Computational Approach to School Geometry

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CONTENTS

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***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 Triangle

1.1 Triangle Examples

1. Do the points $\mathbf{A} = \begin{pmatrix}3\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-2\\-3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\end{pmatrix}$ form a triangle? If so, name the type of triangle formed.

   Solution:
   The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix}-5\\-5\end{pmatrix} \tag{1.1.1.1}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix}-1\\1\end{pmatrix} \tag{1.1.1.2}$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a line, then, AB and AC should have the same direction vector. Hence, there exists a k such that

$$\mathbf{B} - \mathbf{A} = k(\mathbf{C} - \mathbf{B}) \tag{1.1.1.3}$$

$$\implies \mathbf{B} = \frac{k\mathbf{C} + \mathbf{A}}{k+1} \tag{1.1.1.4}$$

Since

$$\mathbf{B} - \mathbf{A} \neq k(\mathbf{C} - \mathbf{A}), \tag{1.1.1.5}$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T \tag{1.1.1.6}$$

If $rank(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \tag{1.1.1.7}$$

$$\implies rank(\mathbf{M}) = 2 \tag{1.1.1.8}$$

as the number of non zero rows is 2. The following code plots Fig. 1.1.1

```
codes/triangle/check_tri.py
```

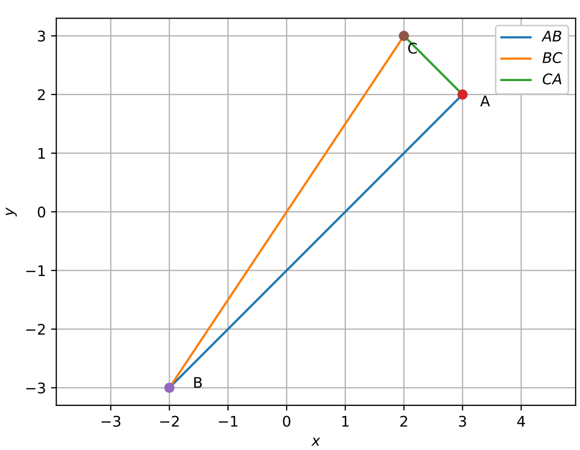

Fig. 1.1.1

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \tag{1.1.1.9}$$

which can be expressed as

$$\begin{aligned} \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} \\ = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T\mathbf{C} \end{aligned} \tag{1.1.1.10}$$

to obtain

$$(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A}) = 0 \tag{1.1.1.11}$$

after simplification. From (1.1.1.1) and (1.1.1.2), it is easy to verify that

$$(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & -5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \tag{1.1.1.12}$$

satisfying (1.1.1.11). Thus, $\triangle ABC$ is right angled at $\mathbf{A}$.

2. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.
Solution: Using Hero's formula, the following code computes the area of the triangle as 24.

```
codes/triangle/area_tri.py
```

3. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$. **Solution:** The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix $\mathbf{M}$ in (1.1.1.6) as
$$\frac{1}{2}\left|\mathbf{M}\right| \tag{1.1.3.1}$$
The computation is done in **area_tri.py**
4. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
Solution: Another formula for the area of $\triangle ABC$ is
$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \tag{1.1.4.1}$$
5. Find the area of a triangle having the points
$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.1.5.1}$$
as its vertices.
Solution: The area of a triangle using the

vector product is obtained as

$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{1.1.5.2}$$

For any two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.1.5.3}$$

The following code computes the area using the vector product.

```
codes/triangle/area_tri_vec.py
```

6. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If the coordinates of $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7 \\ -6 \end{pmatrix}$, respectively, find the coordinates of the point $\mathbf{C}$.
Solution: The centroid of $\triangle ABC$ is given by

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.1.6.1}$$

Thus,

$$\mathbf{C} = 3\mathbf{C} - \mathbf{A} - \mathbf{B} \tag{1.1.6.2}$$

7. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.1.7.1}$$

are the vertices of a right angled triangle.
Solution: The following code plots Fig. 1.1.7

```
codes/triangle/triangle_3d.py
```

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1.7.2}$$

it is proved that the triangle is indeed right angled.

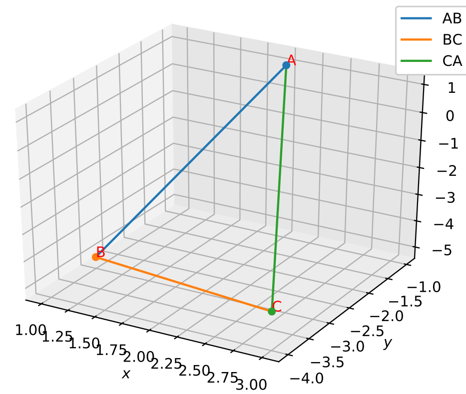

Fig. 1.1.7

8. Are the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \tag{1.1.8.1}$$

the vertices of a right angled triangle?

1.2 Triangle Exercises

1. Draw the graphs of the equations

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.2.1.1}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix}\mathbf{x} - 12 = 0 \tag{1.2.1.2}$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.2.1.3}$$

Substituting in (1.2.1.1),

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = -1 \tag{1.2.1.4}$$

$$\implies a = -1 \tag{1.2.1.5}$$

Simiarly, substituting

$$\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{1.2.1.6}$$

in (1.2.1.1),

$$b = 1 \tag{1.2.1.7}$$

The intercepts on the x and y-axis from above are

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.1.8}$$

Similarly, the intercepts on x and y-axis for (1.2.1.2) are

$$\begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}0\\6\end{pmatrix} \tag{1.2.1.9}$$

The interection of the lines in (1.2.1.1), (1.2.1.1) is obtained from

$$\begin{pmatrix}1 & -1\\3 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}-1\\12\end{pmatrix} \tag{1.2.1.10}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}1 & -1 & -1\\3 & 2 & 12\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2-3R_1}{5}} \begin{pmatrix}1 & -1 & -1\\0 & 1 & 3\end{pmatrix} \tag{1.2.1.11}$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2} \begin{pmatrix}1 & 0 & 2\\0 & 1 & 3\end{pmatrix} \tag{1.2.1.12}$$

$$\implies \mathbf{x} = \begin{pmatrix}2\\3\end{pmatrix} \tag{1.2.1.13}$$

The desired triangle is available in Fig. (1.2.1) with vertices

$$\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}4\\0\end{pmatrix} \tag{1.2.1.14}$$

The equivalent python code for figure (1.2.1)

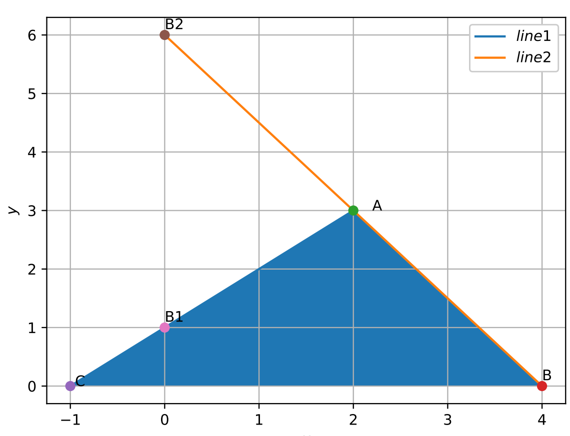

Fig. 1.2.1

is

```
solutions/1/codes/triangle/shaded.py
```

2. In a $\triangle ABC$, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.
Solution:
The given equations result in the matrix equation In vector form:

$$\begin{pmatrix}6 & 0 & -1\\0 & 3 & -1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}A\\B\\C\end{pmatrix} = \begin{pmatrix}0\\0\\180\end{pmatrix} \tag{1.2.2.1}$$

whcih can be solved as

$$\begin{pmatrix}6 & 0 & -1 & 0\\0 & 3 & -1 & 0\\1 & 1 & 1 & 180\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix}1 & 0 & \frac{-1}{6} & 0\\0 & 3 & -1 & 0\\1 & 1 & 1 & 180\end{pmatrix} \tag{1.2.2.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3-R_1} \begin{pmatrix}1 & 0 & \frac{-1}{6} & 0\\0 & 3 & -1 & 0\\0 & 1 & \frac{7}{6} & 180\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix}1 & 0 & -\frac{1}{6} & 0\\0 & 1 & -\frac{1}{3} & 0\\0 & 1 & \frac{7}{6} & 180\end{pmatrix} \tag{1.2.2.3}$$

$$\xleftrightarrow{R_3 \leftarrow\ R_3-R_2} \begin{pmatrix}1 & 0 & -\frac{1}{6} & 0\\0 & 1 & -\frac{1}{3} & 0\\0 & 0 & \frac{3}{2} & 180\end{pmatrix} \xleftrightarrow{R_3 \leftarrow\ \frac{2R_3}{3}} \begin{pmatrix}1 & 0 & -\frac{1}{6} & 0\\0 & 1 & -\frac{1}{3} & 0\\0 & 0 & 1 & 120\end{pmatrix} \tag{1.2.2.4}$$

$$\xleftrightarrow[R_2 \leftarrow\ R_2+\frac{R_3}{3}]{R_1 \leftarrow\ R_1+\frac{R_3}{6}} \begin{pmatrix}1 & 0 & 1 & 20\\0 & 1 & 0 & 40\\0 & 0 & 1 & 120\end{pmatrix} \tag{1.2.2.5}$$

$$\therefore \angle C = 120^\circ \quad \angle A = 20^\circ \quad \angle B = 40^\circ \tag{1.2.2.6}$$

3. Draw the graphs of the equations $5x–y = 5$ and $3x–y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.
Solution:
Line $5x - y = 5$ can be represented in vector form as,

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = 5 \tag{1.2.3.1}$$

Line $3x - y = 3$ can be represented in vector form as,

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 3 \tag{1.2.3.2}$$

Also the equation of y axis is

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{1.2.3.3}$$

Let line (1.2.3.1) and line (1.2.3.2) meet at point $\mathbf{A}$.Then,

$$\begin{pmatrix}5 & -1\\3 & -1\end{pmatrix}\mathbf{A} = \begin{pmatrix}5\\3\end{pmatrix} \tag{1.2.3.4}$$

$$\mathbf{A} = \begin{pmatrix}5 & -1\\3 & -1\end{pmatrix}^{-1}\begin{pmatrix}5\\3\end{pmatrix} \tag{1.2.3.5}$$

$$\mathbf{A} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.2.3.6}$$

Let line (1.2.3.1) and line (1.2.3.3) meet at point $\mathbf{B}$. Then,

$$\begin{pmatrix}5 & -1\\1 & 0\end{pmatrix}\mathbf{B} = \begin{pmatrix}5\\0\end{pmatrix} \tag{1.2.3.7}$$

$$\mathbf{B} = \begin{pmatrix}5 & -1\\1 & 0\end{pmatrix}^{-1}\begin{pmatrix}5\\0\end{pmatrix} \tag{1.2.3.8}$$

$$\mathbf{B} = \begin{pmatrix}0\\-5\end{pmatrix} \tag{1.2.3.9}$$

Let line (1.2.3.2) and line (1.2.3.3) meet at point $\mathbf{C}$. Then,

$$\begin{pmatrix}3 & -1\\1 & 0\end{pmatrix}\mathbf{C} = \begin{pmatrix}3\\0\end{pmatrix} \tag{1.2.3.10}$$

$$\mathbf{C} = \begin{pmatrix}3 & -1\\1 & 0\end{pmatrix}^{-1}\begin{pmatrix}3\\0\end{pmatrix} \tag{1.2.3.11}$$

$$\mathbf{C} = \begin{pmatrix}0\\-3\end{pmatrix} \tag{1.2.3.12}$$

So, $\triangle ABC$ is formed by intersection of (1.2.3.1),(1.2.3.2) and (1.2.3.3). The following Python code generates Fig. 1.2.3 The lines (1.2.3.1) and (1.2.3.2) and the triangle ABC formed by the two lines and y-axis are plotted in the figure below

codes/triangle/linesandtri.py

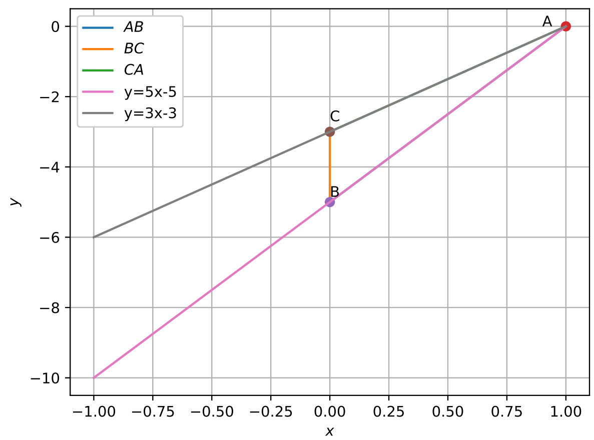

Fig. 1.2.3: Plot of lines and the Triangle ABC

4. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-2\\3\end{pmatrix}, \mathbf{R} = \begin{pmatrix}4\\5\end{pmatrix}$. Find the equation of the median through the vertex $\mathbf{R}$.
Solution: In Fig. 1.2.4, RS is the median. Hence,

$$\mathbf{S} = \frac{\mathbf{P}+\mathbf{Q}}{2} \tag{1.2.4.1}$$

Hence, the equation of the median going through points $\mathbf{S}$ and $\mathbf{R}$ can be given as

$$\mathbf{x} = \mathbf{R} + \lambda(\mathbf{S} - \mathbf{R}) \tag{1.2.4.2}$$

$$\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix} + \lambda\left(\begin{pmatrix}0\\2\end{pmatrix} - \begin{pmatrix}4\\5\end{pmatrix}\right) \tag{1.2.4.3}$$

$$\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix} + \lambda\left(\begin{pmatrix}-4\\-3\end{pmatrix}\right) \tag{1.2.4.4}$$

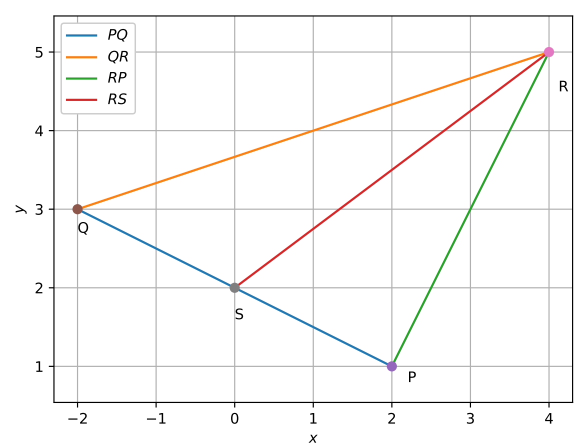

Fig. 1.2.4

solutions/4/codes/triangle/triangle.py

5. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\2\end{pmatrix}$, find the equation and length of the altitude from the vertex $\mathbf{A}$.

Solution: The following python code computes the length of the altitude **AD** in Fig.1.2.5.

```
./solutions/5/codes/triangle/q2.py
```

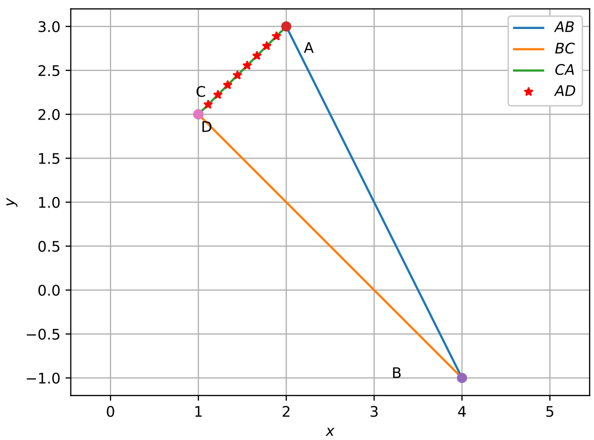

Fig. 1.2.5: Triangle of Q.1.2.5

In $\triangle ABC$,

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.2.5.1}$$

Hence, ABC is a right triangle. The direction vector of BC is

$$(\mathbf{B}-\mathbf{C}) = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.2.5.2}$$

Hence, the equation of AD is

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.2.5.3}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = -1 \tag{1.2.5.4}$$

The length of the altitude is obtained as $\|\mathbf{A}-\mathbf{D}\| = 1.414$

6. Find the area of the triangle whose vertices are
   a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
   b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

   Solution:

   a) See Fig. 1.2.6 generated using the following python code

```
solutions/6/codes/triangle/triangle1.py
```

$$ar(\triangle ABC) = \frac{1}{2}\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\| \tag{1.2.6.1}$$

$$= \frac{1}{2}\left\|\begin{pmatrix} -3 \\ -3 \end{pmatrix}\times\begin{pmatrix} 0 \\ -7 \end{pmatrix}\right\| = \frac{21}{2} \tag{1.2.6.2}$$

and verified by

```
solutions/6/codes/triangle/tri_area_ABC.py
```

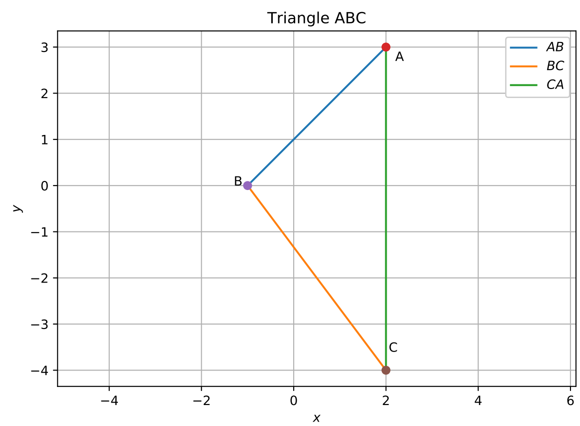

Fig. 1.2.6: Triangle ABC using python

   b) See $\triangle PQR$ in Fig. 1.2.6 generated using the following python code

```
solutions/6/codes/triangle/triangle2.py
```

$$ar(\triangle PQR) = \frac{1}{2}\|(\mathbf{Q}-\mathbf{P})\times(\mathbf{R}-\mathbf{P})\| \tag{1.2.6.3}$$

$$= \frac{1}{2}\left\|\begin{pmatrix} 8 \\ -4 \end{pmatrix}\times\begin{pmatrix} 10 \\ 3 \end{pmatrix}\right\| = \frac{64}{2} \tag{1.2.6.4}$$

and verified by

```
solutions/6/codes/triangle/tri_area_PQR.py
```

7. Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

   Solution: See Fig. 1.2.7. Let the vertices be

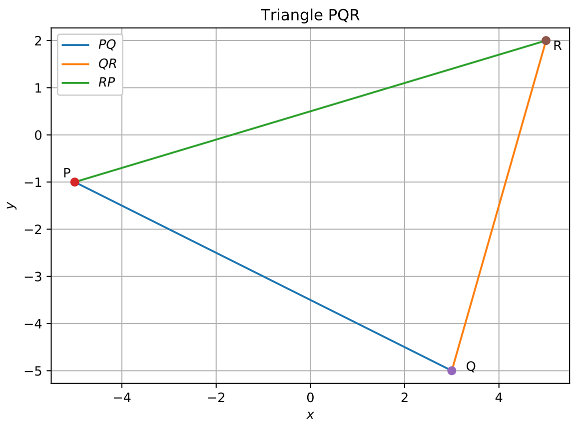

Fig. 1.2.6: Triangle PQR using python

$\mathbf{A}, \mathbf{B}, \mathbf{C}$. The midpoints of each side are

$$\mathbf{D} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.7.1}$$

$$\mathbf{E} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.2.7.2}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.2.7.3}$$

$$\tag{1.2.7.4}$$

Area of a $\triangle$ ABC is given by

$$\frac{1}{2}\left\|(\mathbf{E}-\mathbf{D}) \times (\mathbf{F}-\mathbf{D})\right\|$$
$$= \frac{1}{2}\left\|\begin{pmatrix} 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|$$
$$= 1 \tag{1.2.7.5}$$

Download the python code for finding a triangle's area from

```
solutions/7/codes/triangle/area_tri_area.py
```

and the figure from

```
solutions/7/figs/triangle/draw_triangle.py
```

8. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.
Solution: The following Python code generates Fig. 1.2.8

```
codes/triangle.py
```

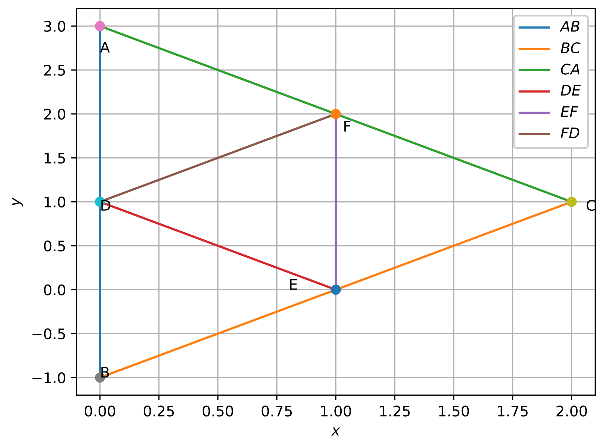

Fig. 1.2.7

From the given information,

$$\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \tag{1.2.8.1}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \tag{1.2.8.2}$$

$$\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{1.2.8.3}$$

$\because$ $\mathbf{M}$ is the midpoint of AB,

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} 7 \\ -8 \end{pmatrix} \tag{1.2.8.4}$$

$\because$ $\mathbf{N}$ is the midpoint of BC,

$$\mathbf{N} = \frac{\mathbf{B}+\mathbf{C}}{2} = \frac{1}{2}\begin{pmatrix} 8 \\ 0 \end{pmatrix} \tag{1.2.8.5}$$

$\because$ $\mathbf{P}$ is the midpoint of CA,

$$\mathbf{P} = \frac{\mathbf{C}+\mathbf{A}}{2} = \frac{1}{2}\begin{pmatrix} 9 \\ -4 \end{pmatrix} \tag{1.2.8.6}$$

The following Python code verifies the determinant values.

```
codes/determinant_check.py
```

For $\triangle ABC$, the vertices are $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. So the area of the triangle $\triangle ABC$ by using determinant

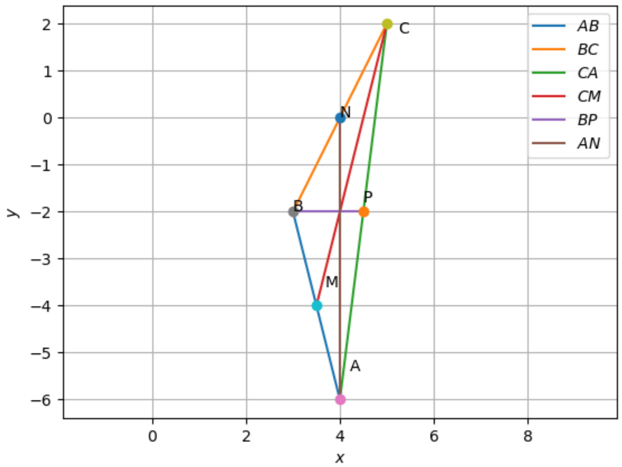

Fig. 1.2.8

will be :

$$
\begin{aligned}
Area = \frac{1}{2}\begin{vmatrix} 4 & -6 & 1\\ 3 & -2 & 1\\ 5 & 2 & 1 \end{vmatrix} &\xleftrightarrow{C_2\leftarrow\frac{C_2}{2}} \frac{2}{2}\begin{vmatrix} 4 & -3 & 1\\ 3 & -1 & 1\\ 5 & 1 & 1 \end{vmatrix}\\
\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{vmatrix} 4 & -3 & 1\\ -1 & 2 & 0\\ 1 & 4 & 0 \end{vmatrix} &\xleftrightarrow{R_3\leftarrow R_3+R_2} \begin{vmatrix} 4 & -3 & 1\\ -1 & 2 & 0\\ 0 & 6 & 0 \end{vmatrix}\\
&\xleftrightarrow{R_3\leftarrow\frac{R_3}{6}} 6\begin{vmatrix} 4 & -3 & 1\\ -1 & 2 & 0\\ 0 & 1 & 0 \end{vmatrix}\\
&= -6
\end{aligned} \tag{1.2.8.7}
$$

Now, we will consider the absolute value of area only. So, $Area = |-6| = 6$.

To verify the problem statement we have to check 3 cases:

Case 1: When **BP** is median, we will consider $\triangle ABP$ triangle. In that case, the vertices will be **A**, **B** and **P**.

Now, the area of $\triangle ABP$ will be :

$$
\begin{aligned}
A1 = \frac{1}{2}\begin{vmatrix} 4 & -6 & 1\\ 3 & -2 & 1\\ 4.5 & -2 & 1 \end{vmatrix} &\xleftrightarrow{C_2\leftarrow\frac{C_2}{(-2)}} \frac{(-2)}{2}\begin{vmatrix} 4 & 3 & 1\\ 3 & 1 & 1\\ 4.5 & 1 & 1 \end{vmatrix}\\
&\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} (-1)\begin{vmatrix} 4 & 3 & 1\\ -1 & -2 & 0\\ 0.5 & -2 & 0 \end{vmatrix}\\
&\xleftrightarrow{R_3\leftarrow R_3-R_2} (-1)\begin{vmatrix} 4 & 3 & 1\\ -1 & -2 & 0\\ 1.5 & 0 & 0 \end{vmatrix}\\
&= -3
\end{aligned} \tag{1.2.8.8}
$$

But, we will consider the absolute value of area only. So, $\mathbf{A1} = |-3| = 3$.

or, $\mathbf{A1} = \frac{1}{2}(\text{Area of } \triangle ABC)$

Case 2: When **AN** is median,we will consider $\triangle ABN$ triangle. In that case, the vertices will be **A**, **B** and **N**.

Now, the area of $\triangle ABN$ will be :

$$
\begin{aligned}
A2 = \frac{1}{2}\begin{vmatrix} 4 & -6 & 1\\ 3 & -2 & 1\\ 4 & 0 & 1 \end{vmatrix} &\xleftrightarrow{C_2\leftarrow\frac{C_2}{(-2)}} \frac{(-2)}{2}\begin{vmatrix} 4 & 3 & 1\\ 3 & 1 & 1\\ 4 & 0 & 1 \end{vmatrix}\\
&\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} (-1)\begin{vmatrix} 4 & 3 & 1\\ -1 & -2 & 0\\ 0 & -3 & 0 \end{vmatrix}\\
&\xleftrightarrow{R_3\leftarrow\frac{R_3}{(-3)}} 3\begin{vmatrix} 4 & 3 & 1\\ -1 & -2 & 0\\ 0 & 1 & 0 \end{vmatrix}\\
&= -3
\end{aligned} \tag{1.2.8.9}
$$

But, we will consider the absolute value of area only. So, $\mathbf{A2} = |-3| = 3$.

or, $\mathbf{A2} = \frac{1}{2}(\text{Area of } \triangle ABC)$

Case 3: When **CM** is median,we will consider $\triangle CAM$ triangle. In that case, the vertices will be **A**, **C** and **M**.

Now, the area of $\triangle CAM$ will be :

$$A3 = \frac{1}{2}\begin{vmatrix}5 & 2 & 1\\4 & -6 & 1\\3.5 & -4 & 1\end{vmatrix} \xleftrightarrow{C_2 \leftarrow \frac{C_2}{2}} \frac{2}{2}\begin{vmatrix}5 & 1 & 1\\4 & -3 & 1\\3.5 & -2 & 1\end{vmatrix}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{vmatrix}5 & 1 & 1\\-1 & -4 & 0\\-1.5 & -3 & 0\end{vmatrix}$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3}{(-1.5)}]{R_2 \leftarrow \frac{R_2}{(-1)}} 1.5\begin{vmatrix}5 & 1 & 1\\1 & 4 & 0\\1 & 2 & 0\end{vmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} 1.5\begin{vmatrix}5 & 1 & 1\\1 & 4 & 0\\0 & -2 & 0\end{vmatrix} = -3 \quad (1.2.8.10)$$

But, we will consider the absolute value of area only. So, $\mathbf{A3} = |-3| = 3$.
or, $\mathbf{A3} = \frac{1}{2}(\text{Area of } \triangle ABC)$

Hence, the above problem statement is verified.

9. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix}4\\6\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}7\\2\end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that
$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \quad (1.2.9.1)$$
Find
$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \quad (1.2.9.2)$$

10. Let $\mathbf{A} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}1\\4\end{pmatrix}$ be the vertices of $\triangle ABC$.
    a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
    b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
    c) Find the coordinates of the points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

11. In $\triangle ABC$, Show that the centroid
$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.2.11.1)$$

12. Show that the points
$$\mathbf{A} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} \quad (1.2.12.1)$$
are the vertices of a right angled triangle.

13. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}0\\1\\2\end{pmatrix}$. Find $\angle B$.

14. Show that the vectors $\begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-4\\-4\end{pmatrix}$ form the vertices of a right angled triangle.

15. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix}$ as its vertices.

16. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix}1\\1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\3\\5\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}1\\5\\5\end{pmatrix}$

17. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix}3\\5\\-4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}-5\\-5\\-2\end{pmatrix}$

18. Without using the Pythagoras theorem, show that the points $\begin{pmatrix}4\\4\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$ and $\begin{pmatrix}-1\\-1\end{pmatrix}$ are the vertices of a right angled triangle.
Solution: The direction vectors of $\mathbf{A}-\mathbf{B}$, $\mathbf{A}-\mathbf{C}$ and $\mathbf{B}-\mathbf{C}$ are
$$\mathbf{A} - \mathbf{B} = \begin{pmatrix}-1\\1\end{pmatrix} \quad (1.2.18.1)$$
$$\mathbf{A} - \mathbf{C} = \begin{pmatrix}-5\\-5\end{pmatrix} \quad (1.2.18.2)$$
$$\mathbf{B} - \mathbf{C} = \begin{pmatrix}-4\\-6\end{pmatrix} \quad (1.2.18.3)$$

a)
$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-4\\-6\end{pmatrix} = -2 \quad (1.2.18.4)$$
$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = -2 \neq 0 \quad (1.2.18.5)$$
Sides $\mathbf{A} - \mathbf{B}$ and $\mathbf{B} - \mathbf{C}$ of triangle are not

perpendicular.

b)

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}-5\\-5\end{pmatrix}\begin{pmatrix}-4\\-6\end{pmatrix} = 50 \tag{1.2.18.6}$$

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 50 \neq 0 \tag{1.2.18.7}$$

Sides $\mathbf{A}-\mathbf{C}$ and $\mathbf{B}-\mathbf{C}$ of triangle are not perpendicular.

c)

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) = \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-5\\-5\end{pmatrix} = 0 \tag{1.2.18.8}$$

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) = 0 \tag{1.2.18.9}$$

Sides $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}-\mathbf{C}$ of triangle are perpendicular to each other and the right angle at vertex $\begin{pmatrix}4\\4\end{pmatrix}$, and the following figure represents the triangle formed by given points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

19. Check whether

$$\begin{pmatrix}5\\-2\end{pmatrix},\begin{pmatrix}6\\4\end{pmatrix},\begin{pmatrix}7\\-2\end{pmatrix} \tag{1.2.19.1}$$

are the vertices of an isosceles triangle.

2 Quadrilateral

2.1 Quadrilateral Examples

1. Show that the points $\mathbf{A} = \begin{pmatrix}1\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\4\end{pmatrix}$ are the vertices of a square.
Solution: By inspection,

$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} = \begin{pmatrix}0\\3\end{pmatrix} \tag{2.1.1.1}$$

Hence, the diagonals AC and BD bisect each other. Also,

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.1.1.2}$$

$\implies AC \perp BD$. Hence $ABCD$ is a square.

2. If the points $\mathbf{A} = \begin{pmatrix}6\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}9\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}p\\3\end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.
Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.

3. If $\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-6\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\5\end{pmatrix}$, find the area of the quadrilateral $ABCD$.
Solution: The area of $ABCD$ is the sum of the areas of trianges ABD and CBD and is given by

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| + \frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| \tag{2.1.3.1}$$

4. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\2\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\7\\6\end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.
Solution: Since the direction vectors

$$\mathbf{A}-\mathbf{B} = \mathbf{D}-\mathbf{C} \tag{2.1.4.1}$$

$$\mathbf{A}-\mathbf{D} = \mathbf{B}-\mathbf{C} \tag{2.1.4.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) \neq 0 \tag{2.1.4.3}$$

Hence, it is not a rectangle. The following code plots Fig. 2.1.4

```
codes/triangle/quad_3d.py
```

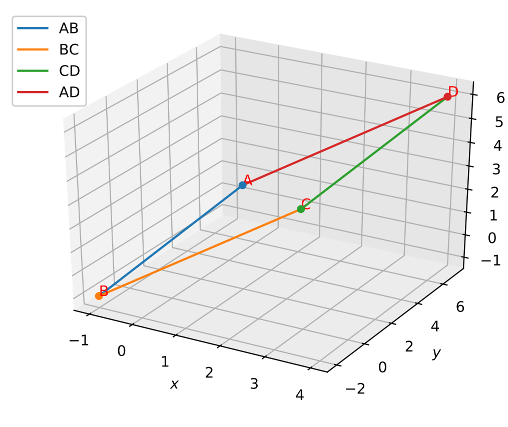

Fig. 2.1.4

5. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
Solution: The area is given by

$$\frac{1}{2}\left\|\begin{pmatrix}3\\1\\4\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\| \tag{2.1.5.1}$$

2.2 *Quadrilateral Geometry*

1. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
   a) square?
   b) rectangle?
   c) rhombus?

   Solution:

$$\mathbf{P} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix}-1 & \frac{3}{2}\end{pmatrix} \tag{2.2.1.1}$$
$$\mathbf{Q} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix}2 & 4\end{pmatrix} \tag{2.2.1.2}$$
$$\mathbf{R} = \frac{\mathbf{C}+\mathbf{D}}{2} = \begin{pmatrix}5 & \frac{3}{2}\end{pmatrix} \tag{2.2.1.3}$$
$$\mathbf{S} = \frac{\mathbf{A}+\mathbf{D}}{2} = \begin{pmatrix}2 & -1\end{pmatrix} \tag{2.2.1.4}$$

$\because$

$$\frac{\mathbf{P}+\mathbf{R}}{2} = \frac{\mathbf{Q}+\mathbf{S}}{2} = \frac{1}{2}\begin{pmatrix}4\\3\end{pmatrix} \tag{2.2.1.5}$$

PQRS is a parallelogram.

$$(\mathbf{P}-\mathbf{R}) = \begin{pmatrix}-6 & 0\end{pmatrix}(\mathbf{Q}-\mathbf{S}) = \begin{pmatrix}0\\5\end{pmatrix} \tag{2.2.1.6}$$
$$\tag{2.2.1.7}$$

$$(\mathbf{P}-\mathbf{R})^T(\mathbf{Q}-\mathbf{S}) = \begin{pmatrix}-6 & 0\end{pmatrix}\begin{pmatrix}0\\5\end{pmatrix} \tag{2.2.1.8}$$
$$(\mathbf{P}-\mathbf{R})^T(\mathbf{Q}-\mathbf{S}) = \begin{pmatrix}0\end{pmatrix} \tag{2.2.1.9}$$
$$\tag{2.2.1.10}$$

Diagonal bisect orthogonally. Thus, PQRS is a rhombus. Se Fig. 2.2.1

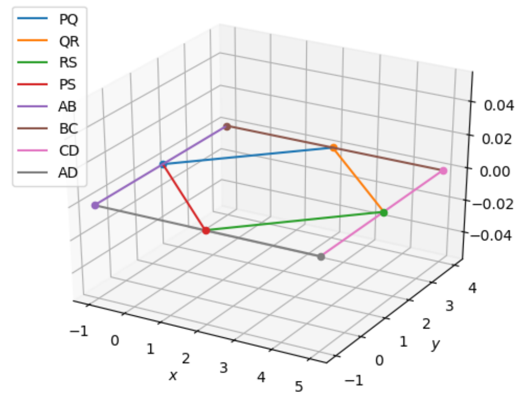

Fig. 2.2.1: Simulation of midpoint of ABCD forms PQRS.

Step4: We will check whether Parallelogram PQRS is Square or not.

$$(\mathbf{P}-\mathbf{Q}) = \frac{1}{2}\begin{pmatrix}-6\\-5\end{pmatrix} \tag{2.2.1.11}$$
$$(\mathbf{P}-\mathbf{S}) = \frac{1}{2}\begin{pmatrix}-6\\5\end{pmatrix} \tag{2.2.1.12}$$
$$\tag{2.2.1.13}$$

If adjacent side of parallelogram are orthogonal to each other then PQRS is a Square.

$$(\mathbf{P}-\mathbf{Q})^T(\mathbf{P}-\mathbf{S}) = \frac{1}{4}\begin{pmatrix}-6 & -5\end{pmatrix}\begin{pmatrix}-6\\5\end{pmatrix} \neq= 0 \tag{2.2.1.14}$$
$$\tag{2.2.1.15}$$

Here the angle between adjacent side is not 90 °. Hence, PQRS is not a Square.

2. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \tag{2.2.2.1}$$
$$\angle B = 3y - 5 \tag{2.2.2.2}$$
$$\angle C = -4x \tag{2.2.2.3}$$
$$\angle D = -7x + 5 \tag{2.2.2.4}$$

   Find its angles.
   Solution: From the given information,

$$\angle A + \angle C = 180° \tag{2.2.2.5}$$
$$\angle B + \angle D = 180° \tag{2.2.2.6}$$

which can be expressed as

$$\begin{pmatrix}-4 & 4\\ -7 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}160\\180\end{pmatrix} \tag{2.2.2.7}$$

and solved as

$$\begin{pmatrix}-4 & 4 & 160\\ -7 & 3 & 180\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{-R_1}{4}} \begin{pmatrix}1 & -1 & -40\\ -7 & 3 & 180\end{pmatrix} \tag{2.2.2.8}$$

$$\xleftrightarrow{R_2\leftarrow R_2+7R_1} \begin{pmatrix}1 & -1 & -40\\ 0 & -4 & -100\end{pmatrix} \xleftrightarrow{R_2\leftarrow\frac{-R_2}{4}} \begin{pmatrix}1 & -1 & -40\\ 0 & 1 & 25\end{pmatrix} \tag{2.2.2.9}$$

$$\xleftrightarrow{R_1\leftarrow\ R_1+R_2} \begin{pmatrix}1 & 0 & -15\\ 0 & 1 & 25\end{pmatrix} \tag{2.2.2.10}$$

Thus,

$$x = -15, y = 25 \tag{2.2.2.11}$$

$$\implies \angle A = 120^\circ, \angle B = 70^\circ, \tag{2.2.2.12}$$

$$\implies \angle C = 60^\circ, \angle D = 110^\circ \tag{2.2.2.13}$$

3. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix}-4\\5\end{pmatrix}$, $\begin{pmatrix}0\\7\end{pmatrix}$, $\begin{pmatrix}5\\-5\end{pmatrix}$ and $\begin{pmatrix}-4\\-2\end{pmatrix}$. Also, find its area.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}-4\\5\end{pmatrix}\mathbf{B} = \begin{pmatrix}0\\7\end{pmatrix}\mathbf{C} = \begin{pmatrix}5\\-5\end{pmatrix}\mathbf{D} = \begin{pmatrix}-4\\-2\end{pmatrix} \tag{2.2.3.1}$$

Quadrilateral ABCD is drawn by joining its vertices **A** and **B**,**B** and **C**, **C** and **D**, **D** and **A**. The following Python code generates Fig. 2.2.3

```
codes/quad/quad.py
```

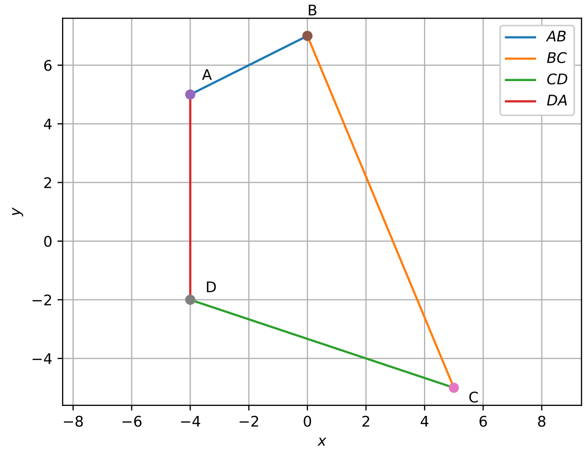

Fig. 2.2.3: Quadrilateral ABCD

From Figure 2.2.3 Area of the Quadrilateral ABCD can be given as

$$Ar(\triangle ABC) + Ar(\triangle BCD) \tag{2.2.3.2}$$

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| + \frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| \tag{2.2.3.3}$$

For two vectors $\mathbf{a} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}b_1\\b_2\end{pmatrix}$

$$\|\mathbf{a}\times\mathbf{b}\| = |a_1b_2 - a_2b_1| \tag{2.2.3.4}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-4\\-2\end{pmatrix} \tag{2.2.3.5}$$

$$\mathbf{A}-\mathbf{D} = \begin{pmatrix}0\\7\end{pmatrix} \tag{2.2.3.6}$$

$$\mathbf{C}-\mathbf{B} = \begin{pmatrix}5\\-12\end{pmatrix} \tag{2.2.3.7}$$

$$\mathbf{C}-\mathbf{D} = \begin{pmatrix}9\\-3\end{pmatrix} \tag{2.2.3.8}$$

Using (2.2.3.4)

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| = \frac{1}{2}\left|(-28)\right| \tag{2.2.3.9}$$

$$= 14 \tag{2.2.3.10}$$

$$\frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| = \frac{1}{2}\left|(-15+108)\right| \tag{2.2.3.11}$$

$$= 46.5 \tag{2.2.3.12}$$

Substituting the above values in equation (2.2.3.3),We get

$$Area = 14 + 46.5 = 60.5 sq.units \tag{2.2.3.13}$$

4. Find the area of a rhombus if its vertices are

$$\mathbf{P} = \begin{pmatrix}3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}4\\5\end{pmatrix}, \tag{2.2.4.1}$$

$$\mathbf{R} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{S} = \begin{pmatrix}-2\\-1\end{pmatrix} \tag{2.2.4.2}$$

taken in order.
Solution: In Fig. 2.2.4,

$$\mathbf{P}-\mathbf{S}=\begin{pmatrix}3+2\\0+1\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix} \tag{2.2.4.3}$$

$$\mathbf{Q}-\mathbf{P}=\begin{pmatrix}4-3\\5-0\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix} \tag{2.2.4.4}$$

Thus, the area of the rhombus can be calculated as

$$\|(\mathbf{P}-\mathbf{S})\times(\mathbf{Q}-\mathbf{P})\| = \left\|\begin{pmatrix}5\\1\end{pmatrix}\times\begin{pmatrix}1\\5\end{pmatrix}\right\| \tag{2.2.4.5}$$

$$\|\Delta\| = 5\times 5 - 1\times 1 = 24 \tag{2.2.4.6}$$

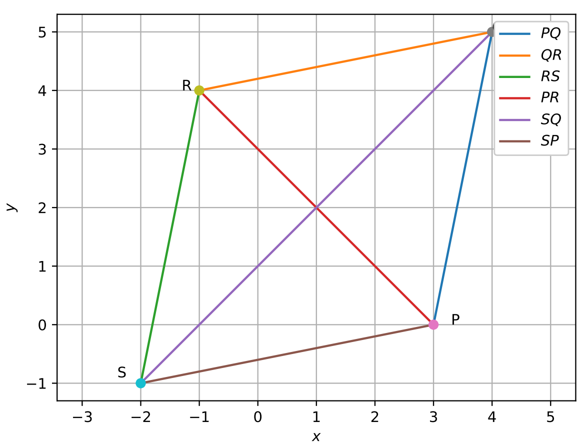

Fig. 2.2.4

```
solutions/4/codes/quadrilateral/quad.py
```

5. Without using distance formula, show that points $\begin{pmatrix}-2\\-1\end{pmatrix}$, $\begin{pmatrix}4\\0\end{pmatrix}$, $\begin{pmatrix}3\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\2\end{pmatrix}$ are the vertices of a parallelogram.
Solution: The following python code plots Fig.2.2.5.

```
./solutions/5/codes/quadrilateral/q4.py
```

$$\because\ \mathbf{A}-\mathbf{B}=\mathbf{D}-\mathbf{C} \tag{2.2.5.1}$$

$$\mathbf{A}-\mathbf{D}=\mathbf{B}-\mathbf{C}, \tag{2.2.5.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence, $ABCD$ is a ||gm.

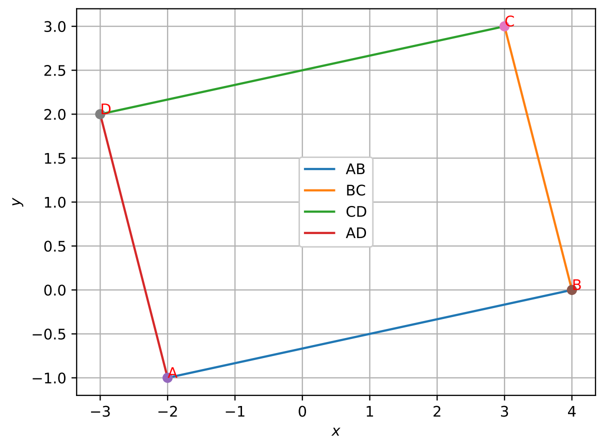

Fig. 2.2.5

6. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}$, $\begin{pmatrix}-3\\-5\end{pmatrix}$, $\begin{pmatrix}3\\-2\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$.
Solution: See quadrilateral $ABCD$ in Fig.2.2.6 is generated using the following python code

```
solutions/6/codes/quadrilateral/quad.py
```

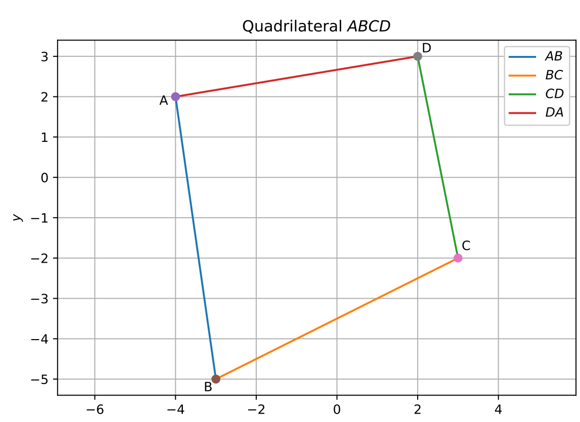

Fig. 2.2.6: Quadrilateral $ABCD$ using python

$$ar(ABCD) = ar(\triangle ABC) + ar(\triangle ACD) \tag{2.2.6.1}$$

$$= \frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{2.2.6.2}$$

$$+\frac{1}{2}\left\|(\mathbf{C}-\mathbf{A})\times(\mathbf{D}-\mathbf{A})\right\| \tag{2.2.6.3}$$

$$= \frac{1}{2}\left\|\begin{pmatrix}1\\-7\end{pmatrix}\times\begin{pmatrix}7\\-4\end{pmatrix}\right\| \tag{2.2.6.4}$$

$$+\frac{1}{2}\left\|\begin{pmatrix}7\\-4\end{pmatrix}\times\begin{pmatrix}6\\1\end{pmatrix}\right\| \tag{2.2.6.5}$$

$$= 38 \tag{2.2.6.6}$$

and verified using the following codes

```
solutions/6/codes/tri_area_ABC.py
```

```
solutions/6/codes/tri_area_ACD.py
```

7. The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.
Solution: See Fig. 2.2.7.

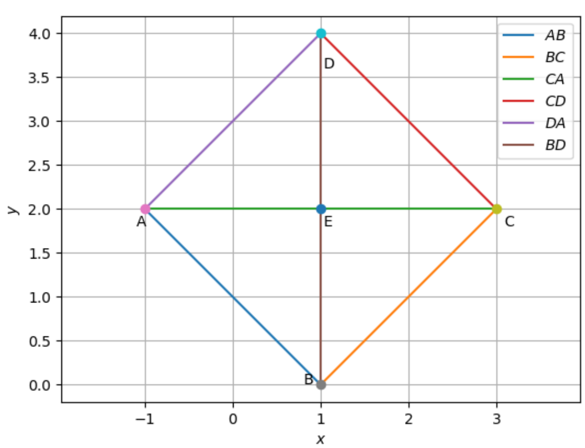

Fig. 2.2.7: Square ABCD

a) From inspection we see that the opposite vertices forms a diagonal which is parallel to x-axis. Then the diagonal formed by other two vertices is parallel to y-axis(i.e. their x coordinates are equal). Let $\mathbf{A} = \begin{pmatrix}-1\\2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\2\end{pmatrix}$.
b) Diagonals bisect each other at 90°. Let **B** and **D** be other two vertices.
c) Using the property that diagonals bisect each other at 90°, we can obtain other vertices by rotating diagonal AC by 90°about **E** in clockwise or anticlockwise direction.
d) The rotation matrix for a rotation of angle 90°about origin in anticlockwise direction is given by

$$\begin{pmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{2.2.7.1}$$

The **E** is given by

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{2.2.7.2}$$

$$= \begin{pmatrix}1\\2\end{pmatrix} \tag{2.2.7.3}$$

e) To make the rotation we need to shift the **E** to origin. So the change in other vectors are

$$\mathbf{A}-\mathbf{E} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{2.2.7.4}$$

$$\mathbf{C}-\mathbf{E} = \begin{pmatrix}2\\0\end{pmatrix} \tag{2.2.7.5}$$

The required matrix now is $\begin{pmatrix}-2 & 2\\ 0 & 0\end{pmatrix}$. Multiplying this with rotation matrix

$$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-2 & 2\\ 0 & 0\end{pmatrix} \tag{2.2.7.6}$$

$$= \begin{pmatrix}0 & 0\\ -2 & 2\end{pmatrix} \tag{2.2.7.7}$$

Now we obtained the coordinates as $\begin{pmatrix}0\\-2\end{pmatrix}$ and $\begin{pmatrix}0\\2\end{pmatrix}$. To obtain the final coordinates we will add **E** to shift to the actual position.

$$\mathbf{B} = \begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} \tag{2.2.7.8}$$

$$\mathbf{D} = \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} \tag{2.2.7.9}$$

Thus

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.2.7.10}$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{2.2.7.11}$$

f) The python code for the figure can be downloaded from

```
solutions/7/codes/quad/quad.py
```

8. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
Solution: The area of a parallelogram is defined as

$$\|\mathbf{a} \times \mathbf{b}\| \tag{2.2.8.1}$$

where

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2.2.8.2}$$

$$= \begin{pmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} \tag{2.2.8.3}$$

Thus, the desired area is

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{5^2 + 1^2 + (-1)^2} \tag{2.2.8.4}$$

$$= 3\sqrt{3} \tag{2.2.8.5}$$

The following Python code generates Fig. 2.2.8

```
codes/parallelogram.py
```

The following Python code verifies the cross-product value.

```
codes/cross_product_check.py
```

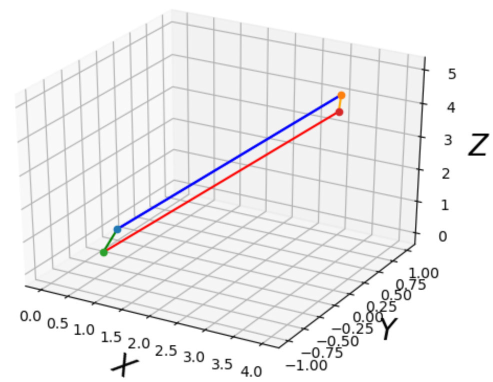

Fig. 2.2.8: Parallelogram generated using python 3D-plot

9. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$.
10. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$.
Solution: Area of rectangle = cross product of vectors of adjacent sides

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \tag{2.2.10.1}$$

Area = cross product of vectors

$$\|(\mathbf{A} - \mathbf{D}) \times (\mathbf{B} - \mathbf{A})\| \tag{2.2.10.2}$$

$$= \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\| \tag{2.2.10.3}$$

$$= \left\| \begin{pmatrix} 0 & -0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\| \tag{2.2.10.4}$$

$$= 2 \tag{2.2.10.5}$$

Area = 2
11. The two adjacent sides of a parallelogram are $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

3 Line

3.1 Geometry: Examples

1. Check whether –2 and 2 are zeroes of the polynomial $x + 2$.

Solution: Let

$$y = x + 2 \implies \begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} \quad = 2 \tag{3.1.1.1}$$

Thus,

$$y = 0 \tag{3.1.1.2}$$
$$\implies x + 2 = 0 \tag{3.1.1.3}$$
$$\text{or, } x = -2 \tag{3.1.1.4}$$

Hence -2 is a zero. This is verified in Fig. 3.1.1.

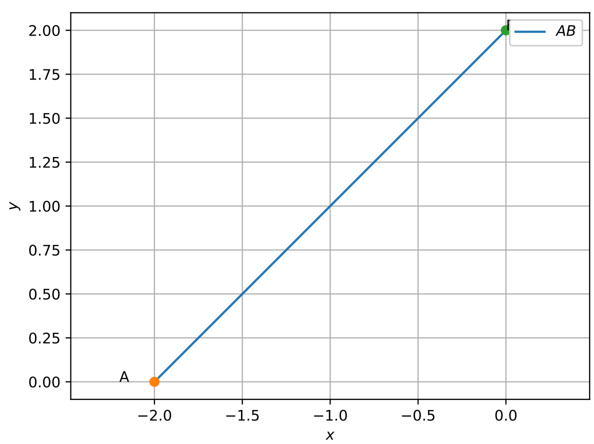

Fig. 3.1.1

2. Find a zero of the polynomial $p(x) = 2x + 1$.
Solution: $p\left(-\frac{1}{2}\right) = 0$.
3. Find four different solutions of the equation

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{3.1.3.1}$$

Solution: Let

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{3.1.3.2}$$

Substituting in (3.1.3.1),

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 6 \tag{3.1.3.3}$$
$$\implies a = 6 \tag{3.1.3.4}$$

Simiarly, substituting

$$\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{3.1.3.5}$$

in (3.1.3.1),

$$b = 3 \tag{3.1.3.6}$$

More solutions can be obtained in a similar fashion.
4. Draw the graph of

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{3.1.4.1}$$

Solution: The intercepts on the x and y-axis can be obtained from Problem 3.1.3 as

$$\mathbf{A} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{3.1.4.2}$$

The following python code can be used to draw the graph in Fig. 3.1.4.

```
codes/line/line_icept.py
```

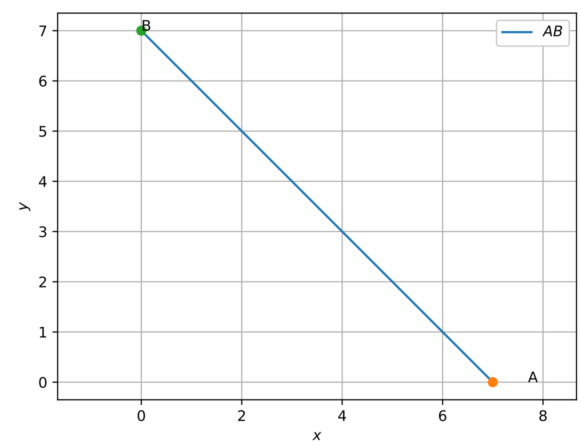

Fig. 3.1.4

5. Two rails are represented by the equations

$$\begin{aligned} \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} &= 4 \text{ and} \\ \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} &= 12. \end{aligned} \tag{3.1.5.1}$$

Will the rails cross each other?
Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{3.1.5.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 6 \end{pmatrix} \tag{3.1.5.3}$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{3.1.5.4}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \tag{3.1.5.5}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{3.1.5.6}$$

is 1, from 3.1.5.4.

$$\because rank \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \neq rank \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix}, \tag{3.1.5.7}$$

(3.1.5.1) has no solution. The equivalent python code is

```
codes/line/line_check_sol.py
```

which plots Fig. 3.1.5, which shows that the rails are parallel.

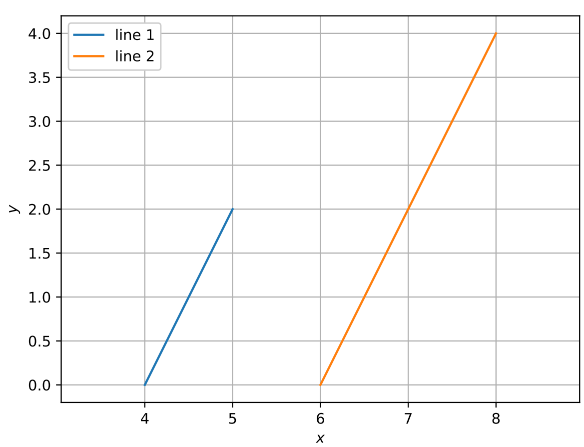

Fig. 3.1.5

6. Check whether the pair of equations

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 6 \text{ and} \\ \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 12 \tag{3.1.6.1}$$

is consistent.
Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \tag{3.1.6.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 3 & 6 \\ 2 & -3 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2 - 2R_1}{-9}} \tag{3.1.6.3}$$

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.1.6.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{3.1.6.5}$$

which is the solution of 3.1.5.1. The python code in Problem 3.1.5 can be used to plot Fig. 3.1.6, which shows that the lines intersect.

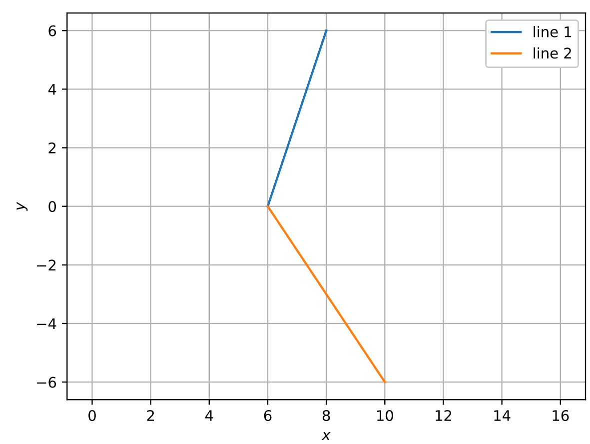

Fig. 3.1.6

7. Find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$\begin{pmatrix} 5 & -8 \end{pmatrix} \mathbf{x} = -1 \text{ and} \\ \begin{pmatrix} 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\frac{3}{5} \tag{3.1.7.1}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\begin{pmatrix} 1 \\ \frac{3}{5} \end{pmatrix} \tag{3.1.7.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 5 & -8 & -1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2} \begin{pmatrix} 5 & -8 & 1 \\ 15 & -24 & -3 \end{pmatrix} \tag{3.1.7.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1.7.4}$$

$$\because rank \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = rank \begin{pmatrix} 5 & -8 & 1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{pmatrix} \tag{3.1.7.5}$$

$$= 1 < dim \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = 2, \tag{3.1.7.6}$$

(3.1.7.1) has infinitely many solutions. The python code in Problem 3.1.5 can be used to plot Fig. 3.1.7, which shows that the lines are the same.

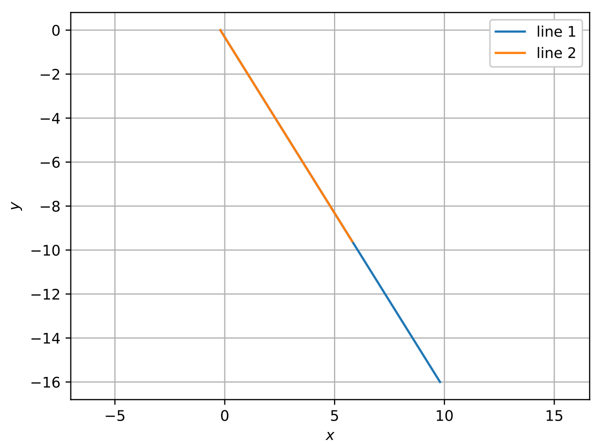

Fig. 3.1.7

8. Solve the following pair of equations

$$\begin{pmatrix} 7 & -15 \end{pmatrix} \mathbf{x} = 2 \tag{3.1.8.1}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 3 \tag{3.1.8.2}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 7 & -15 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{3.1.8.3}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 7 & -15 & 2 \\ 1 & 2 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 7R_2 - R_1} \begin{pmatrix} 7 & -15 & 2 \\ 0 & 29 & 19 \end{pmatrix} \tag{3.1.8.4}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{15R_2 29 + 29R_1}{29}} \begin{pmatrix} 7 & 0 & 2 \\ 0 & 29 & 19 \end{pmatrix} \tag{3.1.8.5}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{2}{7} \\ \frac{19}{29} \end{pmatrix} \tag{3.1.8.6}$$

The python code in Problem 3.1.5 can be used to plot Fig. 3.1.8, which shows that the lines are the same.

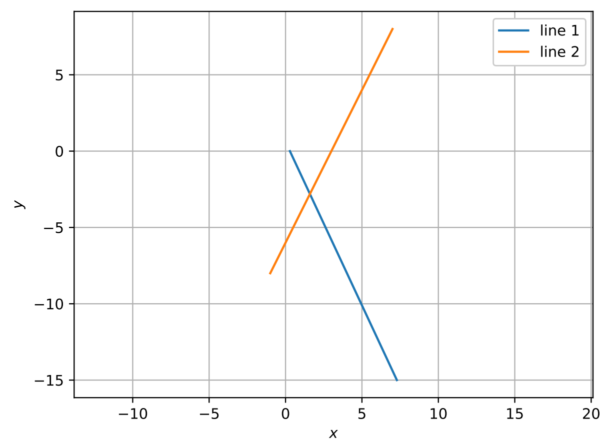

Fig. 3.1.8

9. Find all possibe solutions of

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} &= 8 \\ \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} &= 7 \end{aligned} \tag{3.1.9.1}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \tag{3.1.9.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 2 & -3 & 8 \\ 0 & 0 & -9 \end{pmatrix} \tag{3.1.9.3}$$

$$\implies rank \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 7 \end{pmatrix}. \tag{3.1.9.4}$$

Hence, (3.1.9.1) has no solution. The python code in Problem 3.1.5 can be used to plot Fig. 3.1.9, which shows that the lines are parallel.

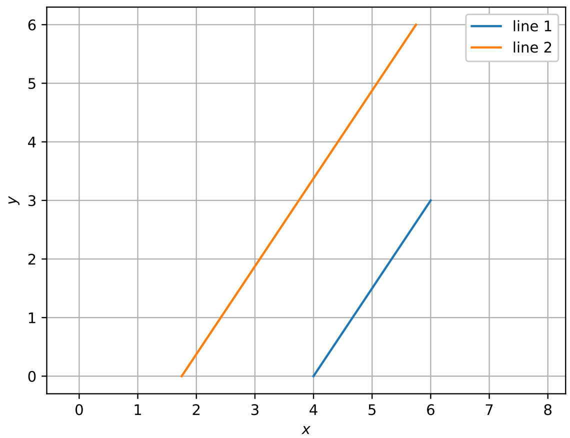

Fig. 3.1.9

10. For which values of p does the pair of equations given below has unique solution?

$$\begin{pmatrix} 4 & p \end{pmatrix} \mathbf{x} = -8 \\ \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} = -2 \tag{3.1.10.1}$$

Solution: (3.1.10.1) has a unique solution

$$\iff \begin{vmatrix} 4 & p \\ 2 & 2 \end{vmatrix} \neq 0 \tag{3.1.10.2}$$

$$\text{or, } p \neq 4 \tag{3.1.10.3}$$

11. For what values of k will the following pair of linear equations have infinitely many solutions?

$$\begin{pmatrix} k & 3 \end{pmatrix} \mathbf{x} = k - 3 \\ \begin{pmatrix} 12 & k \end{pmatrix} \mathbf{x} = k \tag{3.1.11.1}$$

Solution: The first condition for (3.1.11.1) to have infinite solutions is

$$\begin{vmatrix} k & 3 \\ 12 & k \end{vmatrix} = 0 \tag{3.1.11.2}$$

$$\implies k^2 = 36, \text{or, } k = \pm 6 \tag{3.1.11.3}$$

For $k = 6$, the augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 6 & 3 & 3 \\ 12 & 6 & 6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1.11.4}$$

indicating that (3.1.11.1) has infinite number of solutions. For $k = -6$, the augmented matrix is

$$\begin{pmatrix} 6 & 3 & -9 \\ 12 & 6 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & -9 \\ 0 & 0 & 12 \end{pmatrix} \tag{3.1.11.5}$$

indicating that (3.1.11.1) has no solution Thus, (3.1.11.2) is a necessary condition but not sufficient.

12. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \tag{3.1.12.1}$$

Solution: $x = 2, y = 2, z = 1$.

13. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{3.1.13.1}$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$
b) $\mathbf{a} = \mathbf{b}$

Solution:

a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

14. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.1.14.1}$$

15. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

16. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

17. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such

that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k\begin{pmatrix}1\\-2\end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\| = 7 \tag{3.1.17.1}$$

$$\implies |k| = \frac{7}{\sqrt{5}} \tag{3.1.17.2}$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \tag{3.1.17.3}$$

18. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix}2\\2\\-5\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\\3\end{pmatrix}. \tag{3.1.18.1}$$

19. Find a unit vector in the direction of

$$\begin{pmatrix}1\\1\\-2\end{pmatrix}. \tag{3.1.19.1}$$

20. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix} \tag{3.1.20.1}$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix}3\\5\\4\end{pmatrix}, \tag{3.1.20.2}$$

21. Verify if $\mathbf{A} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}8\\6\end{pmatrix}$ are points on a line.
Solution: Refer to Problem 1.1.1.
22. Find the condition for $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ to be equidistant from the points $\begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$.
Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}7\\1\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}3\\5\end{pmatrix}\right\|^2 \tag{3.1.22.1}$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}7\\1\end{pmatrix}\right\|^2 - 2\begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}$$

$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}3\\5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} \tag{3.1.22.2}$$

which can be simplified to obtain

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \tag{3.1.22.3}$$

which is the desired condition. The following code plots Fig. 3.1.22clearly showing that the above equation is the perpendicular bisector of AB.

```
codes/line/line_perp_bisect.py
```

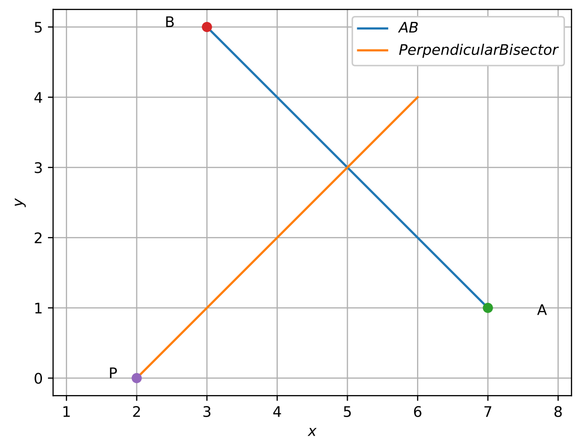

Fig. 3.1.22

23. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix}6\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\3\end{pmatrix}$.
Solution: Choose $\mathbf{x} = \begin{pmatrix}0\\y\end{pmatrix}$ and follow the approach in Problem (3.1.22). Solve for y.
24. Draw a line segement of length 7.6 cm and divide it in the ratio 5 : 8.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7.6\\0\end{pmatrix} \tag{3.1.24.1}$$

From (1.1.1.4), the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{3.1.24.2}$$

If $\mathbf{C}$ divides AB in the ratio

$$m = \frac{5}{8}, \tag{3.1.24.3}$$

then,

$$\frac{\|\mathbf{C}-\mathbf{A}\|^2}{\|\mathbf{B}-\mathbf{C}\|^2} = m^2 \quad (3.1.24.4)$$

$$\implies \frac{\frac{k^2\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}}{\frac{\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}} = m^2 \quad (3.1.24.5)$$

$$\implies k = m \quad (3.1.24.6)$$

upon substituting from (3.1.24.4) and simplifying. (3.1.24.2) is known as the section formula. The following code plots Fig. 3.1.24

```
codes/line/draw_section.py
```

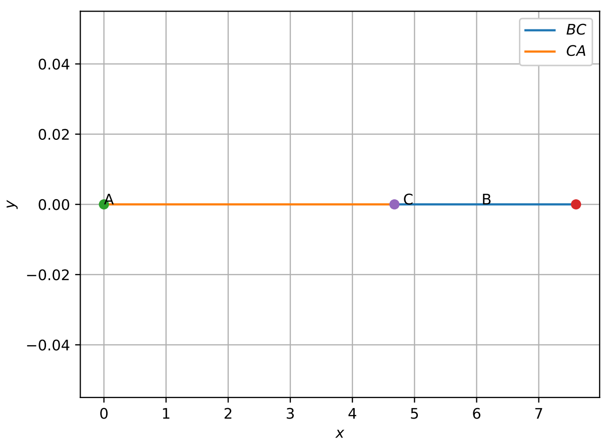

Fig. 3.1.24

25. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.
Solution: Using (3.1.24.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}}{4} \quad (3.1.25.1)$$

26. In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \quad (3.1.26.1)$$

Solution: Use (3.1.24.2).

27. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \quad (3.1.27.1)$$

Solution: Using (3.1.24.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \quad (3.1.27.2)$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \quad (3.1.27.3)$$

28. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.
Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.1.24.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (3.1.28.1)$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \quad (3.1.28.2)$$

29. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.
Solution: Forming the matrix in (1.1.1.6),

$$\mathbf{M} = \begin{pmatrix} \mathbf{B}-\mathbf{A} & \mathbf{B}-\mathbf{A} \end{pmatrix}^T = \begin{pmatrix} 2 & k-3 \\ 4 & -6 \end{pmatrix} \quad (3.1.29.1)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 2 & k-3 \\ 2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & k-3 \\ 0 & -k \end{pmatrix} \quad (3.1.29.2)$$

$$\implies rank(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{or } k = 0 \quad (3.1.29.3)$$

30. Find the direction vectors and slopes of the lines passing through the points
a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.
b) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.
c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.
d) Making an inclination of 60° with the positive direction of the x-axis.

Solution:

a) If the direction vector is

$$\begin{pmatrix}1\\m\end{pmatrix}, \tag{3.1.30.1}$$

the slope is m. Thus, the direction vector is

$$\begin{pmatrix}-1\\4\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}=-\frac{1}{4}\begin{pmatrix}-4\\6\end{pmatrix} \tag{3.1.30.2}$$

$$=\begin{pmatrix}1\\-\frac{3}{2}\end{pmatrix} \implies m=-\frac{3}{2} \tag{3.1.30.3}$$

b) The direction vector is

$$\begin{pmatrix}7\\-2\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix} \tag{3.1.30.4}$$

$$=\begin{pmatrix}1\\0\end{pmatrix} \implies m=0 \tag{3.1.30.5}$$

c) The direction vector is

$$\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix} \tag{3.1.30.6}$$

$$=\begin{pmatrix}1\\\infty\end{pmatrix} \implies m=\infty \tag{3.1.30.7}$$

d) The slope is $m = \tan 60^\circ = \sqrt{3}$ and the direction vector is

$$\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{3.1.30.8}$$

31. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.
Solution: The angle θ between two lines is given by

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2} \tag{3.1.31.1}$$

$$\implies 1=\frac{m_1-\frac{1}{4}}{1+\frac{m_1}{4}} \tag{3.1.31.2}$$

$$\text{or } m_1=\frac{5}{3} \tag{3.1.31.3}$$

32. The line through the points $\begin{pmatrix}-2\\6\end{pmatrix}$ and $\begin{pmatrix}4\\8\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}8\\12\end{pmatrix}$ and $\begin{pmatrix}x\\24\end{pmatrix}$. Find the value of x.
Solution: Using (1.1.1.11)

$$\left\{\begin{pmatrix}-2\\6\end{pmatrix}-\begin{pmatrix}4\\8\end{pmatrix}\right\}^T\left\{\begin{pmatrix}8\\12\end{pmatrix}-\begin{pmatrix}x\\24\end{pmatrix}\right\}=0 \tag{3.1.32.1}$$

which can be used to obtain x.

33. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.
Solution: The equation of the line joining the points $\mathbf{A}=\begin{pmatrix}0\\2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3\\8\end{pmatrix}$ is obtained as

$$\mathbf{x}=\mathbf{A}+\lambda\left(\mathbf{B}-\mathbf{A}\right) \tag{3.1.33.1}$$

$$\implies \begin{pmatrix}T\\D\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}-\lambda\begin{pmatrix}-3\\-6\end{pmatrix} \tag{3.1.33.2}$$

which can be expressed as

$$\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix}=\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{3.1.33.3}$$

$$\implies \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix}=-2 \tag{3.1.33.4}$$

$$\implies D=2+2T \tag{3.1.33.5}$$

34. Find the equations of the lines parallel to the axes and passing through $\mathbf{A}=\begin{pmatrix}-2\\3\end{pmatrix}$.
Solution: The line parallel to the x-axis has direction vector $\mathbf{m}=\begin{pmatrix}1\\0\end{pmatrix}$. Hence,its equation is obtined as

$$\mathbf{x}=\begin{pmatrix}-2\\3\end{pmatrix}+\lambda_1\begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.34.1}$$

Similarly, the equation of the line parallel to the y-axis can be obtained as

$$\mathbf{x}=\begin{pmatrix}-2\\3\end{pmatrix}+\lambda_1\begin{pmatrix}0\\1\end{pmatrix} \tag{3.1.34.2}$$

The following code plots Fig. 3.1.34

```
codes/line/line_parallel_axes.py
```

35. Find the equation of the line through $\mathbf{A}=\begin{pmatrix}-2\\3\end{pmatrix}$ with slope –4.
Solution: The direction vector is $\mathbf{m}=\begin{pmatrix}1\\-4\end{pmatrix}$.

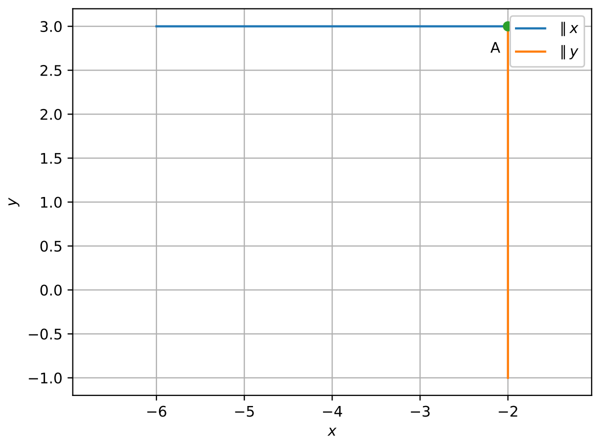

Fig. 3.1.34

Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} \tag{3.1.35.1}$$

$$= \begin{pmatrix}4\\1\end{pmatrix} \tag{3.1.35.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.35.3}$$

$$\implies \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = -5 \tag{3.1.35.4}$$

36. Write the equation of the line through the points $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$.
Solution: Use (3.1.34.1).
37. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and
a) y-intercept is $-\frac{3}{2}$
b) x-intercept is 4.
Solution: From the given information, $\tan\theta = \frac{1}{2} = m$.
a) y-intercept is $-\frac{3}{2} \implies$ the line cuts through the y-axis at $\begin{pmatrix}0\\-\frac{3}{2}\end{pmatrix}$.
b) x-intercept is $4 \implies$ the line cuts through the x-axis at $\begin{pmatrix}4\\0\end{pmatrix}$.
Use the above information get the equations for the lines.
38. Find the equation of a line through the point $\begin{pmatrix}5\\2\\-4\end{pmatrix}$ and parallel to the vector $\begin{pmatrix}3\\2\\-8\end{pmatrix}$.
Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix}5 & 2\\-4\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-8\end{pmatrix} \tag{3.1.38.1}$$

39. Find the equation of a line passing through the points $\begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $\begin{pmatrix}3\\4\\6\end{pmatrix}$.
Solution: Using (3.1.33.1), the desired equation of the line is

$$\mathbf{x} = \begin{pmatrix}-1 & 0\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\4\\4\end{pmatrix} \tag{3.1.39.1}$$

$$= \begin{pmatrix}-1 & 0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{3.1.39.2}$$

40. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \tag{3.1.40.1}$$

find the equation of the line.
Solution: The line can be expressed from (3.1.40.1) as

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3+2\lambda\\5+4\lambda\\-6+2\lambda\end{pmatrix} \tag{3.1.40.2}$$

$$\implies \mathbf{x} = \begin{pmatrix}-3\\5\\-6\end{pmatrix} + \lambda\begin{pmatrix}2\\4\\2\end{pmatrix} \tag{3.1.40.3}$$

$$\implies \mathbf{x} = \begin{pmatrix}-3\\5\\-6\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix} \tag{3.1.40.4}$$

41. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
Solution: See Problem 3.1.37. The line passes through the points $\begin{pmatrix}-3\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\end{pmatrix}$.
42. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°.
Solution: In Fig. 3.1.42, the foot of the perpendicular P is the intersection of the lines L

and M. Thus,

$$\mathbf{n}^T\mathbf{P} = c \tag{3.1.42.1}$$

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{3.1.42.2}$$

$$\text{or, } \mathbf{n}^T\mathbf{P} = \mathbf{n}^T\mathbf{A} + \lambda\left\|\mathbf{n}\right\|^2 = c \tag{3.1.42.3}$$

$$\implies -\lambda = \frac{\mathbf{n}^T\mathbf{A} - c}{\left\|\mathbf{n}\right\|^2} \tag{3.1.42.4}$$

Also, the distance between $\mathbf{A}$ and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{3.1.42.5}$$

$$\implies \left\|\mathbf{P} - \mathbf{A}\right\| = |\lambda|\left\|\mathbf{n}\right\| \tag{3.1.42.6}$$

From (3.1.42.4) and (3.1.42.6)

$$\left\|\mathbf{P} - \mathbf{A}\right\| = \frac{\left|\mathbf{n}^T\mathbf{A} - c\right|}{\left\|\mathbf{n}\right\|} \tag{3.1.42.7}$$

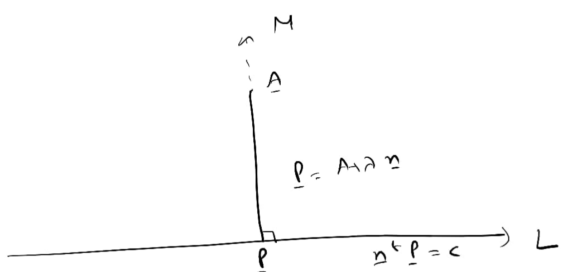

Fig. 3.1.42

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15^\circ \end{pmatrix} \tag{3.1.42.8}$$

$\because \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\left\|\mathbf{n}\right\|} \implies c = \pm 4\sqrt{1 + \tan^2 15^\circ} \tag{3.1.42.9}$$

$$= \pm 4 \sec 15^\circ \tag{3.1.42.10}$$

where

$$\sec\theta = \frac{1}{\cos\theta} \tag{3.1.42.11}$$

This follows from the fact that

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.1.42.12}$$

$$\implies 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{3.1.42.13}$$

It is easy to verify that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \tag{3.1.42.14}$$

$$\implies 1 + \tan^2\theta = \sec^2\theta \tag{3.1.42.15}$$

Thus, the equation of the line is

$$\begin{pmatrix} 1 & \tan 15^\circ \end{pmatrix}\mathbf{c} = \pm 4 \sec 15^\circ \tag{3.1.42.16}$$

43. The Farenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F, when $K = 0$.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \tag{3.1.43.1}$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}$, $\begin{pmatrix} 373 \\ 21 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (3.1.35.3) and (3.1.35.1) as

$$\begin{pmatrix} 11 & -100 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 11 & -100 \end{pmatrix}\begin{pmatrix} 273 \\ 32 \end{pmatrix} \tag{3.1.43.2}$$

$$\implies \begin{pmatrix} 11 & -100 \end{pmatrix}\mathbf{x} = -197 \tag{3.1.43.3}$$

If $\begin{pmatrix} F \\ 0 \end{pmatrix}$ is a point on the line,

$$\begin{pmatrix} 11 & -100 \end{pmatrix}\begin{pmatrix} F \\ 0 \end{pmatrix} = -197 \implies F = -\frac{197}{11} \tag{3.1.43.4}$$

44. Equation of a line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix}\mathbf{x} + 10 = 0. \tag{3.1.44.1}$$

Find its
a) slope,
b) x - and y-intercepts.
Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \tag{3.1.44.2}$$

$$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \tag{3.1.44.3}$$

a) $m = \frac{3}{4}$
b) x-intercept is $-\frac{10}{3}$ and y-intercept is $\frac{10}{4} = \frac{5}{2}$.

45. Find the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ where

$$\left\|\mathbf{a}\right\| = 1, \left\|\mathbf{b}\right\| = 2, \mathbf{a}^T\mathbf{b} = 1. \tag{3.1.45.1}$$

Solution: In Fig. 3.1.45, from the cosine formula,

$$\cos\theta = \frac{\left\|\mathbf{A}-\mathbf{B}\right\|^2 + \left\|\mathbf{B}-\mathbf{C}\right\|^2 - \left\|\mathbf{A}-\mathbf{C}\right\|^2}{2\left\|\mathbf{A}-\mathbf{B}\right\|\left\|\mathbf{B}-\mathbf{C}\right\|} \tag{3.1.45.2}$$

Letting $\mathbf{a} = \mathbf{A}-\mathbf{B}, \mathbf{b} = \mathbf{B}-\mathbf{C}$,

$$\cos\theta = \frac{\left\|\mathbf{a}\right\|^2 + \left\|\mathbf{b}\right\|^2 - \left\|\mathbf{a}+\mathbf{b}\right\|^2}{2\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|} \tag{3.1.45.3}$$

$$= \frac{\left\|\mathbf{a}\right\|^2 + \left\|\mathbf{b}\right\|^2 - \left[\left\|\mathbf{a}\right\|^2 + \left\|\mathbf{b}\right\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|} \tag{3.1.45.4}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|} \tag{3.1.45.5}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|} \tag{3.1.45.6}$$

$$= \frac{1}{2} \tag{3.1.45.7}$$

$$\implies \theta = 60^\circ \tag{3.1.45.8}$$

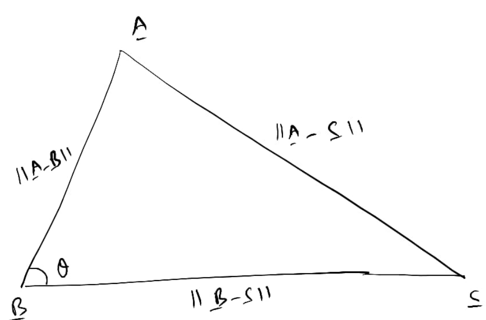

Fig. 3.1.45

46. Find the angle between the lines

$$\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 5 \tag{3.1.46.1}$$

$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = -6. \tag{3.1.46.2}$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\left\|\mathbf{n}_1\right\|\left\|\mathbf{n}_2\right\|} \tag{3.1.46.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30^\circ \tag{3.1.46.4}$$

47. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3 \tag{3.1.47.1}$$

and passes through the point $\begin{pmatrix}1\\-2\end{pmatrix}$.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix}2\\1\end{pmatrix} \tag{3.1.47.2}$$

Thus, the desired equation of the line is

$$\begin{pmatrix}2 & 1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\-2\end{pmatrix}\right) = 0 \tag{3.1.47.3}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.47.4}$$

48. Find the distance of the point $\begin{pmatrix}3\\-5\end{pmatrix}$ from the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 26 \tag{3.1.48.1}$$

Solution: Use (3.1.42.7).

49. If the lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{3.1.49.1}$$

$$\begin{pmatrix}5 & k\end{pmatrix}\mathbf{x} = 3 \tag{3.1.49.2}$$

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 2 \tag{3.1.49.3}$$

are concurrent, find the value of k.

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix}2&1&3\\5&k&3\\3&-1&2\end{pmatrix}\xleftrightarrow{R_2\leftrightarrow R_3}\begin{pmatrix}2&1&3\\3&-1&2\\5&k&3\end{pmatrix}\tag{3.1.49.4}$$

$$\xleftrightarrow[R_3\leftarrow 2R_3-5R_1]{R_2\leftrightarrow 2R_2-3R_1}\begin{pmatrix}2&1&3\\0&-5&-5\\0&2k-5&-9\end{pmatrix}\tag{3.1.49.5}$$

$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{5}}\begin{pmatrix}2&1&3\\0&1&1\\0&2k-5&-9\end{pmatrix}\tag{3.1.49.6}$$

$$\xleftrightarrow{R_3\leftarrow R_3-(2k-5)R_2}\begin{pmatrix}2&1&3\\0&1&1\\0&0&-2k-4\end{pmatrix}\tag{3.1.49.7}$$

$$\implies k=-2\tag{3.1.49.8}$$

50. Find the distance of the line

$$L_1:\quad \begin{pmatrix}4&1\end{pmatrix}\mathbf{x}=0\tag{3.1.50.1}$$

from the point $\begin{pmatrix}4\\1\end{pmatrix}$ measured along the line L_2 making an angle of 135° with the positive x-axis.

Solution: Let P be the point of intersection of L_1 and L_2. The direction vector of L_2 is

$$\mathbf{m}=\begin{pmatrix}1\\\tan 135^\circ\end{pmatrix}\tag{3.1.50.2}$$

Since $\begin{pmatrix}4\\1\end{pmatrix}$ lies on L_2, the equation of L_2 is

$$\mathbf{x}=\begin{pmatrix}4\\1\end{pmatrix}+\lambda\mathbf{m}\tag{3.1.50.3}$$

$$\implies \mathbf{P}=\begin{pmatrix}4\\1\end{pmatrix}+\lambda\mathbf{m}\tag{3.1.50.4}$$

$$\text{or, }\left\|\mathbf{P}-\begin{pmatrix}4\\1\end{pmatrix}\right\|=d=|\lambda|\,\|\mathbf{m}\|\tag{3.1.50.5}$$

Since $\mathbf{P}$ lies on L_1, from (3.1.50.1),

$$\begin{pmatrix}4&1\end{pmatrix}\mathbf{P}=0\tag{3.1.50.6}$$

Substituting from the above in (3.1.50.3),

$$\begin{pmatrix}4&1\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix}+\lambda\begin{pmatrix}4&1\end{pmatrix}\mathbf{m}=0\tag{3.1.50.7}$$

$$\implies \lambda=\frac{\begin{pmatrix}4&1\end{pmatrix}\mathbf{m}}{17}\tag{3.1.50.8}$$

substituting $|\lambda|$ in (3.1.50.5) gives the desired answer.

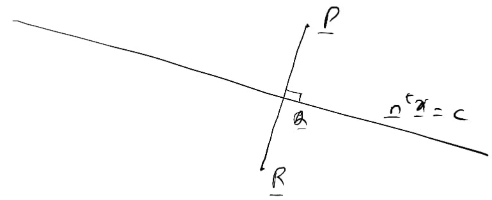

Fig. 3.1.51

51. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix}1\\2\end{pmatrix}$ in the line

$$\begin{pmatrix}1&-3\end{pmatrix}\mathbf{x}=-4.\tag{3.1.51.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q}=\mathbf{P}+\mathbf{R}\tag{3.1.51.2}$$

$$\mathbf{n}^T\mathbf{Q}=c\tag{3.1.51.3}$$

$$\mathbf{m}^T\mathbf{R}=\mathbf{m}^T\mathbf{P}\tag{3.1.51.4}$$

where $\mathbf{m}$ is the direction vector of L. From (3.1.51.2) and (3.1.51.3),

$$\mathbf{n}^T\mathbf{R}=2c-\mathbf{n}^T\mathbf{P}\tag{3.1.51.5}$$

From (3.1.51.5) and (3.1.51.4),

$$\begin{pmatrix}\mathbf{m}&\mathbf{n}\end{pmatrix}^T\mathbf{R}=\begin{pmatrix}\mathbf{m}&-\mathbf{n}\end{pmatrix}^T\mathbf{P}+\begin{pmatrix}0\\2c\end{pmatrix}\tag{3.1.51.6}$$

Letting

$$\mathbf{V}=\begin{pmatrix}\mathbf{m}&\mathbf{n}\end{pmatrix}\tag{3.1.51.7}$$

with the condition that $\mathbf{m},\mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V}=\mathbf{I}\tag{3.1.51.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m}&-\mathbf{n}\end{pmatrix}=\begin{pmatrix}\mathbf{m}&\mathbf{n}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{3.1.51.9}$$

(3.1.51.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.51.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.51.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{3.1.51.12}$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{3.1.51.13}$$

The following code plots Fig. 3.1.51 while computing the reflection

```
codes/line/line_reflect.py
```

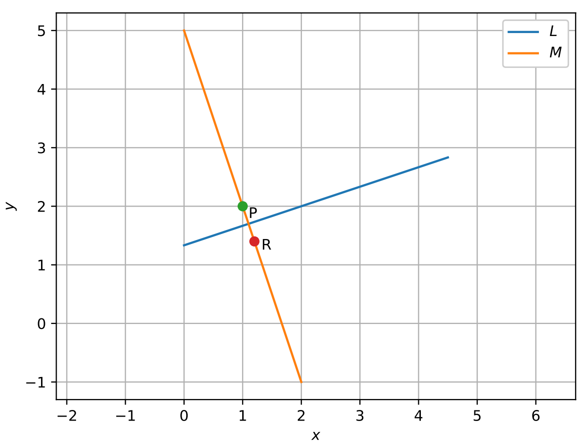

Fig. 3.1.51

52. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix}1\\5\end{pmatrix}$. Obtain its equation.

$$L_1: \quad \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = -4 \tag{3.1.52.1}$$

$$L_2: \quad \begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.1.52.2}$$

Solution: Let

$$L: \quad \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.52.3}$$

If L intersects L_1 and L_2 at $\mathbf{A}$ and $\mathbf{B}$ respectively,

$$\mathbf{A} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.52.4}$$

$$\mathbf{B} = \mathbf{P} - \lambda\mathbf{m} \tag{3.1.52.5}$$

since $\mathbf{P}$ bisects AB. Note that λ is a measure of the distance from P along the line L. From (3.1.52.1), (3.1.52.4) and (3.1.52.5),

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{A} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} + \lambda\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4 \tag{3.1.52.6}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{B} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} - \lambda\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} = 4 \tag{3.1.52.7}$$

yielding

$$19\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} \tag{3.1.52.8}$$

$$\implies \begin{pmatrix}107 & -3\end{pmatrix}\mathbf{m} = 0 \tag{3.1.52.9}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix}107\\-3\end{pmatrix} \tag{3.1.52.10}$$

after simplification. Thus, the equation of the line is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{3.1.52.11}$$

53. Show that the path of a moving point such that its distances from two lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.53.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.53.2}$$

are equal is a straight line.
Solution: Using (3.1.42.7) the point $\mathbf{x}$ satisfies

$$\frac{\left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\-2\end{pmatrix}\right\|} = \frac{\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|} \tag{3.1.53.3}$$

$$\implies \left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right| = \left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right| \tag{3.1.53.4}$$

resulting in

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5 = \pm\left(\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right) \tag{3.1.53.5}$$

leading to the possible lines

$$L_1 : \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.53.6}$$

$$L_2 : \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{5}{3} \tag{3.1.53.7}$$

54. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \tag{3.1.54.1}$$

Solution:
The distance between the two points is given by or,

$$\begin{aligned} d &= \|\mathbf{P} - \mathbf{Q}\| \\ &= \left\| \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} \right\| \\ \implies d &= \sqrt{5^2 + (-4)^2 + 2^2} \\ &= 3\sqrt{5} \end{aligned} \tag{3.1.54.2}$$

The following Python code generates Fig. 3.1.54

```
solutions/line/geometry/examples/54/codes/
    point_distance.py
```

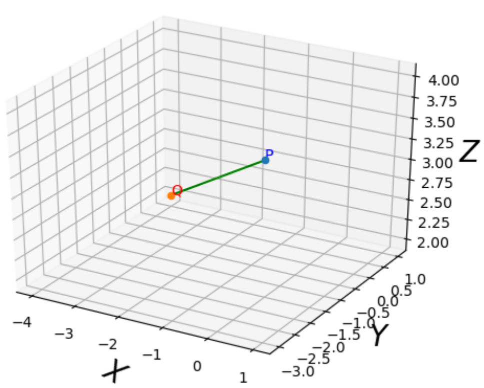

Fig. 3.1.54: Two points and distance between them.

The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

55. Show that the points $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.1.1.6)

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & -2 \\ 9 & -3 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1.55.1}$$

$\implies$ $rank(\mathbf{M}) = 1$. The following code plots Fig. 3.1.55 showing that the points are collinear.

```
codes/line/draw_lines_3d.py
```

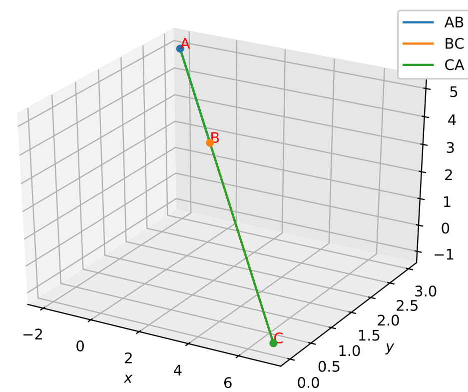

Fig. 3.1.55

56. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$ are collinear.
Solution: Use the approach in Problem (3.1.55).
57. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \tag{3.1.57.1}$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}, \tag{3.1.57.2}$$

respectively.
58. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ in the ratio 2 : 3

a) internally, and
b) externally.

Solution: Use (3.1.24.2).

59. Prove that the three points $\begin{pmatrix}-4\\6\\10\end{pmatrix}$, $\begin{pmatrix}2\\4\\6\end{pmatrix}$ and $\begin{pmatrix}14\\0\\-2\end{pmatrix}$ are collinear.
Solution: Use the approach in Problem 3.1.55.

60. Find the ratio in which the line segment joining the points $\begin{pmatrix}4\\8\\10\end{pmatrix}$ and $\begin{pmatrix}6\\10\\-8\end{pmatrix}$ is divided by the YZ-plane.
Solution: Use (3.1.24.2). The YZ-plane has points $\begin{pmatrix}0\\y\\z\end{pmatrix}$.

61. Find the equation of the set of points $\mathbf{P}$ such that its distances from the points $\mathbf{A} = \begin{pmatrix}3\\4\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$ are equal.
Solution: Use the approach in Problem 3.1.22.

62. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \tag{3.1.62.1}$$
$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \tag{3.1.62.2}$$

find $\mathbf{R}$, which divides PQ in the ratio 2 : 1

a) internally,
b) externally.

Solution: Use (3.1.24.2).

63. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$.
Solution: The angle between 2 vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

Computing the numerator

$$\mathbf{a}^T\mathbf{b} = \begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\implies \mathbf{a}^T\mathbf{b} = -1 \tag{3.1.63.1}$$

Computing the denominator

$$\|\mathbf{a}\|\|\mathbf{b}\| = \sqrt{(1)^2 + (1)^2 + (-1)^2} \tag{3.1.63.2}$$
$$\times\sqrt{(1)^2 + (-1)^2 + (1)^2} \tag{3.1.63.3}$$
$$\implies \|\mathbf{a}\|\|\mathbf{b}\| = (\sqrt{3})^2 \tag{3.1.63.4}$$
$$\implies \|\mathbf{a}\|\|\mathbf{b}\| = 3 \tag{3.1.63.5}$$

So, we get $\cos\theta$ to be,

$$\cos\theta = \frac{-1}{3} \tag{3.1.63.6}$$
$$\tag{3.1.63.7}$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{-1}{3}\right)$$
$$\implies \theta = 1.9106^c$$
$$\implies \theta = 109.47^\circ$$

Therefore, the angle between the 2 vectors is **109.47**°.

64. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix}3\\2\\-4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{3.1.64.1}$$
$$\mathbf{x} = \begin{pmatrix}5\\-2\\0\end{pmatrix} + \lambda_2\begin{pmatrix}3\\2\\6\end{pmatrix} \tag{3.1.64.2}$$

Solution: Looking at the directions of the lines,

$$\mathbf{a} = \begin{pmatrix}1\\2\\2\end{pmatrix} \tag{3.1.64.3}$$
$$\mathbf{b} = \begin{pmatrix}3\\2\\6\end{pmatrix} \tag{3.1.64.4}$$

Clearly over here,

$$\|\mathbf{a}\| = \sqrt{(1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3 \tag{3.1.64.5}$$

$$\tag{3.1.64.6}$$

$$\|\mathbf{b}\| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7 \tag{3.1.64.7}$$

$$\tag{3.1.64.8}$$

$$\mathbf{a}^T\mathbf{b} = (1)(3) + (2)(2) + (2)(6) = 19 \tag{3.1.64.9}$$

$$\tag{3.1.64.10}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{19}{(3)(7)} = \frac{19}{21} \tag{3.1.64.11}$$

$$\tag{3.1.64.12}$$

$$\implies \theta = \arccos\left(\frac{19}{21}\right) \tag{3.1.64.13}$$

$$\tag{3.1.64.14}$$

$$\implies \theta \approx 25.22^\circ \tag{3.1.64.15}$$

65. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \tag{3.1.65.1}$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{3.1.65.2}$$

Solution: Using the definition of a line in coordinate geometry, we see from the above two equations, the direction vectors **a** and **b** of the two lines are

$$\mathbf{a} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \tag{3.1.65.3}$$

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{3.1.65.4}$$

respectively. In order to find the angle between the two direction vectors, we use the definition of dot product,

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.65.5}$$

Which gives us,

$$\mathbf{a}^T\mathbf{b} = 16 \tag{3.1.65.6}$$

$$\|\mathbf{a}\| = \sqrt{50} \tag{3.1.65.7}$$

$$\|\mathbf{b}\| = \sqrt{6} \tag{3.1.65.8}$$

Which gives us

$$\cos\theta = \frac{8}{5\sqrt{3}} \tag{3.1.65.9}$$

$$\implies \theta = \arccos\frac{8}{5\sqrt{3}} \tag{3.1.65.10}$$

$$\implies \theta = 22.517^\circ \tag{3.1.65.11}$$

66. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

Solution:

$$\mathbf{A}^{\mathbf{T}}\mathbf{B} = 0 \tag{3.1.66.1}$$

$$\mathbf{A}^T\mathbf{B} = (\mathbf{a} + \mathbf{b})^T(\mathbf{a} - \mathbf{b}) \tag{3.1.66.2}$$

The transpose of a sum is the sum of transposes so,

$$(\mathbf{a} + \mathbf{b})^T = \left(\mathbf{a}^T + \mathbf{b}^T\right) \tag{3.1.66.3}$$

$$\mathbf{A}^T\mathbf{B} = \left(\mathbf{a}^T + \mathbf{b}^T\right)(\mathbf{a} - \mathbf{b}) \tag{3.1.66.4}$$

$$\mathbf{a}^T(\mathbf{a} - \mathbf{b}) + \mathbf{b}^T(\mathbf{a} - \mathbf{b}) \tag{3.1.66.5}$$

$$\implies \mathbf{a}^T\mathbf{a} - \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{a} - \mathbf{b}^T\mathbf{b} \tag{3.1.66.6}$$

$$\because \mathbf{a}^T\mathbf{a} = \|\mathbf{a}\|^2 \tag{3.1.66.7}$$

$$\because \mathbf{b}^T\mathbf{b} = \|\mathbf{b}\|^2 \tag{3.1.66.8}$$

$$\because \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} \tag{3.1.66.9}$$

Using (3.1.66.7), (3.1.66.8) and (3.1.66.9)

$$\mathbf{A}^T\mathbf{B} = \|\mathbf{a}\|^2 - \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{b} - \|\mathbf{b}\|^T \tag{3.1.66.10}$$

$$\|\mathbf{a}\|^2 = 5^2 + (-1)^2 + (-3)^2 = 35 \tag{3.1.66.11}$$

$$\|\mathbf{b}\|^2 = 1^2 + (3)^2 + (-5)^2 = 35 \tag{3.1.66.12}$$

$$\mathbf{A}^T\mathbf{B} = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \tag{3.1.66.13}$$

Using (3.1.66.11) and (3.1.66.12)

$$\implies \mathbf{A}^T\mathbf{B} = 35 - 35 = 0 \tag{3.1.66.14}$$

Thus the direction vectors of the two lines satisfies the equation 3.1.66.1, hence proved that the lines are **perpendicular**.

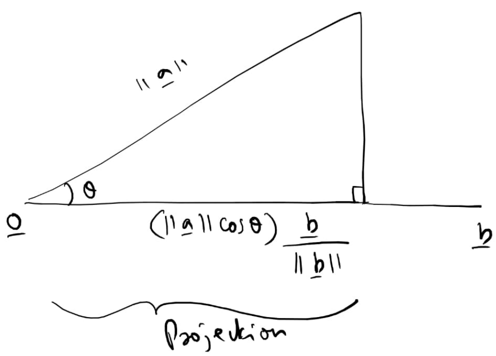

Fig. 3.1.67

67. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{3.1.67.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{3.1.67.2}$$

Solution: The projection of **a** on **b** is shown in Fig. 3.1.67. It has magnitude $\|\mathbf{a}\| \cos\theta$ and is in the direction of **b**. Thus, the projection is defined as

$$(\|\mathbf{a}\| \cos\theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right) \|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{3.1.67.3}$$

68. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{3.1.68.1}$$

Solution:

$$\begin{aligned} \|\mathbf{a} - \mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b} \\ \implies \|\mathbf{a} - \mathbf{b}\|^2 &= 2^2 + 3^2 - 2 \times 4 \\ \implies \|\mathbf{a} - \mathbf{b}\|^2 &= 5 \\ \implies \|\mathbf{a} - \mathbf{b}\| &= \sqrt{5} \end{aligned} \tag{3.1.68.2}$$

69. If **a** is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \tag{3.1.69.1}$$

then find **x**.

Solution:

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{3.1.69.2}$$

$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{3.1.69.3}$$

70. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \tag{3.1.70.1}$$

find $\|\mathbf{a} \times \mathbf{b}\|$.
Solution: Use (1.1.5.3).

71. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{3.1.71.1}$$

Solution: If **x** is the desired vector,

$$(\mathbf{a} + \mathbf{b})^T \mathbf{x} = 0 \tag{3.1.71.2}$$

$$(\mathbf{a} - \mathbf{b})^T \mathbf{x} = 0 \tag{3.1.71.3}$$

resulting in the matrix equation

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.71.4}$$

Performing row operations,

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow R_1 + 3R_2} \begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & -2 \end{pmatrix} \tag{3.1.71.5}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.71.6}$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.71.7}$$

72. Show that $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$, are collinear.
Solution: See Problem 3.1.55.

73. If $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\5\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\2\\-3\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}1\\-6\\-1\end{pmatrix}$, show that $\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.
Solution:

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-1\\-4\\1\end{pmatrix} \tag{3.1.73.1}$$

$$\mathbf{C}-\mathbf{D} = \begin{pmatrix}2\\8\\-2\end{pmatrix} \tag{3.1.73.2}$$

$$\because -2\left(\mathbf{A}-\mathbf{B}\right) = \mathbf{C}-\mathbf{D}, \tag{3.1.73.3}$$

$\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.

74. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|$.
Solution: Given that

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = 0. \tag{3.1.74.1}$$

Then,

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{3.1.74.2}$$

which reduces to

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \tag{3.1.74.3}$$

using (3.1.74.1)

75. Given

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}, \tag{3.1.75.1}$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \tag{3.1.75.2}$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.
Solution: Multiplying (3.1.75.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{c} = 0 \tag{3.1.75.3}$$

$$\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T\mathbf{c} = 0 \tag{3.1.75.4}$$

$$+\mathbf{c}^T\mathbf{a} + \mathbf{b}^T\mathbf{c} + \|\mathbf{c}\|^2 = 0 \tag{3.1.75.5}$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \tag{3.1.75.6}$$

76. Let $\boldsymbol{\alpha} = \begin{pmatrix}3\\-1\\0\end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix}2\\1\\-3\end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
Solution: Let $\beta_1 = k\alpha$. Then,

$$\boldsymbol{\beta} = k\boldsymbol{\alpha} + \boldsymbol{\beta}_2 \tag{3.1.76.1}$$

$$\implies k = \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\|\boldsymbol{\alpha}\|^2} \tag{3.1.76.2}$$

and

$$\boldsymbol{\beta}_2 = \boldsymbol{\beta} - k\boldsymbol{\alpha} \tag{3.1.76.3}$$

This process is known as *Gram-Schmidth orthogonalization*.

77. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix}\cos 90°\\\cos 60°\\\cos 30°\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{3.1.77.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.

78. Find the distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix}1\\2\\-4\end{pmatrix} + \lambda_1\begin{pmatrix}2\\3\\6\end{pmatrix} \tag{3.1.78.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix}3\\3\\-5\end{pmatrix} + \lambda_2\begin{pmatrix}2\\3\\6\end{pmatrix} \tag{3.1.78.2}$$

Solution: Both the lines have the same direction vector, so the lines are parallel. The following code plots

```
codes/line/line_dist_parallel.py
```

Fig. 3.1.78 From Fig. 3.1.78, the distance is

$$\|\mathbf{A_2} - \mathbf{A_1}\| \sin\theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \tag{3.1.78.3}$$

where

$$\mathbf{A}_1 = \begin{pmatrix}1\\2\\-4\end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix}3\\3\\-5\end{pmatrix}, \mathbf{m} = \begin{pmatrix}2\\3\\6\end{pmatrix} \tag{3.1.78.4}$$

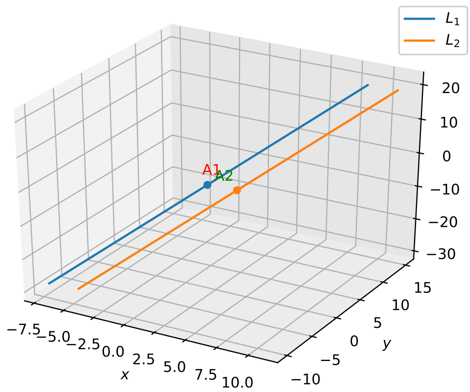

Fig. 3.1.78

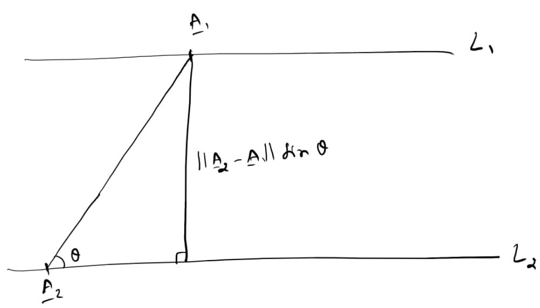

Fig. 3.1.78

79. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{3.1.79.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.79.2}$$

Solution: In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \tag{3.1.79.3}$$

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.79.4}$$

$$\implies \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{3.1.79.5}$$

$$\implies \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.1.79.6}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -5 & 0 \\ 2 & 3 & 1 \end{pmatrix} \tag{3.1.79.7}$$

$$\xleftrightarrow[R_3 = 2R_1 - R_3]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \tag{3.1.79.8}$$

$$\xleftrightarrow{R_3 = 3R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{pmatrix} \tag{3.1.79.9}$$

The above matrix has *rank* = 3. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines. The following code plots Fig. 3.1.79

```
codes/line/line_dist_skew.py
```

The normal to both the lines (and corresponding planes) is

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \tag{3.1.79.10}$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}_2 \tag{3.1.79.11}$$

The distance from $\mathbf{A}_1$ to the above line is then obtained using (3.1.42.7) as

$$\frac{\left|\mathbf{n}^T (\mathbf{A}_2 - \mathbf{A}_1)\right|}{\|\mathbf{n}\|} = \frac{\left|(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2)\right|}{\|\mathbf{m}_1 \times \mathbf{m}_2\|} \tag{3.1.79.12}$$

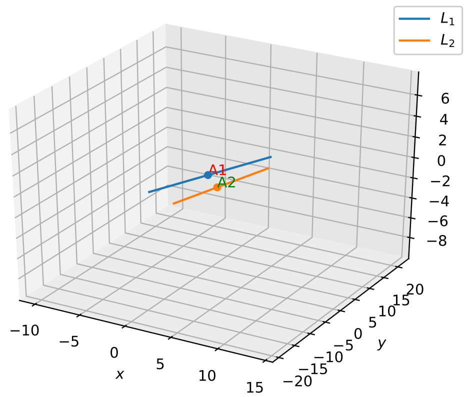

Fig. 3.1.79

80. Find the distance of the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{3.1.80.1}$$

from the origin.
Solution: From (3.1.42.7), the distance is obtained as

$$\frac{|c|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{2^2 + 3^2 + 4^2}} \tag{3.1.80.2}$$

$$= \frac{6}{\sqrt{29}} \tag{3.1.80.3}$$

81. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.
Solution: From the previous problem, the desired equation is

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{3.1.81.1}$$

82. Find the unit normal vector of the plane

$$\begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \mathbf{x} = 1. \tag{3.1.82.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \tag{3.1.82.2}$$

$$\because \|\mathbf{n}\| = 7, \tag{3.1.82.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7} \begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \tag{3.1.82.4}$$

83. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{3.1.83.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \tag{3.1.83.2}$$

Hence, the foot of the perpendicular from the origin is $\lambda \mathbf{n}$. Substituting in (3.1.83.1),

$$\lambda \|\mathbf{n}\|^2 = 6 \implies \lambda = \frac{6}{\|\mathbf{n}\|^2} = \frac{6}{29} \tag{3.1.83.3}$$

Thus, the foot of the perpendicular is

$$\frac{6}{29} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \tag{3.1.83.4}$$

84. Find the equation of the plane which passes through the point $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the line with direction vector $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.
Solution: The normal vector to the plane is $\mathbf{n}$. Hence from (3.1.35.3), the equation of the plane is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.84.1}$$

$$\implies \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} \tag{3.1.84.2}$$

$$= 20 \tag{3.1.84.3}$$

85. Find the equation of the plane passing through $\boldsymbol{R} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \boldsymbol{S} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$.
Solution: If the equation of the plane be

$$\mathbf{n}^T \mathbf{x} = c, \tag{3.1.85.1}$$

$$\mathbf{n}^T \mathbf{R} = \mathbf{n}^T \mathbf{S} = \mathbf{n}^T \mathbf{T} = c, \tag{3.1.85.2}$$

$$\implies \begin{pmatrix} \mathbf{R} - \mathbf{S} & \mathbf{S} - \mathbf{T} \end{pmatrix}^T \mathbf{n} = 0 \tag{3.1.85.3}$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix} 4 & 8 & -8 \\ -7 & -6 & 8 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 2 & -2 \\ -7 & -6 & 8 \end{pmatrix} \tag{3.1.85.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 8 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 4 & -3 \end{pmatrix} \tag{3.1.85.5}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -3 \end{pmatrix} \tag{3.1.85.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ and} \tag{3.1.85.7}$$

$$c = \mathbf{n}^T \mathbf{T} = 7 \tag{3.1.85.8}$$

Thus, the equation of the plane is

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{n} = 7 \tag{3.1.85.9}$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T}) \tag{3.1.85.10}$$

The equation of the plane is then obtained from (3.1.35.3) as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{T}) = [(\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T})]^T (\mathbf{x} - \mathbf{T}) = 0 \tag{3.1.85.11}$$

86. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively. **Solution:** From the given information, the plane passes through the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ respectively. The equation can be obtained using Problem 3.1.85.
87. Find the equation of the plane passing through the intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{3.1.87.1}$$

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = -5 \tag{3.1.87.2}$$

and the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & -5 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{3.1.87.3}$$

$$\xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & 23 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{3.1.87.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.87.5}$$

Thus, $\begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.87.6}$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ -22 & 18 & 1 \end{pmatrix} \xleftrightarrow{R_2 = R_2 + 22R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -26 & 23 \end{pmatrix} \tag{3.1.87.7}$$

$$\xleftrightarrow{R_1 = 13R_1 - R_2} \begin{pmatrix} 13 & 0 & -10 \\ 0 & -26 & 23 \end{pmatrix} \tag{3.1.87.8}$$

$$\implies \mathbf{n} = \begin{pmatrix} \frac{10}{13} \\ \frac{23}{26} \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 23 \\ 26 \end{pmatrix} \tag{3.1.87.9}$$

Since the plane passes through $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, using (3.1.35.3),

$$\begin{pmatrix} 20 & 23 & 26 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = 0 \tag{3.1.87.10}$$

$$\implies \begin{pmatrix} 20 & 23 & 26 \end{pmatrix} \mathbf{x} = 69 \tag{3.1.87.11}$$

Alternatively, the plane passing through the intersection of (3.1.87.1) and (3.1.87.2) has the form

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 6 - 5\lambda \tag{3.1.87.12}$$

Substituting $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in the above,

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 6 - 5\lambda \tag{3.1.87.13}$$

$$\implies 3 + 9\lambda = 6 - 5\lambda \tag{3.1.87.14}$$

$$\implies \lambda = \frac{3}{14} \tag{3.1.87.15}$$

Substituting this value of λ in (3.1.87.12) yields the equation of the plane.

88. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \tag{3.1.88.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \tag{3.1.88.2}$$

are coplanar.

Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (3.1.79), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$(\mathbf{A}_2 - \mathbf{A}_1)^T(\mathbf{m}_1 \times \mathbf{m}_2) = 0 \tag{3.1.88.3}$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix}-3\\1\\5\end{pmatrix} - \begin{pmatrix}-1\\2\\5\end{pmatrix} = \begin{pmatrix}-2\\-1\\0\end{pmatrix} \tag{3.1.88.4}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix}-1 & 2 & 5\\-3 & 1 & 5\end{pmatrix} \xleftrightarrow{R_2 = \frac{R_2 - 3R_1}{5}} \begin{pmatrix}-1 & 2 & 5\\0 & 1 & 2\end{pmatrix} \tag{3.1.88.5}$$

$$\xleftrightarrow{R_1 = -R_1 + 2R_2} \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\end{pmatrix} \implies \begin{pmatrix}-1\\2\\5\end{pmatrix} \times \begin{pmatrix}-3\\1\\5\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{3.1.88.6}$$

The LHS of (3.1.88.3) is

$$\begin{pmatrix}-2 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix} = 0 \tag{3.1.88.7}$$

which completes the proof. Alternatively, the lines are coplanar if

$$\left|\mathbf{A}_1 - \mathbf{A}_2 \quad \mathbf{m}_1 \quad \mathbf{m}_2\right| = 0 \tag{3.1.88.8}$$

89. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.89.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7. \tag{3.1.89.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (3.1.45.6).

90. Find the angle between the two planes

$$\begin{pmatrix}2 & 2 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.90.1}$$

$$\begin{pmatrix}3 & -6 & 2\end{pmatrix}\mathbf{x} = 7. \tag{3.1.90.2}$$

Solution: See Problem (3.1.89).

91. Find the distance of a point $\begin{pmatrix}2\\5\\-3\end{pmatrix}$ from the plane

$$\begin{pmatrix}6 & -3 & 2\end{pmatrix}\mathbf{x} = 4 \tag{3.1.91.1}$$

Solution: Use (3.1.42.7).

92. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{3.1.92.1}$$

and the plane

$$P: \quad \begin{pmatrix}10 & 2 & -11\end{pmatrix}\mathbf{x} = 3 \tag{3.1.92.2}$$

Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left|\begin{pmatrix}10 & 2 & -11\end{pmatrix}\begin{pmatrix}2\\3\\6\end{pmatrix}\right|}{\sqrt{225} \times \sqrt{49}} = \frac{8}{21} \tag{3.1.92.3}$$

Thus, the desired angle is $90° - \theta$.

93. Find the equation of the plane that contains the point $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and is perpedicular to each of the

planes

$$\begin{pmatrix}2 & 3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.93.1}$$
$$\begin{pmatrix}1 & 2 & -3\end{pmatrix}\mathbf{x} = 8 \tag{3.1.93.2}$$

Solution: The normal vector to the desired plane is $\perp$ the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix}2\\3\\-2\end{pmatrix} \times \begin{pmatrix}1\\2\\-3\end{pmatrix} \tag{3.1.93.3}$$

The equation of the plane is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.93.4}$$

94. Find the distance between the point $\mathbf{P} = \begin{pmatrix}6\\5\\9\end{pmatrix}$ and the plane determined by the points $\boldsymbol{A} = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \boldsymbol{B} = \begin{pmatrix}5\\2\\4\end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$.
Solution: Find the equation of the plane using Problem 3.1.85. Find the distance using (3.1.42.7).
95. Find the coordinates of the point where the line through the points $\mathbf{A} = \begin{pmatrix}3\\4\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\1\\6\end{pmatrix}$ crosses the XY plane.
Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{3.1.95.1}$$
$$= \begin{pmatrix}3\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\5\end{pmatrix} \tag{3.1.95.2}$$

The line crosses the XY plane for $x_3 = 0 \implies \lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2\\-3\\5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}13\\23\\0\end{pmatrix} \tag{3.1.95.3}$$

96. Show that the function given by $f(x) = 7x$–3 is increasing on $\mathbf{R}$.
Solution: A function is said to be increasing if

$$x_2 > x_1 \implies f(x_2) > f(x_1) \tag{3.1.96.1}$$
$$\implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \tag{3.1.96.2}$$

Letting $x_1 = x, x_2 = x + h$ in (3.1.96.2), this results in

$$\frac{f(x+h) - f(x)}{h} > 0 \tag{3.1.96.3}$$

In the given problem,

$$\frac{f(x+h) - f(x)}{h} = 7 > 0 \tag{3.1.96.4}$$

Hence, the given function is increasing.
97. A funcion $f(x)$ is increasing if

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} > 0 \tag{3.1.97.1}$$

$f'(x)$ is defined as the *derivative* of $f(x)$. The function is decreasing if $f'(x) < 0$.
98. The funcion $f(x) = ax + b, a \neq 0$ is increasing if $f'(x) = a > 0$. Else, it is decreasing.
99. Find the maximum and minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1). \tag{3.1.99.1}$$

Solution: It is easy to verify that $f'(x) = 1 > 0$ in the given interval. Hence, the function is increasing. The maximum value in the given interval is 1 and the minimum value is 0.
100. Find all points of local maxima and local minima of the function f given by

$$f(x) = 3 + |x|, \quad x \in \mathbf{R} \tag{3.1.100.1}$$

Solution: (3.1.100.1) can be expressed as

$$f(x) = \begin{cases} 3 + x & x > 0 \\ 0 & x = 0 \\ 3 - x & < 0 \end{cases} \tag{3.1.100.2}$$

From Theorem 3.1.98,

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{3.1.100.3}$$

Thus, $f(x)$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$. It is obvious that the minimum value of $f(x) = 0$.
101. Sketch the graph of $y = |x + 3|$ and evaluate its

area for $-6 \le x \le 0$.
Solution: Fig. shows

$$y_1 = |x+3|, -6 \le x \le 0 \quad (3.1.101.1)$$
$$y_2 = |x|, -3 \le x \le 3 \quad (3.1.101.2)$$
$$y_3 = x, 0 < x < 3 \quad (3.1.101.3)$$
$$\implies ar(y_1) = ar(y_2) = 2ar(y_3) \quad (3.1.101.4)$$

From Fig. ,

$$ar(y_3) = h(h + 2h + 3h + \dots + nh), \quad nh = 3 \quad (3.1.101.5)$$
$$= h^2(1 + 2 + 3 + \dots n) = h^2 \sum_{k=1}^{n} k \quad (3.1.101.6)$$

Let

$$S_n = 1 + 2 + 3 + \dots n \quad (3.1.101.7)$$
$$\implies S_n = n + n - 1 + n - 2 + \dots 1 \quad (3.1.101.8)$$
$$\implies 2S_n = (n+1) + (n+1) + \dots + (n+1) \quad n \text{ times} \quad (3.1.101.9)$$
$$\implies 2S_n = n(n+1) \quad (3.1.101.10)$$
$$\text{or, } S_n = \frac{n(n+1)}{2} \quad (3.1.101.11)$$

Substituting (3.1.101.11) in (3.1.101.6),

$$ar(y_3) = \frac{nh(nh+h)}{2} \quad (3.1.101.12)$$
$$= \frac{3(3+h)}{2} \quad (3.1.101.13)$$
$$\text{or, } ar(y_3) = \lim_{h \to 0} ar(y_3) = \frac{9}{2} \quad (3.1.101.14)$$

This result agrees with the area of a triangle calculated using the base and altitude. Thus, from (3.1.101.14) and (3.1.101.1),

$$ar(y_1) = 2ar(y_3) = 9 \quad (3.1.101.15)$$

102. Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.
Solution: See Fig.

$$\because f(1+h) = 2(1+h) + 3 = 5 + h \quad (3.1.102.1)$$
$$f(1) = 5 \quad (3.1.102.2)$$
$$f(1-h) = 2(1+h) + 3 = 5 - h \quad (3.1.102.3)$$
$$\lim_{h \to 0} f(1+h) = f(1) = f(1-h) = 5 \quad (3.1.102.4)$$

Hence, the function is continuous.

103. A function $f(x)$ is defined to be *continuous* at $x = a$ if

$$\lim_{h \to 0} f(a+h) = f(a) = f(a-h) \quad (3.1.103.1)$$

It is possible to draw a continuious function $f(x)$ without lifting a pencil.

104. Discuss the continuity of the function f given by $f(x) = |x|$ at $x = 0$.
Solution: See Fig. . It appears to be continuous. To prove this, we note that

$$\lim_{h \to 0} f(0+h) = f(0) = f(0-h) = 0 \quad (3.1.104.1)$$

105. Check the points where the constant function $f(x) = k$ is continuous.
Solution: $f(x)$ is continuous everywhere.

106. Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x+2 & x < 1 \\ 0 & x = 1 \\ x-2 & x > 1 \end{cases} \quad (3.1.106.1)$$

Solution: From Fig. , the discontinuity appears to be at $x = 1$ and verified by the fact that

$$\because f(1-h) = 3 \ne f(1) \ne f(1+h), \quad (3.1.106.2)$$

107. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x+2 & x < 0 \\ -x+2 & x > 0 \end{cases} \quad (3.1.107.1)$$

The function is not defined at $x = 0$, so it is discontinuous at that point. At all other points it is continuious.

108. Show that the function f defined by

$$f(x) = |1 - x + |x||, \tag{3.1.108.1}$$

where x is any real number, is a continuous function.
Solution: The sum of continuous functions is continuous.

109. If

$$y = f(x), \frac{dy}{dx} = f'(x) \tag{3.1.109.1}$$

110. Find $\frac{dy}{dx}$ if $x - y = \pi$.
Solution:

$$\because y = f(x) = x - \pi, \tag{3.1.110.1}$$

from Theorem 3.1.98,

$$\frac{dy}{dx} = 1. \tag{3.1.110.2}$$

111. Find the derivative at $x = 2$ of the function $f(x) = 3$.
Solution: The derivative is 0.

112. Find the derivative of $f(x) = 3$ at $x = 0$ and $x = 3$.
Solution: $\frac{dy}{dx} = 0$.

113. Find the derivative of $f(x) = 10x$.
Solution: $\frac{dy}{dx} = 10$.

114. Find the derivative of $f(x) = a$ for a fixed real number a.
Solution: $\frac{dy}{dx} = 0$.

115. Form the *differential equation* representing the family of curves $y = mx$, where, m is an arbitrary constant. **Solution:** The desired equation is

$$\frac{dy}{dx} = m \tag{3.1.115.1}$$

3.2 Linear Inequalities: Examples

1. Solve $30x < 200$ when
   a) x is a natural number,
   b) x is an integer.

   Solution: From the given information,

$$30x < 200 \implies x < \frac{20}{3} \tag{3.2.1.1}$$

   If x is a natural number, $x \in \{1, 2, 3, 4, 5, 6\}$. If x is an integer, then the solution set includes 0 as well as all negative integers.

2. Solve $5x - 3 < 3x + 1$ when
   a) x is an integer,
   b) x is a real number.

   Solution:

$$5x - 3 < 3x + 1 \implies x < 2 \tag{3.2.2.1}$$

   If x is real, then $x \in (-\infty, 2)$.

3. Solve the following system of linear inequalities graphically.

$$\begin{aligned} x + y &\geq 5 \\ x - y &\leq 3 \end{aligned} \tag{3.2.3.1}$$

   Solution: Let $u_1 \geq 0, u_2 \geq 0$. This may be expressed as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \geq \mathbf{0} \tag{3.2.3.2}$$

   (3.2.3.1) can then be expressed as

$$\begin{aligned} x + y &\geq 5 \\ -x + y &\geq -3 \end{aligned} \tag{3.2.3.3}$$

$$\implies \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x} \geq \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{3.2.3.4}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x} - \mathbf{u} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{3.2.3.5}$$

$$\text{or, } \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \mathbf{u} \tag{3.2.3.6}$$

   resulting in

$$\mathbf{x} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \mathbf{u} \tag{3.2.3.7}$$

$$\text{or, } \mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{u} \tag{3.2.3.8}$$

   after obtaining the inverse. Fig. 3.2.3 generated using the following python code shows the region satisfying (3.2.3.1)

```
codes/line/line_ineq.py
```

4. Solve

$$\begin{aligned} 2x + y &\geq 4 \\ x + y &\leq 3 \\ 2x - 3y &\leq 6 \end{aligned} \tag{3.2.4.1}$$

   Solution: Fig. 3.2.4 generated using the following python code shows the region satisfying (3.2.4.1)

```
codes/line/line_ineq_mult.py
```

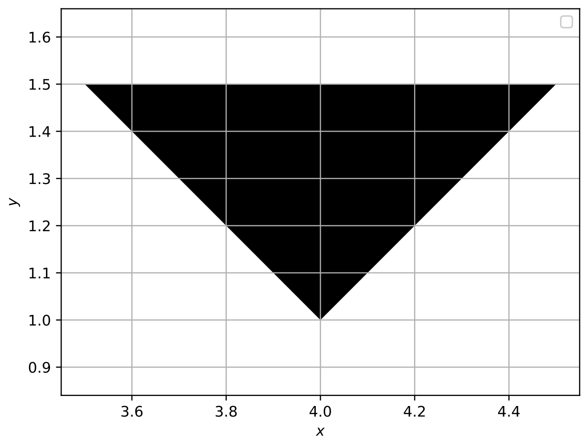

Fig. 3.2.3

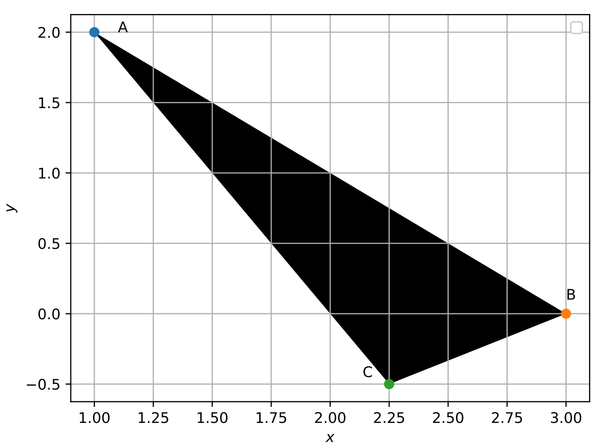

Fig. 3.2.4

5. Solve $x + y < 5$ graphically.
   Solution: The following python code generates Fig. 3.2.5.

   ```
   ./solutions/5/codes/lines/q6.py
   ```

6. Solve

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 150 \\ 80 \\ 15 \\ 0 \\ 0 \end{pmatrix} \tag{3.2.6.1}$$

3.3 Matrix Examples

1. Balance the following chemical equation.

$$Fe + H_2O \rightarrow Fe_3O_4 + H_2 \tag{3.3.1.1}$$

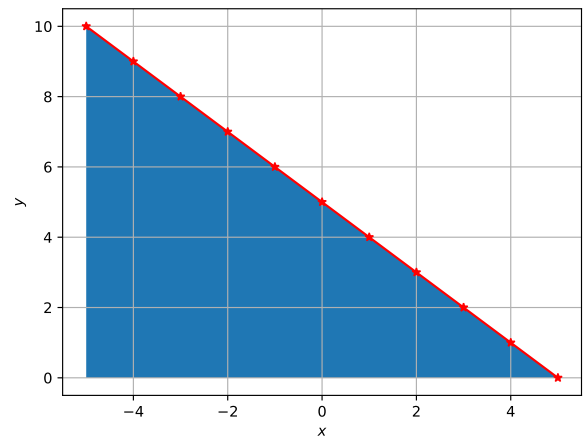

Fig. 3.2.5: x+y<5

Solution: Let the balanced version of (3.3.1.1) be

$$x_1 Fe + x_2 H_2O \rightarrow x_3 Fe_3O_4 + x_4 H_2 \tag{3.3.1.2}$$

which results in the following equations

$$\begin{aligned} (x_1 - 3x_3) \, Fe &= 0 \\ (2x_2 - 2x_4) \, H &= 0 \\ (x_2 - 4x_3) \, O &= 0 \end{aligned} \tag{3.3.1.3}$$

which can be expressed as

$$\begin{aligned} x_1 + 0.x_2 - 3x_3 + 0.x_4 &= 0 \\ 0.x_1 + 2x_2 + 0.x_3 - 2x_4 &= 0 \\ 0.x_1 + x_2 - 4x_3 + 0.x_4 &= 0 \end{aligned} \tag{3.3.1.4}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & -4 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{3.3.1.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{3.3.1.6}$$

(3.3.1.5) can be row reduced as follows

$$\begin{pmatrix}1 & 0 & -3 & 0\\0 & 2 & 0 & -2\\0 & 1 & -4 & 0\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix}1 & 0 & -3 & 0\\0 & 1 & 0 & -1\\0 & 1 & -4 & 0\end{pmatrix} \tag{3.3.1.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix}1 & 0 & -3 & 0\\0 & 1 & 0 & -1\\0 & 0 & -4 & 1\end{pmatrix} \tag{3.3.1.8}$$

$$\xleftrightarrow{R_1 \leftarrow 4R_1 - 3R_3} \begin{pmatrix}4 & 0 & 0 & -3\\0 & 1 & 0 & -1\\0 & 0 & -4 & 1\end{pmatrix} \tag{3.3.1.9}$$

$$\xleftrightarrow[R_3 \leftarrow -\frac{1}{4}R_3]{R_1 \leftarrow \frac{1}{4}} \begin{pmatrix}1 & 0 & 0 & -\frac{3}{4}\\0 & 1 & 0 & -1\\0 & 0 & 1 & -\frac{1}{4}\end{pmatrix} \tag{3.3.1.10}$$

Thus,

$$x_1 = \frac{3}{4}x_4, x_2 = x_4, x_3 = \frac{1}{4}x_4 \tag{3.3.1.11}$$

$$\tag{3.3.1.12}$$

$$\implies \mathbf{x} = x_4 \begin{pmatrix}\frac{3}{4}\\1\\\frac{1}{4}\\1\end{pmatrix} = \begin{pmatrix}3\\4\\1\\4\end{pmatrix} \tag{3.3.1.13}$$

upon substituting $x_4 = 4$. (3.3.1.2) then becomes

$$3Fe + 4H_2O \rightarrow Fe_3O_4 + 4H_2 \tag{3.3.1.14}$$

2. Consider the following information regarding the number of men and women workers in the three factories I,II and III

	Men Workers	Women Workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

3. If a matrix has 8 elements, what are the possible orders it can have?

4. Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$

5. $\begin{pmatrix}x+3 & z+4 & 2y-7\\-6 & a-1 & 0\\b-3 & -21 & 0\end{pmatrix} = \begin{pmatrix}0 & 6 & 3y-2\\-6 & -3 & 2c+2\\2b+4 & -21 & 0\end{pmatrix}$
Find the values of a,b,c,x,y and z.
Solution: As the two matrices are equal their corresponding entries are also equal. Hence

$$x + 3 = 0 \implies x = -3 \tag{3.3.5.1}$$
$$z + 4 = 6 \implies z = 2 \tag{3.3.5.2}$$
$$2y - 7 = 3y - 2 \implies y = -5 \tag{3.3.5.3}$$
$$a - 1 = -3 \implies a = -2 \tag{3.3.5.4}$$
$$2c + 2 = 0 \implies c = -1 \tag{3.3.5.5}$$
$$b - 3 = 2b + 4 \implies b = -7 \tag{3.3.5.6}$$

6. Find the values of a,b,c and d from the following equation:
$\begin{pmatrix}2a+b & a-2b\\5c-d & 4c+3d\end{pmatrix} = \begin{pmatrix}4 & -3\\11 & 24\end{pmatrix}$
Solution: These equations can be written as:

$$\begin{pmatrix}2 & 1 & 0 & 0\\1 & -2 & 0 & 0\\0 & 0 & 5 & -1\\0 & 0 & 4 & 3\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}4\\-3\\11\\24\end{pmatrix} \tag{3.3.6.1}$$

So the coefficient matrix A can be expressed as:

$$A = \begin{pmatrix}2 & 1 & 0 & 0\\1 & -2 & 0 & 0\\0 & 0 & 5 & -1\\0 & 0 & 4 & 3\end{pmatrix} \tag{3.3.6.2}$$

And the augmented matrix B can be expressed as:

$$B = \begin{pmatrix}2 & 1 & 0 & 0 & 4\\1 & -2 & 0 & 0 & -3\\0 & 0 & 5 & -1 & 11\\0 & 0 & 4 & 3 & 24\end{pmatrix} \tag{3.3.6.3}$$

Now, if we express the augmented matrix as Echelon form, then it will be:

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 0 & -3 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & 4 & 3 & 24 \end{pmatrix}$$
$$\xleftrightarrow[R_4 \leftarrow R_4 - R_3]{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 0 & -5 & 0 & 0 & -10 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & -1 & 4 & 13 \end{pmatrix}$$
$$\xleftrightarrow[R_4 \leftarrow 5R_4 + R_3]{R_2 \leftarrow \frac{R_2}{(-5)}} \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & 0 & 19 & 76 \end{pmatrix} \tag{3.3.6.4}$$
$$\xleftrightarrow{R_4 \leftarrow \frac{R_4}{(19)}} \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$
$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_4} \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & 0 & 15 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$
$$\xleftrightarrow[R_1 \leftarrow \frac{R_1}{2}]{R_3 \leftarrow \frac{R_3}{5}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Thus,
$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \tag{3.3.6.5}$$

7. Given A=$\begin{pmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{pmatrix}$, find A+B.
Solution:
$$A + B = \begin{pmatrix} \sqrt{3} + 2 & \sqrt{5} + 1 & 0 \\ 0 & 6 & \frac{1}{2} \end{pmatrix} \tag{3.3.7.1}$$
The python code for matrix addition can be downloaded from

solutions/7/codes/line/matrix/matrix_add.py

8. If A=$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and B=$\begin{pmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$, then find 2A-B.
9. If A=$\begin{pmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{pmatrix}$, then find the matrix X, such that 2A+3X=5B.
10. Find X and Y, if X+Y=$\begin{pmatrix} 5 & 2 \\ 0 & 9 \end{pmatrix}$ and X-Y=$\begin{pmatrix} 3 & 6 \\ 0 & -1 \end{pmatrix}$.
11. Find the values of x and y from the following equation:
$$2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$$
12. Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.
September Sales(in Rupees)
Basmati Permal Naura
A =$\begin{pmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{pmatrix}\begin{pmatrix} Ramakishan \\ GurucharanSingh \end{pmatrix}$
October sales (in Rupees)
Basmati Permal Naura
B=$\begin{pmatrix} 5,000 & 10,000 & 6,000 \\ 20,000 & 10,000 & 10,000 \end{pmatrix}\begin{pmatrix} Ramkishan \\ GurucharanSingh \end{pmatrix}$
(i) Find the combined sales in September and October for each farmer in each variety.
(ii) Find the decrease in sales from September to October.
(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.
13. Find AB, if A=$\begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{pmatrix}$.
14. If A=$\begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$, then find AB,BA.Show that AB≠BA
15. If A=$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and B=$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then find AB,BA. Show that AB≠BA
16. Find AB, if A=$\begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$ and B=$\begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$
17. If A=$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{pmatrix}$, B=$\begin{pmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{pmatrix}$ and

$C=\begin{pmatrix}1 & 2 & 3 & -4\\ 2 & 0 & -2 & 1\end{pmatrix}$, find A(BC),(AB)C and show that (AB)C=A(BC)

18. If $A=\begin{pmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{pmatrix}$,$C=\begin{pmatrix}2\\ -2\\ 3\end{pmatrix}$ Calculate AC,BC and (A+B)C=AC+BC

19. If $A=\begin{pmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{pmatrix}$,then show that $A^3 - 23A - 40I = 0$

20. In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as
Cost per contact
$A=\begin{pmatrix}40\\ 100\\ 50\end{pmatrix}\begin{pmatrix}Telephone\\ Housecall\\ Letter\end{pmatrix}$
The number of contacts of each type made in two cities X and Y is given by
Telephone Housecall Letter
$B=\begin{pmatrix}1000 & 500 & 5000\\ 3000 & 1000 & 10000\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}$. Find the total amount spent by the group in the two cities X and Y.

21. If $A=\begin{pmatrix}3 & \sqrt{3} & 2\\ 4 & 2 & 0\end{pmatrix}$ and $B=\begin{pmatrix}2 & -1 & 2\\ 1 & 2 & 4\end{pmatrix}$, verify that
(i) $(A')' = A$
(ii)$(A + B)' = A' + B'$,
(iii) $(kB)' = kB'$,where k is any constant.

22. If $A=\begin{pmatrix}-2\\ 4\\ 5\end{pmatrix}$,$B=\begin{pmatrix}1 & 3 & -6\end{pmatrix}$, verify that $(AB)' = B'A'$

23. Express the matrix $B=\begin{pmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{pmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

24. By using elementary operations,find the inverse of the matrix
$A=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$.

25. Obtain the inverse of the following matrix using elementary operations
$A=\begin{pmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{pmatrix}$.

26. Find P^{-1}, if it exists, given
$P=\begin{pmatrix}10 & -2\\ -5 & 1\end{pmatrix}$.

27. If $A=\begin{pmatrix}\cos\theta & \sin\theta\\ sin\theta & \cos\theta\end{pmatrix}$,
then prove that $A^n = \begin{pmatrix}\cos\theta & \sin n\theta\\ sinn\theta & \cos n\theta\end{pmatrix}$, n $\in$ N.

28. If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute,that AB = BA.

29. Let $A=\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}$, $B=\begin{pmatrix}5 & 2\\ 7 & 4\end{pmatrix}$, $C=\begin{pmatrix}2 & 5\\ 3 & 8\end{pmatrix}$. Find a matrix D such that CD-AB=0.

3.4 Complex Numbers

1. Find $\begin{pmatrix}5\\ -3\end{pmatrix}^3$

Solution: In general, the complex number $\begin{pmatrix}a_1\\ a_2\end{pmatrix}$ has the matrix representation

$$\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \begin{pmatrix}a_1 & -a_2\\ a_2 & a_1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.4.1.1}$$

$$= \mathbf{T}_a\begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.4.1.2}$$

$$\implies \begin{pmatrix}5\\ -3\end{pmatrix} = \begin{pmatrix}5 & 3\\ -3 & 5\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.4.1.3}$$

Then,

$$\begin{pmatrix}5\\ -3\end{pmatrix}^3 \triangleq \begin{pmatrix}5 & 3\\ -3 & 5\end{pmatrix}^3\begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.4.1.4}$$

$$= \begin{pmatrix}-10 & 198\\ -198 & -10\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.4.1.5}$$

$$= \begin{pmatrix}-10\\ -198\end{pmatrix} \tag{3.4.1.6}$$

The python code for above problem is

```
codes/line/comp.py
```

2. Find $\begin{pmatrix}-\sqrt{3}\\ \sqrt{2}\end{pmatrix}\begin{pmatrix}2\sqrt{3}\\ -1\end{pmatrix}$.
Solution: Using the equivalent matrices for the

complex numbers,

$$\begin{pmatrix}-\sqrt{3}\\ \sqrt{2}\end{pmatrix}\begin{pmatrix}2\sqrt{3}\\ -1\end{pmatrix} = \begin{pmatrix}-\sqrt{3} & -\sqrt{2}\\ \sqrt{2} & -\sqrt{3}\end{pmatrix}\begin{pmatrix}2\sqrt{3} & 1\\ -1 & 2\sqrt{3}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$
$$= \begin{pmatrix}\sqrt{2}-6 & -\sqrt{3}-2\sqrt{6}\\ \sqrt{3}+2\sqrt{6} & \sqrt{2}-6\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\sqrt{2}-6\\ \sqrt{3}+2\sqrt{6}\end{pmatrix} \tag{3.4.2.1}$$

The following code verifies the result.

```
codes/line_ex/complex_ex/complex_ex.py
```

3. Find the multiplicative inverse of $\begin{pmatrix}2\\ -3\end{pmatrix}$.

   Solution: Let $\mathbf{T}_a$ be the matrix for the complex number $\mathbf{a}$. $\mathbf{b}$ is defined to be the multiplicative inverse of $\mathbf{a}$ if

$$\mathbf{T}_a\mathbf{T}_b = \mathbf{T}_b\mathbf{T}_a = \mathbf{I} \tag{3.4.3.1}$$

   Then, from (3.4.1.1)

$$\mathbf{b} = \mathbf{a}^{-1} = \begin{pmatrix}a_1 & -a_2\\ a_2 & a_1\end{pmatrix}^{-1}\begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.4.3.2}$$
$$= \frac{1}{\|\mathbf{a}\|^2}\begin{pmatrix}a_1\\ -a_2\end{pmatrix} \tag{3.4.3.3}$$

   Thus,

$$\begin{pmatrix}2\\ -3\end{pmatrix}^{-1} = \frac{1}{13}\begin{pmatrix}2\\ 3\end{pmatrix} \tag{3.4.3.4}$$

   The python code for above problem is

```
solutions/3/codes/line/comp/comp.py
```

   Note that

$$\mathbf{T}_b = \mathbf{T}_a^{-1} = \frac{\mathbf{T}_a^T}{\left\|\mathbf{a}^2\right\|} \tag{3.4.3.5}$$

4. Find
   a) $\begin{pmatrix}5\\ \sqrt{2}\end{pmatrix}\begin{pmatrix}1\\ -2\sqrt{3}\end{pmatrix}$.
   b) $\begin{pmatrix}0\\ 1\end{pmatrix}^{-35}$.
   c) Show that the polar representation of $\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}$ is $2\angle 60°$.
5. Simplify the complex number $-\frac{16}{\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}}$

   Solution: Using the polar form,

$$\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix} = 2\begin{pmatrix}\cos 60°\\ \sin 60°\end{pmatrix} = 2\underline{/60°} \tag{3.4.5.1}$$
$$\implies \frac{-16}{\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}} = -8\underline{/-60°} = 4\begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix} \tag{3.4.5.2}$$

   The following python code gives the desired answer

```
./solutions/5/codes/lines/q8.py
```

6. Find the conjugate of $\frac{\begin{pmatrix}3\\ -2\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix}}{\begin{pmatrix}1\\ 2\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix}}$.

   Solution: Using the matrix form,

$$\frac{\begin{pmatrix}3\\ -2\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix}}{\begin{pmatrix}1\\ 2\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix}}$$
$$= \begin{pmatrix}3 & 2\\ -2 & 3\end{pmatrix}\begin{pmatrix}2 & -3\\ 3 & 2\end{pmatrix}\left[\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\right]^{-1}\begin{pmatrix}1\\ 0\end{pmatrix}$$
$$= \frac{1}{25}\begin{pmatrix}63\\ -16\end{pmatrix} \tag{3.4.6.1}$$

   The conjugate is given by

$$\frac{1}{25}\begin{pmatrix}63\\ 16\end{pmatrix} \tag{3.4.6.2}$$

7. Find the modulus and argument of the complex numbers
   a) $\frac{\begin{pmatrix}1\\ 1\end{pmatrix}}{\begin{pmatrix}1\\ -1\end{pmatrix}}$.
   b) $\frac{1}{\begin{pmatrix}1\\ 1\end{pmatrix}}$.

   Solution:

$$\begin{pmatrix}1\\ 1\end{pmatrix} = \sqrt{2}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \tag{3.4.7.1}$$
$$= \sqrt{2}\begin{pmatrix}\cos 45°\\ \sin 45°\end{pmatrix} \tag{3.4.7.2}$$

   In the above, the modulus is $\left\|\begin{pmatrix}1\\ 1\end{pmatrix}\right\| = \sqrt{2}$ and the argument is 45°. Similarly,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \cos 45^\circ \\ -\sin 45^\circ \end{pmatrix} \tag{3.4.7.3}$$

$$\implies \begin{pmatrix} 1 \\ -1 \end{pmatrix}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix} \tag{3.4.7.4}$$

Using the matrix representation,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

$$\times \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.7.5}$$

$$= \begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \end{pmatrix} = 1\underline{/90^\circ} \tag{3.4.7.6}$$

In general, if

$$\mathbf{z}_1 = r_1\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{z}_2 = r_2\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \tag{3.4.7.7}$$

$$\mathbf{z}_1\mathbf{z}_2 = r_1 r_2\begin{pmatrix} \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) \end{pmatrix}. \tag{3.4.7.8}$$

Similarly, from (3.4.7.2),

$$\frac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 45^\circ \\ -\sin 45^\circ \end{pmatrix} \tag{3.4.7.9}$$

$$= \frac{1}{\sqrt{2}}\underline{/-45^\circ} \tag{3.4.7.10}$$

8. Find θ such that

$$\frac{\begin{pmatrix} 3 \\ 2\sin\theta \end{pmatrix}}{\begin{pmatrix} 1 \\ -2\sin\theta \end{pmatrix}} \tag{3.4.8.1}$$

is purely real.

9. Convert the complex number

$$\mathbf{z} = \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\begin{pmatrix} \cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \end{pmatrix}} \tag{3.4.9.1}$$

in the polar form.

10. Simplify

$$\mathbf{z} = \left(\frac{1}{\begin{pmatrix} 1 \\ -4 \end{pmatrix}} - \frac{2}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}\right)\frac{\begin{pmatrix} 3 \\ -4 \end{pmatrix}}{\begin{pmatrix} 5 \\ 1 \end{pmatrix}} \tag{3.4.10.1}$$

Solution: Using equivalent matrices for the complex numbers and matrix multiplication,

$$= \left(\begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}^{-1} - 2\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 1 & 5 \end{pmatrix}^{-1}$$

$$= \left(\frac{1}{1^2+4^2}\begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - 2\left(\frac{1}{2^2+1^2}\right)\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \frac{1}{5^2+1^2}\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \left(\frac{1}{1+16}\begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - \frac{2}{4+1}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \frac{1}{25+1}\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \left(\frac{1}{17}\begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \frac{1}{26}\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \left(\begin{pmatrix} \frac{1}{17} & \frac{-4}{17} \\ \frac{4}{17} & \frac{1}{17} \end{pmatrix} - \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{-2}{5} & \frac{4}{5} \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \frac{1}{26}\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{17} - \frac{4}{5} & \frac{-4}{17} - \frac{2}{5} \\ \frac{4}{17} + \frac{2}{5} & \frac{1}{17} - \frac{4}{5} \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \frac{1}{26}\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-63}{85} & \frac{-54}{85} \\ \frac{54}{85} & \frac{-63}{85} \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \frac{1}{26}\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{1}{85}\left(\begin{pmatrix} -63 & -54 \\ 54 & -63 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \frac{1}{26}\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{1}{2210}\left(\begin{pmatrix} -63 & -54 \\ 54 & -63 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix} -189+216 & -162-252 \\ 162+252 & 216-189 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix} 27 & -414 \\ 414 & 27 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix} 27 & -414 \\ 414 & 27 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix} 135+414 & 27-2070 \\ 2070-27 & 414+135 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix} 549 & -2043 \\ 2043 & 549 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix} 549 \\ 2043 \end{pmatrix}$$

$$\implies \mathbf{z} = \begin{pmatrix} \frac{549}{2210} \\ \frac{2043}{2210} \end{pmatrix} \quad (3.4.10.2)$$

11. Convert the following in the polar form:

a) $\dfrac{\begin{pmatrix} 1 \\ 7 \end{pmatrix}}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}^2}$.

b) $\dfrac{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$.

Solution:

a) Below is the solution :

$$\frac{\begin{pmatrix} 1 \\ 7 \end{pmatrix}}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}^2} \quad (3.4.11.1)$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.11.2)$$

$$\implies \begin{pmatrix} 2 \\ -1 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.11.3)$$

$$\implies \begin{pmatrix} 2 \\ -1 \end{pmatrix}^2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (3.4.11.4)$$

$$= \begin{pmatrix} 1 \\ 7 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix}^{-1} \quad (3.4.11.5)$$

$$= \frac{1}{25}\begin{pmatrix} 1 & -7 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.11.6)$$

$$= \frac{1}{25}\begin{pmatrix} -25 & -25 \\ 25 & -25 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.11.7)$$

$$= \frac{25}{25}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.11.8)$$

$$= \sqrt{2}\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (3.4.11.9)$$

$$= \sqrt{2}\begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.11.10)$$

$$= \sqrt{2}\begin{pmatrix} \cos 135^\circ \\ \sin 135^\circ \end{pmatrix} \quad (3.4.11.11)$$

$$= \sqrt{2}\angle 135^\circ \quad (3.4.11.12)$$

b) Below is the solution:

$$\frac{\begin{pmatrix}1\\3\end{pmatrix}}{\begin{pmatrix}1\\-2\end{pmatrix}} \tag{3.4.11.13}$$

$$\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.11.14}$$

$$= \begin{pmatrix}1\\7\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}^{-1} \tag{3.4.11.15}$$

$$= \frac{1}{5}\begin{pmatrix}1 & -3\\3 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.11.16}$$

$$= \frac{1}{5}\begin{pmatrix}-5 & -5\\5 & -5\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.11.17}$$

$$= \frac{5}{5}\begin{pmatrix}-1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.11.18}$$

$$= \sqrt{2}\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \tag{3.4.11.19}$$

$$= \sqrt{2}\begin{pmatrix}\cos 135^\circ & -\sin 135^\circ\\ \sin 135^\circ & \cos 135^\circ\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.11.20}$$

$$= \sqrt{2}\begin{pmatrix}\cos 135^\circ\\ \sin 135^\circ\end{pmatrix} \tag{3.4.11.21}$$

$$= \sqrt{2}\angle 135^\circ \tag{3.4.11.22}$$

12. If $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}1\\1\end{pmatrix}$, find $\left\|\frac{\mathbf{z}_1+\mathbf{z}_1+1}{\mathbf{z}_1-\mathbf{z}_2+1}\right\|$

Solution: Let us consider $\frac{\mathbf{z}_1+\mathbf{z}_1+1}{\mathbf{z}_1-\mathbf{z}_2+1}$, then

$$\mathbf{z}_1 + \mathbf{z}_1 + 1 = \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.12.1}$$

$$= \begin{pmatrix}5\\-2\end{pmatrix} \tag{3.4.12.2}$$

$$\mathbf{z}_1 - \mathbf{z}_2 + 1 = \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.12.3}$$

$$= \begin{pmatrix}2\\-2\end{pmatrix} \tag{3.4.12.4}$$

$$\frac{\mathbf{z}_1 + \mathbf{z}_1 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1} = \frac{\begin{pmatrix}5\\-2\end{pmatrix}}{\begin{pmatrix}2\\-2\end{pmatrix}} \tag{3.4.12.5}$$

The modulus of a complex number $\begin{pmatrix}a\\b\end{pmatrix}$ is defined as $\sqrt{a^2+b^2}$. Therefore,

$$\|\mathbf{z}_1 + \mathbf{z}_1 + 1\| = \sqrt{5^2 + (-2)^2} \tag{3.4.12.6}$$

$$= \sqrt{29} \tag{3.4.12.7}$$

$$\|\mathbf{z}_1 - \mathbf{z}_2 + 1\| = \sqrt{2^2 + (-2)^2} \tag{3.4.12.8}$$

$$= \sqrt{8} \tag{3.4.12.9}$$

Putting together (3.4.12.7) and (3.4.12.9), we have

$$\left\|\frac{\mathbf{z}_1 + \mathbf{z}_1 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1}\right\| = \frac{\sqrt{29}}{\sqrt{8}} \tag{3.4.12.10}$$

13. Let $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}-2\\1\end{pmatrix}$. Find

a) $\mathrm{Re}\left(\frac{\mathbf{z}_1\mathbf{z}_2}{\mathbf{z}_1^*}\right)$.

b) $\mathrm{Im}\left(\frac{1}{\mathbf{z}_1\mathbf{z}_1^*}\right)$.

Solution:

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}-2 & -1\\1 & -2\end{pmatrix}\left[\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}\right]^{-1}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.13.1}$$

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}-2 & -1\\1 & -2\end{pmatrix}\left[\frac{1}{5}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\right]\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.13.2}$$

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \frac{1}{5}\begin{pmatrix}-2 & -11\\11 & -2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.4.13.3}$$

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \frac{1}{5}\begin{pmatrix}-2\\11\end{pmatrix} \tag{3.4.13.4}$$

Hence, the real part of $\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = -\frac{2}{5}$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = (\mathbf{z_1 z_1}^*)^{-1} \tag{3.4.13.5}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \left[\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.13.6}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \left[\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.13.7}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \frac{1}{25}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.13.8}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \frac{1}{25}\begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{3.4.13.9}$$

Hence, the imaginary part of $\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = 0$.

14. Find the modulus and argument of the complex number $\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}}$.

Solution: In general, any complex number can be expressed in matrix representation as follows:

$$\begin{pmatrix} a1 \\ a2 \end{pmatrix} = \begin{pmatrix} a1 & -a2 \\ a2 & a1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.14.1}$$

Converting complex number to matrix form:

$$\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.14.2}$$

$$\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/10 & -3/10 \\ 3/10 & 1/10 \end{pmatrix} \tag{3.4.14.3}$$

Sub (3.4.14.3) in (3.4.14.2),

$$\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/10 & -3/10 \\ 3/10 & 1/10 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.14.4}$$

$$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/10 \\ 3/10 \end{pmatrix} \tag{3.4.14.5}$$

$$= \begin{pmatrix} -5/10 \\ 5/10 \end{pmatrix} \tag{3.4.14.6}$$

$$\implies \boxed{\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}} \tag{3.4.14.7}$$

From (3.4.14.7), The modulus and argument of the complex number is,

$$r = \left\|\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}\right\| = \frac{1}{\sqrt{2}} \tag{3.4.14.8}$$

$$\tan\theta = -1 \implies \theta = 180^\circ - 45^\circ = 135^\circ \tag{3.4.14.9}$$

15. Find the real numbers x, y such that $\begin{pmatrix} x \\ -y \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is the conjugate of $\begin{pmatrix} -6 \\ -24 \end{pmatrix}$.

Solution: The conjugate of $\begin{pmatrix} -6 \\ -24 \end{pmatrix}$ is $\begin{pmatrix} -6 \\ 24 \end{pmatrix}$

$$\implies \begin{pmatrix} x \\ -y \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 24 \end{pmatrix} \tag{3.4.15.1}$$

$$\implies \begin{pmatrix} x \\ -y \end{pmatrix} = \frac{\begin{pmatrix} -6 \\ 24 \end{pmatrix}}{\begin{pmatrix} 3 \\ 5 \end{pmatrix}} \tag{3.4.15.2}$$

Using equivalent matrices for complex numbers, we have

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -6 & -24 \\ 24 & -6 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.15.3)$$

$$= \frac{1}{34} \begin{pmatrix} -6 & -24 \\ 24 & -6 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.15.4)$$

$$= \frac{1}{34} \begin{pmatrix} 102 & -102 \\ 102 & 102 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.15.5)$$

$$= \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.15.6)$$

$$\implies \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad (3.4.15.7)$$

$$\text{Therefore,} x = 3, \quad (3.4.15.8)$$

$$y = -3 \quad (3.4.15.9)$$

16. Find the modulus of $\dfrac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \dfrac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

Solution: In our case,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.16.1)$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.16.2)$$

Now,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.16.3)$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.4.16.4)$$

Similarly,

$$\frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.16.5)$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (3.4.16.6)$$

So,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (3.4.16.7)$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (3.4.16.8)$$

Now, according to the problem statement:

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad (3.4.16.9)$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (3.4.16.10)$$

$\therefore$

$$\left\| \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\| \quad (3.4.16.11)$$

$$= \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\| = \sqrt{0^2 + 2^2} = 2 \quad (3.4.16.12)$$

So, we can say that the modulus value of

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad (3.4.16.13)$$

is 2.

3.5 Points and Vectors

1. Find the distance between the following pairs of points

a)

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (3.5.1.1)$$

b)

$$\begin{pmatrix} -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (3.5.1.2)$$

c)

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -1 \\ b \end{pmatrix} \quad (3.5.1.3)$$

Solution:

1. The distance between two vectors is given by
$$\|\mathbf{A} - \mathbf{B}\| \tag{3.5.1.4}$$
From (3.5.1.4),
   i) $\left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = 2.828$
   ii) $\left\| \begin{pmatrix} -5 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\| = 5.656$
   iii) $\left\| \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} -1 \\ b \end{pmatrix} \right\| = a + 1$

2. Find the distance between the points
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 36 \\ 15 \end{pmatrix} \tag{3.5.2.1}$$
Solution: The desired distance is
$$\|\mathbf{A} - \mathbf{B}\| = \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 36 \\ 15 \end{pmatrix} \right\| = 39 \tag{3.5.2.2}$$
The following Python code generates Fig. 3.5.2

```
solutions/2/codes/line_ex/pts_and_vectors/
    dist_btw_pts.py
```

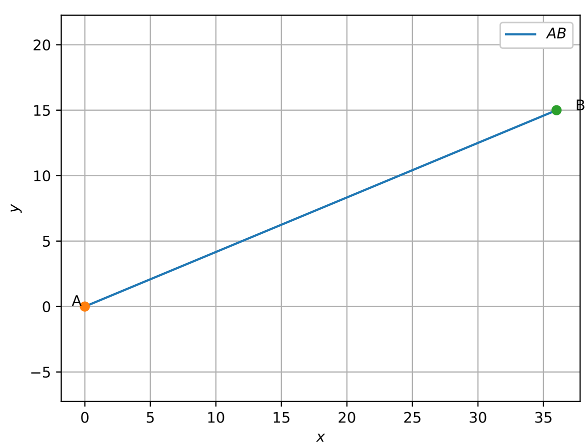

Fig. 3.5.2

3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
Solution: See Fig. 3.5.3.
$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 36 \\ 15 \end{pmatrix} \tag{3.5.3.1}$$
The distance d between A and B is given by
$$\|\mathbf{B} - \mathbf{A}\| = \|\mathbf{B}\| \tag{3.5.3.2}$$
$$= 39km \tag{3.5.3.3}$$
The following Python code generates Fig. 3.5.3.

```
solutions/3/codes/line/towns/towns.py
```

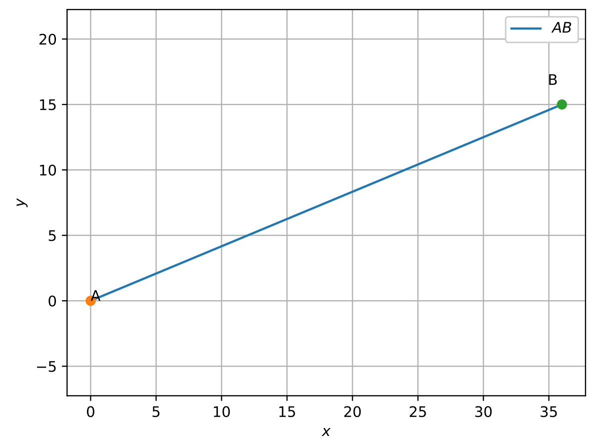

Fig. 3.5.3: Position of Towns A and B

4. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.
   a)
$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{3.5.4.1}$$
   b)
$$\mathbf{P} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{3.5.4.2}$$
   c)
$$\mathbf{P} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \tag{3.5.4.3}$$
$$\mathbf{R} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.5.4.4}$$

Solution:

a) In Fig. 3.5.4.1

$$\mathbf{P}-\mathbf{S}=\mathbf{Q}-\mathbf{R}=\begin{pmatrix}2\\-2\end{pmatrix} \tag{3.5.4.5}$$

$$\mathbf{R}-\mathbf{S}=\mathbf{Q}-\mathbf{P}=\begin{pmatrix}2\\2\end{pmatrix} \tag{3.5.4.6}$$

Hence $PQRS$ is a ||gm $\because$ opposite sides are parallel. Also,

$$\|\mathbf{P}-\mathbf{S}\|=\|\mathbf{Q}-\mathbf{R}\| \tag{3.5.4.7}$$

$$=\|\mathbf{R}-\mathbf{S}\|=\|\mathbf{Q}-\mathbf{P}\|=2\sqrt{2} \tag{3.5.4.8}$$

$\because$ all sides are equal, the ||gm is a rhombus.

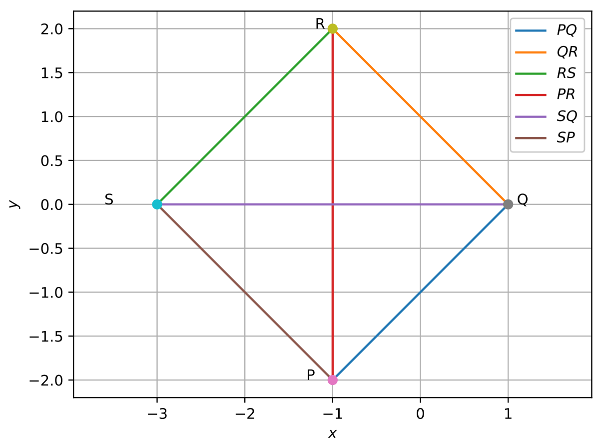

Fig. 3.5.4.1: quadrilateral1

```
solutions/4/codes/line/quad/quad1.py
```

b) In Fig. 3.5.4.2

$$\mathbf{Q}-\mathbf{P}=\begin{pmatrix}6\\-4\end{pmatrix} \tag{3.5.4.9}$$

$$\mathbf{R}-\mathbf{P}=\begin{pmatrix}3\\-2\end{pmatrix} \tag{3.5.4.10}$$

$$\mathbf{Q}-\mathbf{R}=\begin{pmatrix}3\\-2\end{pmatrix} \tag{3.5.4.11}$$

$$(\mathbf{Q}-\mathbf{P})=(\mathbf{R}-\mathbf{P})+(\mathbf{Q}-\mathbf{R})=\begin{pmatrix}6\\-4\end{pmatrix} \tag{3.5.4.12}$$

Hence, $\mathbf{P},\mathbf{Q}$ and $\mathbf{R}$ lie on a straight line, so $PQRS$ is not a quadrilateral.

```
/solutions/4/codes/line/quad/quad2.py
```

c) See Fig. 3.5.4.3.

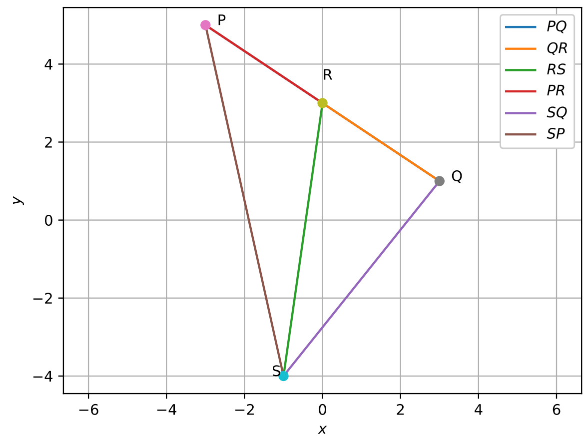

Fig. 3.5.4.2: quadrilateral2

$$\because (\mathbf{Q}-\mathbf{P})=(\mathbf{R}-\mathbf{S})=\begin{pmatrix}3\\1\end{pmatrix} \tag{3.5.4.13}$$

$$(\mathbf{P}-\mathbf{S})=(\mathbf{Q}-\mathbf{R})=\begin{pmatrix}3\\3\end{pmatrix}, \tag{3.5.4.14}$$

$PQRS$ is a parallelogram. Also,

$$\|\mathbf{Q}-\mathbf{P}\|\neq\|\mathbf{P}-\mathbf{S}\| \tag{3.5.4.15}$$

Hence, $PQRS$ is neither a rhombus nor a square.

$$\because (\mathbf{Q}-\mathbf{P})^T(\mathbf{Q}-\mathbf{R})=\begin{pmatrix}3 & 1\end{pmatrix}\begin{pmatrix}3\\3\end{pmatrix}\neq 0, \tag{3.5.4.16}$$

$PQRS$ is not a rectangle. The following code generates Fig. 3.5.4.3.

```
solutions/4/codes/line/quad/quad3.
py
```

5. Find the angle between the x-axis and the line joining the points $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\-2\end{pmatrix}$. **Solution:**

$$\frac{(\mathbf{A}-\mathbf{B})^T\begin{pmatrix}1\\0\end{pmatrix}}{\|\mathbf{A}-\mathbf{B}\|\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|}=\frac{\begin{pmatrix}-1 & 1\end{pmatrix}^T\begin{pmatrix}1\\0\end{pmatrix}}{\left\|\begin{pmatrix}-1\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}-1\\1\end{pmatrix}\right\|} \tag{3.5.5.1}$$

$$=-\frac{1}{\sqrt{2}}=\cos^{-1}(135^\circ) \tag{3.5.5.2}$$

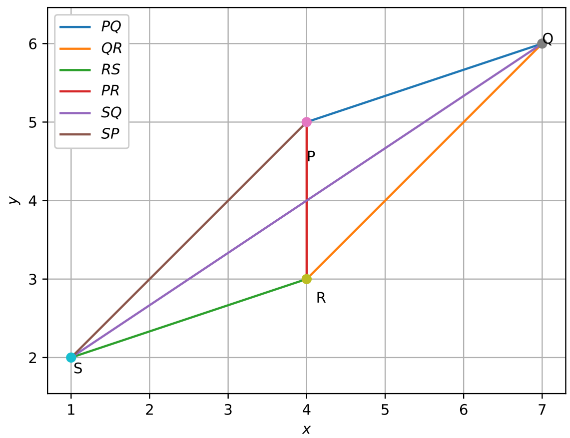

Fig. 3.5.4.3

Thus, the desired angle is 135°. The following python code generates Fig. 3.5.5.

```
./solutions/5/codes/lines/q9.py
```

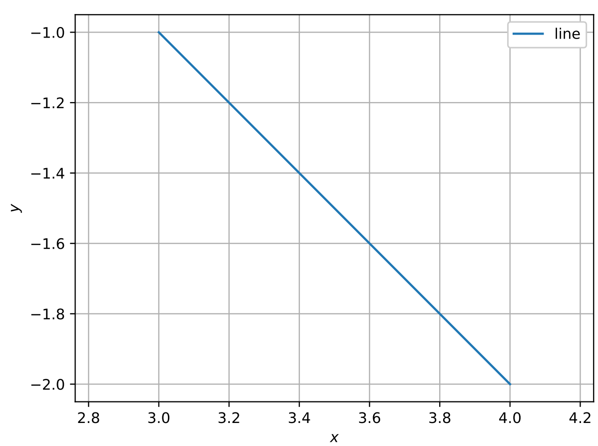

Fig. 3.5.5

6. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix}, \tag{3.5.6.1}$$

Solution: From the given information,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\|^2 = \left\| \mathbf{x} - \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\|^2 \tag{3.5.6.2}$$

$$\implies \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\|^2 - 2\begin{pmatrix} 2 & -5 \end{pmatrix}\mathbf{x}$$

$$= \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\|^2 - 2\begin{pmatrix} -2 & 9 \end{pmatrix}\mathbf{x} \tag{3.5.6.3}$$

which can be simplified to obtain

$$\begin{pmatrix} 8 & -28 \end{pmatrix}\mathbf{x} = -56 \tag{3.5.6.4}$$

Choose $\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ as the point lies on the x-axis

$$\begin{pmatrix} 8 & -28 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = -56 \tag{3.5.6.5}$$

$$\implies x = -7 \tag{3.5.6.6}$$

The desired point is $\begin{pmatrix} -7 \\ 0 \end{pmatrix}$.

See Fig. 3.5.6 generated by the following python code

```
solutions/6/codes/line/point_vector/
    point_vector.py
```

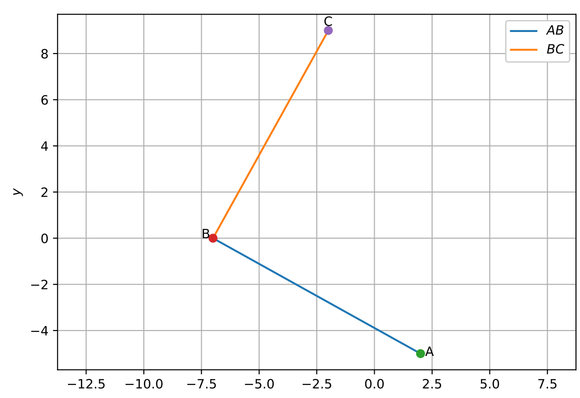

Fig. 3.5.6

7. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 10 \\ y \end{pmatrix} \tag{3.5.7.1}$$

is 10 units. **Solution:** The distance between two points is given by equation

$$(\mathbf{P} - \mathbf{Q})^T (\mathbf{P} - \mathbf{Q}) = 10^2 \tag{3.5.7.2}$$

$$\implies \|P\|^2 - \mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} + \|Q\|^2 = 100 \tag{3.5.7.3}$$

which, upon subsituting the values yields

$$y^2 + 6y - 27 = 0 \tag{3.5.7.4}$$

$$(y+9)(y-3) = 0 \implies y = -9, 3 \tag{3.5.7.5}$$

and

$$\mathbf{Q} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}, \begin{pmatrix} 10 \\ -9 \end{pmatrix} \tag{3.5.7.6}$$

The python code to find the roots of the quadratic equation can be downloaded from

```
solutions/7/codes/line/point_vec/roots.py
```

The python code for Fig. 3.5.7 can be downloaded from

```
solutions/7/codes/line/point_vec/point_vec.py
```

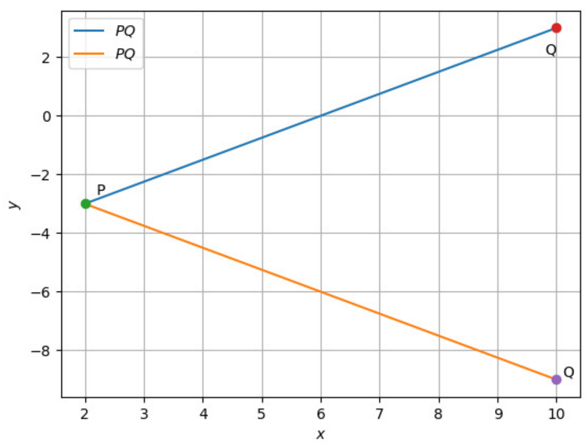

Fig. 3.5.7

8. Show that each of the given three vectors is a unit vector

$$\frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \tag{3.5.8.1}$$

Also, show that they are mutually perpendicular to each other.

Solution: Let $\boldsymbol{A}=\frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \boldsymbol{B}=\frac{1}{7}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \boldsymbol{C}=\frac{1}{7}\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}$

$$\|\boldsymbol{A}\| = \frac{1}{7}\sqrt{2^2 + 3^2 + 6^2} = 1 \tag{3.5.8.2}$$

$$\|\boldsymbol{B}\| = \frac{1}{7}\sqrt{3^2 + -6^2 + 2^2} = 1 \tag{3.5.8.3}$$

$$\|\boldsymbol{C}\| = \frac{1}{7}\sqrt{6^2 + 2^2 + -3^2} = 1 \tag{3.5.8.4}$$

When two vectors are perpendicular to each other their dot product is zero. The dot product of $\boldsymbol{A}$,$\boldsymbol{B}$ and $\boldsymbol{C}$ with each other is

$$\boldsymbol{A}^T\boldsymbol{B} = \frac{1}{7} \times \frac{1}{7}(2 \times 3 + 3 \times -6 + 6 \times 2) = 0 \tag{3.5.8.5}$$

$$\boldsymbol{B}^T\boldsymbol{C} = \frac{1}{7} \times \frac{1}{7}(2 \times 3 + 3 \times -6 + 6 \times 2) = 0 \tag{3.5.8.6}$$

$$\boldsymbol{C}^T\boldsymbol{A} = \frac{1}{7} \times \frac{1}{7}(6 \times 2 + 2 \times 3 + -3 \times 6) = 0 \tag{3.5.8.7}$$

Hence, the three unit vectors are mutually perpendicular to each other.

9. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \tag{3.5.9.1}$$

$(\mathbf{a} + k\mathbf{b}) \perp \mathbf{c}$. Find λ. **Solution:**
The two vectors are perpendicular to each other if their dot product is zero.
So,

$$\mathbf{c}^T(\mathbf{a} + k\mathbf{b}) = 0 \tag{3.5.9.2}$$

$$\mathbf{c}^T\mathbf{a} + k\mathbf{c}^T\mathbf{b} = 0 \tag{3.5.9.3}$$

$$k\mathbf{c}^T\mathbf{b} = -\mathbf{c}^T\mathbf{a} \tag{3.5.9.4}$$

$$\implies k = \frac{-\mathbf{c}^T\mathbf{a}}{\mathbf{c}^T\mathbf{b}} \tag{3.5.9.5}$$

On solving the matrix multiplication,

$$\mathbf{c}^T\mathbf{b} = -1, \tag{3.5.9.6}$$

$$\mathbf{c}^T\mathbf{a} = 8 \tag{3.5.9.7}$$

So,

$$\implies k = \frac{-8}{-1} \tag{3.5.9.8}$$

$$k = 8 \tag{3.5.9.9}$$

10. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \tag{3.5.10.1}$$

Solution: Cross product of two vectors is determined by spanning a vector into skew symmetric matrix

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -7 & -7 \\ 7 & 0 & -1 \\ 7 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 19 \\ 19 \end{pmatrix} \tag{3.5.10.2}$$

11. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \tag{3.5.11.1}$$

Solution: Let $\mathbf{A} = \mathbf{a}+\mathbf{b}$ *and* $\mathbf{B} = \mathbf{a}-\mathbf{b}$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \tag{3.5.11.2}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \tag{3.5.11.3}$$

Let $\mathbf{n}$ be a vector Perpendicular to $\mathbf{A}$ and $\mathbf{B}$ both

$$\mathbf{A}^T\mathbf{n} = 0 \tag{3.5.11.4}$$

$$\mathbf{B}^T\mathbf{n} = 0 \tag{3.5.11.5}$$

The augmented matrix can be represented as follows:

$$\left(\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right) \tag{3.5.11.6}$$

Using row reduction to find an expression for $\mathbf{n}$.

$$\xleftrightarrow[R_2\leftarrow R_2-2R_1]{R_1\leftarrow\frac{R_1}{4}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right) \tag{3.5.11.7}$$

$$\xleftrightarrow[R_1\leftarrow R_1-R_2]{R_2\leftarrow\frac{R_2}{-2}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \tag{3.5.11.8}$$

From above equations we get,

$$\therefore \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -2n_3 \\ 2n_3 \\ n_3 \end{pmatrix} = n_3\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{3.5.11.9}$$

Let us consider n_3 to be 1 which gives us:

$$\therefore \mathbf{n} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{3.5.11.10}$$

$$\|\mathbf{n}\| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3 \tag{3.5.11.11}$$

Let $\mathbf{u}$ be the unit vector of $\mathbf{n}$ which can be found as follows:

$$\mathbf{u} = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{3.5.11.12}$$

Solving the above equation gives the unit vector $\mathbf{u}$ which is perpendicular to vectors $\mathbf{A}$ *and* $\mathbf{B}$

$$\boxed{\therefore \mathbf{u} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}} \tag{3.5.11.13}$$

12. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$.

Solution:

$$\boldsymbol{d} = \boldsymbol{2a} - \boldsymbol{b} + \boldsymbol{3c} \tag{3.5.12.1}$$

$$\boldsymbol{2a} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{3.5.12.2}$$

$$\boldsymbol{-b} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \tag{3.5.12.3}$$

$$\boldsymbol{3c} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \tag{3.5.12.4}$$

From the above,

$$\boldsymbol{d} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{3.5.12.5}$$

$$\|\boldsymbol{d}\| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{22} \tag{3.5.12.6}$$

$$\boldsymbol{e} = \frac{\boldsymbol{d}}{\|\boldsymbol{d}\|} \tag{3.5.12.7}$$

$\boldsymbol{e}$ is the unit vector parallel to given vector Thus,

$$\boxed{\boldsymbol{e} = \frac{1}{\sqrt{22}}\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}} \tag{3.5.12.8}$$

13. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} =$

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$,

Solution: First find resultant $\mathbf{R}$ of $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\mathbf{R} = \mathbf{a} + \mathbf{b} \tag{3.5.13.1}$$

$$\implies \mathbf{R} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.5.13.2}$$

$$\implies \mathbf{R} = \begin{pmatrix} 2+1 \\ 3-2 \\ -1+1 \end{pmatrix} \tag{3.5.13.3}$$

$$\implies \mathbf{R} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}. \tag{3.5.13.4}$$

Magnitude of $\mathbf{R}$ is

$$\|\mathbf{R}\| = \sqrt{3^2 + 1^2 + 0^2} \tag{3.5.13.5}$$

$$\implies \|\mathbf{R}\| = \sqrt{10} \tag{3.5.13.6}$$

$$\tag{3.5.13.7}$$

Then unit vector $\mathbf{r}$ along $\mathbf{R}$ is

$$\mathbf{r} = \frac{\mathbf{R}}{\|\mathbf{R}\|} \tag{3.5.13.8}$$

$$\implies \mathbf{r} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{3.5.13.9}$$

Then vector of magnitude 5 units parallel to resultant $\mathbf{R}$ is given by

$$\mathbf{u} = 5\mathbf{r} \tag{3.5.13.10}$$

$$\implies \mathbf{u} = \frac{5}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{3.5.13.11}$$

$$\implies \mathbf{u} = \begin{pmatrix} 4.7434 \\ 1.5811 \\ 0 \end{pmatrix} \tag{3.5.13.12}$$

14. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

Solution: Let $\mathbf{m}$ be a unit vector such that $\mathbf{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$. Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be the direction vectors of the coordinate axes.
As $\mathbf{m}$ is a unit vector, so $\|\mathbf{m}\| = 1$ and also we are given is that $\mathbf{m}$ is inclined equally to the coordinate axis,

$$\mathbf{e}_1^T\mathbf{m} = \mathbf{e}_2^T\mathbf{m} = \mathbf{e}_3^T\mathbf{m} \tag{3.5.14.1}$$

Now, **??** implies

$$(\mathbf{e}_1 - \mathbf{e}_2)^T\mathbf{m} = 0 \tag{3.5.14.2}$$

$$(\mathbf{e}_2 - \mathbf{e}_3)^T\mathbf{m} = 0 \tag{3.5.14.3}$$

$$(\mathbf{e}_3 - \mathbf{e}_1)^T\mathbf{m} = 0 \tag{3.5.14.4}$$

Thus, converting above system of equations into matrix form, we get

$$\mathbf{A}\mathbf{m} = 0 \tag{3.5.14.5}$$

To find the solution of **??**, we find the echelon form of $\mathbf{A}$.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow r_1 + r_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.5.14.6}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow r_2 + r_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.5.14.7}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{r_1 \leftarrow r_1 + r_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.5.14.8}$$

From **??**, we find out that

$$m_x = m_y = m_z \tag{3.5.14.9}$$

$$\mathbf{m} = \begin{pmatrix} m_z \\ m_z \\ m_z \end{pmatrix} \implies \mathbf{m} = m_z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.5.14.10}$$

Taking $m_z = 1$, then $\|\mathbf{m}\| = \frac{1}{\sqrt{3}}$ and for $\mathbf{m}$ to be a unit vector, we need to divide each element of $\mathbf{m}$ by $\|\mathbf{m}\|$.
Thus, we see that

$$\mathbf{m} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3.5.14.11}$$

is the unit direction vector inclined equally to the coordinate axes.

15. Let $\mathbf{a} = \begin{pmatrix}1\\4\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\7\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T\mathbf{c} = 15$.
Solution: From the given information

$$\mathbf{d}^T\mathbf{a} = 0 \tag{3.5.15.1}$$

Similarly, as $\mathbf{d} \perp \mathbf{b}$

$$\mathbf{d}^T\mathbf{b} = 0 \tag{3.5.15.2}$$

It is given that

$$\mathbf{d}^T\mathbf{c} = 15 \tag{3.5.15.3}$$

Using equations 3.5.15.1, 3.5.15.2, 3.5.15.3, we can represent them in a Matrix Representation of Linear Equations $Ax=B$ form as:

$$\begin{pmatrix}\mathbf{a}^T\\\mathbf{b}^T\\\mathbf{c}^T\end{pmatrix}\mathbf{d} = \begin{pmatrix}0\\0\\15\end{pmatrix} \tag{3.5.15.4}$$

Numerically, using $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ the above equation 3.5.15.4 can be written as,

$$\begin{pmatrix}1&4&2\\3&-2&7\\2&-1&4\end{pmatrix}\mathbf{d} = \begin{pmatrix}0\\0\\15\end{pmatrix} \tag{3.5.15.5}$$

we can use Guassian Elimination Method in order to find the coordinate values of $\mathbf{d}$.

$$\left(\begin{array}{ccc|c}1&4&2&0\\3&-2&7&0\\2&-1&4&15\end{array}\right) \tag{3.5.15.6}$$

$$\xleftrightarrow[R_2\leftarrow R_2-3R_1]{R_3\leftarrow R_3-2R_1}\left(\begin{array}{ccc|c}1&4&2&0\\0&-14&1&0\\0&-9&0&15\end{array}\right) \tag{3.5.15.7}$$

$$\xleftrightarrow{R_3\leftarrow R_3-\frac{9}{14}R_2}\left(\begin{array}{ccc|c}1&4&2&0\\0&-14&1&0\\0&0&\frac{-9}{14}&15\end{array}\right) \tag{3.5.15.8}$$

$$\xleftrightarrow[R_2\leftarrow\frac{-1}{14}R_2]{R_3\leftarrow\frac{-14}{9}R_2}\left(\begin{array}{ccc|c}1&4&2&0\\0&1&\frac{-1}{14}&0\\0&0&1&\frac{-210}{9}\end{array}\right) \tag{3.5.15.9}$$

$$\xleftrightarrow{R_1\leftarrow R_1+\frac{1}{14}R_3}\left(\begin{array}{ccc|c}1&4&2&0\\0&1&0&\frac{-210}{126}\\0&0&1&\frac{-210}{9}\end{array}\right) \tag{3.5.15.10}$$

$$\xleftrightarrow{R_1\leftarrow R_1-4R_3}\left(\begin{array}{ccc|c}1&0&2&\frac{840}{126}\\0&1&0&\frac{-210}{126}\\0&0&1&\frac{-210}{9}\end{array}\right) \tag{3.5.15.11}$$

$$\xleftrightarrow{R_1\leftarrow R_1-2R_3}\left(\begin{array}{ccc|c}1&0&0&\frac{6720}{126}\\0&1&0&\frac{-210}{126}\\0&0&1&\frac{-210}{9}\end{array}\right) \tag{3.5.15.12}$$

By using Guassian Elimination Method, we were able to get the vector $\mathbf{d}$ as $\begin{pmatrix}\frac{6720}{126}\\\frac{-210}{126}\\\frac{-210}{9}\end{pmatrix}$

16. The scalar product of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix}2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}\lambda\\2\\3\end{pmatrix}$ is unity. Find the value of λ.
17. The value of

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right)+\begin{pmatrix}0\\1\\0\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$
$$+\begin{pmatrix}0\\0\\1\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right) \tag{3.5.17.1}$$

is

a) 0 c) 1
b) -1 d) 3

Solution: Given

$$\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{b} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{c} = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{3.5.17.2}$$

Using scalar triple product property we deduce

$$\mathbf{a}^T(\mathbf{b}\times\mathbf{c}) = \mathbf{b}^T(\mathbf{c}\times\mathbf{a}) = \mathbf{c}^T(\mathbf{a}\times\mathbf{b}) \tag{3.5.17.3}$$

Note: Cross product is given by:

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{3.5.17.4}$$

Equating (3.5.17.2) with problem statement we deduce the following:

$$\mathbf{a}^T(\mathbf{b}\times\mathbf{c}) + \mathbf{b}^T(\mathbf{a}\times\mathbf{c}) + \mathbf{c}^T(\mathbf{a}\times\mathbf{b}) \tag{3.5.17.5}$$

As Cross Product is anti-commutative we get:

$$\mathbf{a}^T(\mathbf{b}\times\mathbf{c}) - \mathbf{b}^T(\mathbf{c}\times\mathbf{a}) + \mathbf{c}^T(\mathbf{a}\times\mathbf{b}) \tag{3.5.17.6}$$

$$= \mathbf{a}^T(\mathbf{b}\times\mathbf{c}) - \mathbf{c}^T(\mathbf{a}\times\mathbf{b}) + \mathbf{c}^T(\mathbf{a}\times\mathbf{b}) \tag{3.5.17.7}$$

$$= \mathbf{a}^T(\mathbf{b}\times\mathbf{c}) \tag{3.5.17.8}$$

So instead of calculating each step we just calculate one iteration by referring (3.5.17.4) and (3.5.17.8) i.e.

$$\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{3.5.17.9}$$

$$\implies \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\end{pmatrix} \tag{3.5.17.10}$$

18. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively. **Solution:**

$$\mathbf{m} = \begin{pmatrix}\cos 90°\\ \cos 135°\\ \cos 45°\end{pmatrix} \tag{3.5.18.1}$$

we know that,

$$\mathbf{m} = \frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{3.5.18.2}$$

Also,

$$\|\mathbf{m}\| = \sqrt{0^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \implies \|\mathbf{m}\| = 1 \tag{3.5.18.3}$$

Hence,From (3.5.18.1) and (3.5.18.3) we have the unit vector:

$$\mathbf{m} = \begin{pmatrix}0\\ \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \tag{3.5.18.4}$$

19. Show that the lines with direction vectors $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}4\\12\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-4\\12\end{pmatrix}$ are mutually perpendicular.

20. Show that the line through the points $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}0\\3\\2\end{pmatrix}$, $\begin{pmatrix}3\\5\\6\end{pmatrix}$.

Solution: Let the points be $\mathbf{P}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}3\\4\\-2\end{pmatrix}$, $\mathbf{R}=\begin{pmatrix}0\\3\\2\end{pmatrix}$ and $\mathbf{S}=\begin{pmatrix}3\\5\\6\end{pmatrix}$. The direction vector for the line through the points **P** and **Q** is

$$\mathbf{A} = \mathbf{P} - \mathbf{Q} \tag{3.5.20.1}$$

$$\implies \mathbf{A} = \begin{pmatrix}1\\-1\\2\end{pmatrix} - \begin{pmatrix}3\\4\\-2\end{pmatrix} \tag{3.5.20.2}$$

$$\implies \mathbf{A} = \begin{pmatrix}-2\\-5\\4\end{pmatrix} \tag{3.5.20.3}$$

The direction vector for the line through the points $\mathbf{R}$ and $\mathbf{S}$ is

$$\mathbf{B} = \mathbf{R} - \mathbf{S} \tag{3.5.20.4}$$

$$\implies \mathbf{B} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \tag{3.5.20.5}$$

$$\implies \mathbf{B} = \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} \tag{3.5.20.6}$$

$$\tag{3.5.20.7}$$

To check if the two lines are perpendicular, we perform scalar product of the two direction vectors $\mathbf{A}$ and $\mathbf{B}$ as follows

$$\mathbf{AB} = \mathbf{A}^T\mathbf{B} \tag{3.5.20.8}$$

$$= \begin{pmatrix} -2 & -5 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} \tag{3.5.20.9}$$

$$= 6 + 10 - 16 \tag{3.5.20.10}$$

$$= 0 \tag{3.5.20.11}$$

Thus, the lines are **perpendicular**.

21. Show that the line through the points $\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

Solution: Let the lines be parallel and the first two points pass through $\mathbf{n}^T\mathbf{x} = c1$. i.e.

$$\mathbf{n}^T\mathbf{x}_1 = c_1 => \mathbf{x}_1^T\mathbf{n} = c_1 \tag{3.5.21.1}$$

$$\mathbf{n}^T\mathbf{x}_2 = c_2 => \mathbf{x}_2^T\mathbf{n} = c_2 \tag{3.5.21.2}$$

and the second two points pass through $\mathbf{n}^T\mathbf{x} = c2$ Then

$$\mathbf{n}^T\mathbf{x}_3 = c_3 => \mathbf{x}_3^T\mathbf{n} = c_3 \tag{3.5.21.3}$$

$$\mathbf{n}^T\mathbf{x}_4 = c_4 => \mathbf{x}_4^T\mathbf{n} = c_4 \tag{3.5.21.4}$$

Putting all the equations together, we obtain

$$\begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \mathbf{x}_3^T \\ \mathbf{x}_4^T \end{pmatrix} \mathbf{n} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{3.5.21.5}$$

Now if this equation has a solution, then $\mathbf{n}$ exists and the lines will be parallel. Given the points, $\mathbf{A} = \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, and\ \mathbf{C} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

Applying the row reduction procedure on the coefficient matrix:

$$\begin{pmatrix} 4 & 7 & 8 \\ 2 & 3 & 4 \\ -1 & -2 & 1 \\ 1 & 2 & 5 \end{pmatrix} \tag{3.5.21.6}$$

$$\xleftrightarrow[R_4 \leftarrow R_3 + R_4]{R_2 \leftarrow R_1 - 2R_2} \begin{pmatrix} 4 & 7 & 8 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 6 \end{pmatrix} \tag{3.5.21.7}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - 6R_4]{R_1 \leftarrow R_1 - 7R_2} \begin{pmatrix} 4 & 0 & 8 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \tag{3.5.21.8}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - 8R_4]{R_4 \leftarrow R_4/6} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.5.21.9}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_4]{R_3 \leftarrow (-R_3 - 2R_2)} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.5.21.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 4R_3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.5.21.11}$$

Here, the number of non-zero rows are three and hence the rank of the matrix is 3 which implies that the solution exists. Therefore the lines passing through $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}, \mathbf{D}$ are parallel.

22. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Solution: Given,

$$\mathbf{P} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \quad \mathbf{Q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.5.22.1}$$

A vector on the X-axis $\mathbf{X}$ is equidistant to both

P and **Q**.

$$i.e.\ \mathbf{X} = \frac{\mathbf{P}+\mathbf{Q}}{2} \tag{3.5.22.2}$$

Need to find k. Let $\mathbf{X} = k\begin{pmatrix}1\\0\end{pmatrix}$ be the vector on the X-axis.

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{X} = k \tag{3.5.22.3}$$

$$\implies \mathbf{X} = \frac{\begin{pmatrix}7\\6\end{pmatrix}+\begin{pmatrix}3\\4\end{pmatrix}}{2} \tag{3.5.22.4}$$

$$\implies \mathbf{X} = \begin{pmatrix}5\\5\end{pmatrix} \tag{3.5.22.5}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{X} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}5\\5\end{pmatrix} \tag{3.5.22.6}$$

$$\tag{3.5.22.7}$$

Therefore, k = 5 i.e. $\mathbf{X} = \begin{pmatrix}5\\0\end{pmatrix}$ See Fig. 3.5.22

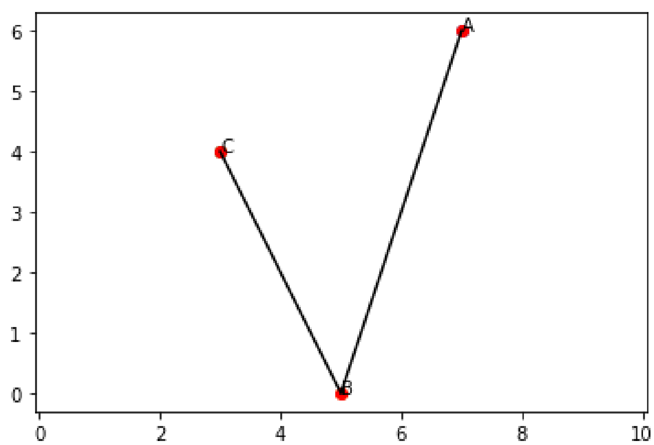

Fig. 3.5.22: Plot representing the Points

23. Find the angle between the vectors

$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{3.5.23.1}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix}1\\-2\\3\end{pmatrix} \mathbf{b} = \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{3.5.23.2}$$

Angle between the vectors is given by,

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right) \tag{3.5.23.3}$$

$$\|\mathbf{a}\| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{14} \tag{3.5.23.4}$$

$$\|\mathbf{b}\| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14} \tag{3.5.23.5}$$

$$\mathbf{a}^T\mathbf{b} = (1)(3)+(-2)(-2)+(3)(1) = 10 \tag{3.5.23.6}$$

$$\theta = \cos^{-1}\left(\frac{10}{(\sqrt{14})(\sqrt{14})}\right) \tag{3.5.23.7}$$

$$= \cos^{-1}\left(\frac{10}{14}\right) \tag{3.5.23.8}$$

$$\tag{3.5.23.9}$$

24. Find the projection of the vector

$$\begin{pmatrix}1\\3\\7\end{pmatrix} \tag{3.5.24.1}$$

on the vector

$$\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{3.5.24.2}$$

Solution:
We have,

$$\mathbf{u} = \begin{pmatrix}1\\3\\7\end{pmatrix}, \mathbf{v} = \begin{pmatrix}7\\-1\\8\end{pmatrix}$$

$$\mathbf{p} = \left[\frac{\begin{pmatrix}1\\3\\7\end{pmatrix}^T\begin{pmatrix}7\\-1\\8\end{pmatrix}}{\left\|\begin{matrix}7\\-1\\8\end{matrix}\right\|^2}\right]\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{3.5.24.3}$$

$$\mathbf{p} = \left[\frac{(7-3+56)}{\left(\sqrt{7^2+(-1)^2+8^2}\right)^2}\right]\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{3.5.24.4}$$

$$\mathbf{p} = \frac{13}{25}\begin{pmatrix}7\\-1\\8\end{pmatrix} = \begin{pmatrix}\frac{92}{25}\\-\frac{13}{25}\\\frac{21}{5}\end{pmatrix} \tag{3.5.24.5}$$

Hence the projection of **u** on **v** is

$$\mathbf{p} = \begin{pmatrix} \frac{92}{25} \\ -\frac{13}{25} \\ \frac{21}{5} \end{pmatrix}$$

25. Write down a unit vector in the xy-plane, makeing an angle of 30° with the positive direction of the x-axis.
Solution:

$$\because m = \tan 30^\circ = \frac{1}{\sqrt{3}}, \tag{3.5.25.1}$$

the direction vector is

$$\mathbf{a} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3.5.25.2}$$

and the unit vector

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|a\|} \tag{3.5.25.3}$$

$$\implies \hat{\mathbf{a}} = \begin{pmatrix} \frac{\frac{1}{2}}{\sqrt{3}} \\ \frac{\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}} \end{pmatrix} \tag{3.5.25.4}$$

$$\hat{\mathbf{a}} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \tag{3.5.25.5}$$

$$\implies \boxed{\hat{\mathbf{a}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \tag{3.5.25.6}$$

26. Find the value of x for which $x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a unit vector.
Solution:

$$\left\| x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \tag{3.5.26.1}$$

$$\implies x \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \tag{3.5.26.2}$$

$$\text{or, } \sqrt{3x^2} = 1 \implies x = \pm\frac{1}{\sqrt{3}} \tag{3.5.26.3}$$

27. Find the angle between the force $\mathbf{F} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ and displacement $\mathbf{d} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$.
Solution: Let the angle between **F** and **d** = θ Then,

$$\cos(\theta) = \frac{\mathbf{F}^T\mathbf{d}}{\|\mathbf{F}\|\,\|\mathbf{d}\|} \tag{3.5.27.1}$$

where $\mathbf{F}^T\mathbf{d}$ is scalar product of vectors **F** and **d**
And, $\|\mathbf{F}\|$ and $\|\mathbf{d}\|$ are their respective magnitudes So,

$$\mathbf{F}^T\mathbf{d} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}^T \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \tag{3.5.27.2}$$

$$\implies \mathbf{F}^T\mathbf{d} = \begin{pmatrix} 3 & 4 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \tag{3.5.27.3}$$

$$= 16 \tag{3.5.27.4}$$

$$\|\mathbf{F}\| = \sqrt{3^2 + 4^2 + (-5)^2} = 5\sqrt{2} \tag{3.5.27.5}$$

$$\|\mathbf{d}\| = \sqrt{5^2 + 4^2 + 3^2} = 5\sqrt{2} \tag{3.5.27.6}$$

Substituting these values in Equation 3.5.27.1,

$$\cos(\theta) = \frac{16}{(5\sqrt{2})(5\sqrt{2})} \tag{3.5.27.7}$$

$$= \frac{8}{25} \tag{3.5.27.8}$$

$$\implies \theta = \arccos\left(\frac{8}{25}\right) \tag{3.5.27.9}$$

$$\implies \theta \approx 71.3^\circ \tag{3.5.27.10}$$

28. A body constrained to move along the z-axis of a coordinate system is subject to a constant force

$$\mathbf{F} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \tag{3.5.28.1}$$

What is the work done by this force in moving the body a distance of 4 m along the z-axis ?
Solution: Work done in moving an object by a distance **s** using an external force **F** is given

by:

$$W = \mathbf{F}^{\mathbf{T}}\mathbf{s} \tag{3.5.28.2}$$

As seen above, work done is the scalar product (dot product) of Force and distance. Here,

$$\mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \tag{3.5.28.3}$$

The scalar product of the variables is given by:

$$\mathbf{F}^{\mathbf{T}}\mathbf{s} = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 12 \tag{3.5.28.4}$$

The work done by the force $\mathbf{F}$ is 12 J

29. Find the scalar and vector products of the two vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \tag{3.5.29.1}$$

Solution:

$$\mathbf{a}^{\mathbf{T}}\mathbf{b} = \begin{pmatrix} 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \tag{3.5.29.2}$$

$$= (3 \times -2) + (-4 \times 1) + (5 \times -3) \tag{3.5.29.3}$$

$$= -25 \tag{3.5.29.4}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & 5 & -4 \\ 5 & 0 & -3 \\ -(-4) & 3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \tag{3.5.29.5}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (0 \times -2) + (-5 \times 1) + (-4 \times -3) \\ (5 \times -2) + (0 \times 1) + (-3 \times -3) \\ (4 \times -2) + (3 \times 1) + (0 \times -3) \end{pmatrix} \tag{3.5.29.6}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix} \tag{3.5.29.7}$$

30. Find the torque of a force $\begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix}$ about the origin. The force acts on a particle whose position vector is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: The torque $\mathbf{T}$ is given by the cross product (vector product) of the position (or distance) vector $\mathbf{r}$ and the force vector $\mathbf{F}$.

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \tag{3.5.30.1}$$

And the vector cross product of vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{3.5.30.2}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{3.5.30.3}$$

can be expressed as the product of a skew-symmetric matrix and a vector:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{3.5.30.4}$$

Torque at the origin is given by,

$$\mathbf{F} \times \mathbf{r} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix} \tag{3.5.30.5}$$

$$\implies \mathbf{F} \times \mathbf{r} = \begin{pmatrix} (0 \times 7) + (-1 \times 3) + (-1 \times -5) \\ (1 \times 7) + (0 \times 3) + (-1 \times -5) \\ (1 \times 7) + (1 \times 3) + (0 \times -5) \end{pmatrix} \tag{3.5.30.6}$$

$$\implies \mathbf{T} = \begin{pmatrix} 2 \\ 12 \\ 10 \end{pmatrix} \tag{3.5.30.7}$$

3.6 *Points on a Line*

1. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{3.6.1.1}$$

in the ratio 2 : 3.

Solution:

1. $\mathbf{A} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Then $\mathbf{C}$ that divides $\mathbf{A}, \mathbf{B}$ in the ratio $k : 1$ is

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{3.6.1.2}$$

For the given problem k=2 : 3
Using the equation 3.6.1.2, the desired point is

$$\mathbf{C} = \frac{\frac{2}{3}\begin{pmatrix}-1\\7\end{pmatrix} + \begin{pmatrix}4\\-3\end{pmatrix}}{\frac{2}{3}+1} \tag{3.6.1.3}$$

$$\therefore \mathbf{C} = \begin{pmatrix}2\\1\end{pmatrix} \tag{3.6.1.4}$$

The following code plots Fig. 3.6.1

```
codes/line/section.py
```

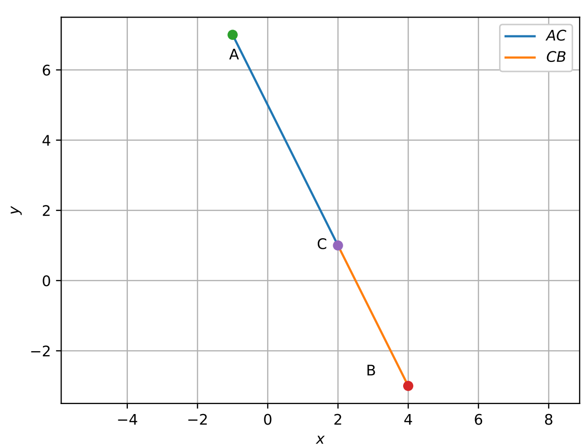

Fig. 3.6.1

2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\end{pmatrix}$.
Solution: The points of trisection are

$$\mathbf{C} = \frac{0.5\mathbf{A} + \mathbf{B}}{0.5+1} \tag{3.6.2.1}$$

$$\mathbf{D} = \frac{2\mathbf{A} + \mathbf{B}}{2+1} \tag{3.6.2.2}$$

$$\implies \therefore \mathbf{C} = \begin{pmatrix}0\\-2.33\end{pmatrix}, \mathbf{D} = \begin{pmatrix}2\\-1.66\end{pmatrix} \tag{3.6.2.3}$$

The following Python code generates Fig. 3.6.2

```
solutions/2/codes/line_ex/pts_on_a_line/
    trisection.py
```

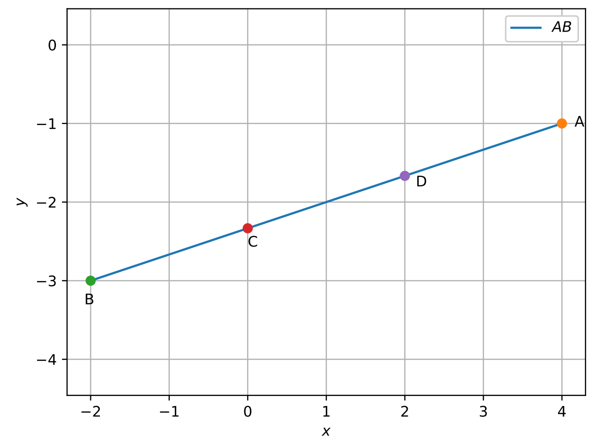

Fig. 3.6.2

3. Find the ratio in which the line segment joining the points $\begin{pmatrix}-3\\10\end{pmatrix}$ and $\begin{pmatrix}6\\-8\end{pmatrix}$ is divided by $\begin{pmatrix}-1\\6\end{pmatrix}$.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}-3\\10\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\-8\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\6\end{pmatrix} \tag{3.6.3.1}$$

Then by section formula,

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{3.6.3.2}$$

$$\begin{pmatrix}-1\\6\end{pmatrix} = \frac{1}{k+1}\begin{pmatrix}6k-3\\-8k+10\end{pmatrix} \tag{3.6.3.3}$$

$$\implies k = \frac{2}{7} \tag{3.6.3.4}$$

The following Python code generates Fig. 3.6.3

```
solutions/3/codes/line/section/section.py
```

4. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix}1\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\5\end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
Solution: Let

$$\mathbf{C}\begin{pmatrix}x\\0\end{pmatrix} \tag{3.6.4.1}$$

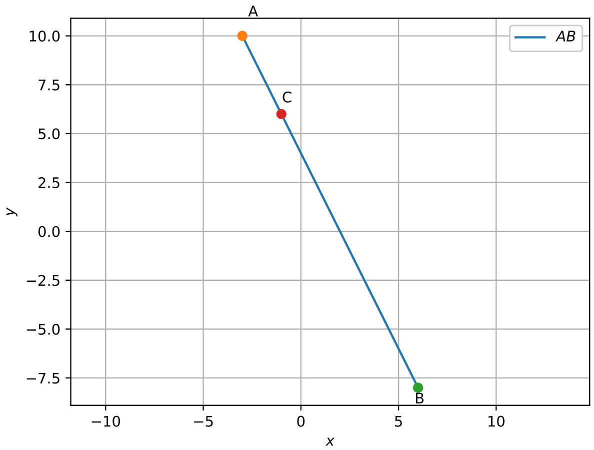

Fig. 3.6.3: C divides AB in ratio k:1

divide **AB** in k:1 ratio. Then,

$$(k+1)\begin{pmatrix} x \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix} \quad (3.6.4.2)$$

$$\implies 0 = -5k + 5 \quad (3.6.4.3)$$

$$\text{or, } k = 1 \quad (3.6.4.4)$$

$$\mathbf{C} = \frac{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} -1.5 \\ 0 \end{pmatrix} \quad (3.6.4.5)$$

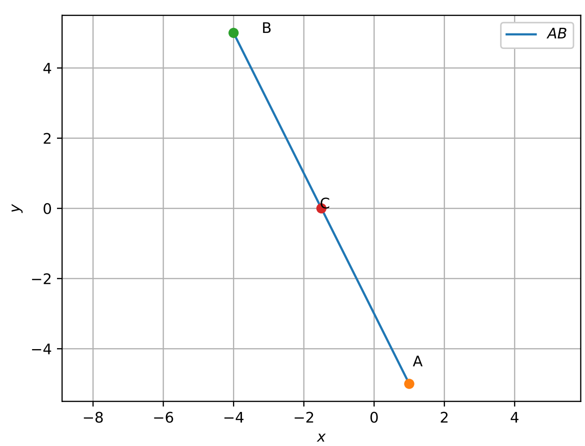

Fig. 3.6.4: line

The following code plots Fig. 3.6.4

```
solutions/4/codes/line/point_on_line/
    points_on_line.py
```

5. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
Solution: See Fig. 3.6.5. In a parallelogram, the diagonals bisect each other. Hence

$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} \quad (3.6.5.1)$$

$$\therefore \frac{1+x}{2} = \frac{7}{2}, \frac{8}{2} = \frac{y+5}{2} \quad (3.6.5.2)$$

$$\implies x = 6, y = 3 \quad (3.6.5.3)$$

The following python code computes the value of x and y used in Fig. 3.6.5.

```
./solutions/5/codes/lines/q10.py
```

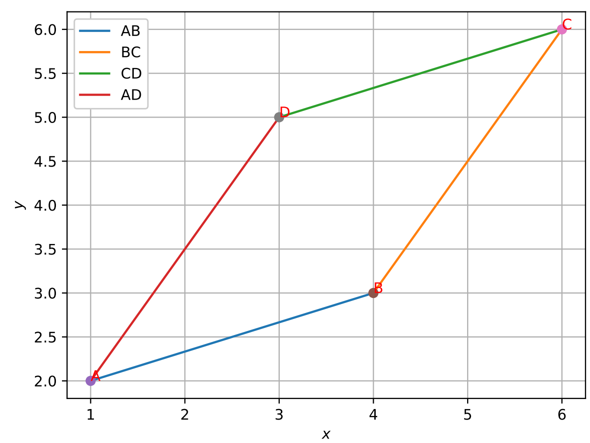

Fig. 3.6.5: Parallelogram of Q.3.6.5

6. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
Solution: The desired point is

$$\mathbf{P} = \frac{\frac{3}{4}\begin{pmatrix} 2 \\ -4 \end{pmatrix} + 1\begin{pmatrix} -2 \\ -2 \end{pmatrix}}{\frac{3}{4}+1} \quad (3.6.6.1)$$

$$\mathbf{P} = \begin{pmatrix} -2/7 \\ -20/7 \end{pmatrix} \quad (3.6.6.2)$$

The following python code plots the Fig. 3.6.6

```
solutions/6/codes/point_line/int_sec.py
```

7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

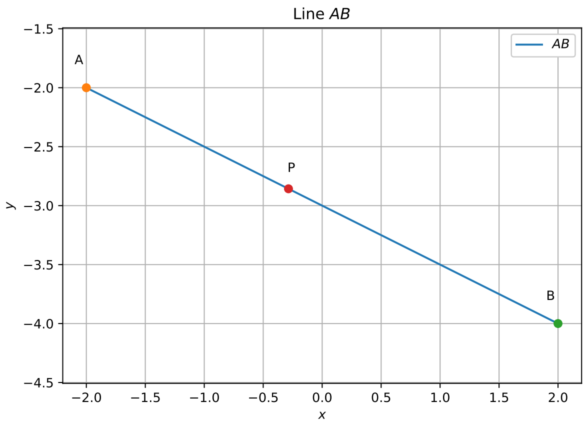

Fig. 3.6.6

into four equal parts.
Solution: The desired coordinates are

$$\mathbf{D} = \frac{1\mathbf{B} + 3\mathbf{A}}{4} = \begin{pmatrix} -1 \\ 7/2 \end{pmatrix} \tag{3.6.7.1}$$

$$\mathbf{E} = \frac{2\mathbf{B} + 2\mathbf{A}}{4} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{3.6.7.2}$$

$$\mathbf{F} = \frac{3\mathbf{B} + 1\mathbf{A}}{4} = \begin{pmatrix} 1 \\ 13/2 \end{pmatrix} \tag{3.6.7.3}$$

The following code plots Fig. 3.6.7

```
solutions/7/codes/line/point_line/
    line_division.py
```

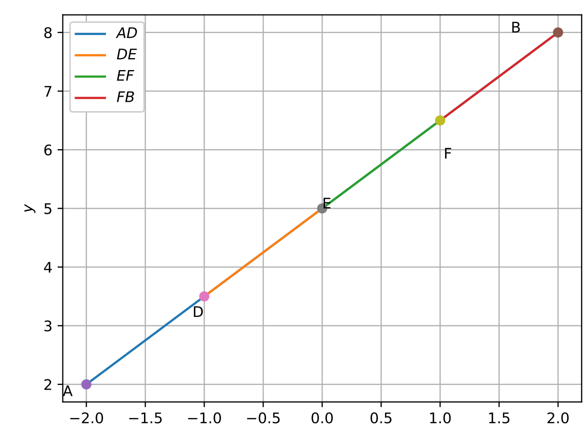

Fig. 3.6.7

8. Determine if the points

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -11 \end{pmatrix} \tag{3.6.8.1}$$

are collinear.

9. By using the concept of equation of a line, prove that the three points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ are collinear.
10. Find the value of x for which the points $\begin{pmatrix} x \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ are collinear.
11. In each of the following, find the value of k for which the points are collinear
    a) $\begin{pmatrix} 7 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ k \end{pmatrix}$
    b) $\begin{pmatrix} 8 \\ 1 \end{pmatrix}, \begin{pmatrix} k \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \end{pmatrix}$
12. Find a condition on $\mathbf{x}$ such that the points $\mathbf{x}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ are collinear.
13. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}$ are collinear.
14. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 11 \\ 3 \\ 7 \end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
15. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$ are collinear.

3.7 Lines and Planes

1. Verify whether the following are zeroes of the polynomial, indicated against them.
    a) $p(x) = 3x + 1, x = \frac{1}{3}$
    b) $p(x) = 5x - \pi, x = \frac{4}{5}$
    c) $p(x) = 5lx + m, x = -\frac{m}{l}$
    d) $p(x) = 2x + 1, x = \frac{1}{2}$

    Solution:
    1. Let

$$y = 3x + 1 \tag{3.7.1.1}$$

$$\implies \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = -1 \tag{3.7.1.2}$$

For $x = \frac{1}{3}$ to be a zero,

$$\mathbf{x} = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \tag{3.7.1.3}$$

should satisfy (3.7.1.2).

$$\because \begin{pmatrix} 3 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} = 1 \neq -1, \tag{3.7.1.4}$$

$\mathbf{x} = \frac{1}{3}$ is not a zero. This is verified in Fig. 3.7.1.1.

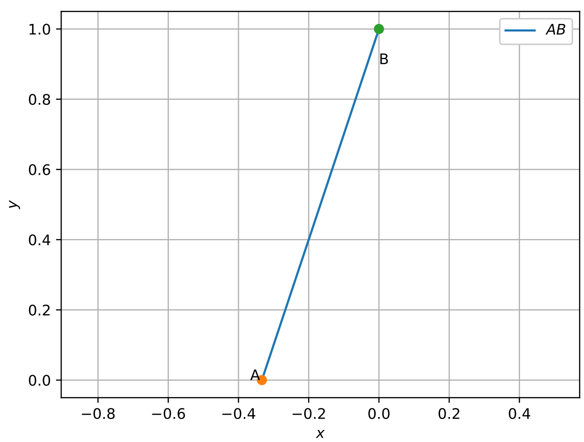

Fig. 3.7.1.1

2. Let

$$y = 5x - \pi \tag{3.7.1.5}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \pi \tag{3.7.1.6}$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} \\ 0 \end{pmatrix} = 4 \neq \pi \tag{3.7.1.7}$$

Hence $\mathbf{x} = \frac{4}{5}$ is not a zero. This is verified in Fig. 3.7.1.2.

3. Let

$$y = 5lx + m \tag{3.7.1.8}$$

$$\implies \begin{pmatrix} 5l & -1 \end{pmatrix} \mathbf{x} = -m \tag{3.7.1.9}$$

Thus,

$$\begin{pmatrix} 5l & -1 \end{pmatrix} \begin{pmatrix} -\frac{m}{l} \\ 0 \end{pmatrix} = -5m \neq -m \tag{3.7.1.10}$$

Hence $\mathbf{x} = -\frac{m}{l}$ is not a zero. This is verified in Fig. 3.7.1.3.

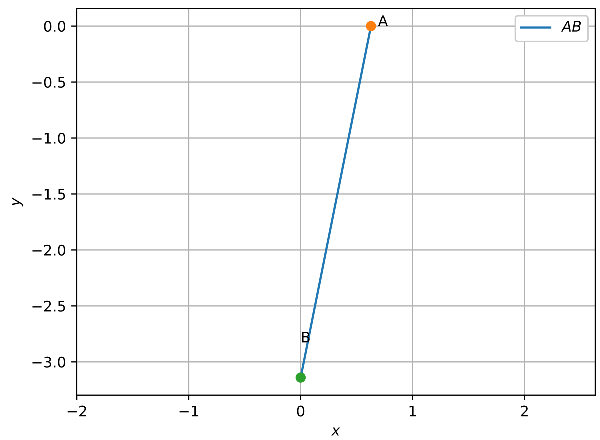

Fig. 3.7.1.2

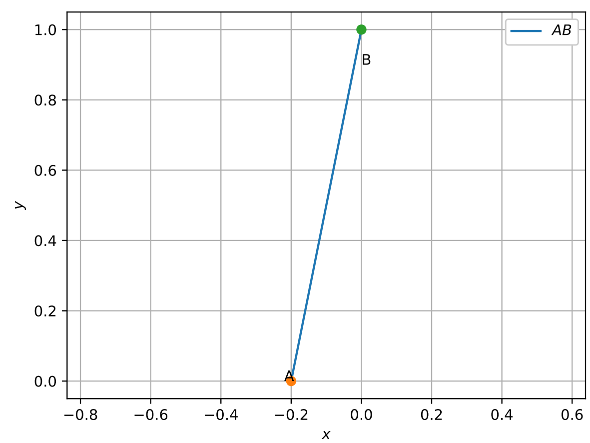

Fig. 3.7.1.3

4. Let

$$y = 2x + 1 \tag{3.7.1.11}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -1 \tag{3.7.1.12}$$

Thus,

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = 2 \neq -1 \tag{3.7.1.13}$$

Hence $\mathbf{x} = \frac{1}{2}$ is not a zero. This is verified in Fig. 3.7.1.4.

2. Find the zero of the polynomial in each of the following cases:

a) $p(x) = x + 5$
b) $p(x) = x – 5$
c) $p(x) = 2x + 5$

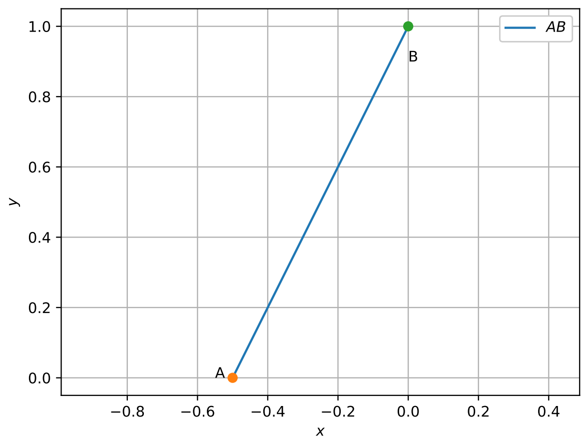

Fig. 3.7.1.4

d) $p(x) = 3x–2$
e) $p(x) = 3x$
f) $p(x) = ax, a \neq 0$
g) $p(x) = cx + d, c \neq 0, c, d$ are real numbers.

Solution:

a) For $p(x) = x + 5$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{3.7.2.1}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{3.7.2.2}$$

$$x_1 + 5 = 0 \tag{3.7.2.3}$$

$$x_1 = -5 \tag{3.7.2.4}$$

b) For $p(x) = x - 5$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} - 5 = 0 \tag{3.7.2.5}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{3.7.2.6}$$

$$x_1 - 5 = 0 \tag{3.7.2.7}$$

$$x_1 = 5 \tag{3.7.2.8}$$

c) For $p(x) = 2x + 5$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{3.7.2.9}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{3.7.2.10}$$

$$2x_1 + 5 = 0 \tag{3.7.2.11}$$

$$x_1 = \frac{-5}{2} \tag{3.7.2.12}$$

d) For $p(x) = 3x - 2$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{3.7.2.13}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{3.7.2.14}$$

$$3x_1 - 2 = 0 \tag{3.7.2.15}$$

$$x_1 = \frac{2}{3} \tag{3.7.2.16}$$

e) For $p(x) = 3x$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.7.2.17}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{3.7.2.18}$$

$$3x_1 = 0 \tag{3.7.2.19}$$

$$x_1 = 0 \tag{3.7.2.20}$$

The following Python code generates Fig 3.7.2.1

```
solutions/2/codes/line_ex/
    lines_and_planes/linear_eq_roots.py
```

3. Find two solutions for each of the following equations:
   a) $\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12$
   b) $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 0$
   c) $\begin{pmatrix} 0 & 3 \end{pmatrix} \mathbf{x} = 4$

   Solution:

3.1. A point $\mathbf{c}$ lying on the line

$$\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} = d \tag{3.7.3.1}$$

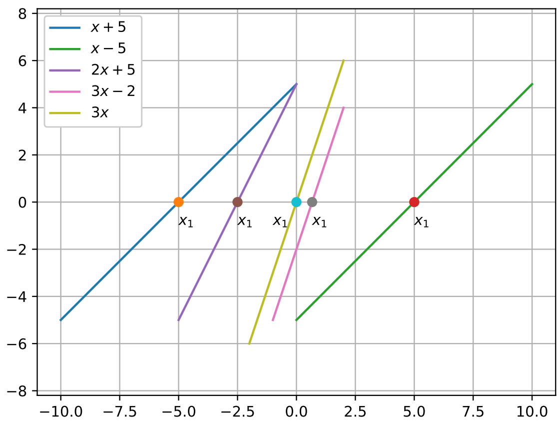

Fig. 3.7.2.1

at a distance λ from point $\mathbf{x}$ lying on the same line is given as

$$\mathbf{c} = \mathbf{x} + \frac{\lambda}{\sqrt{a^2+b^2}}\begin{pmatrix} b \\ -a \end{pmatrix} \quad (3.7.3.2)$$

$$\lambda = \sqrt{a^2+b^2} \implies \mathbf{c} = \mathbf{x} + \begin{pmatrix} b \\ -a \end{pmatrix} \quad (3.7.3.3)$$

Equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0 \quad (3.7.3.4)$$

$$(a)\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \quad (3.7.3.5)$$

The line meets y-axis at point $\mathbf{y_1}$ given using 3.7.3.4 as,

$$\begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}\mathbf{y_1} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (3.7.3.6)$$

$$\mathbf{y_1} = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (3.7.3.7)$$

$$\mathbf{y_1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (3.7.3.8)$$

Another point $\mathbf{c_1}$ on the line is found using equation 3.7.3.3

$$\mathbf{c_1} = \mathbf{y_1} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (3.7.3.9)$$

$$\implies \mathbf{c_1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.7.3.10)$$

$$(b)\begin{pmatrix} 2 & 5 \end{pmatrix}\mathbf{x} = 0 \quad (3.7.3.11)$$

The line meets y-axis at point $\mathbf{y_2}$ given using 3.7.3.4 as,

$$\begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}\mathbf{y_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.7.3.12)$$

$$\mathbf{y_1} = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.7.3.13)$$

$$\mathbf{y_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.7.3.14)$$

Another point $\mathbf{c_2}$ on the line is found using equation 3.7.3.3

$$\mathbf{c_2} = \mathbf{y_2} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (3.7.3.15)$$

$$\implies \mathbf{c_2} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (3.7.3.16)$$

$$(c)\begin{pmatrix} 0 & 3 \end{pmatrix}\mathbf{x} = 4 \quad (3.7.3.17)$$

The line meets y-axis at point $\mathbf{y_2}$ given using 3.7.3.4 as,

$$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}\mathbf{y_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (3.7.3.18)$$

$$\mathbf{y_1} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (3.7.3.19)$$

$$\mathbf{y_1} = \begin{pmatrix} 0 \\ \frac{4}{3} \end{pmatrix} \quad (3.7.3.20)$$

Another point $\mathbf{c_2}$ on the line is found using equation 3.7.3.3

$$\mathbf{c_2} = \mathbf{y_2} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.7.3.21)$$

$$\implies \mathbf{c_2} = \begin{pmatrix} 3 \\ \frac{4}{3} \end{pmatrix} \quad (3.7.3.22)$$

The python code for the above problem , plotting the figure 3.7.3.1 is available at

solutions/3/codes/line/pointonline2/pointonline.py

4. Sketch the following lines

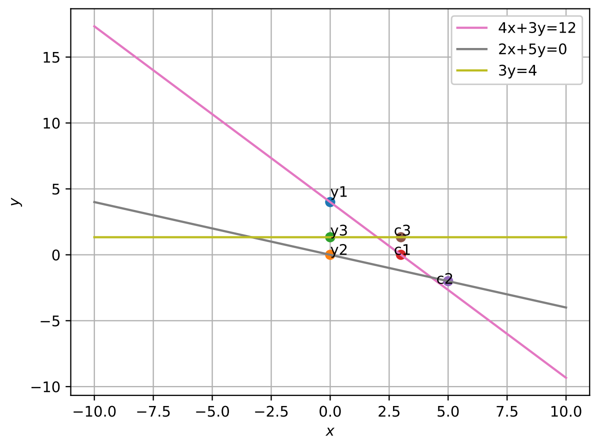

Fig. 3.7.3.1: Plot of the three lines and the points on them

a) $\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 9.35$
b) $\begin{pmatrix} 1 & -\frac{1}{5} \end{pmatrix} \mathbf{x} = 10$
c) $\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = 6$
d) $\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 0$
e) $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 0$
f) $\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} = -2$
g) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2$
h) $\begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} = 5$

Solution:

a) put $\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ in equation

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \frac{187}{20} \tag{3.7.4.1}$$

$$x = \frac{187}{40} \tag{3.7.4.2}$$

put $\mathbf{x} \begin{pmatrix} 0 \\ y \end{pmatrix}$ in equation

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \frac{187}{20} \tag{3.7.4.3}$$

$$y = \frac{187}{60} \tag{3.7.4.4}$$

$$\mathbf{P1} = \begin{pmatrix} \frac{187}{40} \\ 0 \end{pmatrix}, \mathbf{Q1} = \begin{pmatrix} 0 \\ \frac{187}{60} \end{pmatrix} \tag{3.7.4.5}$$

b) put $\mathbf{x} \begin{pmatrix} x \\ 0 \end{pmatrix}$ in equation

$$\begin{pmatrix} 1 & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = 10 \tag{3.7.4.6}$$

$$x = 10 \tag{3.7.4.7}$$

put $\mathbf{x} \begin{pmatrix} 0 \\ y \end{pmatrix}$ in equation

$$\begin{pmatrix} 1 & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = 10 \tag{3.7.4.8}$$

$$y = -50 \tag{3.7.4.9}$$

$$\mathbf{P2} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}, \mathbf{Q2} = \begin{pmatrix} 0 \\ -50 \end{pmatrix} \tag{3.7.4.10}$$

c) put $\mathbf{x} \begin{pmatrix} x \\ 0 \end{pmatrix}$ in equation

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = 6 \tag{3.7.4.11}$$

$$x = -3 \tag{3.7.4.12}$$

put $\mathbf{x} \begin{pmatrix} 0 \\ y \end{pmatrix}$ in equation

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = 6 \tag{3.7.4.13}$$

$$y = 2 \tag{3.7.4.14}$$

$$\mathbf{P3} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{Q3} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{3.7.4.15}$$

d) there is no constant in the line equation thus it passes through the origin.
put $\mathbf{x} \begin{pmatrix} 3 \\ y \end{pmatrix}$ in equation

$$\begin{pmatrix} 1 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ y \end{pmatrix} = 0 \tag{3.7.4.16}$$

$$y = 1 \tag{3.7.4.17}$$

$$\mathbf{P4} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q4} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{3.7.4.18}$$

e) there is no constant in the line equation thus it passes through the origin
put $\mathbf{x} \begin{pmatrix} 1 \\ y \end{pmatrix}$ in equation

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} = 0 \tag{3.7.4.19}$$

$$y = 1 \tag{3.7.4.20}$$

$$\mathbf{P5} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q5} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.7.4.21}$$

f) put $\mathbf{x} \begin{pmatrix} x \\ 0 \end{pmatrix}$ in equation

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = -2 \tag{3.7.4.22}$$

$$x = -\frac{2}{3} \tag{3.7.4.23}$$

we can see in this equation the value of x coordinate does not depend on the y coordinate so we can say that it is parallel to the y-axis.

g) put $\mathbf{x} \begin{pmatrix} x \\ 0 \end{pmatrix}$ in equation

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = 2 \tag{3.7.4.24}$$

$$y = 2 \tag{3.7.4.25}$$

we can see in this equation the value of y coordinate does not depend on the x coordinate so we can say that it is parallel to the x-axis.

h) put $\mathbf{x} \begin{pmatrix} x \\ 0 \end{pmatrix}$ in equation

$$\begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = 5 \tag{3.7.4.26}$$

$$x = -\frac{5}{2} \tag{3.7.4.27}$$

we can see in this equation the value of x coordinate does not depend on the y coordinate so we can say that it is parallel to the y-axis.

```
solutions/4/codes/line/lines_and_planes/
    plane_and_line.py
```

5. Draw the graphs of the following equations

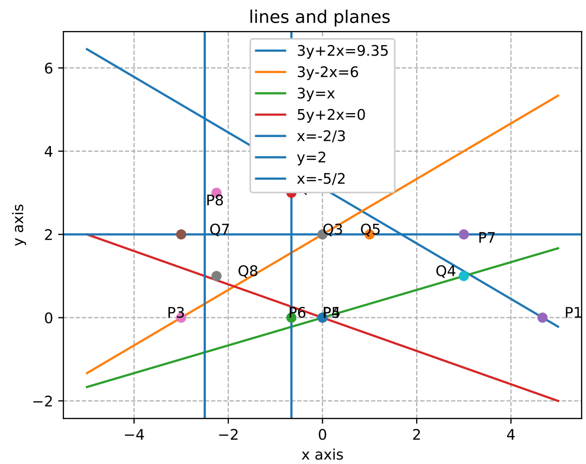

Fig. 3.7.4.1: lines

a) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$

b) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0$

c) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$

d) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -1$

e) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 4$

f) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$

Solution: The following python codes draw the graphs which are represented in Fig.3.7.5.1 and Fig.3.7.5.2.

```
./solutions/5/codes/lines/q11a.py
./solutions/5/codes/lines/q11b.py
```

Solution:

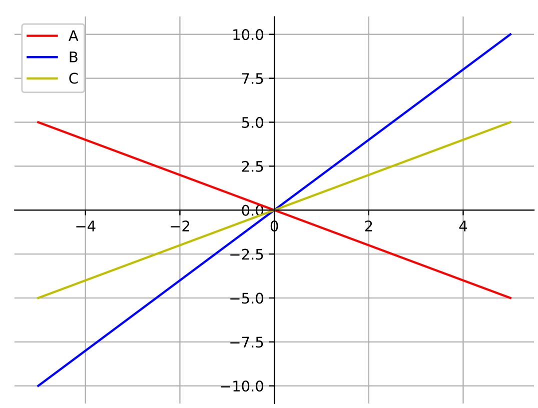

Fig. 3.7.5.1

6. Write four solutions for each of the following equations

a) $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 7$

b) $\begin{pmatrix} \pi & 1 \end{pmatrix} \mathbf{x} = 9$

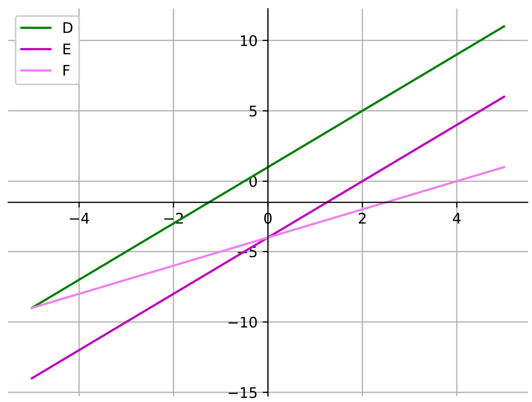

Fig. 3.7.5.2

c) $\begin{pmatrix} 1 & -4 \end{pmatrix} \mathbf{x} = 0$

Solution: The points are obtained by substituting

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{x} \quad = \begin{pmatrix} 0 \\ b \end{pmatrix}, \mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}, \mathbf{x} \quad = \begin{pmatrix} 1 \\ d \end{pmatrix} \tag{3.7.6.1}$$

a)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 7 \tag{3.7.6.2}$$

$$\implies a = \frac{7}{2} \tag{3.7.6.3}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 7 \tag{3.7.6.4}$$

$$\implies b = 7 \tag{3.7.6.5}$$

ii)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 7 \tag{3.7.6.6}$$

$$\implies c = 3 \tag{3.7.6.7}$$

iii)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 7 \tag{3.7.6.8}$$

$$\implies d = 5 \tag{3.7.6.9}$$

b) i)

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 9 \tag{3.7.6.10}$$

$$\implies a = \frac{9}{\pi} \tag{3.7.6.11}$$

ii)

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 9 \tag{3.7.6.12}$$

$$\implies b = 9 \tag{3.7.6.13}$$

iii)

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 9 \tag{3.7.6.14}$$

$$\implies c = \frac{8}{\pi} \tag{3.7.6.15}$$

iv)

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 9 \tag{3.7.6.16}$$

$$\implies d = 9 - \pi \tag{3.7.6.17}$$

c) i)

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 0 \tag{3.7.6.18}$$

$$\implies a = 0 \tag{3.7.6.19}$$

ii)

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 0 \tag{3.7.6.20}$$

$$\implies b = 0 \tag{3.7.6.21}$$

iii)

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 0 \tag{3.7.6.22}$$

$$\implies c = 4 \tag{3.7.6.23}$$

iv)

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 0 \tag{3.7.6.24}$$

$$\implies d = \frac{1}{4} \tag{3.7.6.25}$$

7. Check which of the following are solutions of the equation

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 4 \tag{3.7.7.1}$$

a) $\begin{pmatrix}0\\2\end{pmatrix}$

b) $\begin{pmatrix}2\\0\end{pmatrix}$

c) $\begin{pmatrix}4\\0\end{pmatrix}$

d) $\begin{pmatrix}\sqrt{2}\\4\sqrt{2}\end{pmatrix}$

e) $\begin{pmatrix}1\\1\end{pmatrix}$

8. Find the value of k, if $\begin{pmatrix}2\\1\end{pmatrix}$ is a solution of the equation

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = k \tag{3.7.8.1}$$

9. Draw the graphs of the following equations
   a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4$
   b) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2$
   c) $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 0$
   d) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3$
   e) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$
   f) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
   g) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0$
   h) $\begin{pmatrix}7 & -3\end{pmatrix}\mathbf{x} = 2$
   i) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
   j) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -2$
   k) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$
   l) $\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 6$
10. Give the equations of two lines passing through $\begin{pmatrix}2\\14\end{pmatrix}$. How many more such lines are there, and why?
11. If the point $\begin{pmatrix}3\\4\end{pmatrix}$ lies on the graph of the equation $3y = ax + 7$, find the value of a
12. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident

a)

$$\begin{aligned}\begin{pmatrix}5 & -4\end{pmatrix}\mathbf{x} &= -8\\ \begin{pmatrix}7 & 6\end{pmatrix}\mathbf{x} &= 9\end{aligned} \tag{3.7.12.1}$$

b)

$$\begin{aligned}\begin{pmatrix}9 & 3\end{pmatrix}\mathbf{x} &= -12\\ \begin{pmatrix}18 & 6\end{pmatrix}\mathbf{x} &= -24\end{aligned} \tag{3.7.12.2}$$

c)

$$\begin{aligned}\begin{pmatrix}6 & -3\end{pmatrix}\mathbf{x} &= -10\\ \begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} &= -9\end{aligned} \tag{3.7.12.3}$$

13. Find out whether the following pair of linear equations are consistent, or inconsistent.

a)

$$\begin{aligned}\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} &= 5\\ \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} &= 7\end{aligned} \tag{3.7.13.1}$$

b)

$$\begin{aligned}\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} &= 9\end{aligned} \tag{3.7.13.2}$$

c)

$$\begin{aligned}\begin{pmatrix}\frac{3}{2} & \frac{5}{3}\end{pmatrix}\mathbf{x} &= 7\\ \begin{pmatrix}9 & -10\end{pmatrix}\mathbf{x} &= 14\end{aligned} \tag{3.7.13.3}$$

d)

$$\begin{aligned}\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} &= 11\\ \begin{pmatrix}-10 & 6\end{pmatrix}\mathbf{x} &= -22\end{aligned} \tag{3.7.13.4}$$

e)

$$\begin{aligned}\begin{pmatrix}\frac{4}{3} & 2\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} &= 12\end{aligned} \tag{3.7.13.5}$$

14. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{aligned}\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} &= 5\\ \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} &= 10\end{aligned} \tag{3.7.14.1}$$

b)

$$\begin{aligned}\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}3 & -3\end{pmatrix}\mathbf{x} &= 16\end{aligned} \tag{3.7.14.2}$$

c)

$$\begin{aligned}\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} &= 6\\ \begin{pmatrix}4 & -2\end{pmatrix}\mathbf{x} &= 4\end{aligned} \tag{3.7.14.3}$$

d)

$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2$$
$$\begin{pmatrix}4 & -4\end{pmatrix}\mathbf{x} = 5 \tag{3.7.14.4}$$

15. Given the linear equation $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x}-8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

a) intersecting lines c) coincident lines
b) parallel lines

16. Find the intersection of the following lines

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 14$$
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4 \tag{3.7.16.1}$$

b)

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = 6 \tag{3.7.16.2}$$

c)

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}9 & -3\end{pmatrix}\mathbf{x} = 9 \tag{3.7.16.3}$$

d)

$$\begin{pmatrix}0.2 & 0.3\end{pmatrix}\mathbf{x} = 1.3$$
$$\begin{pmatrix}0.4 & 0.5\end{pmatrix}\mathbf{x} = 2.3 \tag{3.7.16.4}$$

e)

$$\begin{pmatrix}\sqrt{2} & \sqrt{3}\end{pmatrix}\mathbf{x} = 0$$
$$\begin{pmatrix}\sqrt{3} & \sqrt{8}\end{pmatrix}\mathbf{x} = 0 \tag{3.7.16.5}$$

f)

$$\begin{pmatrix}\frac{3}{2} & -\frac{5}{3}\end{pmatrix}\mathbf{x} = -2$$
$$\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = \frac{13}{6} \tag{3.7.16.6}$$

17. Find m if

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 11$$
$$\begin{pmatrix}2 & -4\end{pmatrix}\mathbf{x} = -24 \tag{3.7.17.1}$$
$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = -3$$

Solution: Given, the system of equations in matrix equation format are as below

$$\begin{pmatrix}2 & 3\\2 & -4\\m & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}11\\-24\\-3\end{pmatrix} \tag{3.7.17.2}$$

Step1: Assuming the system of equations are consistent, lets reduce the augmented matrix [A'b], to find the value of m.

$$\begin{pmatrix}2 & 3 & 11\\2 & -4 & -24\\m & -1 & -3\end{pmatrix}
\xleftrightarrow{R_2\leftarrow R_2-R_1}
\begin{pmatrix}2 & 3 & 11\\0 & -7 & -35\\m & -1 & -3\end{pmatrix}
\xleftrightarrow{R_3\leftarrow 2R_3+R_1}
\begin{pmatrix}2 & 3 & 11\\0 & -7 & -35\\2m+2 & 1 & 5\end{pmatrix}
\xleftrightarrow{R_3\leftarrow R_2+7R_3}
\begin{pmatrix}2 & 3 & 11\\0 & -7 & -35\\14m+14 & 0 & 0\end{pmatrix} \tag{3.7.17.3}$$

Since the system of equations are assumed consistent,

$$\Rightarrow 14m + 14 = 0$$
$$\Rightarrow m = -1 \tag{3.7.17.4}$$

Step2:The system of equations can be represented as vectors as below:

18. Solve the following

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{3.7.18.1}$$

b)

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 10$$
$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2 \tag{3.7.18.2}$$

c)

$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 4$$
$$\begin{pmatrix}9 & -2\end{pmatrix}\mathbf{x} = 7 \tag{3.7.18.3}$$

$$\begin{pmatrix}\frac{1}{2} & \frac{2}{3}\end{pmatrix}\mathbf{x} = -1$$
$$\begin{pmatrix}1 & -\frac{1}{3}\end{pmatrix}\mathbf{x} = 3 \tag{3.7.18.4}$$

Solution:

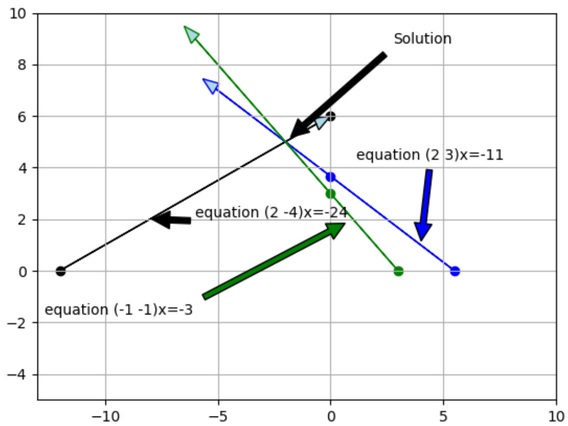

Fig. 3.7.17.1: System of Equations displaying intersecting at a point (-2 5).

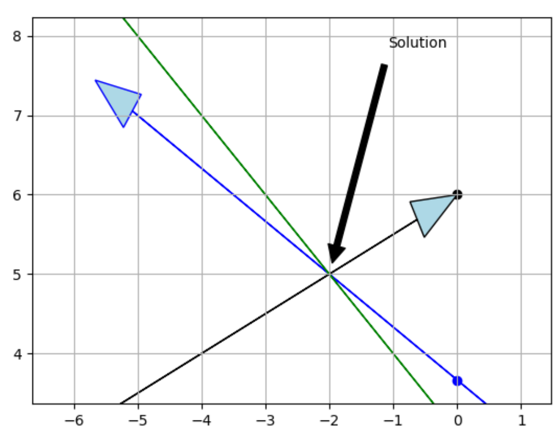

Fig. 3.7.17.2: A zoomed in view of System of Equations displaying intersecting at a point (-2 5).

a) We converted these line vectors in augmented matrix form:

$$\left(\begin{array}{cc|c} 1 & 1 & 14 \\ 1 & -1 & 4 \end{array}\right) \tag{3.7.18.5}$$

$$\tag{3.7.18.6}$$

Now We will apply Row elementary operation to convert left part of matrix to identity matrix.

$$\xleftrightarrow{R_2=R_2-R_1} \left(\begin{array}{cc|c} 1 & 1 & 14 \\ 1 & -2 & -10 \end{array}\right) \tag{3.7.18.7}$$

$$\xleftrightarrow{R_2=\frac{R_2}{-2}} \left(\begin{array}{cc|c} 1 & 1 & 14 \\ 0 & 1 & 5 \end{array}\right) \tag{3.7.18.8}$$

$$\xleftrightarrow{R_1=R_1-R_2} \left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 5 \end{array}\right) \tag{3.7.18.9}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix} 9 \\ 5 \end{pmatrix}$

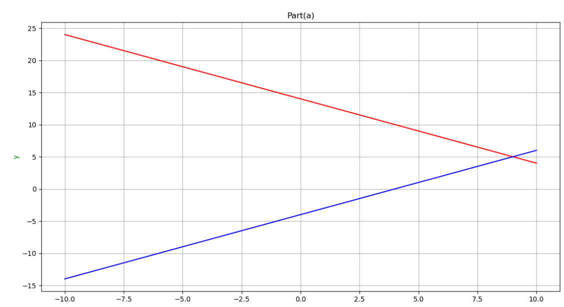

Fig. 3.7.18.1: part(a)

b) We converted these line vectors in augmented matrix form:

$$\left(\begin{array}{cc|c} 1 & -1 & 3 \\ \frac{1}{3} & \frac{1}{2} & 6 \end{array}\right) \tag{3.7.18.10}$$

$$\xleftrightarrow{R_2=6\times R_2} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 3 & 36 \end{array}\right) \tag{3.7.18.11}$$

$$\xleftrightarrow{R_2=R_2-2\times R_1} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 5 & 30 \end{array}\right) \tag{3.7.18.12}$$

$$\xleftrightarrow{R_2=\frac{R_2}{5}} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 6 \end{array}\right) \tag{3.7.18.13}$$

$$\xleftrightarrow{R_1=R_1+R_2} \left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 6 \end{array}\right) \tag{3.7.18.14}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix} 9 \\ 6 \end{pmatrix}$

c) We converted these line vectors in augmented matrix form:

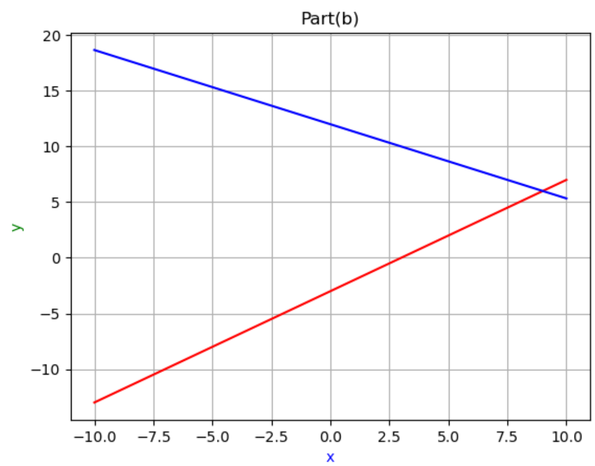

Fig. 3.7.18.2: part(b)

$$\begin{pmatrix} 3 & -1 & | & 3 \\ 9 & -3 & | & 9 \end{pmatrix} \tag{3.7.18.15}$$

$$\xleftrightarrow{R_2=\frac{R_2}{3}} \begin{pmatrix} 1 & -1 & | & 3 \\ 1 & -1 & | & 3 \end{pmatrix} \tag{3.7.18.16}$$

As $R_1 = R_2$, left part can never be converted into a identity matrix , and we can see now both row are same that means both lines are same they intersect at infinitely many points.

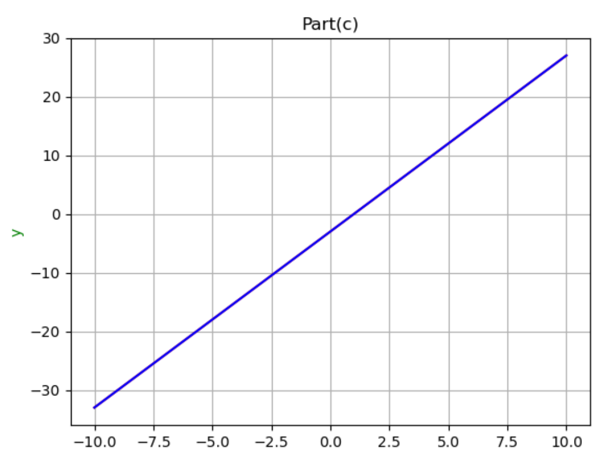

Fig. 3.7.18.3: part(c)

d) We converted these line vectors in augmented matrix form:

$$\begin{pmatrix} 0.2 & 0.3 & | & 1.3 \\ 0.4 & 0.5 & | & 2.3 \end{pmatrix} \tag{3.7.18.17}$$

$$\xleftrightarrow{R_2=R_2-2\times R_1} \begin{pmatrix} 0.2 & 0.3 & | & 1.3 \\ 0 & -0.1 & | & -0.3 \end{pmatrix} \tag{3.7.18.18}$$

$$\xleftrightarrow{R_2=\frac{R_2}{-0.1}} \begin{pmatrix} 0.2 & 0.3 & | & 1.3 \\ 0 & 1 & | & 3 \end{pmatrix} \tag{3.7.18.19}$$

$$\xleftrightarrow{R_1=R_1-0.3\times R_2} \begin{pmatrix} 0.2 & 0 & | & 0.4 \\ 0 & 1 & | & 3 \end{pmatrix} \tag{3.7.18.20}$$

$$\xleftrightarrow{R_1=\frac{R_1}{0.2}} \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3 \end{pmatrix} \tag{3.7.18.21}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

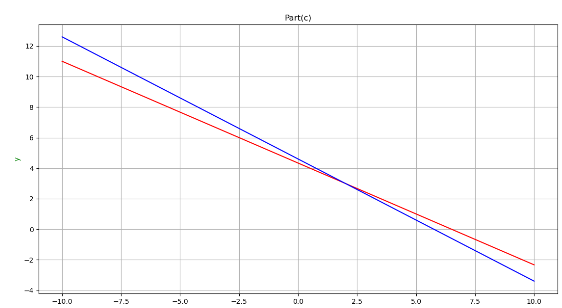

Fig. 3.7.18.4: part(d)

e) We converted these line vectors in augmented matrix form:

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} & | & 0 \\ \sqrt{3} & \sqrt{8} & | & 0 \end{pmatrix} \tag{3.7.18.22}$$

$$\xleftrightarrow{R_2=R_2-\frac{\sqrt{3}}{\sqrt{2}}\times R_1} \begin{pmatrix} \sqrt{2} & \sqrt{3} & | & 0 \\ 0 & \frac{1}{\sqrt{2}} & | & 0 \end{pmatrix} \tag{3.7.18.23}$$

As we see whatever operation we are applying last column of our augmented matrix remains zero.So the lines are homogeneous lines and they always pass through origin , the intersection vector is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

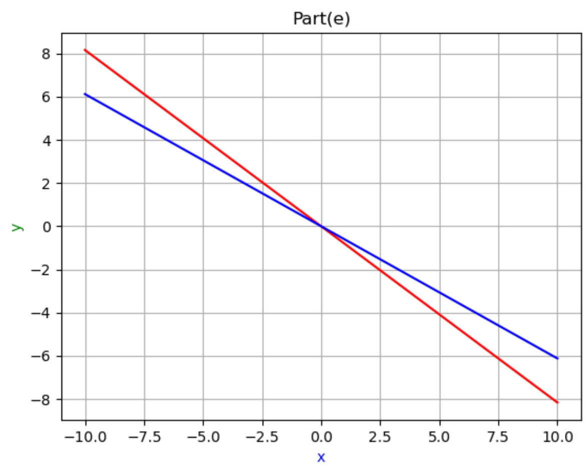

Fig. 3.7.18.5: part(e)

f) We converted these line vectors in augmented matrix form:

$$\left(\begin{array}{cc|c} \frac{3}{2} & \frac{-5}{3} & -2 \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{6} \end{array}\right) \tag{3.7.18.24}$$

$$\xleftrightarrow{R_1=6\times R_1}\xleftrightarrow{R_2=6\times R_2}\left(\begin{array}{cc|c} 9 & -10 & -12 \\ 2 & 3 & 13 \end{array}\right) \tag{3.7.18.25}$$

$$\xleftrightarrow{R_1=R_1-4\times R_2}\left(\begin{array}{cc|c} 1 & -22 & -64 \\ 2 & 3 & 13 \end{array}\right) \tag{3.7.18.26}$$

$$\xleftrightarrow{R_2=R_2-2\times R_1}\left(\begin{array}{cc|c} 1 & -22 & -64 \\ 0 & 47 & 141 \end{array}\right) \tag{3.7.18.27}$$

$$\xleftrightarrow{R_2=\frac{R_2}{47}}\left(\begin{array}{cc|c} 1 & -22 & -64 \\ 0 & 1 & 3 \end{array}\right) \tag{3.7.18.28}$$

$$\xleftrightarrow{R_1=R_1+22\times R_2}\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right) \tag{3.7.18.29}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

19. Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

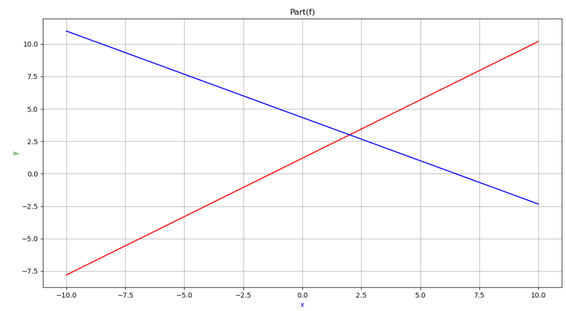

Fig. 3.7.18.6: part(f)

a)

$$\begin{aligned} \begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} &= 3 \\ \begin{pmatrix} 3 & -9 \end{pmatrix}\mathbf{x} &= 2 \end{aligned} \tag{3.7.19.1}$$

b)

$$\begin{aligned} \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} &= 5 \\ \begin{pmatrix} 3 & 2 \end{pmatrix}\mathbf{x} &= 8 \end{aligned} \tag{3.7.19.2}$$

c)

$$\begin{aligned} \begin{pmatrix} 3 & -5 \end{pmatrix}\mathbf{x} &= 20 \\ \begin{pmatrix} 6 & -10 \end{pmatrix}\mathbf{x} &= 40 \end{aligned} \tag{3.7.19.3}$$

d)

$$\begin{aligned} \begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} &= 7 \\ \begin{pmatrix} 3 & -3 \end{pmatrix}\mathbf{x} &= 15 \end{aligned} \tag{3.7.19.4}$$

20. For which values of a and b does the following pair of linear equations have an infinite number of solutions?

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} &= 7 \\ \begin{pmatrix} a-b & a+b \end{pmatrix}\mathbf{x} &= 3a+b-2 \end{aligned} \tag{3.7.20.1}$$

Solution: Constructing the augmented matrix

$$\begin{pmatrix} 2 & 3 & 7 \\ a-b & a+b & 3a+b-2 \end{pmatrix}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2 & 3 & 7 \\ a-b & a+b & 3a+b-2 \end{pmatrix} \xleftrightarrow{R2\leftarrow R1\times\frac{a-b}{2}-R2} \begin{pmatrix} 2 & 3 & 7 \\ 0 & \frac{3a-b}{2}-(a-b) & \frac{7a-b}{2}-(3a+b-2) \end{pmatrix} \tag{3.7.20.2}$$

For the linear equations to have infinite solution,
Rank(Coefficient matrix) =Rank(Augmented matrix)
and both $\neq$ Rank(Full matrix)

$$\begin{pmatrix} 1 & -5 & 0 \\ 1 & -9 & -4 \end{pmatrix} \xleftrightarrow{R2 \leftarrow R2 - R1} \begin{pmatrix} 1 & -5 & 0 \\ 0 & -4 & -4 \end{pmatrix} \tag{3.7.20.3}$$

$$\begin{pmatrix} 1 & -5 & 0 \\ 0 & -4 & -4 \end{pmatrix} \xleftrightarrow{R1 \leftarrow 4 \times R1} \begin{pmatrix} 4 & -20 & 0 \\ 0 & -4 & -4 \end{pmatrix} \tag{3.7.20.4}$$

$$\begin{pmatrix} 4 & -20 & 0 \\ 0 & -4 & -4 \end{pmatrix} \xleftrightarrow{R2 \leftarrow 5 \times R2} \begin{pmatrix} 4 & -20 & 0 \\ 0 & -20 & -20 \end{pmatrix} \tag{3.7.20.5}$$

$$\begin{pmatrix} 4 & -20 & 0 \\ 0 & -20 & -20 \end{pmatrix} \xleftrightarrow{R1 \leftarrow R2 - R1} \begin{pmatrix} -4 & 0 & -20 \\ 0 & -20 & -20 \end{pmatrix} \tag{3.7.20.6}$$

$$\begin{pmatrix} -4 & 0 & -20 \\ 0 & -20 & -20 \end{pmatrix} \xleftrightarrow{R1 \leftarrow R1 \div -4} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -20 & -20 \end{pmatrix} \tag{3.7.20.7}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & -20 & -20 \end{pmatrix} \xleftrightarrow{R2 \leftarrow R2 \div -20} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.7.20.8}$$

Now writing matrix in the form AX=B to obtain solution we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \tag{3.7.20.9}$$

Solving the above equation

$$\implies a = 5 \quad \& \quad b = 1 \tag{3.7.20.10}$$

21. For which value of k will the following pair of linear equations have no solution?

$$\begin{aligned} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} &= 1 \\ \begin{pmatrix} 2k-1 & k-1 \end{pmatrix} \mathbf{x} &= 2k+1 \end{aligned} \tag{3.7.21.1}$$

Solution: Constructing the augmented matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 2k-1 & k-1 & 2k+1 \end{pmatrix}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 3 & 1 & 1 \\ 2k-1 & k-1 & 2k+1 \end{pmatrix} \xleftrightarrow{R2 \leftarrow R1 * \frac{2k-1}{3} - R2} \begin{pmatrix} 3 & 1 & 1 \\ 0 & \frac{2k-1}{3} - (k-1) & \frac{2k-1}{3} - (2k+1) \end{pmatrix} \tag{3.7.21.2}$$

For the linear equations to have no solution, Rank(Coefficient matrix) $\neq$ Rank(Augmented matrix)

$$\implies \frac{2k-1}{3} - (k-1) = 0 \tag{3.7.21.3}$$

and

$$\frac{2k-1}{3} - (2k+1) \neq 0 \tag{3.7.21.4}$$

Solving the above equations,

$$\implies k = 2 \quad \cap \quad k \neq -1 \tag{3.7.21.5}$$

From (3.7.21.5), it is clear that k = 2
Hence, for k = 2, the given set of linear equations will have no solution.

22. Solve the following pair of linear equations

$$\begin{aligned} \begin{pmatrix} 8 & 5 \end{pmatrix} \mathbf{x} &= 9 \\ \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} &= 4 \end{aligned} \tag{3.7.22.1}$$

Solution: Step 1: Construct the Augmented Matrix

$$\begin{pmatrix} 8 & 5 & 9 \\ 3 & 2 & 4 \end{pmatrix} \tag{3.7.22.2}$$

Step 2: Perform row opertions to get a Row Echelon form

$$\begin{pmatrix} 8 & 5 & 9 \\ 3 & 2 & 4 \end{pmatrix} \xleftrightarrow{R_2 \to 8R_2 - 3R_1} \begin{pmatrix} 8 & 5 & 9 \\ 0 & 1 & 5 \end{pmatrix} \tag{3.7.22.3}$$

$$\begin{pmatrix} 8 & 5 & 9 \\ 0 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_1 \to R_1 - 5R_2} \begin{pmatrix} 8 & 0 & -16 \\ 0 & 1 & 5 \end{pmatrix} \tag{3.7.22.4}$$

$$\begin{pmatrix} 8 & 0 & -16 \\ 0 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_1 \to \frac{R_1}{8}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \end{pmatrix} \tag{3.7.22.5}$$

Above final matrix is in the reduced Echelon form and from this matrix we get the solution. Last column represents the solution of the given linear equation. Hence the solution is:

$\begin{pmatrix}-2\\5\end{pmatrix}$ See Fig. 3.7.22.1

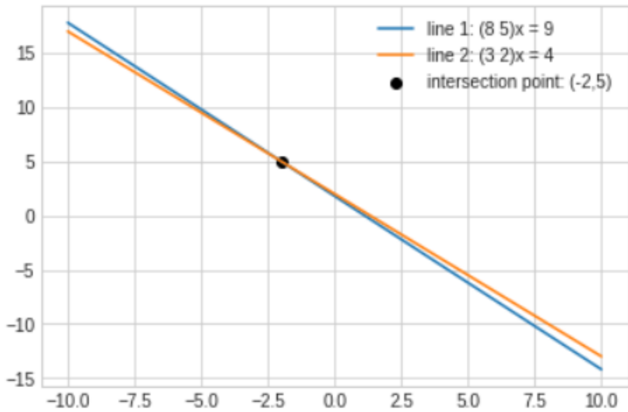

Fig. 3.7.22.1: Linear equations plot generated using python

23. Solve the following pair of linear equations

$$\begin{pmatrix}158 & -378\end{pmatrix}\mathbf{x} = -74 \quad (3.7.23.1)$$
$$\begin{pmatrix}-378 & 152\end{pmatrix}\mathbf{x} = -604$$

Solution: Let

$$\mathbf{A} = \begin{pmatrix}158 & -378\\-378 & 152\end{pmatrix} \quad (3.7.23.2)$$

$$\mathbf{b} = \begin{pmatrix}-74\\-604\end{pmatrix} \quad (3.7.23.3)$$

$$\mathbf{A} = \begin{pmatrix}158 & -378\\-378 & 152\end{pmatrix} \quad (3.7.23.4)$$

$$\mathbf{b} = \begin{pmatrix}-74\\-604\end{pmatrix} \quad (3.7.23.5)$$

$$\mathbf{R} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \quad (3.7.23.6)$$

$$\mathbf{RARAx} = \mathbf{RARb} \quad (3.7.23.7)$$

$$\mathbf{RARA} = k\mathbf{I} \quad (3.7.23.8)$$

I is the Identity Matrix and k is the constant which is equal to 118868 which is shown below

$$k\mathbf{Ix} = \mathbf{RARb} \quad (3.7.23.9)$$

$$k\mathbf{x} = \mathbf{RARb} \quad (3.7.23.10)$$

Finally divide the LHS and RHS by constant k in order to get the value of x. So, by putting the values of $\mathbf{R}, \mathbf{A}$ and $\mathbf{b}$ in equation 3.7.23.7 we can easily find out the value of x as follows:

$$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}158 & -378\\-378 & -152\end{pmatrix}\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}158 & -378\\-378 & -152\end{pmatrix}\mathbf{x}$$
$$= \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}158 & -378\\-378 & -152\end{pmatrix}\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}-74\\-604\end{pmatrix} \quad (3.7.23.11)$$

After doing matrix multiplication in LHS and RHS,

$$\begin{pmatrix}118868 & 0\\0 & 118868\end{pmatrix}\mathbf{x} = \begin{pmatrix}239560\\123404\end{pmatrix} \quad (3.7.23.12)$$

Now, divide both the rows by 118868:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}239560/118868\\123404/118868\end{pmatrix} \quad (3.7.23.13)$$

After further calculations in the fractional result:

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}59890/29717\\30851/29717\end{pmatrix} \quad (3.7.23.14)$$

24. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix}0\\-4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$.

Solution: We are given two points **P** and **B**. Let their mid-point be denoted by **Q**.

$$\therefore \mathbf{Q} = \frac{\mathbf{P}+\mathbf{B}}{2} \quad (3.7.24.1)$$

$$= \frac{1}{2}\left[\begin{pmatrix}0\\-4\end{pmatrix} + \begin{pmatrix}8\\0\end{pmatrix}\right] \quad (3.7.24.2)$$

$$= \begin{pmatrix}4\\-2\end{pmatrix} \quad (3.7.24.3)$$

We know, **O**= Origin= (0,0)
Hence, the directional vector is:

$$\mathbf{m} = \mathbf{Q} - \mathbf{O} \quad (3.7.24.4)$$

$$= \mathbf{Q}, \quad \because \mathbf{O} = \mathbf{0} \quad (3.7.24.5)$$

The direction vector can be expressed in terms of the slope as

$$\mathbf{m} = \begin{pmatrix}1\\m\end{pmatrix} \quad (3.7.24.6)$$

Now using (3.7.24.3), (3.7.24.5) and (3.7.24.6),

$$\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.7.24.7}$$

$$\implies \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.7.24.8}$$

$$\implies \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.7.24.9}$$

Thus, by comparing, we have,

$$m = -\frac{1}{2} \tag{3.7.24.10}$$

25. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines. **Solution:** The direction vector can be represented as below:-

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ m \end{pmatrix} \mathbf{m_2} = \begin{pmatrix} 1 \\ 2m \end{pmatrix} \tag{3.7.25.1}$$

The dot product of the vectors is given by:-

$$\mathbf{m_1}^T \mathbf{m_2} = \|\mathbf{m_1}\| \|\mathbf{m_2}\| \cos\theta \tag{3.7.25.2}$$

Given that

$$\tan\theta = \frac{1}{3} \tag{3.7.25.3}$$

By Baudhayana's theorem, we can obtain

$$\cos\theta = \frac{3}{\sqrt{10}} \tag{3.7.25.4}$$

Therefore,

$$\cos\theta = \frac{\mathbf{m_1}^T \mathbf{m_2}}{\|\mathbf{m_1}\| \|\mathbf{m_2}\|} \tag{3.7.25.5}$$

$$\frac{3}{\sqrt{10}} = \frac{1 \times 1 + m \times 2m}{\sqrt{1+m^2}\sqrt{1+4m^2}} \tag{3.7.25.6}$$

Applying square on both sides:-

$$9 \times (1+m^2)(1+4m^2) = 10(1+2m^2)^2 \tag{3.7.25.7}$$

$$4m^4 - 5m^2 + 1 = 0 \tag{3.7.25.8}$$

$$m_1 = m = 1, -1, \frac{1}{2}, \frac{-1}{2} \tag{3.7.25.9}$$

Substituting the value of m_1 we get value of $m_2 = 2, -2, 1, -1$.

26. Find the slope of the line, which makes an angle of 30° of y-axis measured anticlockwise.
27. Write the equations for the x and y axes.
28. Find the equation of the line satisfying the following conditions
    a) passing through the point $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ with slope $\frac{1}{2}$.
    b) passing through the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m.
    c) passing through the point $\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of 75°.
    d) Intersecting the x-axis at a distance of 3 units to the let of the origin with slope -2.
    e) intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis.
    f) passing through the points $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.
    g) perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is 30°.
29. Find the equation of the line passing through $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$.
30. Find the direction vectors and and y-intercepts of the following lines
    a) $\begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} = 0$.
    b) $\begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} = 5$.
    c) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$.
31. Find the intercepts of the following lines on the axes.
    a) $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 12$.
    b) $\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = 6$.
    c) $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 0$.

    Solution:
    a) Normal vector $\mathbf{n}$ is

    $$\mathbf{n} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \tag{3.7.31.1}$$

    Direction Vector

    $$\mathbf{m} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{3.7.31.2}$$

    Y-intercept $= 0$

b) Normal vector $\mathbf{n}$ is

$$\mathbf{n} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \tag{3.7.31.3}$$

Direction Vector

$$\mathbf{m} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{3.7.31.4}$$

Y-intercept = 5/3

c) Normal vector $\mathbf{n}$ is

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.7.31.5}$$

Direction Vector

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.31.6}$$

Y-intercept = 0

32. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
   a) $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = -8.$
   b) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2.$
   c) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4.$

33. Find the distance of the point $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from the line $\begin{pmatrix} 12 & -5 \end{pmatrix} \mathbf{x} = -82.$

Solution:

The formula for calculating the distance between the point and the given line is

$$d = \frac{\left| c - \mathbf{n}^{\mathbf{T}} \mathbf{A} \right|}{\|\mathbf{n}\|} \tag{3.7.33.1}$$

By substituting the given values

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{n} = \begin{pmatrix} 12 \\ -5 \end{pmatrix} \quad c = -82 \tag{3.7.33.2}$$

we get

$$\left| c - \mathbf{n}^{\mathbf{T}} \mathbf{A} \right| = 99 \tag{3.7.33.3}$$

Thus, the distance between the point and the line is

$$d = \frac{99}{13} \tag{3.7.33.4}$$

See Fig. 3.7.33.1

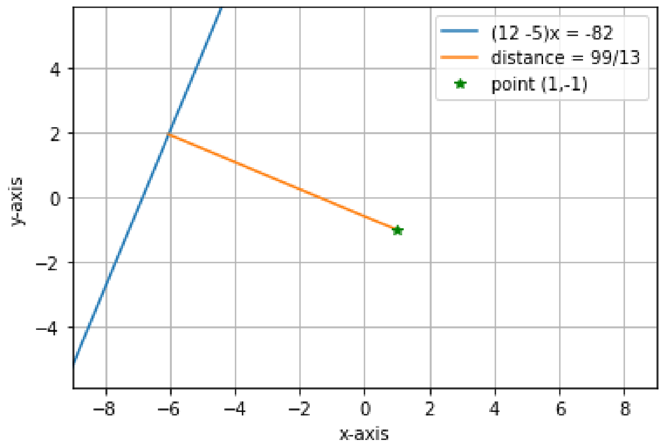

Fig. 3.7.33.1: Plot showing the distance between the point and the line

34. Find the points on the x-axis, whose distances from the line

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{3.7.34.1}$$

are 4 units.

Solution:

First we can find the lines at a distance of 4 from the given line and then it's intersection with the x-axis.

$$\mathbf{n} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

The parallel lines must have the same slope but different intercepts. Hence the lines must be of the form:

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = c_1 \tag{3.7.34.2}$$

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = c_2 \tag{3.7.34.3}$$

These c_1 and c_2 can be easily found by evaluating the distance between the parallel lines:

$$\frac{|(c - 12)|}{\sqrt{4^2 + 3^2}} = 4 \tag{3.7.34.4}$$

$$c = 12 \pm 20 \tag{3.7.34.5}$$

The two parallel lines at a distance of 4 thus obtained are:

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 32 \tag{3.7.34.6}$$

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = -8 \tag{3.7.34.7}$$

Finally the points on x-axis are:

$$\mathbf{x} = 8 \tag{3.7.34.8}$$
$$\mathbf{x} = -2 \tag{3.7.34.9}$$

See Fig. 3.7.34.1

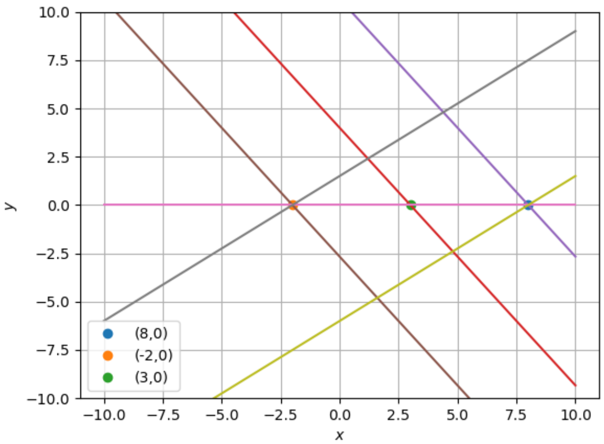

Fig. 3.7.34.1: Points on x-axis at a distance of 4 from the given line

35. Find the distance between the parallel lines

$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = 34 \tag{3.7.35.1}$$
$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = -31 \tag{3.7.35.2}$$

36. Find the equation of the line parallel to the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = -2 \tag{3.7.36.1}$$

and passing through the point $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

37. Find the equation of a line perpendicular to the line

$$\begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} = -5 \tag{3.7.37.1}$$

and having x intercept 3.
Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{3.7.37.2}$$

Thus, the desired equation of the line is

$$\begin{pmatrix} 7 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) = 0$$
$$\implies \begin{pmatrix} 7 & 1 \end{pmatrix} \mathbf{x} = 21$$

See Fig. 3.7.37.1

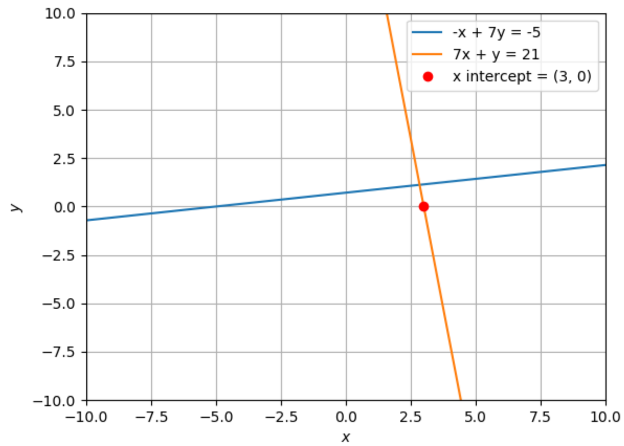

Fig. 3.7.37.1: Plot showing intersection

38. Find angles between the lines

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 1 \tag{3.7.38.1}$$
$$\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \mathbf{x} = 1 \tag{3.7.38.2}$$

Solution: We will make direction vectors from these line vectors form:

$$\mathbf{m}_1 = \begin{pmatrix} -\sqrt{3} & 1 \end{pmatrix} \tag{3.7.38.3}$$
$$\mathbf{m}_2 = \begin{pmatrix} -1 & \sqrt{3} \end{pmatrix} \tag{3.7.38.4}$$

Now we will find out magnitudes of each vectors $\mathbf{m}_1$,$\mathbf{m}_2$:

$$\|\mathbf{m}_1\| = \sqrt{3+1} = 2 \tag{3.7.38.5}$$
$$\|\mathbf{m}_2\| = \sqrt{1+3} = 2 \tag{3.7.38.6}$$

Thus angle between 2 vectors $\mathbf{m}_1$,$\mathbf{m}_2$ can be found using dot-product using the formula below, Let θ be angle between vectors $\mathbf{m}_1$,$\mathbf{m}_2$ then,

$$\theta = \cos^{-1}\left(\frac{\mathbf{m}_1 \mathbf{m}_2^T}{\|\mathbf{m}_1\| \, \|\mathbf{m}_2\|} \right) \tag{3.7.38.7}$$

By,Putting values into above equation we get,

$$\theta = \cos^{-1}\left(\frac{\begin{pmatrix}-\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}-1\\\sqrt{3}\end{pmatrix}}{4}\right) \quad (3.7.38.8)$$

$$= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \quad (3.7.38.9)$$

$$\implies \theta = 30° \quad (3.7.38.10)$$

39. The line through the points $\begin{pmatrix}h\\3\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$ intersects the line

$$\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x} = 19 \quad (3.7.39.1)$$

at right angle. Find the value of h.
Solution:
Let the given points

$$\mathbf{A} = \begin{pmatrix}h\\3\end{pmatrix} \quad (3.7.39.2)$$

$$\mathbf{B} = \begin{pmatrix}4\\1\end{pmatrix} \quad (3.7.39.3)$$

Directional vector of line passing through points **A** and **B** is

$$\mathbf{P} = \mathbf{B} - \mathbf{A} \quad (3.7.39.4)$$

$$\mathbf{P} = \begin{pmatrix}h-4\\2\end{pmatrix} \quad (3.7.39.5)$$

Directional vector of the line $\begin{pmatrix}a & b\end{pmatrix}\mathbf{x} = c$ is

$$\mathbf{Q} = \begin{pmatrix}b\\-a\end{pmatrix} \quad (3.7.39.6)$$

From (3.7.39.6) direction vector of line $\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x} = 19$ is

$$\mathbf{Q} = \begin{pmatrix}-9\\-7\end{pmatrix} \quad (3.7.39.7)$$

If two straight lines intersects at right angles then inner product of their directional vectors is zero.

$$\mathbf{P}^T\mathbf{Q} = 0 \quad (3.7.39.8)$$

$$\begin{pmatrix}h-4\\2\end{pmatrix}^T\begin{pmatrix}-9\\-7\end{pmatrix} = 0 \quad (3.7.39.9)$$

$$\begin{pmatrix}h-4 & 2\end{pmatrix}\begin{pmatrix}-9\\-7\end{pmatrix} = 0 \quad (3.7.39.10)$$

$$(h-4)(-9) + 2(-7) = 0 \quad (3.7.39.11)$$

$$h = \frac{22}{9} \quad (3.7.39.12)$$

Python Plot used to verify the result obtained from (3.7.39.12).

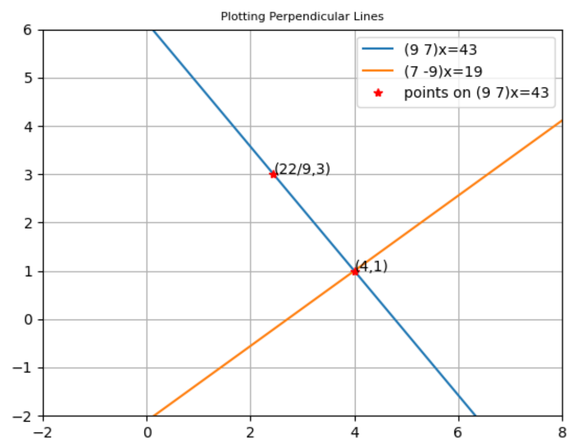

Fig. 3.7.39.1: Figure showing given data and corresponding results

According to the problem statement, equation of line passing through the point $\begin{pmatrix}4\\1\end{pmatrix}$ and perpendicular to the line $\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x} = 19$ is

$$\begin{pmatrix}9 & 7\end{pmatrix}\mathbf{x} = 43 \quad (3.7.39.13)$$

Fig. 3.7.39.1 shows that equation (3.7.39.13) passes through the point $\begin{pmatrix}\frac{22}{9}\\3\end{pmatrix}$

40. Two lines passing through the point $\begin{pmatrix}2\\3\end{pmatrix}$ intersect each other at angle of 60°. If the slope of one line is 2, find the equation of the other line.
Solution: Directional vector of a line1 having slope 2 is $\begin{pmatrix}1\\2\end{pmatrix}$ Hence normal vector of line1 is

given as

$$\mathbf{n_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.7.40.1}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.7.40.2}$$

Similarly, normal vector for line 2

$$\mathbf{n_2} = \begin{pmatrix} -m_2 \\ 1 \end{pmatrix} \tag{3.7.40.3}$$

Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{3.7.40.4}$$

$$\implies \cos 60^\circ = \frac{1}{2} \tag{3.7.40.5}$$

$$= \frac{2m_2 + 1}{\sqrt{5} \times \sqrt{1 + m_2}} \tag{3.7.40.6}$$

$$\implies 11m_2^2 + 16m_2 - 1 = 0 \tag{3.7.40.7}$$

Solving, m_2 yields values $\frac{-8+5\sqrt{3}}{11}$ and $\frac{-8-5\sqrt{3}}{11}$ Equation of line with normal vector $\mathbf{n}$ and passing through point A is given by

$$\mathbf{n}^T(\mathbf{x} - A) = 0 \tag{3.7.40.8}$$

Hence,equation of line with slope $\frac{-8+5\sqrt{3}}{11}$ passing through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is

$$\begin{pmatrix} \frac{8-5\sqrt{3}}{11} & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = 0 \tag{3.7.40.9}$$

$$\implies \begin{pmatrix} \frac{8-5\sqrt{3}}{11} & 1 \end{pmatrix}\mathbf{x} = \frac{49 - 10\sqrt{3}}{11} \tag{3.7.40.10}$$

Similarly,equation of line with slope $\frac{-8-5\sqrt{3}}{11}$ passing through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is

$$\begin{pmatrix} \frac{8+5\sqrt{3}}{11} & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = 0 \tag{3.7.40.11}$$

$$\implies \begin{pmatrix} \frac{8+5\sqrt{3}}{11} & 1 \end{pmatrix}\mathbf{x} = \frac{49 + 10\sqrt{3}}{11} \tag{3.7.40.12}$$

Thus, the required line equations are

$$\begin{pmatrix} \frac{8-5\sqrt{3}}{11} & 1 \end{pmatrix}\mathbf{x} = \frac{49 - 10\sqrt{3}}{11} \tag{3.7.40.13}$$

$$\begin{pmatrix} \frac{8+5\sqrt{3}}{11} & 1 \end{pmatrix}\mathbf{x} = \frac{49 + 10\sqrt{3}}{11} \tag{3.7.40.14}$$

See Fig. 3.7.40.1

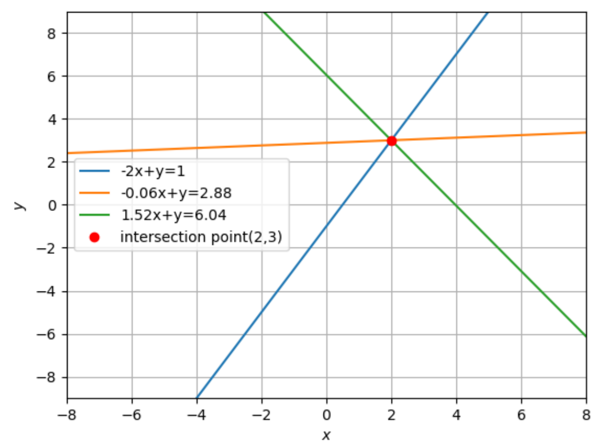

Fig. 3.7.40.1: plot showing intersection of lines

41. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Solution:

Let $\mathbf{M}$ be the midpoint of two points $\mathbf{A}=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} 2 \\ 6 \end{pmatrix} \tag{3.7.41.1}$$

$$\implies \mathbf{M} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The direction vector of line AB is

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{3.7.41.2}$$

The direction vector of line AB is normal vector of right bisector. Then

$$\mathbf{n} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{3.7.41.3}$$

The equation of line in terms of normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{M}) = 0 \tag{3.7.41.4}$$

$$\implies \begin{pmatrix} -4 & -2 \end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}1\\3\end{pmatrix}\right) = 0 \tag{3.7.41.5}$$

$$\implies \begin{pmatrix} -4 & -2 \end{pmatrix}\mathbf{x} = -10 \tag{3.7.41.6}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 5 \tag{3.7.41.7}$$

We got equation of the right bisector of line segment joining points **A** and **B**. The line also passes through point **M**.
See Fig. 3.7.41.1 for plot of line segment and right bisector.

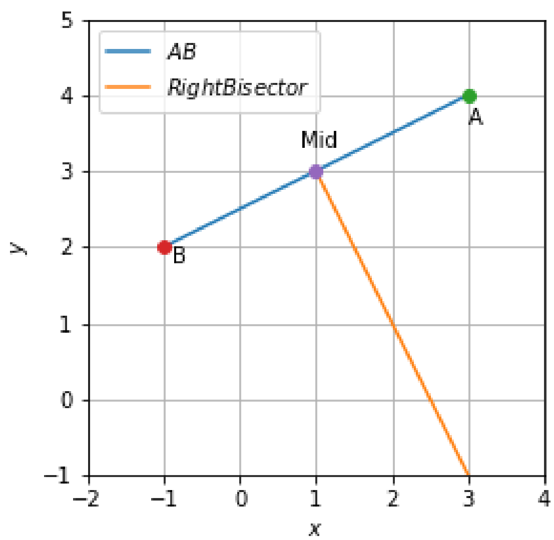

Fig. 3.7.41.1: Right bisector of line AB

42. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix}-1\\3\end{pmatrix}$ to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 16. \tag{3.7.42.1}$$

Solution:
The normal vector to the perpendicular drawn from point (-1 3) is same as the direction vector of the given line:

$$\mathbf{n} = \begin{pmatrix}4\\3\end{pmatrix} \tag{3.7.42.2}$$

The equation of the drawn perpendicular in terms of the normal vector is then obtained as

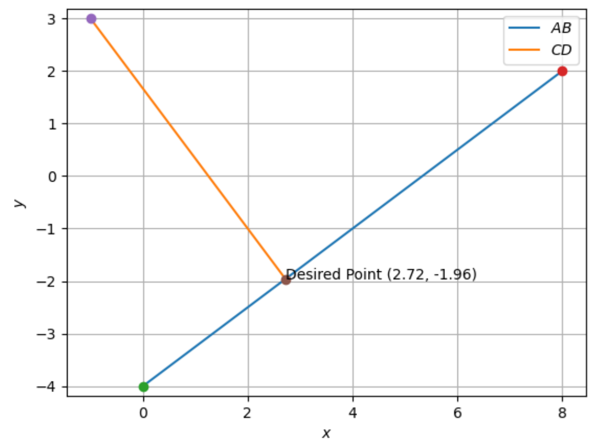

Fig. 3.7.42.1

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{3.7.42.3}$$

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 5 \tag{3.7.42.4}$$

The above two line equations can be expressed as the matrix equation

$$\begin{pmatrix}3 & -4\\4 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}16\\5\end{pmatrix} \tag{3.7.42.5}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}3 & -4 & 16\\4 & 3 & 5\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1/3} \begin{pmatrix}1 & -4/3 & 16/3\\4 & 3 & 5\end{pmatrix} \tag{3.7.42.6}$$

$$\xleftrightarrow{R_2\leftarrow R_2-4R_1} \begin{pmatrix}1 & -4/3 & 16/3\\0 & 25/3 & -49/3\end{pmatrix} \tag{3.7.42.7}$$

$$\xleftrightarrow{R_2\leftarrow R_2\times 3/25} \begin{pmatrix}1 & -4/3 & 16/3\\0 & 1 & -49/25\end{pmatrix} \tag{3.7.42.8}$$

$$\xleftrightarrow{R_1\leftarrow R_1+4/3\times R_1} \begin{pmatrix}1 & 0 & 68/25\\0 & 1 & -49/25\end{pmatrix} \tag{3.7.42.9}$$

Thus, The foot of the perpendicular is at point (68/25 , -49/25) i.e. (2.72, -1.96) See Fig. 3.7.42.1

43. The perpendicular from the origin to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.7.43.1}$$

meets it at the point $\begin{pmatrix}-1\\2\end{pmatrix}$. Find the values of m and c.

Solution: The line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.7.43.2}$$

meets it at the point $\mathbf{P} = \begin{pmatrix}-1\\2\end{pmatrix}$ Since,

$$\mathbf{P} - \mathbf{0} = \mathbf{P} \tag{3.7.43.3}$$

is the normal vector, where $\mathbf{0}$ is the origin, then

$$\mathbf{m} = \begin{pmatrix}1\\m\end{pmatrix} \tag{3.7.43.4}$$

is the direction vector, Hence

$$\mathbf{m}^T\mathbf{P} = 0 \tag{3.7.43.5}$$

$$\implies \begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = 0$$
$$\implies (-1 + 2m) = 0$$
$$\implies m = \frac{1}{2} \tag{3.7.43.6}$$

now, the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c$$

meets it at the point $\mathbf{P}$=$\begin{pmatrix}-1\\2\end{pmatrix}$and using the value of m from 3.7.43.6 we get,

$$\begin{pmatrix}\frac{-1}{2} & 1\end{pmatrix}\mathbf{P} = c$$
$$\implies \begin{pmatrix}\frac{-1}{2} & 1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = c$$
$$\implies c = \frac{5}{2}$$

Hence, the value of m and c are obtained as

$$m = \frac{1}{2},\ c = \frac{5}{2}$$

respectively. See Fig. 3.7.43.1.

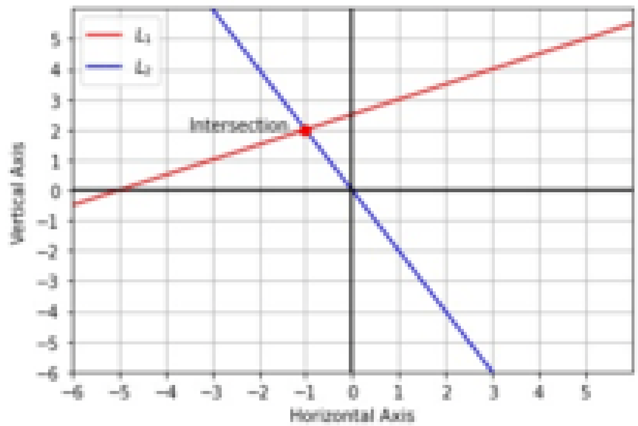

Fig. 3.7.43.1: Perpendicular Lines crossing

44. Find θ and p if

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = -2$$

is equivalent to

$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = p$$

45. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.

Solution: The equation of line in terms of vector notations can be written as

$$\mathbf{n}^T\mathbf{x} = c$$

Let the intercepts be $\begin{pmatrix}a\\0\end{pmatrix}$ and $\begin{pmatrix}0\\b\end{pmatrix}$, respectively.
Given that: $a + b = 1$, and $ab = -6$
The quadratic equation whose roots are the x and y intercepts can be written as:

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\implies \quad x^2 - x - 6 = 0$$
$$\implies \quad x = (3, -2)$$

and corresponding y intercepts are $(-2, 3)$.
The line L_1 passes through $\begin{pmatrix}3\\0\end{pmatrix}$ and $\begin{pmatrix}0\\-2\end{pmatrix}$.
Let direction vector of this line be $\mathbf{m}$.

$$\mathbf{m} = \begin{pmatrix}0\\-2\end{pmatrix} - \begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}-3\\-2\end{pmatrix}$$

The normal vector, $\mathbf{n}$:

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} = \begin{pmatrix}2\\-3\end{pmatrix}$$

The equation of line in terms of normal vector and passing through a point A is

$$\mathbf{n}^{\mathbf{T}}(\mathbf{x}-A)=0 \quad \implies \quad \mathbf{n}^{\mathbf{T}}\mathbf{x}=\mathbf{n}^{\mathbf{T}}A$$

$$\implies \mathbf{n}^{\mathbf{T}}\mathbf{x}=\begin{pmatrix}2 & -3\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix}$$

$$\implies \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x}=6$$

Similarly, the equation of second line L_2, with x and y intercepts $\begin{pmatrix}-2\\0\end{pmatrix}$ and $\begin{pmatrix}0\\3\end{pmatrix}$ and normal vector $\begin{pmatrix}-3\\2\end{pmatrix}$ is

$$\begin{pmatrix}-3 & 2\end{pmatrix}\mathbf{x}=6$$

The equations of lines (3.7.45) and (3.7.45) can be represented collectively as

$$\begin{pmatrix}2 & -3\\-3 & 2\end{pmatrix}\mathbf{x}=\begin{pmatrix}6\\6\end{pmatrix}$$

$x-$intercept	$y-$intercept	$\mathbf{n}$
$\begin{pmatrix}3\\0\end{pmatrix}$	$\begin{pmatrix}0\\-2\end{pmatrix}$	$\begin{pmatrix}2\\-3\end{pmatrix}$
$\begin{pmatrix}-2\\0\end{pmatrix}$	$\begin{pmatrix}0\\3\end{pmatrix}$	$\begin{pmatrix}-3\\2\end{pmatrix}$

TABLE 3.7.45

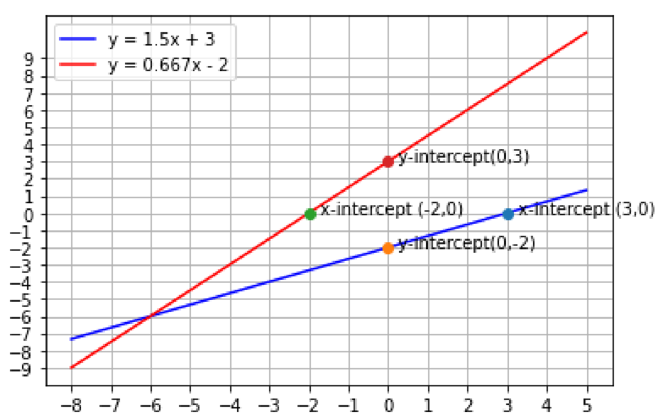

Fig. 3.7.45.1

46. What are the points on the y-axis whose distance from the line

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x}=12$$

4 units.

Solution: Here, direction vectors of the lines are $\begin{pmatrix}3\\-4\end{pmatrix}$

Using the formula for the distance of a point P from a line

$$d=\frac{|\mathbf{n}^T P-c|}{\|\mathbf{n}\|}$$

normal vector n is given by,

$$\mathbf{n}=\begin{pmatrix}4\\3\end{pmatrix}$$

Since the point lies on the y-axis. let

$$P=\begin{pmatrix}0\\k\end{pmatrix}$$

If the equation of the line is :

$$\mathbf{n}^T x=c$$

$$\frac{|\mathbf{n}^T P-c|}{\|\mathbf{n}\|}=4$$

$$\implies 3k-12=\pm 20$$

$$\implies k=\begin{pmatrix}0\\-8\end{pmatrix} and \begin{pmatrix}0\\32/3\end{pmatrix}$$

therefore points on y-axis at distance of P from line are $\begin{pmatrix}0\\-8\end{pmatrix}$ and $\begin{pmatrix}0\\32/3\end{pmatrix}$.

47. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x}=-5$$

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x}=0$$

Solution: consider the equation of the system of lines

$$x-7y=-5$$

$$3x+y=0$$

consider the augmented matrix

$$\begin{pmatrix}1 & -7 & -5\\3 & 1 & 0\end{pmatrix}$$

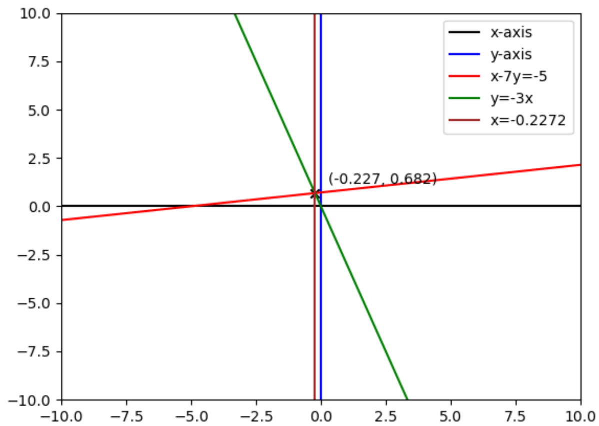

Fig. 3.7.47.1: graphical representation of systems of lines

By applying row reduction reduction technique

$$\begin{pmatrix} 4 & -7 & -5 \\ 3 & 1 & 0 \end{pmatrix}$$
$$\xleftrightarrow[R_2 \leftarrow R_2/22]{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -7 & -5 \\ 0 & 1 & \frac{15}{22} \end{pmatrix}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 + 7R_2} \begin{pmatrix} 1 & 0 & \frac{-5}{22} \\ 0 & 1 & \frac{15}{22} \end{pmatrix}$$

The value of$\mathbf{A}$ is the point of intersection.

$$\mathbf{A} = \begin{pmatrix} \frac{-5}{22} \\ \frac{15}{22} \end{pmatrix}$$

Now the equation of line parallel to y-axis through the point of intersection.

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0$$

where$\mathbf{n}$ is the vector normal to the Y - axis and $\mathbf{A}$ is the point of intersection

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A}$$
$$\text{where } \mathbf{n}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{-5}{22} \\ \frac{15}{22} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = -\frac{5}{22}$$

Shown in Fig. 3.7.47.1 is the equation of the line parallel to the Y-axis drawn through the point of intersection of the lines.

48. Find the alue of p so that the three lines

$$\begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} = 2$$
$$\begin{pmatrix} p & 2 \end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = 3$$

may intersect at one point.

49. In what ratio is the line joining $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ divided by the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 4$$

Solution: The point $\mathbf{X}$ divides the line segment joining the two points $\mathbf{A}$=$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{B}$=$\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ in ratio k : 1. Then,

$$\mathbf{X} = \frac{(k\mathbf{B} + \mathbf{A})}{(k+1)}$$

From the equation (3.7.49)

$$(k+1)\mathbf{X} = k\mathbf{B} + \mathbf{A}$$

Let $\mathbf{n}$=$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\implies (k+1)\,\mathbf{n}^T\mathbf{X} = \mathbf{n}^T\,(k\mathbf{B} + \mathbf{A})$$
$$\implies k\left(\mathbf{n}^T\mathbf{X} - \mathbf{n}^T\mathbf{B}\right) = \mathbf{n}^T\mathbf{A} - \mathbf{n}^T\mathbf{X}$$

$$\implies k = \frac{\mathbf{n}^T\mathbf{A} - \mathbf{n}^T\mathbf{X}}{\mathbf{n}^T\mathbf{X} - \mathbf{n}^T\mathbf{B}}$$

Hence on solving the equation (3.7.49) using

$$\mathbf{n}^T\mathbf{X} = 4$$

The line $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x}$=4 divides the line joining points $\mathbf{A}$=$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{B}$=$\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ in the ratio k=1/2 See Fig. 3.7.49.1

50. Find the distance of the line

$$\begin{pmatrix} 4 & 7 \end{pmatrix}\mathbf{x} = -5$$

from the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ along the line

$$\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = 0.$$

Solution: We need to find the solution of

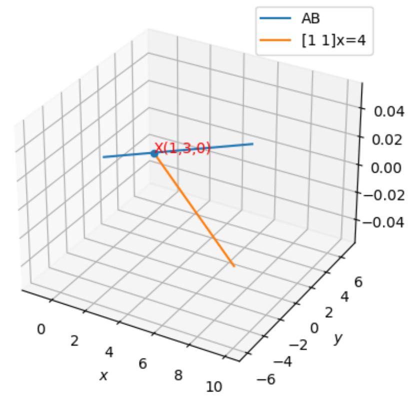

Fig. 3.7.49.1: Line as (1 1)**x**=4 intersecting the line joining points A and B

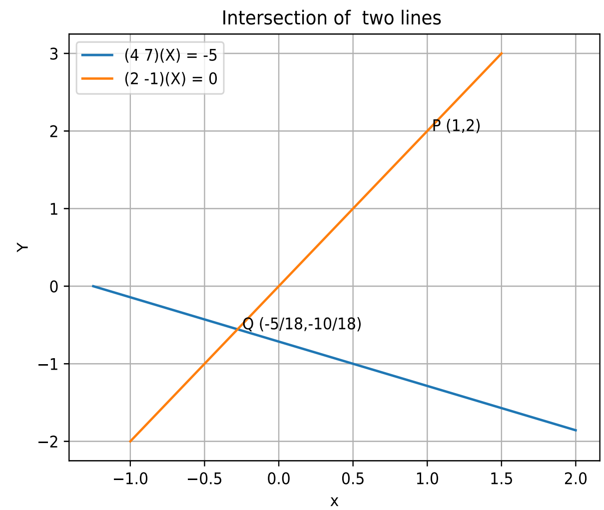

Fig. 3.7.50.1: Intersection of two lines

equations

$$\begin{pmatrix} 4 & 7 \end{pmatrix} \mathbf{x} = -5$$
$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 4 & 7 & -5 \\ 2 & 1 & 0 \end{pmatrix} \xleftrightarrow{R1 \leftarrow \frac{1}{18} * (R1+7\times R2)} \begin{pmatrix} 1 & 0 & -5/18 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -5/18 \\ 2 & -1 & 0 \end{pmatrix} \xleftrightarrow{R2 \leftarrow -(R2-2\times R1)} \begin{pmatrix} 1 & 0 & -5/18 \\ 0 & 1 & -10/18 \end{pmatrix}$$

After solving this two equation we will get the point of intersection, which is intersection of these two lines segments. Thus, point of intersection is $\begin{pmatrix} -5/18 \\ -10/18 \end{pmatrix}$. Now we have point of intersection

$$\mathbf{P} = \begin{pmatrix} -5/18 \\ -10/18 \end{pmatrix}$$

and given point is

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Now the distance between two points is given as :

$$\|\mathbf{P} - \mathbf{Q}\| = \left\| \begin{pmatrix} -5/18 \\ -10/18 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = d = 2.85$$

51. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ so that its point of intersection with the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 4$$

may be at a distance of 3 units from this point.
Solution: The given equation of the line in parametric form:

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m}$$

where,

$$\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If $\mathbf{x}$ be the point of intersection,

$$\|\mathbf{x} - \mathbf{B}\| = 3$$

$$\|\mathbf{A} + \lambda \mathbf{m} - \mathbf{B}\| = 3$$

$$(\mathbf{A} + \lambda \mathbf{m} - \mathbf{B})^T (\mathbf{A} + \lambda \mathbf{m} - \mathbf{B}) = 9$$

$$[(\mathbf{A}-\mathbf{B})^T\mathbf{m} = \mathbf{m}^T(\mathbf{A}-\mathbf{B})]$$

$$\|\mathbf{m}\|^2\lambda^2 + \left[2(\mathbf{A}-\mathbf{B})^T\mathbf{m})\right]\lambda + \|\mathbf{A}-\mathbf{B}\|^2 = 9$$

$$2\lambda^2 + 10\lambda + 8 = 0$$

$$\lambda = -4, \lambda = -1$$

The point of intersection,

$$\boxed{\therefore \mathbf{x} = \begin{pmatrix}-1\\5\end{pmatrix} or \begin{pmatrix}2\\2\end{pmatrix}}$$

The direction vector,

$$\mathbf{v} = \mathbf{B} - \mathbf{x}$$

$$\boxed{\mathbf{v} = \begin{pmatrix}0\\-3\end{pmatrix} or \begin{pmatrix}-3\\0\end{pmatrix}}$$

52. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\1\end{pmatrix}$. Find an equation of the legs of the triangle.
53. Find the image of the point $\begin{pmatrix}3\\8\end{pmatrix}$ with respect to the line

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 7$$

assuming the line to be a plane mirror.
Solution: Let, given vector

$$\mathbf{P} = \begin{pmatrix}3\\8\end{pmatrix} \tag{3.7.53.0}$$

Let, image point be **R**. Let vector,

$$\mathbf{n} = \begin{pmatrix}1\\3\end{pmatrix} \tag{3.7.53.0}$$

Let m be the directional vector along the line, $\begin{pmatrix}1 & 3\end{pmatrix}$ $\mathbf{x}$ =7 Hence m is,

$$\mathbf{m} = \begin{pmatrix}3\\-1\end{pmatrix} \tag{3.7.53.0}$$

By property in Figure 3.7.53.1, the line PR bisects the mirror equation perpendicularly. Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{3.7.53.0}$$

Where, **Q** is the point on the line, $\begin{pmatrix}1 & 3\end{pmatrix}$ $\mathbf{x}$ =7.

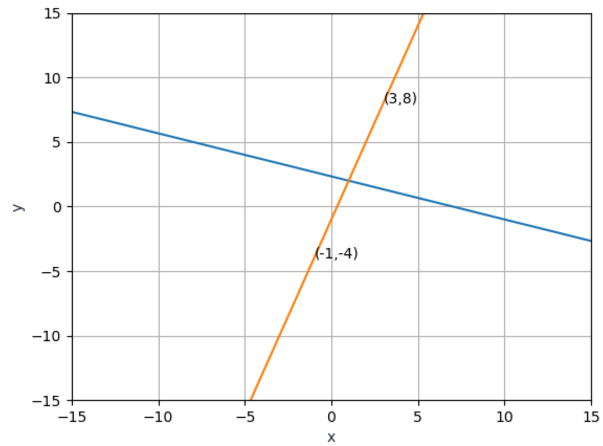

Fig. 3.7.53.1: Image of a point in 2D line

Hence the reflection vector **R** is given as,

$$\begin{pmatrix}\frac{\mathbf{R}}{2}\end{pmatrix} = \begin{pmatrix}\frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\end{pmatrix}\mathbf{P} + c\begin{pmatrix}\frac{\mathbf{n}}{\|\mathbf{n}\|^2}\end{pmatrix} \tag{3.7.53.0}$$

$$\|\mathbf{n}\| = \sqrt{1^2+3^2} = \sqrt{10}$$

Substituting these values in equation (3.7.53) we get,

$$\mathbf{R} = \begin{pmatrix}-1\\-4\end{pmatrix} \tag{3.7.53.0}$$

Hence, it is the required answer for image of **P** in line $\begin{pmatrix}1 & 3\end{pmatrix}$ $\mathbf{x}$ =7.

54. If the lines

$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 1$$
$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 3$$

are equally inclined to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 4,$$

find the value of m.

55. The sum of the perpendicular distances of a variable point **P** from the lines

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$$
$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7$$

is always 10. Show that **P** must move on a line.

56. Find the equation of the line which is equidis-

tant from parallel lines

$$\begin{pmatrix} 9 & 7 \end{pmatrix} \mathbf{x} = 7$$
$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = -6.$$

57. A ray of light passing through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ reflects on the x-axis at point **A** and the reflected ray passes through the point $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Find the coordinates of **A**.

Solution: Let point **P** be $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and point **Q** be $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ Since, point **A** is on x-axis, its y-coordinate is zero. Assume

$$A = \begin{pmatrix} k \\ 0 \end{pmatrix}$$

Incident vector

$$= \mathbf{P} - \mathbf{A}$$

Reflected vector

$$= \mathbf{Q} - \mathbf{A}$$

Vector along y-axis

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Vector along x-axis

$$\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Angle between AP and the x axis = 180°- angle between AQ and the x axis,

$$\frac{(\mathbf{P}-\mathbf{A})^T\mathbf{e_2}}{||\mathbf{P}-\mathbf{A}||} = \frac{(\mathbf{Q}-\mathbf{A})^T\mathbf{e_2}}{||\mathbf{Q}-\mathbf{A}||}$$
$$\frac{\mathbf{P}^T\mathbf{e_2}-\mathbf{A}^T\mathbf{e_2}}{||\mathbf{P}-\mathbf{A}||} = \frac{\mathbf{Q}^T\mathbf{e_2}-\mathbf{A}^T\mathbf{e_2}}{||\mathbf{Q}-\mathbf{A}||}$$

$$\frac{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} k & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 1-k \\ 2 \end{pmatrix}\right\|} = \frac{\begin{pmatrix} 5 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} k & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 5-k \\ 3 \end{pmatrix}\right\|}$$

$$\implies \frac{2}{\sqrt{(1-k)^2+(2)^2}} = \frac{3}{\sqrt{(5-k)^2+(3)^2}}$$

$$\implies 5k^2 + 22k - 91 = 0$$

Solving (3.7.57) we get: k=2.6, -7

Since, incident ray passes through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and reflected ray passes through $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$,

k cannot be negative as reflection takes place in first quadrant.

$$k = 2.6$$

Figure plotted using python code:

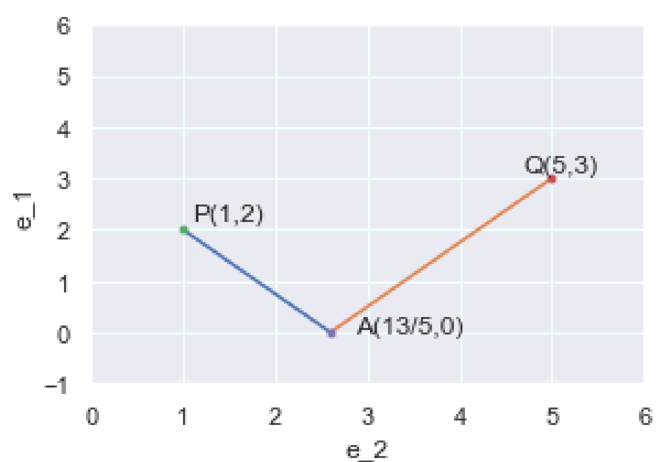

Fig. 3.7.57.1: Incident and reflected ray vectors plotted via Python code

58. A person standing at the junction of two straight paths represented by the equations

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 4$$
$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 5$$

wants to reach the path whose equation is

$$\begin{pmatrix} 6 & -7 \end{pmatrix} \mathbf{x} = -8$$

in the least time. Find the equation of the path that he should follow.

Solution: Step1: we need to find the solution of equation:

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 4$$
$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 5$$

$$\begin{pmatrix} 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \xleftrightarrow{R1 \leftarrow \frac{4}{17} * (R1 + \frac{3}{4} R2)} \begin{pmatrix} 1 & 0 & 31/17 \\ 3 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 31/17 \\ 3 & 4 & 5 \end{pmatrix} \xleftrightarrow{R2 \leftarrow \frac{1}{4}(R2 - 3 * R1)} \begin{pmatrix} 1 & 0 & 31/17 \\ 0 & 1 & -2/17 \end{pmatrix}$$

After solving this two equation we will get the junction point, which is intersection of this line segments. Thus, Junction Point is (31/17 -2/17) To reach in the least time, he should follow the shortest path, i.e, perpendicular from the junction point to the line give by this equation:

$$\begin{pmatrix} 6 & -7 \end{pmatrix} \mathbf{x} = 8$$

normal vector to the given line is:

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$
$$\mathbf{n} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

The equation of the line in terms of normal vector passing through a given point is obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0$$
$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}$$

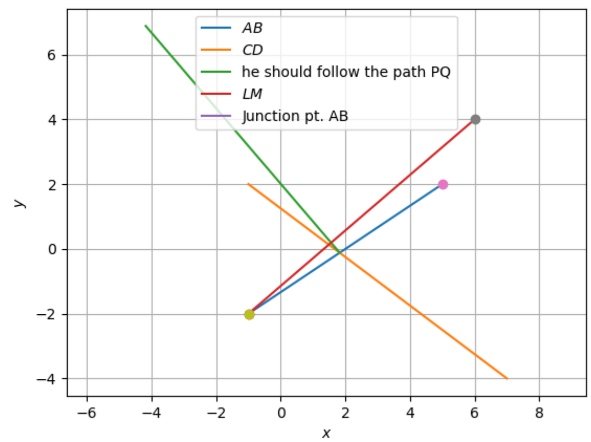

Fig. 3.7.58.1: The Required path is PQ.

Hence, he should follow this path PQ:

$$\begin{pmatrix} 7 & 6 \end{pmatrix} \mathbf{x} = \frac{205}{17}$$

59. Determine the ratio in which the line
$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 4 = 0$$
divides the line segment joining the points $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.
60. A line perpendicular to the line segment joining the points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ divides it in the ratio $1 : n$. Find the equation of the line.
Solution:
61. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
62. Find the equation of the line passing through the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.
63. Find the equation of the line through the point $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.
64. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix} -2 \\ 9 \end{pmatrix}$, find the equation of the line.
65. Find the equation of a line which passes

through the point $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and is parallel to the vector $\begin{pmatrix}3\\2\\-2\end{pmatrix}$.

Solution:

$$\mathbf{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix}3\\2\\-2\end{pmatrix}$$

Equation of the desired line in vector form will be

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + k\begin{pmatrix}3\\2\\-2\end{pmatrix}$$

See Fig. 3.7.65.1

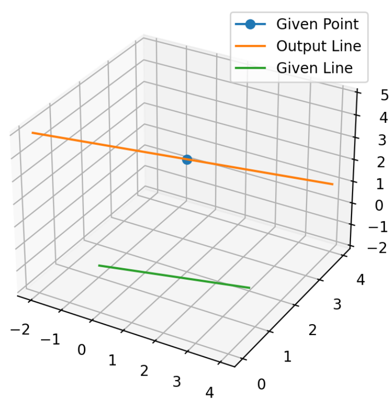

Fig. 3.7.65.1: Figure depicting provided as well as resultant data

66. Find the equaion off the line that passes through $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and is in the direction $\begin{pmatrix}1\\2\\-1\end{pmatrix}$.

Solution:
As the line at the direction of $(1\ 2\ \text{-}1)^T$ is represented by **b**, so **b** is the direction vector and the point is $\mathbf{a} = (2\ \text{-}1\ 4)^T$ through which the line passes through.
From the problem statement, we got

$$\mathbf{a} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$$

So, let us consider another point P with $\mathbf{r} = (x\ y\ z)^T$ on the line that passes through **a**.

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

Fig. 3.7.66.1

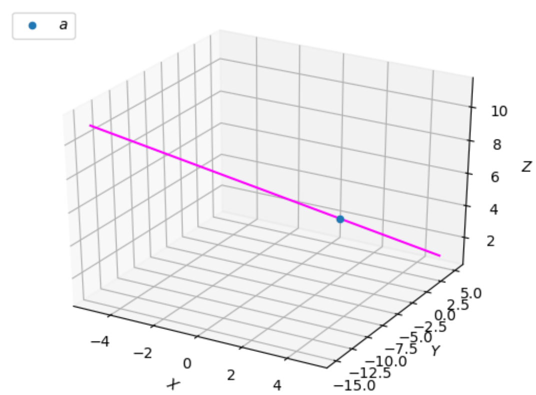

Fig. 3.7.66.1: The Straight line passing through a point at a particular direction

67. Find the equation of the line which passes through the point $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}.$$

Solution: We know that equation of the line passing through given a point a and in a parallel to **b** is given by

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \tag{3.7.67.0}$$

Also we can find direction vector from the Cartesian form of equation

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \tag{3.7.67.0}$$

This can be expressed as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.7.67.0}$$

where $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is a point on given line and $\mathbf{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the direction vector. Writing given equation (**??**) in vector form as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \tag{3.7.67.0}$$

So the direction vector of equation given is

$$\mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \tag{3.7.67.0}$$

and the point on which line passes is

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} \tag{3.7.67.0}$$

Substituting (3.7.67) and (3.7.67) in (3.7.67) we get

$$\mathbf{r} = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \tag{3.7.67.0}$$

which is the line parallel to line (**??**) and passes through point $\begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$.

68. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}.$$

Solution: Let,

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} = t$$

Equation of the line from the above (3.7.68) can be expressed as,

$$\implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3t+5 \\ 7t-4 \\ 2t+6 \end{pmatrix}$$

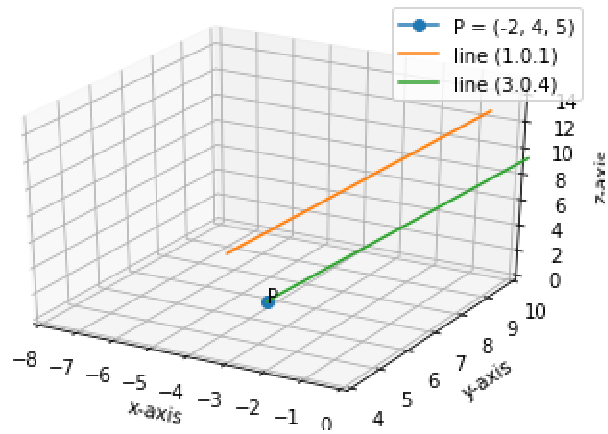

Fig. 3.7.67.1: Equation of line passing through point P and parallel to line (**??**)

It can be further written as,

$$\implies \mathbf{x} = \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} + t \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

where,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Hence, equation 3.7.68 gives the equation of a line and for t=0, the line passes through the point $\begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix}$

Plot of the line which passes through the point when t=0 is given below in Fig. 3.7.68.1

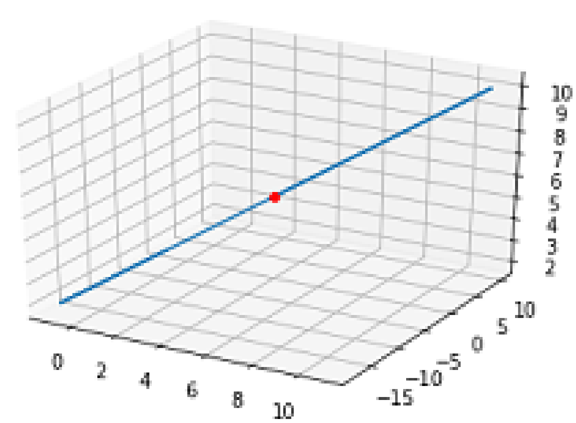

Fig. 3.7.68.1: Line passing through point (5,-4,6)

69. Find the equation of the line passing through

the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

Solution: Let the points be $\mathbf{O}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which is the origin and $\mathbf{P}=\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$. The vector form of the line passing through $\mathbf{O}$ and $\mathbf{P}$, which is the line passing through the point $\mathbf{O}$ and along direction vector $\mathbf{A}$ is given by

$$\mathbf{r} = \mathbf{O} + k\mathbf{A}$$

$$\implies \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

$$\implies \mathbf{r} = k \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

where k is a constant multiple. See Fig. 3.7.69.1

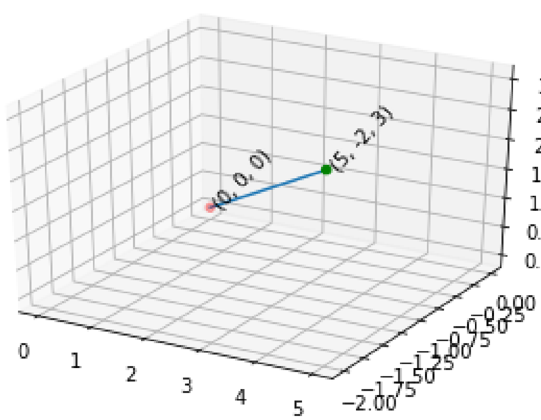

Fig. 3.7.69.1: Line passing through origin and point (5,-2,3)

70. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.

Solution: Let ,

$$\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$$

Direction vector $\mathbf{A}$ of the points a and b is given by,

$$\mathbf{A} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$$

Parametric equation is given by,

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$$

See Fig. 3.7.70.1

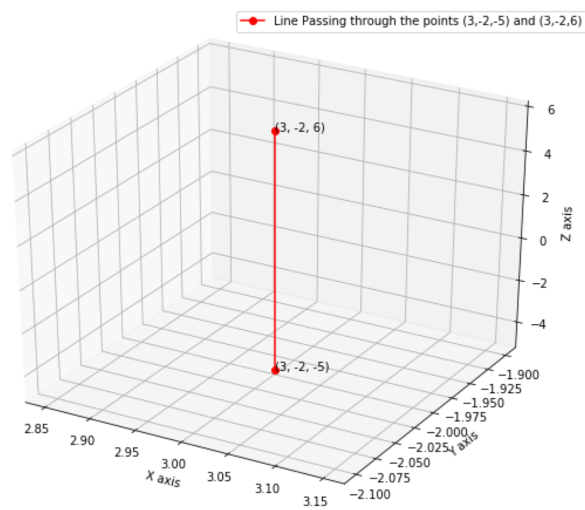

Fig. 3.7.70.1: Line passing through the points (3,-2,-5) and (3,-2,6)

71. Find the angle between the following pair of lines:

a)

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Solution:

i) The direction vectors of the lines are :

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$$

$$\mathbf{m_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Thus, the angle θ between two vectors is given by

$$\begin{aligned} \cos\theta &= \frac{\mathbf{m_1}^T\mathbf{m_2}}{\|\mathbf{m_1}\|\,\|\mathbf{m_2}\|} \\ &= \frac{19}{3 \times 7} \\ \implies \theta &= 25.21^\circ \end{aligned}$$

ii) The direction vectors of the lines are:

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$\mathbf{m_2} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}.$$

Thus, the angle θ between two vectors is given by

$$\begin{aligned} \cos\theta &= \frac{\mathbf{m_1}^T\mathbf{m_2}}{\|\mathbf{m_1}\|\,\|\mathbf{m_2}\|} \\ &= \frac{16}{\sqrt{6} \times \sqrt{50}} \\ \implies \theta &= 22.52^\circ \end{aligned}$$

b)

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}$$

72. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3},$$
$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1},$$
$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

Solution: From theory, we understand that using dot product we can find the angle between the lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3},$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

The above symmetric equations 3.7.72a, 3.7.72a can be represented in the vector form as

$$\mathbf{r_1} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix}$$

As we have to find the angle between the vectors, we will only be taking the direction vectors into consideration. The direction vectors are $\mathbf{u} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix}$. We can find the corresponding magnitude values

$$\|\mathbf{u}\| = \sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{38}$$

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 8^2 + 4^2} = \sqrt{81}$$

Using **??**, 3.7.72a, 3.7.72a we get

$$\begin{aligned} \theta &= \cos^{-1} \frac{\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}^T \begin{pmatrix} -1 \\ 8 \\ 4 \end{pmatrix}}{(\sqrt{38})(\sqrt{81})} \\ \theta &= \cos^{-1} \frac{26}{55.4797} \\ \theta &= \cos^{-1}(0.4686) \\ \theta &= 62.053^\circ \end{aligned}$$

Therefore, the angle between the two lines

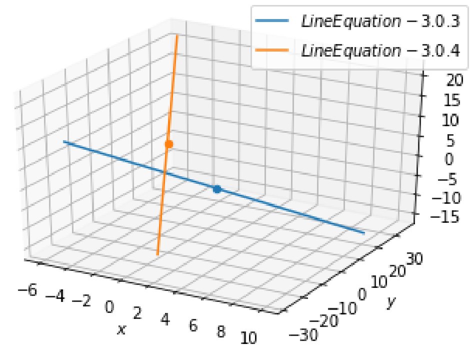

Fig. 3.7.72.1: Graph for equations 3.7.72a

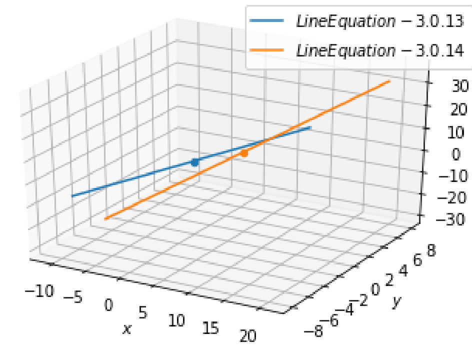

Fig. 3.7.72.2: Graph for equations 3.7.72b

is 62.053°.See Fig. 3.7.72.1

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1},$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

The above symmetric equations 3.7.72b, 3.7.72b can be represented in the vector form as

$$\mathbf{r_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}$$

As we have to find the angle between the vectors, we will only be taking the direction vectors into consideration. The direction vectors are $\mathbf{u} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}$. We can find the corresponding magnitude values

$$\|\mathbf{u}\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9}$$

$$\|\mathbf{v}\| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81}$$

Using **??**, 3.7.72b, 3.7.72b we get

$$\theta = \cos^{-1} \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}}{(\sqrt{9})(\sqrt{81})}$$

$$\theta = \cos^{-1} \frac{18}{27.00}$$

$$\theta = \cos^{-1}(0.667)$$

$$\theta = 48.189°$$

Therefore, the angle between the two lines is 48.189°. See Fig. 3.7.72.2

73. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2},$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles.

Solution: Rewriting the given lines as

$$L_1 : \frac{x-1}{-3} = \frac{y-2}{\frac{2}{7}p} = \frac{z-3}{2}$$

$$L_2 : \frac{x-1}{\frac{-3}{7}p} = \frac{y-5}{1} = \frac{z-6}{-5}$$

Using the definition of a line in co-ordinate geometry, we see from the above two equations, the direction vectors **a** and **b** of the two lines are

$$\mathbf{a} = \begin{pmatrix} -3 \\ \frac{2}{7}p \\ 2 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} \frac{-3}{7}p \\ 1 \\ -5 \end{pmatrix}$$

respectively.
In order for the two lines to be perpendicular, their dot product should be equal to 0 which gives,

$$\frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = 0 \tag{3.7.73.0}$$

Which in turn gives us,

$$\mathbf{a}^T\mathbf{b} = \frac{11}{7}p - 10$$
$$\implies \frac{11}{7}p - 10 = 0$$
$$\implies p = \frac{70}{11}$$
$$\implies p \approx 6.364$$

See Fig. 3.7.73.1

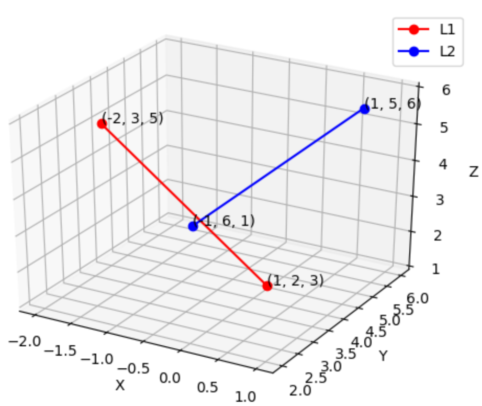

Fig. 3.7.73.1: The two lines plotted by substituting the value of p found

74. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1},$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

are perpendicular to each other.
Solution: Let us consider a parameter t.

Considering the first equation:

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1} = t$$

Line equation of (3.7.74) can be written as,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7t+5 \\ 5t-2 \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix}$$

From (3.7.74), the direction vector is given by

$$\mathbf{d_1} = \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix}$$

Similarly, let us consider second equation:

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t$$

Line equation of (3.7.74) can be written as,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

From (3.7.74), the direction vector is given by

$$\mathbf{d_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Two lines are perpendicular to each other when the dot product of their direction vectors is 0.

Dot product of direction vectors $\mathbf{d_1}$ and $\mathbf{d_2}$ (from equation (3.7.74) and (3.7.74)) is given by:

$$\mathbf{d_1}^T\mathbf{d_2} = (7 \times 1) + (-5 \times 2) + (1 \times 3) = 0$$

$$\implies \boxed{\mathbf{d_1}^T\mathbf{d_2} = 0}$$

From (3.7.74), as the dot product of direction vectors of the lines is 0 ($\mathbf{d_1^T d_2} = 0$), we can say that the lines are perpendicular to each other.

75. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

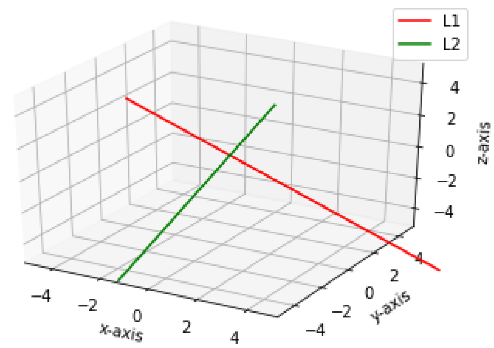

Fig. 1: Lines perpendicular to each other

76. Find the shortest distance between the lines
$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1},$$
$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$
77. Find the shortest distance between the lines
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$
78. Find the shortest distance between the lines
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix}$$

Solution:

$$L_1 : \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$
$$L_2 : \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

We have,
$$L_1 : \mathbf{x} = \mathbf{a_1} + t\mathbf{b_1}$$
$$L_2 : \mathbf{x} = \mathbf{a_2} + s\mathbf{b_2}$$
where, $\mathbf{a_i}$, $\mathbf{b_i}$ are positional and slope vectors of line L_i respectively.

As $\mathbf{b_1} \neq \lambda \mathbf{b_2}$, lines L_1 and L_2 are not parallel to each other.

Now, let us assume that L_1 and L_2 are intersecting at a point. Therefore,
$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$
$$s \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$
$$\begin{pmatrix} -1 & -1 \\ -2 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$
Using Gaussian elimination method:
$$E = E_{32}E_{31}E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-1}{2} & 1 & \frac{3}{4} \end{pmatrix}$$
$$E \begin{pmatrix} -1 & -1 & : & 0 \\ -2 & 1 & : & 1 \\ 2 & -2 & : & -4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & : & 0 \\ 0 & 3 & : & 1 \\ 0 & 0 & : & -2 \end{pmatrix}$$

From (3.7.78) it is clear that the system of linear equations are inconsistent. Therefore L_1 and L_2 are not intersecting at any point.
Hence our assumption was wrong, L_1, L_2 are **skew lines**.
Let d be the shortest distance between L_1, L_2 and $\mathbf{p_1}$, $\mathbf{p_2}$ be the positional vectors of its end points.

For d to be the shortest, we know that,

$$\mathbf{b_1}^T(\mathbf{p_2} - \mathbf{p_1}) = 0$$
$$\mathbf{b_2}^T(\mathbf{p_2} - \mathbf{p_1}) = 0$$
$$\mathbf{b_1}^T\left((\mathbf{a_2} - \mathbf{a_1}) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b_1}\end{pmatrix}\begin{pmatrix}s \\ -t\end{pmatrix}\right) = 0$$
$$\mathbf{b_2}^T\left((\mathbf{a_2} - \mathbf{a_1}) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b_1}\end{pmatrix}\begin{pmatrix}s \\ -t\end{pmatrix}\right) = 0$$
$$\mathbf{B} = \begin{pmatrix}\mathbf{b_2} & \mathbf{b_1}\end{pmatrix}, \mathbf{B^T} = \begin{pmatrix}\mathbf{b_2}^T \\ \mathbf{b_1}^T\end{pmatrix}$$

By combining equation (3.7.78) and (3.7.78) and writing in terms of $\mathbf{B}$ and $\mathbf{B}^T$ using (3.7.78) we get:

$$\mathbf{B}^T\mathbf{B}\begin{pmatrix}s \\ -t\end{pmatrix} = \mathbf{B}^T(\mathbf{a_1} - \mathbf{a_2})$$

By putting the values of $\mathbf{a_1}, \mathbf{a_2}, \mathbf{b_1}$ and $\mathbf{b_2}$ in equation (3.7.78) we get:

$$\begin{pmatrix}5 & 6 \\ 9 & 5\end{pmatrix}\begin{pmatrix}s \\ -t\end{pmatrix} = \begin{pmatrix}-9 \\ -10\end{pmatrix}$$

Solving equation (3.7.78) we get:

$$s = \frac{-15}{29}, t = \frac{31}{29}$$

By putting the values of t and s in equation (3.7.78) and (3.7.78) respectively we get:

$$\mathbf{p_1} = \begin{pmatrix}\frac{-17}{250} \\ \frac{-93}{100} \\ \frac{43}{50}\end{pmatrix}, \mathbf{p_2} = \begin{pmatrix}\frac{12}{25} \\ -2 \\ \frac{17}{500}\end{pmatrix}$$

Hence the shortest distance d between the two skew lines is :

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 1.4855$$

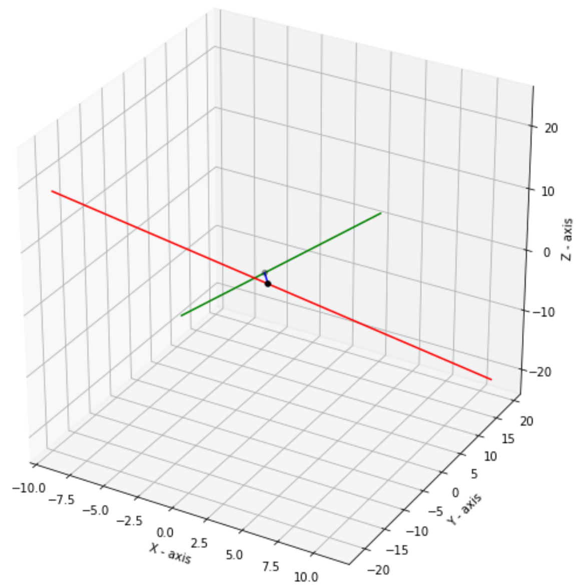

Fig. 1: 3-D plot for the skew lines and the shortest distance between them.

79. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $\begin{pmatrix}0 & 0 & 1\end{pmatrix}\mathbf{x} = 2$ c) $\begin{pmatrix}0 & 5 & 0\end{pmatrix}\mathbf{x} = -8$
b) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$ d) $\begin{pmatrix}2 & 3 & -1\end{pmatrix}\mathbf{x} = 5$

Solution:

(a) $\begin{pmatrix}0 & 0 & 1\end{pmatrix}\mathbf{x} = 2$

$$\mathbf{n} = \begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix}$$
$$c = 2$$

shortest distance from origin =

$$\frac{|2|}{\sqrt{0^2 + 0^2 + 1^2}} = 2$$

(b) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$

$$\mathbf{n} = \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix}$$
$$c = 1$$

shortest distance from origin =

$$\frac{|1|}{\sqrt{1^2 + 1^2 + 1^1}} = \frac{1}{\sqrt{3}}$$

(c) $\begin{pmatrix}0 & 5 & 0\end{pmatrix}\mathbf{x} = -8$

$$\mathbf{n} = \begin{pmatrix}0 \\ 5 \\ 0\end{pmatrix}$$
$$c = -8$$

a) $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 12$

b) $\begin{pmatrix} 3 & 4 & -6 \end{pmatrix} \mathbf{x} = 0$

c) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1$ **Solution:** The equation of a plane is given as

$$\mathbf{n}^T \mathbf{x} = c$$

where $\mathbf{n}$ = normal vector to the plane

Let $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ be the origin and $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ be the foot of the perpendicular drawn from the origin to the plane.

The position vector from $\mathbf{O}$ to $\mathbf{P}$ is $(\mathbf{P} - \mathbf{O})$

Since $(\mathbf{P} - \mathbf{O})$ is perpendicular to the plane, it is parallel to the normal of the plane $\mathbf{n}$. So,

$$\mathbf{P} - \mathbf{O} = k\mathbf{n}$$

$$\mathbf{P} = k\mathbf{n} + \mathbf{O}$$

where k is any scalar quantity.

Since $\mathbf{O}$ is a null vector,

$$\implies \mathbf{P} = k\mathbf{n}$$

Since $\mathbf{P}$ lies on the given plane, it must satisfy the equation of the plane. Therefore,

$$\mathbf{n}^T \mathbf{P} = c$$

$$\implies \mathbf{n}^T (k\mathbf{n}) = c$$

$$\implies k\mathbf{n}^T \mathbf{n} = c$$

a) $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 12$
Comparing with (3.7.81c), we get $\mathbf{n}^T = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$ and $c = 12$. Using this in (3.7.81c)

$$k \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 12$$

$$\implies 29k = 12$$

$$\implies k = \frac{12}{29}$$

$$\mathbf{P} = k\mathbf{n} \implies \mathbf{P} = \frac{12}{29} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\implies \mathbf{P} = \begin{pmatrix} \frac{24}{29} \\ \frac{36}{29} \\ \frac{48}{29} \end{pmatrix}$$

b) $\begin{pmatrix} 3 & 4 & -6 \end{pmatrix} \mathbf{x} = 0$
Comparing with (3.7.81c), we get $\mathbf{n}^T = \begin{pmatrix} 3 & 4 & -6 \end{pmatrix}$ and $c = 0$. Using this in

shortest distance from origin =

$$\frac{|-8|}{\sqrt{0^2+5^2+0^2}} = \frac{8}{5}$$

(d) $\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \mathbf{x} = 5$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$c = 5$$

shortest distance from origin =

$$\frac{|5|}{\sqrt{2^2+3^2+(-1)^2}} = \frac{5}{\sqrt{14}}$$

80. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$. **Solution:**

$$\mathbf{n}^T\mathbf{x} = c$$

where $\mathbf{n}$=normal vector to the plane The distance from the origin is given by:-

$$\frac{|c|}{\|\mathbf{n}\|} = 7$$

$$\|\mathbf{n}\| = \sqrt{3^2+5^2+6^2} = \sqrt{70}$$

Substituting equation (3.7.80) in (3.7.80) we get,

$$\frac{|c|}{\sqrt{70}} = 7$$

$$c = \pm 7\sqrt{70}$$

Substituting equation (**??**),(3.7.80) in (3.7.80) we get two equation of planes,

$$\boxed{\begin{pmatrix} 3 & 5 & 6 \end{pmatrix} \mathbf{x} = 7\sqrt{70}}$$

$$\boxed{\begin{pmatrix} 3 & 5 & 6 \end{pmatrix} \mathbf{x} = -7\sqrt{70}}$$

Equation (3.7.80) and (3.7.80) gives us the equation of two planes which are at a distance of 7 units from origin and normal to $\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$

81. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

(3.7.81c)

$$k\begin{pmatrix} 3 & 4 & -6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix} = 0$$

$$\implies -11k = 0$$

$$\implies k = 0$$

$$\mathbf{P} = k\mathbf{n} \implies \mathbf{P} = 0\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\implies \mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This shows that the plane $\begin{pmatrix} 3 & 4 & -6 \end{pmatrix} \mathbf{x} = 0$ passes through the origin.

c) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1$

Comparing with (3.7.81c), we get $\mathbf{n}^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and $c = 1$. Using this in (3.7.81c)

$$k\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$$

$$\implies 3k = 1$$

$$\implies k = \frac{1}{3}$$

$$\mathbf{P} = k\mathbf{n} \implies \mathbf{P} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\implies \mathbf{P} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

d) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \mathbf{x} = -8$

$$\implies \begin{pmatrix} 0 & -5 & 0 \end{pmatrix} \mathbf{x} = 8$$

Comparing with (3.7.81c), we get $\mathbf{n}^T = \begin{pmatrix} 0 & -5 & 0 \end{pmatrix}$ and $c = 8$. Using this in (3.7.81c)

$$k\begin{pmatrix} 0 & -5 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} = 8$$

$$\implies 25k = 8$$

$$\implies k = \frac{8}{25}$$

$$\mathbf{P} = k\mathbf{n} \implies \mathbf{P} = \frac{8}{25}\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$$

$$\implies \mathbf{P} = \begin{pmatrix} 0 \\ \frac{-8}{5} \\ 0 \end{pmatrix}$$

The Figures for the planes are shown in order: **??** See Fig. **??**.

d) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \mathbf{x} = -8$

82. Find the equation of the planes

a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

b) that passes through the point $\begin{pmatrix}1\\4\\6\end{pmatrix}$ and the normal vetor the plane is $\begin{pmatrix}1\\-2\\1\end{pmatrix}$.

83. Find the equation of the planes that passes through three points

a) $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}6\\4\\-5\end{pmatrix}, \begin{pmatrix}-4\\-2\\3\end{pmatrix}$

b) $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\-1\end{pmatrix}$.

84. Find the intercepts cut off by the plane $\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 5$.

85. Find the equaion of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

86. Find the equation of the plane through the intersection of the planes $\begin{pmatrix}3 & -1 & 2\end{pmatrix}\mathbf{x} = 4$ and $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = -2$ and the pont $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

87. Find the equation of the plane passing through the intersection of the planes $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 7$ and $\begin{pmatrix}2 & 5 & 3\end{pmatrix}\mathbf{x} = 9$ and the pont $\begin{pmatrix}2\\1\\3\end{pmatrix}$.

88. Find the equation of the plane through the intersection of the planes $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 5$ which is perpendicular to the plane $\begin{pmatrix}1 & -1 & 1\end{pmatrix}\mathbf{x} = 0$.

89. Find the angle between the planes whose equations are $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix}3 & -3 & 5\end{pmatrix}\mathbf{x} = 3$

90. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $\begin{pmatrix}7 & 5 & 6\end{pmatrix}\mathbf{x} = -30$ and $\begin{pmatrix}3 & -1 & -10\end{pmatrix}\mathbf{x} = -4$

b) $\begin{pmatrix}2 & 1 & 3\end{pmatrix}\mathbf{x} = 2$ and $\begin{pmatrix}1 & -2 & 5\end{pmatrix}\mathbf{x} = 0$

c) $\begin{pmatrix}2 & -2 & 4\end{pmatrix}\mathbf{x} = -5$ and $\begin{pmatrix}3 & -3 & 6\end{pmatrix}\mathbf{x} = 1$

d) $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = -3$

e) $\begin{pmatrix}4 & 8 & 1\end{pmatrix}\mathbf{x} = 8$ and $\begin{pmatrix}0 & 1 & 1\end{pmatrix}\mathbf{x} = 4$

91. In the following cases, find the distance of each of the given points from the corresponding plane.

Item	Point	Plane
a)	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}3 & -4 & 12\end{pmatrix}\boldsymbol{x} = 3$
b)	$\begin{pmatrix}3\\-2\\1\end{pmatrix}$	$\begin{pmatrix}2 & -1 & 2\end{pmatrix}\boldsymbol{x} = -3$
c)	$\begin{pmatrix}2\\3\\-5\end{pmatrix}$	$\begin{pmatrix}1 & 2 & -2\end{pmatrix}\boldsymbol{x} = 9$
d)	$\begin{pmatrix}-6\\0\\0\end{pmatrix}$	$\begin{pmatrix}2 & -3 & 6\end{pmatrix}\boldsymbol{x} = 2$

TABLE 3.7.91

92. Show that the line joining the origin to the point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix}3\\5\\-1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\end{pmatrix}$.

93. If the coordinates of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\7\end{pmatrix}, \begin{pmatrix}-4\\3\\-6\end{pmatrix}, \begin{pmatrix}2\\9\\2\end{pmatrix}$, then find the angle between the lines AB and CD.

94. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2},$$
$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5},$$

find the value of k.

95. Find the equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and perpendicular to the plane

$$\begin{pmatrix}1 & 2 & -5\end{pmatrix}\mathbf{x} = -9$$

96. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \text{ and}$$

$$\mathbf{x} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$

97. Find the coordinates of the point where the line through $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ crosses the YZ-plane.
98. Find the coordinates of the point where the line through $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ crosses the ZX-plane.
99. Find the coordinates of the point where the line through $\begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} \\ -3 \\ 1 \end{pmatrix}$ crosses the plane

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \mathbf{x} = 7$$

100. Find the equation of the plane passing through the point $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \mathbf{x} = 5$$
$$\begin{pmatrix} 3 & 3 & 1 \end{pmatrix} \mathbf{x} = 0$$

101. If the points $\begin{pmatrix} 1 \\ 1 \\ p \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix} 3 & 4 & -12 \end{pmatrix} \mathbf{x} = -13,$$

then find the value of p.
102. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1 \text{ and}$$
$$\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \mathbf{x} = -4$$

and parallel to the x-axis.
103. If **O** be the origin and the coordinates of **P** be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then find the equation of the plane passing through **P** and perpendicular to OP.
104. Find the equation of the plane which contains the line of intersection of the planes

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \mathbf{x} = 4$$
$$\begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \mathbf{x} = -5$$

and which is perpendicular to the plane

$$\begin{pmatrix} 5 & 3 & -6 \end{pmatrix} \mathbf{x} = -8$$

105. Find the distance of the point $\begin{pmatrix} -1 \\ -5 \\ -10 \end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

and the plane

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 5$$

Solution:
We know that equation of the line passing through given a point and a plane

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{m} \tag{3.7.105.0}$$

Also we can find direction vector from the Cartesian form of equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \tag{3.7.105.0}$$

This can be expressed as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.7.105.0}$$

where $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is a point on given line and $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the direction vector.
Distance between the point and point of intersection.

$$D = \sqrt{(x2 - x1)^2 + (y2 - y1)^2 + (z2 - z1)^2} \tag{3.7.105.0}$$

Writing given equation (1.0.1) in vector form as

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \tag{3.7.105.0}$$

substitute (3.0.1) in (1.0.2) to find the value of λ

$$\begin{pmatrix}1 & -1 & 1\end{pmatrix}\left\{\begin{pmatrix}2\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\2\end{pmatrix}\right\}=5 \quad (3.7.105.0)$$

by multiplying the row vector with the first column vector

$$1(2)-1(-1)+1(2)=5 \quad (3.7.105.0)$$

by multiplying the row vector with the coefficient column vector of lambda

$$1(3\lambda)-1(4\lambda)+1(2\lambda)=\lambda \quad (3.7.105.0)$$

we get as

$$\lambda=0 \quad (3.7.105.0)$$

The line intersects the plane at

$$\mathbf{x}_0=\begin{pmatrix}2\\-1\\2\end{pmatrix} \quad (3.7.105.0)$$

Finally the distance between the point $\mathbf{P}=\begin{pmatrix}-1\\-5\\-10\end{pmatrix}$ and intersection point $\mathbf{x}_0=\begin{pmatrix}2\\-1\\2\end{pmatrix}$ is

$$\|x_0-P\|=\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2} \quad (3.7.105.0)$$

$$\|x_0-P\|=13 \quad (3.7.105.0)$$

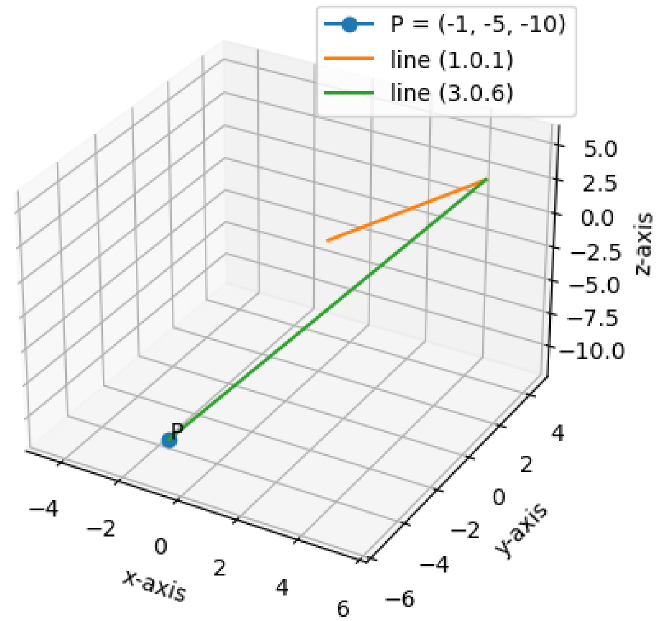

Fig. 1: Equation of line passing through point x_0 and intersection to line (1.0.1)

106. Find the vector equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and parallel to the planes

$$\begin{pmatrix}1 & -1 & 2\end{pmatrix}\mathbf{x}=5$$
$$\begin{pmatrix}3 & 1 & 1\end{pmatrix}\mathbf{x}=6$$

107. Find the vector equation of the line passing through the point $\begin{pmatrix}1\\2\\-4\end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7},$$
$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

Solution: The line passes through $\mathbf{a}=\begin{pmatrix}1\\2\\-4\end{pmatrix}$

$$\mathbf{x}=\begin{pmatrix}8\\-19\\10\end{pmatrix}+\lambda_1\begin{pmatrix}3\\-16\\7\end{pmatrix}$$
$$\mathbf{x}=\begin{pmatrix}15\\29\\5\end{pmatrix}+\lambda_2\begin{pmatrix}3\\8\\-5\end{pmatrix}$$

Let $\mathbf{n}$ be the normal vector to both lines. If $\mathbf{m_1}$ and $\mathbf{m_2}$ are the direction vectors of the lines,then

$$\mathbf{m_1}^T\mathbf{n}=0$$
$$\mathbf{m_2}^T\mathbf{n}=0$$

Let the matrix $\mathbf{M}$ be

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1}^T \\ \mathbf{m_2}^T \end{pmatrix}$$

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\mathbf{Mn} = 0$$

The matrix form is

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix} \xleftrightarrow{R_2=R_1-R_2} \begin{pmatrix} 3 & -16 & 7 \\ 0 & -24 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -16 & 7 \\ 0 & -24 & 12 \end{pmatrix} \xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 3 & -16 & 7 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -16 & 7 \\ 0 & -2 & 1 \end{pmatrix} \xleftrightarrow{R_1=R_1-8R_2} \begin{pmatrix} 3 & 0 & -1 \\ 0 & -2 & 1 \end{pmatrix}$$

We have 2 equations and 3 unknowns , we will have parametric solution

$$n_1 = \frac{k}{3}$$
$$n_2 = \frac{k}{2}$$
$$n_3 = k$$
$$\mathbf{n} = \frac{k}{6}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

The equation of required line is

$$\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

See Fig. 1

108. Distance between the two planes

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 4$$
$$\begin{pmatrix} 4 & 6 & 8 \end{pmatrix} \mathbf{x} = 12$$

Solution: So, the distance between the given

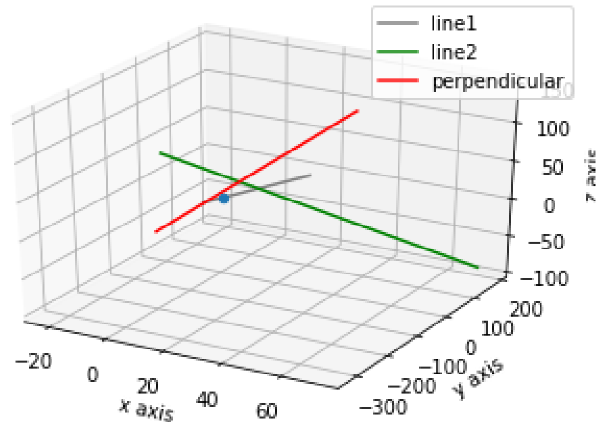

Fig. 1: Perpendicular Line

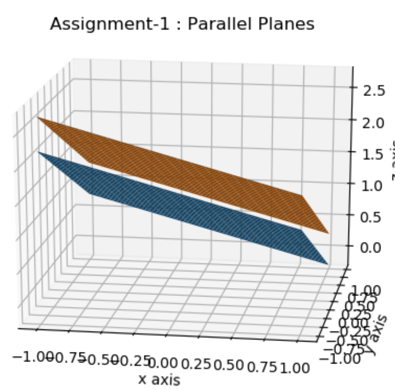

Fig. 1: Example of Two parallel planes

planes is:

$$\frac{|4-6|}{\sqrt{2^2+3^2+4^2}} = \frac{2}{\sqrt{29}}$$

See Fig. 1

a) 2
b) 4
c) 8
d) $\frac{2}{\sqrt{29}}$

109. The planes

$$\begin{pmatrix} 2 & -1 & 4 \end{pmatrix} \mathbf{x} = 5$$
$$\begin{pmatrix} 5 & -\frac{5}{2} & 10 \end{pmatrix} \mathbf{x} = 6$$

are

a) Perpendicular
b) Parallel
c) intersect y-axis
d) passes through $\begin{pmatrix} 0 \\ 0 \\ \frac{5}{4} \end{pmatrix}$

110. Find the maximum and minimum values, if any of the following functions given by
   a) $f(x) = |x+2| - 1$
   b) $f(x) = -|x+1| + 3$
   c) $h(x) = x+1, x \in (-1, 1)$.
111. Using integration find the area of region bounded by the triangle whose vertices are $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}2\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$.
112. Using integration find the area of region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).
113. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -1$, $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -1$ and x = 4.
114. Find the area of the region bounded by the line $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -2$, the x-axis and the ordinates $x = -1, x = 1$.
115. Find the area bounded by the curve $|x|+|y| = 1$.
116. Using the method of integration find the area of $\triangle ABC$, whose vertices are $\mathbf{A} = \begin{pmatrix}2\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}6\\3\end{pmatrix}$.
117. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 4$, $\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 6$ and $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -5$.
118. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?
119. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3 . If building of tank costs ₹70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?
120. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is

$$\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

121. Prove that the function $f(x) = 5x–3$ is continuous at $x = 0, at x = –3$ and at $x = 5$.
122. Examine the following functions for continuity.
   a) $f(x) = x - 5$
   b) $f(x) = |x - 1|$
123. Is the function defined by

$$f(x) = \begin{cases} x, & x \le 1, \\ 5, & x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?
124. Find all points of discontinuity of f, where f is defined by
   a) $f(x) = \begin{cases} 2x+3, & x \le 2, \\ 2x-3, & x > 2 \end{cases}$
   b) $f(x) = \begin{cases} |x|+3, & x \le -3, \\ -2x, & -3 < x < 3 \\ 6x+2, & x \ge 2 \end{cases}$
   c) $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0, \\ 0, & x = 0, \end{cases}$
   d) $f(x) = \begin{cases} \frac{x}{|x|}, & x < 0, \\ -1, & x \ge 0, \end{cases}$
125. Is the function defined by

$$f(x) = \begin{cases} x+5, & x \le 1, \\ x-5, & x > 1 \end{cases}$$

a continuous function?
126. Discuss the continuity of the function f, where f is defined by
   a) $f(x) = \begin{cases} 3, & 0 \le x \le 1, \\ 4, & 0 < x \le 3, \\ 5, & 3 \le x \le 10, \end{cases}$
   b) $f(x) = \begin{cases} 2x, & x < 0, \\ 0, & 0 \le x \le 1 \\ 4x, & x > 1 \end{cases}$
   c) $f(x) = \begin{cases} -2, & x < -1, \\ 2x, & -1 \le x \le 1 \\ 2, & x > 1 \end{cases}$
127. Find the relationship between a and b so that the function defined by

$$f(x) = \begin{cases} ax+1, & x \le 3, \\ bx+3, & x > 3 \end{cases}$$

is continuous at $x = 3$
128. Prove that the function $f(x) = x$ is continiuous at every real number.
129. Is $f(x) = |x|$ a continuous function?
130. Discuss the continuity of the function f defined

by

$$f(x) = \begin{cases} x+2 & x \leq 1 \\ x-2 & x > 1 \end{cases}$$

131. Show that the function defined by $g(x) = x–[x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.
132. For what value of k is the following function continuous at the given point.

$$f(x) = \begin{cases} kx+1, & x \leq 5, \\ 3x-5, & x > 5, \end{cases} \quad x = 5$$

133. Prove that the function f given by

$$f(x) = |x-1|, x \in \mathbf{R}$$

is not differentiable at $x = 1$.

134. Prove that the greatest integer function defined by

$$f(x) = |x|, 0 < x < 3$$

is not differentiable at $x = 1$ and $x = 2$.

135. Examine if Rolle's theorem is applicable to the following functions
   a) $f(x) = [x], x \in [5, 9]$.
   b) $f(x) = [x], x \in [-2, 2]$.

   Can you say some thing about the converse of Rolle's theorem from this example?
136. Examine the applicability of the mean value theorem for all functions in Problem 3.7.135a.
137. Find $\lim_{x\to 5} x + 10$
138. Find $\lim_{x\to 2} 3x$
139. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} 1 & x \leq 0 \\ 2 & x > 0 \end{cases}$$

140. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} x-2 & x < 0 \\ 0 & x = 0 \\ x+2 & x > 0 \end{cases}$$

141. Evaluate the following limits
   a) $\lim_{x\to 3} x + 3$
   b) $\lim_{x\to \pi} \left(x - \frac{22}{7}\right)$
142. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0, & x = 0 \end{cases}$$

143. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0, & x = 0 \end{cases}$$

144. Find $\lim_{x\to 5} |x| - 5$.
145. Suppose

$$f(x) = \begin{cases} a+bx & x \neq 1 \\ 4, & x = 1 \\ b-ax & x > 1 \end{cases}$$

and if $\lim_{x\to 1} f(x) = f(1)$, what are the possible values of a and b?

146. If

$$f(x) = \begin{cases} |x|+1 & x < 0 \\ 0, & x = 0 \\ |x|-1 & x > 0 \end{cases}$$

for what value(s) of a does $\lim_{x\to a} f(x)$ exists?

147. Find the derivative of x at $x = 1$.
148. Find the derivative of $99x$ at $x = 100$.
149. Find the derivative of the following functions:
   a) $-x$
   b) $x + a$
150. Integrate the following as limit of sums:
   (i) $\int_a^b x\,dx$
   (ii) $\int_0^5 (x+1)\,dx$
   (iii) $\int_{-1}^1 (x+1)\,dx$
   (iv) $\int_{-5}^5 |x+2|\,dx$
   (v) $\int_2^8 |x-5|\,dx$
   (vi) $\int_0^4 |x-1|\,dx$
   (vii) $\int_1^4 [|x-1| + |x-2| + |x-3|]\,dx$
151. Form the differential equation representing the following family of curves

$$\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$$

3.8 Motion in a Plane

1. Rain is falling vertically with a speed of 35 ms^{-1} . Winds starts blowing after sometime with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting

at a bus stop hold his umbrella ?

Solution: See Fig. 3.8.1.1. From the given information, the rain velocity is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 35 \end{pmatrix} \tag{3.8.1.1}$$

and the wind velocity is

$$\mathbf{v} = -\begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{3.8.1.2}$$

The resulting rain velocity is

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} -12 \\ 35 \end{pmatrix} \tag{3.8.1.3}$$

The desired angle is

$$-\tan^{-1} \underline{/\mathbf{u} + \mathbf{v}} = \tan^{-1} \frac{12}{35} \tag{3.8.1.4}$$

$$\approx 20.04^\circ \tag{3.8.1.5}$$

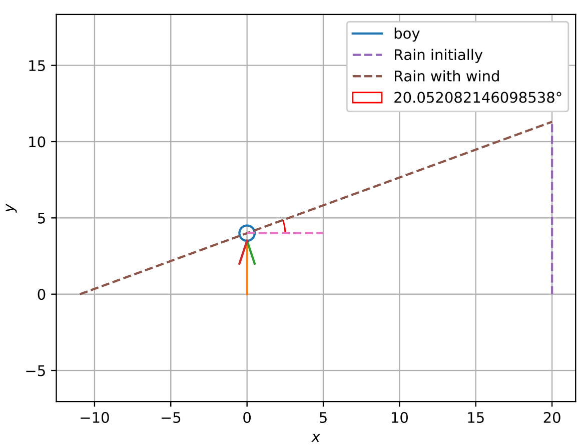

Fig. 3.8.1.1

2. A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.
Solution: In Fig. 3.8.2.1, **A** denotes the velocity of the boat, **B** denotes the water current and **C** represents the resultant velocity.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 25 \end{pmatrix} \tag{3.8.2.1}$$

$$\mathbf{B} = 10 \begin{pmatrix} \cos 30^\circ \\ -\sin 30^\circ \end{pmatrix} \tag{3.8.2.2}$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{3.8.2.3}$$

$$= 5 \begin{pmatrix} \sqrt{3} \\ 4 \end{pmatrix} \tag{3.8.2.4}$$

The following Python code generates Fig. 3.8.2.1

```
solutions/2/codes/line_ex/
    motion_in_a_plane/motion_plane.py
```

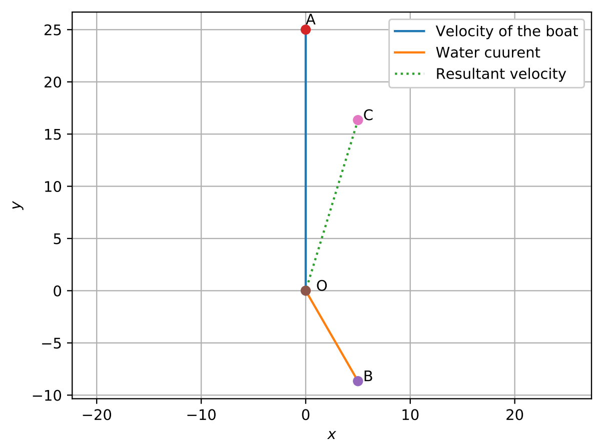

Fig. 3.8.2.1

3. Rain is falling vertically with a speed of 35 ms^{-1} . A woman rides a bicycle with a speed of 12 ms^{-1} in east to west direction. What is the direction in which she should hold her umbrella ?
Solution: See Fig. 3.8.3.1. The velocity of rain and velocity of woman are

$$\mathbf{v_r} = \begin{pmatrix} 0 \\ -35 \end{pmatrix} \tag{3.8.3.1}$$

$$\mathbf{v_w} = \begin{pmatrix} -12 \\ 0 \end{pmatrix} \tag{3.8.3.2}$$

The relative velocity of rain w.r.t woman is given as

$$\mathbf{v_{r_w}} = \mathbf{v_r} - \mathbf{v_w} \tag{3.8.3.3}$$

$$= \begin{pmatrix} 12 \\ -35 \end{pmatrix} \tag{3.8.3.4}$$

So the woman must hold the umbrella along the direction of $-\mathbf{v_{r_w}}$ Thus, the desired angle is

$$\theta = \tan^{-1}\left(\frac{12}{35}\right) \tag{3.8.3.5}$$

The following python code generates Fig. 3.8.3.1.

```
solutions/3/codes/line/rain/rain.py
```

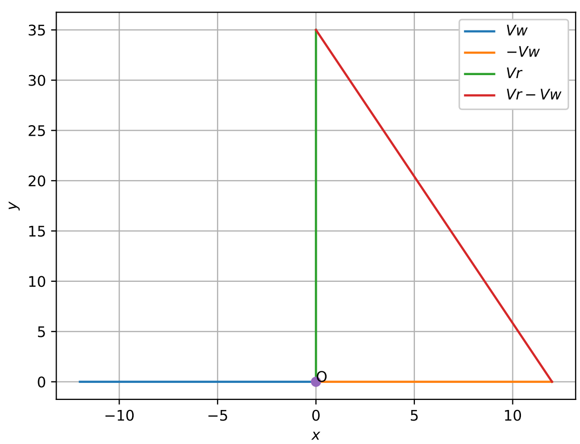

Fig. 3.8.3.1: Direction of umbrella

4. A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 ms^{-1} . Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 ms^{-2}).
Solution: From the given information, the hicker's position vector is

$$\mathbf{A} = \begin{pmatrix} 0 \\ 490 \end{pmatrix} \tag{3.8.4.1}$$

the acceleration of the stone is

$$\mathbf{a} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \tag{3.8.4.2}$$

and the initial velocity of the stone is

$$\mathbf{v}_A = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \tag{3.8.4.3}$$

If **B** be the final position of the stone,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{a}t \tag{3.8.4.4}$$

$$\mathbf{B} = \mathbf{A} + \mathbf{v}_A t + \frac{1}{2}\mathbf{a}t^2 \tag{3.8.4.5}$$

$$\implies \mathbf{B} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 490 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix} t^2 \tag{3.8.4.6}$$

$\because$ the stone finally comes to rest. Thus,

$$490 = \frac{1}{2}9.8t^2 \tag{3.8.4.7}$$

$$\implies t = 10 \tag{3.8.4.8}$$

Substituting in (3.8.4.4),

$$\mathbf{v}_B = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 9.8 \end{pmatrix} 10 \tag{3.8.4.9}$$

$$= \begin{pmatrix} 1.5 \\ 98 \end{pmatrix} \tag{3.8.4.10}$$

The final speed is given by $\|\mathbf{v}_B\|$. The motion of the stone is plotted in Fig. 3.8.4.1 using (3.8.4.6) by varying t through the following code.

```
solutions/4/codes/line/motion/motion.py
```

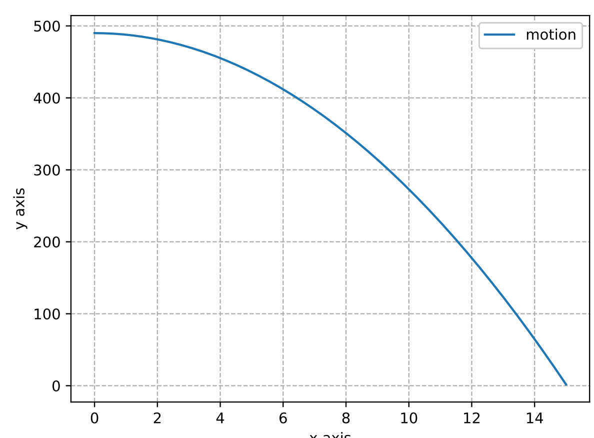

Fig. 3.8.4.1

5. Rain is falling vertically with a speed of 30 ms^{-1}. A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which she should hold her umbrella?
Solution: See Fig. 3.8.5.1. The velocity of rain

and velocity of woman are

$$\mathbf{v_r} = \begin{pmatrix} 0 \\ -30 \end{pmatrix} \tag{3.8.5.1}$$

$$\mathbf{v_w} = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \tag{3.8.5.2}$$

The relative velocity of rain w.r.t woman is given as

$$\mathbf{v_{r_w}} = \mathbf{v_r} - \mathbf{v_w} \tag{3.8.5.3}$$

$$= \begin{pmatrix} 10 \\ -30 \end{pmatrix} \tag{3.8.5.4}$$

So the woman must hold the umbrella along the direction of $-\mathbf{v_{r_w}}$ Thus, the desired angle is

$$\theta = \tan^{-1}\left(\frac{10}{30}\right) \tag{3.8.5.5}$$

The following python code plots Fig. 3.8.5.1.

```
./solutions/5/codes/lines/q12.py
```

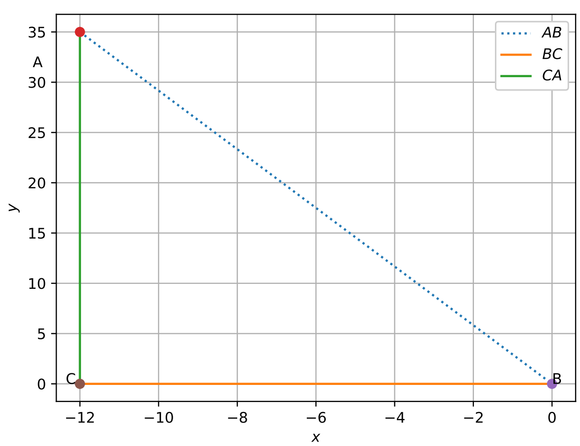

Fig. 3.8.5.1

6. A man can swim with a speed of 4.0 km/h in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km/h and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank ?
Solution: The following code plots Fig. 3.8.6.1

```
solutions/6/codes/line/motion_plane/
    man_river.py
```

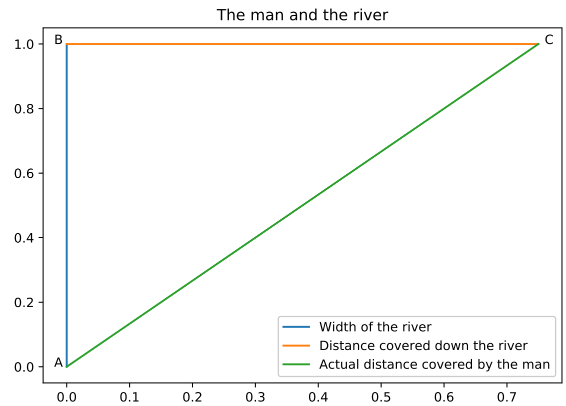

Fig. 3.8.6.1

In Fig. 3.8.6.1, let the man be at

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.8.6.1}$$

The opposite bank of the river is at

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.8.6.2}$$

River current

$$\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.8.6.3}$$

Initial velocity of the man is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{3.8.6.4}$$

The resultant velocity of the man is

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.8.6.5}$$

If the time taken by the man to cross the river be t, then

$$\mathbf{C} = (\mathbf{u} + \mathbf{v})\,t = \begin{pmatrix} 3 \\ 4 \end{pmatrix} t \tag{3.8.6.6}$$

$$= \mathbf{A} + \mathbf{B} = \begin{pmatrix} BC \\ 1 \end{pmatrix} \tag{3.8.6.7}$$

Thus,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} t = \begin{pmatrix} BC \\ 1 \end{pmatrix} \tag{3.8.6.8}$$

$$\implies 4t = 1 \text{or}, \ t = \frac{1}{4} \tag{3.8.6.9}$$

Distance traveled down the river

$$BC = 3t = \frac{3}{4} \tag{3.8.6.10}$$

7. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?
Solution: The velocity of wind and boat are respectively,

$$\mathbf{v_w} = 72\begin{pmatrix}\cos 45^\circ \\ \sin 45^\circ\end{pmatrix} \tag{3.8.7.1}$$

$$\mathbf{v_b} = \begin{pmatrix}0 \\ 51\end{pmatrix} \tag{3.8.7.2}$$

The resulting wind velocity is

$$\mathbf{v_w} - \mathbf{v_b} = \begin{pmatrix}36\sqrt{2} \\ 36\sqrt{2} - 51\end{pmatrix} \tag{3.8.7.3}$$

The direction of the flag is

$$\tan^{-1}\left(\frac{36\sqrt{2} - 51}{36\sqrt{2}}\right) \tag{3.8.7.4}$$

$$= -0.1^\circ \tag{3.8.7.5}$$

The python code for Fig. 3.8.7.1 is

```
solutions/7/codes/line/motion/motion.py
```

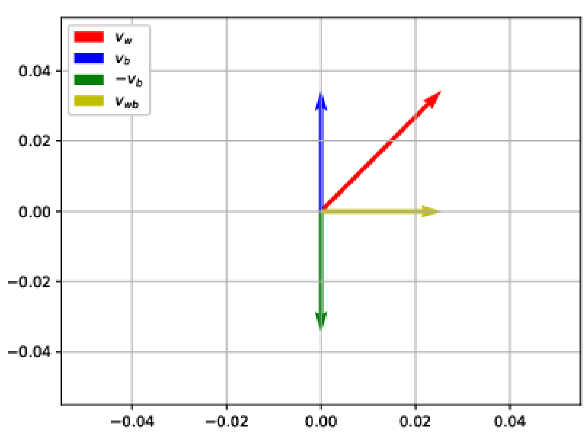

Fig. 3.8.7.1

8. A bullet fired at an angle of 30° with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 km away ? Assume the muzzle speed to be fixed, and neglect air resistance.
9. A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km/h passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 ms^{-1} to hit the plane ? At what minimum altitude should the pilot fly the plane to avoid being hit ? (Take $g = 10ms^{-2}$).
10. Give the magnitude and direction of the net force acting on a stone of mass 0.1 kg,
    a) just after it is dropped from the window of a stationary train,
    b) just after it is dropped from the window of a train running at a constant velocity of 36 km/h,
    c) just after it is dropped from the window of a train accelerating with $1ms^{-2}$
    d) lying on the floor of a train which is accelerating with 1 ms^{-2}, the stone being at rest relative to the train.

    Neglect air resistance throughout.
11. Consider the collision depicted in Fig. 3.8.11.1 to be between two billiard balls with equal masses $m_1 = m_2$. The first ball is called the cue while the second ball is called the target. The billiard player wants to 'sink' the target ball in a corner pocket, which is at an angle $\theta_2 = 37°$. Assume that the collosion is elastic and that friction and rotational motion are not important. Obtain θ_1.

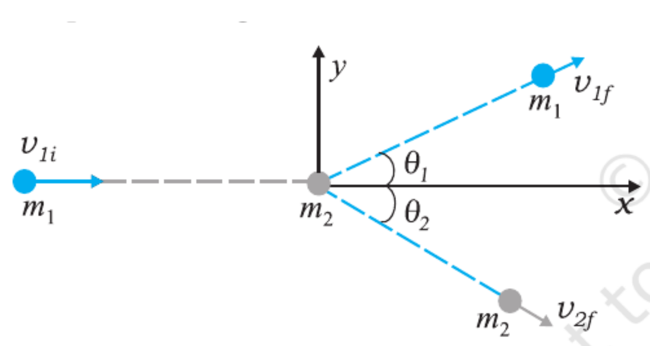

Fig. 3.8.11.1

3.9 Matrix Exercises

1. In the matrix A=$\begin{pmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{pmatrix}$, write

a) The order of the matrix
b) The number of elements
c) Write the elements $a_{31}, a_{21}, a_{33}, a_{24}, a_{23}$.

Solution:

a) The order of matrix for above problem is 3×4.
b) The number of elements=12
c) The elements are

$$a_{31} = \sqrt{3}, a_{21} = 35 \tag{3.9.1.1}$$

$$a_{33} = -5, a_{24} = 12 \tag{3.9.1.2}$$

$$a_{23} = \frac{5}{2} \tag{3.9.1.3}$$

The python code is given in

```
solutions/1/codes/line/matrix.py
```

2. If a matrix has 24 elements,what are the possible orders it can have? What,if it has 13 elements?
Solution: The following Python code generates all possible dimensions for any matrix size

```
solutions/2/codes/line_ex/matrix/matrix.py
```

3. If a matrix has 18 elements,what are the possible orders it can have? What,if it has 5 elements?

Solution:

3.1. A matrix with n elements can be represented as a matrix of order $(r \times c)$ if and only if n,r and c are all natural numbers(Here r is the number of rows and c is the number of coloumns in the matrix.).This is possible only if r is a divisor of n.
3.2. So the total possible orders a matrix with n elements can have is equal to the total number of divisors of n.
The following python code finds the total possible orders (d) for a matrix of n elements.

```
solutions/3/codes/line/matrix/matrix.py
```

So a matrix of 18 elements has 6 possible orders and a matrix of 5 elements can have 2 possible orders.

4. Construct a 2×2 matrix,A=$[a_{ij}]$,whose elements are given by:
(i) $a_{ij}=\frac{(i+j)^2}{2}$ (ii) $a_{ij}=\frac{i}{j}$ (iii) $a_{ij}=\frac{(i+2j)^2}{2}$
Solution: From the following code,

```
solutions/4/codes/line/matrix/matrix.py
```

a) $\mathbf{A} = \begin{pmatrix} 2 & 4.5 \\ 4.5 & 8 \end{pmatrix}$
b) $\mathbf{A} = \begin{pmatrix} 1 & 0.5 \\ 2 & 1 \end{pmatrix}$
c) $\mathbf{A} = \begin{pmatrix} 4.5 & 12.5 \\ 2 & 18 \end{pmatrix}$

5. Construct a 3×4 matrix,whose elements are given by:
(i) $a_{ij}=\frac{1}{2}\left|-3i+j\right|$ (ii) a_{ij}=2i-j
Solution:
The following python code computes the required matrix.

```
./codes/lines/q13.py
```

a) The matrix $A_{ij} = \frac{1}{2}|-3i+j|$ obtained is

$$\begin{pmatrix} 0 & 0.5 & 1 & 1.5 \\ 1.5 & 1 & 0.5 & 0 \\ 3 & 2.5 & 2 & 1.5 \end{pmatrix} \tag{3.9.5.1}$$

b) The matrix $A_{ij} = 2i - j$ obtained is

$$\begin{pmatrix} 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{3.9.5.2}$$

6. Find the values of x,y and z from the following equations:
(i) $\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$
(iii) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix}=\begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

Solution: This problem is solved by comparing the respective elements in both the matrices

a)

$$\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix} \tag{3.9.6.1}$$

$$x = 1, y = 4, z = 3 \tag{3.9.6.2}$$

b)

$$\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix} \tag{3.9.6.3}$$

$$5 + z = 5 \tag{3.9.6.4}$$

$$\implies z = 0 \tag{3.9.6.5}$$

$$x + y = 6 \tag{3.9.6.6}$$

$$xy = 8 \tag{3.9.6.7}$$

$$x = 4, y = 2 \tag{3.9.6.8}$$

$$x = 2, y = 4 \tag{3.9.6.9}$$

$x = 4, y = 2, z = 0$ or $x = 2, y = 4, z = 0$

c) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

Expressing it as $Ax = b$ and $x = A^{-1}b$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} \tag{3.9.6.10}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} \tag{3.9.6.11}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \tag{3.9.6.12}$$

$$x = 2, y = 3, z = 4 \tag{3.9.6.13}$$

7. Find the values of a,b,c and d from the equations: $\begin{pmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 0 & 13 \end{pmatrix}$
Solution: Equate (**??**) in the form:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \\ 13 \end{pmatrix} \tag{3.9.7.1}$$

Using augmented matrix and complete row reduction on (3.9.7.1) we deduce in following steps

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 & 5 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 13 \end{pmatrix} \tag{3.9.7.2}$$

Complete Row elimination:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 & 5 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 13 \end{pmatrix} \tag{3.9.7.3}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - 2R1]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 & 13 \end{pmatrix} \tag{3.9.7.4}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 & 13 \end{pmatrix} \tag{3.9.7.5}$$

$$\xleftrightarrow[R_4 \leftarrow R_4 - 3R3]{R_3 \leftarrow (-)(R_3 - R_2)} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \tag{3.9.7.6}$$

Now equating on (3.9.7.6) we deduce the following:

$$\mathbf{d} = 4, \mathbf{c} = 3, \mathbf{b} = 2, \mathbf{a} = 1 \tag{3.9.7.7}$$

8. A=$[a_{ij}]_{mxn}$ is a square matrix,if
(A) m<n (B)m>n (C) m=n (D) None of these

9. Which of the given values of x and y make the following pair of matrices equal $\begin{pmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{pmatrix}, \begin{pmatrix} 0 & y-2 \\ 8 & 4 \end{pmatrix}$
(A)x=$\frac{-1}{3}$,y=7
(B) Not possible to find
(C) y=7, x=$\frac{-2}{3}$
(D) x=$\frac{-1}{3}$,y=$\frac{-2}{3}$

10. The number of all possible matrices of order 3X3 with each entry 0 or 1 is:
(A) 27 (B)18 (C)81 (D)512

11. Let A=$\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$,B=$\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$,C=$\begin{pmatrix} -2 & 5 \\ 3 & 4 \end{pmatrix}$ Find each of the following:
(i) A+B (ii)A-B (iii)3A-C (iv)AB (v)BA

12. Compute the following:
(i)$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix}$
(ii)$\begin{pmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{pmatrix} + \begin{pmatrix} 2ab & 2bc \\ -2ac & -2ab \end{pmatrix}$

(iii) $\begin{pmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{pmatrix} + \begin{pmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{pmatrix}$

(iv) $\begin{pmatrix} cos^2x & sin^2x \\ sin^2x & cos^2x \end{pmatrix} + \begin{pmatrix} sin^2x & cos^2x \\ cos^2x & sin^2x \end{pmatrix}$

13. Compute the indicated products.

(i) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Solution:

$$\mathbf{C} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{3.9.13.1}$$

$$\mathbf{C} = \begin{pmatrix} a+a & b+b \\ -b+b & a+a \end{pmatrix} \tag{3.9.13.2}$$

$$\mathbf{C} = \begin{pmatrix} 2a & 2b \\ 0 & 2a \end{pmatrix} \tag{3.9.13.3}$$

(ii) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Solution: By applying matrix addition

$$= \begin{pmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{pmatrix} \tag{3.9.13.4}$$

$$= \begin{pmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{pmatrix} \tag{3.9.13.5}$$

(iv) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

(vi) $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}$

14. If,A=$\begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$,B=$\begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$and C=$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$,then compute (A+B) and (B-C).Also ,verify that A+(B-C)=(A+B)-C.

15. If A=$\begin{pmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{pmatrix}$ and B=$\begin{pmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{pmatrix}$,then compute 3A-5B.

16. Simplify $\cos\theta\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$+$\sin\theta\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$

Solution:

$$\cos\theta\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + \sin\theta\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \tag{3.9.16.1}$$

$$= \cos\theta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.9.16.2}$$

$$= \left(\cos\theta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \times \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.9.16.3}$$

$$= \left(\begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{pmatrix}\right) \times \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.9.16.4}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.9.16.5}$$

In (3.9.16.5), the left matrix rotates a vector by angle $+\theta$. Obviously the right matrix rotates a vector by angle -θ. Then the product of matrices is identity matrix.

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{3.9.16.6}$$

Hence it is simplified.

17. Find X and Y,if

(i)X+Y=$\begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$ and X-Y=$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

(ii)2X+3Y=$\begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ and 3X+2Y=$\begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix}$

Solution: Let,

$$X + Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix} = A \tag{3.9.17.1}$$

$$X - Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = B \tag{3.9.17.2}$$

Now, expressing the matrices (3.9.17.1), (3.9.17.2) in vector form,

$$\begin{pmatrix} I & I \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = A$$

$$\begin{pmatrix} I & -I \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = B$$

Combining both the equations into a single matrix equation and constructing the augmented matrix,

$$\begin{pmatrix} I & I & A \\ I & -I & B \end{pmatrix} \tag{3.9.17.3}$$

Transforming the equation (3.9.17.3) using row reduction,

$$\begin{pmatrix} I & I & A \\ I & -I & B \end{pmatrix} \xleftrightarrow{R2 \leftarrow \frac{R1-R2}{2}} \begin{pmatrix} I & I & A \\ 0 & I & \frac{A-B}{2} \end{pmatrix} \xleftrightarrow{R1 \leftarrow R1-R2}$$

$$\begin{pmatrix} I & 0 & \frac{A+B}{2} \\ 0 & I & \frac{A-B}{2} \end{pmatrix} \tag{3.9.17.4}$$

From (3.9.17.4),

$$X = \frac{A+B}{2} = \begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix} \tag{3.9.17.5}$$

$$Y = \frac{A-B}{2} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \tag{3.9.17.6}$$

Let,

$$2X + 3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix} = A \tag{3.9.17.7}$$

$$3X + 2Y = \begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix} = B \tag{3.9.17.8}$$

Now, expressing the matrices (3.9.17.7), (3.9.17.8) in vector form,

$$\begin{pmatrix} 2I & 3I \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = A$$

$$\begin{pmatrix} 3I & 2I \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = B$$

Combining both the equations into a single matrix equation and constructing the augmented matrix,

$$\begin{pmatrix} 2I & 3I & A \\ 3I & 2I & B \end{pmatrix} \tag{3.9.17.9}$$

Transforming the equation (3.9.17.9) using row reduction,

$$\begin{pmatrix} 2I & 3I & A \\ 3I & 2I & B \end{pmatrix} \xleftrightarrow{R2 \leftarrow \frac{3R1-2R2}{5}} \begin{pmatrix} 2I & 3I & A \\ 0 & I & \frac{3A-2B}{5} \end{pmatrix} \xleftrightarrow{R1 \leftarrow \frac{R1-3R2}{2}}$$

$$\begin{pmatrix} I & 0 & \frac{3B-2A}{5} \\ 0 & I & \frac{3A-2B}{5} \end{pmatrix} \tag{3.9.17.10}$$

From (3.9.17.10),

$$X = \frac{3B-2A}{5} = \begin{pmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{pmatrix} \tag{3.9.17.11}$$

$$Y = \frac{3A-2B}{5} = \begin{pmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{pmatrix} \tag{3.9.17.12}$$

18. Find X if Y=$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and 2X+Y=$\begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$

Solution: Let,

$$Y = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = M \tag{3.9.18.1}$$

$$2X + Y = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} = N \tag{3.9.18.2}$$

Expressing the matrices (3.9.18.1), (3.9.18.2) in the vector form,

$$\begin{pmatrix} 0 & I \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = M \tag{3.9.18.3}$$

$$\begin{pmatrix} 2I & I \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = N \tag{3.9.18.4}$$

Combining both the equations into a single matrix equation and constructing the augmented matrix,

$$\begin{pmatrix} 0 & I & M \\ 2I & I & N \end{pmatrix} \tag{3.9.18.5}$$

Transforming (3.9.18.5) using row reduction,

$$\begin{pmatrix} 0 & I & M \\ 2I & I & N \end{pmatrix} \xleftrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 2I & I & N \\ 0 & I & M \end{pmatrix} \tag{3.9.18.6}$$

$$\xleftrightarrow{R1 \leftarrow R1-R2} \begin{pmatrix} 2I & 0 & N-M \\ 0 & I & M \end{pmatrix} \xleftrightarrow{R1 \leftarrow \frac{R1}{2}} \begin{pmatrix} I & 0 & \frac{N-M}{2} \\ 0 & I & M \end{pmatrix} \tag{3.9.18.7}$$

From (3.9.18.7),

$$X = \frac{N-M}{2} = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} \tag{3.9.18.8}$$

19. Find x and y,if $2\begin{pmatrix}1 & 3\\ 0 & x\end{pmatrix}+\begin{pmatrix}y & 0\\ 1 & 2\end{pmatrix}=\begin{pmatrix}5 & 6\\ 1 & 8\end{pmatrix}$

20. Solve the equation for x,y,z and t,if $2\begin{pmatrix}x & z\\ y & t\end{pmatrix}+3\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}=3\begin{pmatrix}3 & 5\\ 4 & 6\end{pmatrix}$

21. If $x=\begin{pmatrix}2\\ 3\end{pmatrix}+y\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}10\\ 5\end{pmatrix}$,find the values of x and y.

22. Given $3\begin{pmatrix}x & y\\ z & w\end{pmatrix}=\begin{pmatrix}x & 6\\ -1 & 2w\end{pmatrix}+\begin{pmatrix}4 & x+y\\ z+w & 3\end{pmatrix}$,find the values of x,y,z and w.

23. If $F(x)=\begin{pmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{pmatrix}$,show that F(x)F(y)=F(x+y)
Solution: Given,

$$F(x)=\begin{pmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3.9.23.1}$$

Replacing x with y in above equation 2.0.1 F(y) is given as,

$$F(y)=\begin{pmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3.9.23.2}$$

From the problem statement,
Consider the LHS,
F(x)F(y)

$$=\begin{pmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3.9.23.3}$$

$$=\begin{pmatrix}\cos x\cos y-\sin x\sin y & -\cos x\sin y-\sin x\cos y & 0\\ \sin x\cos y+\cos x\sin y & -\sin x\sin y+\cos x\cos y & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3.9.23.4}$$

$$=\begin{pmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3.9.23.5}$$

$$=F(x+y) \tag{3.9.23.6}$$

Therefore,

$$F(x)F(y)=F(x+y) \tag{3.9.23.7}$$

24. Show that

(i) $\begin{pmatrix}5 & -1\\ 6 & 7\end{pmatrix}\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}\neq\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & -1\\ 6 & 7\end{pmatrix}$

(ii) $\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{pmatrix}\neq\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{pmatrix}$

Solution: Let the two matrices be $\mathbf{A}=\begin{pmatrix}5 & -1\\ 6 & 7\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$. From the problem we have to prove the following

$$\mathbf{AB}\neq\mathbf{BA} \tag{3.9.24.1}$$

At first we compute left hand side of **??**.

$$\mathbf{AB}=\begin{pmatrix}5 & -1\\ 6 & 7\end{pmatrix}\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix} \tag{3.9.24.2}$$

$$\implies \mathbf{AB}=\begin{pmatrix}5\times2-1\times3 & 5\times1-1\times4\\ 6\times2+7\times3 & 6\times1+7\times4\end{pmatrix} \tag{3.9.24.3}$$

$$\implies \mathbf{AB}=\begin{pmatrix}7 & 1\\ 33 & 34\end{pmatrix} \tag{3.9.24.4}$$

Next, we compute right hand side of **??**.

$$\mathbf{BA}=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & -1\\ 6 & 7\end{pmatrix} \tag{3.9.24.5}$$

$$\implies \mathbf{BA}=\begin{pmatrix}2\times5+1\times6 & 2\times(-1)+1\times7\\ 3\times5+4\times6 & 3\times(-1)+4\times7\end{pmatrix} \tag{3.9.24.6}$$

$$\implies \mathbf{BA}=\begin{pmatrix}16 & 5\\ 39 & 25\end{pmatrix} \tag{3.9.24.7}$$

Clearly we can see from equation **??** and **??** that the resultant matrices are not equal. Hence proved,

$$\begin{pmatrix}5 & -1\\ 6 & 7\end{pmatrix}\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}\neq\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & -1\\ 6 & 7\end{pmatrix}$$

Let's name the ,matrices as $\mathbf{M}=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix}-1 & 1 & 0\\ -1 & 1 & 0\\ 2 & 3 & 4\end{pmatrix}$

To prove that multiplication is non

commutative we have to show that

$$\mathbf{MN} \neq \mathbf{NM} \tag{3.9.24.8}$$

$$\mathbf{MN} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 4 \end{pmatrix} \tag{3.9.24.9}$$

$$= \begin{pmatrix} 1\times -1 + 2\times -1 + 3\times 2 & 1\times 1 + 2\times 1 + 3\times 3 & 1\times 0 + 2\times 0 + 3\times 4 \\ 0\times -1 + 1\times -1 + 0\times 2 & 0\times 1 + 1\times 1 + 0\times 3 & 0\times 0 + 1\times 0 + 0\times 4 \\ 1\times -1 + 1\times -1 + 0\times 2 & 1\times 1 + 1\times 1 + 0\times 3 & 1\times 0 + 1\times 0 + 0\times 4 \end{pmatrix} \tag{3.9.24.10}$$

$$\mathbf{MN} = \begin{pmatrix} 3 & 12 & 12 \\ -1 & 1 & 0 \\ -2 & 2 & 0 \end{pmatrix} \tag{3.9.24.11}$$

$$\mathbf{NM} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{3.9.24.12}$$

$$= \begin{pmatrix} -1\times 1 + 1\times 0 + 0\times 1 & -1\times 2 + 1\times 1 + 0\times 1 & -1\times 3 + 1\times 0 + 0\times 0 \\ -1\times 1 + 1\times 0 + 0\times 1 & -1\times 2 + 1\times 1 + 0\times 1 & -1\times 3 + 1\times 0 + 0\times 0 \\ 2\times 1 + 3\times 0 + 4\times 1 & 2\times 2 + 3\times 1 + 4\times 1 & 2\times 3 + 3\times 0 + 4\times 0 \end{pmatrix} \tag{3.9.24.13}$$

$$\mathbf{NM} = \begin{pmatrix} -1 & -1 & -3 \\ -1 & -1 & -3 \\ 6 & 11 & 6 \end{pmatrix} \tag{3.9.24.14}$$

From the above, we can clearly see that R.H.S $\neq$ L.H.S and Hence, Matrix multiplication is non commutative

25. Find $A^2 - 5A + 6I$,if A $= \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution: Splitting matrix as (A-3I)(A-5I) we have

$$A - 3I = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \tag{3.9.25.1}$$

$$A - 5I = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \tag{3.9.25.2}$$

Multiplying the above,

$$A^2 - 5A + 6I = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix} \tag{3.9.25.3}$$

$$A^2 - 5A + 6I = \begin{pmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{pmatrix} \tag{3.9.25.4}$$

26. If A$=\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$,prove that $A^3 - 6A^2 + 7A + 2I = 0$

Solution: The characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{3.9.26.1}$$

$$\implies \det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{pmatrix} = 0 \tag{3.9.26.2}$$

$$\implies (1-\lambda)(2-\lambda)(3-\lambda) + 2(-2(2-\lambda)) = 0 \tag{3.9.26.3}$$

$$\implies \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0 \tag{3.9.26.4}$$

The above equation is similar to the equation to be proved.

By the Cayley-Hamilton theorem, every square matrix satisfies its own characteristic equation. Hence proved that $\mathbf{A^3 - 6A^2 + 7A + 2I = 0}$.

27. If A$=\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ and I$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,find k so that $A^2 = kA - 2I$

Solution:

For a general square matrix **A** of size n x n, the characteristic equation in variable λ is defined by,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{3.9.27.1}$$

where, **I** is the identity matrix of size n x n. Hence, given $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$, the characteristic equation is computed as follows:

$$\left| \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \tag{3.9.27.2}$$

$$\implies \begin{vmatrix} 3-\lambda & -2 \\ 4 & -2-\lambda \end{vmatrix} = 0 \tag{3.9.27.3}$$

By expanding the above determinant we get,

$$(3-\lambda)(-2-\lambda)+8=0 \quad (3.9.27.4)$$

$$\implies -6+\lambda^2+2\lambda-3\lambda+8=0 \quad (3.9.27.5)$$

$$\implies \lambda^2-\lambda+2=0 \quad (3.9.27.6)$$

$$\implies \lambda^2=\lambda-2 \quad (3.9.27.7)$$

Here, (3.9.27.7) is the required characteristic equation and hence, matrix **A** satisfies the characteristic equation according to the Cayley-Hamilton Theorem. By comparing the coefficients of the equations in (**??**) and (3.9.27.7), we can infer that the value of $k=1$.

28. If A=$\begin{pmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{pmatrix}$ and I is the identity matrix of order 2,show that I+A=(I-A)$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Solution:
Since

$$\mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.9.28.1)$$

$$\mathbf{A}=\tan\frac{\alpha}{2}\begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} \quad (3.9.28.2)$$

$$\mathbf{I}-\mathbf{A}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\tan\frac{\alpha}{2}\begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} \quad (3.9.28.3)$$

$$=\frac{1}{\cos\frac{\alpha}{2}}\begin{pmatrix} \cos\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} \end{pmatrix}-\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} \quad (3.9.28.4)$$

$$=\frac{1}{\cos\frac{\alpha}{2}}\begin{pmatrix} \cos\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} \end{pmatrix}-\frac{1}{\cos\frac{\alpha}{2}}\begin{pmatrix} 0 & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & 0 \end{pmatrix} \quad (3.9.28.5)$$

$$=\frac{1}{\cos\frac{\alpha}{2}}\begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \quad (3.9.28.6)$$

The matrix $\mathbf{I}-\mathbf{A}$ is a rotational Matrix with rotation $-\frac{\alpha}{2}$

The Matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is also a rotational Matrix with an angle $+\alpha$.

Multiplying two rotational matrices gives the resultant rotational matrix $+\alpha-\frac{\alpha}{2}=+\frac{\alpha}{2}$

$$RHS=\left(I-A\right)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & cos\alpha \end{pmatrix} \quad (3.9.28.7)$$

$$=\frac{1}{\cos\frac{\alpha}{2}}\begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & cos\alpha \end{pmatrix} \quad (3.9.28.8)$$

$$=\frac{1}{\cos\frac{\alpha}{2}}\begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \quad (3.9.28.9)$$

(3.9.28.10)

Solving LHS= $\mathbf{I}+\mathbf{A}$

$$\mathbf{I}+\mathbf{A}=\begin{pmatrix} 1 & \tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix} \quad (3.9.28.11)$$

$$=\frac{1}{\cos\frac{\alpha}{2}}\begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \quad (3.9.28.12)$$

This term is a rotational Matrix with angle $+\frac{\alpha}{2}$.Hence both sides evaluates to be a rotational matrix with angle $+\frac{\alpha}{2}$.

29. A trust fund has ₹30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:
(a) ₹1800 (b)₹2000

Solution: Let ₹30000 be divided into two part x_1 and x_2 in part **a**),and into two part y_1 and y_2 in part **b**). Then x_1, x_2, y_1, y_2 satisfies following equations

$$x_1+x_2=30000 \quad (3.9.29.1)$$

$$0.05x_1+0.07x_2=1800 \quad (3.9.29.2)$$

$$y_1+y_2=30000 \quad (3.9.29.3)$$

$$0.05y_1+0.07y_2=2000 \quad (3.9.29.4)$$

From (3.9.29.1) and (3.9.29.2) we get

$$\begin{pmatrix} 1 & 1 \\ 0.05 & 0.07 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 30000 \\ 1800 \end{pmatrix} \quad (3.9.29.5)$$

and from (3.9.29.3) and (3.9.29.4) we get

$$\begin{pmatrix} 1 & 1 \\ 0.05 & 0.07 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 30000 \\ 2000 \end{pmatrix} \quad (3.9.29.6)$$

Combining the two we get

$$\begin{pmatrix}1 & 1\\0.05 & 0.07\end{pmatrix}\begin{pmatrix}x_1 & y_1\\x_2 & y_2\end{pmatrix}=\begin{pmatrix}30000 & 30000\\1800 & 2000\end{pmatrix}$$

$$\xleftrightarrow{R_2=R_2-0.05R_1}\begin{pmatrix}1 & 1\\0 & 0.02\end{pmatrix}\begin{pmatrix}x_1 & y_1\\x_2 & y_2\end{pmatrix}=\begin{pmatrix}30000 & 30000\\300 & 500\end{pmatrix}$$

$$\xleftrightarrow{R_2=50R_2}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}x_1 & y_1\\x_2 & y_2\end{pmatrix}=\begin{pmatrix}30000 & 30000\\15000 & 25000\end{pmatrix}$$

$$\xleftrightarrow{R_1=R_1-R_2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x_1 & y_1\\x_2 & y_2\end{pmatrix}=\begin{pmatrix}15000 & 5000\\15000 & 25000\end{pmatrix} \tag{3.9.29.7}$$

From (3.9.29.7) we get x_1=₹15000, x_2=₹15000, y_1=₹5000 and y_2=₹25000.Therefore to get an annual total interest of ₹1800 trust must invest ₹15000 in first bond and ₹15000 in second bond and to get an annual interest of ₹2000 trust must invest ₹5000 in first bond and ₹25000 in second bond.

30. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹80, ₹60 and ₹40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

    Assume X,Y,Z,W and P are matrices of orders $2\times n$,$3\times k$,$2\times p$,$n\times 3$ and $p\times k$,respectively. Choose the correct answer in Exercise 31 and 32.

31. The restriction on n,k and p so that PY+WY will be defined are:
    (A)k=3,p=n
    (B)k is arbitrary,p=2
    (C)p is arbitrary,k=3
    (D)k=2,p=3

32. If n=p,then the order of the matrix 7X-5Z is:
    (A)$p\times 2$ (B)$2\times n$ (C)$n\times 3$ (D)$p\times n$

33. Find the transpose of each of the following matrices:
    (i)$\begin{pmatrix}5\\\frac{1}{2}\\-1\end{pmatrix}$
    (ii)$\begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}$
    (iii)$\begin{pmatrix}-1 & 5 & 6\\\sqrt{3} & 5 & 6\\2 & 3 & -1\end{pmatrix}$

34. If A=$\begin{pmatrix}-1 & 2 & 3\\5 & 7 & 9\\-1 & 1 & 1\end{pmatrix}$ and B=$\begin{pmatrix}-4 & 1 & -5\\1 & 2 & 0\\1 & 3 & 1\end{pmatrix}$,then verify that
    (i)$(A+B)'=A'+B'$
    (ii) $(A-B)'=A'-B'$

35. If A'=$\begin{pmatrix}3 & 4\\-1 & 2\\0 & 1\end{pmatrix}$ and B=$\begin{pmatrix}-1 & 2 & 1\\1 & 2 & 3\end{pmatrix}$,then verify that
    (i) $(A+B)'=A'+B'$ (ii)$(A-B)'=A'-B'$

36. IfA'=$\begin{pmatrix}-2 & 3\\1 & 2\end{pmatrix}$ and B=$\begin{pmatrix}-1 & 0\\1 & 2\end{pmatrix}$,then find that $(A+2B)'$

37. For the matrices A and B,verify that $(AB)'$=$B'A'$,where
    (i)A=$\begin{pmatrix}1\\-4\\3\end{pmatrix}$,B=$\begin{pmatrix}-1 & 2 & 1\end{pmatrix}$
    (ii)A=$\begin{pmatrix}0\\1\\2\end{pmatrix}$,B=$\begin{pmatrix}1 & 5 & 7\end{pmatrix}$

38. If (i) A=$\begin{pmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{pmatrix}$, then verify that $A'A=I$
    (ii) If A=$\begin{pmatrix}\sin\alpha & \cos\alpha\\-\cos\alpha & \sin\alpha\end{pmatrix}$,then verify that $A'A=I$

    Solution:

    a)

$$A=\begin{pmatrix}\cos\alpha & \sin\alpha\\\text{-}\sin\alpha & \cos\alpha\end{pmatrix} \tag{3.9.38.1}$$

$$\implies A^T=\begin{pmatrix}\cos\alpha & \text{-}\sin\alpha\\\sin\alpha & \cos\alpha\end{pmatrix} \tag{3.9.38.2}$$

$$\begin{pmatrix} a1 \\ a2 \end{pmatrix} = \begin{pmatrix} a1 & \text{-}a2 \\ a2 & a1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.9.38.3)$$

$$i.e\ A = \begin{pmatrix} \cos\alpha \\ \text{-}\sin\alpha \end{pmatrix} \quad (3.9.38.4)$$

$$A^T = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \quad (3.9.38.5)$$

$$\implies A^T A = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \text{-}\sin\alpha \end{pmatrix} \quad (3.9.38.6)$$

$$\implies A^T A = I \quad (3.9.38.7)$$

b)

$$A = \begin{pmatrix} \sin\alpha & \cos\alpha \\ \text{-}\cos\alpha & \sin\alpha \end{pmatrix} \quad (3.9.38.8)$$

$$\implies A^T = \begin{pmatrix} \sin\alpha & \text{-}\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix} \quad (3.9.38.9)$$

$$i.e\ A = \begin{pmatrix} \sin\alpha \\ \text{-}\cos\alpha \end{pmatrix} \quad (3.9.38.10)$$

$$A^T = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \quad (3.9.38.11)$$

$$\implies A^T A = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \begin{pmatrix} \sin\alpha \\ \text{-}\cos\alpha \end{pmatrix} \quad (3.9.38.12)$$

$$\implies A^T A = I \quad (3.9.38.13)$$

Hence Proved for both (1) and (2).

39. (i) Show that the matrix A=$\begin{pmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{pmatrix}$ is a symmetric matrix.

    (ii) Show that the matrix A=$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ is a skew symmetric matrix.

40. For the matrix A=$\begin{pmatrix} 1 & 5 \\ 6 & 7 \end{pmatrix}$,verify that

    (i)$(A + A')$ is a symmetric matrix

    (ii)$(A - A')$ is a skew symmetric matrix

41. Find $\frac{1}{2}(A + A')$and $\frac{1}{2}(A - A')$,when A=$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$

42. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

    (i) $\begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix}$

    (ii) $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

    (iii) $\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$

    (iv) $\begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}$

    Choose the correct answer in question number 43 and 44

43. If A,B are symmetric matrices of same order,then AB-BA is a

    (A)Skew symmetric matrix
    (B)Symmetric matrix
    (C)Zero matrix
    (D)Identity matrix

44. If A=$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$,and $A + A' = I$,then the value of α is

    (A) $\frac{\pi}{6}$
    (B)$\frac{\pi}{3}$
    (C) π
    (D)$\frac{3\pi}{2}$

    Solution:

    The Complex number equivalent to the matrix is:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \implies \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.9.44.1)$$

$$\mathbf{A} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \mathbf{A}^\top = \begin{pmatrix} a \\ -b \end{pmatrix} \quad (3.9.44.2)$$

And addition of $\mathbf{A}$ with $\mathbf{A}^\top$ results in :

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} \tag{3.9.44.3}$$

So, According to the given question $\mathbf{A} + \mathbf{A}^\top$ is :

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} = \begin{pmatrix} 2\cos\alpha \\ 0 \end{pmatrix} \tag{3.9.44.4}$$

Given that $\mathbf{A} + \mathbf{A}^\top = \mathbf{I}$:

$$\begin{pmatrix} 2\cos\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.9.44.5}$$

That Implies,

$$2\cos\alpha = 1 \implies \cos\alpha = \frac{1}{2} \tag{3.9.44.6}$$

As per the cosine values, the angle α is :

$$\alpha = \frac{\pi}{3} = 1.047 \tag{3.9.44.7}$$

Using elementary transforamtions,find the inverse of each of the matrices,if it exists questions 45-61

45. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$
46. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
47. $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$
48. $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$
49. $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$
50. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$
51. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$
52. $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$
53. $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$
54. $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$
55. $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$
56. $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$
57. $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$
58. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$
59. $\begin{pmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}$

Solution:

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 3 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 3 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \tag{3.9.59.1}$$

$$\xleftrightarrow{C_2 \leftarrow C_2 + C_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 1 & 1 \end{array}\right) \tag{3.9.59.2}$$

$$\xleftrightarrow{C_1 \leftarrow C_3 - C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 0 & 0 \\ 1 & 5 & 3 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \end{array}\right) \tag{3.9.59.3}$$

$$\xleftrightarrow{C_3 \leftarrow C_3 - 3C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3 \\ 1 & 5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 5 & 1 & 1 & -2 \end{array}\right) \tag{3.9.59.4}$$

$$\xleftrightarrow{C_3 \leftarrow \frac{1}{5}C_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 & -2/5 \end{array}\right) \tag{3.9.59.5}$$

$$\xleftrightarrow{C_2 \leftarrow \frac{1}{5}C_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 1 & 0 & 0 & 1/5 & 0 \\ -1 & 0 & 1 & 1 & 1/5 & -2/5 \end{array}\right) \tag{3.9.59.6}$$

$$\xleftrightarrow{C_1 \leftarrow C_1 - C_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 1 & 0 & -1/5 & 1/5 & 0 \\ -1 & 0 & 1 & 4/5 & 1/5 & -2/5 \end{array}\right) \tag{3.9.59.7}$$

$$\xleftrightarrow{C_1 \leftarrow C_1 + C_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/5 & 0 & 3/5 \\ 0 & 1 & 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & 1 & 2/5 & 1/5 & -2/5 \end{array}\right) \tag{3.9.59.8}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{pmatrix} \tag{3.9.59.9}$$

60. $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix}$
Solution:

$$Let\ \mathbf{A} = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix} \tag{3.9.60.1}$$

Therefore the augmented matrix can be represented as follows :

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ -3 & 0 & -5 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \tag{3.9.60.2}$$

Applying elementary transformations on **A** as follows:

$$\xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + 3R_1} \left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 9 & -11 & 3 & 1 & 0 \\ 0 & -1 & 4 & -2 & 0 & 1 \end{array}\right) \tag{3.9.60.3}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - 3R_2]{R_2 \leftrightarrow -R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 10 & -5 & 0 & 3 \\ 0 & 1 & -4 & 2 & 0 & -1 \\ 0 & 9 & -11 & 3 & 1 & 0 \end{array}\right) \tag{3.9.60.4}$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3}{25}]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 10 & -5 & 0 & 3 \\ 0 & 1 & -4 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{array}\right) \tag{3.9.60.5}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 + 4R_3]{R_1 \leftarrow R_1 - 10R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{-2}{5} & \frac{-3}{5} \\ 0 & 1 & 0 & \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{array}\right) \tag{3.9.60.6}$$

Therefore $\mathbf{A^{-1}}$ is as follows:

$$\begin{pmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{pmatrix} \tag{3.9.60.7}$$

61. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$
Solution: The augmented matrix $[A|I]$ is as given below:-

$$\begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \tag{3.9.61.1}$$

We apply the elementary row operations on $[A|I]$ as follows :-

$$[A|I] = \begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \tag{3.9.61.2}$$

$$\xrightarrow{R_2 \leftarrow 2R_2 - 5R_1} \begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \tag{3.9.61.3}$$

$$\xrightarrow{R_3 \leftarrow 2R_3 - R_2} \begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{pmatrix} \tag{3.9.61.4}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{pmatrix} \tag{3.9.61.5}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{R_3}{2}]{R_2 \leftarrow R_2 - \frac{5}{2}R_3} \begin{pmatrix} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{pmatrix} \tag{3.9.61.6}$$

By performing elementary transormations on augmented matrix$[A|I]$, we obtained the augmented matrix in the form $[I|B]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \tag{3.9.61.7}$$

62. Matrices Aand B will be inverse of each other only if
(A)AB=BA (B)AB=BA=0
(C)AB=0,BA=I (D)AB=BA=I

63. If A=$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$,
prove that A^n=$\begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}$,$n\epsilon N$
Solution: The above problem can be proven

by the method of induction

$$A^2 = AA = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3^1 & 3^1 & 3^1 \\ 3^1 & 3^1 & 3^1 \\ 3^1 & 3^1 & 3^1 \end{pmatrix} \tag{3.9.63.1}$$

$$A^3 = A^2A = \begin{pmatrix} 3^2 & 3^2 & 3^2 \\ 3^2 & 3^2 & 3^2 \\ 3^2 & 3^2 & 3^2 \end{pmatrix} \tag{3.9.63.2}$$

let this be true for n=k,then

$$A^k = \begin{pmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{pmatrix} \tag{3.9.63.3}$$

$$A^{k+1} = A^kA = \begin{pmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.9.63.4}$$

$$A^{k+1} = \begin{pmatrix} 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \end{pmatrix} \tag{3.9.63.5}$$

$$A^{k+1} = \begin{pmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{pmatrix} \tag{3.9.63.6}$$

therefore it holds for all n$\in \mathbb{N}$

$$A^n = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}, n \in \mathbb{N} \tag{3.9.63.7}$$

64. Let A=$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,show that $(aI + bA)^n = a^nI + na^{n-1}bA$,where I is the identity matrix of order 2 and $n\epsilon N$

65. If A=$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$,
then prove that A^n=$\begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$,where n is any positive integer

66. If A and B are symmetric matrices,prove that AB-BA is a skew symmetric matrix.

67. Show that the matrix $B^{'}AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric

68. Find the values of x,y,z if the matrix A=$\begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$ satisfy the equation $A^{'}A$=I

69. For what values of x: $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ x \end{pmatrix}$=0?

Solution: Below is the solution :

$$\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ x \end{pmatrix} = 0 \tag{3.9.69.1}$$

$$\implies \begin{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 \end{pmatrix}\end{pmatrix}\begin{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} & \begin{pmatrix} 2 \end{pmatrix}\end{pmatrix}\begin{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} x \end{pmatrix}\end{pmatrix} = 0 \tag{3.9.69.2}$$

$$\implies \begin{pmatrix}\begin{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \\ + \\ \begin{pmatrix} 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\end{pmatrix} & \begin{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ + \\ \begin{pmatrix} 1 \end{pmatrix}\begin{pmatrix} 2 \end{pmatrix}\end{pmatrix}\end{pmatrix}\begin{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} x \end{pmatrix}\end{pmatrix} = 0 \tag{3.9.69.3}$$

$$\implies \begin{pmatrix}\begin{pmatrix} 5 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} & \begin{pmatrix} 2+2 \end{pmatrix}\end{pmatrix}\begin{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} x \end{pmatrix}\end{pmatrix} = 0 \tag{3.9.69.4}$$

$$\implies \begin{pmatrix}\begin{pmatrix} 6 & 2 \end{pmatrix} & \begin{pmatrix} 4 \end{pmatrix}\end{pmatrix}\begin{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} x \end{pmatrix}\end{pmatrix} = 0 \tag{3.9.69.5}$$

$$\implies \left(\begin{pmatrix} 6 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \end{pmatrix}\begin{pmatrix} x \end{pmatrix}\right) = 0 \tag{3.9.69.6}$$

$$\implies \begin{pmatrix} 4 + 4 \times x \end{pmatrix} = 0 \tag{3.9.69.7}$$

$$\implies 4 \times x = -4 \tag{3.9.69.8}$$

$$\implies x = -1 \tag{3.9.69.9}$$

70. If A=$\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$,show that A^2-5A+7I=0

71. Find x, if $\begin{pmatrix} x & -5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ 4 \\ 1 \end{pmatrix}=0$

72. A manufactrer produces three products x,y,z which he sells in two markets. Annual sales are indicated below:

Market	Products		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x,y and z are ₹2.50,₹1.50 and ₹1.00 respectively,find the total revenue in each market with the help of matrix algebra.
(b) If the unit cost of the above three commodities are ₹2.00,₹1.00 and 50 paise respectively.Find the gross profit.

73. Find the matrix X so that X$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$=$\begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

74. If A and B are square matrices of the same order such that AB=BA,then prove by indication that $AB^n = B^nA$.Further prove that $(AB)^n = A^nB^n$ for all $n \epsilon N$.
Choose the correct answer in the following questions:

75. If A=$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $A^2 = I$,then
(A)$1 + \alpha^2 + \beta\gamma = 0$ (B)$1 - \alpha 2 + \beta\gamma = 0$
(C)$1 - \alpha^2 - \beta\gamma = 0$ (D)$1 + \alpha^2 - \beta\gamma = 0$

76. If the matrix A is both symmetric and skew symmetric,then
(A) A is a diagonal matrix
(B) A is a zero matriz
(C)A is a square matrix
(D)None of these

77. If A is square matrix such that $A^2 = A$,then $(I + A)^3 - 7A$ is equal to
(A)A
(B)I-A
(C)I
(D)3A

3.10 Determinants

1. Find $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$
Solution: For a 2×2 matrix, the determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (3.10.1.1)$$

$$\implies \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 18 \quad (3.10.1.2)$$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$
Solution:
a)

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1 \quad (3.10.2.1)$$

b) The following python code calculates the determinant

```
solutions/2/codes/line_ex/determinants/det.
    py
```

$$\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = x^3 - x^2 + 2 \quad (3.10.2.2)$$

3. If$\mathbf{A} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$,then show that $|2\mathbf{A}| = 4\,|\mathbf{A}|$
Solution:
Determinant of a (2×2) matrix is calculated as follows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (3.10.3.1)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \quad (3.10.3.2)$$

$$\mathbf{2A} = \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix} \quad (3.10.3.3)$$

Using (3.10.3.1) ,(3.10.3.2), (3.10.3.3)

$$|A| = 2 - 8 = -6 \implies 4|A| = -24 \quad (3.10.3.4)$$

$$|2A| = 8 - 32 = -24 \quad (3.10.3.5)$$

$$\implies |2A| = 4|A| \quad (3.10.3.6)$$

4. If $\mathbf{A} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$, then show that $|3\mathbf{A}| = 27\,|\mathbf{A}|$
Solution:

$$\left|3\mathbf{A}\right| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix} = 108 \quad (3.10.4.1)$$

and

$$\left|\mathbf{A}\right| = 4 \tag{3.10.4.2}$$

Hence,

$$\left|3\mathbf{A}\right| = 27\left|\mathbf{A}\right| \tag{3.10.4.3}$$

In general, for an $n \times n$ matrix $\mathbf{A}$,

$$\left|k\mathbf{A}\right| = k^n\left|\mathbf{A}\right| \tag{3.10.4.4}$$

5. Evaluate the determinants
   a) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$
   b) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

   Solution:
   The following python code computes the required determinant value.

   ./solutions/5/codes/lines/q14.py

   i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = -12$
   ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = -46$
   iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0$
   iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 5$

   c) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$
   d) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6. If A=$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$

   Solution:

$$\begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 5 & 4 & -9 \end{pmatrix} \tag{3.10.6.1}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 5R_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.10.6.2}$$

$$\implies \left|\mathbf{A}\right| = 0 \tag{3.10.6.3}$$

   This is verified by the following python code

   codes/line/determinants/det.py

7. Find the values of x,If
   (i)$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii)$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

   Solution:
   a) Expanding the determinant,

$$-18 = 2x^2 - 24 \tag{3.10.7.1}$$

$$\implies x = \pm\sqrt{3} \tag{3.10.7.2}$$

   b) Following the same steps as above we get,

$$-2 = 5x - 6x \tag{3.10.7.3}$$

$$\implies x = 2 \tag{3.10.7.4}$$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to
   a) 6
   b) ± 6
   c) -6
   d) 0

9. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

   Solution: Given determinant: $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$=0

   Applying transformation:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3 - C_2} \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} \tag{3.10.9.1}$$

$$\begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3 - C_1} \begin{vmatrix} x & a & 0 \\ y & b & 0 \\ z & c & 0 \end{vmatrix} \tag{3.10.9.2}$$

   From 3.10.9.2 If any row or column of determinant is zero ,than it's value is zero

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & 0 \\ y & b & 0 \\ z & c & 0 \end{vmatrix} = 0 \qquad (3.10.9.3)$$

10. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

$$\text{Let, } \left|\mathbf{A}\right| = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \qquad (3.10.10.1)$$

Applying row transformation in above determinant we get,

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \qquad (3.10.10.2)$$

From equation (3.10.10.2) one of the row of $\left|\mathbf{A}\right|$ is zero

$$\implies \left|\mathbf{A}\right| = 0 \qquad (3.10.10.3)$$

11. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3-9C_2} \qquad (3.10.11.1)$$

$$\begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3-C_1} \begin{vmatrix} 2 & 7 & 0 \\ 3 & 8 & 0 \\ 5 & 9 & 0 \end{vmatrix} = 0 \qquad (3.10.11.2)$$

12. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

Solution: Applying transformation:

$$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} \qquad (3.10.12.1)$$

$$= \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ab \\ 1 & ab & ac+bc \end{vmatrix} \qquad (3.10.12.2)$$

$$\xleftrightarrow{C_3 \leftarrow C_3+C_2} \begin{vmatrix} 1 & bc & ab+ac+bc \\ 1 & ca & bc+ab+ca \\ 1 & ab & ac+bc+ab \end{vmatrix} \qquad (3.10.12.3)$$

Taking (ab+bc+ac) common from C3:

$$\Delta = (ab+bc+ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \qquad (3.10.12.4)$$

If any two row or column of determinant is same, then the value of determinant is zero:

$$\Delta = (ab+bc+ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} = 0 \qquad (3.10.12.5)$$

$$\therefore \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0 \qquad (3.10.12.6)$$

Hence proved.

13. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

Solution:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \qquad (3.10.13.1)$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} \qquad (3.10.13.2)$$

$$2 \begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \qquad (3.10.13.3)$$

$$\underset{R_2 \leftarrow R_2-R_1}{\xleftrightarrow{R_3 \leftarrow R_3-R_1}} 2 \begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix} \qquad (3.10.13.4)$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} (-1)\times(-1)\times 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad (3.10.13.5)$$

14. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Solution: From (**??**),We can write $\mathbf{x}$ such that

$$\begin{pmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{pmatrix} \begin{pmatrix} -c \\ b \\ a \end{pmatrix} = \mathbf{0} \qquad (3.10.14.1)$$

So,one of the eigen value is equal to 0. Let

$$\lambda_1 = 0 \tag{3.10.14.2}$$

We know that the

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n \tag{3.10.14.3}$$

From (3.10.14.2) and (3.10.14.3),

$$\det(A) = 0 \tag{3.10.14.4}$$

15. $\begin{vmatrix} -a^2 & ab & ab \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$=$4a^2b^2c^2$

Solution:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \tag{3.10.15.1}$$

$$= abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix} \tag{3.10.15.2}$$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \tag{3.10.15.3}$$

$$\xleftrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2} (-1)(-1)a^2b^2c^2 \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} \tag{3.10.15.4}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} a^2b^2c^2 \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{vmatrix} \tag{3.10.15.5}$$

$$= a^2b^2c^2 \times (1 \times 2 \times 2) \tag{3.10.15.6}$$

$$= 4a^2b^2c^2 \tag{3.10.15.7}$$

By Using properties of determinants, in Exercises 16 to 22,Show that;

16. (i)$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$=(a-b)(b-c)(c-a)

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$=(a-b)(b-c)(c-a)(a+b+c)

Solution: Using column operations to simplify the equation, we get:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \xleftrightarrow{C_1 \to C_1 - C_2} \begin{vmatrix} 0 & 1 & 1 \\ a-b & b & c \\ a^3-b^3 & b^3 & c^3 \end{vmatrix} \tag{3.10.16.1}$$

$$\xleftrightarrow{C_2 \to C_2 - C_3} \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} \tag{3.10.16.2}$$

$$(a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix} \tag{3.10.16.3}$$

$$\xleftrightarrow{C_1 \to C_1 - C_2} (a-b)(b-c) \begin{vmatrix} 0 & 0 \\ 0 & 1 \\ (a^2-c^2)+b(a-c) & b^2+bc+c^2 \end{vmatrix} \tag{3.10.16.4}$$

$$\implies (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ k & b^2+bc+c^2 & c^3 \end{vmatrix} \tag{3.10.16.5}$$

Where

$$k = (a-c)(a+c) + b(a-c) \tag{3.10.16.6}$$

$$(a-b)(b-c)(a-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ a+b+c & b^2+bc+c^2 & c^3 \end{vmatrix} \tag{3.10.16.7}$$

$$= (a-b)(b-c)(a-c)(-1)(a+b+c) \tag{3.10.16.8}$$

$$= (a-b)(b-c)(c-a)(a+b+c) \tag{3.10.16.9}$$

17. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$=(x-y)(y-z)(z-x)(xy+yz+zx)

18. (i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$=$(5x+4)(4-x)^2$

Solution:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \xleftrightarrow{R_1\leftarrow R_1+R_2+R_3} \tag{3.10.18.1}$$

$$(5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \xleftrightarrow[C_2\leftarrow C_2-C_3]{C_1\leftarrow C_1+C_2} \tag{3.10.18.2}$$

$$(5x+4)\begin{vmatrix} 0 & 0 & 1 \\ x-4 & 4-x & 2x \\ 0 & x-4 & x+4 \end{vmatrix} \tag{3.10.18.3}$$

$$(5x+4)(4-x)^2\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 2x \\ 0 & -1 & x+4 \end{vmatrix} \tag{3.10.18.4}$$

Therefore finding the determinant along Row1:

$$= (5x+4)(4-x)^2 \tag{3.10.18.5}$$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} = k^2(3y+k)$

Solution: Given determinant:

$$\Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} \tag{3.10.18.6}$$

Applying transformation:

$$\xleftrightarrow{R_1\leftarrow R_1+R_2+R_3} \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix} \tag{3.10.18.7}$$

$$\xleftrightarrow{C_2\leftarrow C_2-C_1} \begin{vmatrix} 3y+k & 0 & 3y+k \\ y & k & y \\ y & 0 & y+k \end{vmatrix} \tag{3.10.18.8}$$

$$\xleftrightarrow{C_3\leftarrow C_3-C_1} \begin{vmatrix} 3y+k & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix} \tag{3.10.18.9}$$

Taking determinant

$$\implies \Delta = (3y+k)k^2 \tag{3.10.18.10}$$

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} = k^2(3y+k) \tag{3.10.18.11}$$

Hence proved.

19. (i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

20. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

21. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

22. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

Solution:

$$\begin{vmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{vmatrix} \xleftrightarrow[R_3\leftarrow cR_3]{R_1\leftarrow aR_1;R_2\leftarrow bR_2}$$

$$\frac{1}{abc}\begin{vmatrix} a(1+a^2) & a^2b & a^2c \\ ab^2 & b(1+b^2) & b^2c \\ ac^2 & bc^2 & c(1+c^2) \end{vmatrix} \xleftrightarrow[C_3\leftarrow \frac{C_3}{c}]{C_1\leftarrow \frac{C_1}{a};C_2\leftarrow \frac{C_2}{b}}$$

$$\begin{vmatrix} 1+a^2 & a^2 & a^2 \\ b^2 & 1+b^2 & b^2 \\ c^2 & c^2 & 1+c^2 \end{vmatrix} \xleftrightarrow{R_1\leftarrow R_1+R_2+R_3}$$

$$\begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & 1+b^2 & b^2 \\ c^2 & c^2 & 1+c^2 \end{vmatrix} \tag{3.10.22.1}$$

Taking $1 + a^2 + b^2 + c^2$ out from (3.10.22.1),

$$\implies (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & 1 + b^2 & b^2 \\ c^2 & c^2 & 1 + c^2 \end{vmatrix}$$

$$\xleftrightarrow[C_3 \leftarrow C_3 - C_1]{C_2 \leftarrow C_2 - C_1} (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

$= 1 + a^2 + b^2 + c^2$ ($\because$ Determinant of a lower triangle matrix is the product of it's diagonal elements) Choose the correct answer in Exercises 23 and 24.

23. Let A be a square matrix of order 3X3, then $|kA|$ is equal to
    a) $k\,|A|$
    b) $k^2\,|A|$
    c) $k^3\,|A|$
    d) $3k\,|A|$
24. Which of the following is correct
    a) Determinant is a square matrix.
    b) Determinant is a number associated to a matrix.
    c) Determinant is a number associated to a square matrix.
    d) None of these.
25. Find area of the triangle with vertices at the point given in each of the following :
    (i) $\begin{pmatrix} 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 6 & 0 \end{pmatrix}$, $\begin{pmatrix} 4 & 3 \end{pmatrix}$
    (ii) $\begin{pmatrix} 2 & 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 10 & 8 \end{pmatrix}$
    (iii) $\begin{pmatrix} -2 & -3 \end{pmatrix}$, $\begin{pmatrix} 3 & 2 \end{pmatrix}$, $\begin{pmatrix} -1 & -8 \end{pmatrix}$
26. Show that points A=$\begin{pmatrix} a & b + c \end{pmatrix}$, B=$\begin{pmatrix} b & c + a \end{pmatrix}$, C=$\begin{pmatrix} c & a + b \end{pmatrix}$ are collinear.
    Solution: The points **A**, **B** and **C** will be collinear if

$$\begin{pmatrix} \mathbf{A}^{\mathbf{T}} \\ \mathbf{B}^{\mathbf{T}} \\ \mathbf{C}^{\mathbf{T}} \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (3.10.26.1)$$

$$\implies \begin{pmatrix} a & b + c \\ b & c + a \\ c & a + b \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (3.10.26.2)$$

So the augmented matrix of (3.10.26.2) is given by

$$\begin{pmatrix} a & b + c & 1 \\ b & c + a & 1 \\ c & a + b & 1 \end{pmatrix} \qquad (3.10.26.3)$$

Using row reduction we get,

$$\begin{pmatrix} a & b + c & 1 \\ b & c + a & 1 \\ c & a + b & 1 \end{pmatrix} \qquad (3.10.26.4)$$

$$\xleftrightarrow{R_3 = cR_3 - bR_2} \begin{pmatrix} a & b + c & 1 \\ b & c + a & 1 \\ 0 & (b - c)(a + b + c) & b - c \end{pmatrix} \qquad (3.10.26.5)$$

$$\xleftrightarrow{R_3 = \frac{1}{(b - c)}R_3} \begin{pmatrix} a & b + c & 1 \\ b & c + a & 1 \\ 0 & a + b + c & 1 \end{pmatrix} \qquad (3.10.26.6)$$

$$\xleftrightarrow{R_2 = aR_2 - bR_1} \begin{pmatrix} a & b + c & 1 \\ 0 & (a - b)(a + b + c) & a - b \\ 0 & a + b + c & 1 \end{pmatrix} \qquad (3.10.26.7)$$

$$\xleftrightarrow{R_2 = \frac{1}{(a - b)}R_2} \begin{pmatrix} a & b + c & 1 \\ 0 & a + b + c & 1 \\ 0 & a + b + c & 1 \end{pmatrix} \qquad (3.10.26.8)$$

$$\xleftrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} a & b + c & 1 \\ 0 & a + b + c & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad (3.10.26.9)$$

$$(3.10.26.10)$$

From (3.10.26.10) we see that the rank of the augmented matrix is less than 3, hence **A**,**B** and **C** are colinear. We illustrate the concept by an example. Let a=1, b=2 and c=3. The points are **A**=$\begin{pmatrix} 1 & 5 \end{pmatrix}$,**B**=$\begin{pmatrix} 2 & 4 \end{pmatrix}$ and **C**=$\begin{pmatrix} 3 & 3 \end{pmatrix}$.
Below is the diagram of the line passing through the points **A**, **B** and **C**.

27. Find values of k if area of triangle is 4sq.units and vertices are
    (i) $\begin{pmatrix} k & 0 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \end{pmatrix}$
    (ii) $\begin{pmatrix} -2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 0 & k \end{pmatrix}$
28. (i) Find equation of line joining $\begin{pmatrix} 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & 6 \end{pmatrix}$ using determinants.
    (ii) Find equation of line joining $\begin{pmatrix} 3 & 1 \end{pmatrix}$ and

$\begin{pmatrix} 9 & 3 \end{pmatrix}$ using determinants.

29. If the area of triangle is 35 sq.units with vertices $\begin{pmatrix} 2 & -6 \end{pmatrix}$, $\begin{pmatrix} 5 & 4 \end{pmatrix}$ and $\begin{pmatrix} k & 4 \end{pmatrix}$.then k is
    a) 12
    b) -2
    c) -12,-2
    d) 12,-2

Write Minors and Coafactors of the elements of following determinants:

30. (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$
    (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$
31. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$
    (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$
32. Using Cofactors of elements of second row,evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.
33. Using Cofactors of elements of third column ,evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.
34. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij} then value of Δ is given by
    a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$
    b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
    c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$
    d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Find adjoint of each of the matrices

35. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
36. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ Verify A(adjA)=(adjA)A=$|A|\,I$
37. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$
38. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$
39. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$
40. $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$
41. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$
42. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$
43. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$
44. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$
45. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$
46. Let A= $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and B= $\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$
47. Let A $=\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1} **Solution:**
    Multiplying above equation with $\mathbf{A}^{-1}$ on both sides, We get :

$$\implies \mathbf{A}^2\mathbf{A}^{-1} - 5\mathbf{A}\mathbf{A}^{-1} + 7\mathbf{I}\mathbf{A}^{-1} = 0 \tag{3.10.47.1}$$

$$\implies \mathbf{A}\mathbf{A}\mathbf{A}^{-1} - 5\mathbf{I} + 7\mathbf{A}^{-1} = 0 \tag{3.10.47.2}$$

$$\implies \mathbf{A}\mathbf{I} - 5\mathbf{I} + 7\mathbf{A}^{-1} = 0 \tag{3.10.47.3}$$

$$\implies \mathbf{A}^{-1} = \frac{1}{7}(5\mathbf{I} - \mathbf{A}) \tag{3.10.47.4}$$

Solving for $\mathbf{A}^{-1}$, we get :

$$\mathbf{A}^{-1} = \frac{1}{7}\left(5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\right) \tag{3.10.47.5}$$

$$\implies \mathbf{A}^{-1} = \frac{1}{7}\left(\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\right) \tag{3.10.47.6}$$

48. For the matrix A= $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.
    Solution: For a general square matrix (A), the characteristic equation in variable λ is defined

by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{3.10.48.1}$$

$$\implies \begin{vmatrix} 3-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = 0 \tag{3.10.48.2}$$

$$\implies (3-\lambda)(1-\lambda) - 2 = 0 \tag{3.10.48.3}$$

$$\implies \lambda^2 - 4\lambda + 3 - 2 = 0 \tag{3.10.48.4}$$

$$\implies \lambda^2 - 4\lambda + 1 = 0 \tag{3.10.48.5}$$

Now by Cayley-Hamilton Theorem $\mathbf{A}$ satisfies (3.10.48.5),then replacing λ with $\mathbf{A}$ we get

$$\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} = 0 \tag{3.10.48.6}$$

We need to find a and b in

$$\mathbf{A}^2 + a\mathbf{A} + b\mathbf{I} = 0 \tag{3.10.48.7}$$

Comparing (3.10.48.6) and (3.10.48.7) we get a = -4 and b = 1.

49. For the matrix A=$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$.Show that

$$A^3 - 6A^2 + 5A + 11I = 0 \tag{3.10.49.1}$$

and hence find A^{-1}. **Solution:** Given matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix} \tag{3.10.49.2}$$

The characteristic polynomial of the matrix will be According to cayley-hamilton theorem every matrix satisfies it's own characteristic equation. So the matrix A satifies the equation (3.10.49.1)

$$A^3 - 6A^2 + 5A + 11I = 0 \tag{3.10.49.3}$$

The matrix A satisifes the characteristic equation,so

$$A^3 - 6A^2 + 5A + 11I = 0 \tag{3.10.49.4}$$

Multiplying with A^{-1} we get

$$A^2 - 6A + 5I + 11A^{-1} = 0 \tag{3.10.49.5}$$

$$A^{-1} = \frac{1}{11}\left(6A - A^2 - 5I\right) \tag{3.10.49.6}$$

$$A^{-1} = \frac{1}{11}\left[6\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{pmatrix}\right. \tag{3.10.49.7}$$

$$\left. -5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \tag{3.10.49.8}$$

$$A^{-1} = \frac{1}{11}\begin{pmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{pmatrix} \tag{3.10.49.9}$$

$$\implies A^{-1} = \begin{pmatrix} \frac{-3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11} \\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11} \end{pmatrix} \tag{3.10.49.10}$$

50. Let A be a nonsingular square matrix of order 3X3 .Then $|adjA|$ is equal to
   a) $|A|$
   b) $|A|^2$
   c) $|A|^3$
   d) $3|A|$
51. If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to
   a) det(A)
   b) $\frac{1}{det(A)}$
   c) 1
   d) 0

Examine the consistency of the system of given Equations.

52. $x + 2y = 2$ **Solution:** The given system of
$2x + 3y = 3$
equations can be written the matrix equation form as

$$\mathbf{Ax} = \mathbf{b} \tag{3.10.52.1}$$

$$\implies \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{3.10.52.2}$$

The augmented matrix for (3.10.53.2) is row reduced as follows

$$\begin{pmatrix}1 & 2 & 2\\2 & 3 & 3\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-2R_1} \begin{pmatrix}1 & 2 & 2\\0 & -1 & -1\end{pmatrix} \tag{3.10.52.3}$$

$$\xleftrightarrow{R_1\leftarrow R_1+2R_2} \begin{pmatrix}1 & 0 & 0\\0 & -1 & -1\end{pmatrix} \xleftrightarrow{R_2\leftarrow(-1)R_2} \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\end{pmatrix} \tag{3.10.52.4}$$

$$\implies Rank\begin{pmatrix}1 & 2\\2 & 3\end{pmatrix} = Rank\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\end{pmatrix} = 2 \tag{3.10.52.5}$$

Hence, system of equations is consistent and has unique solution $\begin{pmatrix}0\\1\end{pmatrix}$. See Fig. 3.10.52.1

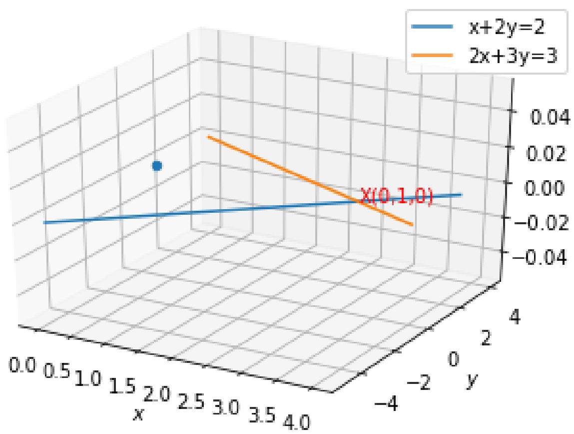

Fig. 3.10.52.1: Intersection of lines x+2y=2 and 2x+3y=3

53. $2x - y = 5$
    $x + y = 4$

Solution: The given system of equations (**??**) can be written the matrix equation form as

$$\mathbf{Ax} = \mathbf{b} \tag{3.10.53.1}$$

$$\implies \begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\4\end{pmatrix} \tag{3.10.53.2}$$

The augmented matrix for (3.10.53.2) is row reduced as follows

$$\begin{pmatrix}2 & -1 & 5\\1 & 1 & 4\end{pmatrix} \tag{3.10.53.3}$$

$$\xleftrightarrow{R_2\leftarrow\frac{2R_2-R_1}{3}} \begin{pmatrix}2 & -1 & 5\\0 & 1 & 1\end{pmatrix} \tag{3.10.53.4}$$

$$\xleftrightarrow{R_1\leftarrow\frac{R_1+R_2}{2}} \begin{pmatrix}1 & 0 & 3\\0 & 1 & 1\end{pmatrix} \tag{3.10.53.5}$$

$$\implies Rank\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix} = Rank\begin{pmatrix}2 & -1 & 5\\1 & 1 & 4\end{pmatrix} \tag{3.10.53.6}$$

$$= 1 < dim\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix} \tag{3.10.53.7}$$

Hence (**??**) is consistent and has unique solution $\begin{pmatrix}3\\1\end{pmatrix}$

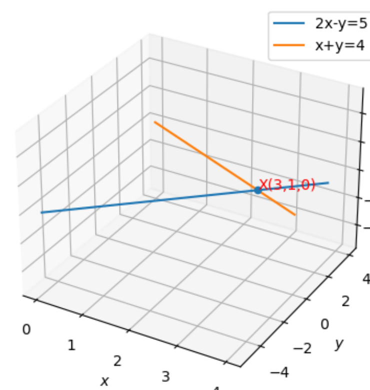

Fig. 3.10.53.1: Intersection of lines 2x-y=5 and x+y=4

54. $x + 3y = 5$
    $2x + 6y = 8$

Solution: If solution exists for the given system of linear equations then they are said to be consistent, otherwise they are inconsistent. we can represent the given lines

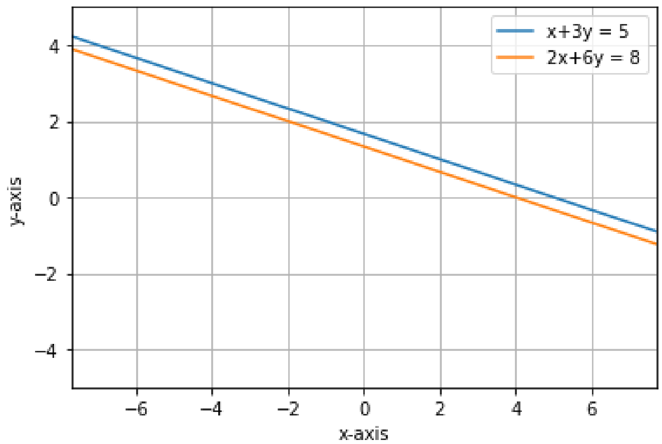

Fig. 3.10.54.1: Plot showing the given two lines are parallel

in the form of

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 5 \quad (3.10.54.1)$$

$$\begin{pmatrix}2 & 6\end{pmatrix}\mathbf{x} = 8 \quad (3.10.54.2)$$

writing the above equations in matrix form

$$\begin{pmatrix}1 & 3 & -5\\ 2 & 6 & -8\end{pmatrix}\mathbf{x} = 0 \quad (3.10.54.3)$$

The matrix equation is row reduced as follows

$$\begin{pmatrix}1 & 3 & -5\\ 2 & 6 & -8\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}1 & 3 & -5\\ 0 & 0 & 2\end{pmatrix} \quad (3.10.54.4)$$

Thus, from the above row reduced form we can conclude that the given system of lines has no solution. Therefore, they are inconsistent. See 3.10.54.1

55. x+y+z=1
2x+3y+2z=2
ax+ay+2az=4

Solution: Assume that a is any real number. The above system of equations can be expressed in the form of matrix:

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ a & a & 2a\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1\\ 2\\ 4\end{pmatrix} \quad (3.10.55.1)$$

This is in the form of:

$$\mathbf{Ax} = \mathbf{B} \quad (3.10.55.2)$$

The system defined above is consistent and has a solution only when

$$rank(\mathbf{A}|\mathbf{B}) = rank(\mathbf{A}) = dim(\mathbf{A}) \quad (3.10.55.3)$$

Reducing the augmented matrix to row echelon form, we get:

$$\left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 2 & 3 & 2 & 2\\ a & a & 2a & 4\end{array}\right) \xleftrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ a & a & 2a & 4\end{array}\right) \quad (3.10.55.4)$$

$$\left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ a & a & 2a & 4\end{array}\right) \xleftrightarrow{R_3 \to R_3 - aR_1} \left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & a & 4-a\end{array}\right) \quad (3.10.55.5)$$

$$\implies rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{B}) = dim(\mathbf{A}) \quad (3.10.55.6)$$

The system of equations is consistent and has a unique solution except at $a = 0$.

56. 3x-y-2z=2
2y-z=-1
3x-5y=3

57. 5x-y+4z=5
2x+3y+5z=2
5x-2y+6z=-1

Solution: The given equations can be written as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix}5 & -1 & 4\\ 2 & 3 & 5\\ 5 & -2 & 6\end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix}5\\ 2\\ -1\end{pmatrix} \quad (3.10.57.1)$$

By row reducing the augmented matrix :

$$\left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 2 & 3 & 5 & 2 \\ 5 & -2 & 6 & -1 \end{array}\right) \tag{3.10.57.2}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow 5R_2 - (R_1 + R_3)} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 15 & 6 \\ 0 & -1 & 2 & -6 \end{array}\right) \tag{3.10.57.3}$$

$$\xleftrightarrow{R_3 \leftarrow 18R_3 + R_2} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 15 & 6 \\ 0 & 0 & 51 & -102 \end{array}\right) \tag{3.10.57.4}$$

$$\xleftrightarrow{R_3 \leftarrow \frac{R_3}{51}} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 15 & 6 \\ 0 & 0 & 1 & -2 \end{array}\right) \tag{3.10.57.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 15R_3} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 0 & 36 \\ 0 & 0 & 1 & -2 \end{array}\right) \tag{3.10.57.6}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{18}} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right) \tag{3.10.57.7}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2 - 4R_3} \left(\begin{array}{ccc|c} 5 & 0 & 0 & 15 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right) \tag{3.10.57.8}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right) \tag{3.10.57.9}$$

$$\implies rank \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix} = rank \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 2 & 3 & 5 & 2 \\ 5 & -2 & 6 & -1 \end{array}\right) = 3 = dim \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix} \tag{3.10.57.10}$$

i.e., the $rank(\mathbf{A}) = rank(\mathbf{A} : \mathbf{b}) = 3$, which is equal to the row size of $\mathbf{x}$. Hence the system of linear equations is consistent, with a unique solution.

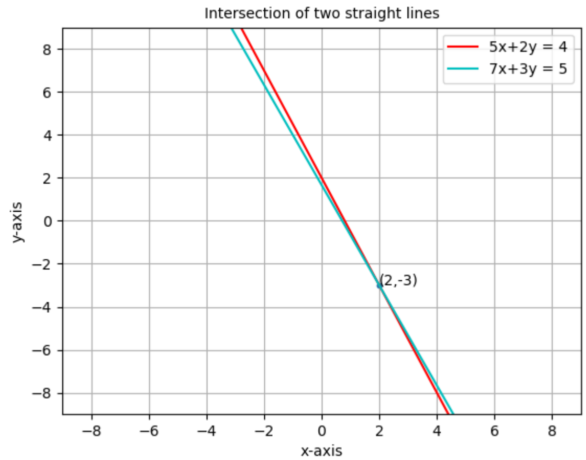

Fig. 3.10.58.1: Intersection of 2 lines

The unique solution is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \tag{3.10.57.11}$$

Solve the system linear equations,using matrix method.

58. $5x + 2y = 4$

$7x + 3y = 5$

Solution: The above equations can be expressed in vector form as,

$$\begin{pmatrix} 5 & 2 \end{pmatrix} \mathbf{x} = 4 \tag{3.10.58.1}$$

$$\begin{pmatrix} 7 & 3 \end{pmatrix} \mathbf{x} = 5 \tag{3.10.58.2}$$

Now, writing it in the matrix form as,

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{3.10.58.3}$$

The augmented matrix of above equation is solved by row reduction as follows

$$\begin{pmatrix} 5 & 2 & 4 \\ 7 & 3 & 5 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - (\frac{7}{5})R_1} \begin{pmatrix} 5 & 2 & 4 \\ 0 & \frac{1}{5} & \frac{-3}{5} \end{pmatrix} \tag{3.10.58.4}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 10R_2} \begin{pmatrix} 5 & 0 & 10 \\ 0 & \frac{1}{5} & \frac{-3}{5} \end{pmatrix} \tag{3.10.58.5}$$

$$\implies \mathbf{x} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{3.10.58.6}$$

See Fig. 3.10.58.1

59. $2x - y = -2$
$3x + 4y = 3$
Solution:
Given equation can be repsented in a matrix form as

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{3.10.59.1}$$

The corresponding augmented matrix is

$$\left(\begin{array}{cc|c} 2 & -1 & -2 \\ 3 & 4 & 3 \end{array}\right) \tag{3.10.59.2}$$

We use the Guass Jordan Elimination method as:

$$\left(\begin{array}{cc|c} 2 & -1 & -2 \\ 3 & 4 & 3 \end{array}\right) \tag{3.10.59.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \left(\begin{array}{cc|c} 2 & -1 & -2 \\ 0 & \frac{11}{2} & 6 \end{array}\right) \tag{3.10.59.4}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2}{11}R_2} \left(\begin{array}{cc|c} 2 & -1 & -2 \\ 0 & 1 & \frac{12}{11} \end{array}\right) \tag{3.10.59.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 2 & 0 & \frac{-10}{11} \\ 0 & 1 & \frac{12}{11} \end{array}\right) \tag{3.10.59.6}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{-5}{11} \\ 0 & 1 & \frac{12}{11} \end{array}\right) \tag{3.10.59.7}$$

Therefore, the values of x and y are:

$$x = \frac{-5}{11} \tag{3.10.59.8}$$

$$y = \frac{12}{11} \tag{3.10.59.9}$$

60. $4x - 3y = 3$
$3x - 5y = 7$
Solution: Writing both euations in matrix form

$$\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = 3 \tag{3.10.60.1}$$

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = 7 \tag{3.10.60.2}$$

Forming the augmented matrix and reducing the matrix to row echelon form:

$$\begin{pmatrix} 4 & -3 & 3 \\ 3 & -5 & 7 \end{pmatrix} \tag{3.10.60.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/4} \begin{pmatrix} 1 & -3/4 & 3/4 \\ 3 & -5 & 7 \end{pmatrix} \tag{3.10.60.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -3/4 & 3/4 \\ 0 & -11/4 & 19/4 \end{pmatrix} \tag{3.10.60.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times -4/11} \begin{pmatrix} 1 & -3/4 & 3/4 \\ 0 & 1 & -19/11 \end{pmatrix} \tag{3.10.60.6}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3/4 \times R_2} \begin{pmatrix} 1 & 0 & -6/11 \\ 0 & 1 & -19/11 \end{pmatrix} \tag{3.10.60.7}$$

Here, $Rank(A) = Rank(A|B)$. Therefore, the system is consistent. Also, there exist a unique solution as $Rank(A) = n$ (number of unknown). From equation 3.10.60.7, we get:

$$\mathbf{x} = \frac{1}{11} \begin{pmatrix} -6 \\ -19 \end{pmatrix} \tag{3.10.60.8}$$

Plotting the lines and the intersection point in Fig. 3.10.60.1

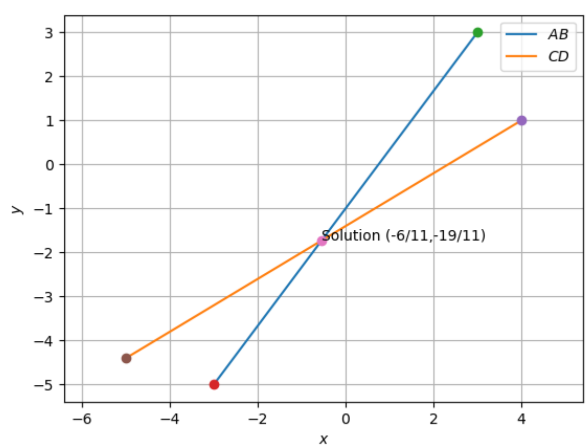

Fig. 3.10.60.1: Lines and their intersection denoting the solution

61. $5x + 2y = 3$
$3x + 2y = 5$

Solution: he given set of equations can be represented in the matrix equation form as

$$\begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{3.10.61.1}$$

The augmented matrix for this system becomes

$$\begin{pmatrix} 5 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix} \tag{3.10.61.2}$$

Row reducing the matrix

$$\begin{pmatrix} 5 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \times \frac{5}{3} - R_1} \begin{pmatrix} 5 & 2 & 3 \\ 0 & \frac{4}{3} & \frac{16}{3} \end{pmatrix} \tag{3.10.61.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{3}{4}} \begin{pmatrix} 5 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix} \tag{3.10.61.4}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2 \times R_2} \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & 4 \end{pmatrix} \tag{3.10.61.5}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \end{pmatrix} \tag{3.10.61.6}$$

$$\implies \text{Rank} \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} = \text{Rank} \begin{pmatrix} 5 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix} = 2 \tag{3.10.61.7}$$

$$= dim \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} \tag{3.10.61.8}$$

So, the given system of equations are consistent with a unique solution of

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{3.10.61.9}$$

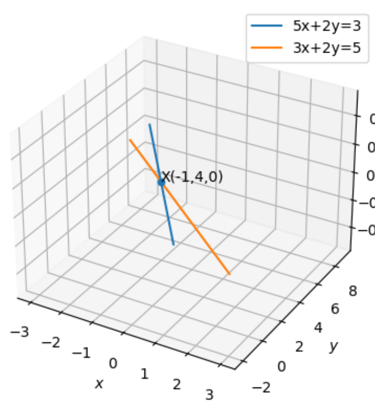

Fig. 3.10.61.1: plot showing intersection of lines

62. 2x+y+z = 1
x-2y-z = $\frac{3}{2}$
3y- 5z = 9

63. x-y+z = 4
2x+y-3z = 0
x+y+z = 2

64. 2x+3y+3z = 5
x-2y+z = -4
3x-y-2z = 3

65. x-y+2z = 7
3x+4y-5z = -5
2x-y+3z = 12

66. If A=$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations
2x-3y+5z = 11,
3x+2y-4z = -5,
x+y-2z =-3.

67. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion,4 kg wheat and 6 kg rice is ₹90.The cost of 6kg onion 2kg wheat and 3kg rice is ₹70.Find the cost of each item per kg by matrix mathod.

68. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ

Solution:

$$\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} \tag{3.10.68.1}$$

Now, Solving the determinant:-

$$\begin{aligned} &= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) \\ &\quad + \cos\theta(-\sin\theta + x\cos\theta) \\ &= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta \\ &\quad - \sin\theta\cos\theta + x\cos^2\theta \end{aligned} \tag{3.10.68.2}$$

$$= -x^3 - x + x\sin^2\theta + x\cos^2\theta \tag{3.10.68.3}$$

$$= -x^3 (\because \sin^2\theta + \cos^2\theta = 1) \tag{3.10.68.4}$$

Hence, the determinant is independent of θ.

69. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

Solution: Proceeding to the solution by taking the left hand side matrix of (**??**) and applying properties of determinants.

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} \tag{3.10.69.1}$$

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} \xleftrightarrow{R_1 \to aR_1} \frac{1}{a}\begin{vmatrix} a^2 & a^3 & abc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} \tag{3.10.69.2}$$

$$\begin{vmatrix} a^2 & a^3 & abc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} \xleftrightarrow{R_2 \to bR_2} \frac{1}{ab}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c & c^2 & ab \end{vmatrix} \tag{3.10.69.3}$$

$$\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c & c^2 & ab \end{vmatrix} \xleftrightarrow{R_3 \to cR_3} \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} \tag{3.10.69.4}$$

$$\frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \leftrightarrow \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \tag{3.10.69.5}$$

$$\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \xleftrightarrow{C_2 \to C_3} -\begin{vmatrix} a^2 & 1 & a^3 \\ b^2 & 1 & b^3 \\ c^2 & 1 & c^3 \end{vmatrix} \xleftrightarrow{C_1 \to C_2} \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \tag{3.10.69.6}$$

The above determinant is equal to right hand side determinant of equation (**??**).
Hence proved:

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & b^3 \end{vmatrix} \tag{3.10.69.7}$$

70. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

Solution:
We first multiply either the rows or the columns, and then try taking the common element out.

$$\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} \underset{C_3 \leftarrow (\sin\alpha)C_3}{\overset{C_3 \leftarrow (\cos\alpha)C_3}{\longleftrightarrow}}$$

$$\left(\frac{1}{\sin\alpha\cos\alpha}\right)\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin^2\alpha\cos\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos^2\alpha\sin\alpha \end{vmatrix} \tag{3.10.70.1}$$

From (3.10.70.1) R_1 we take out common element $\cos\alpha$. And from row R_2 we take out common element $\sin\alpha$

$$\begin{vmatrix} \cos\beta & \sin\beta & -\sin^2\alpha \\ -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & \cos^2\alpha \end{vmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_3}$$
$$\begin{vmatrix} 0 & 0 & -\sin^2\alpha - \cos^2\alpha \\ -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & \cos^2\alpha \end{vmatrix} \tag{3.10.70.2}$$

Now, we can expand the determinant from row 1 in (3.10.70.2), and we get

$$\left(-\sin^2\alpha - \cos^2\alpha\right)\left(-\sin^2\beta - \cos^2\beta\right)$$
$$\implies \left(\sin^2\alpha + \cos^2\alpha\right)\left(\sin^2\beta + \cos^2\beta\right) = 1 \tag{3.10.70.3}$$

Therefore, the determinant of the matrix is 1.

71. If a,b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c=a & a=b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, Show that either a+b+c=0 or a=b=c.

Solution: Given,

$$\triangle = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad (3.10.71.1)$$

$$\xleftrightarrow{C_1 \leftarrow C_1+C_2+C_3} \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} \quad (3.10.71.2)$$

$$= 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} \quad (3.10.71.3)$$

$$\xleftrightarrow{R_1 \leftarrow R_1-R_2; R_2 \leftarrow R_2-R_3} 2(a+b+c) \begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & b+c & c+a \end{vmatrix} = 0 \quad (3.10.71.4)$$

On expanding determinant along first column from equation (3.10.71.4),

$$\implies 2(a+b+c)[(c-b)(b-a)-(a-c)^2] = 0$$

$$\implies 2(a+b+c)(a^2+b^2+c^2 - ab - bc - ca) = 0$$

$$\implies (a+b+c)(2a^2+2b^2+2c^2 - 2ab - 2bc - 2ca) = 0$$

$$\implies (a+b+c) [(a-b)^2+(b-c)^2+(c-a)^2] = 0 \quad (3.10.71.5)$$

From equation (3.10.71.5) we get 2 equations,

$$\implies \boxed{(a+b+c) = 0} \quad (3.10.71.6)$$

or,

$$\implies (a-b)^2+(b-c)^2+(c-a)^2 = 0 \quad (3.10.71.7)$$

Equation (3.10.71.7) is possible only when, $a = b = c$

$$\implies \boxed{a = b = c} \quad (3.10.71.8)$$

From equation (3.10.71.6) and (3.10.71.8) we can say that, $\triangle = 0$ if $a+b+c = 0$ or $a = b = c$.
From equation (3.10.71.6) and (3.10.71.8) we can say that, $\triangle = 0$ if $a+b+c = 0$ or $a = b = c$.

72. Solve the equation

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$$

Solution: Given,

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \quad (3.10.72.1)$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \quad (3.10.72.2)$$

$$(3x+a) \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \quad (3.10.72.3)$$

$$\xleftrightarrow[C_3 \leftarrow C_3-C_1]{C_2 \leftarrow C_2-C_1} (3x+a) \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} \quad (3.10.72.4)$$

$$= (3x+a)(a^2) \quad (3.10.72.5)$$

Since a cannot be equal to zero, 3x+a should be zero for determinant to be zero

$$3x+a = 0 \quad (3.10.72.6)$$

$$a = -3x \quad (3.10.72.7)$$

73. Prove that
$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$\xleftrightarrow{R1\leftarrow\frac{R1+R2+R3}{2}} 2\begin{vmatrix} a^2+ab & b^2+bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$\xleftrightarrow{R1\leftarrow R1-R2} 2\begin{vmatrix} 0 & bc & c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$\xleftrightarrow[C3\leftarrow\frac{C3}{c}]{C1\leftarrow\frac{C1}{a};C2\leftarrow\frac{C2}{b}} 2abc\begin{vmatrix} 0 & c & c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$\xleftrightarrow{R3\leftarrow R3-R1} 2abc\begin{vmatrix} 0 & c & c \\ a+b & b & a \\ b & b & 0 \end{vmatrix}$$

$$\xleftrightarrow{R2\leftarrow R2-R3} 2abc\begin{vmatrix} 0 & c & c \\ a & 0 & a \\ b & b & 0 \end{vmatrix}$$

$$\xleftrightarrow[R3\leftarrow\frac{R3}{b}]{R1\leftarrow\frac{R1}{c};R2\leftarrow\frac{R2}{a}} 2a^2b^2c^2\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad (3.10.73.1)$$

$$\xleftrightarrow{R1\leftarrow R1+R2-R3} 2a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad (3.10.73.2)$$

$$\xleftrightarrow{R1\leftarrow\frac{R1}{2}} 4a^2b^2c^2\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 4a^2b^2c^2(1) \quad (3.10.73.3)$$

$$= 4a^2b^2c^2 \quad (3.10.73.4)$$

74. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

75. Let A= $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that
    (i) $[adjA]^{-1} = adj(A)^{-1}$
    (ii) $(A^{-1})^{-1} = A$

76. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Solution: Given,

$$\left|\mathbf{A}\right| = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \quad (3.10.76.1)$$

$$\xleftrightarrow{R_1\leftarrow R_1+R_2+R_3} \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \quad (3.10.76.2)$$

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \quad (3.10.76.3)$$

$$\xleftrightarrow[C_3\leftarrow C_3-C_1]{C_2\leftarrow C_2-C_1} 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix} \quad (3.10.76.4)$$

Expanding the determinant from (3.10.76.4), we get

$$= 2\left(x+y\right)\left[-x^2-\{(-y)\left(x-y\right)\}\right] \quad (3.10.76.5)$$

$$= 2\left(x+y\right)\left(-x^2+xy-y^2\right) \quad (3.10.76.6)$$

$$= -2x^3+2x^2y-2xy^2-2x^2y+2xy^2-2y^3 \quad (3.10.76.7)$$

$$= -2\left(x^3+y^3\right) \quad (3.10.76.8)$$

$$\therefore \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = -2\left(x^3+y^3\right) \quad (3.10.76.9)$$

77. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$ Using properties of determinants ,prove that:

78. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

Solution: Using transformations and properties of determinants:

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} \quad (3.10.78.1)$$

$$\xleftrightarrow{C_3 \leftarrow C_1+C_3} \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix} \quad (3.10.78.2)$$

$$\implies (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix} \quad (3.10.78.3)$$

$$\xleftrightarrow[R_2 \leftarrow R_2-R_1]{R_3 \leftarrow R_3-R_1} (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix} \quad (3.10.78.4)$$

$$\implies (\beta-\alpha)(\gamma-\alpha)(\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ 1 & \beta+\alpha & 0 \\ 1 & \gamma+\alpha & 0 \end{vmatrix} \quad (3.10.78.5)$$

$$\implies (\beta-\alpha)(\gamma-\alpha)(\alpha+\beta+\gamma)(-1)^{1+3} \begin{vmatrix} 1 & \beta+\alpha \\ 1 & \gamma+\alpha \end{vmatrix} \quad (3.10.78.6)$$

$$\implies (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma) \quad (3.10.78.7)$$

79. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$, where p is any scalar.

Solution:

$$LHS = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} \quad (3.10.79.1)$$

By expanding using sum property

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \quad (3.10.79.2)$$

By using switching of rows(or columns) property

$$= (-1) \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \quad (3.10.79.3)$$

$$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \quad (3.10.79.4)$$

By using scalar multiplication property

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + (pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (3.10.79.5)$$

$$= (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (3.10.79.6)$$

By applying row reduction

$$= (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (3.10.79.7)$$

$$\xleftrightarrow{R_2 \leftarrow R_2-R_3} (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix} \quad (3.10.79.8)$$

$$\xleftrightarrow{R_3 \leftarrow R_3-R_1} (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-z & y^2-z^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \quad (3.10.79.9)$$

By using scalar multiplication property

$$= (1+pxyz)(y-z)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+z \\ 0 & 1 & z+x \end{vmatrix} \quad (3.10.79.10)$$

By applying the determinant formula

$$= (1+pxyz)(y-z)(z-x)(z+x-y-z) \quad (3.10.79.11)$$

$$= (1+pxyz)(x-y)(y-z)(z-x) \quad (3.10.79.12)$$

$$= RHS \quad (3.10.79.13)$$

Hence Proved.

80. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$=3(a+b+c)(ab+bc+ca)

Solution: Applying transformation:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} \tag{3.10.80.1}$$

$$\xleftrightarrow{C_1 \leftarrow C1+C_2+C_3} \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix} \tag{3.10.80.2}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_2} (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & a+2b & a-b \\ 0 & -2b-c & b+2c \end{vmatrix} \tag{3.10.80.3}$$

$$= (a+b+c)\{(a+2b)(b+2c)+(2b+c)(a-b)\}$$

$$= (a+b+c)\{ab+2ac+2b^2+4bc+2ab-2b^2+ca-cb\}$$

$$= (a+b+c)(3ab+3bc+3ca)$$

$$= 3(a+b+c)(ab+bc+ca) \tag{3.10.80.4}$$

81. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$=1
82. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$=0
83. Solve the system of equations
$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$
$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$
$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$
84. Ifa,b,c are in A.P, then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is
    a) 0
    b) 1
    c) x
    d) 2x
85. If x,y,z are nonzero real numbers, then the inverse of matrix A=$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is
    a) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$
    b) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$
    c) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$
    d) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
86. Let A=$\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\Pi$. Then
    a) Det(A)=0
    b) Det(A)$\in (2,\infty)$
    c) Det(A)$\in (2,4)$
    d) Det(A)$\in [2,4]$

3.11 Linear Inequalities: Exercises

1. Solve x ≥ 3, y ≥ 2 graphically.
Solution: From the given information, for

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ge \mathbf{0}, \tag{3.11.1.1}$$

the given conditions can be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} \ge \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{3.11.1.2}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - \mathbf{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{3.11.1.3}$$

$$\text{or, } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mathbf{u} \tag{3.11.1.4}$$

resulting in

$$\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\mathbf{u} \tag{3.11.1.5}$$

$$\text{or, } \mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\mathbf{u} \tag{3.11.1.6}$$

after obtaining the inverse. Fig. 3.11.1.1 generated using the following python code shows the desired region

```
solutions/1/codes/line/line_eq.py
```

2. Solve 7x+3 < 5x+9. Show the graph of the solutions on number line.
Solution:

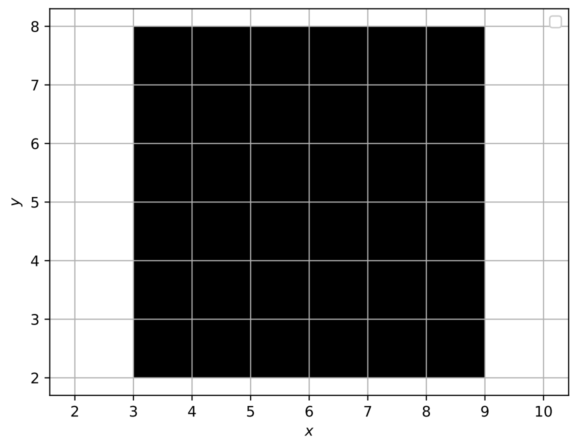

Fig. 3.11.1.1

$$7x + 3 < 5x + 9 \tag{3.11.2.1}$$
$$2x - 6 < 0 \tag{3.11.2.2}$$
$$x < 3 \tag{3.11.2.3}$$
$$\therefore \quad x \in \{3, -\infty\} \tag{3.11.2.4}$$

The following Python code to generate Fig 3.11.2.1

```
solutions/2/codes/line_ex/lin_ineq/
    dist_btw_pts.py
```

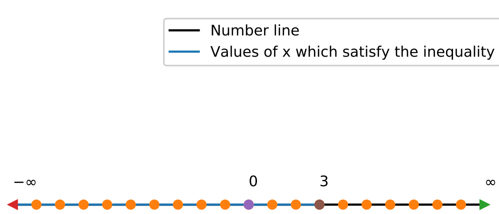

Fig. 3.11.2.1

3. Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.
Solution: Let
$$\frac{3x-4}{2} = \frac{x+1}{4} - 1 + s, \quad s \geq 0 \tag{3.11.3.1}$$
Then,
$$5x - 5 - 4s = 0 \tag{3.11.3.2}$$
$$\implies x = 1 + \frac{4s}{5} \tag{3.11.3.3}$$
$$\implies x \geq 1 \tag{3.11.3.4}$$
The following code marks the solution of inequality on numberline as shown in figure 3.11.3.1

```
codes/line/ineq/ineq.py
```

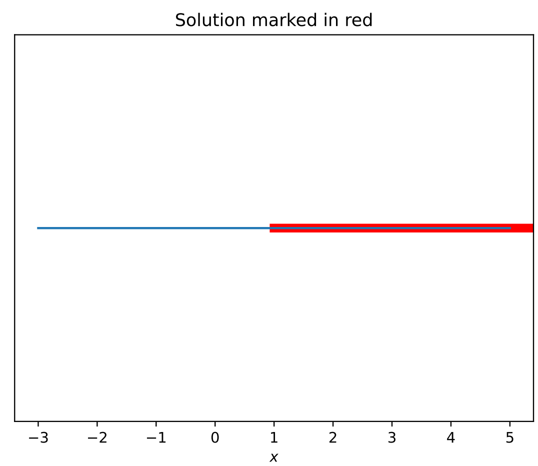

Fig. 3.11.3.1: Solution of the inequality

4. The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.
Solution: If x be the student marks,
$$\frac{62 + 48 + x}{3} \geq 60 \tag{3.11.4.1}$$
$$\implies x \geq 70 \tag{3.11.4.2}$$

5. Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.
Solution:
Let x be an odd natural number and y be the odd natural number consecutive to x.
$$\therefore y = x + 2 \tag{3.11.5.1}$$

We need to find x and y such that

$$x, y > 10 \text{ and } x + y < 40$$
$$\therefore x + x + 2 < 40$$
$$2x + 2 < 40$$
$$x + 1 < 20$$
$$x < 19 \quad (3.11.5.2)$$

Hence the condition is satisfied when $x > 10$ and $x < 19$
The following python code computes the required pairs of consecutive odd natural numbers which satisfy the required condition, shown in Fig.3.11.5.1.

```
./solutions/5/codes/lines/q15.py
```

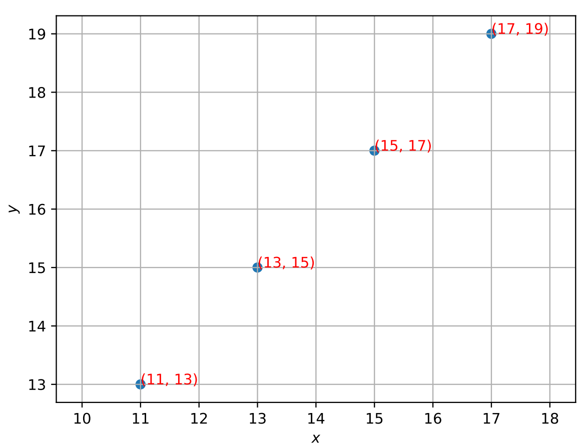

Fig. 3.11.5.1

6. Solve 3x+2y > 6 graphically.
Solution: Let $3x + 2y = 6$ intersects the x-axis and y-axis at **A** and **B** respectively.
a) Let $\mathbf{A} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

$$3x = 6 \quad (3.11.6.1)$$
$$\implies x = 2 \quad (3.11.6.2)$$
$$\mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.11.6.3)$$

b) Let $\mathbf{B} = \begin{pmatrix} 0 \\ y \end{pmatrix}$

$$2y = 6 \quad (3.11.6.4)$$
$$\implies y = 3 \quad (3.11.6.5)$$
$$\mathbf{B} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (3.11.6.6)$$

c) Origin $= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ does not satisfy the equation $3x + 2y < 6$.
$\implies$ The solution is the right side of the line $3x + 2y = 6$

d) The following python code is the diagrammatic representation of the solution in Fig.3.11.6.1

```
solutions/6/codes/linear_inequalities/
    linear_inequalities.py
```

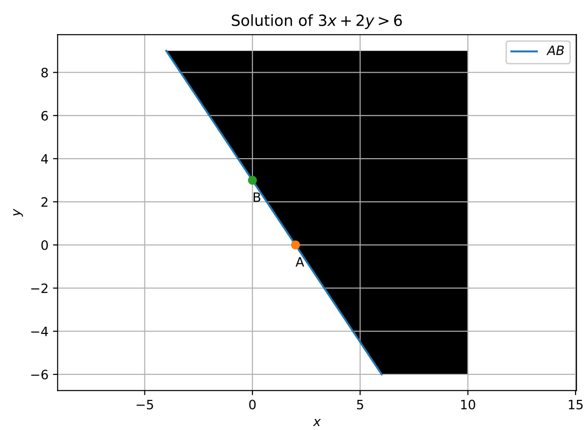

Fig. 3.11.6.1

7. Solve 3x-6 ≥ 0 graphically in a two dimensional plane.
Solution:
The given inequality can be expressed as

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - 6 \geq 0 \implies \mathbf{x} \geq \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.11.7.1)$$

The python code for Fig. 3.11.7.1 is

```
solutions/7/codes/line/lin_ineq/lin_ineq1.py
```

8. Solve y < 2 graphically.
9. Solve the following system of inequalities graphically. 5x+4y ≤ 40 x ≥ 2 y ≥ 3
10. Solve the following system of inequalities graphically. 8x+3y ≤ 100 x ≥ 0 y ≥ 0

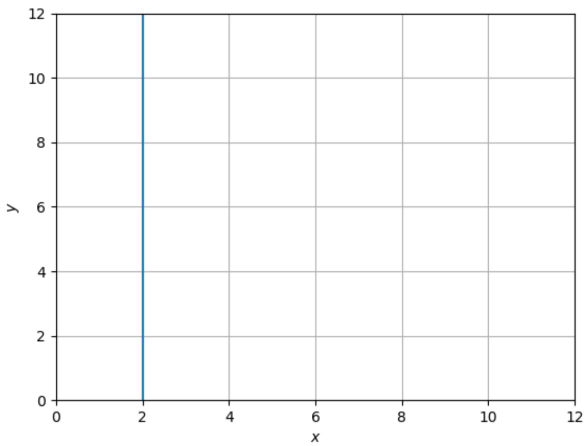

Fig. 3.11.7.1

11. Solve the following system of inequalities graphically. x+2y $\leq$ 8 2x+y $\leq$ 8 x $\geq$ 0 y $\geq$ 0
12. Solve -8 $\leq$ 5x-3 $<$ 7.
13. Solve -5 $\leq \frac{5-3x}{2} \leq$ 8.
14. Solve the system inequalities: 3x-7 $<$ 5+x 11-5x $\leq$ 1 and represent the solutions on the number line.
15. Solve 4x+3 $<$ 6x+7.
16. Solve $\frac{5-2x}{3} \leq \frac{x}{6} - 5$.
17. Solve 24x $<$ 100, when (i) x is a natural number. (ii) x is an integer.
18. Solve -12x $>$ 30, when (i) x is a natural number. (ii) x is an integer.
19. Solve 5x-3 $<$ 7, when (i) x is an integer. (ii) x is a real number.
20. Solve 3x+8 $>$ 2, when (i) x is an integer. (ii) x is a real number
21. 4x+3 $<$ 5x+7.
22. 3x-7 $>$ 5x-1.
23. 3(x-1) $\geq$ 2(x-3).
24. 3(2-x) $\leq$ 2(1-x).
25. x+$\frac{x}{2} + \frac{x}{3}$ $<$ 11.
26. $\frac{x}{3}$ $\frac{x}{2}$+1.
27. $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$.
28. $\frac{1}{2}$($\frac{3x}{5}$+4)$\geq$ $\frac{1}{3}(x-6)$.
29. 2(2x+3)-10 $<$ 6(x-2).
30. 37-(3x+5) $\geq$ 9x-8(x-3).
31. $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$.
32. $\frac{(2x-1)}{3} \geq \frac{(3x-2)}{4} - \frac{(2-x)}{5}$.
33. 3x-2 $<$ 2x+1.
34. 5x-3 $\geq$ 3x-5.
35. 3(1-x) $<$ 2(x+4).
36. $\frac{x}{2} \geq \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$.
37. x+y $<$ 5.
38. 2x+y $\geq$ 6.
39. 3x+4y $\leq$ 12.
40. y+8 $\geq$ 2x.
41. x-y $\leq$ 2.
42. 2x-3y $>$ 6.
43. -3x+2y $\geq$ -6.
44. 3y-5x $<$ 30.
45. y $<$ -2.
46. x $>$ -3.
47. 3x+2y $\leq$ 12, x $\geq$ 1, y $\geq$ 2.
48. 2x+y $\geq$ 6, 3x+4y $\leq$ 12.
49. x+y $\geq$ 4, 2x-y $<$ 0.
50. 2x-y $>$ 1, x-2y $<$ -1.
51. x+y $\leq$ 6, x+y $\geq$ 4.
52. 2x+y $\geq$ 8, x+2y $\geq$ 10.
53. x+y $\leq$ 9, y $>$ x, x $\geq$ 0.
54. 5x+4y $\leq$ 20, x $\geq$ 1, y $\geq$ 2.
55. 3x+4y $\leq$ 60, x+3y $\leq$ 30, x $\geq$ 0, y $\geq$ 0.
56. x-2y $\leq$ 3, 3x+4y $\geq$ 12, x $\geq$ 0, y $\geq$ 1.
57. 4x+3y $\leq$ 60, y $\geq$ 2x, x $\geq$ 3, x,y $\geq$ 0.
58. x+2y $\leq$ 10, x+y $\geq$ 1,x-y $\leq$ 0, x $\geq$ 0, y $\geq$ 0.
59. 2 $\leq$ 3x-4 $\leq$ 5.
60. 6 $\leq$ -3(2x-40) $<$ 12.
61. -3 $\leq$ 4-$\frac{7x}{2}$ $\leq$ 18.
62. -15 $< \frac{3(x-2)}{5} \leq$ 0.
63. -12 $<$ 4-$\frac{3x}{-5}$ $\leq$ 2.
64. 7 $\leq \frac{(3x+11)}{2} \leq$ 11.
65. 5x+1 $>$ -24, 5x-1 $<$ 24.
66. 2(x-1) $<$ x+5, 3(x+2) $>$ 2-x.
67. 3x-7 $>$ 2(x-6), 6-x $>$ 11-2x.
68. 5(2x-7)-3(2x+3) $\leq$ 0, 2x+19 $\leq$ 6x+47.

3.12 Chemistry

1. Balance the following chemical equation.

$$HNO_3 + Ca(OH)_2 \to Ca(NO_3)_2 + H_2O \tag{3.12.1.1}$$

Solution: Let the balanced version of (3.12.1.1) be

$$x_1HNO_3 + x_2Ca(OH)_2 \to x_3Ca(NO_3)_2 + x_4H_2O \tag{3.12.1.2}$$

which results in the following equations:

$$(x_1 + 2x_2 - 2x_4)H = 0 \tag{3.12.1.3}$$
$$(x_1 - 2x_3)N = 0 \tag{3.12.1.4}$$
$$(3x_1 + 2x_2 - 6x_3 - x_4)O = 0 \tag{3.12.1.5}$$
$$(x_2 - x_3)Ca = 0 \tag{3.12.1.6}$$

which can be expressed as

$$x_1 + 2x_2 + 0.x_3 - 2x_4 = 0 \tag{3.12.1.7}$$
$$x_1 + 0.x_2 - 2x_3 + 0.x_4 = 0 \tag{3.12.1.8}$$
$$3x_1 + 2x_2 - 6x_3 - x_4 = 0 \tag{3.12.1.9}$$
$$0.x_1 + x_2 - x_3 + 0.x_4 = 0 \tag{3.12.1.10}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 3 & 2 & -6 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{3.12.1.11}$$

where,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{3.12.1.12}$$

(3.12.1.11) can be reduced as follows:

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 3 & 2 & -6 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow \frac{R_3}{3} - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & -\frac{4}{3} & -2 & \frac{5}{3} \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{3.12.1.13}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & -\frac{4}{3} & -2 & \frac{5}{3} \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{3.12.1.14}$$

$$\xleftrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 + \frac{4}{3} R_2} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -2 & 1 \end{pmatrix} \tag{3.12.1.15}$$

$$\xleftrightarrow[R_3 \leftarrow -\frac{3}{2} R_3]{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -2 & 1 \end{pmatrix} \tag{3.12.1.16}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 + 2R_3} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.12.1.17}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.12.1.18}$$

Thus,

$$x_1 = x_4, x_2 = \frac{1}{2} x_4, x_3 = \frac{1}{2} x_4 \tag{3.12.1.19}$$

$$\implies \quad \mathbf{x} = x_4 \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{3.12.1.20}$$

by substituting $x_4 = 2$.

Hence, (3.12.1.2) finally becomes

$$2HNO_3 + Ca(OH)_2 \to Ca(NO_3)_2 + 2H_2O \tag{3.12.1.21}$$

2. Balance the following chemical equation.

$$NaOH + H_2SO_4 \rightarrow Na_2SO_4 + H_2O \tag{3.12.2.1}$$

Solution: Let $\mathbf{m}$ be a vector consisting of $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ such that

$$x_1NaOH + x_2H_2SO_4 \rightarrow x_3Na_2SO_4 + x_4H_2O \tag{3.12.2.2}$$

For balancing the two equations, we need to calculate number of occurences of each element on left hand side and right hand side of 3.12.3.1 and equate the two. Thus,

$$(x_1 - 2x_3)Na = 0 \tag{3.12.2.3}$$
$$(x_1 + 4x_2 - 4x_3 + x_4)O = 0 \tag{3.12.2.4}$$
$$(x_1 - 2x_4)H = 0 \tag{3.12.2.5}$$
$$(x_2 - x_3)S = 0 \tag{3.12.2.6}$$

From **??**,**??**,**??** and **??**, we get

$$x_1 + 2x_2 + 0x_3 - 2x_4 = 0 \tag{3.12.2.7}$$
$$x_1 + 0x_2 - 2x_3 + 0x_4 = 0 \tag{3.12.2.8}$$
$$0x_1 + x_2 - x_3 + 0x_4 = 0 \tag{3.12.2.9}$$
$$x_1 + 4x_2 - 4x_3 - x_4 = 0 \tag{3.12.2.10}$$

Converting, **??**, **??**, **??** and **??** into matrix form we get,

$$\mathbf{Am} = 0 \tag{3.12.2.11}$$

The matrix $\mathbf{A}$ in above is given as:

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 4 & -4 & -1 \end{pmatrix} \tag{3.12.2.12}$$

To find the solution of **??**, we reduce $\mathbf{A}$ into its Echelon form and solve consequently. The Echolen form of $\mathbf{A}$ can be found as

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 4 & -4 & -1 \end{pmatrix} \xleftrightarrow{r_2 \leftarrow r_2 - r_1} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 4 & -4 & -1 \end{pmatrix} \tag{3.12.2.13}$$

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 4 & -4 & -1 \end{pmatrix} \xleftrightarrow{r_4 \leftarrow r_4 - r_1} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -4 & 1 \end{pmatrix} \tag{3.12.2.14}$$

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_2 \leftarrow r_1 + r_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -4 & 1 \end{pmatrix} \tag{3.12.2.15}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_2 \leftarrow -r_2/2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -4 & 1 \end{pmatrix} \tag{3.12.2.16}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow r_3 - r_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -4 & 1 \end{pmatrix} \tag{3.12.2.17}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_4 \leftarrow r_4 - 2r_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -6 & 3 \end{pmatrix} \tag{3.12.2.18}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow -r_3/2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & -6 & 3 \end{pmatrix} \tag{3.12.2.19}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_2 \leftarrow r_2 - r_3} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & -6 & 3 \end{pmatrix} \tag{3.12.2.20}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_1 \leftarrow r_1 + 2r_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & -6 & 3 \end{pmatrix} \tag{3.12.2.21}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -4 & 1 \end{pmatrix} \xleftrightarrow{r_4 \leftarrow r_4 + 6r_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.12.2.22}$$

?? is the Echelon form of matrix **A** and solving for **m**, we get

$$x_1 = x_4, \quad x_2 = \frac{x_4}{2} \quad and \quad x_3 = \frac{x_4}{2} \tag{3.12.2.23}$$

Hence, we find out that

$$\mathbf{m} = \begin{pmatrix} x_4 \\ x_4/2 \\ x_4/2 \\ x_4 \end{pmatrix} \implies \mathbf{m} = x_4 \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \\ 1 \end{pmatrix} \tag{3.12.2.24}$$

Taking $x_4 = 2$ in **??**, we find out that

$$\mathbf{m} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{3.12.2.25}$$

Thus, by **??** we find out one set of multipliers which balance the given chemical equation and the balanced chemical equation is :

$$2NaOH + H_2SO_4 \rightarrow Na_2SO_4 + 2H_2O \tag{3.12.2.26}$$

3. Balance the following chemical equation

$$BaCl_2 + H_2SO_4 \rightarrow BaSO_4 + HCl \tag{3.12.3.1}$$

Solution: Let the balanced version of (**??**) be:-

$$x_1BaCl_2 + x_2H_2SO_4 \rightarrow x_3BaSO_4 + x_4HCl \tag{3.12.3.2}$$

which results in the following equations:-

$$\begin{aligned} (x_1 - x_3)Ba &= 0 \\ (x_2 - x_4)SO_4 &= 0 \\ (2x_2 - x4)H &= 0 \\ (2x_1 - x_4)Cl &= 0 \end{aligned} \tag{3.12.3.3}$$

which can be expressed as:-

$$\begin{aligned} 1.x_1 + 0.x_2 - 1.x_3 + 0.x_4 &= 0 \\ 0.x_1 + 1.x_2 + 0.x_3 - 1.x_4 &= 0 \\ 0.x_1 + 2.x_2 + 0.x_3 - 1.x4 &= 0 \\ 2.x_1 + 0.x_2 + 0.x_3 - 1.x_4 &= 0 \end{aligned} \tag{3.12.3.4}$$

resulting in the matrix equation:-

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 0 & -1 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{3.12.3.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{3.12.3.6}$$

equation (3.12.3.5) can be row reduced as follows

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & 0 & -1 \end{pmatrix} \xleftrightarrow{R4 \leftarrow R4 - 2R1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \tag{3.12.3.7}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xleftrightarrow{R3 \leftarrow R3 - 2R2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \tag{3.12.3.8}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xleftrightarrow{R4 \leftarrow R4 - R3} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.12.3.9}$$

Thus,

$$x_1 = x_3, x_2 = x_3, 2x_3 = x_4 \quad (3.12.3.10)$$

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \quad (3.12.3.11)$$

Upon substituting $x_3 = 1$ in(3.12.3.11),then (3.12.3.2) becomes,

$$\boxed{BaCl_2 + H_2SO_4 \rightarrow BaSO_4 + 2HCl} \quad (3.12.3.12)$$

(3.12.3.12) is our required balance equation.

4. Balance the following chemical equation.

$$\text{Zinc} + \text{Silver nitrate} \rightarrow \text{Zinc nitrate} + \text{Silver} \quad (3.12.4.1)$$

Solution:
Equation 3.12.4.1 can be written as :

$$Zn + AgNO_3 \rightarrow Ag + Zn(NO_3)_2 \quad (3.12.4.2)$$

Suppose balance form of the equation is :

$$x_1 Zn + x_2 AgNO_3 \rightarrow x_3 Ag + x_4 Zn(NO_3)_2 \quad (3.12.4.3)$$

which results in the following equations:

$$(x_1 - 2x_4)Zn = 0 \quad (3.12.4.4)$$
$$(x_2 - x_3)Ag = 0 \quad (3.12.4.5)$$
$$(x_3 - 2x_4)N = 0 \quad (3.12.4.6)$$
$$(3x_3 - 6x_4)O = 0 \quad (3.12.4.7)$$

which can be expressed as

$$x_1 + 0.x_2 + 0.x_3 - x_4 = 0 \quad (3.12.4.8)$$
$$0.x_1 + x_2 - x_3 + 0.x_4 = 0 \quad (3.12.4.9)$$
$$0.x_1 + 0.x_2 + x_3 - 2.x_4 = 0 \quad (3.12.4.10)$$
$$0.x_1 + 0.x_2 + 3x_3 - 6.x_4 = 0 \quad (3.12.4.11)$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (3.12.4.12)$$

where,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad (3.12.4.13)$$

(3.12.4.12) can be reduced as follows:

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 - 3R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.12.4.14)$$

Thus,

$$x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4 \quad (3.12.4.15)$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} x_4 \\ 2x_4 \\ 2x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad (3.12.4.16)$$

by substituting $x_4 = 1$, we get :

$$\implies \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \quad (3.12.4.17)$$

Hence, (3.12.4.3) finally becomes

$$Zn + 2AgNO_3 \rightarrow 2Ag + Zn(NO_3)_2 \quad (3.12.4.18)$$

5. Write the balanced chemical equations for the following reaction :

$$BaCl_2 + K_2SO_4 \rightarrow BaSO_4 + KCl \quad (3.12.5.1)$$

Solution: We know that the number of atoms of each element remains the same, before and after a chemical reaction.
Equation (3.12.5.1) can be written as:

$$x_1 BaCl_2 + x_2 K_2SO_4 \rightarrow x_3 BaSO_4 + x_4 KCl \quad (3.12.5.2)$$

Element wise contribution in forming the respective chemical compound can be written in the form of equation as :

$$Ba : x_1 + 0x_2 - x_3 - 0x_4 = 0 \quad (3.12.5.3)$$
$$Cl : 2x_1 + 0x_2 - 0x_3 - 1x_4 = 0 \quad (3.12.5.4)$$
$$K : 0x_1 + 2x_2 - 0x_3 - 1x_4 = 0 \quad (3.12.5.5)$$
$$S : 0x_1 + 1x_2 - 1x_3 - 0x_4 = 0 \quad (3.12.5.6)$$
$$O : 0x_1 + 4x_2 - 4x_3 - 0x_4 = 0 \quad (3.12.5.7)$$

In matrix form this can be written as:

$$A\mathbf{x} = 0 \quad (3.12.5.8)$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.12.5.9)$$

Using Gaussian Elimination method

$$\xleftrightarrow{R_2 \leftrightarrow R_5} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 2 & 0 & -1 & : & 0 \\ 0 & 1 & -1 & 0 & : & 0 \\ 2 & 0 & 0 & -1 & : & 0 \end{pmatrix} \quad (3.12.5.10)$$

$$\xleftrightarrow{R_5 \leftarrow 2R_1 - R_5} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 2 & 0 & -1 & : & 0 \\ 0 & 1 & -1 & 0 & : & 0 \\ 0 & 0 & -2 & 1 & : & 0 \end{pmatrix} \quad (3.12.5.11)$$

$$\xleftrightarrow[R_4 \leftarrow 4R_4 - R_2]{R_3 \leftarrow 2R_3 - R_2} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 0 & 4 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & -2 & 1 & : & 0 \end{pmatrix} \quad (3.12.5.12)$$

$$\xleftrightarrow{R_5 \leftrightarrow R_5} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 0 & 4 & -2 & : & 0 \\ 0 & 0 & -2 & 1 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \quad (3.12.5.13)$$

$$\xleftrightarrow{R_4 \leftarrow 2R_4 - R_3} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 0 & 4 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \quad (3.12.5.14)$$

Clearly the system is linearly dependent. Therefore by fixing the value of $x_4 = 2$, one of the possible vector $\mathbf{x}$ is:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \quad (3.12.5.15)$$

Hence by putting the values of x_1, x_2, x_3, x_4 in equation (3.12.5.1) we get our balanced chemical equation as followes :

$$BaCl_2 + K_2SO_4 \rightarrow BaSO_4 + 2KCl \quad (3.12.5.16)$$

3.13 Miscellaneous

1. Solve the following pair of linear equations

a)

$$\begin{aligned} \begin{pmatrix} p & q \end{pmatrix} \mathbf{x} &= p - q \\ \begin{pmatrix} q & -p \end{pmatrix} \mathbf{x} &= p + q \end{aligned} \quad (3.13.1.1)$$

b)

$$\begin{aligned} \begin{pmatrix} a & b \end{pmatrix} \mathbf{x} &= c \\ \begin{pmatrix} b & a \end{pmatrix} \mathbf{x} &= 1 + c \end{aligned} \quad (3.13.1.2)$$

c)

$$\begin{aligned} \begin{pmatrix} \frac{1}{a} & -\frac{1}{b} \end{pmatrix} \mathbf{x} &= 0 \\ \begin{pmatrix} a & b \end{pmatrix} \mathbf{x} &= a^2 + b^2 \end{aligned} \quad (3.13.1.3)$$

Solution:

a) The above equations can be expressed as the matrix equation

$$\begin{pmatrix} p & q \\ q & -p \end{pmatrix} \mathbf{x} = \begin{pmatrix} p - q \\ p + q \end{pmatrix} \quad (3.13.1.4)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} p & -q & p - q \\ q & -p & p + q \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{pR_2 - qR_1}{-(p^2+q^2)}} \begin{pmatrix} p & q & p - q \\ 0 & 1 & -1 \end{pmatrix} \quad (3.13.1.5)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1 - qR_2}{p}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix} \quad (3.13.1.6)$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b) The equations can be expressed as the matrix equation

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mathbf{x} = \begin{pmatrix} c \\ 1+c \end{pmatrix} \tag{3.13.1.7}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} a & b & c \\ b & a & 1+c \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{aR_2 - bR_1}{\left(a^2-b^2\right)}} \begin{pmatrix} a & b & c \\ 0 & 1 & \frac{a+ac-bc}{a^2-b^2} \end{pmatrix} \tag{3.13.1.8}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1 - bR_2}{a}} \begin{pmatrix} 1 & 0 & \frac{c^2-ab-abc}{\left(a^3-ab^2\right)} \\ 0 & 1 & \frac{a+ac-bc}{a^2-b^2} \end{pmatrix} \tag{3.13.1.9}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{c^2-ab-abc}{\left(a^2-b^2\right)a} \\ \frac{a+ac-bc}{a^2-b^2} \end{pmatrix}$$

c) The equations can be expressed as the matrix equation

$$\begin{pmatrix} \frac{1}{a} & -\frac{1}{b} \\ a & b \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ a^2+b^2 \end{pmatrix} \tag{3.13.1.10}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} \frac{1}{a} & -\frac{1}{b} & 0 \\ a & b & a^2+b^2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2 - a^2 R_1}{\frac{a^2+b^2}{b}}} \begin{pmatrix} \frac{1}{a} & -\frac{1}{b} & 0 \\ 0 & 1 & b \end{pmatrix} \tag{3.13.1.11}$$

$$\xleftrightarrow{R_1 \leftarrow aR_1 + \frac{a}{b}R_2} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \tag{3.13.1.12}$$

$$\implies \mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$$

2. Solve the following pair of equations

$$\begin{aligned} \begin{pmatrix} a-b & a+b \end{pmatrix} \mathbf{x} &= a^2 - 2ab - b^2 \\ \begin{pmatrix} a+b & a+b \end{pmatrix} \mathbf{x} &= a^2 + b^2 \end{aligned} \tag{3.13.2.1}$$

Solution:
The given equations can be expressed as

$$\begin{pmatrix} a-b & a+b \\ a+b & a+b \end{pmatrix} \mathbf{x} = \begin{pmatrix} a^2-2ab-b^2 \\ a^2+b^2 \end{pmatrix} \tag{3.13.2.2}$$

Using row operations,

$$\begin{pmatrix} a-b & a+b & a^2-2ab-b^2 \\ a+b & a+b & a^2+b^2 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{a+b}]{R_1 \leftarrow \frac{R_1}{a-b}} \begin{pmatrix} 1 & \frac{a+b}{a-b} & \frac{a^2-2ab-b^2}{a-b} \\ 1 & 1 & \frac{a^2+b^2}{a+b} \end{pmatrix} \tag{3.13.2.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & \frac{a+b}{a-b} & \frac{a^2-2ab-b^2}{a-b} \\ 0 & \frac{-2b}{a-b} & \frac{4ab^2}{a^2-b^2} \end{pmatrix} \tag{3.13.2.4}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-(a-b)R_2}{2b}} \begin{pmatrix} 1 & \frac{a+b}{a-b} & \frac{a^2-2ab-b^2}{a-b} \\ 0 & 1 & \frac{-2ab}{a+b} \end{pmatrix} \tag{3.13.2.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & a+b \\ 0 & 1 & \frac{-2ab}{a+b} \end{pmatrix} \tag{3.13.2.6}$$

$$\therefore \mathbf{x} = \begin{pmatrix} a+b \\ \frac{2ab}{a-b} \end{pmatrix} \tag{3.13.2.7}$$

3. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{3.13.3.1}$$

Solution:

3.1. Let the medians BE and CF in Fig. 3.13.3.1 intersect at O, such that

$$\begin{aligned} \frac{OB}{OE} &= k_1 \\ \frac{OC}{OF} &= k_2 \end{aligned} \tag{3.13.3.2}$$

Using (3.1.24.2),

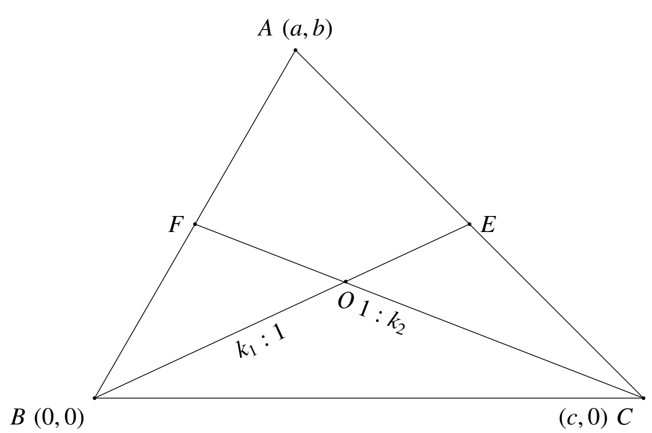

Fig. 3.13.3.1: Medians BE and CF

$$E = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{3.13.3.3}$$

$$F = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{3.13.3.4}$$

Similarly, since O divides BE in the ratio $k_1 : 1$ and CF in the ratio $k_2 : 1$.

$$O = \frac{k_1\mathbf{E} + \mathbf{B}}{k_1 + 1} = \frac{k_2\mathbf{F} + \mathbf{C}}{k_2 + 1} \quad (3.13.3.5)$$

$$\implies \frac{k_1\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right) + B}{k_1 + 1} = \frac{k_2\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) + C}{k_2 + 1} \quad (3.13.3.6)$$

upon substituting from (3.13.3.4). Simplifying (3.13.3.6),

$$\frac{k_1(\mathbf{A} + \mathbf{C}) + 2\mathbf{B}}{k_1 + 1} = \frac{k_2(\mathbf{A} + \mathbf{B}) + 2\mathbf{C}}{k_2 + 1} \quad (3.13.3.7)$$

which can be expressed as

$$\implies [k_1(k_2 + 1) - k_2(k_1 + 1)]\mathbf{A} + [2(k_2 + 1) - k_2(k_1 + 1)]\mathbf{B} + [k_1(k_2 + 1) - 2(k_1 + 1)]\mathbf{C} = 0 \quad (3.13.3.8)$$

resulting in

$$\mathbf{B} = \frac{(k_1 - k_2)\mathbf{A} + (k_1k_2 - k_1 - 2)}{k_1k_2 - k_2 - 2} \quad (3.13.3.9)$$

If the above equation has a solution, then $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ lie on a straight line. Since that is not the case, the only possibility is

$$k_1 - k_2 = 0 \quad (3.13.3.10)$$
$$k_1k_2 - k_1 - 2 = 0 \quad (3.13.3.11)$$
$$k_1k_2 - k_2 - 2 = 0 \quad (3.13.3.12)$$
$$\implies k_1 = k_2 = 2 \quad (3.13.3.13)$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *lie on a triangle, they are linearly independent.* In which case,

$$x_1\mathbf{A} + x_2\mathbf{B} + x_3\mathbf{C} = 0 \quad (3.13.3.14)$$
$$\implies x_1 = x_2 = x_3 = 0, \quad (3.13.3.15)$$

Else, they are linearly dependent and lie on a straight line.

3.2. In Fig. 3.13.3.1,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \quad \text{and} \quad (3.13.3.16)$$
$$\mathbf{O} = \frac{\mathbf{B} + 2\mathbf{E}}{3} \quad (3.13.3.17)$$
$$= \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (3.13.3.18)$$

4. (Cauchy-Schwarz Inequality:) Show that

$$\left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\| \|\mathbf{b}\| \quad (3.13.4.1)$$

Solution:

$$\|\lambda\mathbf{a} - \mu\mathbf{b}\|^2 = \lambda^2 \|\mathbf{a}\|^2 + \mu^2 \|\mathbf{b}\|^2 - 2\lambda\mu\mathbf{a}^T\mathbf{b} \ge 0 \quad (3.13.4.2)$$

$$\implies \mathbf{a}^T\mathbf{b} \le \frac{1}{2}\left[\frac{\lambda}{\mu}\|\mathbf{a}\|^2 + \frac{\mu}{\lambda}\|\mathbf{b}\|^2\right] \quad (3.13.4.3)$$

Substituting

$$\frac{\lambda}{\mu} = \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}, \quad (3.13.4.4)$$
$$\mathbf{a}^T\mathbf{b} \le \|\mathbf{a}\| \|\mathbf{b}\| \quad (3.13.4.5)$$

after some algebra.

5. (Triangle Inequality:) Show that

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \quad (3.13.5.1)$$

Solution:

$$\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 \quad (3.13.5.2)$$
$$\le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\| + \|\mathbf{b}\|^2 \quad (3.13.5.3)$$
$$\implies \|\mathbf{a} + \mathbf{b}\|^2 \le (\|\mathbf{a}\| + \|\mathbf{b}\|)^2 \quad (3.13.5.4)$$
$$\text{or, } \|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \quad (3.13.5.5)$$

using Cauchy-Schwarz inequality.

6. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle. **Solution:** See Fig. 3.13.6.1. Let AB be the base and $\mathbf{M}$ be the midpoint. Then,

$$\mathbf{A} = \begin{pmatrix} 0 \\ a \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -a \end{pmatrix} \quad (3.13.6.1)$$

If

$$\mathbf{C} = \begin{pmatrix} p \\ 0 \end{pmatrix}, \quad (3.13.6.2)$$

$$\|\mathbf{C} - \mathbf{A}\| = 2a \implies \sqrt{p^2 + a^2} = 2a \quad (3.13.6.3)$$

$$\text{or, } p = \pm\sqrt{3}a \quad (3.13.6.4)$$

resulting in two possible triangles as in Fig. 3.13.6.1. $\triangle ABC$ in Fig.3.13.6.1 is generated using the following python code

```
solutions/6/codes/line/miscellaneous/tri_equi.
```

```
py
```

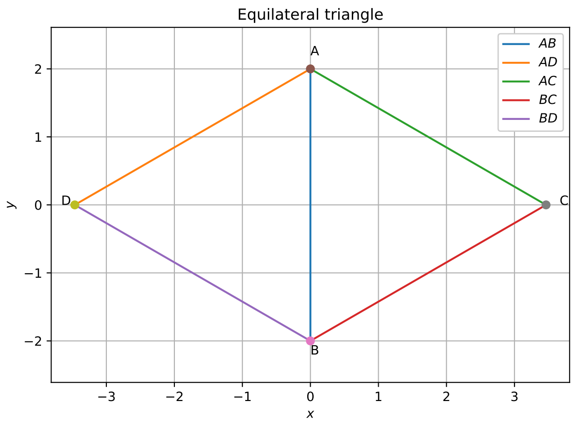

Fig. 3.13.6.1: Triangles ABC and ABD using python

7. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 y_2 \end{pmatrix}$ when
   a) PQ is parallel to the y-axis.
   b) PQ is parallel to the x-axis.
8. If three points $\begin{pmatrix} h 0 \end{pmatrix}$, $\begin{pmatrix} a b \end{pmatrix}$ and $\begin{pmatrix} 0 k \end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
9. $\mathbf{P} = \begin{pmatrix} a b \end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is
$$\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 2 \tag{3.13.9.1}$$
10. Point $\mathbf{R} = \begin{pmatrix} h k \end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.
11. Show that two lines
$$\begin{pmatrix} a_1 & b_1 \end{pmatrix} \mathbf{x} + c_1 = 0 \tag{3.13.11.1}$$
$$\begin{pmatrix} a_2 & b_2 \end{pmatrix} \mathbf{x} + c_2 = 0 \tag{3.13.11.2}$$
are
   a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and
   b) perpendicular if $a_1 a_2 - b_1 b_2 = 0$.
12. Find the distance between the parallel lines
$$l \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -p \tag{3.13.12.1}$$
$$l \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = r \tag{3.13.12.2}$$
13. Find th equation of the line through the point $\mathbf{x}_1$ and parallel to the line
$$\begin{pmatrix} A & B \end{pmatrix} \mathbf{x} = -C \tag{3.13.13.1}$$
14. If p and q are the lengths of perpendiculars from the origin to the lines
$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \mathbf{x} = k \cos 2\theta \tag{3.13.14.1}$$
$$\begin{pmatrix} \sec\theta & \operatorname{cosec}\theta \end{pmatrix} \mathbf{x} = k \tag{3.13.14.2}$$
respectively, prove that $p^2 + 4q^2 = k^2$.
15. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{3.13.15.1}$$
16. Show that the area of the triangle formed by the lines
$$\begin{pmatrix} -m_1 & 1 \end{pmatrix} \mathbf{x} = c_1 \tag{3.13.16.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{3.13.16.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.13.16.3}$$
is $\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.
17. Find the values of k for which the line
$$\begin{pmatrix} k-3 & -\left(4-k^2\right) \end{pmatrix} \mathbf{x} + k^2 - 7k + 6 = 0 \tag{3.13.17.1}$$
is
   a) parallel to the x-axis
   b) parallel to the y-axis
   c) passing through the origin.
18. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix} \cos\theta \sin\theta \end{pmatrix}$ and $\begin{pmatrix} \cos\phi \sin\phi \end{pmatrix}$.
19. Find the area of the triangle formed by the lines
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.13.19.1}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.13.19.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{3.13.19.3}$$
20. If three lines whose equations are
$$\begin{pmatrix} -m_1 & 1 \end{pmatrix} \mathbf{x} = c_1 \tag{3.13.20.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{3.13.20.2}$$
$$\begin{pmatrix} -m_3 & 1 \end{pmatrix} \mathbf{x} = c_3 \tag{3.13.20.3}$$
are concurrent, show that
$$m_1 (c_2 - c_3) + m_2 (c_3 - c_1) + m_3 (c_1 - c_2) = 0 \tag{3.13.20.4}$$

21. Find the equation of the line passing through the origin and making an angle θ with the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.13.21.1}$$

22. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix}\sqrt{a^2-b^2}0\end{pmatrix}$ and $\begin{pmatrix}\sqrt{a^2-b^2}0\end{pmatrix}$ to the line

$$\begin{pmatrix}\frac{\cos\theta}{a} & \frac{\sin\theta}{b}\end{pmatrix}\mathbf{x} = 1 \tag{3.13.22.1}$$

is b^2.

23. If $\begin{pmatrix}l_1 m_1 n_1\end{pmatrix}$ and $\begin{pmatrix}l_2 m_2 n_2\end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix}m_1n_2 - m_2n_1n_1l_2 - n_2l_1l_1m_2 - l_2m_1\end{pmatrix}$.

24. A line makes angles $\alpha,\beta,\gamma,\delta$ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \tag{3.13.24.1}$$

25. Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \tag{3.13.25.1}$$

$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \tag{3.13.25.2}$$

are coplanar.

26. Find $\mathbf{R}$ which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \tag{3.13.26.1}$$

$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \tag{3.13.26.2}$$

externally in the ratio 1 : 2.

27. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}-\mathbf{b}) = 8 \tag{3.13.27.1}$$

$$\|\mathbf{a}\| = 8\|\mathbf{b}\| \tag{3.13.27.2}$$

28. Evaluate the product

$$(3\mathbf{a}-5\mathbf{b})^T(2\mathbf{a}+7\mathbf{b}) \tag{3.13.28.1}$$

29. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \tag{3.13.29.1}$$

$$\mathbf{a}^T\mathbf{b} = \frac{1}{2} \tag{3.13.29.2}$$

and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

30. Show that

$$(\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a}) \perp (\|\mathbf{a}\|\,\mathbf{b} - \|\mathbf{b}\|\,\mathbf{a}) \tag{3.13.30.1}$$

$$\tag{3.13.30.2}$$

31. If $\mathbf{a}^T\mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?

32. If $\mathbf{a},\mathbf{b},\mathbf{c}$ are unit vectors such that

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = 0, \tag{3.13.32.1}$$

find the value of

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{3.13.32.2}$$

33. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda\mathbf{a}\| = 1$ if
   a) $\lambda = 1$
   b) $\lambda = -1$
   c) $\|\mathbf{a}\| = |\lambda|$
   d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

34. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.

35. Show that

$$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b}) \tag{3.13.35.1}$$

36. If $\mathbf{a}^T\mathbf{b} = 0$ and $\mathbf{a}\times\mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?

37. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that

$$(\mathbf{x}-\mathbf{a})^T(\mathbf{x}+\mathbf{a}) = 12. \tag{3.13.37.1}$$

38. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a}\times\mathbf{b}$ is a unit vector if the angle between $\mathbf{a}$ and $\mathbf{b}$ is
   a) $\frac{\pi}{6}$
   b) $\frac{\pi}{4}$
   c) $\frac{\pi}{3}$
   d) $\frac{\pi}{2}$

39. Prove that

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \tag{3.13.39.1}$$

$$\iff \mathbf{a} \perp \mathbf{b}. \tag{3.13.39.2}$$

40. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when
   a) $0 < \theta < \frac{\pi}{2}$
   b) $0 \leq \theta \leq \frac{\pi}{2}$
   c) $0 < \theta < \pi$
   d) $0 \leq \theta \leq \pi$

41. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

42. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \|\mathbf{a} \times \mathbf{b}\|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π.

43. Find the angle between the lines whose direction vectors are $\begin{pmatrix} abc \end{pmatrix}$ and $\begin{pmatrix} b-cc-aa-b \end{pmatrix}$.
44. Find the equation of a line parallel to the x-axis and passing through the origin.
45. Find the equation of a plane passing through $\begin{pmatrix} abc \end{pmatrix}$ and parallel to the plane

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} x = 2 \tag{3.13.45.1}$$

46. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \tag{3.13.46.1}$$

47. In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by C = $\frac{5}{9}(F-32)$, where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively.
48. A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?
49. Ravi obtained 70 and 75 marks in first two unit test. Find the minimum marks he should get in the third test to have an average of at least 60 marks.
50. To receive Grade A in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.
51. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.
52. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.
53. A man wants to cut three lengths from a single piece of board of length 91cm.The second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second?
54. A solution is to be kept between 68°F and 77°F. What is the range in temperature in degree Celsius(C) if the Celsius / Fahrenheit(F) conversion formula is given by F= $\frac{9}{5}$ C+32?
55. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?
56. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?
57. IQ of a person is given by the formula IQ = $\frac{MA}{CA} \times 100$, where MA is mental age and CA is chronological age.If $80 \le IQ \le 140$ for a group of 12 years old children, find the range of their mental age.

4 Circle

4.1 Circle Geometry Examples

1. Find the equation of a circle with centre $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and radius 4.
Solution: From the given information, the desired equation is

$$\left\| \mathbf{x} - \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\|^2 = 4^2 \tag{4.1.1.1}$$

$$\implies \mathbf{x}^T\mathbf{x} + \begin{pmatrix} 6 & -4 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{4.1.1.2}$$

The python code for Fig. 4.1.1.1 is

```
solutions/1/codes/circle/circle1.py
```

2. Find the centre and radius of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 8 \\ 10 \end{pmatrix} \mathbf{x} - 8 = 0 \tag{4.1.2.1}$$

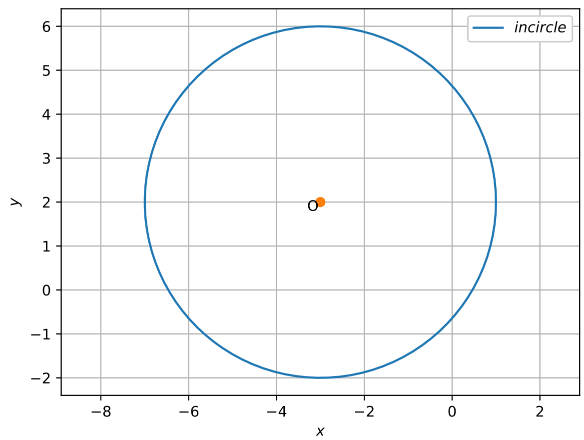

Fig. 4.1.1.1: Circle using python

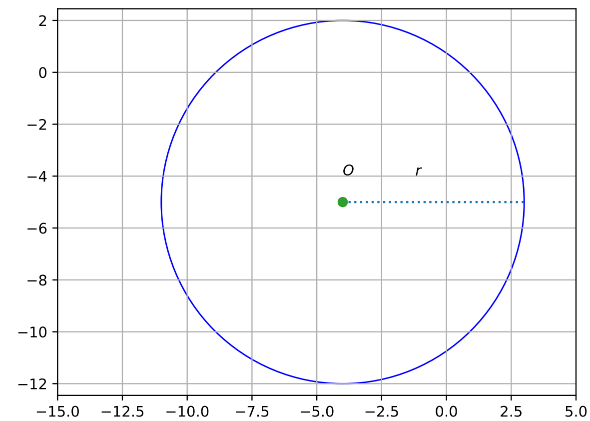

Fig. 4.1.2.1

Solution:
The general equation of a circle is

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + \|\mathbf{O}\|^2 - r^2 = 0 \quad (4.1.2.2)$$

Comparing equation (4.1.2.2) with the given circle equation:

$$\mathbf{O} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \quad (4.1.2.3)$$

$$\|\mathbf{O}\|^2 = 41 \quad (4.1.2.4)$$

$$r^2 = 41 + 8 \quad (4.1.2.5)$$

$$\therefore r = 7 \quad (4.1.2.6)$$

The following Python code generates Fig. 4.1.2.1

```
solutions/2/codes/circle_exam.py
```

3. Find the equation of the circle which passes through the points $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and whose centre lies on the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2. \quad (4.1.3.1)$$

Solution:

4. Let $\mathbf{O}$ be the centre and r be the radius. For

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad (4.1.4.1)$$

$$\implies \|\mathbf{A} - \mathbf{O}\| = \|\mathbf{B} - \mathbf{O}\| = r \quad (4.1.4.2)$$

$$\implies \|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \quad (4.1.4.3)$$

$$\implies (\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (4.1.4.4)$$

$$\text{or, } \begin{pmatrix} 1 & 6 \end{pmatrix}\mathbf{O} = \frac{17}{2} \quad (4.1.4.5)$$

Also centre O lies on the line in (4.1.3.1)

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{O} = 2 \quad (4.1.4.6)$$

(4.1.4.5) and (4.1.4.6) result in the matrix equation

$$\begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix}\mathbf{O} = \begin{pmatrix} \frac{17}{2} \\ 2 \end{pmatrix} \quad (4.1.4.7)$$

The following code calculates centre and radius and plots figure 4.1.4.1

```
solutions/3/codes/circle1/circle1.py.py
```

5. Find the area enclosed by the circle $\|\mathbf{x}\| = a$
Solution: The area is $2\pi a^2$.
6. Find the area of the region in the first quadrant enclosed by the x-axis, the line $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$, and the circle $\|\mathbf{x}\| = 1$.
Solution: The circle in Fig. 4.1.6.1 is generated using the following python code

```
solutions/6/codes/circle/example/circle.py
```

The angle that the line makes with the x-axis

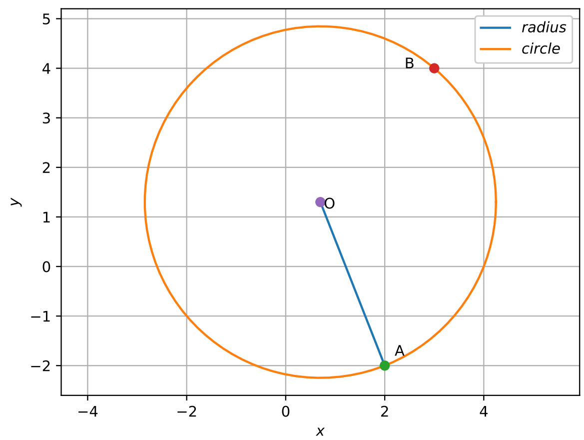

Fig. 4.1.4.1: Circle with centre at **O** and radius r

is given by

$$\cos\theta = \frac{\begin{pmatrix}1 & -1\end{pmatrix}^T \begin{pmatrix}1\\0\end{pmatrix}}{\left\|\begin{pmatrix}1 & -1\end{pmatrix}\right\| \left\|\begin{pmatrix}1 & -1\end{pmatrix}\right\|} \tag{4.1.6.1}$$

$$= \frac{1}{\sqrt{2}} \tag{4.1.6.2}$$

$$\implies \theta = 45^\circ. \tag{4.1.6.3}$$

The area of the sector is then obtained as

$$\frac{\theta}{360^\circ}\pi r^2 = \frac{45^\circ}{360^\circ}\pi r^2 \tag{4.1.6.4}$$

$$= \frac{\pi}{8} \tag{4.1.6.5}$$

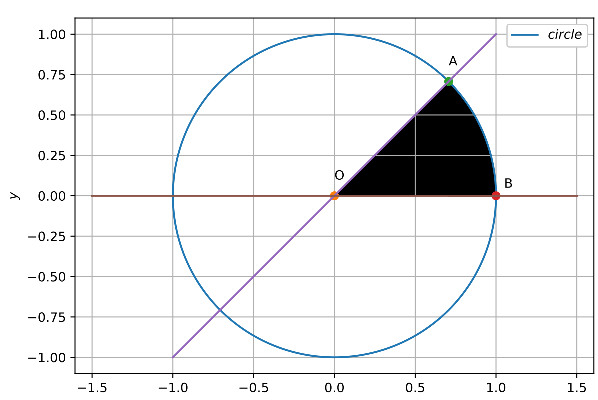

Fig. 4.1.6.1: Circle generated using python

7. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix}2\\0\end{pmatrix}\right\| = 2$.
8. Form the differntial equation of the fmaily of circles touching the x-axis at the origin.
9. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

4.2 Circle Geometry Exercises

1. Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $\left(2, -3\right)$ and $\mathbf{B} = \begin{pmatrix}1\\4\end{pmatrix}$.

   Solution:
   The input values for the question are given in the table (4.2.1) The **A** is at the end of

Input values	
Parameters	Values
O	$\begin{pmatrix}2\\-3\end{pmatrix}$
A	$\begin{pmatrix}1\\4\end{pmatrix}$

TABLE 4.2.1: Input Values

   diameter, so the centre(**O**) is the midpoint of **AB**.

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{4.2.1.1}$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{B} \tag{4.2.1.2}$$

$$\therefore \mathbf{A} = \begin{pmatrix}3\\-10\end{pmatrix} \tag{4.2.1.3}$$

   The python code for the figure (4.2.1.1) is

```
solutions/1/codes/circle/circle.py
```

2. Find the centre O of a circle passing through the points $\begin{pmatrix}6\\-6\end{pmatrix}$, $\begin{pmatrix}3\\-7\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$. **Solution:** The general of a circle equation is:

$$\|\mathbf{x} - \mathbf{O}\| = r \tag{4.2.2.1}$$

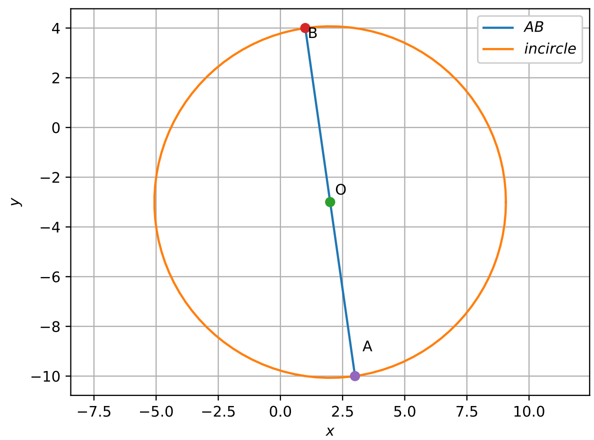

Fig. 4.2.1.1

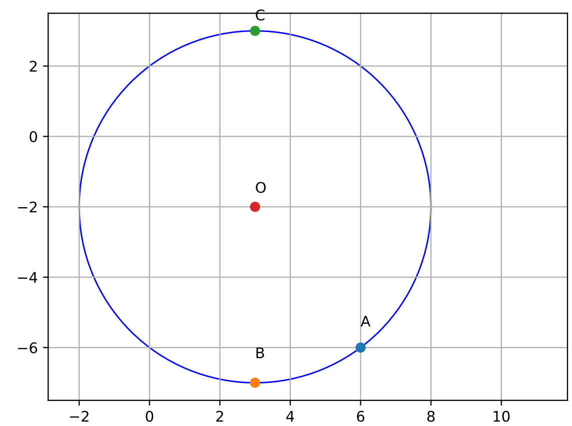

Fig. 4.2.2.1

Substituting the given coordinates

$$\left\|\begin{pmatrix}6\\-6\end{pmatrix}-\mathbf{O}\right\|^2 = r^2 \tag{4.2.2.2}$$

$$\left\|\begin{pmatrix}3\\-7\end{pmatrix}-\mathbf{O}\right\|^2 = r^2 \tag{4.2.2.3}$$

$$\left\|\begin{pmatrix}3\\3\end{pmatrix}-\mathbf{O}\right\|^2 = r^2 \tag{4.2.2.4}$$

From (4.2.2.2), (4.2.2.3), (4.2.2.4):

$$\left\|\begin{pmatrix}3\\-7\end{pmatrix}-\mathbf{O}\right\|^2 - \left\|\begin{pmatrix}6\\-6\end{pmatrix}-\mathbf{O}\right\|^2 = 0 \tag{4.2.2.5}$$

$$\left\|\begin{pmatrix}3\\3\end{pmatrix}-\mathbf{O}\right\|^2 - \left\|\begin{pmatrix}6\\-6\end{pmatrix}-\mathbf{O}\right\|^2 = 0 \tag{4.2.2.6}$$

Simplifying equations (4.2.2.5) and (4.2.2.6):

$$\begin{pmatrix}3 & 1\\1 & -3\end{pmatrix}\mathbf{O} = \begin{pmatrix}7\\9\end{pmatrix} \tag{4.2.2.7}$$

$$\begin{pmatrix}3 & 1 & 7\\1 & -3 & 9\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{R_1}{3}} \begin{pmatrix}1 & \frac{1}{3} & \frac{7}{3}\\1 & -3 & 9\end{pmatrix} \tag{4.2.2.8}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}1 & \frac{1}{3} & \frac{7}{3}\\1 & -\frac{10}{3} & \frac{20}{3}\end{pmatrix} \tag{4.2.2.9}$$

$$\xleftrightarrow{R_2\leftarrow\frac{-3R_2}{10}} \begin{pmatrix}1 & \frac{1}{3} & \frac{7}{3}\\1 & 1 & -2\end{pmatrix} \tag{4.2.2.10}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{R_2}{3}} \begin{pmatrix}1 & 0 & 3\\0 & 1 & -2\end{pmatrix} \tag{4.2.2.11}$$

$$\therefore \mathbf{O} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{4.2.2.12}$$

The following Python code generates Fig. 4.2.2.1

```
solutions/2/codes/circle_ex/circumcircle.py
```

3. Sketch the circles with
   a) centre $\begin{pmatrix}0\\2\end{pmatrix}$ and radius 2
   b) centre $\begin{pmatrix}-2\\32\end{pmatrix}$ and radius 4
   c) centre $\begin{pmatrix}\frac{1}{2}\\\frac{1}{4}\end{pmatrix}$ and radius $\frac{1}{12}$.
   d) centre $\begin{pmatrix}1\\1\end{pmatrix}$ and radius $\sqrt{2}$.
   e) centre $\begin{pmatrix}-a\\-b\end{pmatrix}$ and radius $\sqrt{a^2-b^2}$.

   Solution:

   a) Let $\mathbf{O}$ be the centre , r be the radius of the circle.Any point $\mathbf{X}$ lying on the circle is at a distance r from $\mathbf{O}$.
   Therefore the equation of the circle is

$$\|\mathbf{X}-\mathbf{O}\| = r \tag{4.2.3.1}$$

   b)

$$(a)\,\mathbf{O} = \begin{pmatrix}0\\2\end{pmatrix}, r = 2 \tag{4.2.3.2}$$

   The following code sketches the circle (4.2.3.2) in figure 4.2.3.1 using the equation (4.2.3.1)

```
solutions/3/codes/circle2/circle2a.py
```

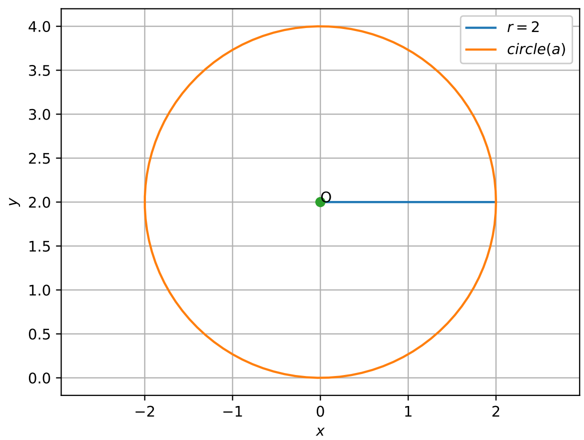

Fig. 4.2.3.1: Circle with centre at $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and radius 2

c)

$$(b)\, \mathbf{O} = \begin{pmatrix} -2 \\ 32 \end{pmatrix}, r = 4 \tag{4.2.3.3}$$

The following code sketches the circle (4.2.3.3) in figure 4.2.3.2 using the equation (4.2.3.1)

```
solutions/3/codes/circle2/circle2b.py
```

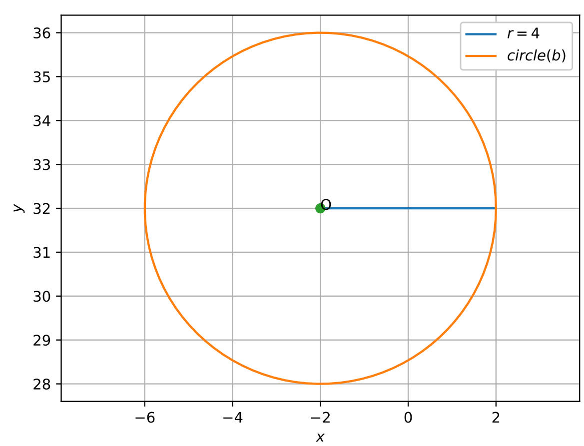

Fig. 4.2.3.2: Circle with centre at $\begin{pmatrix} -2 \\ 32 \end{pmatrix}$ and radius 4

d)

$$(c)\, \mathbf{O} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, r = \frac{1}{12} \tag{4.2.3.4}$$

The following code sketches the circle (4.2.3.4) in figure 4.2.3.3 using the equation (4.2.3.1)

```
solutions/3/codes/circle2/circle2c.py
```

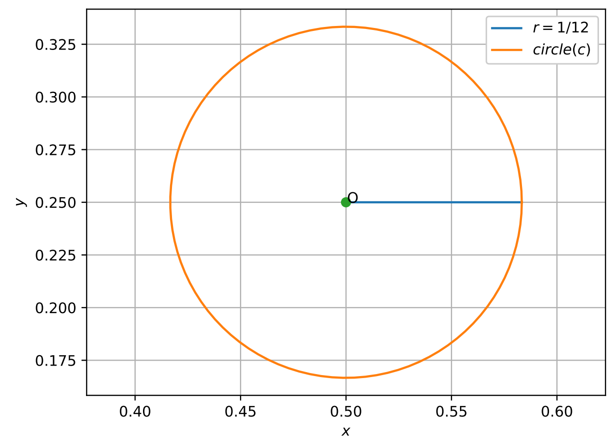

Fig. 4.2.3.3: Circle with centre at $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ and radius $\frac{1}{12}$

e)

$$(d)\, \mathbf{O} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, r = \sqrt{2} \tag{4.2.3.5}$$

The following code sketches the circle (4.2.3.5) in figure 4.2.3.4 using the equation (4.2.3.1)

```
solutions/3/codes/circle2/circle2d.py
```

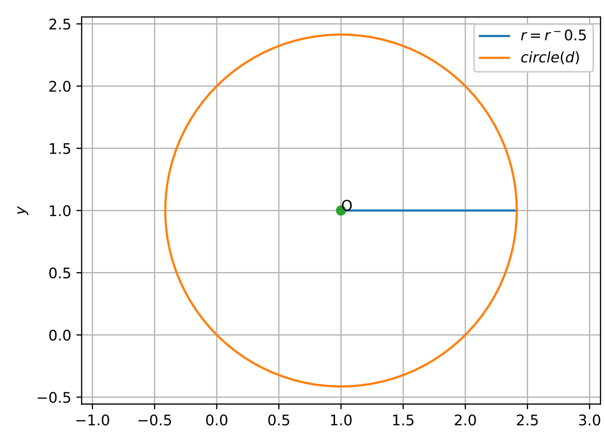

Fig. 4.2.3.4: Circle with centre at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and radius $\sqrt{2}$

f)

$$(e)\,\mathbf{O} = \begin{pmatrix}-a\\-b\end{pmatrix}, r = \sqrt{a^2 - b^2} \tag{4.2.3.6}$$

The parameters used to sketch the circle are taken as

$$a = 5, b = 4 \implies \mathbf{O} = \begin{pmatrix}-5\\-4\end{pmatrix} \tag{4.2.3.7}$$

$$r = \sqrt{5^2 - 4^2} = 3 \tag{4.2.3.8}$$

The following code sketches the circle (4.2.3.8) in figure 4.2.3.5 using the equation (4.2.3.1)

```
solutions/3/codes/circle2/circle2e.py
```

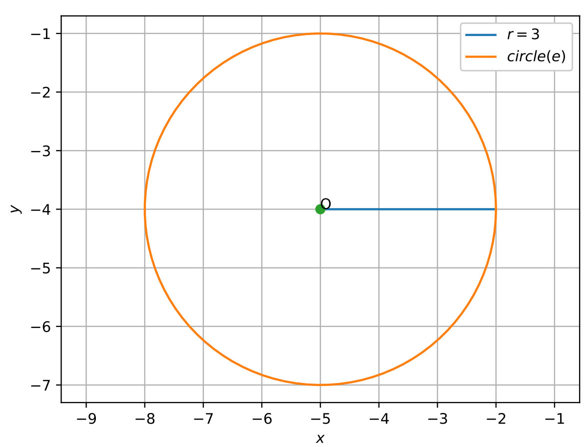

Fig. 4.2.3.5: Circle with centre at $\begin{pmatrix}-5\\-4\end{pmatrix}$ and radius 3

4. Does the point $\begin{pmatrix}-2.5\\3.5\end{pmatrix}$ lie inside, outside or on the circle $\mathbf{x}^T\mathbf{x} = 25$?

Solution: See Fig. 4.2.4.1. The general equation for the circle can be given as

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{O}^T\mathbf{x} + \|O\|^2 - \mathbf{r}^2 = 0 \tag{4.2.4.1}$$

given equation of circle

$$\mathbf{x}^T\mathbf{x} - 25 = 0 \tag{4.2.4.2}$$

comparing bothe of equation we can find the value of r and value of $\mathbf{O}$

$$\mathbf{r} = 4 \tag{4.2.4.3}$$

$$\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix} \tag{4.2.4.4}$$

$$\implies \mathbf{B} - \mathbf{O} = \begin{pmatrix}-2.5\\3.5\end{pmatrix} \tag{4.2.4.5}$$

$$\implies \|\mathbf{B} - \mathbf{O}\|^2 = 18.5 < 25 \tag{4.2.4.6}$$

$$\text{or, } OB < r \tag{4.2.4.7}$$

Hence, $\mathbf{B}$ lies inside the circle.
The following code plots Fig. 4.2.4.1

```
solutions/4/codes/circle/circle2.py
```

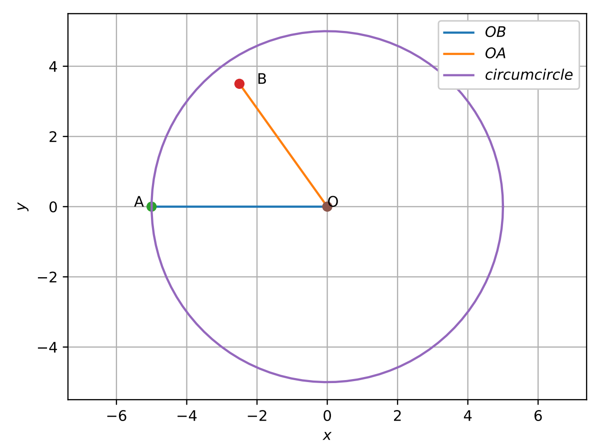

Fig. 4.2.4.1: circle

5. Sketch the circles with equation
   a) $\left\|\mathbf{x} - \begin{pmatrix}5\\-3\end{pmatrix}\right\|^2 = 36$
   b) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4\\8\end{pmatrix}\mathbf{x} - 45 = 0$
   c) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}8\\-10\end{pmatrix}\mathbf{x} - 12 = 0$
   d) $2\mathbf{x}^T\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\mathbf{x} = 0$

Solution: The following python codes generate the required circle

```
./solutions/5/codes/circle/q18abc.py
./solutions/5/codes/circle/q18d.py
```

a)

$$\mathbf{x^T}\mathbf{x} - \begin{pmatrix}4\\8\end{pmatrix}\mathbf{x} - 45 = 0 \tag{4.2.5.1}$$

See Fig. 4.2.5.1.

$$\mathbf{O} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, r = \sqrt{65} \tag{4.2.5.2}$$

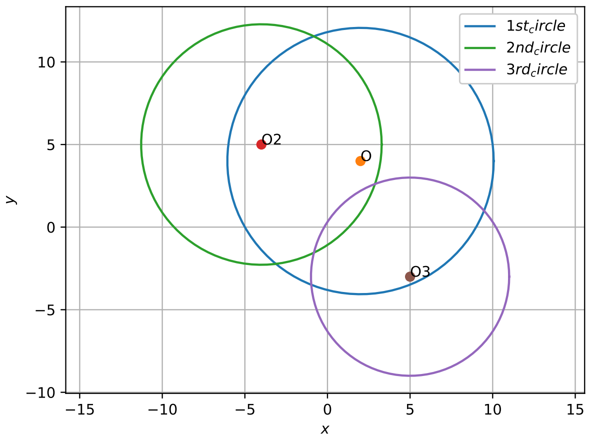

Fig. 4.2.5.1

b) See Fig. 4.2.5.1.

$$\mathbf{O} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}, r = \sqrt{53} \tag{4.2.5.3}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix} 8 \\ -10 \end{pmatrix}\mathbf{x} - 12 = 0 \tag{4.2.5.4}$$

c) See Fig. 4.2.5.1.

$$\mathbf{O} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, r = 6 \tag{4.2.5.5}$$

$$\left\| x - \begin{pmatrix} 5 \\ -3 \end{pmatrix} \right\| = 36 \tag{4.2.5.6}$$

d) See Fig. 4.2.5.2.

$$\mathbf{O} = \frac{1}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, r = \frac{1}{4} \tag{4.2.5.7}$$

$$2\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbf{x} = 0 \tag{4.2.5.8}$$

6. Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{x} = 16$.
Solution: The vector form of general equation

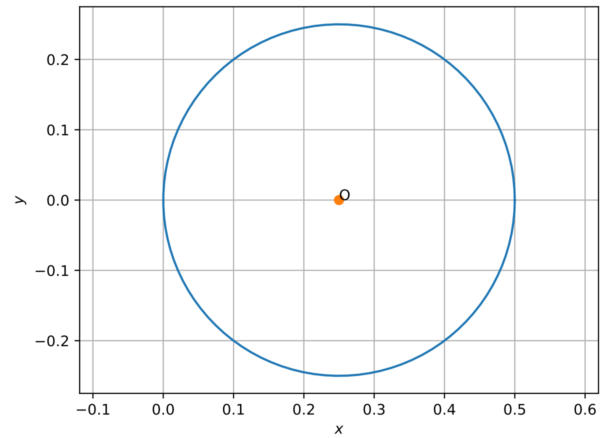

Fig. 4.2.5.2

of circle is,

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \tag{4.2.6.1}$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + \|\mathbf{O}\|^2 - r^2 = 0 \tag{4.2.6.2}$$

whose centre is $\mathbf{O}$ and radius r. $\because \mathbf{A} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on the circle. Letting

$$F = \|\mathbf{O}\|^2 - r^2, \tag{4.2.6.3}$$

$$\begin{pmatrix} 4 & 1 \end{pmatrix}^T \begin{pmatrix} 4 \\ 1 \end{pmatrix} - 2\mathbf{O}^T \begin{pmatrix} 4 \\ 1 \end{pmatrix} + F = 0 \tag{4.2.6.4}$$

$$\implies 2\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{O} - F = 17 \tag{4.2.6.5}$$

Similarly,

$$\begin{pmatrix} 6 & 5 \end{pmatrix}^T \begin{pmatrix} 6 \\ 5 \end{pmatrix} - 2\mathbf{O}^T \begin{pmatrix} 6 \\ 5 \end{pmatrix} + F = 0 \tag{4.2.6.6}$$

$$\implies 2\begin{pmatrix} 6 \\ 5 \end{pmatrix}\mathbf{O} - F = 61 \tag{4.2.6.7}$$

Subtracting 4.2.6.5 from 4.2.6.7,

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{O} = 11 \tag{4.2.6.8}$$

Also, from the given information,

$$\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{O} = 16 \tag{4.2.6.9}$$

From 4.2.6.9 and 4.2.6.8

$$\mathbf{O} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, F = 15 \tag{4.2.6.10}$$

and the vector form of the circle is

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} + 15 = 0 \tag{4.2.6.11}$$

The following code generates Fig. 4.2.6.1

```
solutions/6/codes/circle/exercise/circle.py
```

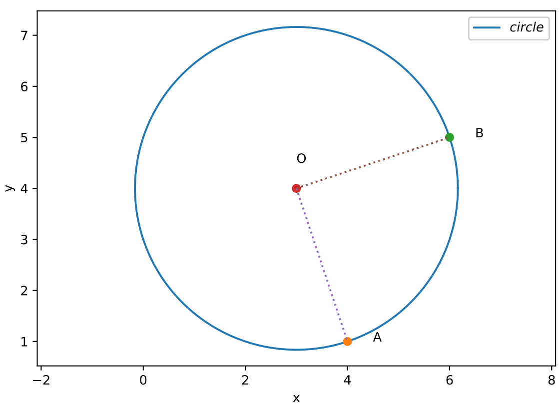

Fig. 4.2.6.1

7. Find the equation of the circle passing through the points $\mathbf{P} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}-1\\1\end{pmatrix}$ and whose centre is on the line $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 11$. **Solution:** Let $\mathbf{O}$ be the centre of the circle and r be the radius of the circle. Since centre lies on the given line

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{O} = 11 \tag{4.2.7.1}$$

Also

$$\|\mathbf{P} - \mathbf{O}\|^2 = \|\mathbf{Q} - \mathbf{O}\|^2 = r^2 \tag{4.2.7.2}$$

$$\implies \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{O} = 11 \tag{4.2.7.3}$$

From (4.2.7.1) and (4.2.7.3),

$$\begin{pmatrix}1 & -3\\6 & 4\end{pmatrix}\mathbf{O} = \begin{pmatrix}11\\11\end{pmatrix} \tag{4.2.7.4}$$

$$\implies \mathbf{O} = \begin{pmatrix}\frac{7}{2}\\\frac{-5}{2}\end{pmatrix} \tag{4.2.7.5}$$

From $\mathbf{O}$ we get r= 5.7. This is verified in Fig. 4.2.7.1 by the following python code.

```
solutions/7/codes/circle/circle.py
```

8. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.

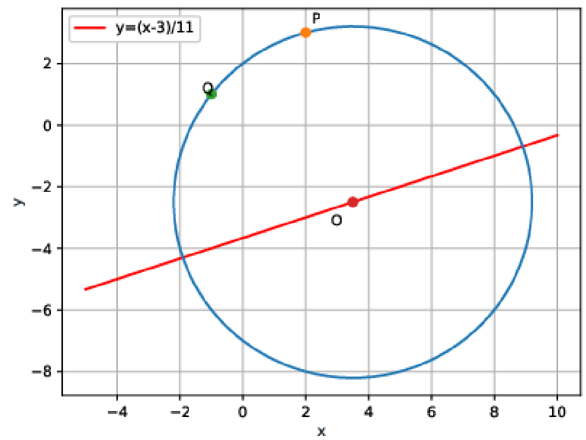

Fig. 4.2.7.1

9. Find the equation of the circle passing through $\begin{pmatrix}0\\0\end{pmatrix}$ and making intercepts a and b on the coordinate axes.
10. Find the equation of a circle with centre $\begin{pmatrix}2\\2\end{pmatrix}$ and passes through the point $\begin{pmatrix}4\\5\end{pmatrix}$.
11. Find the locus of all the unit vectors in the xy-plane.
12. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
13. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^T\mathbf{x} = 4$.
14. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
15. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.
16. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
17. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.
18. If $(x-a)^2+(y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{(1+y_2)^{\frac{3}{2}}}{y_2} \tag{4.2.18.1}$$

is a constant independent of a and b.
19. Form the differential equation of the family of circles touching the y-axis at origin.

20. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

5 Conics

5.1 Examples

1. Find the value of the following polynomial at the indicated value of variables

$$p(x) = 5x^2 – 3x + 7 \text{ at } x = 1. \tag{5.1.1.1}$$

2. Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.
Solution: The given polynomial can be expressed as the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{5.1.2.1}$$

$$\because \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \tag{5.1.2.2}$$

0 is a root.

$$\because \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 0 \tag{5.1.2.3}$$

2 is also a root. This is verified by plotting Fig. 5.1.2.1 through the following code.

```
solutions/2/codes/conics_example/conics.py
```

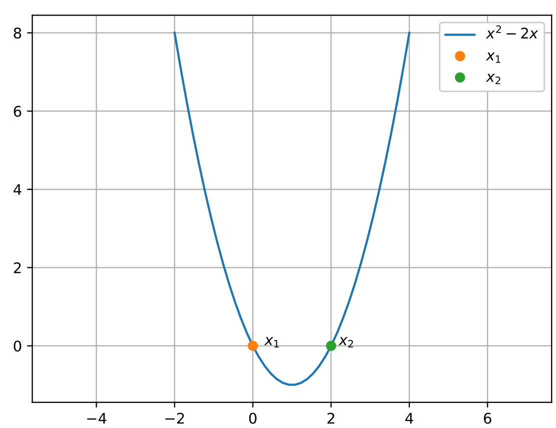

Fig. 5.1.2.1

3. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:

a) $p(y) = y^2$.
b) $p(x) = (x–1)(x + 1)$.

Solution:

a) To find p(0) we substitute 0 in place of the variable y in p(y). Similarly we find p(1) and p(2)

$$p(y) = y^2 \tag{5.1.3.1}$$
$$\implies p(0) = 0 \tag{5.1.3.2}$$
$$p(1) = 1 \tag{5.1.3.3}$$
$$p(2) = 4 \tag{5.1.3.4}$$

The following code sketches the graph of 5.1.3.1 in Fig. 5.1.3.1

```
solutions/3/codes/conic1/conic1a.py
```

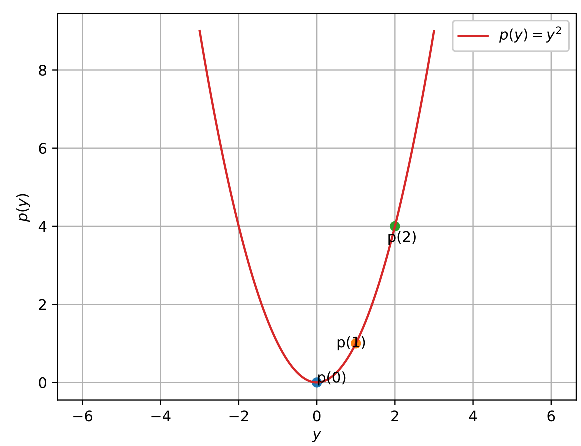

Fig. 5.1.3.1: Graph of p(y)

b) Similarly we find p(0), p(1) and p(2) of p(x) by replacing x

$$p(x) = (x - 1)(x + 1) \tag{5.1.3.5}$$
$$\implies p(0) = -1 \tag{5.1.3.6}$$
$$p(1) = 0 \tag{5.1.3.7}$$
$$p(2) = 3 \tag{5.1.3.8}$$

The following code sketches the graph of 5.1.3.5 in Fig. 5.1.3.2

```
solutions/3/codes/conic1/conic1b.py
```

4. Find the roots of the equation $2x^2–5x + 3 = 0$.
5. Find the roots of the quadratic equation $6x^2–x–2 = 0$.
Solution:

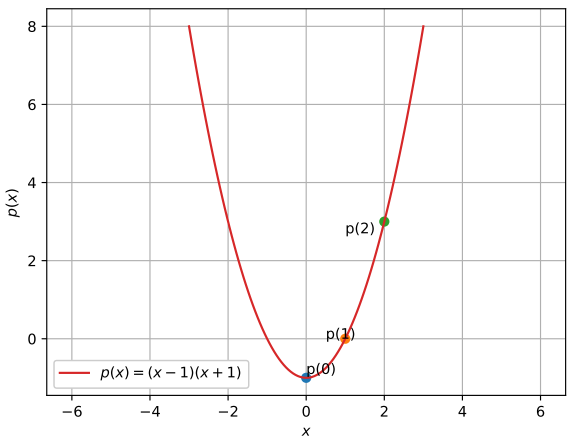

Fig. 5.1.3.2: Graph of p(x)

The vector form of

$$y = 6x^2 - x - 2 \tag{5.1.5.1}$$

is

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{5.1.5.2}$$

Thus,

$$y = 0 \quad \implies 6x^2 - x - 2 = 0 \tag{5.1.5.3}$$

$$\left(x + \frac{1}{2}\right)\left(x - \frac{2}{3}\right) = 0 \tag{5.1.5.4}$$

$$x = \frac{-1}{2}, \frac{2}{3} \tag{5.1.5.5}$$

The following python code computes roots of the quadratic equation represented in Fig. 5.1.5.1.

```
./solutions/5/codes/conics/q19.py
```

6. Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.
Solution: The vector form of the equation is

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2\sqrt{6} & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{5.1.6.1}$$

The values of $\mathbf{x}$ are found in the following python code

```
solutions/6/codes/conics/example/conics.py
```

$$\mathbf{x} = \begin{pmatrix} 0.81649658 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.81649658 \\ 0 \end{pmatrix}$$

which can be verified from Fig. 5.1.6.1 generated by the following python code

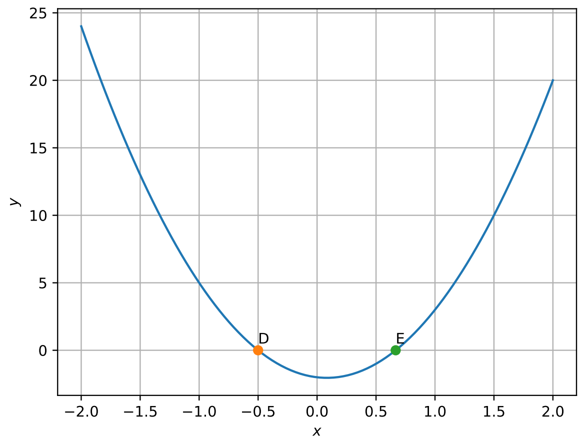

Fig. 5.1.5.1

```
codes/conics/example/conics.py
```

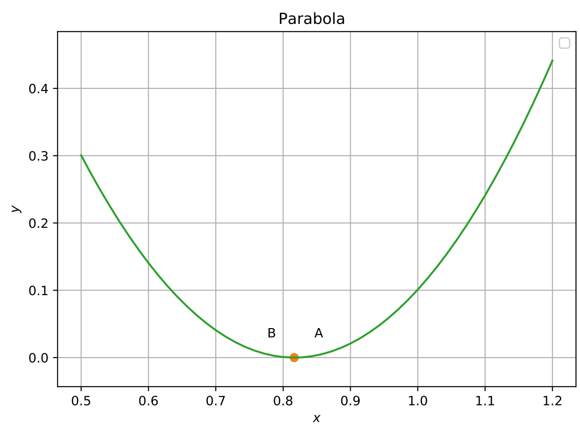

Fig. 5.1.6.1: Parabola

7. Factorise $6x^2 + 17x + 5$.
8. Factorise $y^2 – 5y + 6$.
9. Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$ and verify the relationship between the zeroes and the coefficients.
10. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.
11. Find a quadratic polynomial, the sum and product of whose zeroes are – 3 and 2, respectively.
12. Find the roots of the equation $5x^2 – 6x – 2 = 0$.
13. Find the roots of $4x^2 + 3x + 5 = 0$.
14. Find the roots of the following quadratic equations, if they exist.
    a) $3x^2 - 5x + 2 = 0$

b) $x^2 + 4x + 5 = 0$
c) $2x^2 - 2\sqrt{2}x + 1 = 0$

15. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$ hence find the nature of its roots.
16. Find the discriminant of the quadratic equation $3x^2 - 2x + \frac{1}{3} = 0$ hence find the nature of its roots.
17. Solve $x^2 + 2 = 0$.
18. Solve $x^2 + x + 1 = 0$.
19. Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$.
20. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.
21. Find the equation of the parabola with focus $\begin{pmatrix}2\\0\end{pmatrix}$ and directrix $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -2$.
22. Find the equation of the parabola with vertex at $\begin{pmatrix}0\\0\end{pmatrix}$ and focus at $\begin{pmatrix}0\\2\end{pmatrix}$.
23. Find the equation of the parabola which is symmetric about the y-axis, and passes through the point $\begin{pmatrix}2\\-3\end{pmatrix}$.
24. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse
$$\mathbf{x}^T\begin{pmatrix}\frac{1}{25} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1 \tag{5.1.24.1}$$
25. Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse
$$\mathbf{x}^T\begin{pmatrix}9 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 36 \tag{5.1.25.1}$$
26. Find the equation of the ellipse whose vertices are $\begin{pmatrix}\pm 13\\0\end{pmatrix}$ and foci are $\begin{pmatrix}\pm 5\\0\end{pmatrix}$.
27. Find the equation of the ellipse, whose length of the major axis is 20 and foci are $\begin{pmatrix}0\\\pm 5\end{pmatrix}$
28. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$.
29. Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas
    a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & -\frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
    b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -16\end{pmatrix}\mathbf{x} = 16$
30. Find the equation of the hyperbola with vertices $\begin{pmatrix}0\\\pm\frac{\sqrt{11}}{2}\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm 3\end{pmatrix}$
31. Find the equation of the hyperbola with foci $\begin{pmatrix}0\\\pm 12\end{pmatrix}$ and length of latus rectum 36.
32. Find the equation of all lines having slope 2 and being tangent to the curve
$$y + \frac{2}{x-3} = 0 \tag{5.1.32.1}$$
33. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.
34. Find the roots of the following equations:
    a) $x + \frac{1}{x} = 3, x \neq= 0$
    b) $\frac{1}{x} + \frac{1}{x-2} = 3, x \neq= 0, 2$
35. Find points on the curve $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\0 & \frac{1}{25}\end{pmatrix}\mathbf{x} = 1$ at which the tangents are
    a) parallel to x-axis
    b) parallel to y-axis
36. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $\begin{pmatrix}1\\2\end{pmatrix}$.
37. Find the area enclosed by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1$
38. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.
39. Find the area bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1$ and $x = ae$, where, $b^2 = a^2(1-e^2)$ and $e < 1$.
40. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.
41. Find the area of the region
$$\left\{(x, y) = 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\right\} \tag{5.1.41.1}$$
42. Find the intervals in which the function
$$f(x) = x^2 - 4x + 6 \tag{5.1.42.1}$$
is
    a) increasing

b) decreasing.

43. Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.
44. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x & x \geq 0 \\ x^2 & x < 0 \end{cases} \tag{5.1.44.1}$$

45. Verify Rolle's theorem for the function $y = x^2 + 2, a = -2$ and $b = 2$.
46. Verify Mean Value Theorem for the function $f(x) = x^2$ in the interval $[2, -4]$.
47. Find the derivative of $f(x) = x^2$.
48. Find the derivative of $x^2 - 2$ at $x = 10$.
49. Find the derivative of $(x-1)(x-2)$.
50. Find

$$\int_0^2 \left(x^2 + 1\right) dx \tag{5.1.50.1}$$

as a limit of a sum.

51. Evaluate the following integral:

$$\int_2^3 x^2\, dx \tag{5.1.51.1}$$

52. Form the differntial equation representing the family of ellipses having foci on x-axis and cenre at the origin.
53. Form the differntial equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.
54. Form a differntial equation representing the following family of curves

$$y^2 = a\left(b^2 - x^2\right) \tag{5.1.54.1}$$

55. A cricket ball is thrown at a speed of 28 ms^{-1} in a direction 30° above the horizontal. Calculate
    a) the maximum height,
    b) the time taken by the ball to return to the same level, and
    c) the distance from the thrower to the point where the ball returns to the same level.

5.2 *Exercises*

1. Verify whether the following are zeroes of the polynomial, indicated against them.
    a) $p(x) = x^2 - 1, x = 1, -1$
    b) $p(x) = (x+1)(x-2), x = -1, 2$
    c) $p(x) = x^2, x = 0.$
    d) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}.$

Solution: For a general polynomial equation of degree 2,
$p(x, y) \implies Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \tag{5.2.1.1}$$

a)

$$y = x^2 - 1 \tag{5.2.1.2}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{5.2.1.3}$$

For $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 = 0 \tag{5.2.1.4}$$

For $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 = 0 \tag{5.2.1.5}$$

Hence +1,-1 are zeros, which can be verified from Fig. 5.2.1.1 The python code for Fig. 5.2.1.1 is

```
solutions/1/codes/conics/parab1.py
```

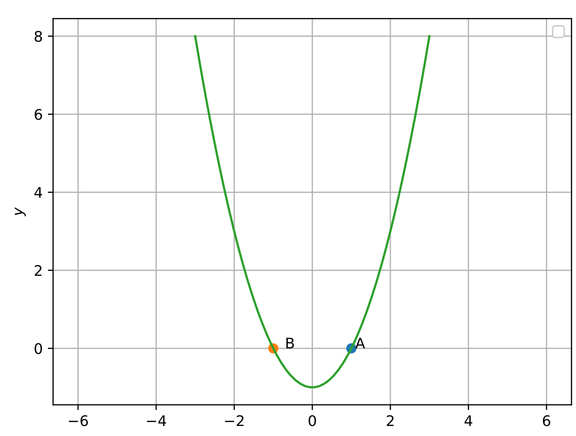

Fig. 5.2.1.1

b)

$$y = (x+1)(x-2) \tag{5.2.1.6}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{5.2.1.7}$$

For $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{5.2.1.8}$$

Similarly, for For $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{5.2.1.9}$$

Hence -1,+2 are zeros, which can be verified from Fig. 5.2.1.2 The python code is

solutions/1/codes/conics/parab2.py

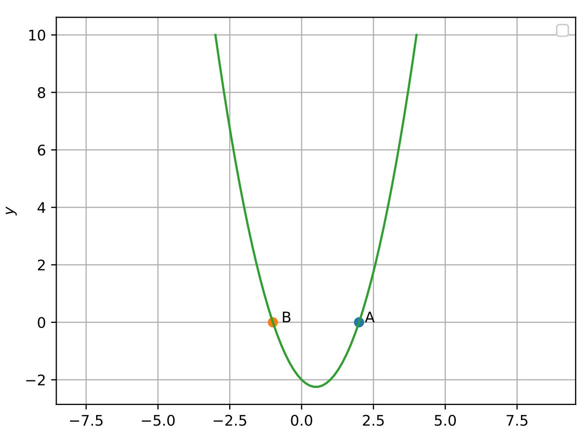

Fig. 5.2.1.2

c)

$$y = x^2 \tag{5.2.1.10}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.2.1.11}$$

For $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \tag{5.2.1.12}$$

Hence 0 is the zero, which can be verified from the Fig. 5.2.1.3. The python code is

codes/conics/parab3.py

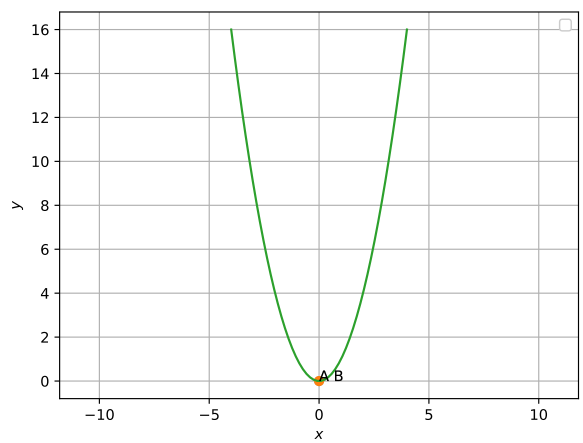

Fig. 5.2.1.3

d)

$$y = 3x^2 - 1 \tag{5.2.1.13}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{5.2.1.14}$$

For $\mathbf{x} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} - 1 = 0 \tag{5.2.1.15}$$

For $\mathbf{x} = \begin{pmatrix} \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} \frac{2}{\sqrt{3}} & 0 \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} - 1 \neq 0 \tag{5.2.1.16}$$

Hence $\frac{1}{\sqrt{3}}$ is a zero, but not $-\frac{2}{\sqrt{3}}$, which can be verified from Fig. 5.2.1.4 generated through the python code

```
solutions/1/codes/conics/parab4.py
```

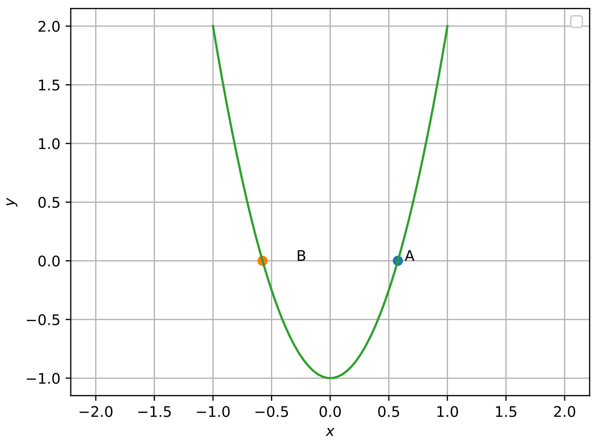

Fig. 5.2.1.4

2. Solve each of the following equations
   a) $3x^2 - 4x + \frac{20}{3} = 0$
   b) $x^2 - 2x + \frac{3}{2} = 0$
   c) $27x^2 - 10x + 1 = 0$
   d) $21x^2 - 28x + 10 = 0$

   Solution:

   a) To solve the equation - $3x^2 - 4x + \frac{20}{3} = 0$
   The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + \frac{20}{3} = 0 \quad (5.2.2.1)$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (5.2.2.2)$$

$$3x^2 - 4x + \frac{20}{3} = 0 \quad (5.2.2.3)$$

$$\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{2\sqrt{14}}{3} \end{pmatrix}\right)\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{-2\sqrt{14}}{3} \end{pmatrix}\right) = 0 \quad (5.2.2.4)$$

$$x = \begin{pmatrix} \frac{2}{3} \\ \frac{2\sqrt{14}}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2\sqrt{14}}{3} \end{pmatrix} \quad (5.2.2.5)$$

   Figure 5.2.2.1 show that the equation does not intersect the x-axis hence there are no real roots.

   b) To solve the equation - $x^2 - 2x + \frac{3}{2} = 0$
   The given equation can be represented as follows in the vector form

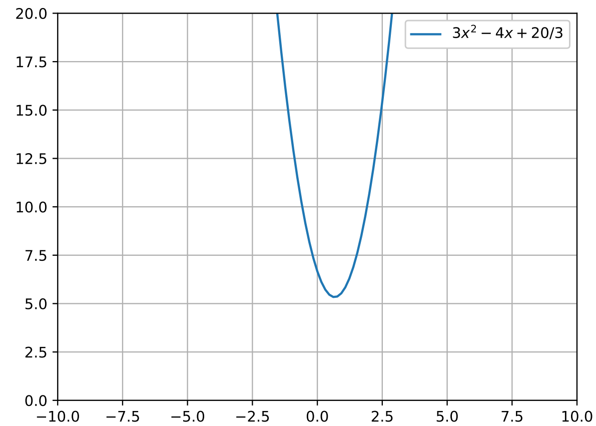

Fig. 5.2.2.1: $3x^2 - 4x + \frac{20}{3}$ generated using python

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + \frac{3}{2} = 0 \quad (5.2.2.6)$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (5.2.2.7)$$

$$x^2 - 2x + \frac{3}{2} = 0 \quad (5.2.2.8)$$

$$\left(x - \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right)\left(x - \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\right) = 0 \quad (5.2.2.9)$$

$$x = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad (5.2.2.10)$$

Figure 5.2.2.2 show that the equation does not intersect the x-axis hence there are no real roots.

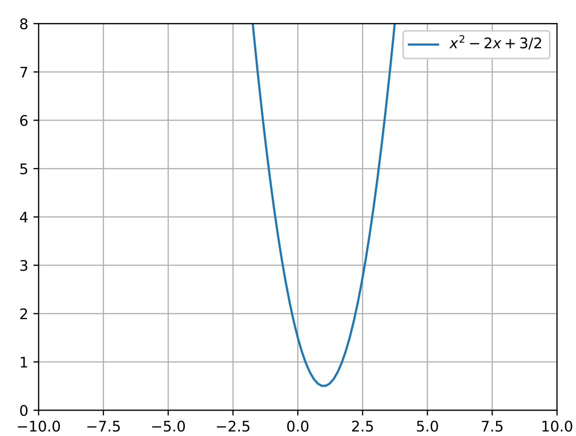

Fig. 5.2.2.2: $x^2 - 2x + \frac{3}{2}$ generated using python

c) To solve the equation - $27x^2 - 10x + 1 = 0$
The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 27 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -10 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.2.2.11}$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{5.2.2.12}$$

$$27x^2 - 10x + 1 = 0 \tag{5.2.2.13}$$

$$\left(x - \begin{pmatrix} \frac{5}{27} \\ \frac{\sqrt{2}}{27} \end{pmatrix}\right)\left(x - \begin{pmatrix} \frac{5}{27} \\ \frac{-\sqrt{2}}{27} \end{pmatrix}\right) = 0 \tag{5.2.2.14}$$

$$x = \begin{pmatrix} \frac{5}{27} \\ \frac{\sqrt{2}}{27} \end{pmatrix}, \begin{pmatrix} \frac{5}{27} \\ \frac{-\sqrt{2}}{27} \end{pmatrix} \tag{5.2.2.15}$$

Figure 5.2.2.3 show that the equation does not intersect the x-axis hence there are no real roots.

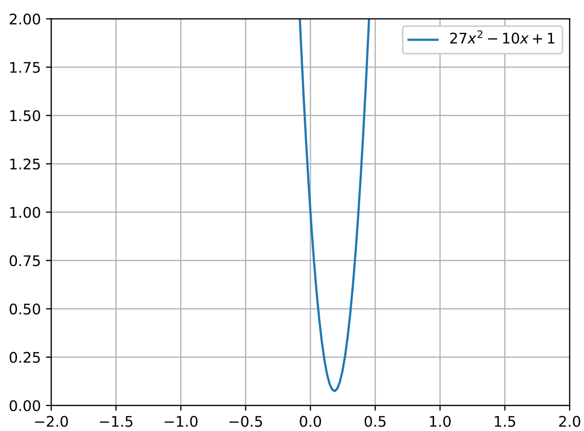

Fig. 5.2.2.3: $27x^2 - 10x + 1$ generated using python

d) To solve the equation - $21x^2 - 28x + 10 = 0$
The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -28 & 0 \end{pmatrix} \mathbf{x} + 10 = 0 \tag{5.2.2.16}$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{5.2.2.17}$$

$$21x^2 - 28x + 10 = 0 \tag{5.2.2.18}$$

$$\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{\sqrt{14}}{21} \end{pmatrix}\right)\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{-\sqrt{14}}{21} \end{pmatrix}\right) = 0 \tag{5.2.2.19}$$

$$x = \begin{pmatrix} \frac{2}{3} \\ \frac{\sqrt{14}}{21} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-\sqrt{14}}{21} \end{pmatrix} \tag{5.2.2.20}$$

Figure 5.2.2.4 show that the equation does not intersect the x-axis hence there are no real roots.

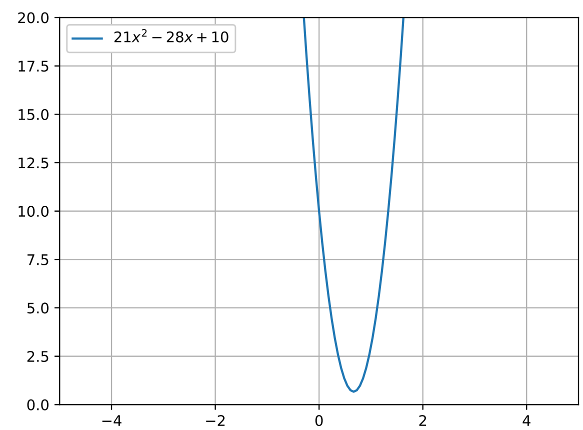

Fig. 5.2.2.4: $21x^2 - 28x + 10$ generated using python

The following Python code generates Fig.5.2.2.1, 5.2.2.2, 5.2.2.3 and 5.2.2.4

```
solutions/2/codes/conics_ex/conics_ex.py
```

3. Factorise
   a) $12x^2–7x + 1$
   b) $6x^2 + 5x–6$
   c) $2x^2 + 7x + 3$
   d) $3x^2–x–4$

Solution:

a)

$$(a)\, 12x^2 - 7x + 1 \tag{5.2.3.1}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 12 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.2.3.2}$$

To find roots using 5.2.3.2, substitute

$$y = 0 \tag{5.2.3.3}$$

$$\implies 12x^2 - 7x + 1 = 0 \tag{5.2.3.4}$$

$$x = \frac{1}{3}, \frac{1}{4} \tag{5.2.3.5}$$

Hence $\left(x - \frac{1}{3}\right)$ and $\left(x - \frac{1}{4}\right)$ are the factors

$$\implies (3x - 1)(4x - 1) = 12x^2 - 7x + 1 \tag{5.2.3.6}$$

The following code sketches the graph of 5.2.3.1 in figure 5.2.3.1

solutions/3/codes/conic2/conic2a.py

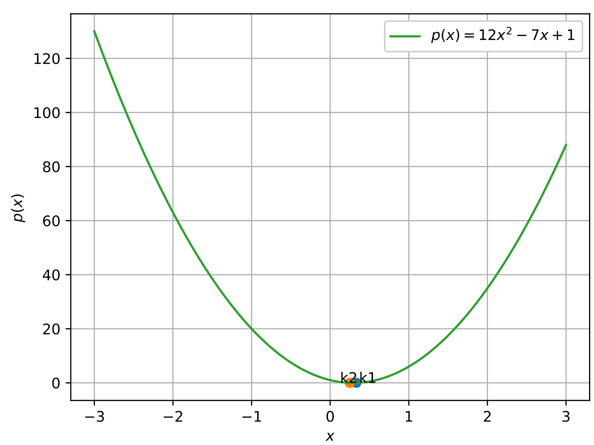

Fig. 5.2.3.1: Graph of $12x^2 - 7x + 1$

b)

$$(b)\, 6x^2 + 5x - 6 \tag{5.2.3.7}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{5.2.3.8}$$

Substituting $y = 0$ in equation 5.2.3.8 to find roots,

$$\implies 6x^2 + 5x - 6 = 0 \tag{5.2.3.9}$$

$$x = \frac{-3}{2}, \frac{2}{3} \tag{5.2.3.10}$$

$$(2x + 3)(3x - 2) = 6x^2 + 5x - 6 \tag{5.2.3.11}$$

The following code sketches the graph of 5.2.3.7 in figure 5.2.3.2

solutions/3/codes/conic2/conic2b.py

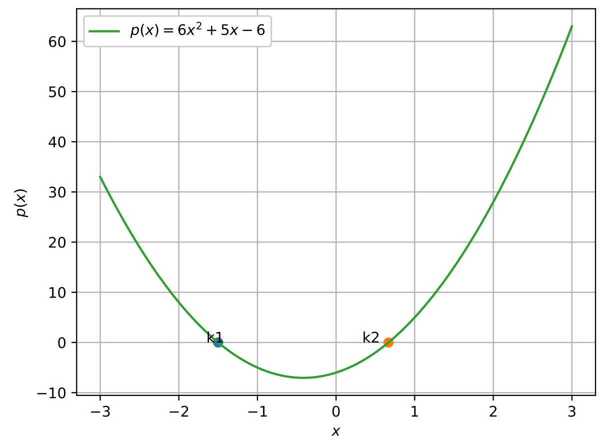

Fig. 5.2.3.2: Graph of $6x^2 + 5x - 6$

c)

$$(c)\, 2x^2 + 7x + 3 \tag{5.2.3.12}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{5.2.3.13}$$

Substituting $y = 0$ in equation 5.2.3.13,

$$\implies 2x^2 + 7x + 3 = 0 \tag{5.2.3.14}$$

$$x = \frac{-1}{2}, -3 \tag{5.2.3.15}$$

$$(2x + 1)(x + 3) = 2x^2 + 7x + 3 \tag{5.2.3.16}$$

The following code sketches the graph of 5.2.3.12 in figure 5.2.3.3

solutions/3/codes/conic2/conic2c.py

d)

$$(d)\, 3x^2 - x - 4 \tag{5.2.3.17}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{5.2.3.18}$$

Substituting $y = 0$ in equation 5.2.3.13,

$$\implies 3x^2 - x - 4 = 0 \tag{5.2.3.19}$$

$$x = \frac{4}{3}, -1 \tag{5.2.3.20}$$

$$(3x - 4)(x + 1) = 3x^2 - x - 4 \tag{5.2.3.21}$$

The following code sketches the graph of 5.2.3.17 in figure 5.2.3.4

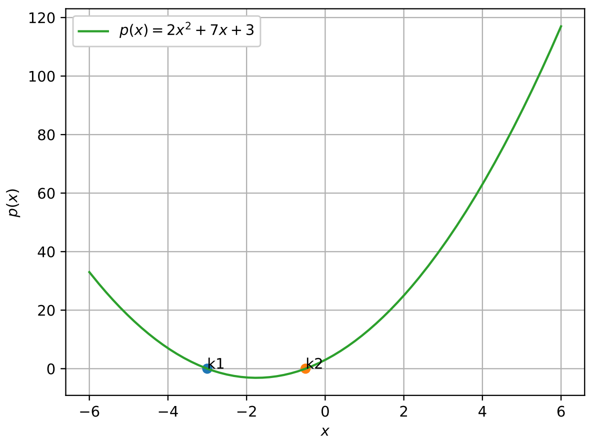

Fig. 5.2.3.3: Graph of $2x^2 + 7x + 3$

```
solutions/3/codes/conic2/conic2d.py
```

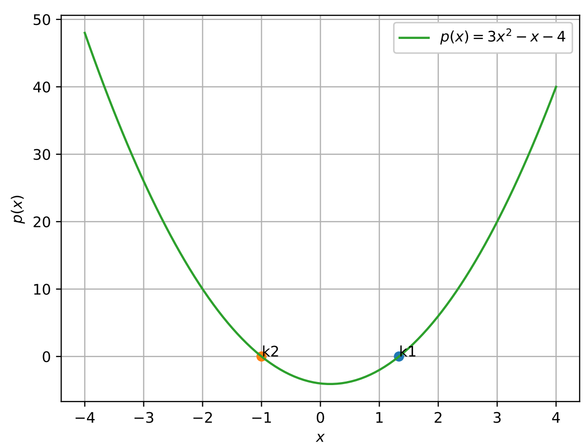

Fig. 5.2.3.4: Graph of $3x^2 - x - 4$

4. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.
   a) $x^2–2x–8$
   b) $4u^2 + 8u$
   c) $4s^2–4s + 1$
   d) $t^2–15$
   e) $6x^2–3–7x$
   f) $3x^2–x–4$

   Solution:

1. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} - 8 = 0 \quad (5.2.4.1)$$

$$x^2 - 2x - 8 = 0 \quad (5.2.4.2)$$

$$(x - 4)(x + 2) = 0 \quad (5.2.4.3)$$

$$\alpha = 4, \beta = -2 \quad (5.2.4.4)$$

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \quad (5.2.4.5)$$

$$\alpha + \beta = -\frac{b}{a} = 2 \quad (5.2.4.6)$$

$$\alpha \times \beta = \frac{c}{a} = -8 \quad (5.2.4.7)$$

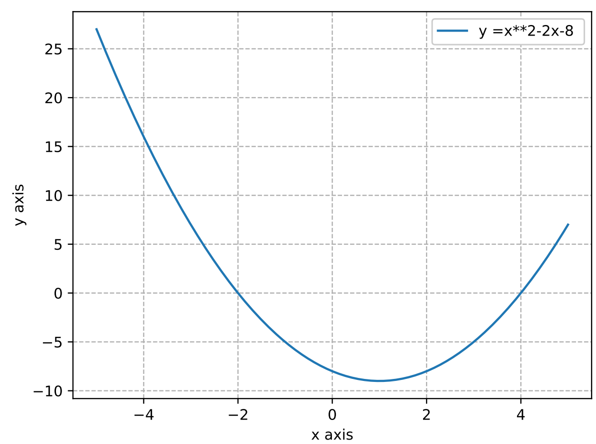

Fig. 5.2.4.1

```
solutions/4/codes/conics/perabola2.py
```

2. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (5.2.4.8)$$

$$4u^2 + 8u = 0 \quad (5.2.4.9)$$

$$(4u)(u + 2) = 0 \quad (5.2.4.10)$$

$$\alpha = 0, \beta = -2 \quad (5.2.4.11)$$

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \quad (5.2.4.12)$$

$$\alpha + \beta = -\frac{b}{a} = -2 \quad (5.2.4.13)$$

$$\alpha \times \beta = \frac{c}{a} = 0 \quad (5.2.4.14)$$

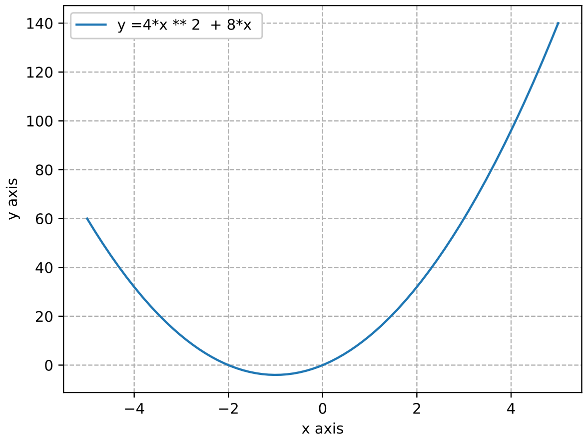

Fig. 5.2.4.2: equation 2

solutions/4/codes/conics/perabola2.py

3. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.2.4.15}$$

$$4s^2 - 4s + 1 = 0 \tag{5.2.4.16}$$

$$(2s - 1)(2s - 1) = 0 \tag{5.2.4.17}$$

$$\alpha = \frac{1}{2}, \beta = -\frac{1}{2} \tag{5.2.4.18}$$

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \tag{5.2.4.19}$$

$$\alpha + \beta = -\frac{b}{a} = 1 \tag{5.2.4.20}$$

$$\alpha \times \beta = \frac{c}{a} = \frac{1}{4} \tag{5.2.4.21}$$

solutions/4/codes/conics/perabola3.py

4. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 15 = 0 \tag{5.2.4.22}$$

$$t^2 - 15 = 0 \tag{5.2.4.23}$$

$$\alpha = \sqrt{15}, \beta = -\sqrt{15} \tag{5.2.4.24}$$

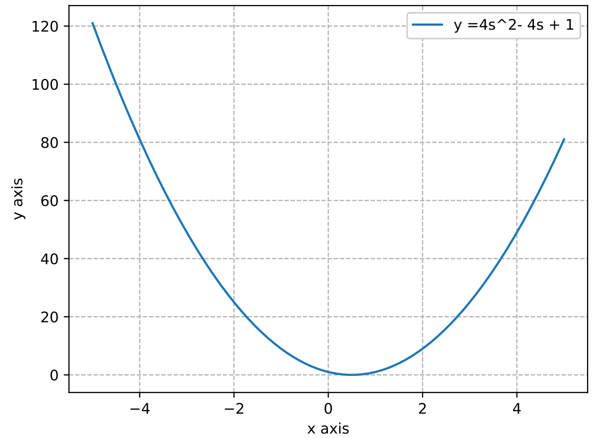

Fig. 5.2.4.3: equation 3

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \tag{5.2.4.25}$$

$$\alpha + \beta = -\frac{b}{a} = 0 \tag{5.2.4.26}$$

$$\alpha \times \beta = \frac{c}{a} = -15 \tag{5.2.4.27}$$

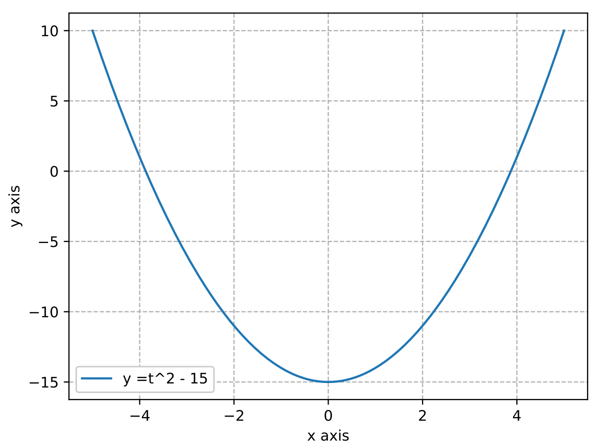

Fig. 5.2.4.4: equation 4

solutions/4/codes/conics/perabola4.py

5. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} - 3 = 0 \quad (5.2.4.28)$$

$$6x^2 - 3 - 7x = 0 \quad (5.2.4.29)$$

$$(2x - 3)(3x \ + 1) = 0 \quad (5.2.4.30)$$

$$\alpha = \frac{3}{2}, \beta = -\frac{1}{3} \quad (5.2.4.31)$$

$$\alpha + \beta = -\frac{b}{a} = \frac{7}{6} \quad (5.2.4.32)$$

$$\alpha \times \beta = \frac{c}{a} = -\frac{1}{2} \quad (5.2.4.33)$$

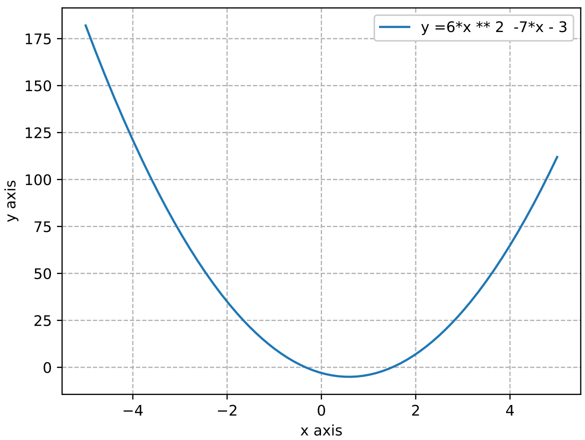

Fig. 5.2.4.5: equation 5

solutions/4/codes/conics/perabola5.py

6. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (5.2.4.34)$$

$$3x^2 - 2x - 8 = 0 \quad (5.2.4.35)$$

$$(3x + 4)(x + 1) = 0 \quad (5.2.4.36)$$

$$\alpha = -1, \beta = -\frac{4}{3} \quad (5.2.4.37)$$

solutions/4/codes/conis/perabola6.py

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \quad (5.2.4.38)$$

$$\alpha + \beta = -\frac{b}{a} = \frac{2}{3} \quad (5.2.4.39)$$

$$\alpha \times \beta = \frac{c}{a} = -\frac{8}{3} \quad (5.2.4.40)$$

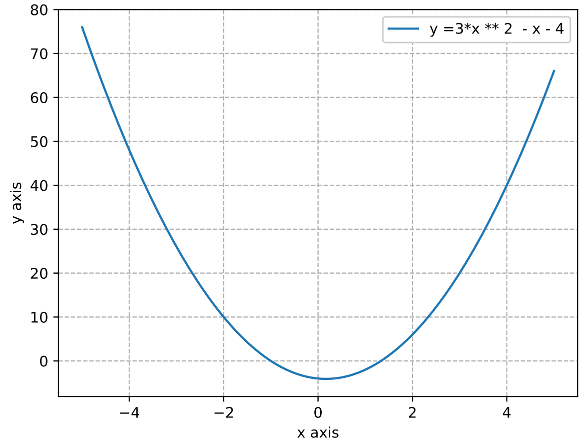

Fig. 5.2.4.6: equation 6

5. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

a) -1 , $\frac{1}{4}$
b) 1, 1
c) 0, $\sqrt{5}$
d) 4, 1
e) $\frac{1}{4}, \frac{1}{4}$
f) $\sqrt{2}, \frac{1}{3}$

Solution: The following python code computes roots of the quadratic equation obtained:

./solutions/5/codes/conics/q20a.py ./solutions/5/codes/conics/q20b.py ./solutions/5/codes/conics/q20c.py ./solutions/5/codes/conics/q20d.py ./solutions/5/codes/conics/q20e.py ./solutions/5/codes/conics/q20f.py

a) -1,$\frac{1}{4}$

For a general polynomial equation of degree 2,

$$p(x, y) =$$
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (5.2.5.1)$$

Here, sum of zeroes = D = -1
Product of zeroes = F =$\frac{1}{4}$
Substituing the values in 5.2.5.1,

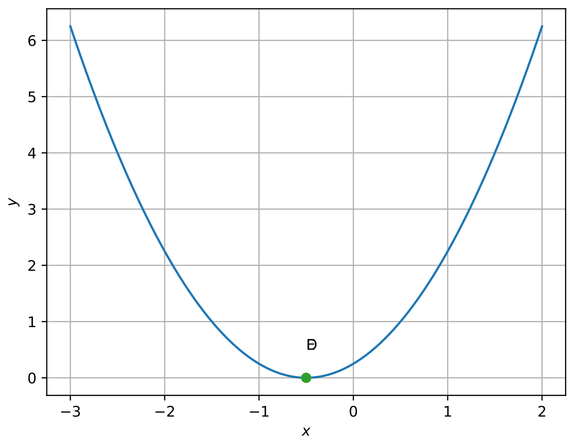

Fig. 5.2.5.1

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{1}{4} = 0 \tag{5.2.5.2}$$

$$\implies y = x^2 + x + \frac{1}{4} \tag{5.2.5.3}$$

The roots are -0.5 and -0.5 as represented in Fig. 5.2.5.1

b) 1,1

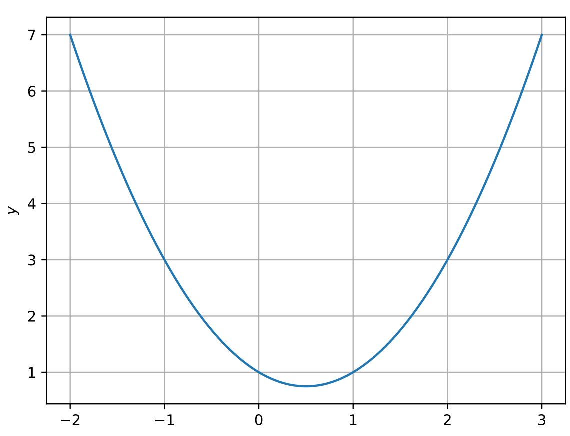

Fig. 5.2.5.2

Here, sum of zeroes = D = 1
Product of zeroes = F =1
Substituing the values in 5.2.5.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.2.5.4}$$

$$\implies y = x^2 - x + 1 \tag{5.2.5.5}$$

Since the curve doesn't meet the x-axis, real roots don't exist for this parabola as represented in Fig. 5.2.5.2

c) $0, \sqrt{5}$

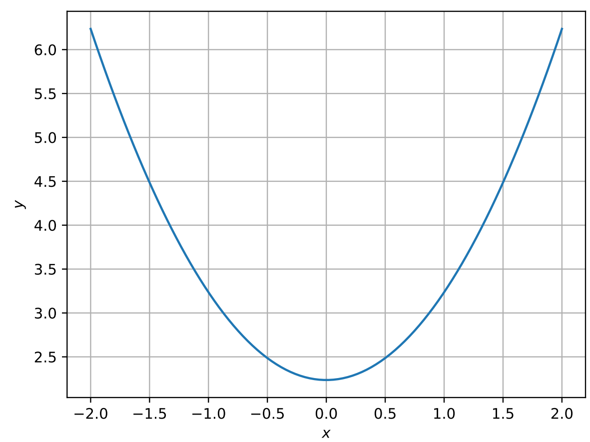

Fig. 5.2.5.3

Here, sum of zeroes = D = 0
Product of zeroes = F = $\sqrt{5}$
Substituing the values in 5.2.5.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + \sqrt{5} = 0 \tag{5.2.5.6}$$

$$\implies y = x^2 + \sqrt{5} \tag{5.2.5.7}$$

Since the curve doesn't meet the x-axis, real roots don't exist for this parabola as represented in Fig. 5.2.5.3

d) 4,1

Here, sum of zeroes = D = 4
Product of zeroes = F = 1
Substituing the values in 5.2.5.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.2.5.8}$$

$$\implies y = x^2 - 4x + 1 \tag{5.2.5.9}$$

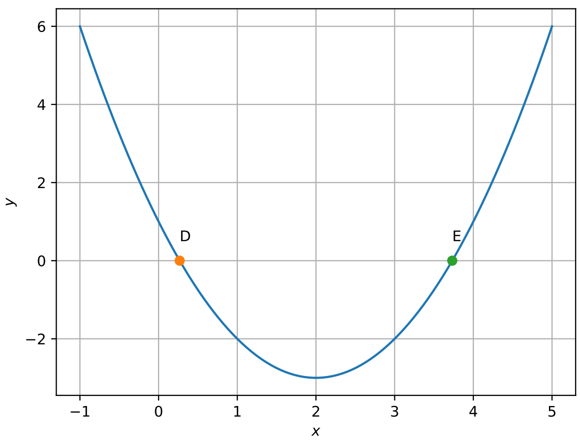

Fig. 5.2.5.4

The roots are 3.73 and 0.26 as represented in Fig. 5.2.5.4

e) $\frac{1}{4}, \frac{1}{4}$

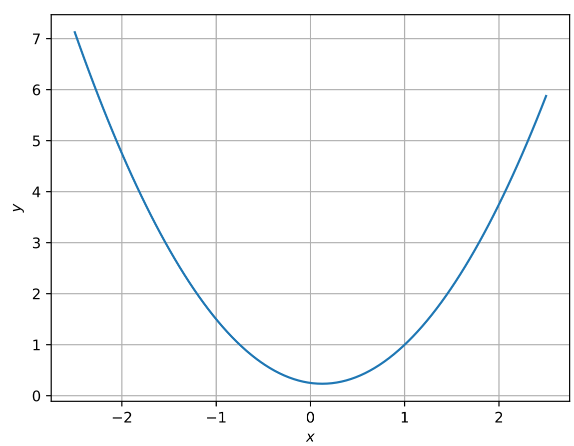

Fig. 5.2.5.5

Here, sum of zeroes = D = $\frac{1}{4}$
Product of zeroes = F = $\frac{1}{4}$
Substituing the values in 5.2.5.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\frac{1}{4} & -1 \end{pmatrix} \mathbf{x} + \frac{1}{4} = 0 \tag{5.2.5.10}$$

$$\implies y = x^2 - \frac{1}{4}x + \frac{1}{4} \tag{5.2.5.11}$$

Since the curve doesn't meet the x-axis, real roots don't exist for this parabola as represented in Fig. 5.2.5.5

f) $\sqrt{2}, \frac{1}{3}$

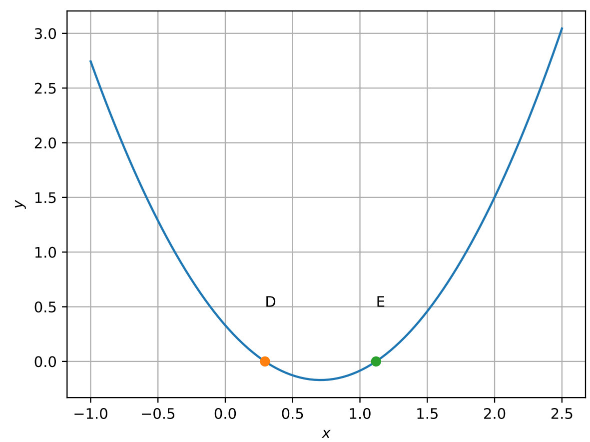

Fig. 5.2.5.6

Here, sum of zeroes = D = $\sqrt{2}$
Product of zeroes = F = $\frac{1}{3}$
Substituing the values in 5.2.5.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\sqrt{2} & -1 \end{pmatrix} \mathbf{x} + \frac{1}{3} = 0 \tag{5.2.5.12}$$

$$\implies y = x^2 - \sqrt{2}x + \frac{1}{3} \tag{5.2.5.13}$$

The roots are 1.11 and 0.29 as represented in Fig. 5.2.5.6

6. Find the roots of the following quadratic equations:
   a) $x^2–3x–10 = 0$
   b) $2x^2 + x - 6 = 0$
   c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
   d) $2x^2–x + \frac{1}{8} = 0$
   e) $100x^2–20x + 1 = 0$

   Solution:

   a) $x^2 - 3x - 10 = 0$
   The vector form from the equation is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} - 10 = 0 \tag{5.2.6.1}$$

The values of $\mathbf{x}$ are found in the following python code solutions/6/codes/conics/exercise/conics_1.py

$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
which can be verified from the Fig.5.2.6.1. The following python code generates the fig.5.2.6.1

solutions/6/codes/conics/exercise/conics_1.py

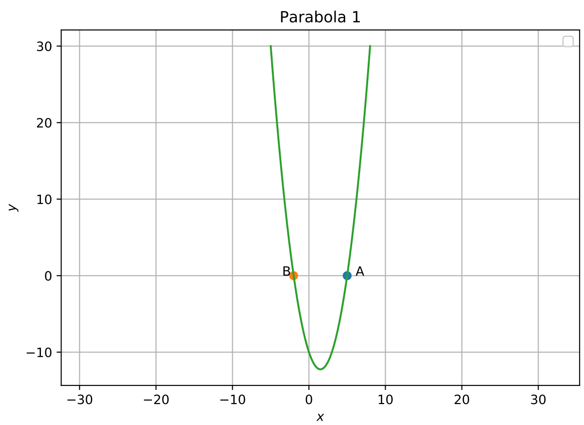

Fig. 5.2.6.1: Parabola 1

b) $2x^2 + x - 6 = 0$
The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 6 = 0 \quad (5.2.6.2)$$

The values of $\mathbf{x}$ are found in the following python code

solutions/6/codes/conics/exercise/conics_2.py

$\mathbf{x} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
which can be verified from the Fig.5.2.6.2. The following python code generates the fig.5.2.6.2

solutions/6/codes/conics/exercise/conics_2.py

c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 5\sqrt{2} = 0 \quad (5.2.6.3)$$

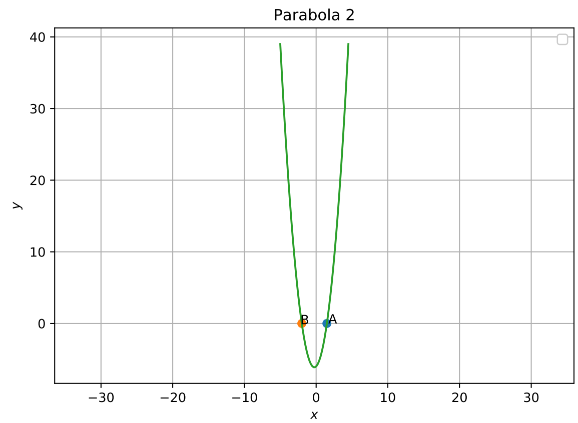

Fig. 5.2.6.2: Parabola 2

The values of $\mathbf{x}$ are found in the following python code

solutions/6/codes/conics/exercise/conics_3.py

$\mathbf{x} = \begin{pmatrix} -1.414 \\ 0 \end{pmatrix}, \begin{pmatrix} -3.535 \\ 0 \end{pmatrix}$ which can be verified from the Fig.5.2.6.3. The following python code generates the fig.5.2.6.3

solutions/6/codes/conics/exercise/conics_3.py

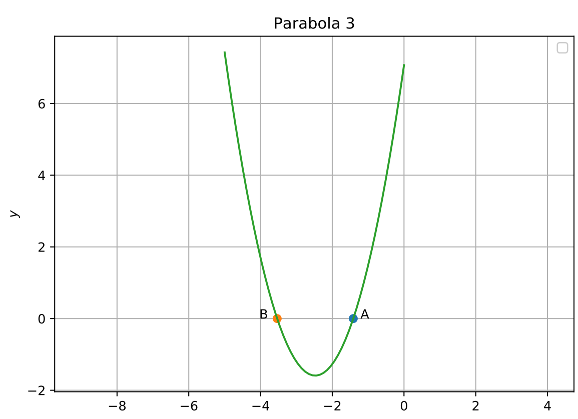

Fig. 5.2.6.3: Parabola 3

d) $2x^2 - x + \frac{1}{8} = 0$
The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + \frac{1}{8} = 0 \quad (5.2.6.4)$$

The values of $\mathbf{x}$ are found in the following python code solutions/6/codes/conics/exercise/conics_4.py

$\mathbf{x} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$
which can be verified from the Fig.5.2.6.4. The following python code generates the fig.5.2.6.4

solutions/6/codes/conics/exercise/conics_4.py

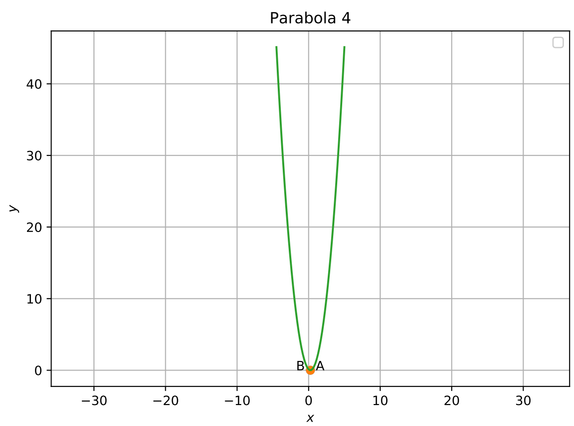

Fig. 5.2.6.4: Parabola 4

e) $100x^2 - 20x + 1 = 0$
The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -20 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.2.6.5}$$

The values of $\mathbf{x}$ are found in the following python code

solutions/6/codes/conics/exercise/conics_5.py

$\mathbf{x} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$ which can be verified from the Fig.5.2.6.5. The following python code generates the fig.5.2.6.5

solutions/6/codes/conics/exercise/conics_5.py

7. Find the roots of the following quadratic equations
   a) $2x^2 - 7x + 3 = 0$
   b) $2x^2 + x - 4 = 0$

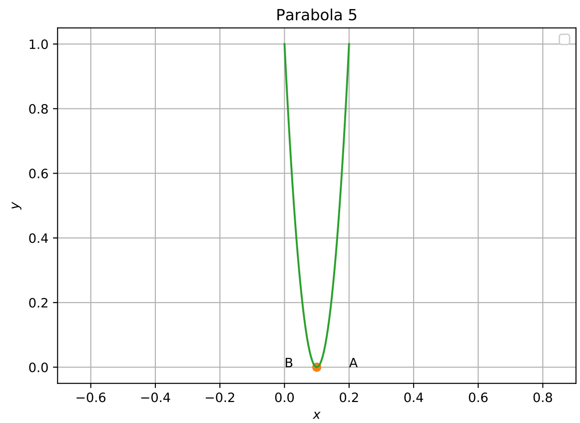

Fig. 5.2.6.5: Parabola 5

   c) $4x^2 + 4\sqrt{3}x + 3 = 0$
   d) $2x^2 + x + 4 = 0$

Solution:

a) $2x^2 - 7x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{5.2.7.1}$$

If $\begin{pmatrix} k \\ 0 \end{pmatrix}$ satisfies (5.2.7.1) then k is the root of the equation (5.2.7.1).
From graph, the roots are the points where the quadratic equation cuts the x-axis. A quadratic equation can have a maximum of two distinct roots.

$$2k^2 - 7k + 3 = 0 \tag{5.2.7.2}$$
$$(k-3)(2k-1) = 0 \tag{5.2.7.3}$$

From the graph in 5.2.7.1, the roots are 3 and $\frac{1}{2}$. The python code can be downloaded from

solutions/7/codes/conics/parabola1.py

b) $2x^2 + x - 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{5.2.7.4}$$

From the 5.2.7.2, the roots are 1.186 and 1.686. The python code can be downloaded from

solutions/7/codes/conics/parabola2.py

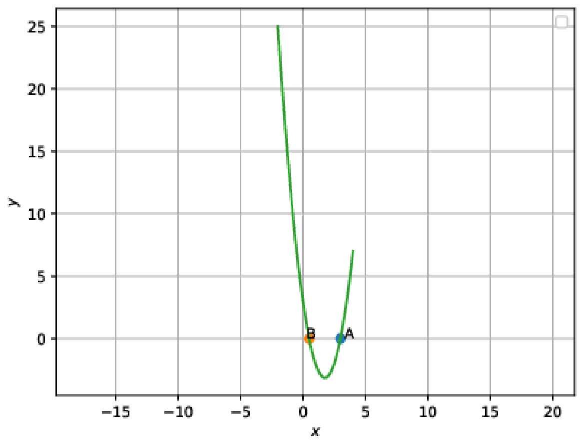

Fig. 5.2.7.1: Roots of $2x^2 - 7x + 3 = 0$

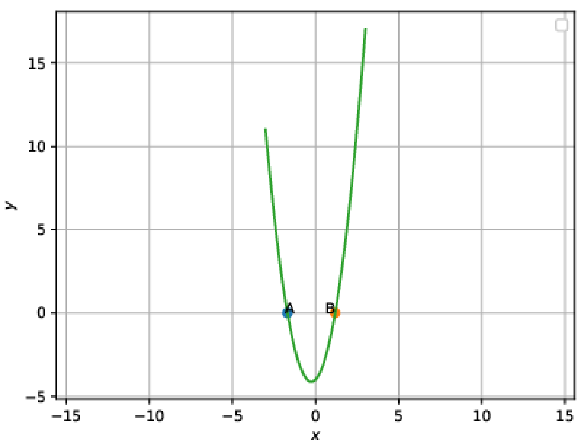

Fig. 5.2.7.2: Roots of $2x^2 + x - 4 = 0$

c) $4x^2 + 4\sqrt{3}x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4\sqrt{3} & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (5.2.7.5)$$

From the graph in 5.2.7.3, the roots are real and equal. The root is $\frac{-\sqrt{3}}{2}$. The python code can be downloaded from

solutions/7/codes/conics/parabola3.py

d) $2x^2 + x + 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 4 = 0 \quad (5.2.7.6)$$

From the graph 5.2.7.4, the quadratic equation doesn't intersect x-axis. Thus it doesn't have real roots. It has ccomplex and conju-

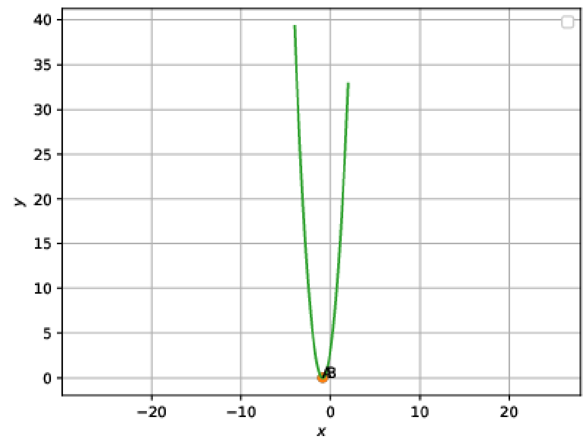

Fig. 5.2.7.3: Roots of $4x^2 + 4\sqrt{3}x + 3 = 0$

gate roots. The python code can be down-

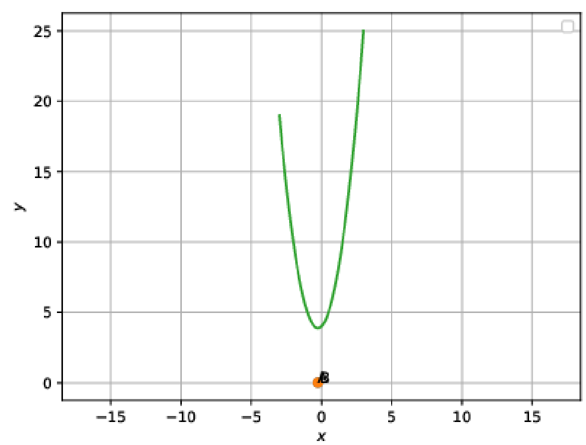

Fig. 5.2.7.4: Roots of $2x^2 + x + 4 = 0$

loaded from

solutions/7/codes/conics/parabola4.py

8. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:
   a) $2x^2 - 3x + 5 = 0$
   b) $2x^2 - 6x + 3 = 0$
   c) $3x^2 - 4\sqrt{3}x + 4 = 0$
9. Solve each of the following equations
   a) $x^2 + 3 = 0$
   b) $2x^2 + x + 1 = 0$
   c) $x^2 + 3x + 9 = 0$
   d) $-x^2 + x - 2 = 0$

e) $x^2 + 3x + 5 = 0$
f) $x^2 - 3x + 2 = 0$
g) $\sqrt{2}x^2 + x + \sqrt{2} = 0$
h) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
i) $x^2 + x + \frac{1}{\sqrt{2}} = 0$
j) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

10. In each of the following exercises, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum
a) $y^2 = 12x$
b) $x^2 = 6y$
c) $y^2 = -8x$
d) $x^2 = -16y$
e) $y^2 = 10x$
f) $x^2 = -9y$

11. In each of the following exercises, find the equation of the parabola that satisfies the following conditions:
a) Focus $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$, directrix $\begin{pmatrix} 1 & 0 \end{pmatrix} = -6$.
b) Focus $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$, directrix $\begin{pmatrix} 0 & 1 \end{pmatrix} = 3$.
c) Focus $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$.
d) Focus $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$.
e) vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$ passing through $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and axis is along the x-axis
f) vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$ passing through $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and symmetric with respect to the y-axis.

12. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.
a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{36} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$
b) $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \mathbf{x} = 1$
c) $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
d) $\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{100} \end{pmatrix} \mathbf{x} = 1$
e) $\mathbf{x}^T \begin{pmatrix} \frac{1}{49} & 0 \\ 0 & \frac{1}{36} \end{pmatrix} \mathbf{x} = 1$
f) $\mathbf{x}^T \begin{pmatrix} \frac{1}{100} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$
g) $\mathbf{x}^T \begin{pmatrix} 36 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 144$
h) $\mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 16$
i) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 36$

13. In each of the following, find the equation for the ellipse that satisfies the given conditions:
a) Vertices $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$
b) Vertices $\begin{pmatrix} 0 \\ \pm 13 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$
c) Vertices $\begin{pmatrix} \pm 6 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$
d) Ends of major axis $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} 0 \\ \pm 2 \end{pmatrix}$
e) Ends of major axis $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$
f) Length of major axis 26, foci $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$
g) Length of minor axis 16, foci $\begin{pmatrix} 0 \\ \pm 6 \end{pmatrix}$.
h) Foci $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$, a = 4
i) b = 3, c = 4, centre at the origin; foci on the x axis.
j) Centre at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, major axis on the y-axis and passes through the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$.
k) Major axis on the x-axis and passes through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

14. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.
a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & -\frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
b) $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{27} \end{pmatrix} \mathbf{x} = 1$
c) $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x} = 36$

d) $\mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 576$

e) $\mathbf{x}^T \begin{pmatrix} 5 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 36$

f) $\mathbf{x}^T \begin{pmatrix} 49 & 0 \\ 0 & -16 \end{pmatrix} \mathbf{x} = 784$

15. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$

b) Vertices $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 8 \end{pmatrix}$

c) Vertices $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$

d) Transverse axis length 8, foci $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$.

e) Conjugate axis length 24, foci $\begin{pmatrix} 0 \\ \pm 13 \end{pmatrix}$.

f) Latus rectum length 8, foci $\begin{pmatrix} \pm 3\sqrt{5} \\ 0 \end{pmatrix}$.

g) Latus rectum lenght 12, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$.

h) Ends of major axis $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$

i) Vertices $\begin{pmatrix} \pm 7 \\ 0 \end{pmatrix}$, $e = \frac{4}{3}$

j) Foci $\begin{pmatrix} 0 \\ \pm\sqrt{10} \end{pmatrix}$, passing through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

16. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

17. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

18. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

19. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.

20. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at which tangents are

a) parallel to x-axis

b) parallel to y-axis.

21. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

22. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$

a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$

b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.

23. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0$.

24. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.

25. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.

26. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$

27. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

28. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

29. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

30. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

31. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

32. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

33. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

34. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

35. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.

36. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

37. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.

38. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.

39. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.

40. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.
41. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
42. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
43. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
44. Find the area between $y = x^2$ and $y = x$.
45. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
46. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.
47. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$
48. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.
49. Find the area bounded by the curves
$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{5.2.49.1}$$
50. Find the area of the region
$$\left\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T \mathbf{x} = 9\right\} \tag{5.2.50.1}$$
51. Find the area of the circle $\mathbf{x}^T \mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.
52. Find the intervals in which the function given by
$$f(x) = 2x^2 - 3x \tag{5.2.52.1}$$
is
    a) increasing
    b) decreasing.
53. Find the intervals in which the following functions are strictly increasing or decreasing
    a) $x^2 + 2x - 5$
    b) $10 - 6x - 2x^2$
    c) $6 - 9x - x^2$
54. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(1, -1)$.
55. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.
56. Find all points of discontinuity of f, where f is defined by
$$f(x) = \begin{cases} x + 1, & x \geq 1, \\ x^2 + 1, & x < 1, \end{cases} \tag{5.2.56.1}$$
57. For what value of $\lambda$is the function defined by
$$f(x) = \begin{cases} \lambda\left(x^2 - 2x\right), & x \leq 0, \\ 4x + 1, & x > 0 \end{cases} \tag{5.2.57.1}$$
continuous at $x = 0$? What about continuity at $x = 1$?
58. For what value of k is the following function continuous at the given point.
$$f(x) = \begin{cases} kx^2, & x \leq 2, \\ 3, & x > 2, \end{cases} \quad x = 2 \tag{5.2.58.1}$$
59. Find $\frac{dy}{dx}$ in the following
$$x^2 + xy + y^2 = 100 \tag{5.2.59.1}$$
60. Verify Rolle's theorem for the function . $f(x) = x^2 + 2x - 8, x \in [-4, 2]$
61. Examine if Rolle's theorem is applicable to the following function $f(x) = x^2 - 1, x \in [1, 2]$. Can you say some thing about the converse of Rolle's theorem from this example?
62. Examine the applicability of the mean value theorem for the function in Problem 5.2.60.
63. Find $\lim_{x \to 1} \pi r^2$.
64. Find $\lim_{x \to 0} f(x)$ where
$$f(x) = \begin{cases} x^2 - 1 & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases} \tag{5.2.64.1}$$
65. For some constans a and b, find the derivative of
$$(x - a)(x - b) \tag{5.2.65.1}$$
66. Integrate the following as limit of sums:
    (i) $\int_2^3 x^2 \, dx$
    (ii) $\int_1^4 \left(x^2 - x\right) dx$
67. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.
68. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

69. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.
70. The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 ms^{-1} can go without hitting the ceiling of the hall ?
71. A cricketer can throw a ball to a maximum horizontal distance of 100 m. How much high above the ground can the cricketer throw the same ball ?