22 Mat220 Assignment - 2

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1. $f(x) = \frac{1}{2}||Ax-b||^2$

$$\Rightarrow f(x) = \frac{1}{2}\sum_{i=1}^{n}(Ax_i - b)^2$$

$$\frac{\partial f(x)}{\partial x} = \nabla f(x) = \frac{1}{2}\sum_{i=1}^{n} 2 \times (Ax_i - b) \times A$$

$$= A\sum_{i=1}^{n}(Ax_i - b)$$

$$\Rightarrow A^T(Ax-b)$$

$$\boxed{\nabla f(x) = A^T(Ax-b)}$$

The gradient can be used in Gradient descent to minimize error in Regression fitting

Gradient Descent

$$x_{k+1} = x_k - \lambda \nabla f(x_k)$$

The Gradient $\nabla f(x)$ is used to reach the optimal solution $\hat{x}$ iteratively by changing the preset x_0 value by small step values $\lambda \nabla f(x_k)$ which is controlled by a learning rate λ

λ ensures stability in descent

2. $f(x,y,z) = 2y^2z - x^2z^2$ at $P(1,2,3)$

$$\frac{\partial f(x,y,z)}{\partial x} = -2xz^2$$

$$\frac{\partial f(x,y,z)}{\partial y} = 4yz$$

$$\frac{\partial f(x,y,z)}{\partial z} = 2y^2 - 2x^2z$$

$$\nabla f(x,y,z) = \begin{bmatrix} -2xz^2 \\ 4yz \\ 2y^2 - 2x^2z \end{bmatrix}$$

$$\nabla f(1,2,3) = \begin{bmatrix} -2(1)(3)^2 \\ 4(2)(3) \\ 2(2)^2 - 2(1)^2(3) \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \\ 2 \end{bmatrix}$$

$$\nabla f(1,2,3) = \begin{bmatrix} -18 \\ 24 \\ 2 \end{bmatrix}$$ is the Gradient in the direction of increase

$$\begin{bmatrix} -18 \\ 24 \\ 2 \end{bmatrix} \Big/ \sqrt{(-18)^2 + (24)^2 + (2)^2} = \begin{bmatrix} -18 \\ 24 \\ 2 \end{bmatrix} \Big/ 30.06$$

$$\hat{u} = \begin{bmatrix} -0.598 \\ 0.798 \\ 0.066 \end{bmatrix}$$ is the direction of increase

3. $f(x,y,z) = x(y-3) + z(z^2-3)$

$\Rightarrow x(y-3) + z^3 - 3z$

$$\frac{\partial f(x,y,z)}{\partial x} = y-3$$

$$\frac{\partial f(x,y,z)}{\partial y} = x$$

$$\frac{\partial f(x,y,z)}{\partial z} = 3z^2-3$$

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix} = \begin{bmatrix} y-3 \\ x \\ 3z^2-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y-3=0$

$\boxed{y=3}$

$\boxed{x=0}$

$3z^2-3=0$

$3z^2=3$

$z^2=1$

$\boxed{z=\pm 1}$

$\therefore$ Critical points $= \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \right\}$

4. $f(x,y) = 3x + xy + 2y^2$

$$H(f(x,y)) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$$

$|H| = -1$

$\therefore f(x,y)$ is a Concave function

Constrained by

$$x^2 - y = 3$$

$$x^2 - y - 3 = 0$$

$$\boxed{g(x,y) = x^2 - y - 3}$$

The Lagrangian function

$$L(x,y,\lambda) = -f(x,y) - \lambda g(x,y)$$

To maximize a function $\rightarrow$ negate objective func and minimize

$$L(x,y,\lambda) = -3x - xy - 2y^2 - \lambda(x^2 - y - 3)$$

$$\frac{\partial L(x,y,\lambda)}{\partial x} = -3 - y - 2\lambda x = 0$$

$$\boxed{2\lambda x + y - 3 = 0} \quad - (1)$$

$$\frac{\partial L(x,y,\lambda)}{\partial y} = -x - 4y + \lambda = 0$$

$$\boxed{x + 4y + \lambda = 0} \quad - (2)$$

$$\frac{\partial L(x,y,\lambda)}{\partial \lambda} = -x^2 + y + 3 = 0$$

$$\boxed{x^2 - y - 3 = 0} \quad - (3)$$

(2) $\Rightarrow$ $\lambda = -x - 4y$

substitute for λ in (1)

$$2(-x-4y)x + y - 3 = 0$$

$$-2x^2 - 8xy + \cancel{y} - 3 = 0$$

$$x^2 \qquad - \cancel{y} - 3 = 0 \quad (+)$$

$$-x^2 - 8xy - 6 = 0$$

4. $f(x,y) = 3x + xy + 2y^2$

$$H(f) = \begin{bmatrix} \partial^2 f/\partial x^2 & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$$

$|H| = -1$

$|H| < 0$

$f(x,y)$ is ~~strictly~~ neither convex or concave function

Constraind by

$x^2 - y = 3$

$x^2 - y - 3 = 0$

$g(x,y) = x^2 - y - 3$

Lagrangian function

$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y))$

$L(x,y,\lambda) = 3x + xy + 2y^2 - \lambda(x^2 - y - 3)$

$\frac{\partial L(x,y,\lambda)}{\partial x} = 3 + y - 2\lambda x = 0$

$\frac{\partial L(x,y,\lambda)}{\partial y} = x + 4y - -\lambda = 0$

$\frac{\partial L(x,y,\lambda)}{\partial \lambda} = -(x^2 - y - 3) = 0$

$x^2 - y = 3$

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$x^2 - y = 3$

$x^2 = y + 3$

① $- \quad 3 + y - 2\lambda x = 0$

$x^2 - 2\lambda x = 0$

$x = 0$ or $x = 2\lambda$

$x^2 - y = 3$

$0 - y = 3$

$\boxed{y = -3}$

$(x, y) = (0, -3)$

$x + 4y + \lambda = 0$

$2\lambda + 4y + \lambda = 0$

$4y = -3\lambda$

$y = \frac{-3\lambda}{4}$

$3 + y - 2\lambda x = 0$

$3 - \frac{3\lambda}{4} - 2\lambda(2\lambda) = 0$

$+4\lambda^2 + \frac{3\lambda}{4} - 3 = 0$

$16\lambda^2 + 3\lambda - 12 = 0$

$\lambda = \frac{-3 \pm \sqrt{9 + 768}}{32}$

$= \frac{-3 \pm \sqrt{777}}{32}$

$(x, y) = \left\{ \left(2\lambda, \frac{-3\lambda}{4}\right) \right.$

$(x, y) = \left(\frac{-3 - \sqrt{777}}{16}, \frac{-9 - 3\sqrt{777}}{128}\right)$

$(x, y) = \left(\frac{-3 + \sqrt{777}}{16}, \frac{-9 + 3\sqrt{777}}{128}\right)$

$\Downarrow$

$(x, y) = (-1.929, -0.723)$

$= (1.554, 0.583)$

$f(0,-3) = 18$

$f(-1.929, -0.723) = -3.346$

$f(1.554, 0.583) = 6.247$

$\therefore$ The functions Global maximum lies in

$f(0,-3)$

$(x,y) = (0,-3)$

$\max = 18$ //

5. $f(x,y,z) = x^2 + 6y^2 - 10z^2 + 4xy + 6yz - 2xz + 5y$

subject to

$x + 2y + 3z \geq 5$

$x, y, z \geq 0$

$x + 2y + 3z - 5 \geq 0$

~~$-x - 2y - 3z + 5$~~

$-(x + 2y + 3z - 5) \leq 0$

$g(x,y,z) = -(x + 2y + 3z - 5)$

$L(x,y,z,\mu) = f(x,y,z) + \mu(g(x,y,z) + s^2)$

$= x^2 + 6y^2 - 10z^2 + 4xy + 6yz - 2xz + 5y - \mu(x + 2y + 3z - 5 - s^2)$

$\frac{\partial L}{\partial x} = 2x + 4y - 2y + \lambda = 0$

$\frac{\partial L}{\partial y} = 12y + 4x + 6z - 2z + 5 + 2\lambda = 0$

$\frac{\partial L}{\partial z} = -20z + 6y + 3\lambda = 0$

$$\lambda(g(x,y,z)) = \lambda(x+2y+3z-5) = 0$$

Case I

$\lambda = 0$

$\therefore \quad 2x + 4y - 2y + 0 = 0$

$2x + 2y = 0$

$\boxed{x + y = 0}$

$12y + 4x + 6z - 2x + 5 + 2 \cdot 0 = 0$

$\boxed{2x + 12y + 6z + 5 = 0}$ — ①

$-20z + 6y + 3 \cdot 0 = 0$

$6y = 20z$

$\cancel{y = x}$

$x = -y$

$z = \frac{3\cancel{6}y}{\cancel{20}\,10} = \frac{3y}{10}$

$\therefore$ ① $= -2y + 12y + 6 \times \frac{3y}{10} + 5 = 0$

$10y + \frac{18y}{10} + 5 = 0$

$f(x,y,z) = 2.926 \approx 3$

$118y + 50 = 0$

$y = \frac{-50}{118}$

$\boxed{x = \frac{25}{59}}$ $\boxed{y = \frac{-25}{59}}$ $\boxed{z = \frac{-15}{118}}$

6. $f(x,y) = \frac{1}{2}x^2 - y$

$g_1(x,y) \Rightarrow x + 2y - 6 \le 0$

$g_2(x,y) \Rightarrow 1 - x \le 0$

$g_3(x,y) \Rightarrow x^2 + y^2 - 25 \le 0$

$$L(x, y, \mu_1, \mu_2, \mu_3) = f(x,y) + \mu_1\left(g_1(x,y) + s_1^2\right) + \mu_2\left(g_2(x,y) + s_2^2\right) + \mu_3\left(g_3(x,y) + s_3^2\right)$$

$$L(x, y, \mu_1, \mu_2, \mu_3) = \frac{1}{2}x^2 - y + \mu_1\left(x + 2y - 6 + s_1^2\right) + \mu_2\left(1 - x + s_2^2\right) + \mu_3\left(x^2 + y^2 - 25 + s_3^2\right)$$

$$\frac{\partial L}{\partial x} = x + \mu_1 - \mu_2 + 2x\mu_3 = 0 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial y} = -1 + 2\mu_1 + 2y\mu_3 = 0 \quad \text{— (2)}$$

$\mu_1(x + 2y - 6) = 0$, $\mu_2(1 - x) = 0$, $\mu_3(x^2 + y^2 - 25) = 0$

$\mu_1 = 0$, $\mu_2 \neq 0$, $\mu_3 \neq 0$

$\cancel{x + 2y = 6}$

$2y\mu_3 = 1$

$$\boxed{y = \frac{1}{2\mu_3}}$$

$\boxed{x=1} \Rightarrow \quad x+2y=6$

$$1+2y=6$$
$$2y=5$$
$$y=\frac{5}{2} \Rightarrow 2.5$$
$$\boxed{y=2.5}$$

$(x,y) = (1, 2.5)$

Constraints

$x+2y \leq 6 \Rightarrow 1+2\cdot 2.5 \leq 6$

$1+5 \leq 6$

$6 \leq 6 \rightarrow T$

$x \geq 0 \Rightarrow 1 \geq 0 \rightarrow T$

$x^2+y^2 \leq 25 \rightarrow 1^2+(2.5)^2 \leq 25$

$1+6.25 \leq 25$

$7.25 \leq 25 \rightarrow T$

all Constraints are satisfied

$\therefore$ minima at $(1, 2.5)$

$f(1, 2.5) = \frac{1}{2}(1)^2 - 2.5$

$= \frac{1}{2} - 2.5 = -2$

$\boxed{f(1, 2.5) = -2} \rightarrow$ minima minimize

7. $f(r,h) = 2\pi r h + \pi r^2$

Constrained by $\pi r^2 h \le 500$

$\therefore \quad \pi r^2 h - 500 \le 0$

$g(r,h) = \pi r^2 h - 500$

Lagrangian function

$L(r,h,\mu) = f(r,h) + \mu(g(r,h) + s^2)$

$L(r,h,\mu) = 2\pi r h + \pi r^2 + \mu(\pi r^2 h - 500 + s^2)$

$\frac{\partial L}{\partial r} = 2\pi h + 2\pi r + 2\pi r h \mu = 0 \quad \div 2\pi$

$h + r + rh\mu \Rightarrow \boxed{\mu = \frac{-h-r}{hr}}$ — 1

$\frac{\partial L}{\partial h} = 2\pi r + \mu \pi r^2 = 0 \quad \div \pi$

$\Rightarrow 2r + \mu r^2 = 0 \quad [r \ne 0] \Rightarrow 2 + \mu r = 0$

$\boxed{\mu = -\frac{2}{r}}$ — 2

$\mu(\pi r^2 h - 500) = 0$

$\mu \ne 0$

$\pi r^2 h - 500 = 0$

$\pi r^2 h = 500$ — 3

by 1 & 2

$\frac{-h-r}{hr} = \frac{-2}{\cancel{r}}$

$-h - r = -2h$

$\boxed{h = r}$

3 $\Rightarrow \pi r^2 h = 500 \Rightarrow \pi r^3 = 500$

$\boxed{r = \sqrt[3]{\frac{500}{\pi}} = h}$

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$$f\left(\sqrt[3]{\frac{500}{\pi}}, \sqrt[3]{\frac{500}{\pi}}\right) = 2\pi r h + \pi r^2$$

$$= 2\pi r^2 + \pi r^2 \quad [\because r = h]$$

$$= 3\pi r^2$$

$$= 3\pi \times \sqrt[3]{\frac{500}{\pi}}$$

$$= \sqrt[3]{\frac{27\pi^{\cancel{3}} \times 500}{\cancel{\pi}}} = \boxed{\sqrt[3]{13500\pi^2} \ ft^2}$$

minimum surface Area

8.

Transition Matrix

$$A = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/5 & 2/5 & 1/5 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{array}{l} \rightarrow \frac{6}{6} = 1 \\ \rightarrow 4/5 \neq 1 \\ \rightarrow \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \end{array}$$

$\therefore$ The second row does not satisfy complete probability

all rows should add up to 1 since they are probabilities