Part II: Lagrange Multiplier Method & Karush-Kuhn-Tucker (KKT) Conditions

KKT Conditions

General Non-Linear Constrained Minimum:
>   Min: f[x]
>   Constrained by: h[x] = 0 (m equality constraints)
>   g[x] ≤ 0 (k inequality constraints)

Introduce slack variables s_i for the inequality contraints: $g_i[x] + s_i^2 == 0$
and construct the monster Lagrangian:

> $L[x,\lambda,\mu] = f[x] + \lambda h[x] + \sum \mu_i (g_i[x] + s_i^2)$

Recall the geometry of the Lagrange multiplier conditions: The gradient of the objective function must be orthogonal to the tangent plane of the (active) constraints. That is the projection of the gradient of f onto the space of directions tangent to the constraint "surface" is zero. The KKT conditions are analogous conditions in the case of constraints.

The KKT conditions are the following:

1) Gradient of the Lagrangian = 0
2) Constraints: h[x] = 0 (m equality constraints) & g[x] ≤ 0 (k inequality constraints)
3) Complementary Slackness (for the s_i variables) $\mu.s == 0$
4) Feasibility for the inequality constraints: $s_i^2 \geq 0$
5) Sign condition on the inequality multipliers: $\mu \geq 0$

One final requirement for KKT to work is that the gradient of f at a feasible point must be a linear combination of the gradients for the equality constraints and the gradients of the active constraints: this is often called regularity of a feasible point.

At a feasible point for the constraints, the active constraints are those components of g with $g_i[x] = 0$ (if the value of the constraining function is < 0, that constraint is said to be inactive).

The solution of a set of KKT equations proceeds by cases according to which inequality constraints are Active & Inactive.

Example: Chong Zak Example 20.2

Consider a circuit with a 20V battery and two resistances in series: R and 10 ohm.
We will investigate the maximization of the power in each of the resistors separately

▪ Subproblem One: Max in resistor R

Maximize power absorbed by resistance R
Equivalently: minimized: - Power absorbed by R= - i^2 R = $- 400 \, R / (10 + R)\text{^}2$ subject to -R ≤ 0.

The lagrangian is L[R,λ] =

```
L[R_, μ_] := ─── - 400 R ───  + μ (-R)
                (10 + R)²

D[L[R, μ], R]
D[L[R, μ], μ]

  800 R        400
─────────  -  ─────────  - μ
(10 + R)³     (10 + R)²

-R
```

KKT also gives us the complementary slackness: $\mu . R = 0$ and the sign condition for the inequality constraints: $\mu \geq 0$.
 But, if $\mu > 0$, then R==0 which is gives no power absorbed by R.
 If the constraint is inactive we must solve

```
Solve[(D[L[R, μ], R] /. {μ → 0}) == 0, R]

{{R → 10}}
```

This gives the maximum

▪ Subproblem Two: Maximization of power in 10 ohm resistor

Find the value of R such that maximal power is delivered to the 10 ohm resistor, i.e:
Min: - $\frac{4000}{(10+R)\text{^}2}$ subject to -R ≤ 0

```
L[R_, μ_] := ─── -4000 ───  + μ (-R)
               (10 + R)²
```

Again, KKT gives us a complementary slackness condition: $\mu . R = 0$ and the sign condition for the inequality constraints: $\mu \geq 0$.
 But, if $\mu = 0$, we must solve

```
Solve[(D[L[R, μ], R] /. {μ → 0}) == 0, R]

{}
```

This first case has no solutions. However, R = 0 is possible in the case $\mu > 0$

KKT conditions for a Linear Program: (Franklin: p. 197)

■ **Lagrangian**

Min: c^T .x subject to the linear constraints:

   A.x == b, x ≥ 0,

the associated Lagrangian is:

   L[x,λ] = c^T .x + Transpose[λ] (A.x - b) + μ.(-x+$s^{[2]}$)

The KKT conditions yield:

 c^T - Transpose[λ].A + Transpose[μ] == 0

   A.x - b ==0

   x ≥ 0,

   2 Transpose[μ].s == 0

   μ≥ 0

Eliminating μ, we obtain the following: c^T - Transpose[λ].A ≤ 0, (c^T - Transpose[λ].A).x= - Transpose[μ]).($s^{[2]}$) = 0
yielding

   c^T .x = Transpose[λ].A.x= Transpose[λ].b which is the **equilibrium condition** in mild disquise!

Example: Pedregal Example 3.10, p. 82

A certain electrical networks is designed to supply power x_i thru 3 channels.

   p[x_, y_, z_] = z + (x^2 + y^2 + z^2 / 10) / 2;

Contraint: h[x,y,z]=x+y+z== 5, x,y,z ≥0
Note that this gives a "slanted" triangles with vertices on the axes 5 units from the origin.

   h[x_, y_, z_] = x + y + z

The KKT conditions give:

   1)   ∇f + λ ∇h + μ ∇g = {x,y,1+z/10} + λ {1,1,1} + {μ_1 ,μ_2 ,μ_3 } =={0,0,0}
   2)   Constraint: h==5
   3) μ_1 x = 0, μ_2 y = 0, μ_3 z = 0,

Checking for active constraints will divide in the consideration of 8 cases for the three inequality constraints.

KKT: Necessary Conditions for Quad. Program

▪ Review: Quadratic Programs

The general quadratic program proposes to minimize an objective function of the form: Min: $x.Q.x/2 + p.x$ subject to the linear constraints:

        $A.x == b, \ x \geq 0$

Note that we may assume that Q is a symmetric matrix (and that Q is the Hessian of the objective function.) This is a non-linear program problem, for the objective function is a quadratic function (if Q is non-zero.)

▪ Wolfe's Reduction to LP

Given the Quadratic Program as above, the associated Linear Program is:

      $A.x == b$
      $Q.x + Transpose[A].u - v == -p$
      $x \geq 0, \ v \geq 0, \ x.v == 0$

Note that the vector u is not necessarily positive. The only additional feature is the **exclusion rule**: $x.v == 0$, this requirement states that the ith components of x and v can not both be positive simultaneously.

Finally, note that the existence of a solution of the associated LP is only necessary for the existence of a solution to the QP and is sufficient in case Q is positive semi-definite.

In practice we solve the LP program:
       $A.x==b$
       $Q.x + Transpose[A].u -v + D.z == -p$
       $x \geq 0l, \ u$ free, $v \geq 0, \ z \geq 0$
       Min: $\{1, \ldots, 1\}.z$

Initial Feasible Solution: x, u=0, v=0, z where x is a basic feasible solution of $A.x == b, \ x \geq 0$, D is a diagonal matrix with entries ± 1 to correct the signs of z and z is a chosen such that $Q.x + D.z == - p, \ z \geq 0$.

▪ KKT conditions for a Quadratic Program

▪ Lagrangian

Given: $x.Q.x/2 + p.x$ subject to the linear constraints:
        $A.x == b, \ x \geq 0,$

the associated Lagrangian is:

      $L[x,\lambda] = x.Q.x/2 + p.x + \lambda \ (A.x - b) + \mu.x$

The KKT conditions yield:
 $Q.x + p - Transpose[A].\lambda + \mu \ == 0$
     $Transpose[\lambda].(A.x - b) == 0$
     $Transpose[\mu].x \ == 0$
        $A.x == b, \ x \geq 0,$

The KKT conditions yield:

$\quad Q.x + p - \text{Transpose}[A].\lambda + \mu\ == 0$

$\qquad \text{Transpose}[\lambda].(A.x - b) == 0$

$\qquad \text{Transpose}[\mu].x\ == 0$

$\qquad\qquad A.x == b,\ \ x \ge 0,$

Exercises

I. Write down the KKT conditions for the problem:

Min $f[x] = - x_1^{\,3} + x_2^{\,2} - 2\, x_1\, x_3^{\,2}$ subject to the constraints:

$2\, x_1 + x_2^{\,2} + x_3 - 5 == 0$

$5\, x_1^{\,2} - x_2^{\,2} - x_3 \ge\ 2$

$x_i \ge\ 0\ \ \text{for i} = 1,2,3.$

Verify that the KKT conditions are satisfied at (1,0,3).

II. Write down the KKT conditions for the problem:

Min $f[x] = x_1^{\,2} + x_2^{\,2} + x_3^{\,2}$ subject to the constraints:

$-x_1 + x_2 -\ x_3 \ge\ -10$

$x_1 + x_2 + 4\ x_3 \ge\ 20$

Find all the solutions.

III. Here's an exercise from Grieg p. 142):

Minimize: $f[x] = (x_1 - 1)\verb|^|2 + (x_2 - 2)\verb|^|2 + (x_3 - 3)\verb|^|2 + (x_4 - 4)\verb|^|2$

subject to: $- x_1\ - x_2\ - x_3\ - x_4 + 5 \ge\ 0$

$\qquad\qquad\qquad - 3\, x_1\ - 3\, x_2\ - 2\, x_3\ - x_4 + 10 \ge\ 0$

$\qquad\qquad\qquad x \ge 0$

Verify the KT conditions and then expand f to obtains a Quadratic Program and compare the KT conditions with Wolfe's.

Solve the problem using Wolfe's method .

Bonus: Solve the problem using a projected gradient methods (mimic affine scaling) and a projected Newton's method. Compare the results.