SIGNAL PROCESSING OPTIMIZATION TECHNIQUES

7. SECOND ORDER CONE PROGRAM

Second-order Cone Program (SOCP)

- In the simplest case, SOCP has a standard form

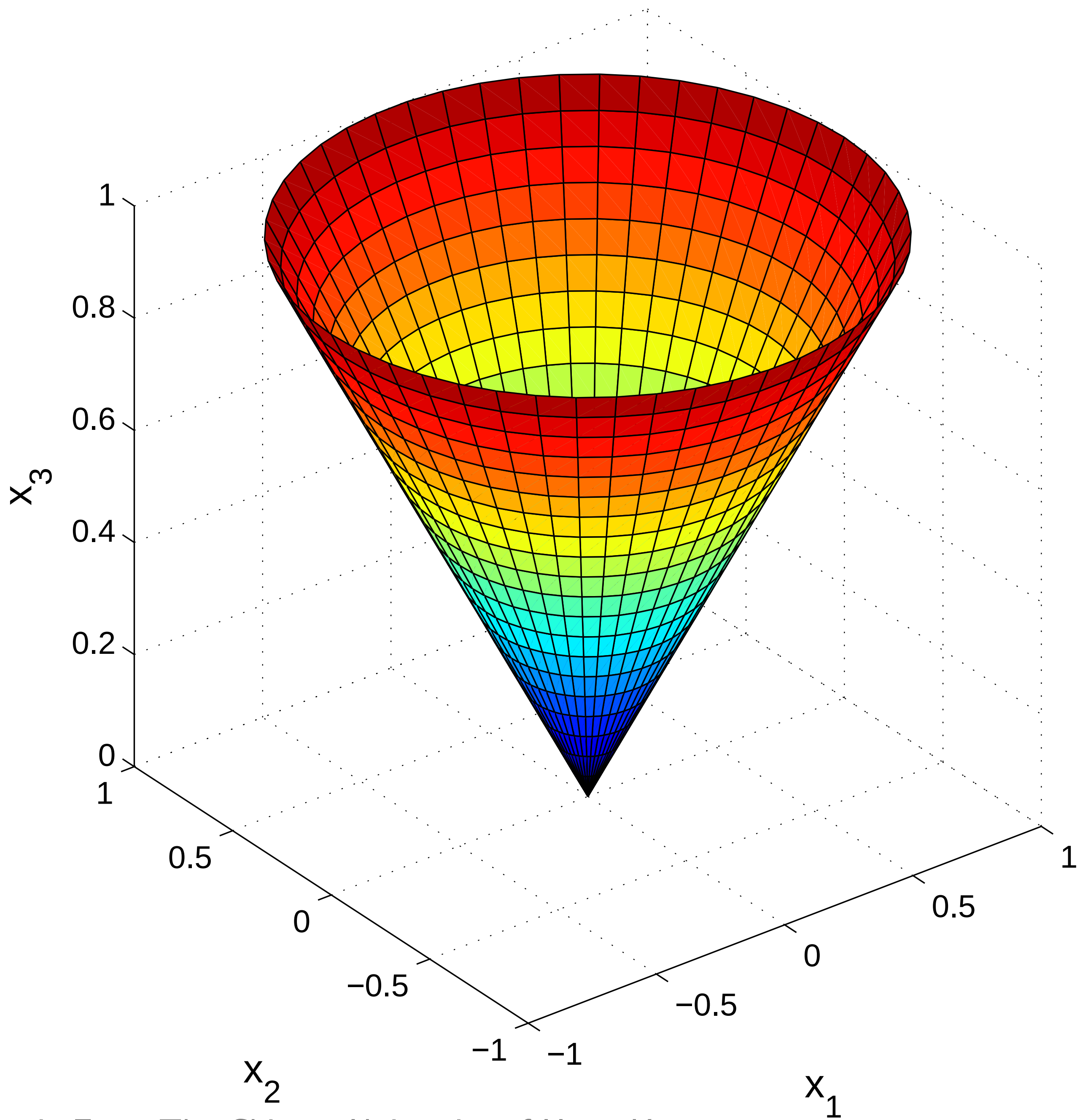
$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b, \\ & x \in \mathcal{K} \end{aligned}$$

where $\mathcal{K} = \{ \, x \in \mathbf{R}^{n+1} \mid \sqrt{\sum_{i=1}^{n} x_i^2} \leq x_{n+1} \, \}$ is an SOC.

- A more general (and useful) standard SOCP formulation has

$$\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_m$$

where $\mathcal{K}_i$ is an SOC.



1
0.8
0.6
0.4
0.2
0
x_3
1
0.5
0
−0.5
−1
x_2
−1
−0.5
0
0.5
1
x_1

A very general way (or most?) of writing an SOCP:

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t.}\ \|A_i x + b_i\|_2 \leq f_i^T x + d_i, \quad i = 1, \ldots, L \\ &\qquad Fx = g \end{aligned}$$

The inequality constraints are generalized inequalities

$$\|A_i x + b_i\|_2 \leq f_i^T x + d_i \Longleftrightarrow \begin{bmatrix} A_i x + b_i \\ f_i^T x + d_i \end{bmatrix} \succeq_{K_i} 0,$$

where each K_i denotes an SOC of appropriate dimension.

Some class of QCQPs may be regarded as a special case of the SOCP.

For example, consider a QCQP in the form of

$$\begin{aligned} \min\ & \|A_0 x + b_0\|_2^2 \\ \text{s.t.}\ & \|A_i x + b_i\|_2^2 \le r_i, \quad i = 1, \ldots, L \end{aligned}$$

The problem can be reformulated as

$$\begin{aligned} \min\ & t \\ \text{s.t.}\ & \|A_0 x + b_0\|_2 \le t \\ & \|A_i x + b_i\|_2 \le \sqrt{r_i}, \quad i = 1, \ldots, L \end{aligned}$$

which is an SOCP.

Robust Linear Programming

Recall the standard LP problem:

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{aligned}$$

Consider that there is uncertainty in a_i:

$$a_i \in \{\ \bar{a}_i + P_i u \mid \|u\|_2 \leq 1\ \} \triangleq \mathcal{E}_i$$

where we only have knowledge of $\bar{a}_i$ & P_i.

Worst-Case Robust LP formulation:

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } a_i^T x \le b_i, \quad \text{for all } a_i \in \mathcal{E}_i \quad i = 1, \ldots, m \end{aligned}$$

Since

$$a_i^T x \le b_i \text{ for all } a_i \in \mathcal{E}_i \Longleftrightarrow \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i,$$

the robust LP problem is equiv. to

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i, \quad i = 1, \ldots, m \end{aligned}$$

which is an SOCP.

Probabilistically Robust LP formulation:

- Sometimes we may want $99.9\ldots\%$ okay, rather than the worst case (worst case can be quite worse, and yet it rarely happens).
- Robust LP formulation using probabilistic constraints:

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } \mathbf{Prob}(a_i^T x \le b_i) \ge \eta, \quad i = 1, \ldots, m \end{aligned}$$

  where a_i's are modeled as random variables, and $0 \le \eta \le 1$ is the minimum probability requirement.

- Assume $a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$ (Gaussian distributed with mean $\bar{a}_i$ and covariance Σ_i).
- As $\mathbf{Prob}(a_i^T x \le b_i) = \Phi((b_i - \bar{a}_i^T x)/\sqrt{x^T \Sigma_i x})$, where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$, the probabilistically robust LP can be formulated as

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \le b_i - \bar{a}_i^T x, \quad i = 1, \ldots, m \end{aligned}$$

  The above problem is an SOCP for $Q^{-1}(\eta) \ge 0$, or, equivalently, $\eta \ge 0.5$.

Robust Least Squares

Standard LS:

$$\min_{x} \|Ax - b\|_2^2$$

Consider that there is uncertainty in a_i:

$$A \in \{ \bar{A} + U \mid \|U\|_2 \leq \alpha \} \triangleq \mathcal{A}$$

and we only have knowledge of $\bar{A}$ & α.

(Worst-case) robust LS formulation:

$$\min_{x} \sup_{A \in \mathcal{A}} \|Ax - b\|_2$$

For $A = \bar{A} + U$, $\|U\|_2 \leq \alpha$,

$$\begin{aligned} \|Ax - b\|_2 &= \|\bar{A}x - b + Ux\|_2 \\ &\leq \|\bar{A}x - b\|_2 + \|Ux\|_2 \\ &\leq \|\bar{A}x - b\|_2 + \alpha\|x\|_2 \end{aligned}$$

The equality is shown to be achievable for some $\|U\|_2 \leq \alpha$.

The robust LS problem then becomes

$$\begin{aligned} &\min \|\bar{A}x - b\|_2 + \alpha\|x\|_2 \\ \Longleftrightarrow &\min t_1 + \alpha t_2 \\ &\;\text{s.t. } \|\bar{A}x - b\|_2 \leq t_1, \quad \|x\|_2 \leq t_2 \end{aligned}$$

Robust Beamforming

Background: Minimum Variance Beamforming

Recall the average energy minimization design:

$$\begin{aligned} \min_{w \in \mathbf{C}^P} \;& w^H P w \\ \text{s.t. } \;& w^H a(\theta_{\mathsf{des}}) = 1 \end{aligned}$$

Here $P = \sum_i a(\theta_i) a^H(\theta_i)$, where θ_i are directions that are not of interest.

- In the previous lectures, θ_i are chosen to be a discretized set of directions outside a certain beamwidth.
- We can also set θ_i to be the interfering source directions, if we knew them.
- The resultant beamformer will then focus on minimizing energies at the interfering source directions, resulting in better interference suppression than the sidelobe energy minimization design.
- In practice, those interfering source directions are not known.

Received signal model:

$$y(t) = a(\theta_{\mathsf{des}})s(t) + \sum_{i=1}^{K} a(\theta_i)u_i(t) + \nu(t)$$

If $s(t)$ & $u_i(t)$ are uncorrelated and wide-sense stationary, and $\nu(t)$ is spatially white,

$$\begin{aligned} R &= \mathbf{E}\{y(t)y^H(t)\} \\ &= \sigma_s^2 a(\theta_{\mathsf{des}})a^H(\theta_{\mathsf{des}}) + \sum_{i=1}^{K} \sigma_{u_i}^2 a(\theta_i)a^H(\theta_i) + \sigma_\nu^2 I \end{aligned}$$

R can be reliably estimated from $y(t)$ via averaging: $\hat{R} = \sum_{t=1}^{N} y(t)y^H(t)$.

The minimization problem

$$\begin{aligned} &\min \; w^H R w \\ &\text{s.t. } w^H a(\theta_{\mathsf{des}}) = 1 \end{aligned} \qquad (*)$$

is equiv. to

$$\begin{aligned} &\min \; \sum_{i=1}^{K} \sigma_{u_i}^2 |w^H a(\theta_i)|^2 + \sigma_\nu^2 \|w\|_2^2 \\ &\text{s.t. } w^H a(\theta_{\mathsf{des}}) = 1 \end{aligned}$$

which we minimize the output interference and noise power.

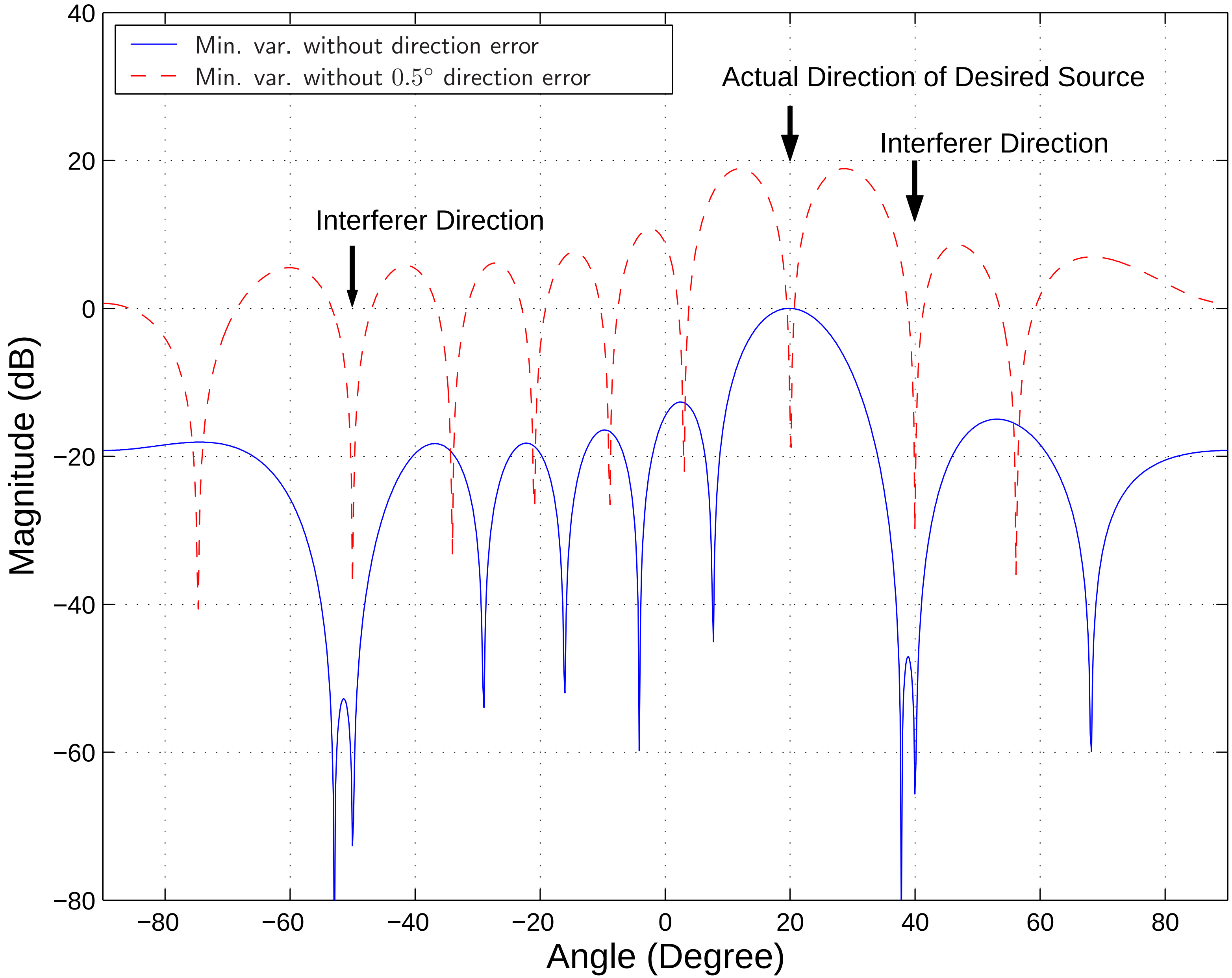
In the signal processing literature, $(*)$ is known as the **minimum variance beamformer** design.

Problem with imperfectly known steering vector

- Consider situations where there is uncertainty with the desired direction θ_{des}, or the desired steering vector $a(\theta_{\text{des}})$ is imperfectly known.
- Let $a = a(\theta_{\text{des}})$ for simplicity. The uncertain effect can be modeled as

$$a = \bar{a} - u$$

  where $\bar{a}$ is the true steering vector & u is the uncertainty.
- The min. variance beamformer design can be very sensitive to uncertainty in a .



Direction patterns of min. variance beamformers. $P = 10$.

Robust Beamforming via SOCP [VGL03]

- Robust beamforming problem formulation:

$$\begin{aligned} &\min\ w^H R w \\ &\text{s.t.}\ |w^H(a+u)| \geq 1,\ \text{for all } \|u\|_2 \leq \epsilon \end{aligned}$$

- Or we can write

$$\begin{aligned} &\min\ w^H R w \\ &\text{s.t.}\ \inf_{\|u\|_2 \leq \epsilon} |w^H(a+u)| \geq 1 \end{aligned}$$

- At first look this is a nonconvex problem.

- By triangular inequality:

$$
\begin{aligned}
|w^H(\bar{a}+u)| &\geq |w^H\bar{a}| - |w^H u| \\
&\geq |w^H\bar{a}| - \epsilon\|w\|_2, \quad \forall \|u\|_2 \leq \epsilon \qquad (*)
\end{aligned}
$$

where we assume $|w^H a| > \epsilon\|w\|_2$. (What happens if $|w^H a| \leq \epsilon\|w\|_2$?)

- Choose

$$u = -\frac{\epsilon e^{j\angle(w^H a)}}{\|w\|_2} w.$$

Then equality in $(*)$ is achieved. Thus

$$\inf_{\|u\|_2 \leq \epsilon} |w^H(a+u)| = |w^H\bar{a}| - \epsilon\|w\|_2$$

- The robust beamforming problem can be rewritten as

$$\begin{aligned} &\min\ w^H R w \\ &\text{ s.t. } |w^H\ a| - \epsilon \|w\|_2 \geq 1 \end{aligned}$$

which is still nonconvex.

- We note the following: If $w^\star$ is a solution, then $e^{j\psi} w^\star$ is also a solution for any phase shift ψ.

- Without losing optimality, let us add additional constraints:

$$\mathbf{Re}\{w^H a\} \geq 0, \qquad \mathbf{Im}\{w^H a\} = 0$$

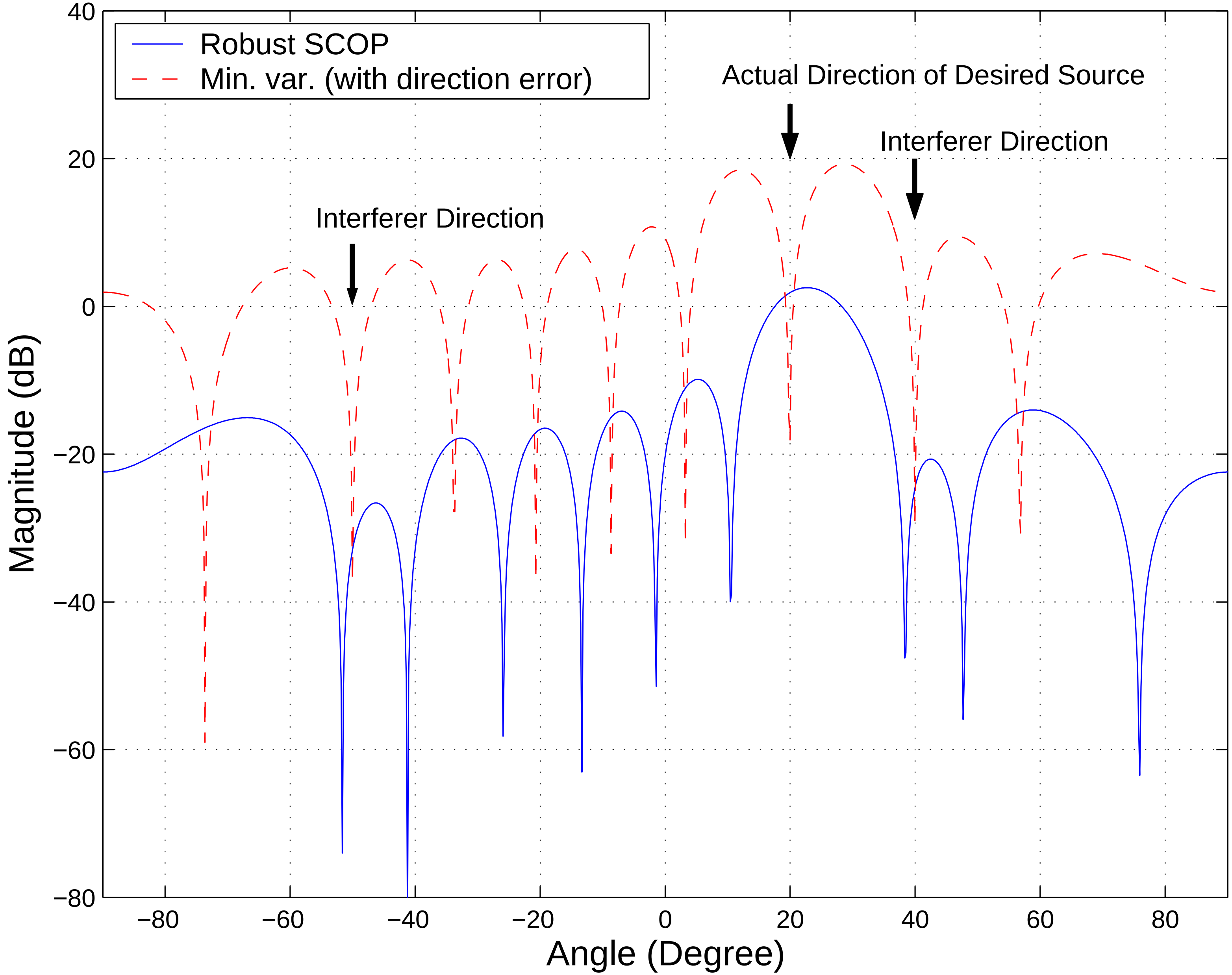
- By adding those constraints,

$$
\begin{aligned}
\min\ & w^H R w \\
\text{s.t. } & w^H a \geq 1 + \epsilon \|w\|_2 \\
& \mathbf{Im}\{w^H a\} = 0
\end{aligned}
$$

- Finally, by the epigraph reformulation, the robust beamforming problem is rewritten as a SOCP:

$$
\begin{aligned}
\min\ & t \\
\text{s.t. } & \|Vw\|_2 \leq t, \quad \epsilon \|w\|_2 \leq w^H a - 1 \\
& \mathbf{Im}\{w^H a\} = 0
\end{aligned}
$$

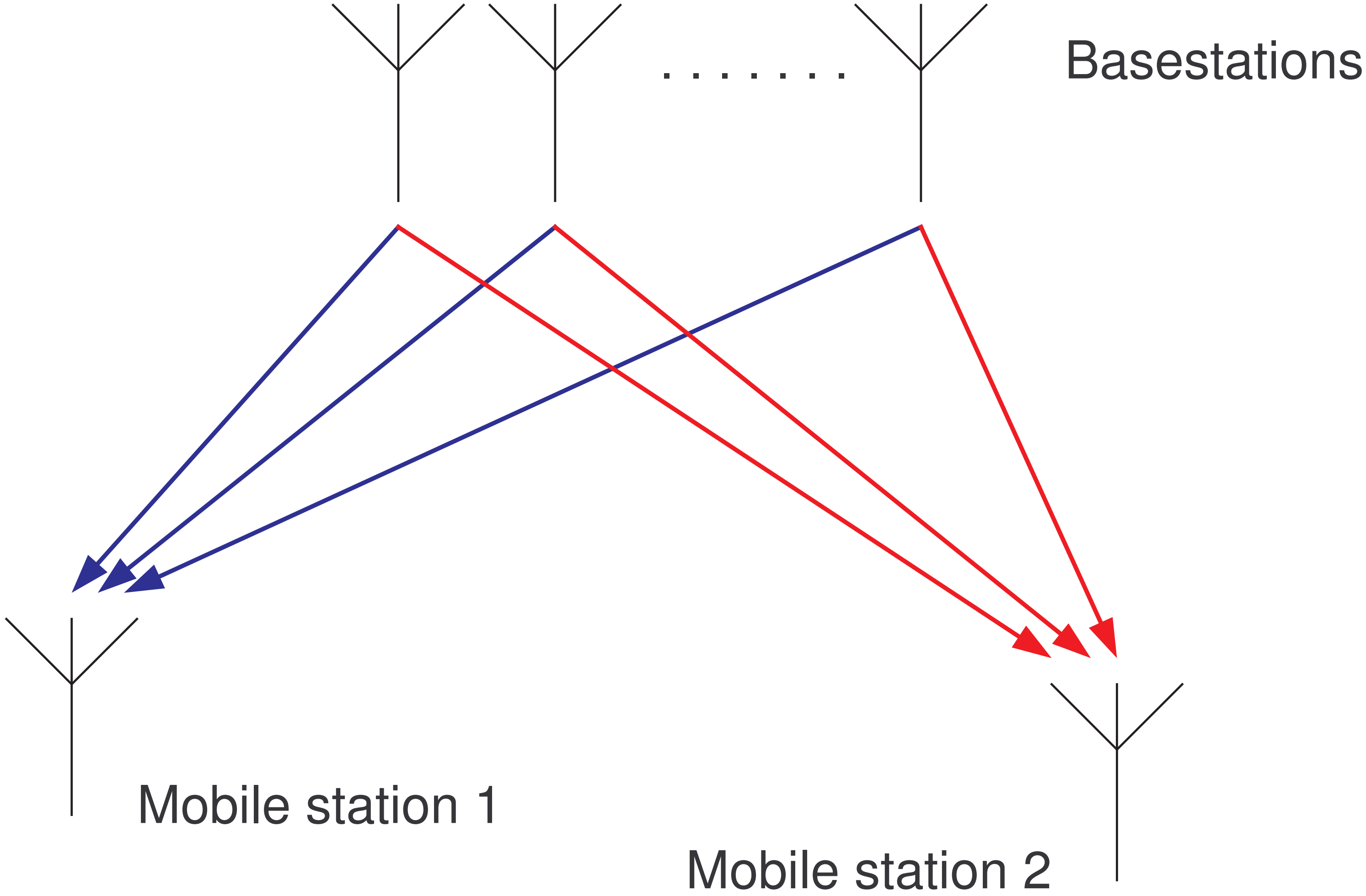
where V is a square root factor of R (i.e., $R = V^H V$).



Direction patterns of the robust beamformer. $\bar{\theta}_{\text{des}} = 20^\circ$, $\theta_{\text{des}} = 20.5^\circ$; $\epsilon = 0.2\|\bar{a}\|_2$.

Transmit Downlink Beamforming

- The basestation (BS) has m antennas.
- It sends data to n mobile stations (MSs) each of which has 1 antenna.
- The BS uses transmit beamforming to simultaneously transmit signals to the n MSs, over the same channel.



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Basestations
Mobile station 1
Mobile station 2

- Assuming frequency flat channel fading, the received signal at MS i at each time instant may expressed as

$$y_i = h_i^T x + v_i$$

$h_i \in \mathbf{C}^m$	multiple-input-single-output (MISO) channel for MS i;
$v_i \in \mathbf{C}$	noise;
$x \in \mathbf{C}^n$	BS transmitted signal vector, with x_i being the tx signal of the ith antenna of the BS.

- The BS transmitted signal:

$$x = \sum_{i=1}^{n} f_i s_i = Fs$$

where

$s_i \in \mathbf{C}$	information carrying signal for MS i;
$f_i \in \mathbf{C}^m$	the corresponding transmit beamforming vector.

- Assume that $\mathrm{E}\{|s_i|^2\} = 1$ for all i, & $\mathrm{E}\{|v_i|^2\} = \sigma_i^2$.
- The SINR of MS i is

$$\gamma_i(F) = \frac{|h_i^T f_i|^2}{\sum_{j \neq i} |h_i^T f_j|^2 + \sigma_i^2}$$

- **Problem: [BO02],[WES06]** given a min. SINR requirement γ_o, find a beamformer matrix F that minimizes the total transmit power:

$$\begin{aligned} \min \ & \sum_{i=1}^{n} \|f_i\|_2^2 \\ \text{s.t. } & \gamma_i(F) \geq \gamma_o, \qquad i = 1, \ldots, n \end{aligned}$$

- The constraints can be re-expressed as

$$\frac{1}{\gamma_o}|h_i^T f_i|^2 \geq \sum_{j\neq i} |h_i^T f_j|^2 + \sigma_i^2 \qquad (*)$$

and w.lo.g. we can add extra constraints:

$$h_i^T f_i \geq 0 \qquad (**)$$

- With $(**)$, $(*)$ can be re-expressed as

$$\left\| \begin{bmatrix} h_i^T & & 0 \\ & \ddots & \\ 0 & & h_i^T \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_i \end{bmatrix} \right\|_2 \leq \frac{1}{\gamma_o} h_i^T f_i$$

- Hence, the transmit beamformer design problem can be cast as a SOCP:

$$
\begin{aligned}
&\min\ t \\
&\text{s.t.}\ \|[\ f_1^T \cdots f_n^T\ ]^T\|_2 \le t \\
&\qquad h_i^T f_i \ge 0, \qquad i = 1, \ldots, n \\
&\qquad \left\| \begin{bmatrix} h_i^T & & 0 \\ & \ddots & \\ 0 & & h_i^T \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_i \end{bmatrix} \right\|_2 \le \frac{1}{\gamma_o} h_i^T f_i, \\
&\qquad i = 1, \ldots, n
\end{aligned}
$$

- We may also consider the following design:
- **Problem:** given a power limit P_o, find a beamformer matrix F that maximizes the smallest (or worst-case) SINR:

$$\begin{aligned} \max \quad & \min_{i=1,\dots,n} \gamma_i(F) \\ \text{s.t.} \quad & \sum_{i=1}^{n} \|f_i\|_2^2 \le P_o \end{aligned}$$

- This problem is not convex. But it can be solved under a quasi-convex opt. framework.

References

[VGL03] S. Vorobyov, A. Gershman, & Z.-Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem", *IEEE Trans. Signal Proc.*, 2003.

[BO02] M. Bengtsson and B. Ottersten, "Optimal and suboptimal transmit beamforming," in *Handbook of Antennas in Wireless Communications*, edited by Lal Chand Godara, CRC Press, 2002.

[WES06] A. Wiesel, Y. C. Eldar, & S. Shamai, "Linear precoding via conic opt. for fixed MIMO receivers," *IEEE Trans. Signal Proc.*, 2006.