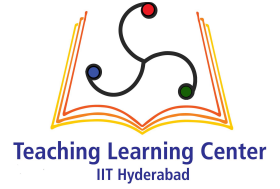




Coordinate Geometry Exercises



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Abstract—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.

1 CONICS

1.1. Find the area of the region enclosed between the two circles: $\mathbf{x}^T \mathbf{x} = 4$ and $\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.1.1)$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (1.1.2)$$

$$\mathbf{x}^T \mathbf{x} - 4 = 0 \quad (1.1.3)$$

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.1.4)$$

$$f_1 = -4 \quad (1.1.5)$$

$$\mathbf{O}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.1.6)$$

Taking equation of the second circle to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\|^2 = 2^2 \quad (1.1.7)$$

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}_2^T \mathbf{x} = 0 \quad (1.1.8)$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.1.9)$$

$$f_2 = 0 \quad (1.1.10)$$

$$\mathbf{O}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (1.1.11)$$

Now, Subtracting equation (1.1.8) from (1.1.3)
We get,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{u}_2^T \mathbf{x} + f_1 - \mathbf{x}^T \mathbf{x} = 0 \quad (1.1.12)$$

$$2\mathbf{u}_2^T \mathbf{x} = -4 \quad (1.1.13)$$

$$\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} = -4 \quad (1.1.14)$$

Which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \quad (1.1.15)$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.16)$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (1.1.17)$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.1.18)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.19)$$

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Substituting (1.1.17) in (1.1.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (1.1.20)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 + f_1 = 0 \quad (1.1.21)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (1.1.22)$$

$$\mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) + f_1 = 0 \quad (1.1.23)$$

$$\|\mathbf{q}\|^2 + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (1.1.24)$$

$$\|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 + f_1 = 0 \quad (1.1.25)$$

$$\lambda(\lambda \|\mathbf{m}\|^2 + 2\mathbf{q}^T \mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \quad (1.1.26)$$

$$\lambda^2 \|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \quad (1.1.27)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.1.28)$$

$$\lambda^2 = 3 \quad (1.1.29)$$

$$\lambda = +\sqrt{3}, -\sqrt{3} \quad (1.1.30)$$

Substituting the value of λ in (1.1.17)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (1.1.31)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (1.1.32)$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (1.1.33)$$

Now finding the direction vector \mathbf{m}_{O_1A} , \mathbf{m}_{O_1B} , \mathbf{m}_{O_2A} and \mathbf{m}_{O_2B} .

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \quad (1.1.34)$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad (1.1.35)$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (1.1.36)$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (1.1.37)$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B} = \|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\| \cos \theta_1 \quad (1.1.38)$$

$$\frac{\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B}}{\|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\|} = \cos \theta_1 \quad (1.1.39)$$

$$\frac{-2}{4} = \cos \theta_1 \quad (1.1.40)$$

$$\frac{-1}{2} = \cos \theta_1 \quad (1.1.41)$$

$$\theta_1 = 120^\circ \quad (1.1.42)$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B} = \|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\| \cos \theta_2 \quad (1.1.43)$$

$$\frac{\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B}}{\|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\|} = \cos \theta_2 \quad (1.1.44)$$

$$\frac{-2}{4} = \cos \theta_2 \quad (1.1.45)$$

$$\frac{-1}{2} = \cos \theta_2 \quad (1.1.46)$$

$$\theta_2 = 120^\circ \quad (1.1.47)$$

Finding area of $\mathbf{O}_1\mathbf{AB}$ and $\mathbf{O}_2\mathbf{AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2\sqrt{3} \quad (1.1.48)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \quad (1.1.49)$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360} r^2 - \frac{1}{2} 2\sqrt{3} \quad (1.1.50)$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \quad (1.1.51)$$

Area of $\mathbf{O}_1\mathbf{AO}_2\mathbf{B}$

$$A_{O_1AO_2B} = \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} + \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \quad (1.1.52)$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \quad (1.1.53)$$

- 1.2. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre (h, k) is given by,

$$x^2 + y^2 + 2ux + 2vy + f = 0 \quad (1.2.1)$$

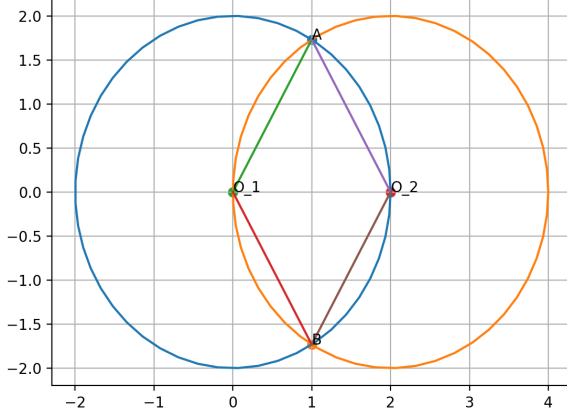


Fig. 1.1: Figure depicting intersection points of circle

where,

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (1.2.2)$$

The radius and centre are respectively given by,

$$r = 5 \quad (1.2.3)$$

$$\mathbf{c} = -\mathbf{u} = k\mathbf{e} \quad (1.2.4)$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.2.5)$$

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.2.6)$$

From the given data , we modify equation 1.2.1 as,

$$\mathbf{x}_1^T \mathbf{x}_1 + 2 \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \quad (1.2.7)$$

$$\|\mathbf{x}_1\|^2 + 2(k^2) + f = 0 \quad (1.2.8)$$

$$2k^2 + f = -\|\mathbf{x}_1\|^2 \quad (1.2.9)$$

Substituting \mathbf{u} in equation 1.2.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix} \begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \quad (1.2.10)$$

$$f = (k^2) - r^2 \quad (1.2.11)$$

$$k^2 - f = r^2 \quad (1.2.12)$$

From equations 1.2.9 and 1.2.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x}_1\|^2 \\ r^2 \end{pmatrix} \quad (1.2.13)$$

Here $\|\mathbf{x}_1\|$ is given by ,

$$\|\mathbf{x}_1\| = \sqrt{2^2 + 3^2} \quad (1.2.14)$$

$$\|\mathbf{x}_1\| = \sqrt{13} \quad (1.2.15)$$

Substituting equation 1.2.6, 1.2.3 in equation 1.2.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \quad (1.2.16)$$

The augmented matrix of 1.2.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \quad (1.2.17)$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array} \right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \quad (1.2.18)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array} \right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \quad (1.2.19)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array} \right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \quad (1.2.20)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array} \right) \xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array} \right) \quad (1.2.21)$$

Equation 1.2.16 can be rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \quad (1.2.22)$$

Expanding the above equation 1.2.22 we get ,

$$k^2 = 4 \quad (1.2.23)$$

$$k = \pm 2 \quad (1.2.24)$$

$$f = -21 \quad (1.2.25)$$

To get the centre substitute equation 1.2.24 in equation 1.2.4 To verify the above results we plot the circle with centre \mathbf{c} as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, From the above figure 1 it is clear that circle with centre $\mathbf{c} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ passes through the point \mathbf{x}_1

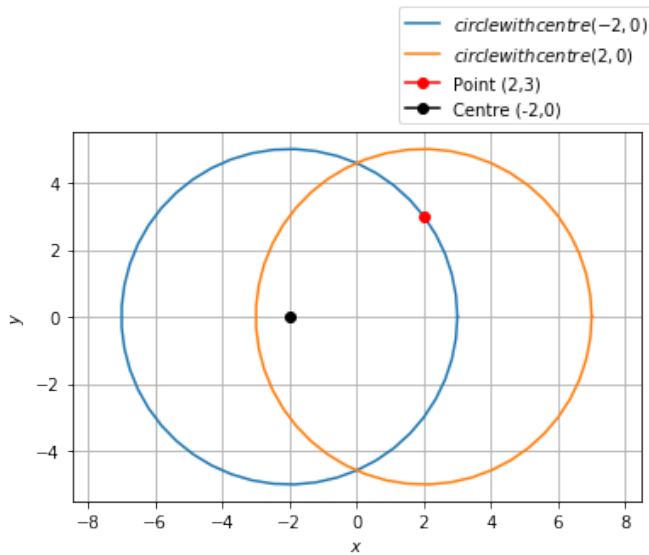


Fig. 1: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

Desired equation of circle is given by ,

$$c = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.2.26)$$

$$f = -21 \quad (1.2.27)$$

1.3. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.

1.4. Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Solution: he general equation of a circle is

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.4.1)$$

$$\text{If } r \text{ is radius, } f = \mathbf{u}^T \mathbf{u} - r^2 \quad (1.4.2)$$

$$\text{center } \mathbf{c} = -\mathbf{u} \quad (1.4.3)$$

Given centre is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (1.4.4)$$

$$\Rightarrow \mathbf{u} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (1.4.5)$$

Equation (1.4.1) becomes

$$\mathbf{x}^T \mathbf{x} + (-4 \ -4) \mathbf{x} + f = 0 \quad (1.4.6)$$

This passes through point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Substituting $\mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ in (1.4.6)

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} + (-4 \ -4) \begin{pmatrix} 4 \\ 5 \end{pmatrix} + f = 0 \quad (1.4.7)$$

$$\Rightarrow f = -5 \quad (1.4.8)$$

Also, radius can be determined as follows

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (1.4.9)$$

$$\Rightarrow -5 = \begin{pmatrix} -2 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} - r^2 \quad (1.4.10)$$

$$\Rightarrow -5 = 8 - r^2 \quad (1.4.11)$$

$$\Rightarrow r = \sqrt{13} \quad (1.4.12)$$

The equation of required circle is

$$\mathbf{x}^T \mathbf{x} + (-4 \ -4) \mathbf{x} - 5 = 0 \quad (1.4.13)$$

See Fig. 1

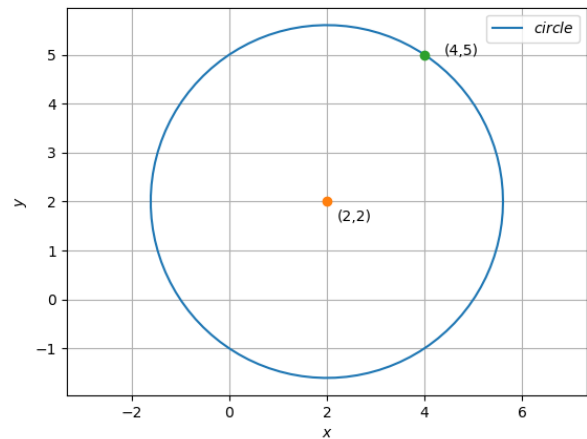


Fig. 1: plot showing the circle

1.5. Find the locus of all the unit vectors in the xy-plane.

1.6. Find the points on the curve $\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.

1.7. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 0$ and the circle $\mathbf{x}^T \mathbf{x} = 4$.

Solution: The equation of a circle can be expressed as,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \quad (1.7.1)$$

where \mathbf{c} is the center.

Comparing equation (1.7.1) with the circle equation given,

$$\mathbf{x}^T \mathbf{x} = 4 \quad (1.7.2)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad f = -4 \quad (1.7.3)$$

$$r = \sqrt{\mathbf{c}^T \mathbf{c} - f} = \sqrt{4} \quad (1.7.4)$$

$$\Rightarrow \boxed{r = 2} \quad (1.7.5)$$

From equation (1.7.5), the point at which circle touches x -axis is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The direction vector of x -axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The direction vector of the given line $(1 - \sqrt{3})\mathbf{x} = 0$ is $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$.

The angle that the line makes with the x -axis is given by,

$$\cos \theta = \frac{\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 & 0 \end{pmatrix} \right\|} = \frac{\sqrt{3}}{2} \quad (1.7.6)$$

$$\Rightarrow \boxed{\theta = 30^\circ} \quad (1.7.7)$$

Using equation (1.7.5) and (1.7.7), the area of the sector is obtained as,

$$\Rightarrow \boxed{\frac{\theta}{360^\circ} \pi r^2 = \frac{30^\circ}{360^\circ} \pi (2)^2 = \frac{\pi}{3}} \quad (1.7.8)$$

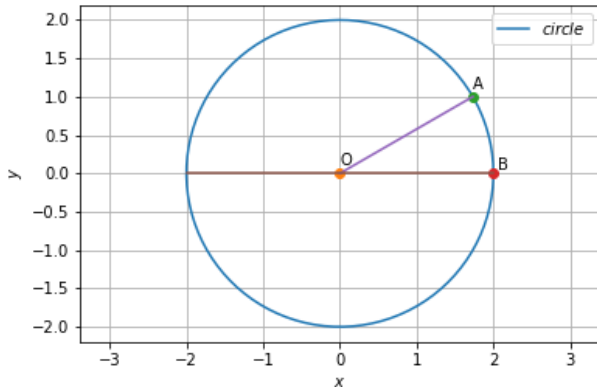


Fig. 4: Region enclosed by x -axis, line and circle

To find points \mathbf{A} and \mathbf{B} ,

The parametric form of x -axis is,

$$\mathbf{B} = \mathbf{q} + \lambda \mathbf{m} \quad (1.7.9)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.7.10)$$

From the intersection of circle and line, the value of λ can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.7.11)$$

$$= \frac{4 - 0}{1} = 4 \quad (1.7.12)$$

$$\Rightarrow \lambda = \pm 2 \quad (1.7.13)$$

Sub equation (1.7.13) in (1.7.10),

$$\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.7.14)$$

As given in question as first quadrant,

$$\Rightarrow \boxed{\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}} \quad (1.7.15)$$

Similarly, to find point \mathbf{A} , The parametric form of line is,

$$\mathbf{A} = \mathbf{q} + \lambda \mathbf{m} \quad (1.7.16)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad (1.7.17)$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.7.18)$$

$$= \frac{4 - 0}{4} = 1 \quad (1.7.19)$$

$$\Rightarrow \lambda = \pm 1 \quad (1.7.20)$$

$$\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \quad (1.7.21)$$

$$\Rightarrow \boxed{\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \quad (1.7.22)$$

1.8. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.

1.9. Find the area of the circle $4\mathbf{x}^T \mathbf{x} = 9$.

1.10. Find the area bounded by curves $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$ and $\|\mathbf{x}\| = 1$

1.11. Find the smaller area enclosed by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$.

1.12. Find the slope of the tangent to the curve $y =$

$\frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

Solution:

$$y = \frac{x-1}{x-2} \quad (1.12.1)$$

Equation (1.12.1) can be expressed as

$$y(x-2) = x-1 \quad (1.12.2)$$

$$yx - 2y - x + 1 = 0 \quad (1.12.3)$$

From above we can say,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.12.4)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} & -1 \end{pmatrix} \quad (1.12.5)$$

$$f = 1 \quad (1.12.6)$$

Now,

$$\because |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0, \quad (1.12.7)$$

(1.12.1) is the equation of a hyperbola. To verify that this we will find the characteristic equation of \mathbf{V} .

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} = 0 \quad (1.12.8)$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (1.12.9)$$

The eigenvalues are the roots of (1.12.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (1.12.10)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (1.12.11)$$

$$\implies (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (1.12.12)$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.12.13)$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.12.14)$$

Now, λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.12.15)$$

$$\implies \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.12.16)$$

From Equations,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (1.12.17)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (1.12.18)$$

We can say that

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (1.12.19)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (1.12.20)$$

$\because \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f > 0$, there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (1.12.21)$$

$$\text{axes} = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (1.12.22)$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad (1.12.23)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (1.12.24)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \quad (1.12.25)$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \quad (1.12.26)$$

Let us assume slope to be 1, now finding the direction vector and normal vector of the tangent with slope 1.

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.12.27)$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.12.28)$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (1.12.29)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (1.12.30)$$

By using (1.12.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \quad (1.12.31)$$

Now substituting this κ in (1.12.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{-\frac{1}{4l}} + 1 \end{pmatrix} \quad (1.12.32)$$

We know that $x=10$.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \quad (1.12.33)$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \quad (1.12.34)$$

$$\sqrt{-\frac{1}{4l}} = 4 \quad (1.12.35)$$

$$-\frac{1}{4l} = 16 \quad (1.12.36)$$

$$l = -\frac{1}{64} \quad (1.12.37)$$

The slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x=10$ is $\frac{1}{64}$. So, from the above we can say that $\kappa=4, -4$ and from equation (1.12.27) and (1.12.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \quad (1.12.38)$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \quad (1.12.39)$$

Now using equation (1.12.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1}(\kappa_1 \mathbf{n} - \mathbf{u}) \quad (1.12.40)$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left(-4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (1.12.41)$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \quad (1.12.42)$$

$$\mathbf{q}_2 = \mathbf{V}^{-1}(\kappa_2 \mathbf{n} - \mathbf{u}) \quad (1.12.43)$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left(4 \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \right) \quad (1.12.44)$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \quad (1.12.45)$$

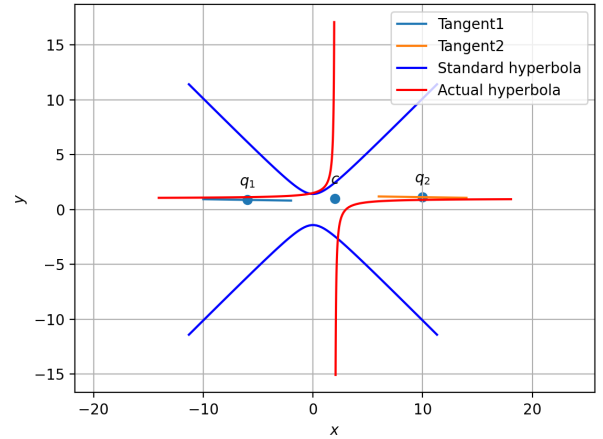


Fig. 5: Tangent 2 shows the tangent

1.13. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

1.14. Find the equation of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}$, $x \neq 1$

Solution: The given curve

$$y = \frac{1}{x-1} \quad (1.14.1)$$

can be expressed as

$$xy - y - 1 = 0 \quad (1.14.2)$$

Hence, we have

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix}, f = -1 \quad (1.14.3)$$

Since $|\mathbf{V}| < 0$, the equation (1.14.2) represents hyperbola. To find the values of λ_1 and λ_2 ,

consider the characteristic equation,

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (1.14.4)$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \right| = 0 \quad (1.14.5)$$

$$\Rightarrow \left| \begin{pmatrix} \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \lambda \end{pmatrix} \right| = 0 \quad (1.14.6)$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (1.14.7)$$

In addition, given the slope -1, the direction and normal vectors are given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.14.8)$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.14.9)$$

The parameters of hyperbola are as follows:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (1.14.10)$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \quad (1.14.11)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.14.12)$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \end{cases} \quad (1.14.13)$$

which represents the standard hyperbola equation,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (1.14.14)$$

The points of contact are given by

$$K = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} = \pm \frac{1}{2} \quad (1.14.15)$$

$$\mathbf{q} = \mathbf{V}^{-1}(K\mathbf{n} - \mathbf{u}) \quad (1.14.16)$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left[\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right] \quad (1.14.17)$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.14.18)$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \left[\frac{-1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right] \quad (1.14.19)$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (1.14.20)$$

\therefore The tangents are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0 \quad (1.14.21)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0 \quad (1.14.22)$$

The desired equations of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$ are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (1.14.23)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -1 \quad (1.14.24)$$

The above results are verified in the following figure.

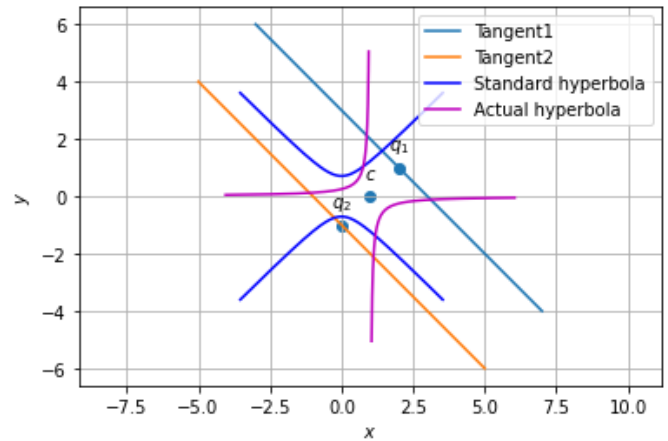


Fig. 6: The standard and actual hyperbola.

1.15. Find the equation of all lines having slope -2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.

Solution: Given the curve,

$$y = \frac{1}{x-3} \quad (1.15.1)$$

$$\Rightarrow xy - 3y - 1 = 0 \quad (1.15.2)$$

From (1.15.2) we get,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = -1 \quad (1.15.3)$$

Now,

$$\therefore |V| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4} < 0 \quad (1.15.4)$$

(1.15.1) is equation of hyperbola. Now,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \quad (1.15.5)$$

$$\Rightarrow \lambda^2 - \frac{1}{4} = 0 \quad (1.15.6)$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \quad (1.15.7)$$

The eigen vector \mathbf{p} is given by,

$$(\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (1.15.8)$$

For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (1.15.9)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.15.10)$$

Similarly for λ_2 ,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow -2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.15.11)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.15.12)$$

Now,

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (1.15.13)$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \quad (1.15.14)$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 \quad (1.15.15)$$

$\because \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 > 0$, there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.15.16)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (1.15.17)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_1}} = \sqrt{2} \quad (1.15.18)$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (1.15.19)$$

The direction and normal vectors of the tangent with slope -2 are given as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.15.20)$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \quad (1.15.21)$$

$$k = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (1.15.22)$$

Thus,

$$\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n} = 8 \quad (1.15.23)$$

$$k = \pm \frac{1}{2\sqrt{2}} \quad (1.15.24)$$

$$\mathbf{q}_1 = \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \quad (1.15.25)$$

$$\mathbf{q}_2 = \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \quad (1.15.26)$$

The desired tangents are,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \right\} = 0 \quad (1.15.27)$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6 + 2\sqrt{2} \quad (1.15.28)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \right\} = 0 \quad (1.15.29)$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6 - 2\sqrt{2} \quad (1.15.30)$$

Below figure corresponds to the tangents on the hyperbola, represented by (1.15.28) and (1.15.30) each having slope of -2 .

1.16. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at

which tangents are

a) parallel to x-axis

b) parallel to y-axis.

1.17. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

1.18. Find the equation of the tangent line to the

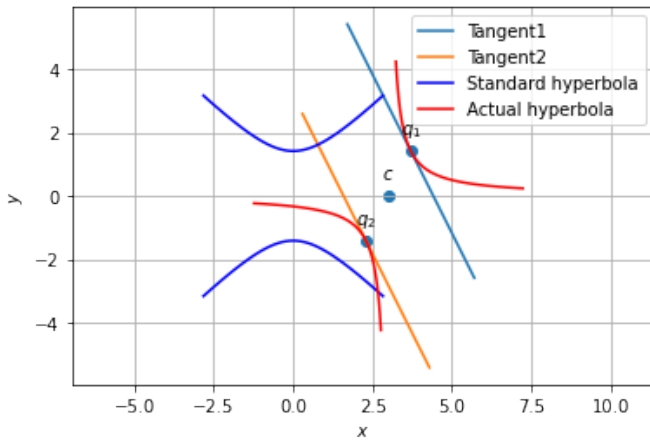


Fig. 7: Tangents to the hyperbola

curve $y = x^2 - 2x + 7$

a) parallel to the line $(2 \ -1)\mathbf{x} = -9$

b) perpendicular to the line $(-15 \ 5)\mathbf{x} = 13$.

1.19. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $(4 \ 2)\mathbf{x} + 5 = 0$.

1.20. Find the point at which the line $(-1 \ 1)\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.

1.21. The line $(-m \ 1)\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m .

1.22. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$

1.23. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

1.24. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

1.25. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

1.26. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

1.27. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.28. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

1.29. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the

value of a .

1.30. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

1.31. Find the area bounded by the curve $x^2 = 4y$ and the line $(1 \ -1)\mathbf{x} = -2$.

1.32. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

1.33. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.

1.34. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.

1.35. Find the area lying above x-axis and included between the circle $\mathbf{x}^T \mathbf{x} - 8(1 \ 0)\mathbf{x} = 0$ and inside of the parabola $y^2 = 4x$.

1.36. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB .

1.37. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.

1.38. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.

1.39. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.

1.40. Find the area between $y = x^2$ and $y = x$.

1.41. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

1.42. Find the area enclosed by the parabola $4y = 3x^2$ and the line $(-3 \ 2)\mathbf{x} = 12$.

1.43. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$

1.44. Find the area of the region enclosed by the parabola $x^2 = y$, the line $(-1 \ 1)\mathbf{x} = 2$ and the x-axis.

1.45. Find the area bounded by the curves

$$\{(x, y) : y > x^2, y = |x|\} \quad (1.45.1)$$

1.46. Find the area of the region

$$\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T \mathbf{x} = 9\} \quad (1.46.1)$$

1.47. Find the area of the circle $\mathbf{x}^T \mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.

2 QR DECOMPOSITION

2.1. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution: Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.1.1)$$

$$\beta = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (2.1.2)$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \quad (2.1.3)$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (2.1.4)$$

where

$$k_1 = \|\alpha\| \quad (2.1.5)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \quad (2.1.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (2.1.7)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (2.1.8)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (2.1.9)$$

From (2.1.3) and (2.1.4),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.1.10)$$

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \mathbf{Q} \mathbf{R} \quad (2.1.11)$$

From above we can see that \mathbf{R} is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (2.1.12)$$

Now by using equations (2.1.5) to (2.1.9)

$$k_1 = \sqrt{5} \quad (2.1.13)$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (2.1.14)$$

$$r_1 = \sqrt{5} \quad (2.1.15)$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (2.1.16)$$

$$k_2 = \sqrt{5} \quad (2.1.17)$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \quad (2.1.18)$$

2.2. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

2.3. $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

2.4. $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

2.5. $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$

2.6. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

2.7. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

2.8. $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

2.9. $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

2.10. $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2.11. $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$

2.12. $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

2.13. $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

2.14. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

2.15. Find QR decomposition of $\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$

Solution: Let \mathbf{a} and \mathbf{b} be the column vectors of the given matrix.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (2.15.1)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (2.15.2)$$

The column vectors can be expressed as follows,

$$\mathbf{a} = k_1 \mathbf{u}_1 \quad (2.15.3)$$

$$\mathbf{b} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (2.15.4)$$

Here,

$$k_1 = \|\mathbf{a}\| \quad (2.15.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \quad (2.15.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.15.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1 \mathbf{u}_1}{\|\mathbf{b} - r_1 \mathbf{u}_1\|} \quad (2.15.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{b} \quad (2.15.9)$$

The (2.15.3) and (2.15.4) can be written as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.15.10)$$

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \mathbf{QR} \quad (2.15.11)$$

Now, \mathbf{R} is an upper triangular matrix and also,

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (2.15.12)$$

Now using equations (2.15.5) to (2.15.9) we get,

$$k_1 = \sqrt{2^2 + 3^2} = \sqrt{13} \quad (2.15.13)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (2.15.14)$$

$$r_1 = \left(\frac{2}{\sqrt{13}} \quad \frac{3}{\sqrt{13}} \right) \begin{pmatrix} 3 \\ -4 \end{pmatrix} = -\frac{6}{\sqrt{13}} \quad (2.15.15)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.15.16)$$

$$k_2 = \left(\frac{3}{\sqrt{13}} \quad -\frac{2}{\sqrt{13}} \right) \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{17}{\sqrt{13}} \quad (2.15.17)$$

Thus putting the values from (2.15.13) to (2.15.17) in (2.15.11) we obtain QR decomposition,

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \sqrt{13} & -\frac{6}{\sqrt{13}} \\ 0 & \frac{17}{\sqrt{13}} \end{pmatrix} \quad (2.15.18)$$

2.16. Find the QR decomposition of $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

Solution:

Let \mathbf{c}_1 and \mathbf{c}_2 be the column vectors of the given

matrix.

$$\mathbf{c}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.16.1)$$

$$\mathbf{c}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad (2.16.2)$$

The column vectors can be represented as,

$$\mathbf{c}_1 = k_1 \mathbf{u}_1 \quad (2.16.3)$$

$$\mathbf{c}_2 = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (2.16.4)$$

where,

$$k_1 = \|\mathbf{c}_1\| \quad (2.16.5)$$

$$\mathbf{u}_1 = \frac{\mathbf{c}_1}{k_1} \quad (2.16.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{c}_2}{\|\mathbf{u}_1\|^2} \quad (2.16.7)$$

$$\mathbf{u}_2 = \frac{\mathbf{c}_2 - r_1 \mathbf{u}_1}{\|\mathbf{c}_2 - r_1 \mathbf{u}_1\|} \quad (2.16.8)$$

$$k_2 = \mathbf{u}_2^T \mathbf{c}_2 \quad (2.16.9)$$

From (2.16.3) and (2.16.4),

$$\begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.16.10)$$

$$\begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} = \mathbf{QR} \quad (2.16.11)$$

Where \mathbf{R} is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (2.16.12)$$

Using equations (2.16.5) to (2.16.9) we get,

$$k_1 = \sqrt{3^2 + 1^2} = \sqrt{10} \quad (2.16.13)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \quad (2.16.14)$$

$$r_1 = \left(\frac{3}{\sqrt{10}} \quad \frac{1}{\sqrt{10}} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \sqrt{10} \quad (2.16.15)$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{-1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \quad (2.16.16)$$

$$k_2 = \left(\frac{-1}{\sqrt{10}} \quad \frac{3}{\sqrt{10}} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \sqrt{10} \quad (2.16.17)$$

Now putting the values from (2.16.13) to (2.16.17), we obtain the QR decomposition of given matrix,

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \sqrt{10} \\ 0 & \sqrt{10} \end{pmatrix} \quad (2.16.18)$$

2.17. Find QR decomposition of $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$A = QR \quad (2.17.1)$$

where Q is an orthogonal matrix and R is an upper triangular matrix Given

$$A = \begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} \quad (2.17.2)$$

Let \mathbf{a} and \mathbf{b} be the column vectors of the given matrix

$$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (2.17.3)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (2.17.4)$$

The above column vectors (2.17.3) ,(2.17.4) can be expressed as ,

$$\mathbf{a} = t_1 \mathbf{u}_1 \quad (2.17.5)$$

$$\mathbf{b} = s_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 \quad (2.17.6)$$

Where,

$$t_1 = \|\mathbf{a}\| \quad (2.17.7)$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{t_1} \quad (2.17.8)$$

$$s_1 = \frac{\mathbf{u}_1^T \mathbf{b}}{\|\mathbf{u}_1\|^2} \quad (2.17.9)$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - s_1 \mathbf{u}_1}{\|\mathbf{b} - s_1 \mathbf{u}_1\|} \quad (2.17.10)$$

$$t_2 = \mathbf{u}_2^T \mathbf{b} \quad (2.17.11)$$

The (2.17.5) and (2.17.6) can be written as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} t_1 & s_1 \\ 0 & t_2 \end{pmatrix} \quad (2.17.12)$$

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = QR \quad (2.17.13)$$

Here, R is an upper triangular matrix and Q is an orthogonal matrix such that

$$Q^T Q = I \quad (2.17.14)$$

Now using equations from (2.17.7) to (2.17.11)

we get,

$$t_1 = \sqrt{4^2 + 5^2} = \sqrt{41} \quad (2.17.15)$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{41}} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (2.17.16)$$

$$s_1 = \left(\frac{4}{\sqrt{41}} \quad \frac{5}{\sqrt{41}} \right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{2}{\sqrt{41}} \quad (2.17.17)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{41}} \begin{pmatrix} 5 \\ -4 \end{pmatrix} \quad (2.17.18)$$

$$t_2 = \left(\frac{5}{\sqrt{41}} \quad \frac{-4}{\sqrt{41}} \right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{23}{\sqrt{41}} \quad (2.17.19)$$

Substituting the values from (2.17.15) to (2.17.19) in (2.17.13) we obtain QR decomposition as,

$$\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \\ \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}} \end{pmatrix} \begin{pmatrix} \sqrt{41} & \frac{2}{\sqrt{41}} \\ 0 & \frac{23}{\sqrt{41}} \end{pmatrix} \quad (2.17.20)$$