The University of Georgia

Mathematics Education
EMAT 4600/6600, Jim Wilson

Using the Arithmetic Mean-Geometric Mean Inequality in Problem Solving

by

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A Presentation to the Annual Meeting of School Mathematics and Science Association, Birmingham, November 8, 2012, was prepared using some parts of this paper.

PDF version

The Arithmetic Mean-Geometric Mean Inequality (AM-GM Inquality) is a fundamental relationship in mathematics. It is a useful tool for problems solving and building relationships with other mathematics. It should find more use in school mathematics than currently. It what follows I present an introduction to the theorem, some background and generalizations, alternative demonstrations of the proof, and examples of problems that can be explored by using the AM-GM Inquality. I will rely heavily on a collection of problems and essays on my web site:

Http://jwilson.coe.uga.edu

and the sub-directory for my mathematics problem solving course at the University of Georgia.

I will concentrate on the theorem for two positive numbers in my examples, but I mention the generalizations below and occasionally use the case for three positive numbers.

Arithmetic Mean – Geometric Mean Inequality

For real positive numbers a and b, the AM-GM Inequality for two numbers is:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

with equality if and only if $a = b$

Equivalently we can write

$$a + b \geq 2\sqrt{ab}$$

or

$$\left(\frac{a+b}{2}\right)^2 \geq ab$$

Arithmetic Mean

For a set of positive numbers $a_1, a_2, \ldots a_n$, the Arithmetic Mean is

$$AM = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

The arithmetic mean of two positive numbers a and b is

$$AM = \frac{a + b}{2}$$

Geometric Mean

For a set of positive numbers $a_1, a_2, \ldots a_n$, the Geometric Mean is

$$GM = \sqrt[n]{a_1 . a_2 . \ldots . a_n}$$

The Geometric Mean of two positive numbers a and b is

$$GM = \sqrt{ab}$$

Clearly, the AM-GM Inequality can be generalized for n positive numbers. The link poses the problem of generalizing the proof following the lines of argument advanced by Courant and Robbins (1942). That is,

For 3 positive quantities a, b, and c

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

with equality if and only if $a = b = c$

Equivalently we can write $$a + b + c \geq 3\sqrt[3]{abc}$$

or $$\left(\frac{a+b+c}{3}\right)^3 \geq abc$$

For n positive quantities $a_1, a_2, \ldots, a_n$

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$

Equivalently we can write $$a_1 + a_2 + \ldots + a_n \geq n\sqrt[n]{a_1 a_2 \ldots a_n}$$

or $$\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^n \geq a_1 a_2 \ldots a_n$$

The AM-GM Inequality can also be generalized to its inclusion in relation to other means such as the [Harmonic Mean] (HM) or the [Root Mean Square] (RMS -- sometimes called the Quadratic Mean). In particular, for two positive numbers a and b,

$$HM = \frac{2ab}{a+b} \qquad RMS = \sqrt{\frac{a^2+b^2}{2}}$$

$$RMS \geq AM \geq GM \geq HM$$

with equality if and only if $a = b$.

Geometric demonstration of the RMS-AM-GM-HM Inequality

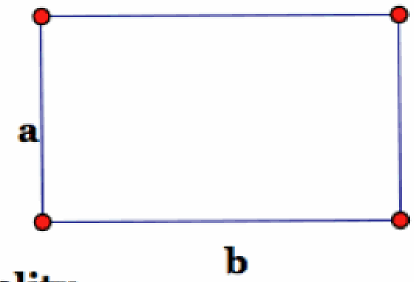
At a more advanced level (perhaps more fundamental?) all of these means are instances of Power Means where the power parameter p takes on different values for the different means. These may also be called Generalized Means.

What is the Value of this Theorem? An Example.

The AM-GM for two positive numbers can be a useful tool in examining some optimization problems. For example, it is well known that for rectangles with a fixed perimeter, the maximum area is given by a square having that perimeter.

Given an a by b rectangle with fixed perimeter P = 2a + 2b.

Show that the minimum area occurs when the rectangle is a square.



Area = ab

$$\text{Area} = ab \le \left(\frac{a+b}{2}\right)^2 \quad \text{by the AM-GM inequality}$$

$$\text{Substitute} \quad b = \frac{P}{2} - a$$

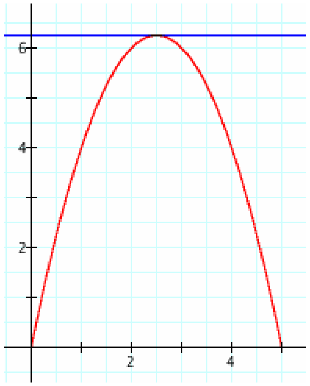
$$\text{Area} = ab \le \left(\frac{a+b}{2}\right)^2 = \left(\frac{a + \left(\frac{P}{2} - a\right)}{2}\right)^2 = \left(\frac{P}{4}\right)^2$$

Thus the area is always less than a constant $\left(\frac{P}{4}\right)^2$ and is equal to that value if and only if $a = b$. Thus the maximum area occurs when the rectangle is a square.

Perhaps some insight is given by the graph at the right. The fixed perimeter is

$$2a + 2b = 10.$$

The red graph represents the area function of the graph as either of a or b varies from 0 to 5. Let b represent the length of one side and 5 - b the length of the other.



Area = ab = b(5 - b) for 0 < b < 5

When the AM-GM Inequality is applied we get Area ≤ $\left(\frac{b+5-b}{2}\right)^2 = (2.5)^2 = 6.25$

Points on the blue curve, Area = 6.25 are always greater than points on the red curve (That is, the area of the rectangle is always less that 6.25) and the blue curve (a horizontal line representing a constant) is tangent to the red curve if and only if b = (5 - b), i.e., b =2.5.

Setting this up with the usual function notation, let the Perimeter of the rectangle equal 10 and let one side be x. The the other side is 5 - x.

$$\begin{aligned} \text{Area} = f(x) &= x(5 - x) \\ &\le \left(\frac{x+5-x}{2}\right)^2 \qquad \text{(by AM-GM)} \\ &= \left(\frac{5}{2}\right)^2 \\ &= (2.5)^2 \\ &= 6.25 \end{aligned}$$

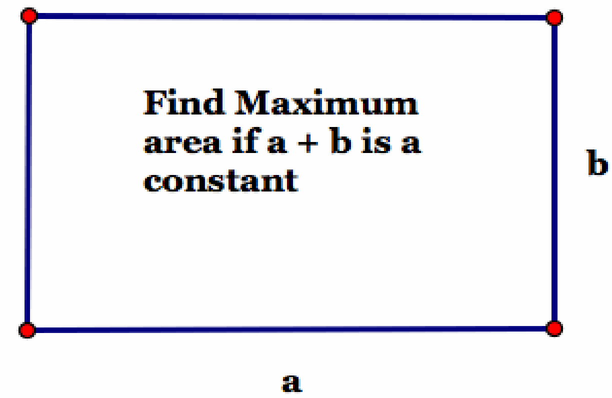
$$\begin{aligned} &\text{Equality occurs when } x = 5 - x, \\ &\text{i.e., when } x = 2.5 \end{aligned}$$

The utility of the AM-GM Inequality is that the replacement function after the application of the AM-GM inequality is a constant line tangent to the previous function. The area is always less than the constant AND it is equal to that constant when a = b. In our example for P = 10, a = b = 2.5 when the rectangle is a square.

From my web page, here is a problem posed for students:

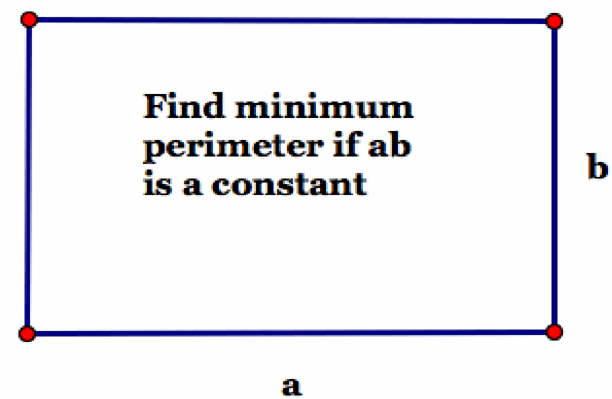
Maximum area -- Rectangle

Use the Arithmetic Mean -- Geometric Mean Inequality to show that the maximum area of a rectangular region with a given perimeter is a square.



Minimum Perimeter -- Rectangle

Use the Arithmetic Mean -- Geometric Mean Inequality to show that the minimum perimeter of a rectangle with a given area is a square.



The second problem, where ab is a constant rather than a + b, is a nice contrast to the first and it also follows quickly from the AM-GM Inequality:

Given ab is a constant, that is, the area $A = ab$ is fixed but the perimeter can vary.

$$\text{Perimeter} = 2a + 2b = 2a + 2\left(\frac{A}{a}\right)$$

By the AM-GM Inequality

$$2a + 2\left(\frac{A}{a}\right) \geq 2\sqrt{(2a)\left(\frac{2A}{a}\right)}$$

$$= 2\sqrt{4A}$$

$$= 4\sqrt{A}$$

Thus, the perimeter of rectangles having the same area A will always be greater than or equal to 4 times the square root of the area AND equality occurs if and only if $a = b$.

Alternative Proofs and Demonstrations

of the

Arithmetic Mean-Geometric Mean Inequality

In this section, I will limit the exploration to the simplest case: The arithmetic mean and geometric mean of two positive numbers. In my Problem Solving course I pose AS AN EXPLORATION that the student find a least 5 demonstrations or proofs. Seeing multiple approaches to this relation can help with understanding it, seeing its importance, and finding it useful as a problem solving tool.

Some algebra

For positive numbers a and b, $(a - b)^2$ is always positive.

Consider

$$(a - b)^2 \geq 0$$

This value is always positive or zero and equal to zero only when a = b. So by expanding the LHS, we have

$$a^2 - 2ab + b^2 \geq 0$$

Since a and b are positive, we can add 4ab to each side

$$a^2 - 2ab + b^2 + 4ab \geq 0 + 4ab$$

$$a^2 + 2ab + b^2 \geq 4ab$$

Now $$(a + b)^2 \geq 4ab$$

or $$a + b \geq 2\sqrt{ab}$$

$$\frac{a + b}{2} \geq \sqrt{ab}$$

The equality if and only if a = b carries along at each step. So the tautology in the original $(a - b)^2 \geq 0$ leads to the proof of the Arithmetic Mean-Geometric Mean inequality.

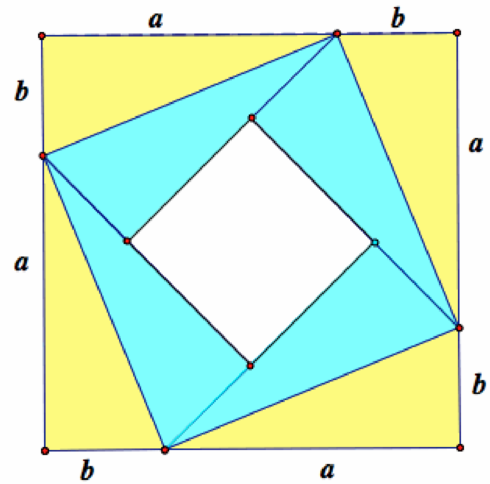
A Geometric Demonstration

The area of the large square is $(a + b)^2$, $a + b$ is constant.

The total area of the yellow triangles is 2ab and since the blue triangles are reflections of the yellow ones, the total blue area is also 2ab; total triangles area is 4ab.

The area of the white square is $(a - b)^2$

The illustration is set up with $a > b$. As a and b are changed so that they approach $a = b$, the area of the white square goes to 0.



$$(a + b)^2 \geq 4ab$$

$$\sqrt{(a + b)^2} \geq \sqrt{4ab}$$

. . . with equality if and only if

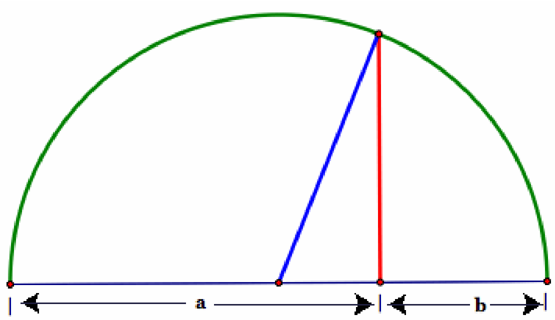
$$a + b \geq 2\sqrt{ab}$$

$a = b$.

$$\frac{a + b}{2} \geq \sqrt{ab}$$

Another Geometric Suggestion

Construct a semicircle with a diameter a + b. The radius will be the Arithmetic Mean of a and b. Construct a perpendicular to the diameter from common endpoint of the segments of length a and b. From the intersection of this perpendicular with the semicircle construct the red segment. This segment will always have a length less than or equal to the radius of the circle and it will be equal only if $a = b$.



The blue segment has length

$$\frac{a+b}{2}$$

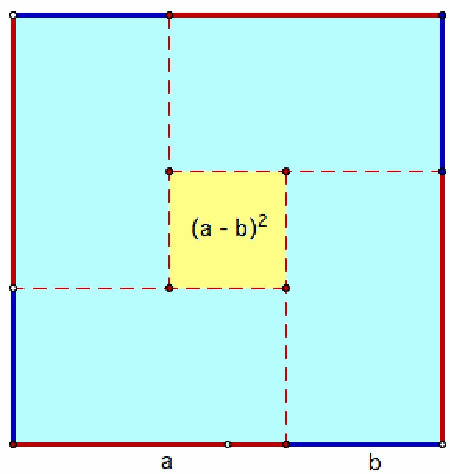
The red segment has length $\sqrt{ab}$

$$\text{So,} \quad \frac{a+b}{2} \geq \sqrt{ab}$$

with equality if and only if $a = b$.

This example is closely related to the well known geometric theorem that the altitude of a right triangle from the 90 degree vertex to the hypothenuse will be the geometric mean of the two segments cut off on the hypotenuse.

Another Geometry Example



$$(a+b)^2 = (a-b)^2 + 4ab$$

Another Algebra Demonstration

This demonstration begins with the identity: $(a + b)^2 - 4ab = (a - b)^2$

Since $(a - b)^2 \geq 0$ **we have** $(a + b)^2 - 4ab \geq 0$ **with equality if and only if a = b.**

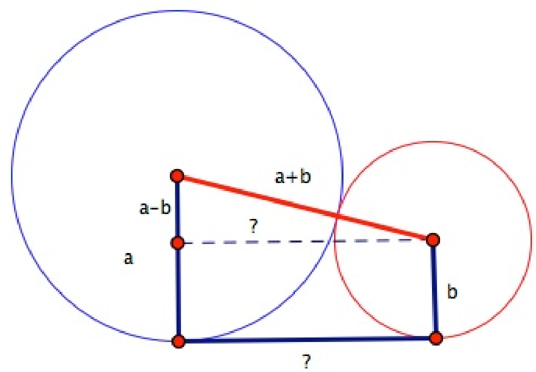
Another Geometry Example

Given two tangent circles of radii a and b. Construct a common external tangent to the two circl3s and draw radii of each circle to the common tangent. Find the length indicatec by ? along the common tanget in terms of a and b.

$$(a+b)^2 = (a-b)^2 + (?)^2$$

$$(?)^2 = 4ab$$

$$? = 2\sqrt{ab}$$

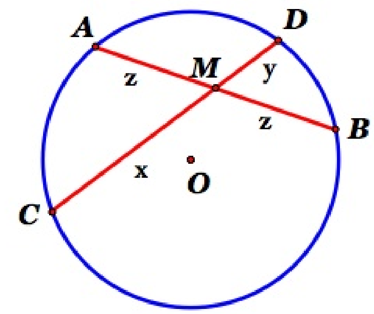


Solution: by constructing a segment parallel to the one under study, a right triangle with legs of length ? and a-b, and hypotenuse of length a+b. the segment along the common tangent has length twice the geometric mean of a and b.

More Geometry

Consider a circle with chords AB and CD. Let the CD pass through the midpoint M of AB.

Let AM = MB = z and let CM = x, MD = y. The lengths are all positive values.



By an elementary theorem of geometry the products of the parts of the chords are equal.

We know that $x + y \geq 2z$ with equality only if M is the midpoint of CD.

Algebra

$$(\sqrt{a} - \sqrt{b})^2 \geq 0 \qquad \text{with equality if and only if } a = b.$$

$\therefore$

$$a + b - 2\sqrt{ab} \geq 0$$

so

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad \text{with equality if and only if } a = b.$$

Something Different

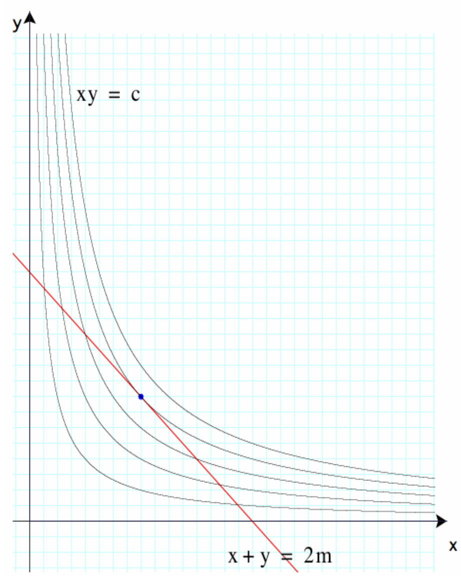
For any (x,y) in the plane, if the line $x + y = 2m$ intersects a curve in the family hyperbolas, it does so at two points or it is tangent to exactly one of the curves.

If the line $x + y = 2m$ is tangent to a curve $xy = c$, it does so at the point (m,m) since both the line and the hyperbola are symmetric with respect to the line $y = x$.

So for the tangent case, $c = xy = m^2$ and we have

$$xy = \left(\frac{x+y}{2}\right)^2$$

Any other point (x_0,y_0) on the line will intersect one of the family of $xy = c$ hyperbola curves with a value of $c < m^2$.



Solving Problems with the AM-GM Inequality

Cost of Fencing a Field

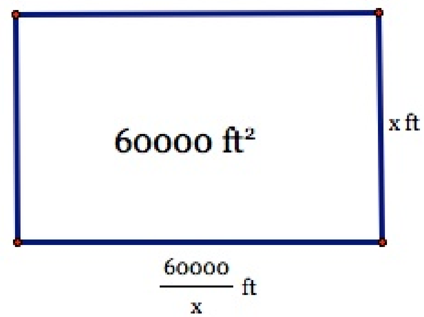
Problem: A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs \$2 per foot, while the fence for the other three sides costs \$1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?

Solution: (This is a typical calculus problem but no calculus is needed!)

Let x = the distance along the highway. The area = 60000 sq ft means $\frac{60000}{x}$ is the distance along one side of the field that is not parallel to the highway. The cost function is

$$\text{cost}(x) = 2x + x + \frac{60000}{x} + \frac{60000}{x}$$

$$\text{cost}(x) = 3x + \frac{120000}{x}$$



By the AM-GM Inequality,

$$\text{cost}(x) \geq 2\sqrt{(3x)\left(\frac{120000}{x}\right)} = 2\sqrt{360000} = 2(600) = \$1200$$

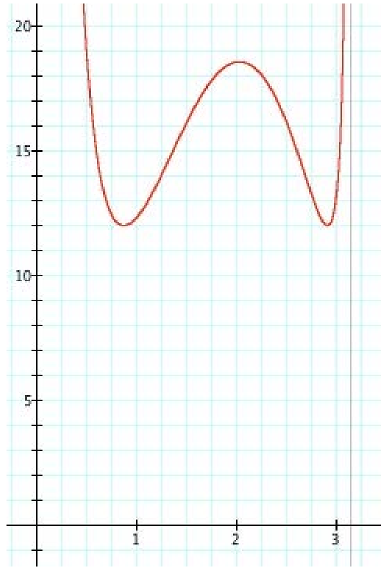
The cost fencing for any such rectangular plot is always greater that \$1200 so \$1200 is the minimum cost. It reaches this amount when

$$3x = \frac{120000}{x}$$

which means x = 200 feet. So the plot of land is 200 feet along the highway and 300 feet along the side perpendicular to the highway.

Minimum of $f(x) = \dfrac{9x^2(\sin x)^2 + 4}{x \sin x}$

The problem is to find minimum values for this function in the range or $0 < x < \pi$. This one requires some change in the form of the equation in order to apply the AM-GM Inequality. First, however, it is helpful to see a graph and have some sense of the equation. The graph at the right suggests that there may be two minimum values in the range $0 < x < \pi$.



Dividing both the numerator and the denominator by $x \sin x$ gives

$$f(x) = 9x \sin x + \frac{4}{x \sin x}$$

Applying the AM-GM Inequality,

$$f(x) \geq 2\sqrt{\left(9x \sin x\left(\frac{4}{x \sin x}\right)\right)}$$

$$= 2\sqrt{9(4)} = 12$$

The minimum value of $f(x)$ is 12. The minimum occurs when

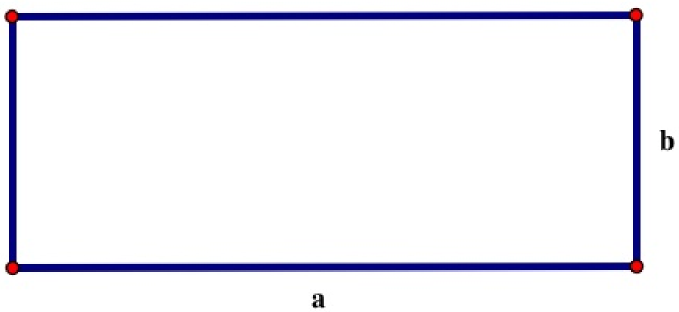
$$9x \sin x = \frac{4}{x \sin x}$$

That is, $x \sin x = \frac{2}{3}$. Thus $x \approx .871$ or $x \approx 2.91$

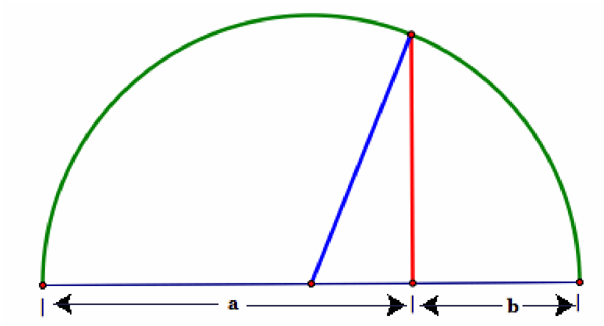
Construct a Square with Same Area as a Given Rectangle

The problem is to construct a square with straightedge and compass that is the same area as a given rectangle.

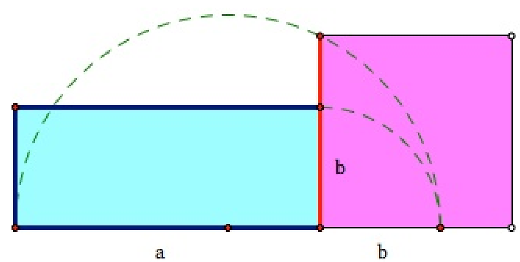
The area will be ab and so the length of the side of the square is the geometric mean.



Consider:



The red segment is the geometric mean of a and b.



[Maximum Area of a Sector of a Circle With Fixed Perimeter]

Understanding the problem. A sector of a circle has a perimeter made up of two radii and an arc of the circle connecting the endpoints of the two radii. Compare the areas of three sectors -- each with P = 100 -- central angles of 45 degrees, 90 degrees, and 180 degrees. These are sectors that are an eighth circle, quarter circle, and semicircle. As the angles increas, the radii become shorter because more of the fixed perimeter is in the arc. Note these angles are

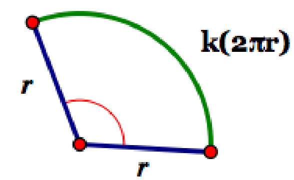
When these three areas are computed we get

$Area_{45} \approx 506.157$ sq units
$Area_{90} \approx 615.970$ sq units
$Area_{180} \approx 594.189$ sq units

Clearly, as the angle increases from 45 to 90 to 180 the area increases and then decreases. Somewhere there is a maximum area. Where?

Sector as Fraction of a Circle

Some students (and teachers) have an aversion to working with radian measure. For them the approach might be as follows:



Let k be the fraction of a circle represented by the sector. $0 < k < 1$. The Perimeter is a constant and so we can represent it as twice the radius plus the fraction of the circumference.

$$P = 2r + k(2\pi r)$$

$$r = \frac{P}{2(1 + k\pi)} \qquad k = \frac{P - 2r}{2\pi r}$$

The area of the sector in the fraction notation can be formed as either a function of r or a function of k, depending on a subsitution of r or k from the perimeter equation. Either one can be used with the AM-GM Inequality to reach closure on the problem:

$$\text{Area} = k\pi r^2$$

$$\text{Area}(r) = \left(\frac{P - 2r}{2\pi r}\right)\pi r^2 = \left(\frac{P}{2} - r\right)r$$

By the AM-GM Inequality

$$\text{Area}(r) \le \left(\frac{\frac{P}{2} - r + r}{2}\right)^2 = \left(\frac{P}{4}\right)^2$$

with equality if and only if

$$\frac{P}{2} - r = r$$

$$\therefore \quad r = \frac{P}{4} \quad \text{and} \quad k = \frac{1}{\pi}$$

$$\text{Area}(k) = k\pi\left(\frac{P}{1 + k\pi}\right)^2 = k\pi\left(\frac{P^2}{1 + 2k\pi + k^2\pi^2}\right)$$

Divide numerator and denominator by $k\pi$

$$\text{Area}(k) = \frac{P^2}{\frac{1}{k\pi} + 2 + k\pi}$$

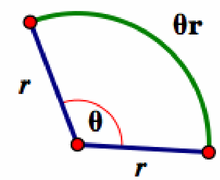
Now $\frac{1}{k\pi} + k\pi \ge 2\sqrt{\frac{1}{k\pi}k\pi} = 2$ by the AM-GM

So the denominator is always greater than 4 and

Therfore the Area(k) is LESS than $\left(\frac{P}{4}\right)^2$.

The equality is obtained when $\frac{1}{k\pi} = k\pi$. $k = \frac{1}{\pi}$

Looking back at the example when P = 100, the maximum area would be when r = 25 and therefore the arc length is 50. The angle is a little less than 120 degrees.



Sector in Radian Measure

If we take the sector in radian measure and let θ be the central angle then the equation for the fixed perimeter is $P = 2r + \theta r$ and we can solve for either r or θ to substitute in the area function, Area $= \frac{\theta}{2}r^2$.

$$r = \frac{P}{2+\theta} \qquad\qquad \theta = \frac{P - 2r}{r}$$

$$\text{Area}(r) = \frac{1}{2}\left(\frac{P-2r}{r}\right)r^2 = \left(\frac{P}{2} - r\right)r$$

By the AM-GM,

$$\text{Area}(r) \le \left(\frac{\frac{P}{2} - r + r}{2}\right)^2 = \left(\frac{P}{4}\right)^2$$

with equality if and only if

$$\frac{P}{2} - r = r \qquad \text{That is,} \quad r = \frac{P}{4}$$

and $\theta = 2$ radians.

$$\text{Area}(\theta) = \frac{\theta}{2}\left(\frac{P}{2+\theta}\right)^2 = \frac{\theta}{2}\left(\frac{P^2}{4 + 4\theta + \theta^2}\right)$$

Divide the numerator and denominator by θ

$$\text{Area}(\theta) = \frac{1}{2}\left(\frac{P^2}{\frac{4}{\theta} + 4 + \theta}\right)$$

By AM-GM $\frac{4}{\theta} + \theta \ge 2\sqrt{\frac{4}{\theta}\theta} = 4$

Thus $\text{Area}(\theta) \le \left(\frac{P}{4}\right)^2$

Equality is reached if and only if

$\frac{4}{\theta} = \theta$ **so** $\theta = 2$ **radians.**

The parallel of this solution with the one where the square was the maximum area for rectangles with fixed perimeter is worth noting.

[Inequalities Problems]

1. For positive real numbers $a, b,$ and c

   Prove: $(a + b)(b + c)(c + a) \geq 8abc$

2. For positive real numbers $a, b,$ and c

   Prove: $a^2 + b^2 + c^2 \geq ab + bc + ca$

3. Find the maximum value of

   $S = ab(72 - 3a - 4b)$ for $a > 0, b > 0$

4. Find the minimum value for $x > 0$

   $$y = 5x + \frac{16}{x} + 21$$

5. When $x > 0$, find the minimum for

   $$y = x^2 + \frac{1}{x}$$

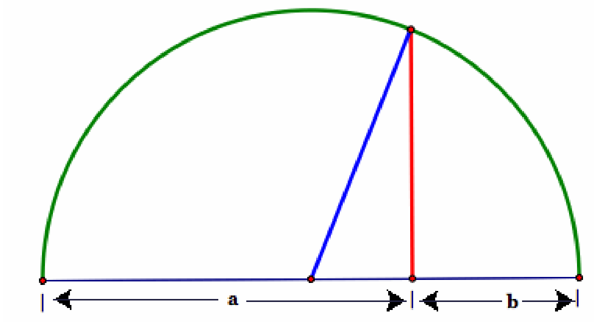
1. By AM-GM, $a+b \geq 2\sqrt{ab}$, etc.

$$(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca})$$

$$= 8\sqrt{a^2b^2c^2} = 8abc$$

Comparison of altitude and median in a right triangle

Construct segments from the endpoints of the diameter to complete a right triangle.



Maximum and Minimum of $y = \frac{x}{1+x^2}$

Rewrite

$$y = \frac{x}{1+x^2} = \frac{1}{\frac{1}{x} + x}$$

For x > 0, using the arithmetic mean-geometric mean inequality,

$$\frac{1}{x} + x \geq 2\sqrt{\left(\frac{1}{x}\right)(x)} = 2$$

with equality iff $x = \frac{1}{x}$ of, that is $x = 1$

Therefore the value of the function is always less than or equal to .5 and it is equal to .5 only when x = 1.

For x < 0, a similar argument leads to finding the minimum of the function at x = -1. Since the Arithmetic Mean -- Geometric Mean Inequality holds only for

postitive values, when x < 0 we have to apply the inequality to - x and - 1/x. We know

$$-x + \frac{-1}{x} \geq 2\sqrt{(-x)\left(\frac{-1}{x}\right)} = 2$$

Keep in mind this is for x< 0 so -x and -1/x are postive. Multiplying each side of the inequality by -1 gives

$$x + \frac{1}{x} \leq -2$$

and equality occurs when x = -1. Therefore the value of the function is always more than or equal to -0.5 and it is equal to -0.5 only when x = -1.

See Graphs

Maximum of f(x) = (1–x)(1+x)(1+x)

We have three factors in the function and we want to know when the function reaches a maximum in the interval [0,1].

$$f(x) = (1 - x)(1 + x)(1 + x)$$

In order to take advantage of the AM-GM Inequality, we need the sum of these factors to equal a constant. That is we need -2x rather than -x. We can get that by the following

$$\begin{aligned} f(x) &= (1 - x)(1 + x)(1 + x) \\ &= \left(\frac{1}{2}\right)(2)(1 - x)(1 + x)(1 + x) \\ &= \left(\frac{1}{2}\right)(2 - 2x)(1 + x)(1 + x) \\ &\le \left(\frac{1}{2}\right)\left(\frac{2 - 2x + 1 + x + 1 + x}{3}\right)^3 \quad \text{(by AM-GM)} \\ &= \left(\frac{1}{2}\right)\left(\frac{4}{3}\right)^3 = \frac{32}{27} = 1.185185185\ldots \end{aligned}$$

The function reaches this maximum value for x in [0,1] if and only if

$$\begin{aligned} 2(1 - x) &= 1 + x \\ 2 - 2x &= 1 + x \\ 1 &= 3x \end{aligned}$$

That is,

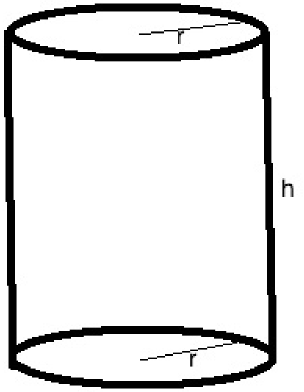
$$f(x) \le \frac{32}{27} = 1.185185185\ldots$$

with equality iff

$$x = \frac{1}{3}$$

Minimum Surface Area of a Can of Fixed Volume

In packaging a product in a can the shape of right circular cylinder, various factors such as tradition and supposed customer preferences may enter into decisions about what shape (e.g. short and fat vs. tall and skinny) can might be used for a fixed volume. Note, for example, all 12 oz. soda cans have the same shape -- a height of about 5 inches



and a radius of about 1.25 inches. Why?

What if the decison was based on minimizing the material used to make the can? This would mean that for a fixed volume V the shape of the can (e.g. the radius and the height) would be determined by the minimum surface area for the can. What is the relationship between the radius and the height in order to minimize the surface area for a fixed volume?

What would be the shape of a 12 ounce soda can that minimizes the amount of aluminum in the can?

The Volume is fixed.

$$V = \pi r^2 h$$

The Surface area is a function of r and h.

$$S = 2\pi r^2 + 2\pi rh$$

What is the minimum S?

We want to minimize the surface area for a cylinder of fixed Volume $V = \pi r^2 h$, r being the radius of the cylinder and h being the height. The surface area is $S = 2\pi r^2 + 2\pi r h$

$$S = 2\pi r^2 + 2\pi r h$$

Use $h = \dfrac{V}{\pi r^2}$ and substitute for h.

$$S = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right)$$

$$S = 2\pi r^2 + 2\left(\frac{V}{r}\right). \quad \text{Rewrite}$$

$$S = 2\pi r^2 + \left(\frac{V}{r}\right) + \left(\frac{V}{r}\right).$$

By the AM-GM inequality we have

(Note that we have use the AM-GM Inequality for three positive numbers)

$$S \geq 3\sqrt[3]{2\pi r^2\left(\frac{V}{r}\right)\left(\frac{V}{r}\right)} = 3\sqrt[3]{2\pi V^2} \quad \text{(a constant)}$$

with equality iff $2\pi r^2 = \dfrac{V}{r}$. So $2\pi r^3 = V$

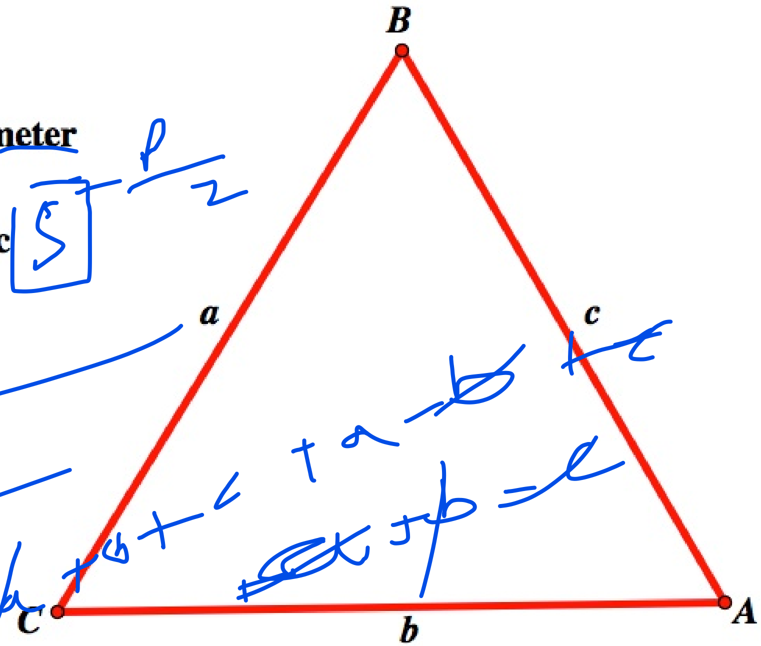
Thus, $2\pi r^3 = \pi r^2 h$ or, that is, $2r = h$

Maximum area of a Pen, fixed length of fencing, with Partitions Parallel to a Side

Suppose you have 100 feet of fencing. You want a rectangular pen with a partition parallel to a side. The 100 feet of fencing must enclose the four sides and the partition. What is the shape of the pen and what is the maximum area?

Show that the dimensions of the maximum area pen is 25 ft. by 16.67 ft. with maxim area of approximately 417.7 sq. ft.

Prove the Maximum Area of a Triangle with Fixed Perimeter is Equilateral



Maximum Area of a Triangle with Fixed Perimeter

Given a $\triangle ABC$ with sides of length a, b, and c with a fixed perimeter $P = a + b + c = 2s$ where s is the semiperimeter

$\boxed{s} = \frac{P}{2}$

$$s = \frac{a + b + c}{2}$$

The area, using Heron's Formula is

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

s is a constant and so the issue of maximum area is in the product (s - a)(s - b)(s - c)

We will use the AM-GM Inequality for three positive number a, b, and c.

$$(s - a)(s - b)(s - c) \le \left(\frac{s - a + s - b + s - c}{3}\right)^3 \quad \text{by AM-GM Inequality}$$

$$= \left(\frac{3s - (a + b + c)}{3}\right)^3 = \left(\frac{3s - 2s}{3}\right)^3 = \left(\frac{s}{3}\right)^3$$

The maximum area is

$$\text{Area} = \sqrt{s\left(\frac{s}{3}\right)^3} = \sqrt{\frac{s^4}{27}} = \frac{s^2}{\sqrt{27}} \rightarrow$$

By the AM-GM Inequality, the maximum occurs if and only if

$s - a = s - b = s - c$. This means $a = b = c$ and the triangle is equilateral.

Minimum Distance from (0,0) to $y = \frac{1}{x^2}$

Find a point on the graph of $y = \frac{1}{x^2}$ that is closest to the origin.

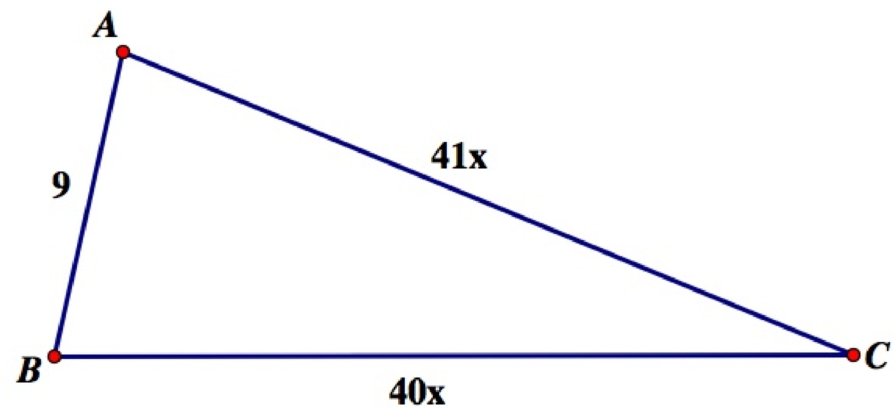
(follow the link to the discussion)

Maximum Area of a Triangle with Sides of 9, 40x, and 41x

Given a triangle with one side of length 9 units and the ratio of the other two sides is 40/41.

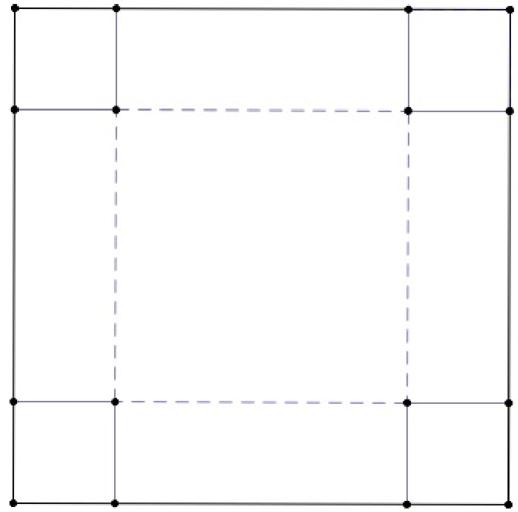
Find the Maximum Area.

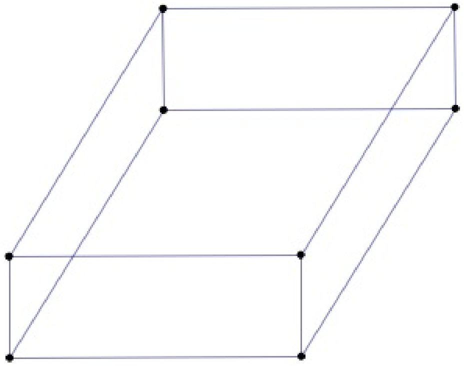
(use link for discussion)



The Box Problem

Use the Arithmetic Mean-Geometric Mean Inequality to find the maximum volume of a box made from a 25 by 25 square sheet of cardboard by removing a small square from each corner and folding up the sides to form a lidless box.





Let x be the length of the side of the square cut from each corner.

$$\text{Volume} = x(25 - 2x)(25 - 2x)$$

We can write

$$\text{Volume} = \frac{1}{4}(4x)(25 - 2x)(25 - 2x)$$

By AM-GM Inequality

$$\text{Volume} \le \frac{1}{4}\left(\frac{4x + 25 - 2x + 25 - 2x}{3}\right)^3$$

$$= \frac{1}{4}\left(\frac{50}{3}\right)^3 \approx 1157.41 \text{ cu. in.}$$

with the maximum reached if and only if

$$4x = 25 - 2x \quad \Rightarrow \quad x = \frac{25}{6}$$

Theorem: In the Product of n positive numbers is equal to 1 then the sum if the numbers is greater than or equal to n.

1. Let a and b be two positive numbers such that $ab = 1$.

   Prove that $a + b \geq 2$

2. Let r, s, and t be three positive real numbers such that $rst = 1$.

   Prove that $r + s + t \geq 3$

3. Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers such that

   $a_1 a_2 a_3 \ldots a_n = 1$

   Prove that $a_1 + a_2 + a_3 + \ldots + a_n \geq n$

Clearly, the proof of 3 implies the first two.

Given: $a_1 . a_2 . a_3 . \ldots . a_n = 1$

But by the AM-GM Inequality for n positive numbers,

$$\frac{a_1+a_2+a_3+\ldots+a_n}{n} \geq \sqrt[n]{a_1 . a_2 . a_3 . \ldots . a_n}$$

Given the product,

$$\frac{a_1+a_2+a_3+\ldots+a_n}{n} \geq \sqrt[n]{1}$$

That is,

$$\frac{a_1+a_2+a_3+\ldots+a_n}{n} \geq 1$$

and so $a_1+a_2+a_3+\ldots+a_n \geq n$