Lab - 3

3. C & D are circulant matrices, Prove their Product is also a circulant matrix

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 4 \\ 4 & 5 & 0 \\ 0 & 4 & 5 \end{bmatrix}$$

$$C \cdot D = \begin{bmatrix} 13 & 22 & 19 \\ 19 & 13 & 22 \\ 22 & 19 & 13 \end{bmatrix}$$

$\therefore$ C.D is also a circulant matrix

[C.D has Constant Diagonals,]

[Sum of Row is same = 55]

4. $a = [0, 1, 2]$, $b = [3, 1, 2]$

$$A = a_0 I_{3\times3} + a_2 P_{3\times3} + a_3 P^2_{3\times3}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

b.A gives convolution of a & b

$$\begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 4 & 7 & 7 \end{bmatrix}$ is the cyclic convolution of the two vectors a & b.

5. $x = (0, 1, 0, 1)$

$y = (0, 1, 2, 3]$

$Y = y_0 I + y_1 P + y_2 P^2 + y_3 P^4$

$$Y = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

$x . Y =$ Circular Convolution

$$\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 & 2 \end{bmatrix}$$

$[4, 2, 4, 2]$ is the circular convolution of the two vectors x & y

9. Eigen Decomposition of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} = \frac{6}{2}, \frac{2}{2}$$

$$\boxed{\lambda_1 = 3}$$
$$\boxed{\lambda_2 = 1}$$

$$(A - \lambda I)x = 0$$

$$\lambda = 1 \Rightarrow \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x = 0$$

[$Ax = 0$, x gives the Nullspace]

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \qquad x_1 + x_2 = 0$$

$$x_2 = x_2$$

$$x_1 = -x_2$$
$$x_2 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$$

$\therefore$ Eigen space $= \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \longrightarrow v_1$

$\lambda = 3 \Big|$ $(A - \lambda I)x = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

$$-x_1 + x_2 = 0$$

$$x_2 = x_2$$

$$x_1 = x_2$$
$$x_2 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

$\therefore$ Eigenspace $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \longrightarrow v_2$

Eigen Decomposition

$$A = X \Lambda X^{-1}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}$$

$$x_1 = \| v_1 \|$$

$$x_2 = \| v_2 \|$$

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Big/ \sqrt{1+1} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big/ \sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$X = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$X^{-1} = \frac{-1}{4} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\cancel{A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & \\ 0 & 1 \end{bmatrix}}$$

$$A = X \Lambda X^{-1}$$

$$= \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1/4\sqrt{2} & -1/4\sqrt{2} \\ -1/4\sqrt{2} & -1/4\sqrt{2} \end{bmatrix}}_{X^{-1}}$$

6. Fourier Matrix $\boxed{F_2}$

$$F_{jk} = \frac{1}{\sqrt{n}} e^{-2\pi i \frac{j \cdot k}{2}}$$

$$F_2 = \begin{bmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{bmatrix}$$

$$F_{00} = \frac{1}{\sqrt{2}} e^{-2\pi i \frac{0 \cdot 0}{2}} = 1/\sqrt{2}$$

$$F_{01} = \frac{1}{\sqrt{2}} e^{-2\pi i \frac{0 \cdot 1}{2}} = \frac{1}{\sqrt{2}}$$

$$F_{10} = \frac{1}{\sqrt{2}} e^{-2\pi i \frac{1 \cdot 0}{2}} = \frac{1}{\sqrt{2}}$$

$$F_{11} = \frac{1}{\sqrt{2}} e^{-\cancel{2}\pi i \frac{1 \cdot 1}{\cancel{2}}} = \frac{e^{-\pi i}}{\sqrt{2}} = \frac{(-1)}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$F_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} //$$