}$$

$$yx - 2y - x + 1 = 0 \tag{1.12.3}$$

From above we can say,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.12.4}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} & -1 \end{pmatrix} \tag{1.12.5}$$

$$f = 1 \tag{1.12.6}$$

Now,

$$\because |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0, \tag{1.12.7}$$

(1.12.1) is the equation of a hyperbola. To verify that this we will find the the characteristic equation of $\mathbf{V}$.

$$\left| \lambda\mathbf{I} - \mathbf{V} \right| = \begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} = 0 \tag{1.12.8}$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{1.12.9}$$

The eigenvalues are the roots of (1.12.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \tag{1.12.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{1.12.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{1.12.12}$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.12.13}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.12.14}$$

Now,λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.12.15}$$

$$\implies \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.12.16}$$

From Equations,

$$\mathbf{V} = \mathbf{PDP}^{-1} = \mathbf{PDP}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{1.12.17}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{VP} \tag{1.12.18}$$

We can say that

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.12.19}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{1.12.20}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f > 0$,there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-}1\mathbf{u} \tag{1.12.21}$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{1.12.22}$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{1.12.23}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.12.24}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{1.12.25}$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \tag{1.12.26}$$

Let us assume slope to be l,now finding the direction vector and normal vector of the tangent with slope l.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \tag{1.12.27}$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \tag{1.12.28}$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{1.12.29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.12.30}$$

By using (1.12.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \tag{1.12.31}$$

Now substituting this κ in (1.12.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{\frac{-l}{4}} + 1 \end{pmatrix} \tag{1.12.32}$$

We know that x=10.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \tag{1.12.33}$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \tag{1.12.34}$$

$$\sqrt{-\frac{1}{4l}} = 4 \tag{1.12.35}$$

$$-\frac{1}{4l} = 16 \tag{1.12.36}$$

$$l = -\frac{1}{64} \tag{1.12.37}$$

The slope of the tangent to the curve y=$\frac{x-1}{x-2}$, x≠2 at x=10 is $\frac{1}{64}$. So,from the above we can say that κ=4,-4 and from equation (1.12.27) and (1.12.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \tag{1.12.38}$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \tag{1.12.39}$$

Now using equation (1.12.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1}\left(\kappa_1 \mathbf{n} - \mathbf{u}\right) \tag{1.12.40}$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(-4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.12.41}$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{1.12.42}$$

$$\mathbf{q}_2 = \mathbf{V}^{-1}\left(\kappa_2 \mathbf{n} - \mathbf{u}\right) \tag{1.12.43}$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.12.44}$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \tag{1.12.45}$$

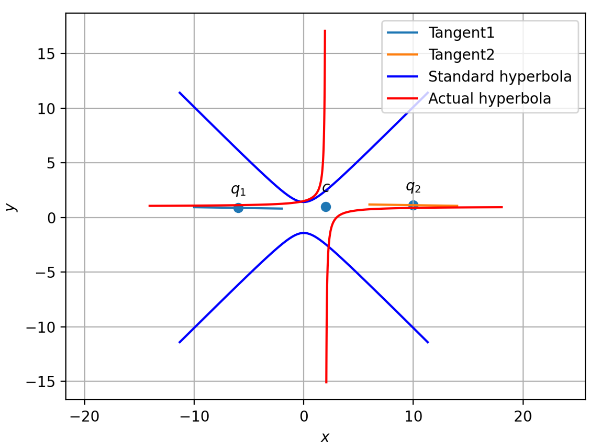

Fig. 1.12: Tangent 2 shows the tangent

1.13. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Solution: $y = (x-2)^2$ can be written as,

$$x^2 - 4x - y + 4 = 0 \tag{1.13.1}$$

From (1.13.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \mathbf{u} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}; f = 4 \tag{1.13.2}$$

$$\left|V\right| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{1.13.3}$$

(1.13.3) implies that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left|V - \lambda I\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\implies \lambda = 0, 1 \tag{1.13.4}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.13.5}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.13.6}$$

By Eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$$

$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.13.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.13.8}$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{1.13.9}$$

$$where, \eta = \mathbf{u}^T\mathbf{p}_1 = -\frac{1}{2} \tag{1.13.10}$$

Substituting values from (1.13.2), (1.13.5) and (1.13.10) in (1.13.9),

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \tag{1.13.11}$$

Removing last row and representing (1.13.11) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix}-2&-1&-4\\1&0&2\end{pmatrix}\xleftrightarrow{R_1\leftarrow\frac{R_1}{-2}}\begin{pmatrix}1&\frac{1}{2}&2\\1&0&2\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2-R_1}$$
$$\begin{pmatrix}1&\frac{1}{2}&2\\0&-\frac{1}{2}&0\end{pmatrix}\xleftrightarrow{R_2\leftarrow(-2R_2)}\begin{pmatrix}1&\frac{1}{2}&2\\0&1&0\end{pmatrix}\xleftrightarrow{R_1\leftarrow R_1-\frac{R_2}{2}}$$

$$\begin{pmatrix}1&0&2\\0&1&0\end{pmatrix} \tag{1.13.12}$$

From (1.13.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix}2\\0\end{pmatrix} \tag{1.13.13}$$

Direction vector of the chord joining A(4,4) and B(2,0) can be calculated as,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix}4\\4\end{pmatrix} - \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix}1\\2\end{pmatrix} \tag{1.13.14}$$

We know that,

$$\mathbf{m}^T\mathbf{n} = 0; \implies \mathbf{n} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{1.13.15}$$

To find the point of contact $\mathbf{q}$, which is intersection point for normal of the chord AB and also tangent of the curve,

$$\begin{pmatrix}\mathbf{u}^T + \kappa\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f\\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix} \tag{1.13.16}$$

$$where, \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = \frac{1}{2} \tag{1.13.17}$$

Substituting the values from (1.13.2),(1.13.15) and (1.13.17) in (1.13.16),

$$\begin{pmatrix}-1&1\\1&0\\0&0\end{pmatrix}\mathbf{q} = \begin{pmatrix}-4\\3\\0\end{pmatrix} \tag{1.13.18}$$

Removing last row and representing (1.13.18) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix}-1&-1&-4\\1&0&3\end{pmatrix}\xleftrightarrow{R_1\leftarrow(-R_1)}\begin{pmatrix}1&1&4\\1&0&3\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2-R_1}$$
$$\begin{pmatrix}1&1&4\\0&-1&-1\end{pmatrix}\xleftrightarrow{R_2\leftarrow(-R_2)}\begin{pmatrix}1&1&4\\0&1&1\end{pmatrix}\xleftrightarrow{R_1\leftarrow R_1-R_2}$$

$$\begin{pmatrix}1&0&3\\0&1&1\end{pmatrix} \tag{1.13.19}$$

From (1.13.19), it can be observed,

$$\mathbf{q} = \begin{pmatrix}3\\1\end{pmatrix} \tag{1.13.20}$$

which is the required point of contact

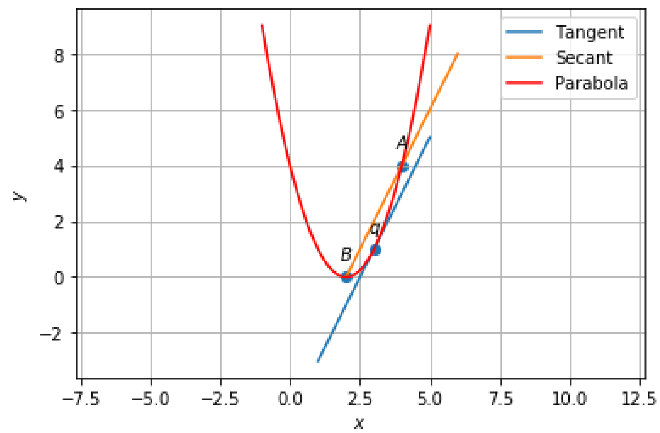

Fig. 1.13: Parabola with AB as chord, a tangent parallel to the chord

1.14. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

Solution: The given curve

$$y = \frac{1}{x-1} \tag{1.14.1}$$

can be expressed as

$$xy - y - 1 = 0 \tag{1.14.2}$$

Hence, we have

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix}, f = -1 \tag{1.14.3}$$

Since $\left|\mathbf{V}\right| < 0$, the equation (1.14.2) represents hyperbola. To find the values of λ_1 and λ_2, consider the characteristic equation,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{1.14.4}$$

$$\implies \left|\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix} - \begin{pmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{pmatrix}\right| = 0 \tag{1.14.5}$$

$$\implies \begin{vmatrix}\lambda&\frac{-1}{2}\\\frac{-1}{2}&\lambda\end{vmatrix} = 0 \tag{1.14.6}$$

$$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.14.7}$$

In addition, given the slope -1, the direction and normal vectors are given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.14.8)$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.14.9)$$

The parameters of hyperbola are as follows:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (1.14.10)$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \quad (1.14.11)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.14.12)$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} = \sqrt{2} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \end{cases} \quad (1.14.13)$$

which represents the standard hyperbola equation,

$$\frac{x^2}{2} - \frac{x^2}{2} = 1 \quad (1.14.14)$$

The points of contact are given by

$$K = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} = \pm\frac{1}{2} \quad (1.14.15)$$

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \quad (1.14.16)$$

$$\mathbf{q_1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \quad (1.14.17)$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.14.18)$$

$$\mathbf{q_2} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{-1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \quad (1.14.19)$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (1.14.20)$$

$\therefore$ The tangents are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 \quad (1.14.21)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = 0 \quad (1.14.22)$$

The desired equations of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$ are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \quad (1.14.23)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = -1 \quad (1.14.24)$$

The above results are verified in the following figure.

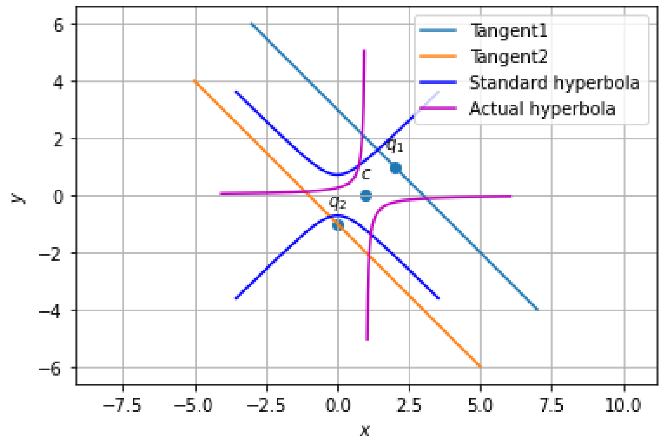

Fig. 1.14: The standard and actual hyperbola.

1.15. Find the equation of all lines having slope -2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.
Solution: Given the curve,

$$y = \frac{1}{x-3} \quad (1.15.1)$$

$$\implies xy - 3y - 1 = 0 \quad (1.15.2)$$

From (1.15.2) we get,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{-3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = -1 \quad (1.15.3)$$

Now,

$$\because \left|V\right| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = \frac{-1}{2} < 0 \quad (1.15.4)$$

(1.15.1) is equation of hyperbola. Now,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \quad (1.15.5)$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \quad (1.15.6)$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \quad (1.15.7)$$

The eigen vector $\mathbf{p}$ is given by,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (1.15.8)$$

For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I}-\mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1\leftarrow 2R_1]{R_2\leftarrow R_2+R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.15.9}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.15.10}$$

Similarly for λ_2,

$$(\lambda_2\mathbf{I}-\mathbf{V}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1\leftarrow 2R_1]{R_2\leftarrow R_-R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.15.11}$$

$$\implies \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.15.12}$$

Now,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.15.13}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{1.15.14}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{1.15.15}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0$, there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.15.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.15.17}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = \sqrt{2} \tag{1.15.18}$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \tag{1.15.19}$$

The direction and normal vectors of the tangent with slope -2 are given as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.15.20}$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{1.15.21}$$

$$k = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.15.22}$$

Thus,

$$\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n} = 8 \tag{1.15.23}$$

$$k = \pm\frac{1}{2\sqrt{2}} \tag{1.15.24}$$

$$\mathbf{q_1} = \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \tag{1.15.25}$$

$$\mathbf{q_2} = \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \tag{1.15.26}$$

The desired tangents are,

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix}\right\} = 0 \tag{1.15.27}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 + 2\sqrt{2} \tag{1.15.28}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix}\right\} = 0 \tag{1.15.29}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 - 2\sqrt{2} \tag{1.15.30}$$

Below figure corresponds to the tangents on the hyperbola, represented by (1.15.28) and (1.15.30) each having slope of -2.

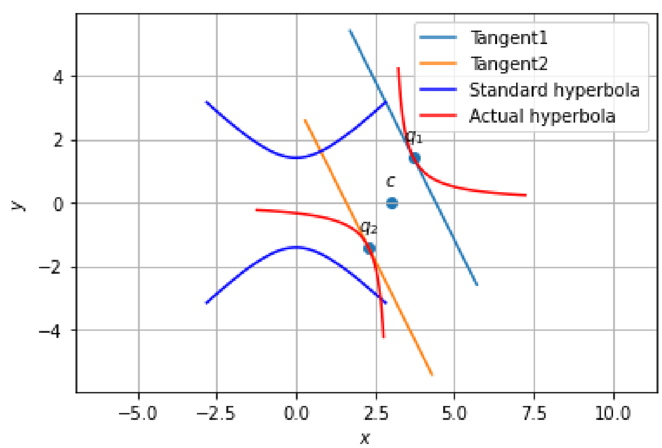

Fig. 1.15: Tangents to the hyperbola

1.16. Find points on the curve $\mathbf{x}^T\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\mathbf{x} = 1$ at which tangents are

a) parallel to x-axis
b) parallel to y-axis.

Solution:
General equation of conics is

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.16.1}$$

Comparing with the equation given,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \tag{1.16.2}$$

$$\mathbf{u} = \mathbf{0} \tag{1.16.3}$$

$$f = -1 \tag{1.16.4}$$

$$\left|\mathbf{v}\right| = \left|\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\right| > 0 \tag{1.16.5}$$

$\because$ $|\mathbf{V}| > 0$, the given equation is of ellipse.
a)The tangents are parallel to the x-axis, hence, their direction and normal vectors, $\mathbf{m_1}$ and $\mathbf{n_1}$ are respectively,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.6}$$

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16.7}$$

For an ellipse, given the normal vector $\mathbf{n}$, the tangent points of contact to the ellipse are given by

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) = \mathbf{V}^{-1}\kappa\mathbf{n} \tag{1.16.8}$$

where

$$\kappa = \pm\sqrt{\frac{\mathbf{u^T}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n^T}\mathbf{V}^{-1}\mathbf{n}}} \tag{1.16.9}$$

$$= \pm\sqrt{\frac{-f}{\mathbf{n^T}\mathbf{V}^{-1}\mathbf{n}}} \tag{1.16.10}$$

$$\mathbf{V}^{-1} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \tag{1.16.11}$$

$$\kappa_1 = \pm\sqrt{\frac{-(-1)}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{1.16.12}$$

$$\implies \kappa_1 = \pm\sqrt{\frac{1}{16}} \tag{1.16.13}$$

$$\implies \kappa_1 = \pm\frac{1}{4} \tag{1.16.14}$$

From (1.16.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_1} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}\frac{1}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16.15}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} \tag{1.16.16}$$

$$= \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{1.16.17}$$

$$\mathbf{q_2} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} -\frac{1}{4} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16.18}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix} \tag{1.16.19}$$

$$= \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{1.16.20}$$

b) The tangents are parallel to the y-axis, hence, their direction and normal vectors, $\mathbf{m_2}$ and $\mathbf{n_2}$ are respectively,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16.21}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.22}$$

Using equation (1.16.9), the values of κ for this case are

$$\kappa_2 = \pm\sqrt{\frac{-(-1)}{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}} \tag{1.16.23}$$

$$\implies \kappa_2 = \pm\sqrt{\frac{1}{9}} \tag{1.16.24}$$

$$\implies \kappa_2 = \pm\frac{1}{3} \tag{1.16.25}$$

and from (1.16.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_3} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.26}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \tag{1.16.27}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.16.28}$$

$$\mathbf{q_4} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \left(-\frac{1}{3}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.29}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} \tag{1.16.30}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.16.31}$$

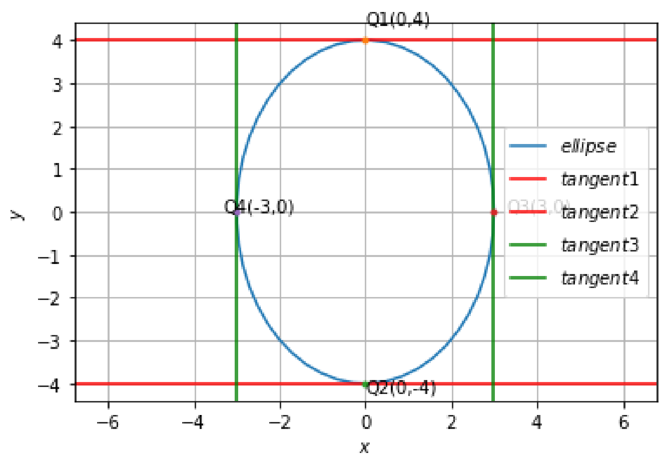

Fig. 1.16: Figure depicting point of contact of tangents of ellipse parallel to x-axis and y-axis

1.17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

1.18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$

a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$

b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.

Solution:

Given equation

$$y = x^2 - 2x + 7 \tag{1.18.1}$$

The equation (1.18.1) can be written as,

$$x^2 - 2x - y + 7 = 0 \tag{1.18.2}$$

Comparing it with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.18.3}$$

$\therefore$ a = 1, b = 0, c = 0, d = -1, e = $\frac{-1}{2}$, f = 7.

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.18.4}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{-1}{2} \end{pmatrix} \tag{1.18.5}$$

$$\text{Now,} \left|V\right| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{1.18.6}$$

$\implies$ that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left|\mathbf{V} - \lambda \mathbf{I}\right| = 0 \tag{1.18.7}$$

$$\begin{vmatrix} 1 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \tag{1.18.8}$$

$$\implies \lambda = 0, 1. \tag{1.18.9}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{Vx} = \lambda \mathbf{x} \tag{1.18.10}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.18.11}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.18.12}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{PDP^T} \tag{1.18.13}$$

$$\text{where,} \mathbf{P} = \begin{pmatrix} \mathbf{p_1} \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.18.14}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.18.15}$$

Hence equation (1.18.5) becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.18.16}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.18.17}$$

a) The given parallel line equation is

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9 \tag{1.18.18}$$
$$\implies 2x - y + 9 = 0 \tag{1.18.19}$$

Now the tangent to parabola is parallel to the line equation (1.18.19), the general straight line equation is of the form

$$ax + by + c = 0 \tag{1.18.20}$$

The normal vector ($\mathbf{n}$) and direction ($\mathbf{m}$) are given by,

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.18.21}$$
$$\mathbf{m} = \begin{pmatrix} b \\ -a \end{pmatrix} \tag{1.18.22}$$

Comparing (1.18.19), (1.18.13), (1.18.21), the direction vectors ($\mathbf{m}$) and normal ($\mathbf{n}$) vectors are,

$$\mathbf{m} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{1.18.23}$$
$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.18.24}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.18.25}$$
$$\text{where, } \kappa = \frac{\mathbf{p_1}^T \mathbf{u}}{\mathbf{p_1}^T \mathbf{n}} = \frac{1}{2} \tag{1.18.26}$$

Hence substituting the values of (1.18.5), (1.18.24), (1.18.13), (1.18.26) in equation (1.18.25) we get,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -7 \\ 2 \\ 0 \end{pmatrix} \tag{1.18.27}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (1.18.27) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} 0 & -1 & -7 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -7 \end{pmatrix} \tag{1.18.28}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \end{pmatrix} \tag{1.18.29}$$

Hence from equation (1.18.29) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \tag{1.18.30}$$

Now $\mathbf{q}$ is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \tag{1.18.31}$$

where c is,

$$c = \mathbf{n}^T \mathbf{q} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = -3 \tag{1.18.32}$$

Hence equation of tangent to the curve (1.18.1) parallel to (1.18.19) is given by substituting the value of c and $\mathbf{n}$ from equation (1.18.32) and (1.18.24) respectively to the equation (1.18.31),

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -3 \tag{1.18.33}$$

Figure 1.18 verifies that the $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -3$ is a tangent to parabola $y = x^2 - 2x + 7$

b) The given perpendicular line equation is

$$\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13 \tag{1.18.34}$$
$$\implies -15x + 5y - 13 = 0 \tag{1.18.35}$$

Now the tangent to parabola is perpendicular to the line equation (1.18.35), the general straight line equation is of the form

$$ax + by + c = 0 \tag{1.18.36}$$

Therefore, if we find the line that is parallel to the line (1.18.35), it will be parallel to the tangent itself. For the given line the normal vector ($\mathbf{n}$) and direction ($\mathbf{m}$) are given by,

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.18.37}$$
$$\mathbf{m} = \begin{pmatrix} b \\ -a \end{pmatrix} \tag{1.18.38}$$

Comparing (1.18.35), (1.18.37), (1.18.38), the direction vectors ($\mathbf{m}$) and normal ($\mathbf{n}$)

vectors are,

$$\mathbf{m} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \tag{1.18.39}$$

$$\mathbf{n} = \begin{pmatrix} -15 \\ 5 \end{pmatrix} \tag{1.18.40}$$

The parallel line for this vector will have the normal vector ($\mathbf{n_1}$) and direction ($\mathbf{m_1}$) are given by

$$\mathbf{m_1} = \begin{pmatrix} 15 \\ -5 \end{pmatrix} \tag{1.18.41}$$

$$\mathbf{n_1} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \tag{1.18.42}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n_1} - \mathbf{u} \end{pmatrix} \tag{1.18.43}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T \mathbf{u}}{\mathbf{p_1}^T \mathbf{n_1}} = \frac{-1}{30} \tag{1.18.44}$$

Hence substituting the values of (1.18.5), (1.18.42), (1.18.13), (1.18.44) in equation (1.18.43) we get,

$$\begin{pmatrix} \frac{-7}{6} & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -7 \\ \frac{5}{6} \\ 0 \end{pmatrix} \tag{1.18.45}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (1.18.45) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} \frac{-7}{6} & -1 & -7 \\ 1 & 0 & \frac{5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ \frac{-7}{6} & -1 & -7 \end{pmatrix} \tag{1.18.46}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \left(\frac{7}{6}\right) R_1} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ 0 & -1 & \frac{-217}{36} \end{pmatrix} \tag{1.18.47}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ 0 & 1 & \frac{217}{36} \end{pmatrix} \tag{1.18.48}$$

Hence from equation (1.18.48) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} \tag{1.18.49}$$

Now $\mathbf{q}$ is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n_1}^T \mathbf{x} = c \tag{1.18.50}$$

where c is,

$$c = \mathbf{n_1}^T \mathbf{q} = \begin{pmatrix} 5 & 15 \end{pmatrix} \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} = \frac{3405}{36} \tag{1.18.51}$$

Hence equation of tangent to the curve (1.18.1) parallel to (1.18.35) is given by substituting the value of c and $\mathbf{n_1}$ from equation (1.18.51) and (1.18.42) respectively to the equation (1.18.50),

$$\implies \begin{pmatrix} 5 & 15 \end{pmatrix} \mathbf{x} = \frac{3405}{36} \tag{1.18.52}$$

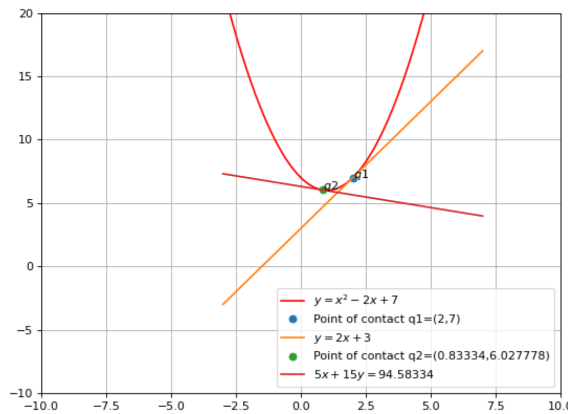

Fig. 1.18: Tangents to parabola $y = x^2 - 2x + 7$

Figure 1.18 verifies that the $\begin{pmatrix} 5 & 15 \end{pmatrix} \mathbf{x} = \frac{3405}{36}$ is a tangent to parabola $y = x^2 - 2x + 7$

1.19. Find the equation of the tangent to the curve,

$$y = \sqrt{3x - 2} \tag{1.19.1}$$

which is parallel to the line,

$$\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.19.2}$$

Solution: The equation (1.19.1) can be written as,

$$y^2 - 3x + 2 = 0 \tag{1.19.3}$$

Comparing it with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.19.4}$$

$\therefore$ a = b = e = 0, d = $\frac{-3}{2}$, c = 1, f = 2.

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.19.5}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix} \tag{1.19.6}$$

$$\text{Now, } \left|V\right| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \tag{1.19.7}$$

$\implies$ that the curve is a parabola. Now, finding the eigen values corresponding to the **V**,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{1.19.8}$$

$$\begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \tag{1.19.9}$$

$$\implies \lambda = 0, 1. \tag{1.19.10}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{1.19.11}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.19.12}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.19.13}$$

Now by eigen decomposition on **V**,

$$\mathbf{V} = \mathbf{PDP^T} \tag{1.19.14}$$

$$\text{where,}\mathbf{P} = \begin{pmatrix} \mathbf{p_1} \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.19.15}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.19.16}$$

Hence equation (1.19.14) becomes,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.19.17}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.19.18}$$

Now the tangent to parabola is parallel to the line equation (1.19.2), Hence the direction vectors (**m**) and normal (**n**) vectors are,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.19.19}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.19.20}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.19.21}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = \frac{-3}{4} \tag{1.19.22}$$

Hence substituting the values of (1.19.22), (1.19.20), (1.19.14) and (1.19.6) in equation (1.19.21) we get,

$$\begin{pmatrix} -3 & \frac{-3}{4} \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -2 \\ 0 \\ \frac{-3}{4} \end{pmatrix} \tag{1.19.23}$$

Solving for **q** by removing the zero row and representing (1.19.23) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} -3 & \frac{-3}{4} & -2 \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \left(\frac{-R_1}{3}\right)} \begin{pmatrix} 1 & \frac{1}{4} & \frac{2}{3} \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \tag{1.19.24}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{1}{4}R_2} \begin{pmatrix} 1 & 0 & \frac{41}{48} \\ 0 & 1 & \frac{-3}{4} \end{pmatrix} \tag{1.19.25}$$

Hence from equation (1.19.25) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{-3}{4} \end{pmatrix} \tag{1.19.26}$$

Now **q** is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \tag{1.19.27}$$

where c is,

$$c = \mathbf{n}^T\mathbf{q} = \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{41}{48} \\ \frac{-3}{4} \end{pmatrix} = \frac{23}{24} \tag{1.19.28}$$

Hence equation of tangent to the curve (1.19.1) parallel to (1.19.2) is given by substituting the value of c and **n** from equation (1.19.28) and

(1.19.20) respectively to the equation (1.19.27),

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{23}{24} \tag{1.19.29}$$

Figure 1.19 verifies that the $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{23}{24}$ is a tangent to parabola $y = \sqrt{3x-2}$

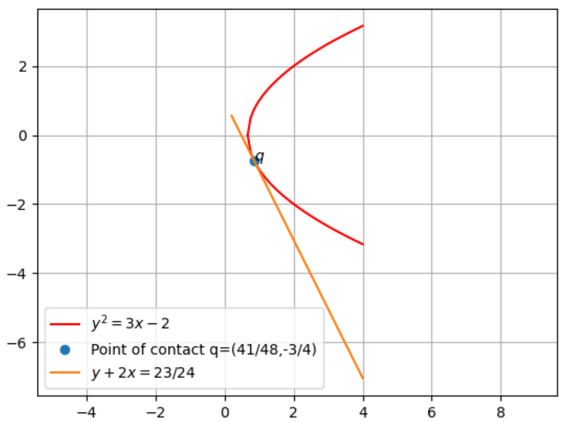

Fig. 1.19: Tangent to parabola $y = \sqrt{3x-2}$

1.20. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.
Solution: Comparing $y^2 = 4x$ to standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.20.1}$$

$\therefore$ a = b = e = 0, d =-2, c = 1, f = 0.

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.20.2}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.20.3}$$

$$\text{Now, } \left|V\right| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \tag{1.20.4}$$

$\implies$ That the curve is a parabola.

$$\text{Since } \mathbf{V}\mathbf{p_1} = 0 \tag{1.20.5}$$

$$\therefore \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.20.6}$$

Since the slope of the line is 1 The direction vector $\mathbf{m}$ is as follows:

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.20.7}$$

$$\text{Since } \mathbf{m}^T\mathbf{n} = 0 \tag{1.20.8}$$

$$\therefore \mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.20.9}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.20.10}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = -2 \tag{1.20.11}$$

By substituting the values ,we get:

$$\begin{pmatrix} -4 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \tag{1.20.12}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (1.20.12) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} -4 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \left(\frac{-R_1}{4}\right)} \begin{pmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.20.13}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.20.14}$$

Threrefore the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

1.21. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.

1.22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$

1.23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

1.24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

1.25. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first

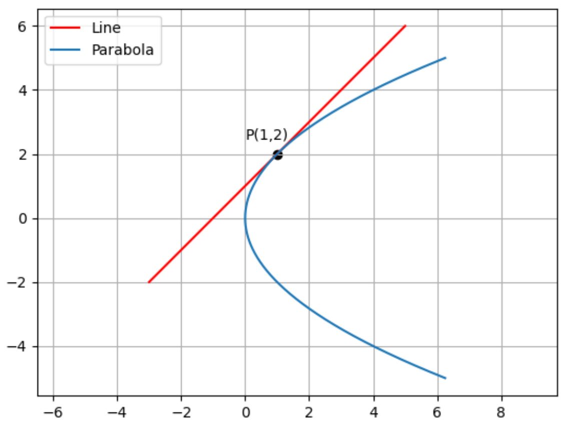

Fig. 1.20: Figure depicting the point at which the line is tangent to the parabola

quadrant.

1.26. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

1.27. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.28. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

1.30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

1.31. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.

1.32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

1.33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.

1.34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.

1.35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.

1.36. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.

1.37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.

1.38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.

1.39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.

1.40. Find the area between $y = x^2$ and $y = x$.

1.41. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

1.42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.

1.43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$

1.44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.

1.45. Find the area bounded by the curves

$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{1.45.1}$$

1.46. Find the area of the region

$$\left\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{1.46.1}$$

1.47. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.

2 QR Decomposition

2.1. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution: Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.1.1}$$

$$\beta = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.1.2}$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \tag{2.1.3}$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{2.1.4}$$

where

$$k_1 = \|\alpha\| \tag{2.1.5}$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\beta}{\|\mathbf{u}_1\|^2} \tag{2.1.7}$$

$$\mathbf{u}_2 = \frac{\beta - r_1\mathbf{u}_1}{\|\beta - r_1\mathbf{u}_1\|} \tag{2.1.8}$$

$$k_2 = \mathbf{u}_2^T\beta \tag{2.1.9}$$

From (2.1.3) and (2.1.4),

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\ 0 & k_2\end{pmatrix} \tag{2.1.10}$$

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \mathbf{QR} \tag{2.1.11}$$

From above we can see that $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.1.12}$$

Now by using equations (2.1.5) to (2.1.9)

$$k_1 = \sqrt{5} \tag{2.1.13}$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}}\begin{pmatrix}1\\2\end{pmatrix}, \tag{2.1.14}$$

$$r_1 = \sqrt{5} \tag{2.1.15}$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}}\begin{pmatrix}-2\\1\end{pmatrix} \tag{2.1.16}$$

$$k_2 = \sqrt{5} \tag{2.1.17}$$

Thus obtained QR decomposition is

$$\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & \sqrt{5}\\ 0 & \sqrt{5}\end{pmatrix} \tag{2.1.18}$$

2.2. $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$

2.3. $\begin{pmatrix}1 & 3\\ 2 & 7\end{pmatrix}$

2.4. $\begin{pmatrix}2 & 3\\ 5 & 7\end{pmatrix}$

2.5. $\begin{pmatrix}2 & 1\\ 7 & 4\end{pmatrix}$

2.6. $\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}$

2.7. $\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}$

2.8. $\begin{pmatrix}4 & 5\\ 3 & 4\end{pmatrix}$

2.9. $\begin{pmatrix}3 & 10\\ 2 & 7\end{pmatrix}$

2.10. $\begin{pmatrix}3 & -1\\ -4 & 2\end{pmatrix}$

2.11. $\begin{pmatrix}2 & -6\\ 1 & -2\end{pmatrix}$

2.12. $\begin{pmatrix}6 & -3\\ -2 & 1\end{pmatrix}$

2.13. $\begin{pmatrix}2 & -3\\ -1 & 2\end{pmatrix}$

2.14. $\begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix}$

2.15. Find QR decomposition of $\begin{pmatrix}2 & 3\\ 3 & -4\end{pmatrix}$

Solution: Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix.

$$\mathbf{a} = \begin{pmatrix}2\\3\end{pmatrix} \tag{2.15.1}$$

$$\mathbf{b} = \begin{pmatrix}3\\-4\end{pmatrix} \tag{2.15.2}$$

The column vectors can be expressed as follows,

$$\mathbf{a} = k_1\mathbf{u}_1 \tag{2.15.3}$$

$$\mathbf{b} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.15.4}$$

Here,

$$k_1 = \|\mathbf{a}\| \tag{2.15.5}$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \tag{2.15.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\mathbf{b}}{\|\mathbf{u}_1\|^2} \tag{2.15.7}$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1\mathbf{u}_1}{\|\mathbf{b} - r_1\mathbf{u}_1\|} \tag{2.15.8}$$

$$k_2 = \mathbf{u}_2^T\mathbf{b} \tag{2.15.9}$$

The (2.15.3) and (2.15.4) can be written as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\ 0 & k_2\end{pmatrix} \tag{2.15.10}$$

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \mathbf{QR} \tag{2.15.11}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.15.12}$$

Now using equations (2.15.5) to (2.15.9) we get,

$$k_1 = \sqrt{2^2+3^2} = \sqrt{13} \tag{2.15.13}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{13}}\begin{pmatrix}2\\3\end{pmatrix} \tag{2.15.14}$$

$$r_1 = \begin{pmatrix}\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix} = -\frac{6}{\sqrt{13}} \tag{2.15.15}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\-2\end{pmatrix} \tag{2.15.16}$$

$$k_2 = \begin{pmatrix}\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix} = \frac{17}{\sqrt{13}} \tag{2.15.17}$$

Thus putting the values from (2.15.13) to (2.15.17) in (2.15.11) we obtain QR decomposition,

$$\begin{pmatrix}2 & 3\\3 & -4\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}\sqrt{13} & -\frac{6}{\sqrt{13}}\\ 0 & \frac{17}{\sqrt{13}}\end{pmatrix} \tag{2.15.18}$$

2.16. Find the QR decomposition of $\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix}$

Solution:
Let $\mathbf{c_1}$ and $\mathbf{c_2}$ be the column vectors of the given matrix.

$$\mathbf{c_1} = \begin{pmatrix}3\\1\end{pmatrix} \tag{2.16.1}$$

$$\mathbf{c_2} = \begin{pmatrix}2\\4\end{pmatrix} \tag{2.16.2}$$

The column vectors can be represented as,

$$\mathbf{c_1} = k_1\mathbf{u}_1 \tag{2.16.3}$$

$$\mathbf{c_2} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.16.4}$$

where,

$$k_1 = \|\mathbf{c_1}\| \tag{2.16.5}$$

$$\mathbf{u_1} = \frac{\mathbf{c_1}}{k_1} \tag{2.16.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\mathbf{c_2}}{\|\mathbf{u}_1\|^2} \tag{2.16.7}$$

$$\mathbf{u_2} = \frac{\mathbf{c_2} - r_1\mathbf{u_1}}{\|\mathbf{c_2} - r_1\mathbf{u_1}\|} \tag{2.16.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{c_2} \tag{2.16.9}$$

From (2.16.3) and (2.16.4),

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{2.16.10}$$

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \mathbf{QR} \tag{2.16.11}$$

Where $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.16.12}$$

Using equations (2.16.5) to (2.16.9) we get,

$$k_1 = \sqrt{3^2+1^2} = \sqrt{10} \tag{2.16.13}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{pmatrix} \tag{2.16.14}$$

$$r_1 = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \sqrt{10} \tag{2.16.15}$$

$$\mathbf{u_2} = \begin{pmatrix}\frac{-1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{pmatrix} \tag{2.16.16}$$

$$k_2 = \begin{pmatrix}\frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \sqrt{10} \tag{2.16.17}$$

Now putting the values from (2.16.13) to (2.16.17), we obtain the QR decomposition of given matrix,

$$\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & \sqrt{10}\\ 0 & \sqrt{10}\end{pmatrix} \tag{2.16.18}$$

2.17. Find QR decomposition of $\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix}$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$\mathbf{A} = \mathbf{QR} \tag{2.17.1}$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix Given

$$\mathbf{A} = \begin{pmatrix}4 & 3\\5 & -2\end{pmatrix} \tag{2.17.2}$$

Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix

$$\mathbf{a} = \begin{pmatrix}4\\5\end{pmatrix} \tag{2.17.3}$$

$$\mathbf{b} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{2.17.4}$$

The above column vectors (2.17.3) ,(2.17.4) can be expressed as ,

$$\mathbf{a} = t_1\mathbf{u}_1 \quad (2.17.5)$$

$$\mathbf{b} = s_1\mathbf{u}_1 + t_2\mathbf{u}_2 \quad (2.17.6)$$

Where,

$$t_1 = \|\mathbf{a}\| \quad (2.17.7)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{t_1} \quad (2.17.8)$$

$$s_1 = \frac{\mathbf{u}_1^T\mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.17.9)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - s_1\mathbf{u}_1}{\|\mathbf{b} - s_1\mathbf{u}_1\|} \quad (2.17.10)$$

$$t_2 = \mathbf{u}_2^T\mathbf{b} \quad (2.17.11)$$

The (2.17.5) and (2.17.6) can be written as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}t_1 & s_1\\ 0 & t_2\end{pmatrix} \quad (2.17.12)$$

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \mathbf{QR} \quad (2.17.13)$$

Here, $\mathbf{R}$ is an upper triangular matrix and $\mathbf{Q}$ is an orthogonal matrix such that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad (2.17.14)$$

Now using equations from (2.17.7) to (2.17.11) we get,

$$t_1 = \sqrt{4^2 + 5^2} = \sqrt{41} \quad (2.17.15)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{41}}\begin{pmatrix}4\\5\end{pmatrix} \quad (2.17.16)$$

$$s_1 = \begin{pmatrix}\frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \frac{2}{\sqrt{41}} \quad (2.17.17)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{41}}\begin{pmatrix}5\\-4\end{pmatrix} \quad (2.17.18)$$

$$t_2 = \begin{pmatrix}\frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \frac{23}{\sqrt{41}} \quad (2.17.19)$$

Substituting the values from (2.17.15) to (2.17.19) in (2.17.13) we obtain QR decomposition as,

$$\begin{pmatrix}4 & 3\\5 & -2\end{pmatrix} = \begin{pmatrix}\frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}}\\ \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}}\end{pmatrix}\begin{pmatrix}\sqrt{41} & \frac{2}{\sqrt{41}}\\ 0 & \frac{23}{\sqrt{41}}\end{pmatrix} \quad (2.17.20)$$

2.18. Perform the QR decomposition of matrix

$$\mathbf{A} = \begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} \quad (2.18.1)$$

Solution:
If α and β are the columns of a (2×2) matrix $\mathbf{A}$,
then $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \mathbf{QR} \quad (2.18.2)$$

$$\text{where, } \mathbf{U} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}, \quad (2.18.3)$$

$$\text{uppertriangular matrix } \mathbf{R} = \begin{pmatrix}k_1 & r_1\\ 0 & k_2\end{pmatrix} \quad (2.18.4)$$

$$k_1 = \|\alpha\|, \mathbf{u_1} = \frac{\alpha}{k_1} \quad (2.18.5)$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} \quad (2.18.6)$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|}, k_2 = \mathbf{u_2}^T\beta \quad (2.18.7)$$

$$\alpha = \begin{pmatrix}1\\3\end{pmatrix}, \beta = \begin{pmatrix}2\\1\end{pmatrix} \quad (2.18.8)$$

$$\text{From, (2.18.5), } k_1 = \|\alpha\| = \sqrt{10} \quad (2.18.9)$$

$$\text{and } \mathbf{u_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \quad (2.18.10)$$

$$\text{From (2.18.6), } r_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.11)$$

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix}2\\1\end{pmatrix} - \frac{5}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \quad (2.18.12)$$

$$= \begin{pmatrix}\frac{3}{2}\\ \frac{-1}{2}\end{pmatrix} \quad (2.18.13)$$

$$\text{From (2.18.7), } \mathbf{u_2} = \frac{\begin{pmatrix}\frac{3}{2}\\ \frac{-1}{2}\end{pmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4}}} \quad (2.18.14)$$

$$\implies \mathbf{u_2} = \begin{pmatrix}\frac{3}{\sqrt{10}}\\ \frac{-1}{\sqrt{10}}\end{pmatrix}, \quad (2.18.15)$$

$$k_2 = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{5}{\sqrt{10}} \quad (2.18.16)$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \quad (2.18.17)$$

The matrix $\mathbf{A}$ can now be rewritten using (2.18.2) as

$$\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & \frac{5}{\sqrt{10}}\\ 0 & \frac{5}{\sqrt{10}}\end{pmatrix} \quad (2.18.18)$$

2.19. Find the QR decomposition of the given matrix.

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \quad (2.19.1)$$

Solution: QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \quad (2.19.2)$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \quad (2.19.3)$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix} \quad \mathbf{b} = \begin{pmatrix}2\\-2\end{pmatrix} \quad (2.19.4)$$

Equation (2.19.3) can be written in form of (2.19.4) as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}u_1 & u_3\\0 & u_2\end{pmatrix} = \mathbf{QR} \quad (2.19.5)$$

where,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (2.19.6)$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix} \quad (2.19.7)$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \frac{-2}{\sqrt{5}} \quad (2.19.8)$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ -\frac{1}{\sqrt{5}}\end{pmatrix} \quad (2.19.9)$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \frac{6}{\sqrt{5}} \quad (2.19.10)$$

Substituting equation (2.19.6) to (2.19.10) in (2.19.5),

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & -\frac{2}{\sqrt{5}}\\ 0 & \frac{6}{\sqrt{5}}\end{pmatrix} \quad (2.19.11)$$

The QR decomposition is,

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & -\frac{2}{\sqrt{5}}\\ 0 & \frac{6}{\sqrt{5}}\end{pmatrix} \quad (2.19.12)$$

2.20. Find the QR decomposition on a given 2×2 matrix.

$$\begin{pmatrix}2 & 1\\1 & -2\end{pmatrix} \quad (2.20.1)$$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \quad (2.20.2)$$

Here $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix}2 & 1\\1 & -2\end{pmatrix} \quad (2.20.3)$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix}2\\1\end{pmatrix} \quad \mathbf{b} = \begin{pmatrix}1\\-2\end{pmatrix} \quad (2.20.4)$$

Equation (2.20.3) can be written in **QR** form as:

$$\mathbf{QR} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}u_1 & u_3\\0 & u_2\end{pmatrix} \quad (2.20.5)$$

Now,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad (2.20.6)$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix} \quad (2.20.7)$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = 0 \quad (2.20.8)$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}}\end{pmatrix} \quad (2.20.9)$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} = \sqrt{5} \quad (2.20.10)$$

Substituting equation (2.20.6) to (2.20.10) in (2.20.5),to obtain the QR Decomposition of the given matrix as:

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \quad (2.20.11)$$

In equation (2.20.11) **R** is diagonal because the columns and rows are orthogonal to each other.

2.21. Perform QR decomposition on matrix **A**

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} \quad (2.21.1)$$

Solution:
The columns of matrix **A** can be represented in α and β as

$$\implies \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.21.2)$$

$$\beta = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad (2.21.3)$$

For QR decomposition, matrix **A** can be expressed as

$$\mathbf{A} = \mathbf{QR} \quad (2.21.4)$$

where, **Q** and **R** are expressed as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \quad (2.21.5)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.21.6)$$

Note that **R** is an upper triangular matrix.
Now,we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \quad (2.21.7)$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.21.8)$$

$$r_1 = \frac{\mathbf{u_1}^T \beta}{\|\mathbf{u_1}\|^2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad (2.21.9)$$

$$\implies r_1 = -\frac{11}{\sqrt{10}} \quad (2.21.10)$$

$$\mathbf{u_2} = \frac{\beta - r_1 \mathbf{u_1}}{\|\beta - r_1 \mathbf{u_1}\|} \quad (2.21.11)$$

Consider

$$\beta - r_1 \mathbf{u_1} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} + \frac{11}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.21.12)$$

$$\implies \beta - r_1 \mathbf{u_1} = \begin{pmatrix} \frac{51}{10} \\ -\frac{17}{10} \end{pmatrix} \quad (2.21.13)$$

$$\|\beta - r_1 \mathbf{u_1}\| = \frac{17}{\sqrt{10}} \quad (2.21.14)$$

Substitute (2.21.13),(2.21.14) in (2.21.11), we get

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix} \quad (2.21.15)$$

$$k_2 = \mathbf{u_2}^T \beta = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad (2.21.16)$$

$$\implies k_2 = \frac{17}{\sqrt{10}} \quad (2.21.17)$$

Therefore, from (2.21.5) and (2.21.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \quad (2.21.18)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \quad (2.21.19)$$

Note that,

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (2.21.20)$$

Now matrix **A** can be written as (2.21.4)

$$\begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \quad (2.21.21)$$

2.22. Perform QR decomposition on matrix **A**

$$\mathbf{A} = \begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} \quad (2.22.1)$$

Solution: The columns of matrix **A** can be represented in α and β as

$$\implies \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (2.22.2)$$

$$\beta = \begin{pmatrix} -7 \\ 1 \end{pmatrix} \quad (2.22.3)$$

For QR decomposition, matrix **A** can be expressed as

$$\mathbf{A} = \mathbf{QR} \quad (2.22.4)$$

where, **Q** and **R** are expressed as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \quad (2.22.5)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.22.6)$$

Note that **R** is an upper triangular matrix.

Now,we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \tag{2.22.7}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \tag{2.22.8}$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}-7\\1\end{pmatrix} \tag{2.22.9}$$

$$\implies r_1 = -\frac{4}{\sqrt{10}} \tag{2.22.10}$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{2.22.11}$$

Consider

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix}-7\\1\end{pmatrix} + \frac{4}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \tag{2.22.12}$$

$$\implies \beta - r_1\mathbf{u_1} = \begin{pmatrix}\frac{-66}{10}\\\frac{22}{10}\end{pmatrix} \tag{2.22.13}$$

$$\|\beta - r_1\mathbf{u_1}\| = \frac{22}{\sqrt{10}} \tag{2.22.14}$$

Substitute (2.22.13),(2.22.14) in (2.22.11), we get

$$\mathbf{u_2} = \begin{pmatrix}-\frac{3}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\end{pmatrix} \tag{2.22.15}$$

$$k_2 = \mathbf{u_2}^T\beta = \begin{pmatrix}-\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}-7\\1\end{pmatrix} \tag{2.22.16}$$

$$\implies k_2 = \frac{22}{\sqrt{10}} \tag{2.22.17}$$

Therefore, from (2.22.5) and (2.22.6)

$$\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix} \tag{2.22.18}$$

$$\mathbf{R} = \begin{pmatrix}\sqrt{10} & -\frac{4}{\sqrt{10}}\\0 & \frac{22}{\sqrt{10}}\end{pmatrix} \tag{2.22.19}$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\-\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \tag{2.22.20}$$

Now matrix $\mathbf{A}$ can be written as (2.22.4)

$$\begin{pmatrix}1 & -7\\3 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & -\frac{4}{\sqrt{10}}\\0 & \frac{22}{\sqrt{10}}\end{pmatrix} \tag{2.22.21}$$

2.23. Given a matrix $\mathbf{A} = \begin{pmatrix}3 & -2\\4 & -2\end{pmatrix}$, find its **QR** decomposition **Solution:**
Given

$$\mathbf{A} = \begin{pmatrix}3 & -2\\4 & -2\end{pmatrix} \tag{2.23.1}$$

Let us use the Gram-Schmidt approach to obtain QR decomposition of $\mathbf{A}$. Consider column vectors say $\mathbf{a}_1$ and $\mathbf{a}_2$ of $\mathbf{A}$ which is given by

$$\mathbf{a}_1 = \begin{pmatrix}3\\4\end{pmatrix} \tag{2.23.2}$$

$$\mathbf{a}_2 = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{2.23.3}$$

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{pmatrix}3\\4\end{pmatrix} \tag{2.23.4}$$

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix} \tag{2.23.5}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - \left(\mathbf{a}_2^T.\mathbf{e}_1\right)\mathbf{e}_1 \tag{2.23.6}$$

$$= \begin{pmatrix}-2\\-2\end{pmatrix} - \left(-\frac{14}{5}\right)\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix} \tag{2.23.7}$$

$$= \begin{pmatrix}-2\\-2\end{pmatrix} - \begin{pmatrix}-\frac{42}{25}\\-\frac{56}{25}\end{pmatrix} = \begin{pmatrix}-\frac{8}{25}\\\frac{6}{25}\end{pmatrix} \tag{2.23.8}$$

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{pmatrix}-\frac{4}{5}\\\frac{3}{5}\end{pmatrix} \tag{2.23.9}$$

The matrix $\mathbf{Q}$ and $\mathbf{R}$ is given by,

$$\mathbf{Q} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\\frac{4}{5} & \frac{3}{5}\end{pmatrix} \tag{2.23.10}$$

$$\mathbf{R} = \begin{pmatrix}\mathbf{a}_1^T.\mathbf{e}_1 & \mathbf{a}_2^T.\mathbf{e}_1\\0 & \mathbf{a}_2^T.\mathbf{e}_2\end{pmatrix} = \begin{pmatrix}5 & -\frac{14}{5}\\0 & \frac{2}{5}\end{pmatrix} \tag{2.23.11}$$

Hence, the **QR** decomposition of matrix $\mathbf{A}$ is as follows:

$$\begin{pmatrix}3 & -2\\4 & -2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}5 & -\frac{14}{5}\\0 & \frac{2}{5}\end{pmatrix} \tag{2.23.12}$$

2.24. Perform the QR decomposition of the matrix $\mathbf{A} = \begin{pmatrix}3 & -1\\-4 & 2\end{pmatrix}$. **Solution:**
Let $\mathbf{a}$ and $\mathbf{b}$ are the columns of matrix $\mathbf{A}$. The matrix $\mathbf{A}$ can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \tag{2.24.1}$$

$$\mathbf{U} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \tag{2.24.2}$$

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.24.3}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.24.4}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.24.5}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|u_1\|^2} \tag{2.24.6}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.24.7}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{2.24.8}$$

Then the given matrix can be represented as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.24.9}$$

The the columns of matrix $\mathbf{A} = \begin{pmatrix}3 & -1 \\ -4 & 2\end{pmatrix}$ are $\mathbf{a}$ and $\mathbf{b}$ where

$$\mathbf{a} = \begin{pmatrix}3 \\ -4\end{pmatrix} \tag{2.24.10}$$

$$\mathbf{b} = \begin{pmatrix}-1 \\ 2\end{pmatrix} \tag{2.24.11}$$

Now for the given matrix, From (2.24.4) and (2.24.5)

$$k_1 = \|\mathbf{a}\| = 5 \tag{2.24.12}$$

$$\mathbf{u_1} = \frac{1}{5}\begin{pmatrix}3 \\ -4\end{pmatrix} \tag{2.24.13}$$

From (2.24.6)

$$r_1 = \frac{1}{5}\begin{pmatrix}3 & -4\end{pmatrix}\begin{pmatrix}-1 \\ 2\end{pmatrix} = \frac{-11}{5} \tag{2.24.14}$$

From (2.24.7)

$$\mathbf{b} - r_1\mathbf{u_1} = \begin{pmatrix}-1 \\ 2\end{pmatrix} - \frac{-11}{5}\begin{pmatrix}\frac{3}{5} \\ \frac{-4}{5}\end{pmatrix} \tag{2.24.15}$$

$$\|\mathbf{b} - r_1\mathbf{u_1}\| = \frac{2}{5} \tag{2.24.16}$$

$$\implies \mathbf{u_2} = \frac{5}{2}\begin{pmatrix}\frac{8}{25} \\ \frac{6}{25}\end{pmatrix} \tag{2.24.17}$$

From (2.24.8)

$$k_2 = \mathbf{u_2}^T\mathbf{b} = \begin{pmatrix}\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}-1 \\ 2\end{pmatrix} = \frac{2}{5} \tag{2.24.18}$$

Now we can observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = I$

$$\begin{pmatrix}\frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}\frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5}\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} \tag{2.24.19}$$

From (2.24.9), The matrix $\mathbf{A}$ can now be written as,

$$\mathbf{A} = \begin{pmatrix}3 & -1 \\ -4 & 2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}-5 & \frac{-11}{5} \\ 0 & \frac{2}{5}\end{pmatrix} \tag{2.24.20}$$

2.25. Perform QR decomposition on matrix A given by

$$A = \begin{pmatrix}3 & -4 \\ -4 & 3\end{pmatrix}$$

Solution: Representing matrix A in terms of its column vectors as

$$A = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} \tag{2.25.1}$$

Let

$$\mathbf{q_1} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{2.25.2}$$

An orthonormal vector to $\mathbf{q_1}$ can be obtained by subtracting the projection of $\mathbf{b}$ on $\mathbf{q_1}$ from $\mathbf{b}$. Thus

$$\mathbf{q_2} = \frac{\mathbf{b} - k\mathbf{q_1}}{\|\mathbf{b} - k\mathbf{q_1}\|} \tag{2.25.3}$$

where

$$k = \frac{\mathbf{b}^T\mathbf{q_1}}{\|\mathbf{q_1}\|^2} \tag{2.25.4}$$

From (2.25.2) and (2.25.3)

$$\mathbf{a} = \|\mathbf{a}\|\,\mathbf{q_1} \tag{2.25.5}$$

$$\mathbf{b} = k\mathbf{q_1} + \|\mathbf{b} - k\mathbf{q_1}\|\,\mathbf{q_2} \tag{2.25.6}$$

$$\implies \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}\|\mathbf{a}\| & k \\ 0 & \|\mathbf{b} - k\mathbf{q_1}\|\end{pmatrix} \tag{2.25.7}$$

$$\implies \mathbf{A} = \mathbf{QR} \tag{2.25.8}$$

QR decomposition of a matrix A is essentially representation of column vectors of matrix A in terms of linear combination of orthonormal basis of column space of A. For matrix A

$$\mathbf{a} = \begin{pmatrix}3 \\ -4\end{pmatrix}, \ \mathbf{b} = \begin{pmatrix}-4 \\ 3\end{pmatrix} \tag{2.25.9}$$

$$\tag{2.25.10}$$

Let

$$\mathbf{q_1} = \frac{1}{\sqrt{5}}\begin{pmatrix}3\\-4\end{pmatrix} \tag{2.25.11}$$

$$\tag{2.25.12}$$

From (2.25.3) and (2.25.4)

$$\mathbf{q_2} = \frac{1}{\sqrt{5}}\begin{pmatrix}-4\\3\end{pmatrix} \tag{2.25.13}$$

$$\implies \mathbf{Q} = \frac{1}{\sqrt{5}}\begin{pmatrix}3 & -4\\-4 & 3\end{pmatrix} \tag{2.25.14}$$

$$\mathbf{R} = \begin{pmatrix}\sqrt{5} & 0\\0 & \sqrt{5}\end{pmatrix} \tag{2.25.15}$$

Therefore the matrix A can be decomposed as

$$\mathbf{A} = \begin{pmatrix}\frac{3}{\sqrt{5}} & -\frac{4}{\sqrt{5}}\\-\frac{4}{\sqrt{5}} & \frac{3}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & 0\\0 & \sqrt{5}\end{pmatrix} \tag{2.25.16}$$

2.26. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix}2 & -6\\1 & -2\end{pmatrix} \tag{2.26.1}$$

Solution: Let c_1 and c_2 be the column vectors of given matrix **A**

$$c_1 = \begin{pmatrix}2\\1\end{pmatrix} \tag{2.26.2}$$

$$c_2 = \begin{pmatrix}-6\\-2\end{pmatrix} \tag{2.26.3}$$

We can express the matrix **A** as,

$$\mathbf{A} = \mathbf{QR} \tag{2.26.4}$$

Where, **Q** is an orthogonal matrix given as,

$$\mathbf{Q} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \tag{2.26.5}$$

and **R** is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{2.26.6}$$

Now, we can express α and β as,

$$c_1 = k_1\mathbf{u_1} \tag{2.26.7}$$

$$c_2 = r_1\mathbf{u_1} + k_2\mathbf{u_2} \tag{2.26.8}$$

$$\text{where, } k_1 = \|c_1\| = \sqrt{2^2 + (1^2} = \sqrt{5} \tag{2.26.9}$$

Solving equation (2.26.7) for $\mathbf{u_1}$,

$$\mathbf{u_1} = \frac{c_1}{k_1} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix} \tag{2.26.10}$$

$$\text{Now, } r_1 = \frac{\mathbf{u_1}^T c_2}{\|\mathbf{u_1}\|^2} \tag{2.26.11}$$

$$\implies \frac{\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}-6\\-2\end{pmatrix}}{1} \tag{2.26.12}$$

$$\text{Hence, } r_1 = -\frac{-14}{\sqrt{5}} \tag{2.26.13}$$

$$\mathbf{u_2} = \frac{c_2 - r_1\mathbf{u_1}}{\|c_2 - r_1\mathbf{u_1}\|} \tag{2.26.14}$$

$$\implies \frac{\begin{pmatrix}-6\\-2\end{pmatrix} - \left(\frac{-14}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}\right)}{\left\|\begin{pmatrix}-6\\-2\end{pmatrix} - \left(-\frac{14}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}\right)\right\|} \tag{2.26.15}$$

$$\implies \mathbf{u_2} = \frac{1}{\sqrt{5}}\begin{pmatrix}-1\\2\end{pmatrix} \tag{2.26.16}$$

$$\text{Now, } k_2 = u_2^T c_2 \tag{2.26.17}$$

$$\implies \frac{1}{\sqrt{5}}\begin{pmatrix}-1 & 2\end{pmatrix}\begin{pmatrix}-6\\-2\end{pmatrix} \tag{2.26.18}$$

$$\implies k_2 = \frac{2}{\sqrt{5}} \tag{2.26.19}$$

Hence substituting the values of unknown parameter from equations (2.26.9), (2.26.19), (2.26.10), (2.26.16) and (2.26.13) to equation (2.26.5) and (2.26.6) we get,

$$\mathbf{Q} = \frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix} \tag{2.26.20}$$

$$\mathbf{R} = \begin{pmatrix}\sqrt{5} & \frac{-14}{\sqrt{5}}\\0 & \frac{2}{\sqrt{5}}\end{pmatrix} \tag{2.26.21}$$

2.27. Find the QR decompostion of

$$\mathbf{A} = \begin{pmatrix}4 & 7\\3 & 5\end{pmatrix} \tag{2.27.1}$$

Solution: If $\mathbf{A} \in \mathbf{R}^{m\times n}$ has linearly independent columns then it can be factored as

$$\mathbf{A} = \mathbf{QR}$$

where **Q** is a orthogonal matrix and **R** is a upper triangular matrix with non zero diagonal elements

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \quad (2.27.2)$$

The column vectors of **A** are,

$$\mathbf{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (2.27.3)$$

(2.27.2) can be written as,

$$\mathbf{QR} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \quad (2.27.4)$$

Now,

$$u_1 = \|a\| = \sqrt{4^2 + 7^2} = \sqrt{65} \quad (2.27.5)$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{4}{\sqrt{65}} \\ \frac{7}{\sqrt{65}} \end{pmatrix} \quad (2.27.6)$$

$$u_3 = \frac{\mathbf{q_1}^T \mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix} \frac{4}{\sqrt{65}} & \frac{7}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{47}{\sqrt{65}} \quad (2.27.7)$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3 \mathbf{q_1}}{\|\mathbf{b} - u_3 \mathbf{q_1}\|} = \begin{pmatrix} \frac{7}{\sqrt{65}} \\ -\frac{4}{\sqrt{65}} \end{pmatrix} \quad (2.27.8)$$

$$u_2 = \mathbf{q_2}^T \mathbf{b} = \begin{pmatrix} \frac{7}{\sqrt{65}} & -\frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{65}} \quad (2.27.9)$$

Substituting (2.27.5) to (2.27.9) in (2.27.4),

$$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{65}} & \frac{7}{\sqrt{65}} \\ \frac{7}{\sqrt{65}} & -\frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} \sqrt{65} & \frac{47}{\sqrt{65}} \\ 0 & \frac{1}{\sqrt{65}} \end{pmatrix} \quad (2.27.10)$$

Which can also be written as,

$$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{4}{\sqrt{65}} & -\frac{7}{\sqrt{65}} \\ -\frac{7}{\sqrt{65}} & \frac{4}{\sqrt{65}} \end{pmatrix} \begin{pmatrix} -\sqrt{65} & -\frac{47}{\sqrt{65}} \\ 0 & -\frac{1}{\sqrt{65}} \end{pmatrix} \quad (2.27.11)$$

2.28. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 6 & -2 \end{pmatrix} \quad (2.28.1)$$

Solution: Let c_1 and c_2 be the column vectors of given matrix **A**

$$c_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad (2.28.2)$$

$$c_2 = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad (2.28.3)$$

We can express the matrix **A** as,

$$\mathbf{A} = \mathbf{QR} \quad (2.28.4)$$

Where, **Q** is an orthogonal matrix given as,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \quad (2.28.5)$$

and **R** is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.28.6)$$

Now, we can express α and β as,

$$c_1 = k_1 \mathbf{u_1} \quad (2.28.7)$$

$$c_2 = r_1 \mathbf{u_1} + k_2 \mathbf{u_2} \quad (2.28.8)$$

$$\text{where, } k_1 = \|c_1\| = \sqrt{4^2 + (6^2} = \sqrt{52} \quad (2.28.9)$$

Solving equation (2.28.7) for $\mathbf{u_1}$,

$$\mathbf{u_1} = \frac{c_1}{k_1} = \frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad (2.28.10)$$

$$\text{Now, } r_1 = \frac{\mathbf{u_1}^T c_2}{\|\mathbf{u_1}\|^2} \quad (2.28.11)$$

$$\implies \frac{\frac{1}{\sqrt{52}} \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix}}{1} \quad (2.28.12)$$

$$\text{Hence, } r_1 = -\frac{-24}{\sqrt{52}} \quad (2.28.13)$$

$$\mathbf{u_2} = \frac{c_2 - r_1 \mathbf{u_1}}{\|c_2 - r_1 \mathbf{u_1}\|} \quad (2.28.14)$$

$$\implies \frac{\begin{pmatrix} -3 \\ -2 \end{pmatrix} - \left(\frac{-24}{\sqrt{52}}\right) \left(\frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix}\right)}{\left\| \begin{pmatrix} -3 \\ -2 \end{pmatrix} - \left(-\frac{24}{\sqrt{52}} \frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix}\right) \right\|} \quad (2.28.15)$$

$$\implies \mathbf{u_2} = \frac{1}{\sqrt{335}} \begin{pmatrix} -15 \\ 10 \end{pmatrix} \quad (2.28.16)$$

$$\text{Now, } k_2 = u_2^T c_2 \quad (2.28.17)$$

$$\implies \frac{1}{\sqrt{335}} \begin{pmatrix} -15 & 10 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad (2.28.18)$$

$$\implies k_2 = \frac{25}{\sqrt{335}} \quad (2.28.19)$$

Hence substituting the values of unknown parameter from equations (2.28.9), (2.28.19), (2.28.10), (2.28.16) and (2.28.13) to equation (2.28.5) and (2.28.6) we get,

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{\sqrt{52}} & \frac{-15}{\sqrt{335}} \\ \frac{6}{\sqrt{52}} & \frac{10}{\sqrt{335}} \end{pmatrix} \tag{2.28.20}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{52} & \frac{-24}{\sqrt{52}} \\ 0 & \frac{25}{\sqrt{335}} \end{pmatrix} \tag{2.28.21}$$

2.29. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} \tag{2.29.1}$$

2.30. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \tag{2.30.1}$$

Solution:
The matrix $\mathbf{A}$ can be written as,

$$\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \tag{2.30.2}$$

where $\mathbf{a}$ and $\mathbf{b}$ are column vectors. From (2.29.1)

$$\mathbf{a} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \tag{2.30.3}$$

$$\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.30.4}$$

The QR decomposition of the given matrix is given by

$$\mathbf{A} = \mathbf{QR} \tag{2.30.5}$$

here $\mathbf{R}$ is a upper triangular matrix and

$$\mathbf{Q^T Q} = \mathbf{I} \tag{2.30.6}$$

where

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \tag{2.30.7}$$

The above values are given by,

$$r_1 = \|\mathbf{a}\| \tag{2.30.8}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{r_1} \tag{2.30.9}$$

$$r_2 = \frac{\mathbf{q_1^T b}}{\|q_1\|^2} \tag{2.30.10}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - r_2\mathbf{q_1}}{\|\mathbf{b} - r_2\mathbf{q_1}\|} \tag{2.30.11}$$

$$r_3 = \mathbf{q_2^T b} \tag{2.30.12}$$

Subtituting (2.30.3) and (2.30.4) we get

$$r_1 = \sqrt{8^2 + 3^2} = \sqrt{73} \tag{2.30.13}$$

$$\mathbf{q_1} = \frac{1}{\sqrt{73}}\begin{pmatrix} 8 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{73}} \\ \frac{3}{\sqrt{73}} \end{pmatrix} \tag{2.30.14}$$

$$r_2 = \frac{1}{\left(\sqrt{\frac{64}{73} + \frac{9}{73}}\right)^2}\begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{3}{\sqrt{73}} \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{46}{\sqrt{73}} \tag{2.30.15}$$

$$\mathbf{q_2} = \frac{1}{\sqrt{73}}\left(\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \frac{46}{\sqrt{73}}\begin{pmatrix} \frac{8}{\sqrt{73}} \\ \frac{3}{\sqrt{73}} \end{pmatrix}\right) = \begin{pmatrix} \frac{-3}{73\sqrt{73}} \\ \frac{8}{73\sqrt{73}} \end{pmatrix} \tag{2.30.16}$$

$$r_3 = \begin{pmatrix} \frac{-3}{73\sqrt{73}} & \frac{8}{73\sqrt{73}} \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{73\sqrt{73}} \tag{2.30.17}$$

Hence subtituting these values in (2.30.7) and then back in (2.30.5) we get,

$$\mathbf{A} = \begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{-3}{73\sqrt{73}} \\ \frac{3}{\sqrt{73}} & \frac{8}{73\sqrt{73}} \end{pmatrix}\begin{pmatrix} \sqrt{73} & \frac{46}{\sqrt{73}} \\ 0 & \frac{1}{73\sqrt{73}} \end{pmatrix} \tag{2.30.18}$$

Hence QR decomposition is,

$$\begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{8}{\sqrt{73}} & \frac{-3}{73\sqrt{73}} \\ \frac{3}{\sqrt{73}} & \frac{8}{73\sqrt{73}} \end{pmatrix}\begin{pmatrix} \sqrt{73} & \frac{46}{\sqrt{73}} \\ 0 & \frac{1}{73\sqrt{73}} \end{pmatrix} \tag{2.30.19}$$

2.31. Perform QR decomposition on the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \tag{2.31.1}$$

Solution: The columns of the matrix $\mathbf{A}$ can be represented as:

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.31.2}$$

$$\beta = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.31.3}$$

For QR decomposition, matrix A is represented in the form:

$$\mathbf{A} = \mathbf{QR} \tag{2.31.4}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{2.31.5}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.31.6}$$

Here $\mathbf{R}$ is a upper triangular matrix and Q is a orthogonal matrix such that,

$$\mathbf{Q^T Q} = \mathbf{I} \tag{2.31.7}$$

Now we calculate the above values,

$$k_1 = \|\alpha\| \tag{2.31.8}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} \tag{2.31.9}$$

$$r_1 = \frac{\mathbf{u_1^T}\beta}{\|\mathbf{u_1}\|^2} \tag{2.31.10}$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{2.31.11}$$

$$k_2 = \mathbf{u_2^T}\beta \tag{2.31.12}$$

Substituting (2.31.2) and (2.31.3) in the above equations, we get

$$k_1 = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{2.31.13}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.31.14}$$

$$r_1 = \frac{1}{\left(\sqrt{\frac{1}{5} + \frac{4}{5}}\right)^2}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.31.15}$$

$$\implies r_1 = \frac{11}{\sqrt{5}} \tag{2.31.16}$$

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{11}{5} \\ \frac{22}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{-2}{5} \end{pmatrix} \tag{2.31.17}$$

$$\mathbf{u_2} = \frac{\begin{pmatrix} \frac{4}{5} \\ \frac{-2}{5} \end{pmatrix}}{\sqrt{\frac{4}{5}^2 + \frac{-2}{5}^2}} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{2.31.18}$$

$$k_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{5}} \tag{2.31.19}$$

Therefore, from (2.31.5) and (2.31.6) we get,

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{2.31.20}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.31.21}$$

where

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{2.31.22}$$

Therefore matrix $\mathbf{A}$ in QR decomposed form is,

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.31.23}$$

2.32. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \tag{2.32.1}$$

Solution: Let α and β be the column vectors of given matrix $\mathbf{A}$

$$\alpha = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{2.32.2}$$

$$\beta = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.32.3}$$

We can express these as,

$$\alpha = k_1\mathbf{u_1} \tag{2.32.4}$$

$$\beta = r_1\mathbf{u_1} + k_2\mathbf{u_2} \tag{2.32.5}$$

Where,

$$k_1 = \|\alpha\| \tag{2.32.6}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} \tag{2.32.7}$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} \tag{2.32.8}$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{2.32.9}$$

$$k_2 = \mathbf{u_2}^T\beta \tag{2.32.10}$$

From (2.32.4) and (2.32.5)

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.32.11}$$

$$\mathbf{A} = \mathbf{QR} \tag{2.32.12}$$

From the above equation we can see that $\mathbf{R}$ is an upper triangular matrix and $\mathbf{Q}$ is an orthogonal matrix

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.32.13}$$

Now by using equations (2.32.2) to (2.32.10)

$$k_1 = \sqrt{9+16} = 5 \tag{2.32.14}$$

$$\mathbf{u_1} = \frac{1}{5}\begin{pmatrix}3 \\ -4\end{pmatrix} = \begin{pmatrix}\frac{3}{5} \\ \frac{-4}{5}\end{pmatrix} \tag{2.32.15}$$

$$r_1 = \frac{\begin{pmatrix}\frac{3}{5} & \frac{-4}{5}\end{pmatrix}\begin{pmatrix}-1 \\ 2\end{pmatrix}}{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{-11}{5} \tag{2.32.16}$$

$$\mathbf{u_2} = \frac{\begin{pmatrix}-1 \\ 2\end{pmatrix} - \frac{-11}{5}\begin{pmatrix}\frac{3}{5} \\ \frac{-4}{5}\end{pmatrix}}{\left\|\begin{pmatrix}-1 \\ 2\end{pmatrix} - \frac{-11}{5}\begin{pmatrix}\frac{3}{5} \\ \frac{-4}{5}\end{pmatrix}\right\|} = \begin{pmatrix}\frac{4}{5} \\ \frac{3}{5}\end{pmatrix} \tag{2.32.17}$$

$$k_2 = \begin{pmatrix}\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}-1 \\ 2\end{pmatrix} = \frac{2}{5} \tag{2.32.18}$$

From equations (2.32.11) and (2.32.12) the obtained $\mathbf{QR}$ decomposition is

$$\begin{pmatrix}3 & -1 \\ -4 & 2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}5 & \frac{-11}{5} \\ 0 & \frac{2}{5}\end{pmatrix} \tag{2.32.19}$$

2.33. Perform QR decomposition of matrix $\begin{pmatrix}6 & 1 \\ -8 & 2\end{pmatrix}$

Solution: Let $\mathbf{a}$ and $\mathbf{b}$ are the columns of matrix $\mathbf{A}$. The matrix $\mathbf{A}$ can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \tag{2.33.1}$$

$$\mathbf{U} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \tag{2.33.2}$$

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.33.3}$$

where

$$k_1 = \|\mathbf{a}\| \tag{2.33.4}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{2.33.5}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|u_1\|^2} \tag{2.33.6}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{2.33.7}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{2.33.8}$$

The given matrix can be represented as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{2.33.9}$$

The columns of matrix $\mathbf{A} = \begin{pmatrix}6 & 1 \\ -8 & 2\end{pmatrix}$ are $\mathbf{a}$ and $\mathbf{b}$ where

$$\mathbf{a} = \begin{pmatrix}6 \\ -8\end{pmatrix} \tag{2.33.10}$$

$$\mathbf{b} = \begin{pmatrix}1 \\ 2\end{pmatrix} \tag{2.33.11}$$

Now for the given matrix, From (2.33.4) and (2.33.5)

$$k_1 = \|\mathbf{a}\| = 10 \tag{2.33.12}$$

$$\mathbf{u_1} = \frac{1}{10}\begin{pmatrix}6 \\ -8\end{pmatrix} \tag{2.33.13}$$

From (2.33.6)

$$r_1 = \frac{1}{10}\begin{pmatrix}6 & -8\end{pmatrix}\begin{pmatrix}1 \\ 2\end{pmatrix} = -1 \tag{2.33.14}$$

From (2.33.7)

$$\mathbf{b} - r_1\mathbf{u_1} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}\frac{6}{10}\\\frac{-8}{10}\end{pmatrix} = \begin{pmatrix}\frac{16}{10}\\\frac{12}{10}\end{pmatrix} \tag{2.33.15}$$

$$\|\mathbf{b} - r_1\mathbf{u_1}\| = \frac{20}{10} = 2 \tag{2.33.16}$$

$$\implies \mathbf{u_2} = \frac{1}{10}\begin{pmatrix}8\\6\end{pmatrix} \tag{2.33.17}$$

From (2.33.8)

$$k_2 = \mathbf{u_2}^T\mathbf{b} = \begin{pmatrix}\frac{8}{10} & \frac{6}{10}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \frac{20}{10} = 2 \tag{2.33.18}$$

Now we can observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = I$

$$\begin{pmatrix}\frac{6}{10} & \frac{8}{10}\\\frac{-8}{10} & \frac{6}{10}\end{pmatrix}\begin{pmatrix}\frac{6}{10} & \frac{-8}{10}\\\frac{8}{10} & \frac{6}{10}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.33.19}$$

From (2.33.9), The matrix $\mathbf{A}$ can now be written as,

$$\mathbf{A} = \begin{pmatrix}6 & 1\\-8 & 2\end{pmatrix} = \begin{pmatrix}\frac{6}{10} & \frac{8}{10}\\\frac{-8}{10} & \frac{6}{10}\end{pmatrix}\begin{pmatrix}10 & -1\\0 & 2\end{pmatrix} \tag{2.33.20}$$

2.34. Find QR decomposition of $\begin{pmatrix}3 & 1\\-4 & 1\end{pmatrix}$ **Solution:** Let α and β be transpose of column vectors of the given matrix.

$$\alpha = \begin{pmatrix}3\\-4\end{pmatrix} \tag{2.34.1}$$

$$\beta = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.34.2}$$

We can express these as

$$\alpha = k_1\mathbf{u}_1 \tag{2.34.3}$$

$$\beta = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.34.4}$$

where

$$k_1 = \|\alpha\| \tag{2.34.5}$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \tag{2.34.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\beta}{\|\mathbf{u}_1\|^2} \tag{2.34.7}$$

$$\mathbf{u}_2 = \frac{\beta - r_1\mathbf{u}_1}{\|\beta - r_1\mathbf{u}_1\|} \tag{2.34.8}$$

$$k_2 = \mathbf{u}_2^T\beta \tag{2.34.9}$$

From (2.34.3) and (2.34.4),

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{2.34.10}$$

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \mathbf{QR} \tag{2.34.11}$$

From above we can see that $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.34.12}$$

Now by using equations (2.34.5) to (2.34.9)

$$k_1 = 5 \tag{2.34.13}$$

$$\mathbf{u}_1 = \frac{1}{5}\begin{pmatrix}3\\-4\end{pmatrix}, \tag{2.34.14}$$

$$r_1 = \frac{-1}{5} \tag{2.34.15}$$

$$\mathbf{u}_2 = \frac{5}{7}\begin{pmatrix}\frac{28}{25}\\\frac{21}{25}\end{pmatrix} \tag{2.34.16}$$

$$k_2 = \frac{7}{5} \tag{2.34.17}$$

Thus obtained QR decomposition is

$$\begin{pmatrix}3 & 1\\-4 & 1\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & \frac{4}{5}\\-\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}5 & -\frac{1}{5}\\0 & \frac{7}{5}\end{pmatrix} \tag{2.34.18}$$

3 Singular Value Decomposition

3.1. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{3.1.1}$$

$$\mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{3.1.2}$$

Solution:
The lines will intersect if

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{3.1.3}$$

$$\begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{3.1.4}$$

$$\mathbf{Mx} = \mathbf{b} \tag{3.1.5}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{USV}^T \tag{3.1.6}$$

Where the columns of **V** are the eigenvectors of $\mathbf{A}^T\mathbf{A}$,the columns of **U** are the eigenvectors of $\mathbf{A}\mathbf{A}^T$ and **S** is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{3.1.7}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{3.1.8}$$

Calculating eigen value of $\mathbf{M}^T\mathbf{M}$.

$$\begin{vmatrix} 6-\lambda & 13 \\ 13 & 38-\lambda \end{vmatrix} \lambda^2 - 44\lambda + 59 = 0 \tag{3.1.9}$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22 \tag{3.1.10}$$

Eigen vectors of $\mathbf{M}\mathbf{M}^T$.

$$\begin{vmatrix} 13-\lambda & -17 & 8 \\ 17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \tag{3.1.11}$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \tag{3.1.12}$$

Hence,The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{5} \\ \frac{-3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix} \tag{3.1.13}$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12-\sqrt{17}}{5}\right)^2 + \left(\frac{1-3\sqrt{17}}{5}\right)^2 + 1^2} \tag{3.1.14}$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} \end{pmatrix} \tag{3.1.15}$$

$$\tag{3.1.16}$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \tag{3.1.17}$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+30\sqrt{7}}} \begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \tag{3.1.18}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} \\ \frac{5}{\sqrt{340+30\sqrt{17}}} \end{pmatrix} \tag{3.1.19}$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \tag{3.1.20}$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}} \begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \end{pmatrix} \tag{3.1.21}$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.22}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.23}$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{3.1.24}$$

Now, $\mathbf{V} = \mathbf{M}^T \frac{\mathbf{u_i}}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.25}$$

So,from equation (3.1.6)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} = \tag{3.1.26}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{3.1.27}$$

$$\begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{3.1.28}$$

$$\begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}^T \tag{3.1.29}$$

Now, Finding Moore-Penrose Pseudo inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \tag{3.1.30}$$

We,know that, $\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+0\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix} \tag{3.1.31}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.32}$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.33}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{3.1.34}$$

$$\mathbf{x} = \begin{pmatrix} \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \\ \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \end{pmatrix} \tag{3.1.35}$$

Simplifying the values of x_1 and x_2

$$x_2 = \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \tag{3.1.36}$$

$$= \frac{-702100}{591700} \tag{3.1.37}$$

$$= -\frac{7}{59} \tag{3.1.38}$$

$$x_1 = \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \tag{3.1.39}$$

$$= \frac{2507500}{591700} \tag{3.1.40}$$

$$= \frac{25}{59} \tag{3.1.41}$$

Now, Verifying the values using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.1.42}$$

Solving R.H.S

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.1.43}$$

Now using equation (3.1.7) in (3.1.43)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.1.44}$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xleftrightarrow{R_2 - \frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \tag{3.1.45}$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \tag{3.1.46}$$

$$6x_1 + 13x_2 = 1 \tag{3.1.47}$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \tag{3.1.48}$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{3.1.49}$$

3.2. Find the distance of the point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane $\begin{pmatrix} 6 & -3 & 2 \end{pmatrix}\mathbf{x} = 4$

Solution:

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{3.2.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \tag{3.2.2}$$

$$\implies 6a - 3b + 2c = 0 \tag{3.2.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \tag{3.2.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{3.2.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{3.2.6}$$

Putting values in (3.2.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & \frac{3}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \tag{3.2.7}$$

Now, to solve (3.2.7), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{3.2.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix} \tag{3.2.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4} \end{pmatrix} \tag{3.2.10}$$

From (3.2.6) putting (3.2.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.2.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{3.2.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{3.2.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4}-\lambda \end{pmatrix} = 0 \tag{3.2.14}$$

$$\implies \lambda^3 - \frac{53}{4}\lambda^2 + \frac{49}{4}\lambda = 0 \tag{3.2.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{4} \tag{3.2.16}$$

$$\lambda_2 = 1 \tag{3.2.17}$$

$$\lambda_3 = 0 \tag{3.2.18}$$

Hence the eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{15} \\ \frac{2}{15} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -3 \\ -\frac{3}{2} \\ 1 \end{pmatrix} \tag{3.2.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{7\sqrt{5}} \\ \frac{2}{7\sqrt{5}} \\ \frac{3\sqrt{5}}{7} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{6}{7} \\ -\frac{3}{7} \\ \frac{2}{7} \end{pmatrix} \quad (3.2.20)$$

Hence we obtain $\mathbf{U}$ of (3.2.8) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \quad (3.2.21)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (3.2.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.2.22)$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (3.2.23)$$

$$\implies \begin{pmatrix} 10 - \lambda & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} - \lambda \end{pmatrix} = 0 \quad (3.2.24)$$

$$\implies \lambda^2 - \frac{53}{4}\lambda + \frac{49}{4} = 0 \quad (3.2.25)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{49}{4} \quad (3.2.26)$$

$$\lambda_5 = 1 \quad (3.2.27)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \quad (3.2.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (3.2.29)$$

Hence we obtain $\mathbf{V}$ of (3.2.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (3.2.30)$$

Finally from (3.2.8) we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (3.2.31)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (3.2.32)$$

From (3.2.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{27}{7\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \\ -\frac{33}{7} \end{pmatrix} \quad (3.2.33)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{54}{49\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \end{pmatrix} \quad (3.2.34)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \quad (3.2.35)$$

Verifying the solution of (3.2.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (3.2.36)$$

Evaluating the R.H.S in (3.2.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \quad (3.2.37)$$

$$\implies \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \quad (3.2.38)$$

Solving the augmented matrix of (3.2.38) we get,

$$\begin{pmatrix} 10 & \frac{9}{2} & -7 \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{10}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \quad (3.2.39)$$

$$\xleftrightarrow{R_2 = R_2 - \frac{9}{2}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & \frac{49}{40} & \frac{73}{20} \end{pmatrix} \quad (3.2.40)$$

$$\xleftrightarrow{R_2 = \frac{40}{49}R_2} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \quad (3.2.41)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{9}{20}R_1} \begin{pmatrix} 1 & 0 & -\frac{100}{49} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \quad (3.2.42)$$

Hence, Solution of (3.2.36) is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \quad (3.2.43)$$

Comparing results of $\mathbf{x}$ from (3.2.35) and (3.2.43) we conclude that the solution is verified.

3.3. Check whether the given line equations intersect. If they do not intersect find the closest points on the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.3.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.2}$$

Solution:
Given

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.3.3}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.4}$$

The above equations (3.3.3), (3.3.4) are in the form

$$L_1: \quad \mathbf{x} = \mathbf{a_1} + \lambda_1 \mathbf{b_1} \tag{3.3.5}$$

$$L_2: \quad \mathbf{x} = \mathbf{a_2} + \lambda_2 \mathbf{b_2} \tag{3.3.6}$$

Here ,

$$\mathbf{a_1} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \tag{3.3.7}$$

$$\mathbf{a_2} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} \tag{3.3.8}$$

$$\mathbf{b_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.3.9}$$

$$\mathbf{b_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.10}$$

Now let us assume the lines L_1 and L_2 are intersecting at a point. Therefore ,

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.3.11}$$

$$\lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.3.12}$$

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{3.3.13}$$

The augumented matrix of (3.3.13) is given by

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array}\right) \tag{3.3.14}$$

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array}\right) \xleftrightarrow{R_2 = R_2 - \frac{2}{3}R_1} \left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array}\right) \tag{3.3.15}$$

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array}\right) \xleftrightarrow{R_3 = R_3 - 2R_1} \left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{array}\right) \tag{3.3.16}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.Hence our assumptions is wrong
Equation (3.3.13) can be expressed as

$$\mathbf{Mx} = \mathbf{b} \tag{3.3.17}$$

By singular value decomposition $\mathbf{M}$ can be expressed as

$$\mathbf{M} = \mathbf{USV}^T \tag{3.3.18}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix} \tag{3.3.19}$$

$$\mathbf{MM}^T = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \tag{3.3.20}$$

The characteristic equation of $\mathbf{M}^T\mathbf{M}$ is obtained by evaluating the determinant

$$\begin{vmatrix} 49 - \lambda & -19 \\ -19 & 9 - \lambda \end{vmatrix} = 0 \tag{3.3.21}$$

$$\implies \lambda^2 - 58\lambda + 80 = 0 \tag{3.3.22}$$

The eigenvalues are the roots of equation 3.3.22 is given by

$$\lambda_{11} = 29 + \sqrt{761} \tag{3.3.23}$$

$$\lambda_{12} = 29 - \sqrt{761} \tag{3.3.24}$$

The eigen vectors comes out to be ,

$$\mathbf{u_{11}} = \begin{pmatrix} \frac{-20-\sqrt{761}}{19} \\ 1 \end{pmatrix}, \mathbf{u_{12}} = \begin{pmatrix} \frac{-20+\sqrt{761}}{19} \\ 1 \end{pmatrix} \quad (3.3.25)$$

Normalising the eigen vectors,

$$l_{11} = \sqrt{\left(\frac{-20-\sqrt{761}}{19}\right)^2 + 1^2} \quad (3.3.26)$$

$$\implies l_{11} = \frac{\sqrt{1522+40\sqrt{761}}}{19} \quad (3.3.27)$$

$$\mathbf{u_{11}} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} \end{pmatrix} \quad (3.3.28)$$

$$l_{12} = \sqrt{\left(\frac{-20+\sqrt{761}}{19}\right)^2 + 1^2} \quad (3.3.29)$$

$$\implies l_{12} = \frac{\sqrt{1522-40\sqrt{761}}}{19} \quad (3.3.30)$$

$$\mathbf{u_{12}} = \begin{pmatrix} \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \quad (3.3.31)$$

$$\mathbf{V} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} & \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} & \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \quad (3.3.32)$$

S is given by

$$\mathbf{S} = \begin{pmatrix} \sqrt{29+\sqrt{761}} & 0 \\ 0 & \sqrt{29-\sqrt{761}} \\ 0 & 0 \end{pmatrix} \quad (3.3.33)$$

The characteristic equation of $\mathbf{MM}^T$ is obtained by evaluating the determinant

$$\begin{vmatrix} 10-\lambda & 8 & 20 \\ 8 & 8-\lambda & 16 \\ 20 & 16 & 40-\lambda \end{vmatrix} = 0 \quad (3.3.34)$$

$$\implies \lambda^3 - 58\lambda^2 + 80\lambda = 0 \quad (3.3.35)$$

The eigenvalues are the roots of equation 3.3.35 is given by

$$\lambda_{21} = 29 + \sqrt{761} \quad (3.3.36)$$

$$\lambda_{22} = 29 - \sqrt{761} \quad (3.3.37)$$

$$\lambda_{23} = 0 \quad (3.3.38)$$

The eigen vectors comes out to be ,

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-\sqrt{761}+21}{16} \\ -1 \end{pmatrix}, \mathbf{u_{22}} = \begin{pmatrix} \frac{1}{2} \\ \frac{-\sqrt{761}-21}{16} \\ 1 \end{pmatrix}, \mathbf{u_{23}} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad (3.3.39)$$

Normalising the eigen vectors,

$$l_{21} = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{21-\sqrt{761}}{16}\right)^2 + (-1)^2} \quad (3.3.40)$$

$$\implies l_{21} = \frac{\sqrt{1522-42\sqrt{761}}}{16} \quad (3.3.41)$$

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \quad (3.3.42)$$

$$l_{22} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-21-\sqrt{761}}{16}\right)^2 + 1^2} \quad (3.3.43)$$

$$\implies l_{22} = \frac{\sqrt{1522+42\sqrt{761}}}{16} \quad (3.3.44)$$

$$\mathbf{u_{22}} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} \end{pmatrix} \quad (3.3.45)$$

$$l_{23} = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \quad (3.3.46)$$

$$\mathbf{u_{23}} = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.3.47)$$

$$\mathbf{U} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.3.48}$$

From equation (3.3.18) we rewrite $\mathbf{M}$ as follows,

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.3.49}$$

$$\begin{pmatrix} \sqrt{29+\sqrt{761}} & 0 \\ 0 & \sqrt{29-\sqrt{761}} \\ 0 & 0 \end{pmatrix} \tag{3.3.50}$$

$$\begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} & \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} & \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix}^T \tag{3.3.51}$$

By substituting the equation (3.3.18) in equation (3.3.17) we get

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{3.3.52}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{3.3.53}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{29+\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29-\sqrt{761}}} & 0 \end{pmatrix} \tag{3.3.54}$$

From (3.3.53) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{761}-45}{\sqrt{1522-42\sqrt{761}}} \\ \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ -\frac{11}{\sqrt{5}} \end{pmatrix} \tag{3.3.55}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{761\sqrt{15}-761-45\sqrt{11415}+45\sqrt{761}}{10654} \\ \frac{45\sqrt{11415}+45\sqrt{761}+761\sqrt{15}+761}{10654} \end{pmatrix} \tag{3.3.56}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{3.3.57}$$

Verifying the solution of (3.3.57) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.3.58}$$

Evaluating the R.H.S in (3.3.58) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{3.3.59}$$

$$\implies \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{3.3.60}$$

Solving the augmented matrix of (3.3.60) we get,

$$\begin{pmatrix} 49 & -19 & 7 \\ -19 & 9 & -1 \end{pmatrix} \xleftrightarrow{R_2=R_2+\frac{19}{49}R_1} \begin{pmatrix} 49 & -19 & 7 \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{3.3.61}$$

$$\xleftrightarrow{R_1=\frac{1}{49}R_1} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{3.3.62}$$

$$\xleftrightarrow{R_2=\frac{80}{49}R_2} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{3.3.63}$$

$$\xleftrightarrow{R_1=R_1+\frac{19}{49}R_2} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{3.3.64}$$

Hence, Solution of (3.3.58) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{3.3.65}$$

Comparing results of $\mathbf{x}$ from (3.3.57) and (3.3.65) we conclude that the solution is verified.

3.4. Check if the lines L_1, L_2 are skew.If so, find the closest points on those lines using Singular Value Decomposition(SVD)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.4.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.4.2}$$

Solution:
The matrix $\mathbf{M}$ of dimensions $(m \times n)$ can be decomposed using SVD as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{3.4.3}$$

where,columns of $\mathbf{U}_{(m\times m)}$ are eigen vectors of

$\mathbf{MM}^T$
columns of $\mathbf{V}_{(n\times n)}$ are eigen vectors of $\mathbf{M}^T\mathbf{M}$
$\mathbf{S}$ is a diagonal matrix containing singular values of $\mathbf{M}$. Also, $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices

$$\mathbf{UU}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{3.4.4}$$
$$\mathbf{VV}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{3.4.5}$$

Given line equations intersect if

$$\begin{pmatrix}2\\-5\\1\end{pmatrix} + \lambda_1\begin{pmatrix}3\\2\\6\end{pmatrix} = \begin{pmatrix}7\\-6\\0\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{3.4.6}$$

This can be written as

$$\begin{pmatrix}3 & 1\\2 & 2\\6 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}5\\-1\\-1\end{pmatrix} \tag{3.4.7}$$
$$\mathbf{Mx} = \mathbf{b} \tag{3.4.8}$$
$$\text{where, } \mathbf{x} = \begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} \tag{3.4.9}$$

The augmented matrix is

$$\begin{pmatrix}3 & 1 & 5\\2 & 2 & -1\\6 & 2 & -1\end{pmatrix} \xleftrightarrow[R_2\leftarrow R_2-R_1\times\frac{2}{3}]{R_3\leftarrow R_3-2\times R_1} \begin{pmatrix}3 & 1 & 5\\0 & \frac{5}{3} & -\frac{13}{3}\\0 & 0 & -11\end{pmatrix} \tag{3.4.10}$$

So, the given pair of lines do not intersect and also their direction vectors are not parallel. Hence they are skew lines.
To find $\mathbf{U}$,

$$\mathbf{MM}^T = \begin{pmatrix}3 & 1\\2 & 2\\6 & 2\end{pmatrix}\begin{pmatrix}3 & 2 & 6\\1 & 2 & 2\end{pmatrix} = \begin{pmatrix}10 & 8 & 20\\8 & 8 & 16\\20 & 16 & 40\end{pmatrix} \tag{3.4.11}$$

To calculate its Eigen values,

$$\begin{vmatrix}10-\lambda & 8 & 20\\8 & 8-\lambda & 16\\20 & 16 & 40-\lambda\end{vmatrix} = 0 \tag{3.4.12}$$
$$\implies \lambda^3 + 58\lambda^2 + 80\lambda = 0 \tag{3.4.13}$$
$$\lambda_1 = 29 - \sqrt{761}, \lambda_2 = 29 + \sqrt{761}, \lambda_3 = 0 \tag{3.4.14}$$

with corresponding Eigen vectors as

$$\mathbf{u_1} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{21+\sqrt{761}}{16}\right)^2}}\begin{pmatrix}\frac{1}{2}\\\frac{-21-\sqrt{761}}{16}\\1\end{pmatrix} \tag{3.4.15}$$

$$\mathbf{u_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{-21+\sqrt{761}}{16}\right)^2}}\begin{pmatrix}\frac{1}{2}\\\frac{-21+\sqrt{761}}{16}\\1\end{pmatrix} \tag{3.4.16}$$

$$\mathbf{u_3} = \frac{1}{\sqrt{(-2)^2+1}}\begin{pmatrix}-2\\0\\1\end{pmatrix} \tag{3.4.17}$$

Solving, the $\mathbf{U}$ matrix becomes

$$\mathbf{U} = \begin{pmatrix}\frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}}\\\frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0\\\frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{3.4.18}$$

Also, from the obtained Eigen values,the $\mathbf{S}$ matrix becomes

$$\mathbf{S} = \begin{pmatrix}\sqrt{29-\sqrt{761}} & 0\\0 & \sqrt{29+\sqrt{761}}\\0 & 0\end{pmatrix} \tag{3.4.19}$$

The Moore-Penrose pseudo inverse of $\mathbf{S}$ is given by

$$\mathbf{S}_+ = \begin{pmatrix}\frac{1}{\sqrt{29-\sqrt{761}}} & 0 & 0\\0 & \frac{1}{\sqrt{29+\sqrt{761}}} & 0\end{pmatrix} \tag{3.4.20}$$

Now to find $\mathbf{V}$,
Rewriting (3.4.3)

$$\mathbf{V} = (\mathbf{M}^T\mathbf{U})\mathbf{S_+}^T \tag{3.4.21}$$

$\mathbf{M}^T\mathbf{U}$ becomes

$$\begin{pmatrix}3 & 2 & 6\\1 & 2 & 2\end{pmatrix} \tag{3.4.22}$$

$$\begin{pmatrix}\frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}}\\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0\\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{3.4.23}$$

$$= \begin{pmatrix}\frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0\\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0\end{pmatrix} \tag{3.4.24}$$

Therefore from (3.4.20),(3.4.21),(3.4.24),

$$\mathbf{V} = \begin{pmatrix}\frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}}\\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}}\end{pmatrix} \tag{3.4.25}$$

Now, to calculate $\mathbf{x}$

$$\mathbf{Mx} = \mathbf{b} \tag{3.4.26}$$

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.4.27}$$

$$\implies \mathbf{SV}^T\mathbf{x} = \mathbf{U}^T\mathbf{b} \tag{3.4.28}$$

$$\implies \mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b})) \tag{3.4.29}$$

Calculating $\mathbf{U}^T\mathbf{b}$, we have

$$\begin{pmatrix}\frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}}\\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}}\end{pmatrix} \tag{3.4.30}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix}\frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}}\\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}}\end{pmatrix} \tag{3.4.31}$$

$\mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$= \begin{pmatrix}\frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}}\\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}}\end{pmatrix} \tag{3.4.32}$$

$$\begin{pmatrix}\frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}}\\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}}\end{pmatrix} \tag{3.4.33}$$

Solving,

$$\mathbf{x} = \begin{pmatrix}\frac{8371}{15220}\\ \frac{-15981}{15220}\end{pmatrix} = \begin{pmatrix}\frac{11}{20}\\ \frac{-21}{20}\end{pmatrix} \tag{3.4.34}$$

Verifying the solution,

$$\mathbf{Mx} = \mathbf{b} \tag{3.4.35}$$

$$\implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{3.4.36}$$

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix}3 & 2 & 6\\1 & 2 & 2\end{pmatrix}\begin{pmatrix}5\\-1\\-1\end{pmatrix} \tag{3.4.37}$$

$$= \begin{pmatrix}7\\1\end{pmatrix} \tag{3.4.38}$$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}3 & 2 & 6\\1 & 2 & 2\end{pmatrix}\begin{pmatrix}3 & 1\\2 & 2\\6 & 2\end{pmatrix} \tag{3.4.39}$$

$$= \begin{pmatrix}49 & 19\\19 & 9\end{pmatrix} \tag{3.4.40}$$

$$\text{From, (3.4.36)} \begin{pmatrix}49 & 19\\19 & 9\end{pmatrix}\mathbf{x} = \begin{pmatrix}7\\1\end{pmatrix} \tag{3.4.41}$$

Solving for $\mathbf{x}$

$$\begin{pmatrix}49 & 19 & 7\\19 & 9 & 1\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1 \times \frac{19}{49}} \begin{pmatrix}49 & 19 & 7\\0 & \frac{80}{49} & \frac{-84}{49}\end{pmatrix} \tag{3.4.42}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 \times \frac{1}{49}} \begin{pmatrix}1 & \frac{19}{49} & \frac{7}{49}\\0 & \frac{80}{49} & \frac{-84}{49}\end{pmatrix} \tag{3.4.43}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2 \times \frac{19}{80}} \begin{pmatrix}1 & 0 & \frac{11}{20}\\0 & \frac{80}{49} & \frac{-84}{49}\end{pmatrix} \tag{3.4.44}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{49}{80}} \begin{pmatrix}1 & 0 & \frac{11}{20}\\0 & 1 & \frac{-21}{20}\end{pmatrix} \tag{3.4.45}$$

$$\implies \mathbf{x} = \begin{pmatrix}\frac{11}{20}\\ \frac{-21}{20}\end{pmatrix} \tag{3.4.46}$$

3.5. Find the point on the plane closest to the point $\begin{pmatrix}6\\5\\9\end{pmatrix}$ and the plane is determined by the points

$$\mathbf{A} = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\2\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$$

Solution: The equation of plane is given by,

$$\mathbf{n}^T\mathbf{x} = c \tag{3.5.1}$$

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = \mathbf{n}^T\mathbf{C} = c \tag{3.5.2}$$

$$\implies \begin{pmatrix}\mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C}\end{pmatrix}^T \mathbf{n} = 0 \tag{3.5.3}$$

Using row reduction on above matrix,

$$\begin{pmatrix}-2 & -3 & -2\\ 6 & 3 & -2\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{R_1}{-2}} \begin{pmatrix}1 & \frac{3}{2} & 1\\ 6 & 3 & -2\end{pmatrix} \tag{3.5.4}$$

$$\xleftrightarrow{R_2\leftarrow R_2-6R_1} \begin{pmatrix}1 & \frac{3}{2} & 1\\ 0 & -6 & -8\end{pmatrix} \xleftrightarrow{R_2\leftarrow\frac{R_2}{-2}} \begin{pmatrix}1 & \frac{3}{2} & 1\\ 0 & 3 & 4\end{pmatrix} \tag{3.5.5}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{R_2}{2}} \begin{pmatrix}1 & 0 & -1\\ 0 & 3 & 4\end{pmatrix} \tag{3.5.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix}1\\ \frac{-4}{3}\\ 1\end{pmatrix} = \begin{pmatrix}3\\ -4\\ 3\end{pmatrix} \tag{3.5.7}$$

$$c = \mathbf{n}^T\mathbf{A} = 19 \tag{3.5.8}$$

Thus the equation of the plane is,

$$\begin{pmatrix}3 & -4 & 3\end{pmatrix}\mathbf{x} = 19 \tag{3.5.9}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be the two orthogonal vectors to the given normal. Let, $\mathbf{m} = \begin{pmatrix}a\\ b\\ c\end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0 \tag{3.5.10}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}3\\ -4\\ 3\end{pmatrix} = 0 \tag{3.5.11}$$

$$\implies 3a - 4b + 3c = 0 \tag{3.5.12}$$

Let $a = 1$, $b = 0$ we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\ 0\\ -1\end{pmatrix} \tag{3.5.13}$$

Let $a = 0$, $b = 1$ we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\ 1\\ \frac{4}{3}\end{pmatrix} \tag{3.5.14}$$

Solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{3.5.15}$$

Putting the values in (3.5.15),

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ -1 & \frac{4}{3}\end{pmatrix}\mathbf{x} = \begin{pmatrix}6\\ 5\\ 9\end{pmatrix} \tag{3.5.16}$$

To solve (3.5.16), we perform Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{USV}^T \tag{3.5.17}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}2 & \frac{-4}{3}\\ \frac{-4}{3} & \frac{25}{9}\end{pmatrix} \tag{3.5.18}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & \frac{4}{3}\\ -1 & \frac{4}{3} & \frac{25}{9}\end{pmatrix} \tag{3.5.19}$$

Putting (3.5.17) in (3.5.15) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.5.20}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{3.5.21}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Now, calculating eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{3.5.22}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & -1\\ 0 & 1-\lambda & \frac{4}{3}\\ -1 & \frac{4}{3} & \frac{25}{9}-\lambda\end{pmatrix} = 0 \tag{3.5.23}$$

$$\implies \lambda^3 - \frac{43}{9}\lambda^2 + \frac{34}{9}\lambda = 0 \tag{3.5.24}$$

Thus the eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{34}{9} \tag{3.5.25}$$

$$\lambda_2 = 1 \tag{3.5.26}$$

$$\lambda_3 = 0 \tag{3.5.27}$$

The eigen vectors comes out to be,

$$\mathbf{u_1} = \begin{pmatrix}\frac{-9}{25}\\ \frac{12}{25}\\ 1\end{pmatrix}, \mathbf{u_2} = \begin{pmatrix}\frac{4}{3}\\ 1\\ 0\end{pmatrix}, \mathbf{u_3} = \begin{pmatrix}1\\ \frac{-4}{3}\\ 1\end{pmatrix} \tag{3.5.28}$$

Normalising the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} \\ \frac{12}{5\sqrt{34}} \\ \frac{5}{\sqrt{34}} \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} \frac{3}{\sqrt{34}} \\ \frac{-4}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{pmatrix} \quad (3.5.29)$$

Hence we obtain $\mathbf{U}$ matrix as,

$$\mathbf{U} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \quad (3.5.30)$$

Now,

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.5.31)$$

Calculating the eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (3.5.32)$$

$$\implies \begin{pmatrix} 2-\lambda & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9}-\lambda \end{pmatrix} = 0 \quad (3.5.33)$$

$$\implies \lambda^2 - \frac{43}{9}\lambda + \frac{34}{9} = 0 \quad (3.5.34)$$

The eigen values are,

$$\lambda_1 = \frac{34}{9} \quad (3.5.35)$$

$$\lambda_2 = 1 \quad (3.5.36)$$

The eigen vectors are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \quad (3.5.37)$$

Normalising the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \quad (3.5.38)$$

Hence we obtain $\mathbf{V}$ matrix as,

$$\mathbf{V} = \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \quad (3.5.39)$$

Thus we get the Singular Value Decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}^T \quad (3.5.40)$$

The Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (3.5.41)$$

From (3.5.21) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{231}{5\sqrt{34}} \\ \frac{39}{5} \\ \frac{25}{\sqrt{34}} \end{pmatrix} \quad (3.5.42)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{693}{170} \\ \frac{39}{5} \end{pmatrix} \quad (3.5.43)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \quad (3.5.44)$$

Verifying the solution of (3.5.44) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (3.5.45)$$

Evaluating the R.H.S in (3.5.45) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \quad (3.5.46)$$

$$\implies \begin{pmatrix} 2 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \quad (3.5.47)$$

Solving the augmented matrix of (3.5.47) we get,

$$\begin{pmatrix} 2 & \frac{-4}{3} & -3 \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \quad (3.5.48)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{4}{3}R-1} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \quad (3.5.49)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \quad (3.5.50)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{9}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & 1 & \frac{135}{17} \end{pmatrix} \quad (3.5.51)$$

Hence, solution of (3.5.45) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \quad (3.5.52)$$

Comparing results of $\mathbf{x}$ from (3.5.44) and (3.5.52) we conclude that the solution is verified. Find the foot of the perpendicular using svd drawn from $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ to the plane

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{3.5.53}$$

Solution:

3.6. Find the distance of the given point $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ from the plane $\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} = 3$.

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{3.6.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0 \tag{3.6.2}$$

$$\implies 2a - b + 2c = 0 \tag{3.6.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.6.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \tag{3.6.5}$$

Let us solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{3.6.6}$$

Substituting (3.6.4) and (3.6.5) in (3.6.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{3.6.7}$$

To solve (3.6.7), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{3.6.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \tag{3.6.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \tag{3.6.10}$$

Substituting (3.6.8) in (3.6.6),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.6.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{3.6.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.

Let us calculate eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{3.6.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{3.6.14}$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \tag{3.6.15}$$

From equation (3.6.15) eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{3.6.16}$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \tag{3.6.17}$$

Normalizing the eigen vectors in equation (3.6.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{3.6.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \tag{3.6.19}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.6.20}$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{3.6.21}$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{3.6.22}$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{3.6.23}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \tag{3.6.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{3.6.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{3.6.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{3.6.27}$$

From (3.6.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \tag{3.6.28}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.6.29}$$

From (3.6.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{11}{3\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ \frac{10}{3} \end{pmatrix} \tag{3.6.30}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{22}{9\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.6.31}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \tag{3.6.32}$$

Verifying the solution of (3.6.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.6.33}$$

Evaluating the R.H.S in (3.6.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \tag{3.6.34}$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \tag{3.6.35}$$

Solving the augmented matrix of (3.6.35) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & 2 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \tag{3.6.36}$$

$$\xleftrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{9}{8} & -1 \end{pmatrix} \tag{3.6.37}$$

$$\xleftrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \tag{3.6.38}$$

$$\xleftrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \tag{3.6.39}$$

From equation (3.6.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \tag{3.6.40}$$

Comparing results of $\mathbf{x}$ from (3.6.32) and (3.6.40), we can say that the solution is verified.

3.7. Find the distance of the point $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ from the plane $\begin{pmatrix} 1 & 2 & -2 \end{pmatrix}\mathbf{x} = 9$

Solution:

Find the distance of the point $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ from the plane $\begin{pmatrix}1 & 2 & -2\end{pmatrix}\mathbf{x} = 9$ First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{3.7.1}$$
$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}1\\2\\-2\end{pmatrix} = 0 \tag{3.7.2}$$
$$\implies a + 2b - 2c = 0 \tag{3.7.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix} \tag{3.7.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\1\end{pmatrix} \tag{3.7.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{3.7.6}$$

Putting values in (3.7.6),

$$\begin{pmatrix}1 & 0\\0 & 1\\\frac{1}{2} & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\3\\-5\end{pmatrix} \tag{3.7.7}$$

In order to solve (3.7.7), perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{3.7.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}\frac{5}{4} & \frac{1}{2}\\\frac{1}{2} & 2\end{pmatrix} \tag{3.7.9}$$
$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & 1\\\frac{1}{2} & 1 & \frac{5}{4}\end{pmatrix} \tag{3.7.10}$$

From (3.7.6) putting (3.7.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{3.7.11}$$
$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{3.7.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{3.7.13}$$
$$\implies \begin{pmatrix}1-\lambda & 0 & \frac{1}{2}\\0 & 1-\lambda & 1\\\frac{1}{2} & 1 & \frac{5}{2}-\lambda\end{pmatrix} = 0 \tag{3.7.14}$$
$$\implies -4\lambda^3 + 13\lambda^2 - 9\lambda = 0 \tag{3.7.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \tag{3.7.16}$$
$$\lambda_2 = 1 \tag{3.7.17}$$
$$\lambda_3 = 0 \tag{3.7.18}$$

Hence the eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{2}{5}\\\frac{4}{5}\\1\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-2\\1\\0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}\frac{-1}{2}\\-1\\1\end{pmatrix} \tag{3.7.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{2}{\sqrt{45}}\\\frac{4}{\sqrt{45}}\\\frac{5}{\sqrt{45}}\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\\0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-\frac{1}{3}\\-\frac{2}{3}\\\frac{2}{3}\end{pmatrix} \tag{3.7.20}$$

Hence we obtain $\mathbf{U}$ of (3.7.8) as follows,

$$\begin{pmatrix}\frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3}\\\frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & -\frac{2}{3}\\\frac{5}{\sqrt{45}} & 0 & \frac{2}{3}\end{pmatrix} \tag{3.7.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (3.7.8) as follows,

$$\mathbf{S} = \begin{pmatrix}\frac{3}{2} & 0\\0 & 1\\0 & 0\end{pmatrix} \tag{3.7.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{3.7.23}$$
$$\implies \begin{pmatrix}\frac{5}{4}-\lambda & \frac{1}{2}\\\frac{1}{2} & 2-\lambda\end{pmatrix} = 0 \tag{3.7.24}$$
$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{3.7.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{9}{4} \tag{3.7.26}$$
$$\lambda_5 = 1 \tag{3.7.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-2\\1\end{pmatrix} \tag{3.7.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix}\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix} \tag{3.7.29}$$

Hence we obtain $\mathbf{V}$ of (3.7.8) as follows,

$$\mathbf{V} = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{3.7.30}$$

From (3.7.8) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix}\frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3}\\\frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & \frac{-2}{3}\\\frac{5}{\sqrt{45}} & 0 & \frac{2}{3}\end{pmatrix}\begin{pmatrix}\frac{3}{2} & 0\\0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}^T \tag{3.7.31}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix}\frac{2}{3} & 0 & 0\\0 & 1 & 0\end{pmatrix} \tag{3.7.32}$$

From (3.7.11) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix}-\frac{3\sqrt{5}}{5}\\-\frac{\sqrt{5}}{5}\\-6\end{pmatrix} \tag{3.7.33}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix}-\frac{2\sqrt{5}}{5}\\-\frac{\sqrt{5}}{5}\end{pmatrix} \tag{3.7.34}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix}0\\-1\end{pmatrix} \tag{3.7.35}$$

Verifying the solution of (3.7.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.7.36}$$

Evaluating the R.H.S in (3.7.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix}-\frac{1}{2}\\-2\end{pmatrix} \tag{3.7.37}$$

$$\implies \begin{pmatrix}\frac{5}{4} & \frac{1}{2}\\\frac{1}{2} & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}-\frac{1}{2}\\-2\end{pmatrix} \tag{3.7.38}$$

Solving the augmented matrix of (3.7.38) we get,

$$\begin{pmatrix}\frac{5}{4} & \frac{1}{2} & -\frac{1}{2}\\\frac{1}{2} & 2 & -2\end{pmatrix} \xleftrightarrow{R_1=\frac{4}{5}R_1} \begin{pmatrix}1 & \frac{2}{5} & \frac{-2}{5}\\\frac{1}{2} & 2 & -2\end{pmatrix} \tag{3.7.39}$$

$$\xleftrightarrow{R_2=R_2-\frac{1}{2}R_2} \begin{pmatrix}1 & -\frac{2}{5} & -\frac{2}{5}\\0 & \frac{9}{5} & -\frac{9}{5}\end{pmatrix} \tag{3.7.40}$$

$$\xleftrightarrow{R_2=\frac{5}{9}R_2} \begin{pmatrix}1 & \frac{2}{5} & -\frac{2}{5}\\0 & 1 & -1\end{pmatrix} \tag{3.7.41}$$

$$\xleftrightarrow{R_1=R_1-\frac{2}{5}R_2} \begin{pmatrix}1 & 0 & 0\\0 & 1 & -1\end{pmatrix} \tag{3.7.42}$$

From equation (3.7.42), solution is given by,

$$\mathbf{x} = \begin{pmatrix}0\\-1\end{pmatrix} \tag{3.7.43}$$

Comparing results of $\mathbf{x}$ from (3.7.35) and (3.7.43), we can say that the solution is verified.

4 Linear Programming

1. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} \tag{4.1.1}$$

$$s.t. \quad \begin{pmatrix}-1 & -1\\3 & 5\end{pmatrix}\mathbf{x} \le \begin{pmatrix}-8\\15\end{pmatrix} \tag{4.1.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{4.1.3}$$

2. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix}200 & 500\end{pmatrix}\mathbf{x} \tag{4.2.1}$$

$$s.t. \quad \begin{pmatrix}-1 & -2\\3 & 4\end{pmatrix}\mathbf{x} \le \begin{pmatrix}-10\\24\end{pmatrix} \tag{4.2.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{4.2.3}$$

3. Maximise Z=3x+4y
   subject to the constraints : x+y≤4, x≥0, y≥ 0.

4. Minimise Z=-3x+4y
   subject to x+2y≤8, 3x+2y≤12, x≥0, y≥0.

5. Maximise Z=5x+3y subject to 3x+5y≤15, 5x+2y≤10, x≥0, y≥0.
   Solution:

$$Z - 5x - 3y = 0 \tag{4.5.1}$$

$$3x + 5y + s_1 = 15 \tag{4.5.2}$$

$$5x + 2y + s_2 = 10 \tag{4.5.3}$$

We will write the simplex tableau

$$\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & c \end{array} \\ \left(\begin{array}{ccccc} 3 & 5 & 1 & 0 & 15 \\ \boxed{5} & 2 & 0 & 1 & 10 \\ \hline -5 & -3 & 0 & 0 & 0 \end{array}\right) \end{array} \tag{4.5.4}$$

Keeping the pivot element as 5, we will use gauss-jordan elimination.

$$\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & c \end{array} \\ \left(\begin{array}{ccccc} 0 & \boxed{\frac{19}{5}} & 1 & \frac{-3}{5} & 9 \\ 1 & \frac{2}{5} & 0 & \frac{1}{5} & 2 \\ \hline 0 & -1 & 0 & 1 & 10 \end{array}\right) \end{array} \tag{4.5.5}$$

Keeping the pivot element as $\frac{19}{5}$, we will use gauss-jordan elimination.

$$\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & c \end{array} \\ \left(\begin{array}{ccccc} 0 & 1 & \frac{5}{19} & \frac{-3}{19} & \frac{45}{19} \\ 1 & 0 & \frac{-2}{19} & \frac{5}{19} & \frac{20}{19} \\ \hline 0 & 0 & \frac{5}{19} & \frac{16}{19} & \frac{235}{19} \end{array}\right) \end{array} \tag{4.5.6}$$

In this tableau, there are no negative elements in the bottom row. We have therefore determined the optimal solution to be:

$$(x, y, s_1, s_2) = \left(\frac{20}{19}, \frac{45}{19}, 0, 0\right) \tag{4.5.7}$$

$$Z = 5x + 3y \tag{4.5.8}$$

$$Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} \tag{4.5.9}$$

$$Z = \frac{235}{19} \tag{4.5.10}$$

The given problem can be expressed in general as matrix inequality as:

$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \tag{4.5.11}$$

$$s.t. \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \tag{4.5.12}$$

$$\mathbf{x} \geq \mathbf{0} \tag{4.5.13}$$

$$\mathbf{y} \geq \mathbf{0} \tag{4.5.14}$$

where

$$\mathbf{c} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{4.5.15}$$

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 5 & 2 \end{pmatrix} \tag{4.5.16}$$

$$\mathbf{b} = \begin{pmatrix} 15 \\ 10 \end{pmatrix} \tag{4.5.17}$$

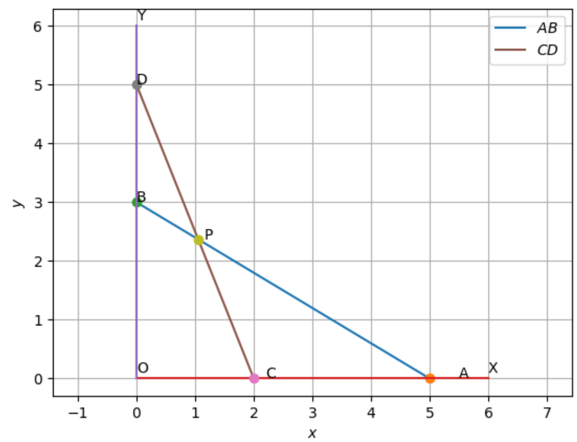

Fig. 4.5: optimal point through the intersection of various lines

and can be solved using *cvxpy*. Hence,

$$\mathbf{x} = \begin{pmatrix} 1.05263158 \\ 2.36842105 \end{pmatrix}, Z = 12.36842102 \tag{4.5.18}$$

6. Minimise Z=3x+5y such that x+3y≥3, x+y≥2, x,y≥0.

7. Maximise Z=3x+2y subject to x+2y≤10, 3x+y≤15, x,y≥0.
Solution:

$$Maximize : 3x_1 + 2x_2 \tag{4.7.1}$$

$$Subject-to : x_1 + 2x_2 \leq 10 \tag{4.7.2}$$

$$3x_1 + x_2 \leq 15 \tag{4.7.3}$$

The Problem is converted into canonical form by adding slack variables.Then Problem becomes,

$$Maximize : 3x_1 + 2x_2 + 0s_1 + 0s_2 \tag{4.7.4}$$

$$Constraints : x_1 + 2x_2 + s_1 = 10 \tag{4.7.5}$$

$$3x_1 + x_2 + s_2 = 15 \tag{4.7.6}$$

we write the Simplex tableau ,

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & c \end{array}$$
$$\left(\begin{array}{ccccc} 1 & 2 & 1 & 0 & 10 \\ \boxed{3} & 1 & 0 & 1 & 15 \\ \hline -3 & -2 & 0 & 0 & 0 \end{array}\right) \tag{4.7.7}$$

Keeping the Pivot element as 3 and by using gauss-jordan Elimination we get

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & c \end{array}$$
$$\left(\begin{array}{ccccc} 0 & \boxed{\frac{5}{3}} & 1 & \frac{-1}{3} & 5 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 5 \\ \hline 0 & -1 & 0 & 1 & 15 \end{array}\right) \tag{4.7.8}$$

Keeping the Pivot element as $\frac{5}{3}$ and by using gauss-jordan Elimination we get

$$\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & c \end{array}$$
$$\left(\begin{array}{ccccc} 0 & 1 & \frac{3}{5} & \frac{-1}{5} & 2 \\ 1 & 0 & \frac{-1}{5} & \frac{2}{5} & 3 \\ \hline 0 & 0 & \frac{3}{5} & \frac{4}{5} & 18 \end{array}\right) \tag{4.7.9}$$

In this tableau Since all indicators in last row are non-negative ,we found optimal solution to given problem. Therefore Optimal Solution will be:

$$(x_1, x_2) = (4, 3) \tag{4.7.10}$$
$$Z = 3x_1 + 2x_2 \tag{4.7.11}$$
$$Z = 3 \times 4 + 2 \times 3 \tag{4.7.12}$$
$$Z = 18 \tag{4.7.13}$$

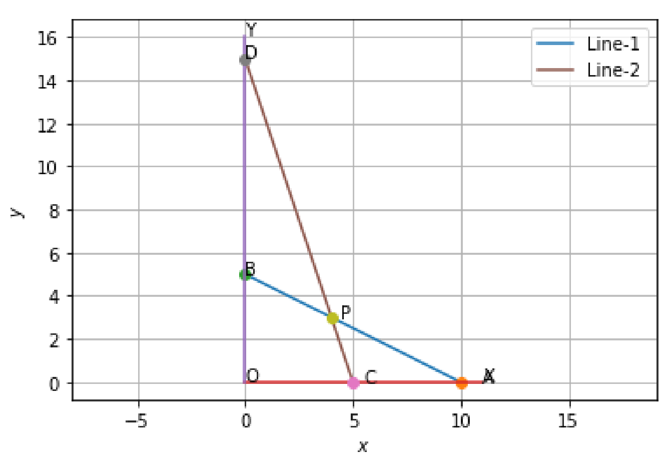

Fig. 4.7: Region OBPC is Valid region

This Problem can be represented in matrix form as follows,

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} \tag{4.7.14}$$
$$s.t. \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 10 \\ 15 \end{pmatrix} \tag{4.7.15}$$
$$\mathbf{x} \succeq \mathbf{0} \tag{4.7.16}$$
$$\mathbf{y} \succeq \mathbf{0} \tag{4.7.17}$$

this is solved using cvxpy in python,we get

$$\mathbf{x} = \begin{pmatrix} 3.99999999 \\ 2.99999999 \end{pmatrix}, Z = 17.99999996 \tag{4.7.18}$$

8. Minimise Z=x+2y subject to 2x+y≥3, x+2y≥6, x,y≥0.
   Show that the minimum of Z occurs at more than two points.

9. Minimise and Maximise Z=5x+10y subject to x+2y≤120, x+y≥60, x-2y≥0, x,y≥0.

10. Minimise and Maximise Z=x+2y subject to x+2y≥100, 2x-y≤0, 2x+y≤200; x,y≥0.

11. Maximise Z=-x+2y, subject to the constraints: x≥3, x+y≥5, x+2y≥6, y≥0.

12. Maximise Z=x+y, subject to x-y≤-1,-x+y≤0, x,y≥0.

13. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.

14. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which

can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.

15. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
(i) What number of rackets and bats must be made if the factory is to work at full capacity?
(ii)If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.

16. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?

17. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

18. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

19. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?

20. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

21. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foodsF_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost

for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

22. There are two types of fertilisers F_1 and F_2.F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

23. The corner points of the feasible region determined by the following system of linear inequalities: $2x+y\leq10$, $x+3y\leq15$, $x,y\geq0$ are (0,0), (5,0),(3,4) and (0,5).Let Z=px+qy, where p,q>0.Condition on p and q so that the maximum of Z occurs at both (3,4) and (0,5) is
(A) p = q
(B) p = 2q
(C) p = 3q
(D) q = 3p

24. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

25. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

26. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	Vitamin A	Vitamin B	VitaminC
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

27. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

	Machines		
Types of toys	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

28. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

29. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per qunital (in Rs)		
From/To	A	B
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

30. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From/To	**A**	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

31. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	**Brand Q**
Nitrogen	3	3.5
Phospheric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

32. Refer to Question 29. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

33. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?

34. Find the shortest distance of the point $\begin{pmatrix} 0 \\ c \end{pmatrix}$ from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.

35. Find the maximum area of an isosceles triangle inscribed in the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{4.35.1}$$

with its vertex at one end of the major axis.

36. **(Manufacturing problem)** A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?

37. **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

38. Solve:

$$\max_{\{x\}} Z = \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} \tag{4.38.1}$$

$$s.t \quad \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \leq \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{4.38.2}$$

$$\mathbf{x} \geq 0 \tag{4.38.3}$$

Solution: Adding slack variables to the left side of (4.38.2) and (4.38.3),we get

$$Z - 4x - y = 0 \tag{4.38.4}$$

$$x + y + s_1 = 50 \tag{4.38.5}$$

$$3x + y + s_2 = 90 \tag{4.38.6}$$

Forming simplex tableau,

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & b \\ \hline 1 & 1 & 1 & 0 & 50 \\ 3 & 1 & 0 & 1 & 90 \\ \hline -4 & -1 & 0 & 0 & 0 \end{array}\right) \tag{4.38.7}$$

-4 is the smallest entry in the bottom row. Therefore, we determine that x is the starting variable.
Also, the smallest positive ratio is 30, therefore, we chose s_2 as the departing variable.
Hence, keeping the pivot element as 3, we perform Gauss Jordan elimination,

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & b \\ 1 & 1 & 1 & 0 & 50 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 30 \\ \hline -4 & -1 & 0 & 0 & 0 \end{array}\right) \tag{4.38.8}$$

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & b \\ 0 & \frac{2}{3} & 1 & \frac{-1}{3} & 20 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 30 \\ \hline 0 & \frac{1}{3} & 0 & \frac{4}{3} & 120 \end{array}\right) \tag{4.38.9}$$

Note that x has replaced in the basis column s_2 and the improved solution

$$(x, y, s_1, s_2) = (30, 0, 20, 0) \tag{4.38.10}$$

maximizes Z to value

$$Z = 4(30) + 3(0) \tag{4.38.11}$$

$$Z = 120 \tag{4.38.12}$$

The given problem can be expressed in the form of matrix inequality as:

$$\max_{\{x\}} \mathbf{c}^T \mathbf{x} \tag{4.38.13}$$

$$s.t \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{4.38.14}$$

$$\mathbf{x} \geq 0 \tag{4.38.15}$$

$$\tag{4.38.16}$$

where

$$\mathbf{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{4.38.17}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \tag{4.38.18}$$

$$\mathbf{b} = \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{4.38.19}$$

can be solved using Python. The plot obtained from python is attached below:

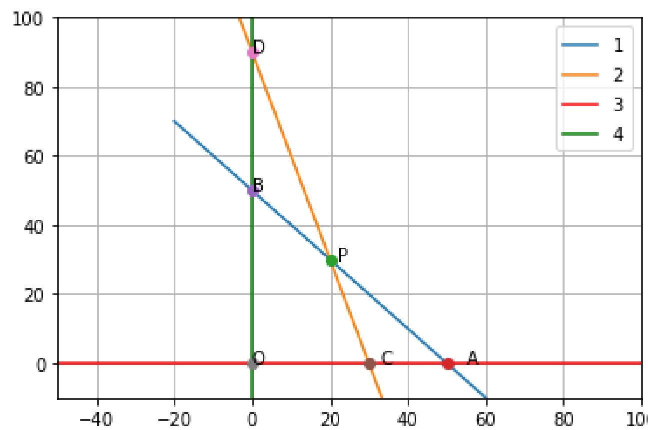

Fig. 4.38: Plot obtained from python code

5 Gradient Descent

1. Find the maximum and minimum values, if any, of the following functions given by
   a) $f(x) = (2x - 1)^2 + 3$
   b) $f(x) = 9x^2 + 12x + 2$
   c) $f(x) = -(x - 1)^2 + 10$
   d) $f(x) = x^2$.
2. Find the absolute maximum and absoute minimum value of the following functions in the given intervals
   a) $f(x) = 4x - \frac{1}{2}x^2, x \in \left(-2, \frac{9}{2}\right)$
   b) $f(x) = (x - 1)^2 + 3, x \in (-3, 1)$

3. Find the maximum profit that a company can make, if the profit function is given by
$$p(x) = 41 - 72x - 18x^2 \tag{5.0.3.1}$$
4. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.
5. Find two numbers whose sum is 24 and whose product is as large as possible.
6. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.
7. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.
8. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.
9. Find the shortest distance of the point $\begin{pmatrix} 0 \\ c \end{pmatrix}$ from the parabola $y = x^2$, where $\frac{1}{2} \leq c \leq 5$.
10. Find the maximum area of an isosceles triangle inscribed in the ellipse
$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{5.0.10.1}$$
with its vertex at one end of the major axis.