Geometry through Linear Algebra

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***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on William Dresden's book on solid geometry.**

1 PLANES AND LINES

1.1 Distance from a plane to a point

1.1.1. Solve the following

a) Find the foot of perpendicular from the point $\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ on the plane $\begin{pmatrix} 3 & 2 & -6 \end{pmatrix}\mathbf{x} = 2$.

Solution: Consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^Tn} = 0 \tag{1.1.1.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 0 \tag{1.1.1.2}$$

$$\implies 3a + 2b - 6c = 0 \tag{1.1.1.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.1.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \tag{1.1.1.5}$$

Solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.1.6}$$

Substituting (1.1.1.4) and (1.1.1.5) in (1.1.1.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \tag{1.1.1.7}$$

Solving (1.1.1.7) using Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U\Sigma V}^T \tag{1.1.1.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. We

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have,

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \tag{1.1.1.9}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36} \end{pmatrix} \tag{1.1.1.10}$$

Substituting (1.1.1.8) in (1.1.1.6),

$$\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.1.1.11}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{1.1.1.12}$$

Where $\mathbf{\Sigma}^{-1}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$ and is obtained by inversing only non-zero elements in $\mathbf{\Sigma}$

Calculating eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.13}$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda \end{vmatrix} = 0 \tag{1.1.1.14}$$

$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \tag{1.1.1.15}$$

From the characteristic equation (1.1.1.15), the eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{1.1.1.16}$$

The eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{13} \\ \frac{12}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{3} \\ 1 \end{pmatrix} \tag{1.1.1.17}$$

Normalizing the eigen vectors in equation (1.1.1.17)

$$\mathbf{u_1} = \begin{pmatrix} \frac{18}{7\sqrt{13}} \\ \frac{12}{7\sqrt{13}} \\ \frac{\sqrt{13}}{7} \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{-7}{12} \\ \frac{-7}{18} \\ \frac{7}{6} \end{pmatrix} \tag{1.1.1.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \tag{1.1.1.19}$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.1.20}$$

Calculating eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.21}$$

$$\implies \begin{vmatrix} \frac{5}{4}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9}-\lambda \end{vmatrix} = 0 \tag{1.1.1.22}$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \tag{1.1.1.23}$$

From the characteristic equation, the eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \tag{1.1.1.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.1.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.27}$$

From (1.1.1.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{18}{7\sqrt{13}} & \frac{-2}{\sqrt{13}} & \frac{-7}{12} \\ \frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{-7}{18} \\ \frac{\sqrt{13}}{7} & 0 & \frac{7}{6} \end{pmatrix} \begin{pmatrix} \frac{49}{36} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.1.28}$$

And, the Moore-Penrose Pseudo inverse of $\mathbf{\Sigma}$ is given by,

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.1.29}$$

From (1.1.1.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-17}{7\sqrt{13}} \\ \frac{12}{\sqrt{13}} \\ \frac{77}{36} \end{pmatrix} \tag{1.1.1.30}$$

$$\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-102}{49\sqrt{13}} \\ \frac{12}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.31}$$

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b} \quad = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \tag{1.1.1.32}$$

Now we verify the solution (1.1.1.32) using,

$$\mathbf{Mx} = \mathbf{b} \implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{1.1.1.33}$$

On evaluating the R.H.S in (1.1.1.33) we get,

$$\mathbf{M}^T\mathbf{Mx} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \tag{1.1.1.34}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-5}{2} \\ \frac{7}{3} \end{pmatrix} \tag{1.1.1.35}$$

On solving the augmented matrix of (1.1.1.35) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & \frac{-5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xleftrightarrow{R_1=\frac{4R_1}{5}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \tag{1.1.1.36}$$

$$\xleftrightarrow{R_2=R_2-\frac{R_1}{6}} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{49}{45} & \frac{8}{3} \end{pmatrix} \tag{1.1.1.37}$$

$$\xleftrightarrow{R_2=\frac{45}{49}R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \tag{1.1.1.38}$$

$$\xleftrightarrow{R_1=R_1-\frac{2R_2}{15}} \begin{pmatrix} 1 & 0 & \frac{-114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \tag{1.1.1.39}$$

From equation (1.1.1.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{-114}{49} \\ \frac{120}{49} \end{pmatrix} \tag{1.1.1.40}$$

From the equations (1.1.1.32) and (1.1.1.40), the solution $\mathbf{x}$ is verified.

b) Find the foot of perpendicular from point $B = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ to the plane $\begin{pmatrix} 2 & 3 & -4 \end{pmatrix}\mathbf{x} = -5$.

Solution: Let us consider orthogonal vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ to the given normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then,

$$\mathbf{m}^T\mathbf{n} = 0 \tag{1.1.1.41}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 0 \tag{1.1.1.42}$$

$$\implies 2a + 3b - 4c = 0 \tag{1.1.1.43}$$

Let $a = 1, \ b = 0$, so that

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.1.44}$$

and $a = 0, \ b = 1$, so that

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \tag{1.1.1.45}$$

We, now, solve the equation

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.1.46}$$

which, upon substitution, becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.1.1.47}$$

Any $m \times n$ matrix $\mathbf{M}$ can be factorized in SVD form as

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.1.48}$$

where $\mathbf{U}$ and $\mathbf{V}$ are matrices of eigen vectors which are orthogonal. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is the diagonal matrix of singular values of $\mathbf{M}$ of the eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \tag{1.1.1.49}$$

Putting (1.1.1.48) into (1.1.1.46), we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.1.50}$$

$$\implies \mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} \tag{1.1.1.51}$$

where $\mathbf{S}_+$ is the Moore-Penrose Pseudoinverse of $\mathbf{S}$.

The eigenvalues of $\mathbf{M}^T\mathbf{M}$:

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.1.52}$$

$$\implies \begin{vmatrix} \frac{10}{8} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{vmatrix} = 0 \tag{1.1.1.53}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{116}{64} = 0 \tag{1.1.1.54}$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \tag{1.1.1.55}$$

$$\lambda_2 = 1 \tag{1.1.1.56}$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector $\mathbf{v_1}$ can be calculated using row reduction as :

$$\begin{pmatrix} -\frac{9}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{16}{9}R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{3}{8} & -\frac{4}{16} \end{pmatrix} \tag{1.1.1.57}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \tag{1.1.1.58}$$

Hence,

$$\mathbf{v_1} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.59}$$

Similarly, for $\lambda_2 = 1$,

$$\mathbf{v_2} = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.60}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.1.61}$$

Now,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.1.62}$$

Now, calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$

$$\begin{vmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \lambda \end{vmatrix} = 0 \tag{1.1.1.63}$$

So, the eigenvalues are

$$\lambda_1 = \frac{29}{16} \tag{1.1.1.64}$$

$$\lambda_2 = 1 \tag{1.1.1.65}$$

$$\lambda_3 = 0 \tag{1.1.1.66}$$

For $\lambda_1 = \frac{29}{16}$, the eigen vector can be computed as:

$$\begin{pmatrix} 1 - \frac{29}{16} & 0 & \frac{1}{2} \\ 0 & 1 - \frac{29}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - \frac{29}{16} \end{pmatrix} \tag{1.1.1.67}$$

$$\leftrightarrow \begin{pmatrix} -\frac{13}{16} & 0 & \frac{1}{2} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \tag{1.1.1.68}$$

$$\xleftrightarrow{R_1 \leftarrow -\frac{16}{13}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -1 \end{pmatrix} \tag{1.1.1.69}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & -\frac{13}{16} & \frac{3}{4} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \tag{1.1.1.70}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{16}{13}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & \frac{3}{4} & -\frac{9}{13} \end{pmatrix} \tag{1.1.1.71}$$

$$\xleftrightarrow{R_2 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & -\frac{8}{3} \\ 0 & 1 & -\frac{12}{13} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.72}$$

Hence, the eigen vector $\mathbf{u}_1$:

$$\mathbf{u_1} = \begin{pmatrix} \frac{8}{\sqrt{377}} \\ \frac{12}{\sqrt{377}} \\ \frac{13}{\sqrt{377}} \end{pmatrix} \tag{1.1.1.73}$$

For $\lambda_2 = 1$, the eigen vector is:

$$\begin{pmatrix} 1-1 & 0 & \frac{1}{2} \\ 0 & 1-1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} - 1 \end{pmatrix} \tag{1.1.1.74}$$

$$\leftrightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{3}{16} \end{pmatrix} \tag{1.1.1.75}$$

Hence, the eigen vector $\mathbf{u}_2$:

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{1.1.1.76}$$

Similarly, for $\lambda_3 = 0$, the eigen vector is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.1.77}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1 - \frac{3}{4}R_2} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.78}$$

Hence, the eigen vector $\mathbf{u}_3$:

$$\mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.1.79}$$

So, the orthonormal matrix $\mathbf{U}$ of eigen vectors is:

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & -\frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.1.80}$$

The matrix of singular values of $\mathbf{M}$ is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.1.81}$$

The Moore-Penrose pseudoinverse of $\mathbf{S}$ is computed as

$$\mathbf{S}_+ = (\mathbf{S}\mathbf{S}^T)^{-1}\mathbf{S}^T \tag{1.1.1.82}$$

$$= \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.1.83}$$

To solve for $\mathbf{x}$ in (1.1.1.51), noting that $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \\ 0 \end{pmatrix} \tag{1.1.1.84}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{1.1.1.85}$$

Thus, the foot of perpendicular is:

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{13} \end{pmatrix} \tag{1.1.1.86}$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.87}$$

The solution can be verified using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.1.88}$$

The LHS gives

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.89}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.90}$$

Now, finding $\mathbf{x}$ from

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.91}$$

Solving the augmented matrix, we get

$$\begin{pmatrix} \frac{10}{8} & \frac{3}{8} & -3 \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -\frac{3}{10}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ \frac{3}{8} & \frac{25}{16} & 2 \end{pmatrix} \tag{1.1.1.92}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & \frac{29}{20} & \frac{58}{20} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & -\frac{24}{10} \\ 0 & 1 & 2 \end{pmatrix} \tag{1.1.1.93}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.1.1.94}$$

Hence, the solution is given by

$$\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.1.1.95}$$

Comparing the results in Eq.(1.1.1.87) and (1.1.1.95), it is concluded that the solution is verified.

1.1.2. Solve the following

a) Find the foot of the perpendicular from,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.1}$$

to the plane,

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.2.2}$$

Solution: The equation of plane is given as,

$$\mathbf{n}^T\mathbf{x} = c \tag{1.1.2.3}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.4}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.2.5}$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \tag{1.1.2.6}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.7}$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.8}$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.9}$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.2.10}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.11}$$

$$\text{and, } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.12}$$

To solve (1.1.2.10) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.2.13}$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.13) to (1.1.2.10),

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.2.14}$$

$$\implies \mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} \tag{1.1.2.15}$$

where, $\mathbf{S}_+$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.16}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left| \mathbf{M}^T\mathbf{M} - \lambda\mathbf{I} \right| = 0 \tag{1.1.2.17}$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \tag{1.1.2.18}$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \tag{1.1.2.19}$$

$$\therefore \lambda_1 = 14 \tag{1.1.2.20}$$

$$\lambda_2 = 1 \tag{1.1.2.21}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.2.22}$$

$$\mathbf{v_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.2.23}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{1.1.2.24}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.1.2.25}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \tag{1.1.2.26}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.2.27}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.28}$$

Eigenvalues corresponding to $\mathbf{MM}^T$ is,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \quad (1.1.2.29)$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \quad (1.1.2.30)$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (1.1.2.31)$$

$$\implies -\lambda(\lambda-1)(\lambda-14) = 0 \quad (1.1.2.32)$$

$$\therefore \lambda_3 = 14 \quad (1.1.2.33)$$

$$\lambda_4 = 1 \quad (1.1.2.34)$$

$$\lambda_5 = 0 \quad (1.1.2.35)$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad (1.1.2.36)$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad (1.1.2.37)$$

$$\mathbf{v_5} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (1.1.2.38)$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (1.1.2.39)$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (1.1.2.40)$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.41)$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (1.1.2.42)$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.1.2.43)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.44)$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}}\begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}\begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (1.1.2.45)$$

Now substituting the values of (1.1.2.26), (1.1.2.44), (1.1.2.42) and (1.1.2.12) in (1.1.2.15),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \quad (1.1.2.46)$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.47)$$

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.1.2.48)$$

$$\implies \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}}\begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \quad (1.1.2.49)$$

$\therefore$ from equation (1.1.2.15),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}}\begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix}\begin{pmatrix} \frac{-29}{\sqrt{182}} \\ \frac{11}{\sqrt{13}} \\ \frac{-13}{\sqrt{14}} \end{pmatrix} \quad (1.1.2.50)$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \tag{1.1.2.51}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.52}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \tag{1.1.2.53}$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \tag{1.1.2.54}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & 7 \\ -6 & 10 & -5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ -6 & 10 & -5 \end{pmatrix} \tag{1.1.2.55}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & \frac{14}{5} & \frac{17}{5} \end{pmatrix} \tag{1.1.2.56}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14}R_2} \begin{pmatrix} 1 & -\frac{6}{5} & \frac{7}{5} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \tag{1.1.2.57}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5}R_2} \begin{pmatrix} 1 & 0 & \frac{20}{7} \\ 0 & 1 & \frac{17}{14} \end{pmatrix} \tag{1.1.2.58}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{20}{7} \\ \frac{17}{14} \end{pmatrix} \tag{1.1.2.59}$$

Hence from equations (1.1.2.51) and (1.1.2.59) we conclude that the solution is verified.

b) Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.60}$$

to the plane,

$$\tag{1.1.2.61}$$

$$\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.1.2.62}$$

Solution: The equation of plane is give

$$\mathbf{n}^T\mathbf{x} = c \tag{1.1.2.63}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.64}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.2.65}$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \tag{1.1.2.66}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.67}$$

Let, a=0 and b=1 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.68}$$

Let, a=1 and b=0,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.69}$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.1.2.70}$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.71}$$

To solve (1.1.2.70) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.1.2.72}$$

substituting the value of $\mathbf{M}$ from equation (1.1.2.72) to (1.1.2.70),

$$\implies \mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.1.2.73}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} \tag{1.1.2.74}$$

where, $\mathbf{S}_+$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{M}\mathbf{M}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.75}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.76}$$

$$\implies \begin{pmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{pmatrix} \tag{1.1.2.77}$$

$$\implies (\lambda - 14)(\lambda - 1) = 0 \tag{1.1.2.78}$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \tag{1.1.2.79}$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.2.80}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{1.1.2.81}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.1.2.82}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \tag{1.1.2.83}$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.1.2.84}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.85}$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.86}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{pmatrix} \tag{1.1.2.87}$$

$$\implies -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \tag{1.1.2.88}$$

$$\implies -\lambda(\lambda - 1)(\lambda - 14) = 0 \tag{1.1.2.89}$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1 \tag{1.1.2.90}$$

$$\lambda_5 = 0 \tag{1.1.2.91}$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_5} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.92}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \tag{1.1.2.93}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \tag{1.1.2.94}$$

$$\mathbf{v_5} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \tag{1.1.2.95}$$

$$\implies \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \tag{1.1.2.96}$$

Now $\mathbf{S}$ corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.97}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.98}$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{13}}\begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}\begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \tag{1.1.2.99}$$

Now substituting the values of (1.1.2.83), (1.1.2.98), (1.1.2.96) and (1.1.2.71) in

(1.1.2.74),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.100}$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \tag{1.1.2.101}$$

$$\mathbf{V}\mathbf{S}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.102}$$

$$\implies \mathbf{V}\mathbf{S}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \tag{1.1.2.103}$$

$\therefore$ from equation (1.1.2.74),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3\sqrt{14} & 0 \\ 3 & 2\sqrt{14} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \tag{1.1.2.104}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.2.105}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.106}$$

$$\implies \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \tag{1.1.2.107}$$

$$\implies \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{1.1.2.108}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix} \tag{1.1.2.109}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix} \tag{1.1.2.110}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5}{14} R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.2.111}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{5} R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.1.2.112}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.2.113}$$

Hence from equations (1.1.2.105) and (1.1.2.113) we conclude that the solution is verified.

c) Find the foot of the perpendicular from $\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$ on the plane $\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \mathbf{x} = 0$

Solution: Let orthogonal vectors be $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.2.114}$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \tag{1.1.2.115}$$

$$\implies -5a + b + 3c = 0 \tag{1.1.2.116}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.117}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.118}$$

From (1.1.2.117) and (1.1.2.118),

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.119}$$

Now solving the equation

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.2.120}$$

Substituting the given point and (1.1.2.119) in (1.1.2.120)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.121}$$

Using the Singular value decomposition to solve (1.1.2.121) as follows,

$$\mathbf{M} = \mathbf{U\Sigma V}^T \tag{1.1.2.122}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{\Sigma}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.123}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.124}$$

Substituting (1.1.2.122) in (1.1.2.120)

$$\mathbf{U\Sigma V}^T\mathbf{x} = \mathbf{b} \tag{1.1.2.125}$$

$$\mathbf{x} = \mathbf{V\Sigma^{-1}U^Tb} \tag{1.1.2.126}$$

where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$.
Now finding the eigen values of $\mathbf{MM}^T$

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.2.127}$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0 \tag{1.1.2.128}$$

$$\implies \lambda^3 - 15\lambda^2 + 14\lambda = 0 \tag{1.1.2.129}$$

Hence eigen values of $\mathbf{MM}^T$,

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad \lambda_3 = 0 \tag{1.1.2.130}$$

Therefore eigen vectors of $\mathbf{MM}^T$,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.131}$$

Normalizing the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.132}$$

Hence from the above we get,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{-3}{\sqrt{14}} \\ 0 & \frac{13}{\sqrt{182}} & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.133}$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 14 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.134}$$

Now calculating eigen values of $\mathbf{M}^T\mathbf{M}$

$$\left|\mathbf{M}^T\mathbf{M} - \lambda I\right| = 0 \tag{1.1.2.135}$$

$$\begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0 \tag{1.1.2.136}$$

$$\implies \lambda^2 - 15\lambda + 14 = 0 \tag{1.1.2.137}$$

hence the eigen values of $\mathbf{M}^T\mathbf{M}$

$$\lambda_1 = 1 \quad \lambda_2 = 14 \tag{1.1.2.138}$$

Therefore eigen vectors $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{1.1.2.139}$$

Normalizing the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.140}$$

Hence $\mathbf{V}$ is given as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.141}$$

Moore Pseudo inverse of Σ is,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \tag{1.1.2.142}$$

Substituting (1.1.2.133), (1.1.2.141) and (1.1.2.142) in (1.1.2.126),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{182}} & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{1}} \end{pmatrix} \tag{1.1.2.143}$$

$$\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} \tag{1.1.2.144}$$

$$\mathbf{V}\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{1.1.2.145}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{1.1.2.146}$$

Now verifying (1.1.2.146) using (1.1.2.120)

$$\mathbf{Mx} = \mathbf{b} \implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.147}$$

Substituting (1.1.2.119), (1.1.2.123) and given point in (1.1.2.147)

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 \\ 10 \end{pmatrix} \tag{1.1.2.148}$$

$$\tag{1.1.2.149}$$

Solving the augmented matrix.

$$\begin{pmatrix} 5 & -6 & -11 \\ -6 & 10 & 10 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ -6 & 10 & 10 \end{pmatrix} \tag{1.1.2.150}$$

$$\xleftrightarrow{R_2=R_2+6R_1} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & \frac{14}{5} & \frac{-16}{5} \end{pmatrix} \tag{1.1.2.151}$$

$$\xleftrightarrow{R_2=\frac{5R_2}{14}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \tag{1.1.2.152}$$

$$\xleftrightarrow{R_1=R_1+\frac{6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{-25}{7} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \tag{1.1.2.153}$$

From (1.1.2.153) we get,

$$\mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{1.1.2.154}$$

Hence from (1.1.2.146) and (1.1.2.154) the $\mathbf{x}$ is verified

d) Find the coordinates of foot of perpendicular from $\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ to the plane

$$2x - 3y + z = 0 \tag{1.1.2.155}$$

using SVD

Solution: First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given plane $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^Tn} = 0$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$\implies 2a - 3b + c = 0 \tag{1.1.2.156}$$

By substituting $a = 1; b = 0$ in (1.1.2.156),

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{1.1.2.157}$$

By substituting $a = 0; b = 1$ in (1.1.2.156),

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{1.1.2.158}$$

Now $\mathbf{M}$ can be written as,

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{1.1.2.159}$$

such that solving $\mathbf{Mx} = \mathbf{b}$ gives the required solution.

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \tag{1.1.2.160}$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.2.161}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{1.1.2.162}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{1.1.2.163}$$

From (1.1.2.160) and (1.1.2.161),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{1.1.2.164}$$

Where $\mathbf{S}_{+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^3 + 15\lambda^2 - 14\lambda = 0$$

Hence eigenvalues of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = 14; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{1.1.2.165}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{1.1.2.166}$$

Normalizing the above eigenvectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.167}$$

From (1.1.2.167) we obtain $\mathbf{U}$ as,

$$\mathbf{U} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{1.1.2.168}$$

Using values from (1.1.2.165),

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.169}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$

$$\implies \begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 - 15\lambda + 14 = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 14; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Normalizing the above eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.170}$$

From(1.1.2.170) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.2.171}$$

From (1.1.2.161) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.2.172}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_{+} = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.173}$$

From (1.1.2.164),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{2}}{\sqrt{91}} \\ \frac{3}{\sqrt{13}} \\ \frac{2\sqrt{2}}{7} \end{pmatrix}$$

$$\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.1.2.174}$$

To verify the solution obtained from (1.1.2.174),

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.175}$$

Substituting the values from (1.1.2.162) in (1.1.2.175),

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

Converting the above equation into augmented form and solving for $\mathbf{x}$,

$$\begin{pmatrix} 5 & -6 & -3 \\ -6 & 10 & 6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{5R_2+6R_1}{14}} \begin{pmatrix} 5 & -6 & -3 \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1+6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \tag{1.1.2.176}$$

From (1.1.2.176) it can be observed that,

$$\mathbf{x} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.1.2.177}$$

a) Determine the distance from the Y-axis to the plane $5x - 2z - 3 = 0$
Solution: Equation of plane can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \tag{1.1.2.178}$$

Rewriting given equation of plane in (1.1.2.178) form

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \tag{1.1.2.179}$$

where : $\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 3$
We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1 \mathbf{q} + \lambda_2 \mathbf{r} \tag{1.1.2.180}$$

Here p is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Find two vectors that are $\perp$ to $\mathbf{n}$

$$\begin{pmatrix} 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{1.1.2.181}$$

Put $a = 0$ and $b = 1$ in (1.1.2.180), $\implies c = 0$
Put $a = 1$ and $b = 0$ in (1.1.2.180), $\implies c = \frac{5}{2}$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
Let us find point $\mathbf{p}$ on the plane. Put $x = 1, y = 0$ in (1.1.2.179), we get $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
Since given plane is parallel to y-axis, we can use any point P on y-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.2.182}$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (1.1.2.181) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{1.1.2.183}$$

Equation $\mathbf{P}$ and $\mathbf{Q}$, and computing pseudo inverse using SVD should give the value of λ_1 and λ_2 (since plane and y-axis never intersect pseudo inverse should give the points which are closest)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.2.184}$$

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \tag{1.1.2.185}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \tag{1.1.2.186}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.1.2.187}$$

$$\mathbf{x} = \mathbf{M}^{+}\mathbf{b} \tag{1.1.2.188}$$

where $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{5}{2} & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

Diagonalize $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix}1 & 0\\0 & 1\\\frac{5}{2} & 0\end{pmatrix}\begin{pmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & 0\\\frac{5}{2} & 0 & \frac{25}{4}\end{pmatrix} \quad (1.1.2.189)$$

$$= \begin{pmatrix}0 & \frac{2}{5} & -\frac{5}{2}\\1 & 0 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{29}{4} & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\\frac{2}{5} & 0 & 1\\-\frac{5}{2} & 0 & 1\end{pmatrix} \quad (1.1.2.190)$$

$$= \mathbf{U}\Sigma^T\Sigma\mathbf{U}^T \quad (1.1.2.191)$$

Verify (1.1.2.190) from,

```
codes/diagonalize1.py
```

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\\frac{5}{2} & 0\end{pmatrix} = \begin{pmatrix}\frac{29}{4} & 0\\0 & 1\end{pmatrix} \quad (1.1.2.192)$$

$$= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \frac{29}{4}\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \quad (1.1.2.193)$$

$$= \mathbf{V}\Sigma^T\Sigma\mathbf{V}^T \quad (1.1.2.194)$$

Verify (1.1.2.193) from,

```
codes/diagonalize2.py
```

Compute SVD of $\mathbf{M}$ from (1.1.2.190) and (1.1.2.195),

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T \quad (1.1.2.195)$$

$$\begin{pmatrix}1 & 0\\0 & 1\\\frac{5}{2} & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{2}{5} & -\frac{5}{2}\\1 & 0 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{29}}{2} & 0\\0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad (1.1.2.196)$$

$$\mathbf{M}^+ = \mathbf{V}\Sigma^T\mathbf{U}^T \quad (1.1.2.197)$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{29}}{2} & 0 & 0\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\\frac{2}{5} & 0 & 1\\-\frac{5}{2} & 0 & 1\end{pmatrix} \quad (1.1.2.198)$$

$$= \begin{pmatrix}\frac{4}{29} & 0 & \frac{10}{29}\\0 & 1 & 0\end{pmatrix} \quad (1.1.2.199)$$

Verify (1.1.2.199) from,

```
codes/pseudo_inverse.py
```

Substitute (1.1.2.199) in (1.1.2.188),

$$\mathbf{x} = \begin{pmatrix}\frac{4}{29} & 0 & \frac{10}{29}\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\0\\-1\end{pmatrix} = \begin{pmatrix}-\frac{14}{29}\\0\end{pmatrix} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \quad (1.1.2.200)$$

Substituting λ_1, λ_2 in (1.1.2.183)

$$\mathbf{Q} = \begin{pmatrix}\frac{15}{29}\\0\\-\frac{6}{29}\end{pmatrix} \quad (1.1.2.201)$$

Distance between point $\mathbf{P}$ and $\mathbf{Q}$ is

$$\|\mathbf{P}-\mathbf{Q}\| = \sqrt{\left(\frac{15}{29}\right)^2 + 0 + \left(-\frac{6}{29}\right)^2} = \frac{3}{\sqrt{29}} \quad (1.1.2.202)$$

Hence, distance from y-axis to $5x-2z-3 = 0$ is $\frac{3}{\sqrt{29}}$.

Verifying solution to (1.1.2.187) by least squares method

$$\mathbf{M}^T(\mathbf{b}-\mathbf{M}\mathbf{x}) = 0 \quad (1.1.2.203)$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.1.2.204)$$

Substituting $\mathbf{M}, \mathbf{b}$ from (1.1.2.186) in (1.1.2.204)

$$\begin{pmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\\frac{5}{2} & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\0\\-1\end{pmatrix} \quad (1.1.2.205)$$

$$\begin{pmatrix}\frac{29}{4} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}-\frac{7}{2}\\0\end{pmatrix} \quad (1.1.2.206)$$

$$\implies \frac{29}{4}\lambda_1 = -\frac{7}{2} \quad (1.1.2.207)$$

$$\lambda_1 = -\frac{7}{2}\times\frac{4}{29} = -\frac{14}{29} \quad (1.1.2.208)$$

$$\text{and } \lambda_2 = 0 \quad (1.1.2.209)$$

$$\mathbf{x} = \begin{pmatrix}-\frac{14}{29}\\0\end{pmatrix} \quad (1.1.2.210)$$

Comparing (1.1.2.200) and (1.1.2.210) solution is verified.

b) Determine the distance from the Z-axis to the plane 5x - 12y - 8 = 0
Solution: Equation of plane can be ex-

pressed as

$$\mathbf{n}^T\mathbf{x} = c \tag{1.1.2.211}$$

Rewriting given equation of plane in (1.1.2.211) form

$$\begin{pmatrix} 5 & -12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8 \tag{1.1.2.212}$$

where the value of

$$\mathbf{n} = \begin{pmatrix} 5 \\ -12 \\ 0 \end{pmatrix} \tag{1.1.2.213}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.1.2.214}$$

$$c = 8 \tag{1.1.2.215}$$

We need to represent the equation of plane in parametric form,

$$\mathbf{Q} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \tag{1.1.2.216}$$

Here p is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Now, we need to find these two vectors $\mathbf{q}$ and $\mathbf{r}$ which are $\perp$ to $\mathbf{n}$

$$\begin{pmatrix} 5 & -12 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \implies 5a - 12b = 0 \tag{1.1.2.217}$$

Put $a = 0$ and $c = 1$ in (1.1.2.217), $\implies b = 0$

Put $a = 1$ and $c = 0$ in (1.1.2.217), $\implies b = \frac{5}{12}$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, z = 0$ in (1.1.2.212), we get $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Since given plane is parallel to Z-axis, we can use any point P on Z-axis to compute shortest distance.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.2.218}$$

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (1.1.2.217) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.2.219}$$

Computation of Pseudo Inverse using SVD in order to determine the value of λ_1 and λ_2 :

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.2.220}$$

$$\lambda_1\begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{1.1.2.221}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{1.1.2.222}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.1.2.223}$$

$$\implies \mathbf{x} = \mathbf{M}^{+}\mathbf{b} \tag{1.1.2.224}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.2.225}$$

$$\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{1.1.2.226}$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{1.1.2.227}$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.1.2.228}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of Singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{169}{144} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.2.229}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1.2.230}$$

As we know that,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b}$$
$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.1.2.231}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & \frac{5}{12} & 0 \\ \frac{5}{12} & \frac{25}{144}-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^3 - \frac{313}{144}\lambda^2 + \frac{169}{144}\lambda = 0$$

Hence eigenvalues of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{169}{144}; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{1.1.2.232}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ \frac{5}{12} \\ 0 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} -\frac{5}{12} \\ 1 \\ 0 \end{pmatrix} \tag{1.1.2.233}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.2.234}$$

Using values from (1.1.2.232),

$$\mathbf{S} = \begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.2.235}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} \frac{169}{144}-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - \frac{313}{144}\lambda + \frac{169}{144} = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{169}{144}; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.1.2.236}$$

From (1.1.2.236) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.2.237}$$

Now, we can compute *SVD* of $\mathbf{M}$:

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.2.238}$$
$$= \begin{pmatrix} 1 & 0 & -\frac{5}{12} \\ \frac{5}{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{13}{12} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.2.239}$$
$$\mathbf{M}^+ = \mathbf{VS}^T\mathbf{U}^T \tag{1.1.2.240}$$
$$= \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1.2.241}$$

Substitute (1.1.2.241) in (1.1.2.224),

$$\mathbf{x} = \begin{pmatrix} \frac{144}{169} & \frac{60}{169} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{1.1.2.242}$$
$$\mathbf{x} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{1.1.2.243}$$
$$\implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{204}{169} \\ 0 \end{pmatrix} \tag{1.1.2.244}$$

Substituting λ_1, λ_2 in (1.1.2.219)

$$\mathbf{Q} = \begin{pmatrix} -\frac{204}{169} \\ -\frac{85}{169} \\ 0 \end{pmatrix} \tag{1.1.2.245}$$

Distance between point $\mathbf{P}$ and $\mathbf{Q}$ is

$$\|\mathbf{P} - \mathbf{Q}\| = \sqrt{\left(-\frac{204}{169}\right)^2 + \left(-\frac{85}{169}\right)^2 + 0} \tag{1.1.2.246}$$
$$\|\mathbf{P} - \mathbf{Q}\| = \frac{17}{13} \tag{1.1.2.247}$$

Hence, the distance from the Z-axis to the plane $5x - 12y - 8 = 0$ is $\frac{17}{13}$. Now, we can verify the solution using Least Squares Method,

$$\mathbf{M}^T(\mathbf{b} - \mathbf{Mx}) = 0 \tag{1.1.2.248}$$
$$\implies \mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{1.1.2.249}$$

Substituting $\mathbf{M}, \mathbf{b}$ from (1.1.2.222) in

(1.1.2.249)

$$\begin{pmatrix}1 & 0\\ \frac{5}{12} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{5}{12} & 0\\ 0 & 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & \frac{5}{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1\\ -1\\ 0\end{pmatrix} \tag{1.1.2.250}$$

$$\begin{pmatrix}\frac{169}{144} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}-\frac{17}{12}\\ 0\end{pmatrix} \tag{1.1.2.251}$$

$$\implies \frac{169}{144}\lambda_1 = -\frac{17}{12} \tag{1.1.2.252}$$

$$\lambda_1 = -\frac{17}{12} \times \frac{144}{169} = -\frac{204}{169} \tag{1.1.2.253}$$

$$\text{and } \lambda_2 = 0 \tag{1.1.2.254}$$

$$\implies \mathbf{x} = \begin{pmatrix}-\frac{204}{169}\\ 0\end{pmatrix} \tag{1.1.2.255}$$

Comparing (1.1.2.242) and (1.1.2.255) solution is verified.

1.1.3. Find the foot of the perpendicular using svd drawn from $\begin{pmatrix}-3\\ 1\\ 2\end{pmatrix}$ to the plane

$$\begin{pmatrix}2 & -1 & -2\end{pmatrix}\mathbf{x} + 4 = 0 \tag{1.1.3.1}$$

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\ b\\ c\end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.1.3.2}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}2\\ -1\\ -2\end{pmatrix} = 0 \tag{1.1.3.3}$$

$$\implies 2a - b - 2c = 0 \tag{1.1.3.4}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\ 0\\ 1\end{pmatrix} \tag{1.1.3.5}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\ 1\\ -\frac{1}{2}\end{pmatrix} \tag{1.1.3.6}$$

Let us solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.3.7}$$

Substituting (1.1.3.5) and (1.1.3.6) in (1.1.3.7),

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & -\frac{1}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}-3\\ 1\\ 2\end{pmatrix} \tag{1.1.3.8}$$

To solve (1.1.3.8), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.3.9}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}2 & \frac{-1}{2}\\ \frac{-1}{2} & \frac{5}{4}\end{pmatrix} \tag{1.1.3.10}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -\frac{1}{2}\\ 1 & -\frac{1}{2} & \frac{5}{4}\end{pmatrix} \tag{1.1.3.11}$$

Substituting (1.1.3.9) in (1.1.3.7),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.3.12}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.1.3.13}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.
Let us calculate eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.3.14}$$

$$\implies \begin{vmatrix}1-\lambda & 0 & 1\\ 0 & 1-\lambda & -\frac{1}{2}\\ 1 & -\frac{1}{2} & \frac{5}{4}-\lambda\end{vmatrix} = 0 \tag{1.1.3.15}$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \tag{1.1.3.16}$$

From equation (1.1.3.16) eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{1.1.3.17}$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix}-\frac{4}{5}\\ \frac{2}{5}\\ -1\end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{2}\\ -1\\ 0\end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix}-1\\ \frac{1}{2}\\ 1\end{pmatrix} \tag{1.1.3.18}$$

Normalizing the eigen vectors in equation

(1.1.3.18)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ -\frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.1.3.19}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \tag{1.1.3.20}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.3.21}$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.3.22}$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{1.1.3.23}$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{1.1.3.24}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \tag{1.1.3.25}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \tag{1.1.3.26}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \tag{1.1.3.27}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \tag{1.1.3.28}$$

From (1.1.3.7), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ -\frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}^T \tag{1.1.3.29}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.3.30}$$

From (1.1.3.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{4}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{11}{3} \end{pmatrix} \tag{1.1.3.31}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{8}{9\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.1.3.32}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \tag{1.1.3.33}$$

Verifying the solution of (1.1.3.33) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.3.34}$$

Evaluating the R.H.S in (1.1.3.34) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.1.3.35}$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.1.3.36}$$

Solving the augmented matrix of (1.1.3.36) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} & 0 \end{pmatrix} \tag{1.1.3.37}$$

$$\xleftrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{9}{8} & -\frac{1}{4} \end{pmatrix} \tag{1.1.3.38}$$

$$\xleftrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \tag{1.1.3.39}$$

$$\xleftrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & -\frac{5}{9} \\ 0 & 1 & -\frac{2}{9} \end{pmatrix} \tag{1.1.3.40}$$

From equation (1.1.3.40), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{2}{9} \end{pmatrix} \tag{1.1.3.41}$$

Comparing results of $\mathbf{x}$ from (1.1.3.33) and (1.1.3.41), we can say that the solution is verified.

1.1.4. Find the foot of the perpendicular from

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.1.4.1}$$

to the given plane

$$2x + 3y - 4z + 5 = 0 \tag{1.1.4.2}$$

Solution: The given equation of plane can be represented as

$$\begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \mathbf{x} = -5 \tag{1.1.4.3}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \tag{1.1.4.4}$$

We need to find two vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{1.1.4.5}$$

Put $a = 1$ and $b = 0$ in (1.1.4.5),we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.4.6}$$

Put $a = 0$ and $b = 1$ in (1.1.4.5),we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \tag{1.1.4.7}$$

Now, solving the equation

$$\mathbf{Mx} = \mathbf{b} \tag{1.1.4.8}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{1.1.4.9}$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.1.4.10}$$

Now, to solve equation (1.1.4.8), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.1.4.11}$$

Substituting the value of $\mathbf{M}$ from equation (1.1.4.11) to(1.1.4.8),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.1.4.12}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} \tag{1.1.4.13}$$

Where, $\mathbf{S_+}$ is the Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \tag{1.1.4.14}$$

Eigen values corresponding to $\mathbf{M}^T\mathbf{M}$ are given by,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.1.4.15}$$

$$\implies \left|\begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{pmatrix}\right| = 0 \tag{1.1.4.16}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{29}{16} = 0 \tag{1.1.4.17}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{29}{16} \tag{1.1.4.18}$$

$$\lambda_2 = 1 \tag{1.1.4.19}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \tag{1.1.4.20}$$

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \tag{1.1.4.21}$$

Normalizing the eigen vectors, we obtain $\mathbf{V}$ of (1.1.4.11) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.1.4.22}$$

$\mathbf{S}$ of the diagonal matrix of (1.1.4.11) is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.1.4.23}$$

Now, calculating eigen value of $\mathbf{MM}^T$,

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{1.1.4.24}$$

Eigen values corresponding to $\mathbf{MM}^T$ are given by

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.1.4.25}$$

$$\implies \left|\begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{pmatrix}\right| = 0 \tag{1.1.4.26}$$

$$\implies \lambda^3 - \frac{45}{16}\lambda^2 + \frac{29}{16}\lambda = 0 \tag{1.1.4.27}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_3 = \frac{29}{16} \tag{1.1.4.28}$$

$$\lambda_4 = 1 \tag{1.1.4.29}$$

$$\lambda_5 = 0 \tag{1.1.4.30}$$

Hence we obtain $\mathbf{U}$ of (1.1.4.11) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{29} \\ \sqrt{\frac{13}{29}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \tag{1.1.4.31}$$

Finally from (1.1.4.11) we get the Singular Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{29} \\ \sqrt{\frac{13}{29}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.1.4.32}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.1.4.33}$$

Substituting the values of (1.1.4.31),(1.1.4.22),(1.1.4.33) in (1.1.4.13) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \\ 0 \end{pmatrix} \tag{1.1.4.34}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -\sqrt{13} \end{pmatrix} \tag{1.1.4.35}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.4.36}$$

Verifying the solution of (1.1.4.36) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.1.4.37}$$

Evaluating the R.H.S in (1.1.4.37) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.4.38}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.4.39}$$

The augmented matrix of (1.1.4.39) is,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & 3 \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \tag{1.1.4.40}$$

Solving the augmented matrix into Row reduced echelon form of (1.1.4.40) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & 3 \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{8} & \frac{25}{16} & -2 \end{pmatrix} \tag{1.1.4.41}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ 0 & \frac{29}{20} & -\frac{29}{10} \end{pmatrix} \tag{1.1.4.42}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{5} \\ 0 & 1 & -2 \end{pmatrix} \tag{1.1.4.43}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \tag{1.1.4.44}$$

Therefore,

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.1.4.45}$$

Comparing results of $\mathbf{x}$ from (1.1.4.36) and (1.1.4.45) we conclude that the solution is verified.