

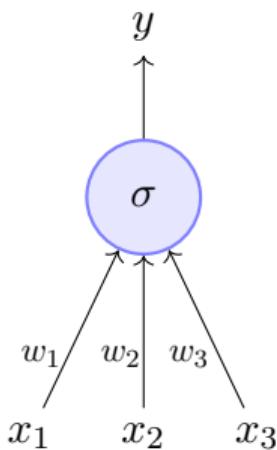
CS7015 (Deep Learning) : Lecture 2

McCulloch Pitts Neuron, Thresholding Logic, Perceptrons, Perceptron Learning Algorithm and Convergence, Multilayer Perceptrons (MLPs), Representation Power of MLPs

Mitesh M. Khapra

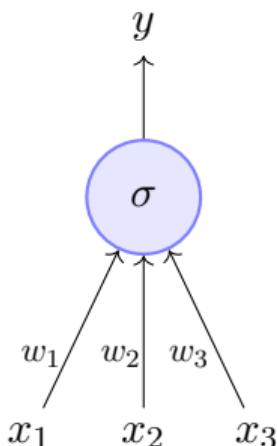
Department of Computer Science and Engineering
Indian Institute of Technology Madras

Module 2.1: Biological Neurons



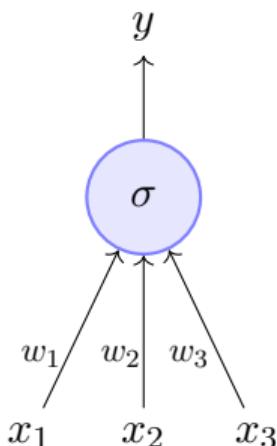
Artificial Neuron

- The most fundamental unit of a deep neural network is called an *artificial neuron*



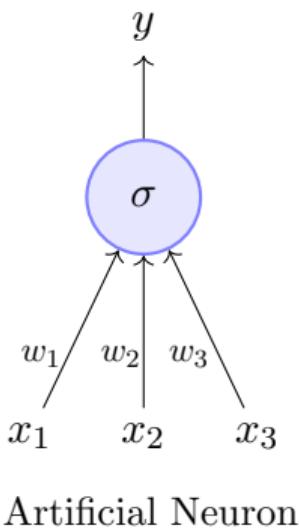
Artificial Neuron

- The most fundamental unit of a deep neural network is called an *artificial neuron*
- Why is it called a neuron ? Where does the inspiration come from ?

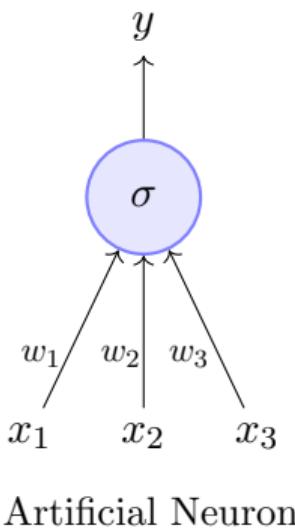


Artificial Neuron

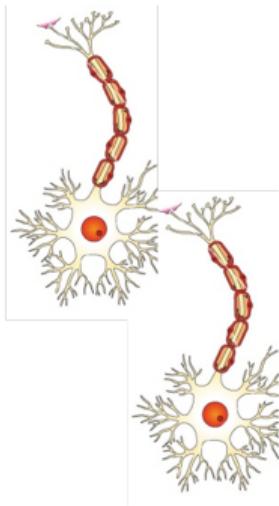
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- The inspiration comes from biology (more specifically, from the *brain*)



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- *biological neurons = neural cells = neural processing units*



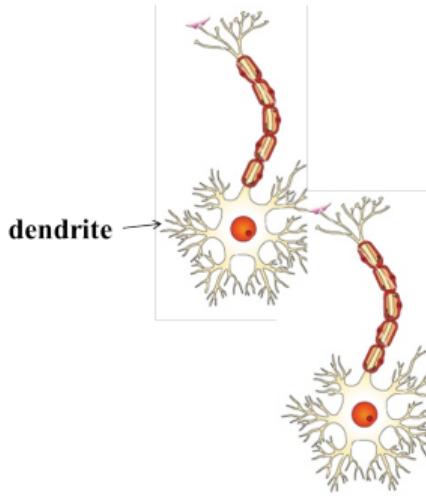
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- Why is it called a neuron ? Where does the inspiration come from ?
- The inspiration comes from biology (more specifically, from the *brain*)
- *biological neurons = neural cells = neural processing units*
- We will first see what a biological neuron looks like ...



Biological Neurons*

*Image adapted from

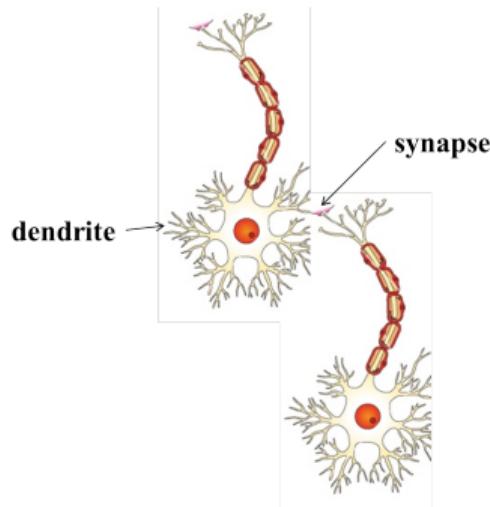
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Biological Neurons*

- **dendrite:** receives signals from other neurons

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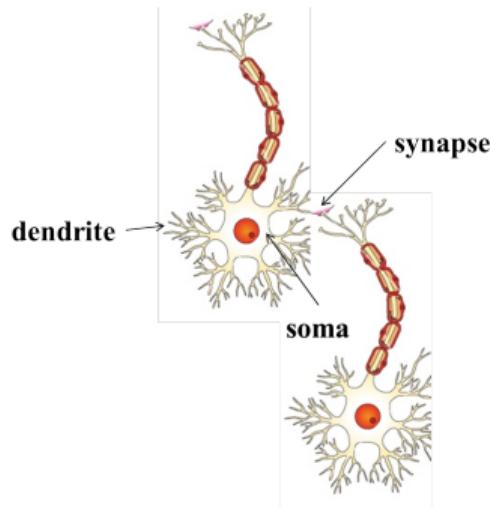


Biological Neurons*

- **dendrite:** receives signals from other neurons
- **synapse:** point of connection to other neurons

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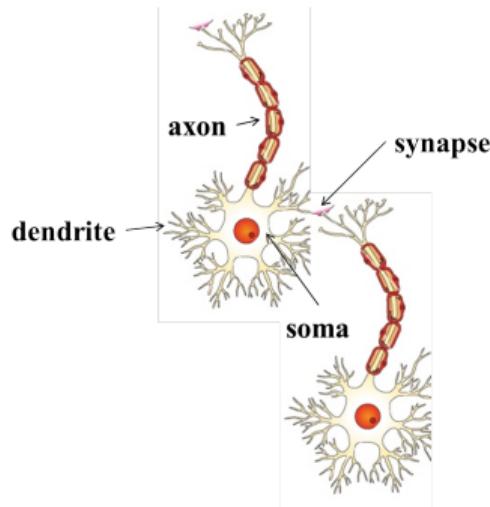


Biological Neurons*

- **dendrite:** receives signals from other neurons
- **synapse:** point of connection to other neurons
- **soma:** processes the information

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Biological Neurons*

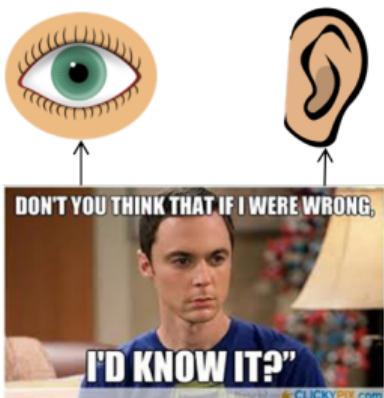
- **dendrite:** receives signals from other neurons
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- **soma:** processes the information
- **axon:** transmits the output of this neuron

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- Let us see a very cartoonish illustration of how a neuron works

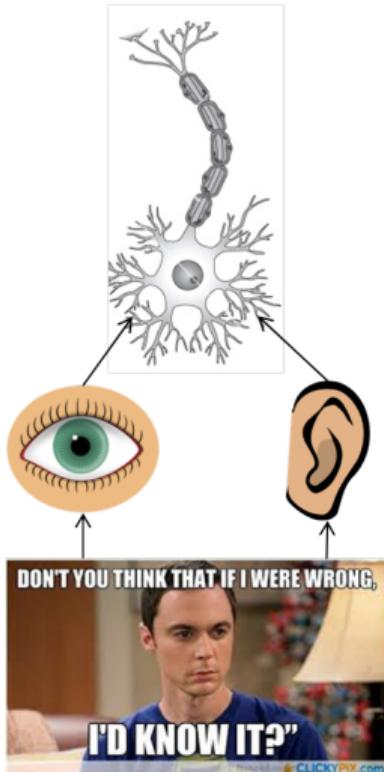
- Let us see a very cartoonish illustration of how a neuron works
- Our sense organs interact with the outside world

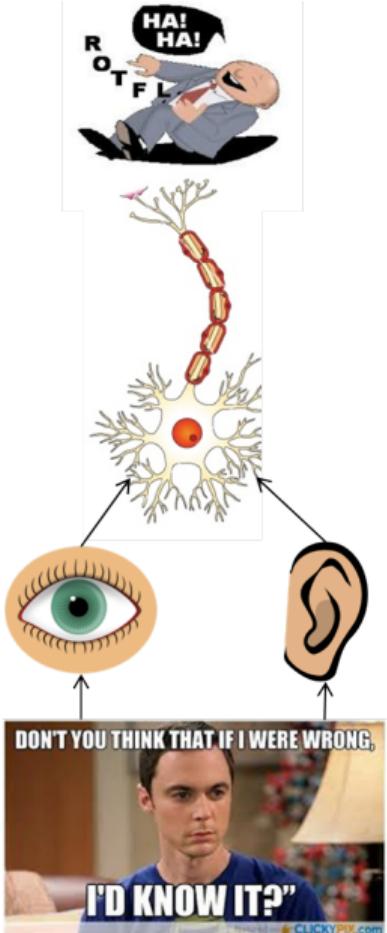


DON'T YOU THINK THAT IF I WERE WRONG,

I'D KNOW IT?"

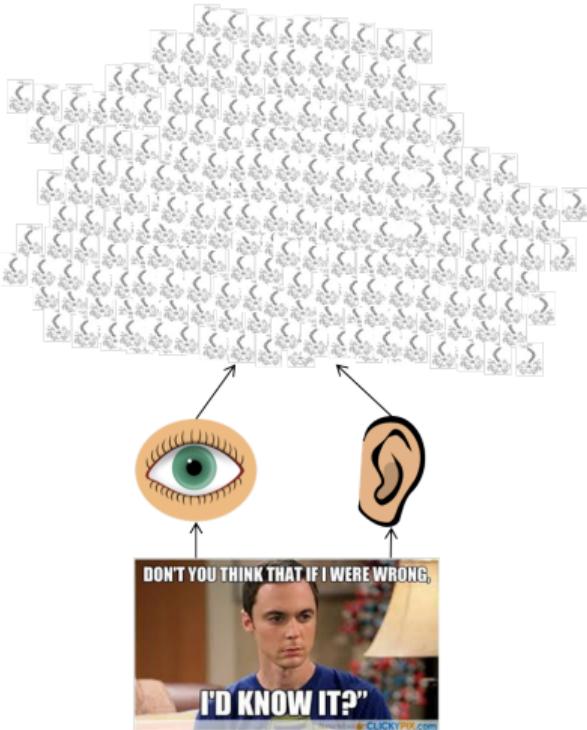
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- Our sense organs interact with the outside world
- They relay information to the neurons



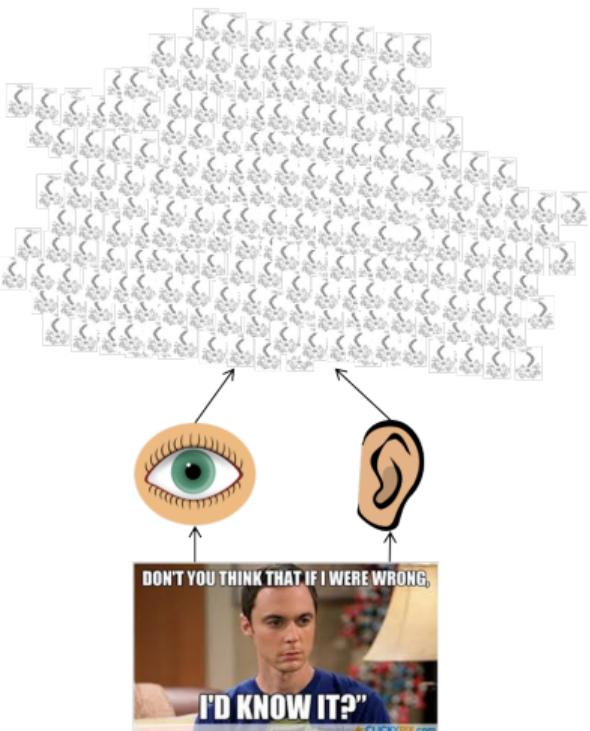


- Let us see a very cartoonish illustration of how a neuron works
- Our sense organs interact with the outside world
- They relay information to the neurons
- The neurons (may) get activated and produce a response (laughter in this case)

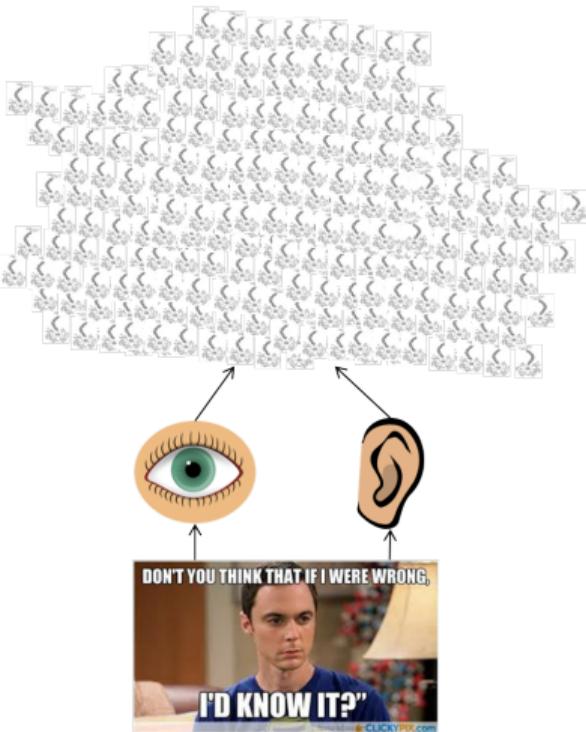
- Of course, in reality, it is not just a single neuron which does all this



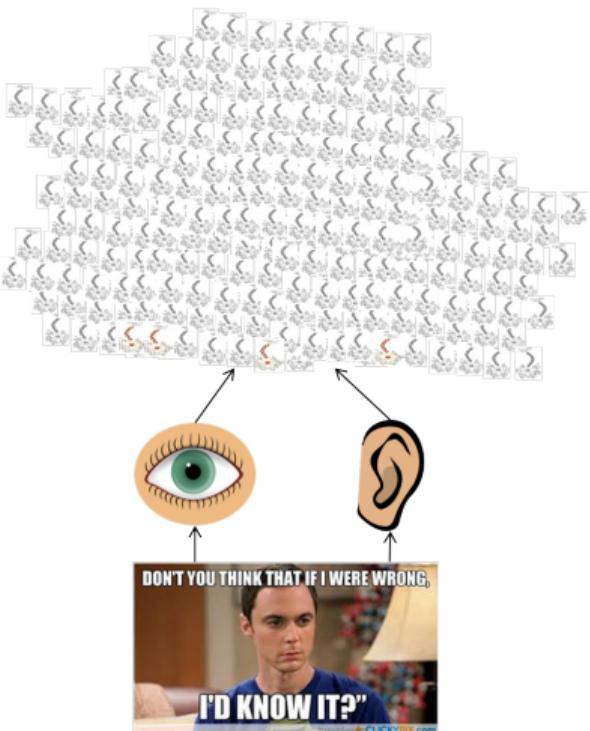
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- There is a massively parallel interconnected network of neurons

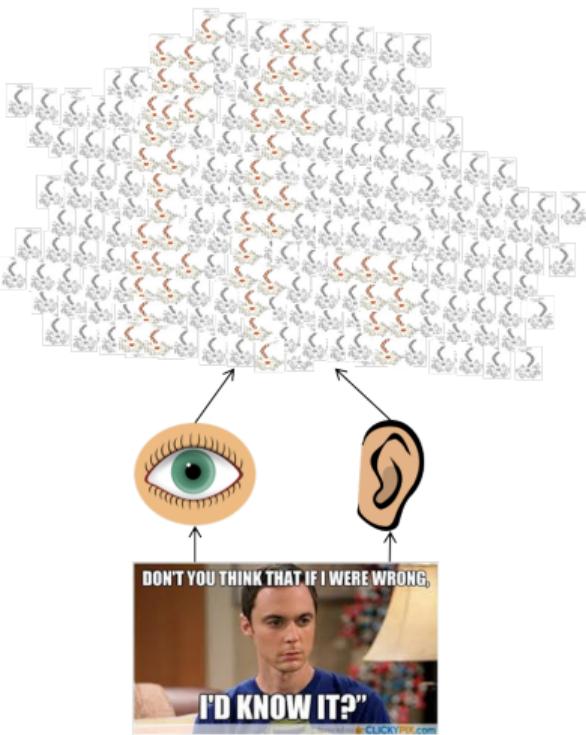


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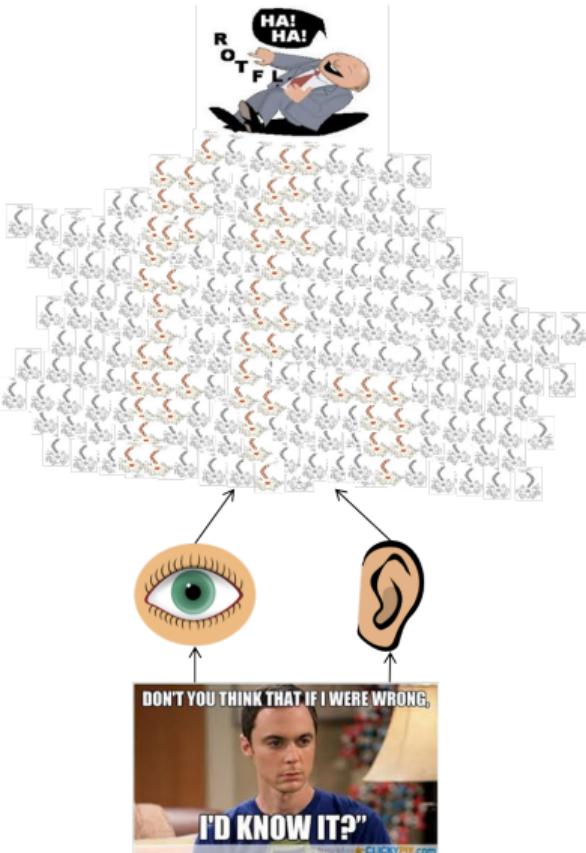


- Of course, in reality, it is not just a single neuron which does all this
- There is a massively parallel interconnected network of neurons
- The sense organs relay information to the lowest layer of neurons
- Some of these neurons may fire (in red) in response to this information and in turn relay information to other neurons they are connected to

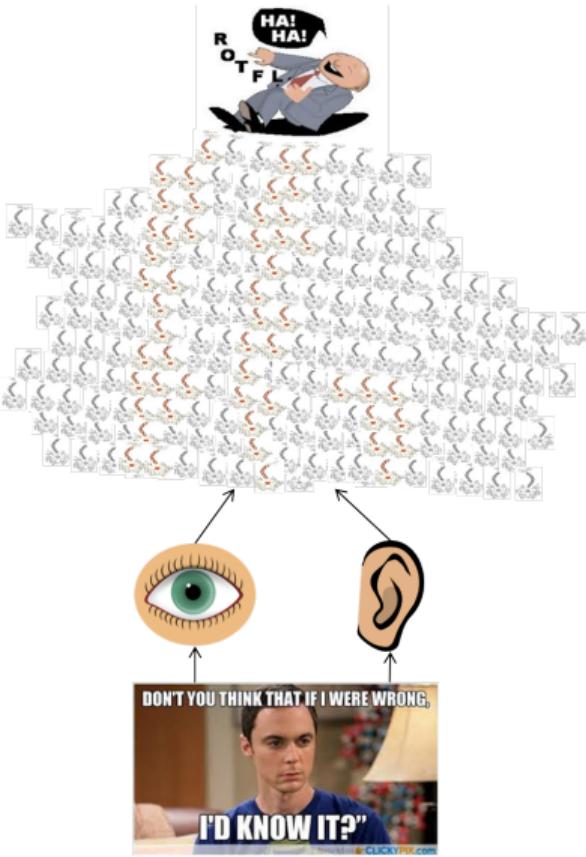




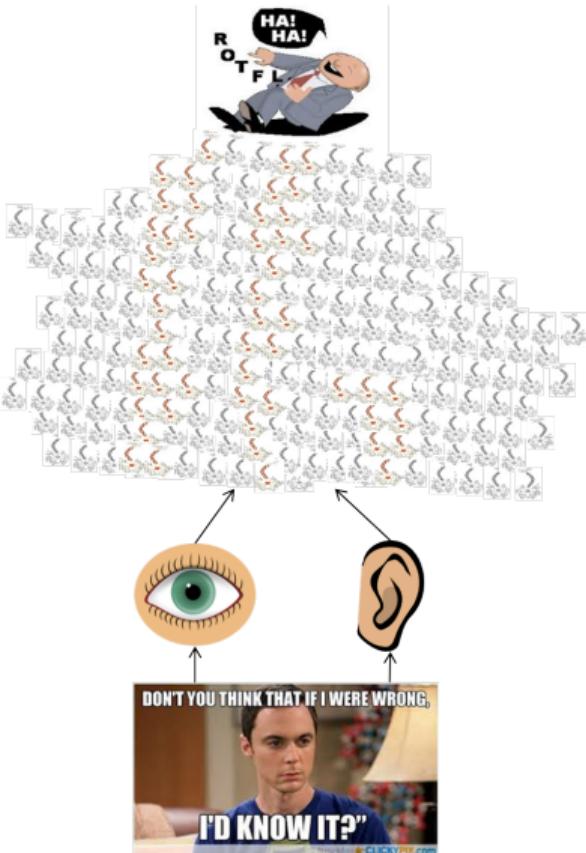
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- These neurons may also fire (again, in red) and the process continues



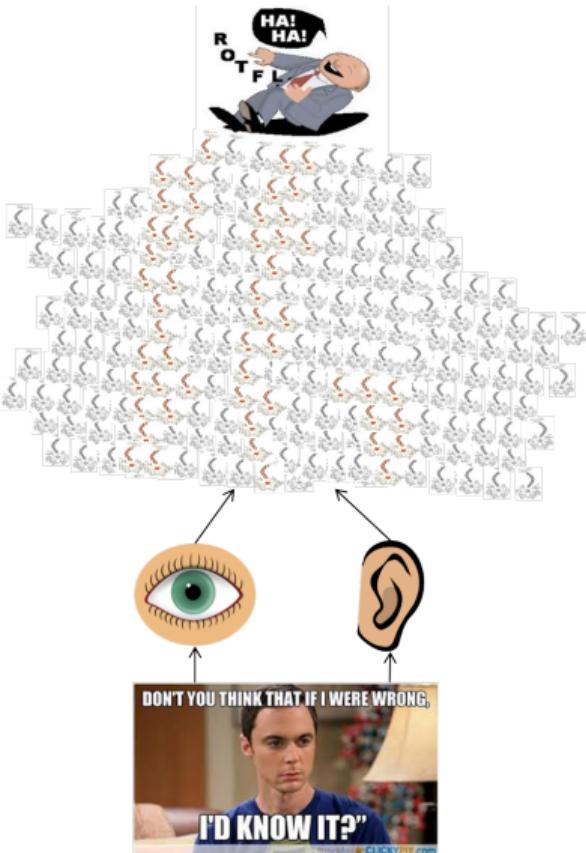
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- An average human brain has around 10^{11} (100 billion) neurons!

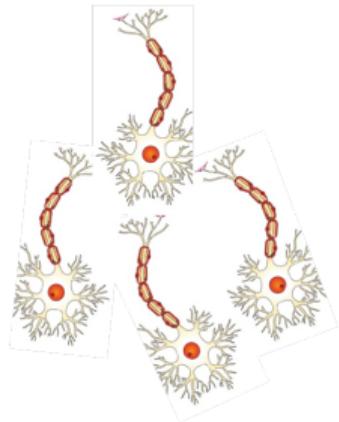


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A simplified illustration

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A simplified illustration

*fires if at least
2 of the 3 inputs fired*

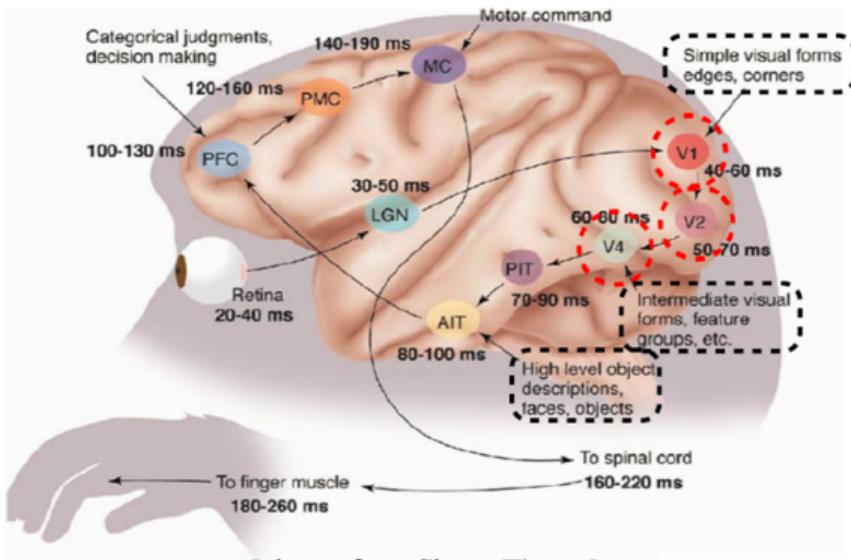


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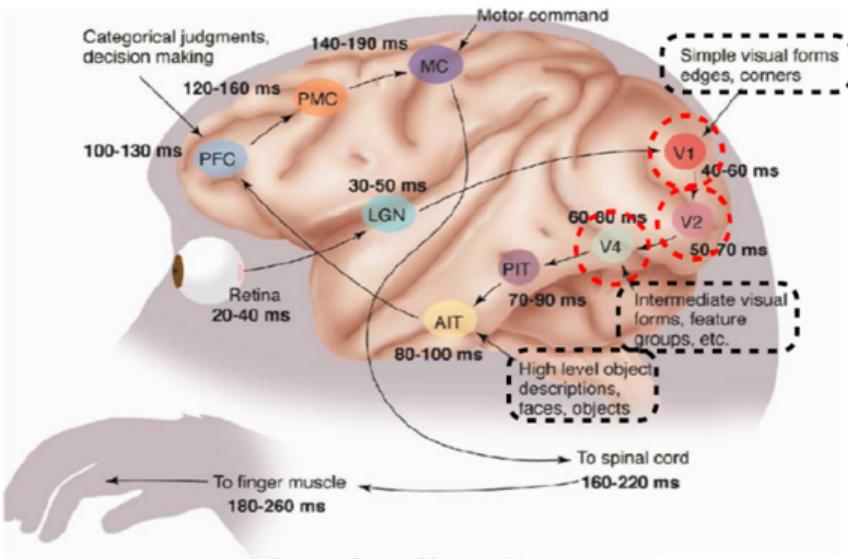
A simplified illustration

- The neurons in the brain are arranged in a hierarchy



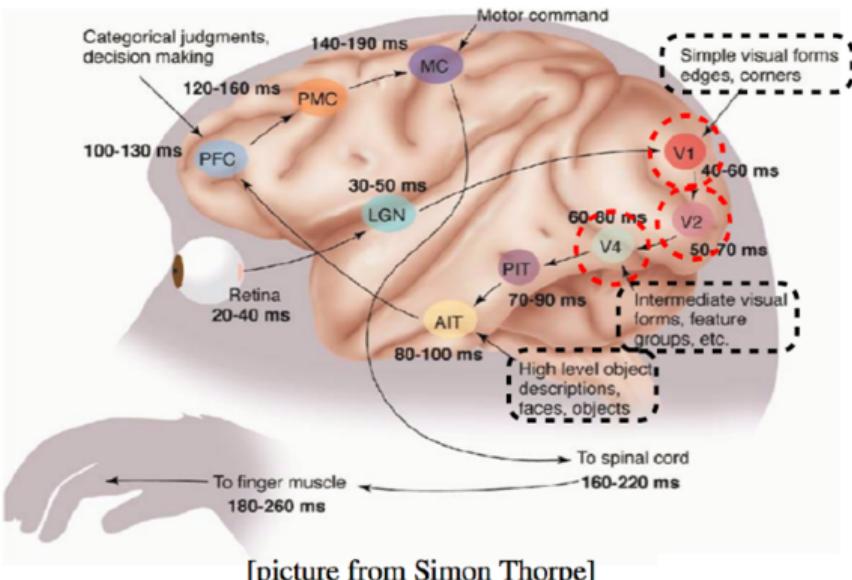
[picture from Simon Thorpe]

- The neurons in the brain are arranged in a hierarchy
- We illustrate this with the help of visual cortex (part of the brain) which deals with processing visual information



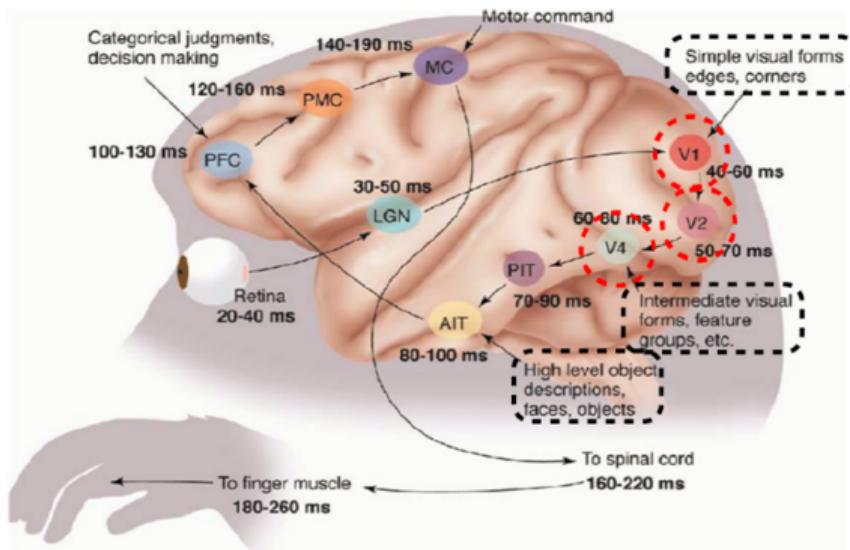
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- Starting from the retina, the information is relayed to several layers (follow the arrows)



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- The neurons in the brain are arranged in a hierarchy
- We illustrate this with the help of visual cortex (part of the brain) which deals with processing visual information
- Starting from the retina, the information is relayed to several layers (follow the arrows)
- We observe that the layers $V1$, $V2$ to AIT form a hierarchy (from identifying simple visual forms to high level objects)



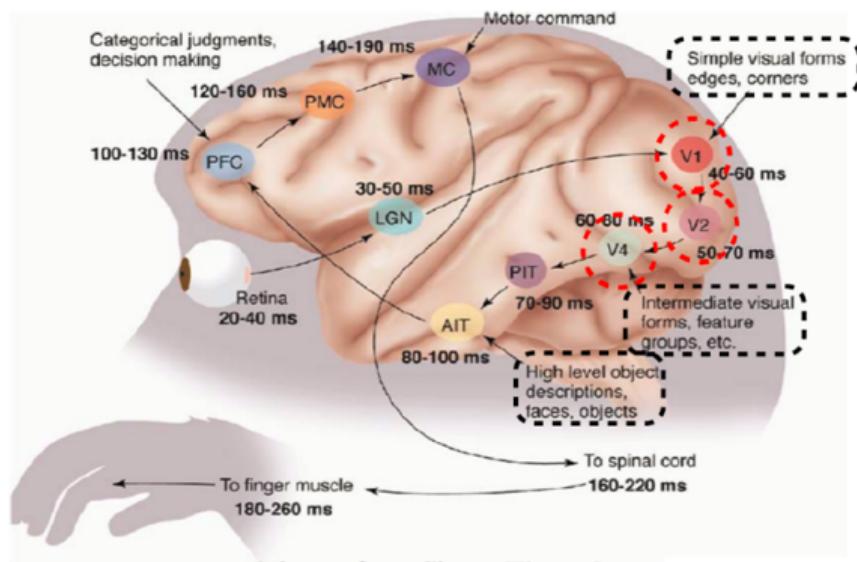
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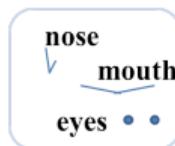
Layer 1: detect edges & corners

Sample illustration of hierarchical processing*

* Idea borrowed from Hugo Larochelle's lecture slides



Layer 1: detect edges & corners

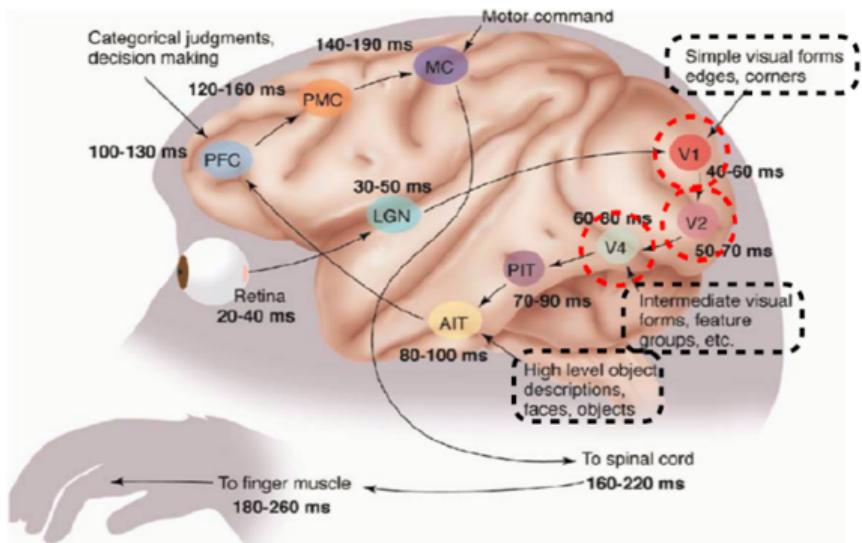


Layer 2: form feature groups

[picture from Simon Thorpe]

Sample illustration of hierarchical processing*

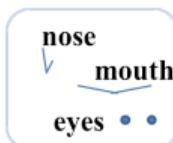
^{*}Idea borrowed from Hugo Larochelle's lecture slides



[picture from Simon Thorpe]



Layer 1: detect edges & corners



Layer 2: form feature groups



Layer 3: detect high level objects, faces, etc.

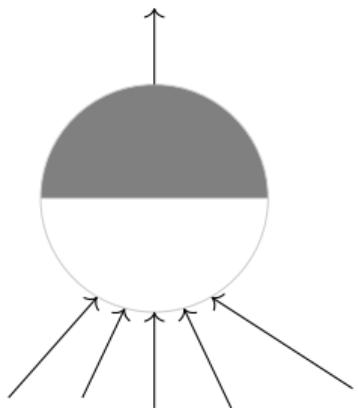
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Disclaimer

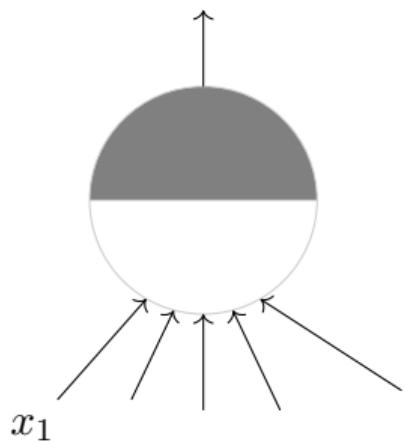
- I understand very little about how the brain works!
- What you saw so far is an overly simplified explanation of how the brain works!
- But this explanation suffices for the purpose of this course!

Module 2.2: McCulloch Pitts Neuron

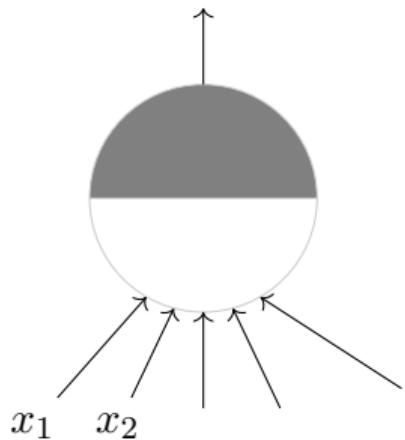


- McCulloch (neuroscientist) and Pitts (logician) proposed a highly simplified computational model of the neuron (1943)

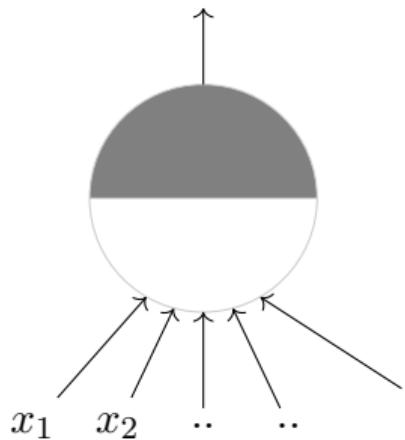
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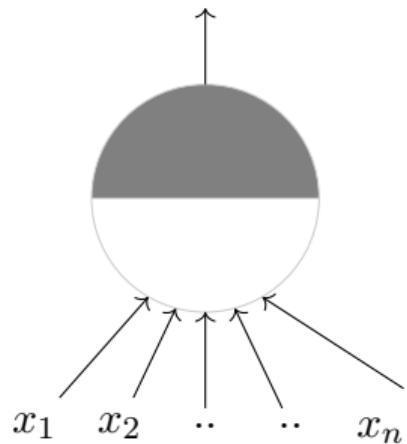
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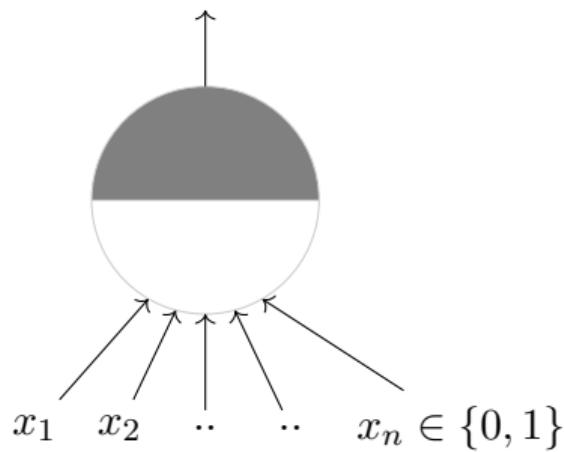
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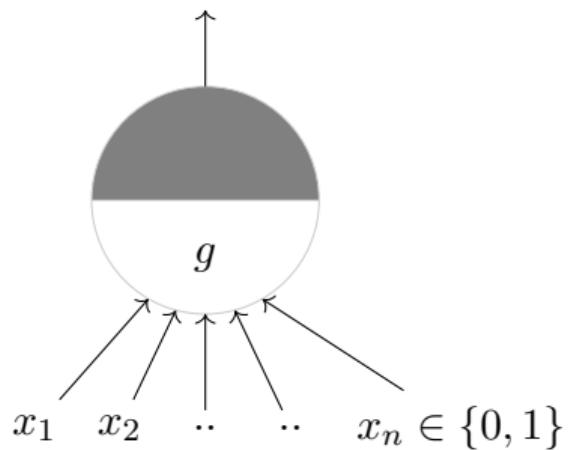
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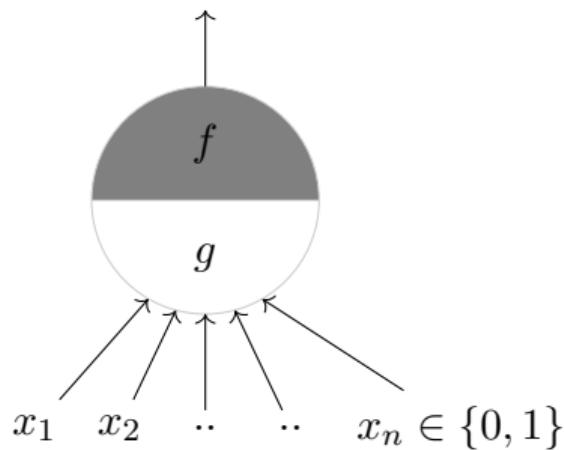


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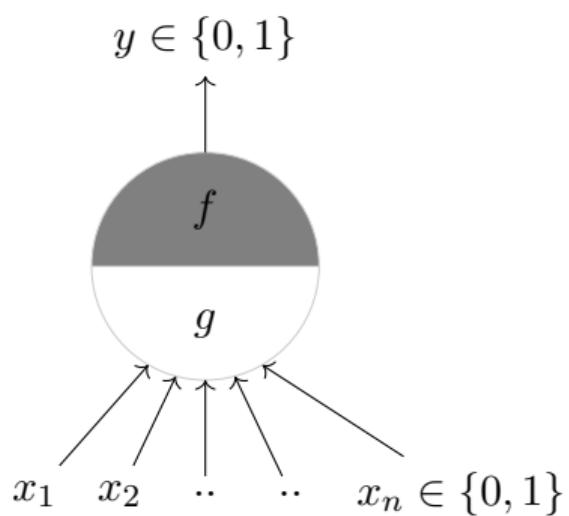


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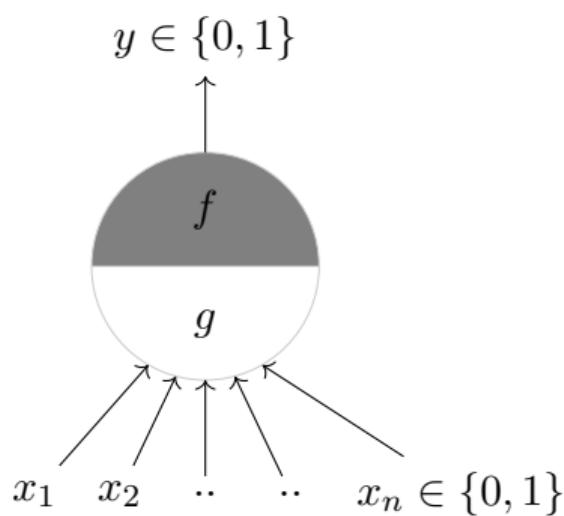




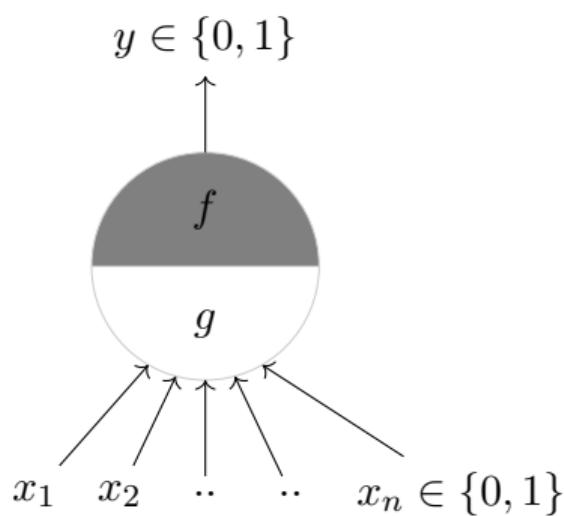
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- g aggregates the inputs and the function f takes a decision based on this aggregation



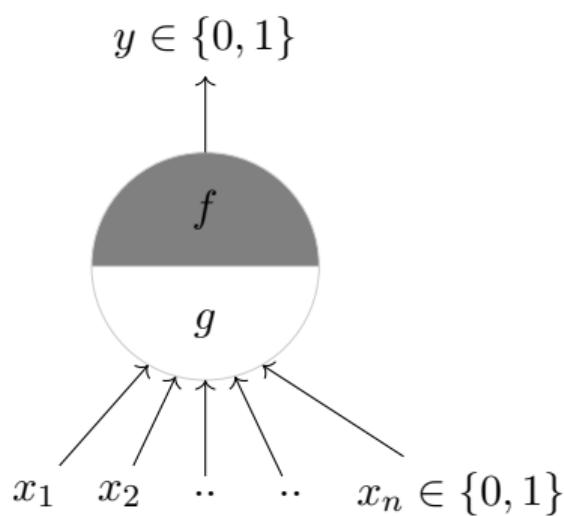
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- The inputs can be excitatory or inhibitory

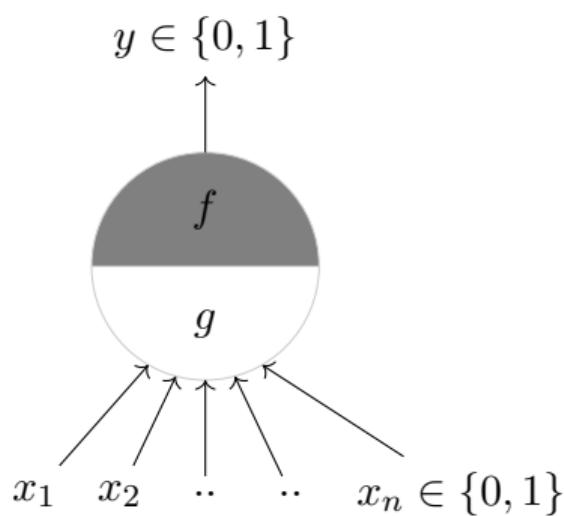


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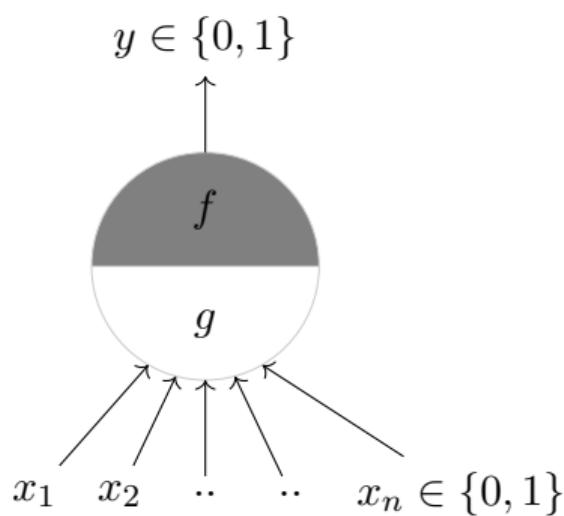
$$g(x_1, x_2, \dots, x_n) = g(\mathbf{x}) = \sum_{i=1}^n x_i$$



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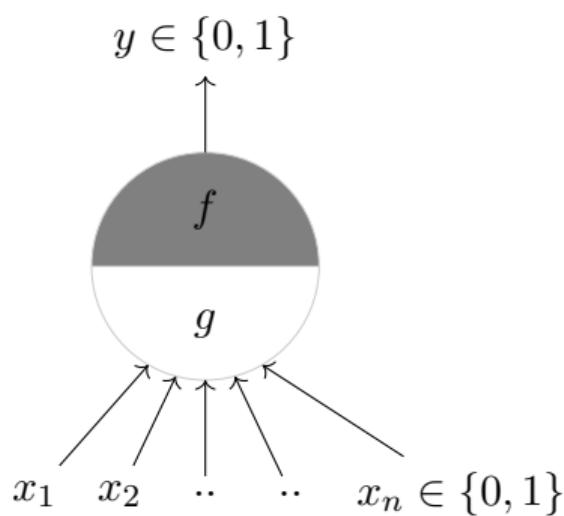
$$y = f(g(\mathbf{x})) = 1 \quad \text{if} \quad g(\mathbf{x}) \geq \theta$$



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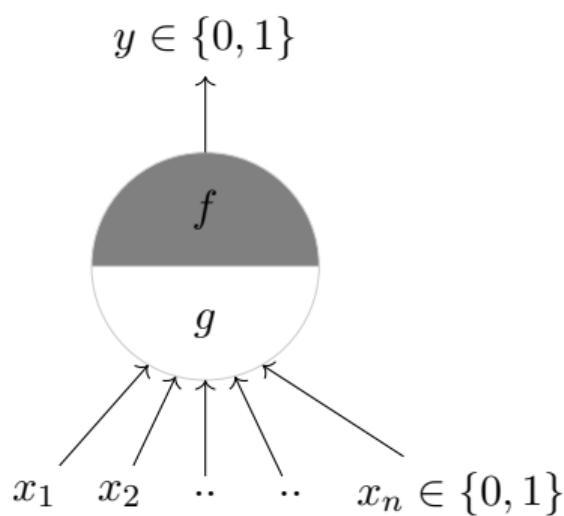


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- θ is called the thresholding parameter



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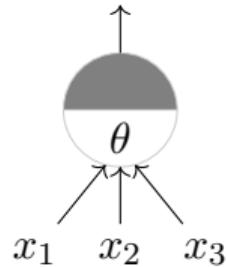
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- θ is called the thresholding parameter
- This is called Thresholding Logic

Let us implement some boolean functions using this McCulloch Pitts (MP) neuron

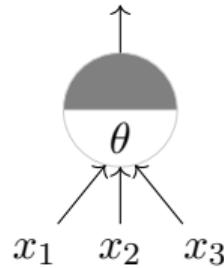
...

$$y \in \{0, 1\}$$



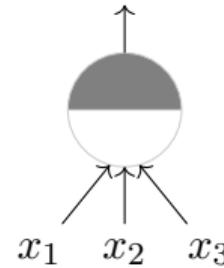
A McCulloch Pitts unit

$$y \in \{0, 1\}$$



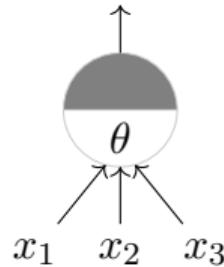
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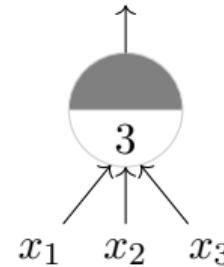
AND function

$$y \in \{0, 1\}$$



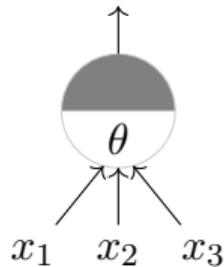
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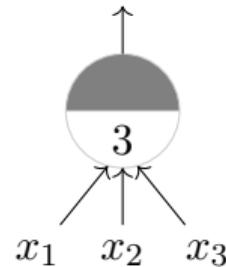
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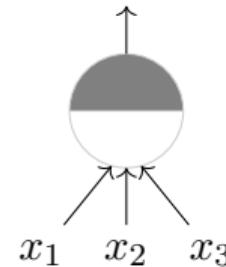
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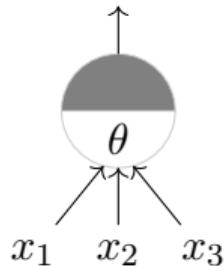
AND function

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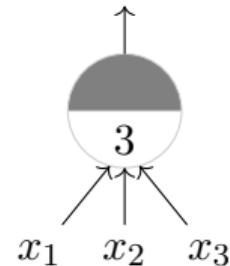
OR function

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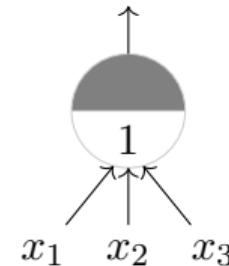
A McCulloch Pitts unit

$$y \in \{0, 1\}$$

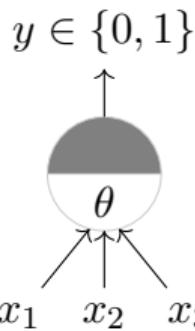


AND function

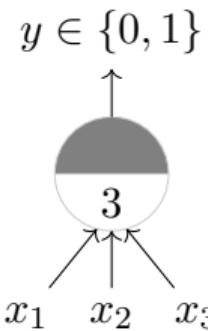
$$y \in \{0, 1\}$$



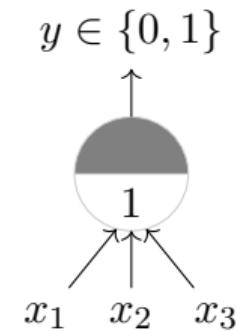
OR function



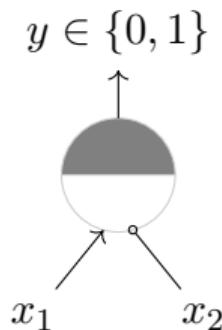
A McCulloch Pitts unit



AND function

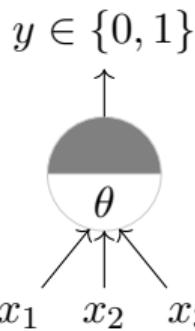


OR function

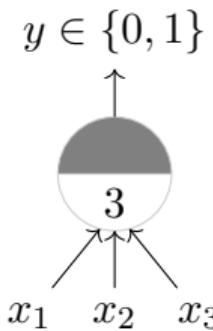


$x_1 \text{ AND } !x_2^*$

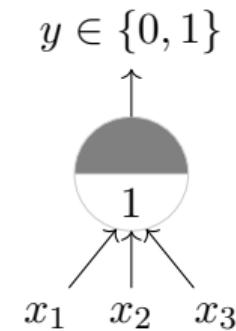
*circle at the end indicates inhibitory input: if any inhibitory input is 1 the output will be 0



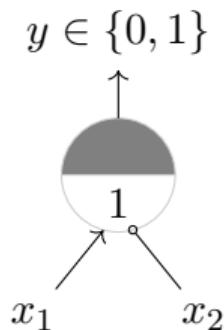
A McCulloch Pitts unit



AND function

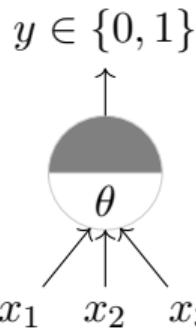


OR function

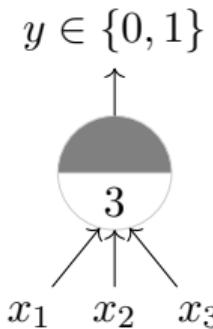


$x_1 \text{ AND } !x_2^*$

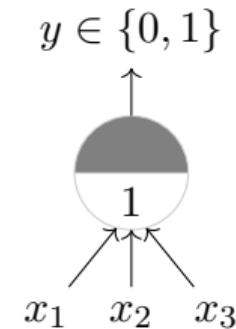
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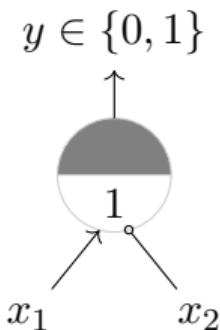
A McCulloch Pitts unit



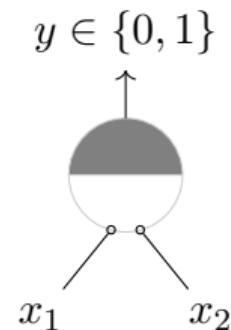
AND function



OR function

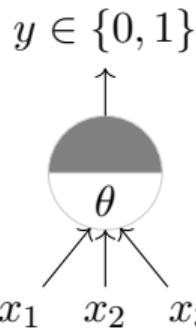


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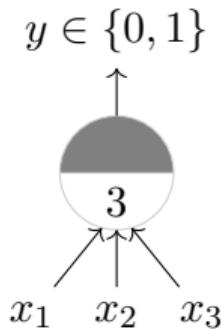


NOR function

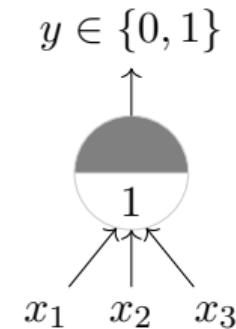
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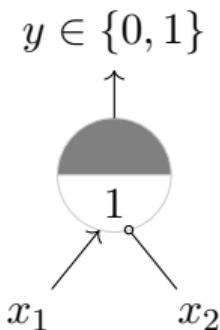
A McCulloch Pitts unit



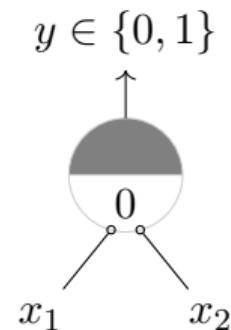
AND function



OR function

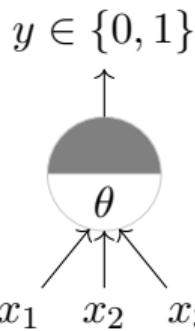


x_1 AND $\neg x_2$ *

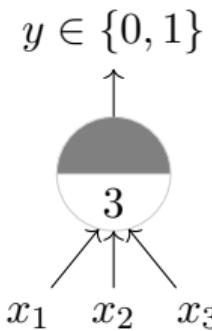


NOR function

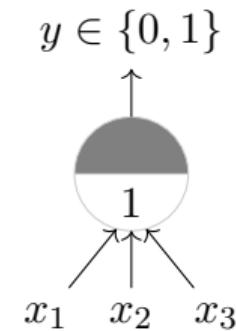
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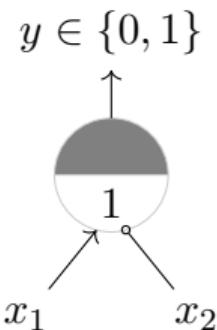
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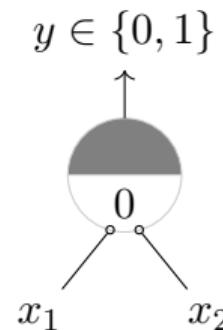
AND function



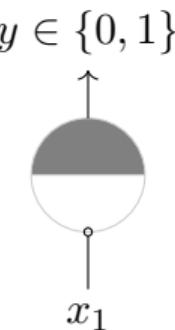
OR function



x_1 AND $\neg x_2$ *

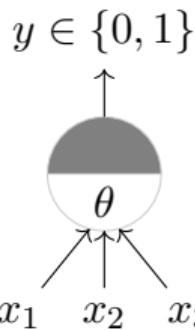


NOR function

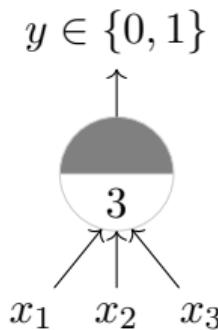


NOT function

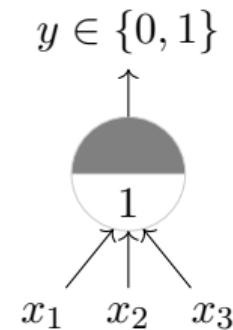
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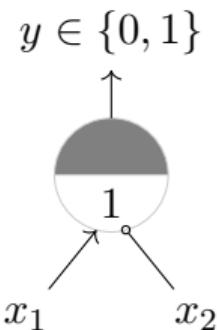
A McCulloch Pitts unit



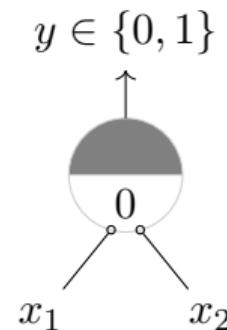
AND function



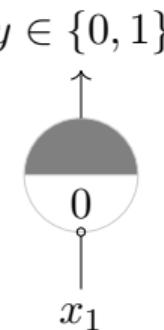
OR function



x_1 AND $\neg x_2^*$



NOR function



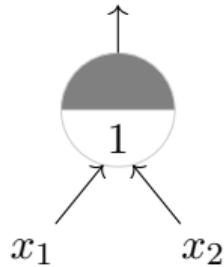
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- Can any boolean function be represented using a McCulloch Pitts unit ?

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- Before answering this question let us first see the geometric interpretation of a MP unit ...

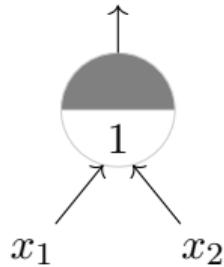
$$y \in \{0, 1\}$$



OR function

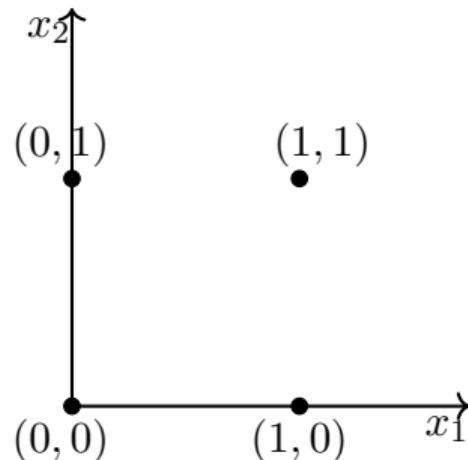
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 1$$

$$y \in \{0, 1\}$$

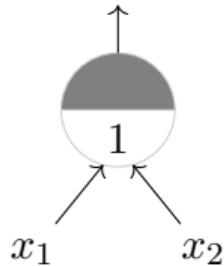


OR function

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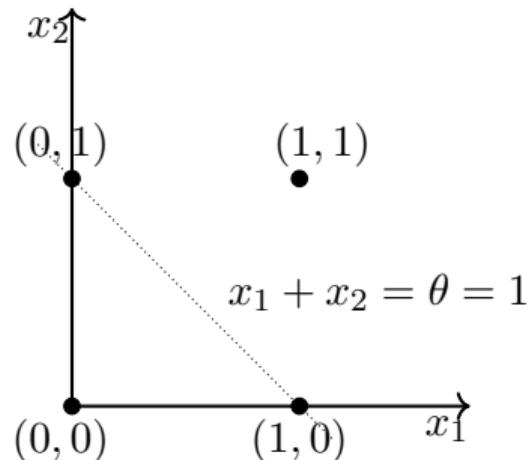


$$y \in \{0, 1\}$$

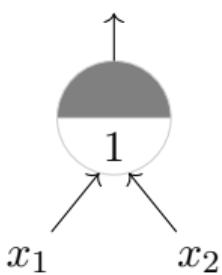


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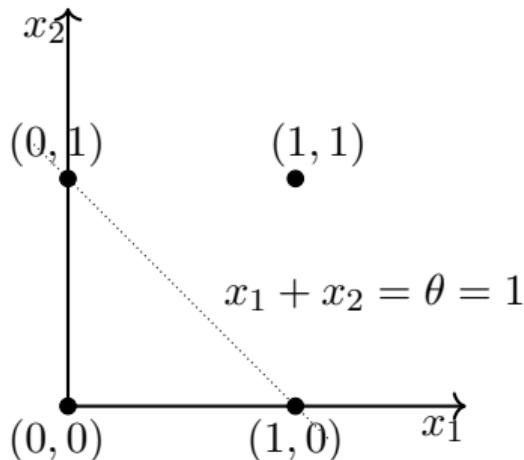
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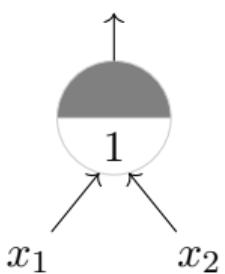
- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves

OR function

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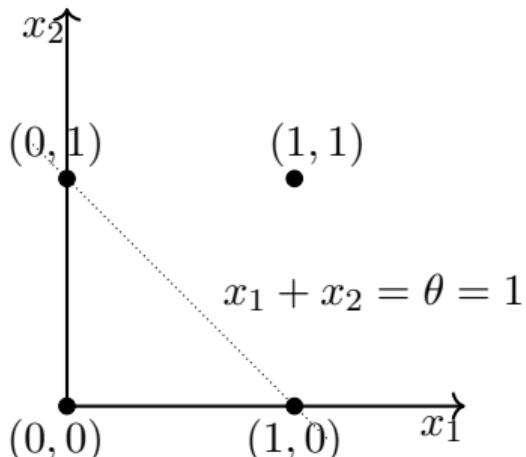
$$y \in \{0, 1\}$$

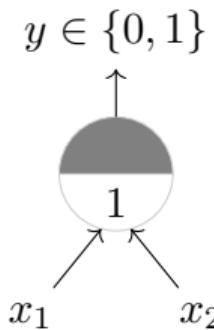


- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves
- Points lying on or above the line $\sum_{i=1}^n x_i - \theta = 0$ and points lying below this line

OR function

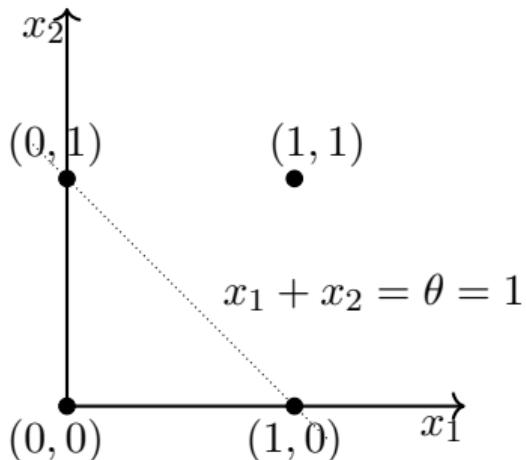
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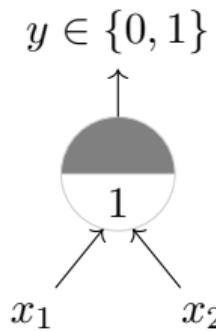


OR function

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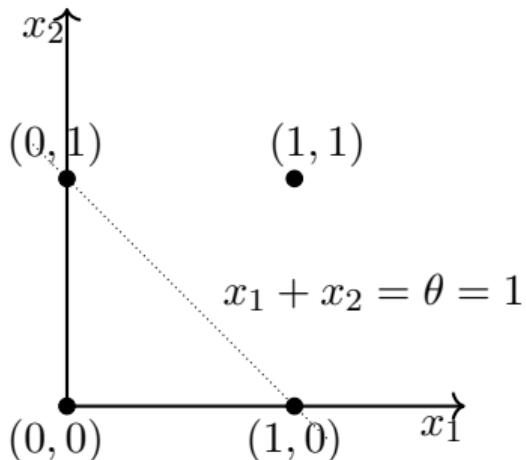


- A single MP neuron splits the input points (4 points for 2 binary inputs) into two halves
- Points lying on or above the line $\sum_{i=1}^n x_i - \theta = 0$ and points lying below this line
- In other words, all inputs which produce an output 0 will be on one side ($\sum_{i=1}^n x_i < \theta$) of the line and all inputs which produce an output 1 will lie on the other side ($\sum_{i=1}^n x_i \geq \theta$) of this line



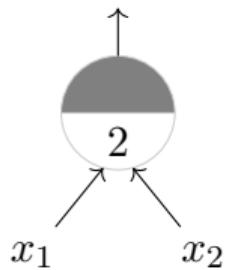
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- Let us convince ourselves about this with a few more examples (if it is not already clear from the math)

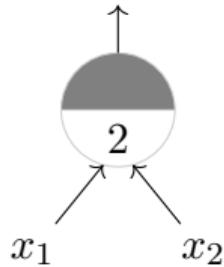
$$y \in \{0, 1\}$$



AND function

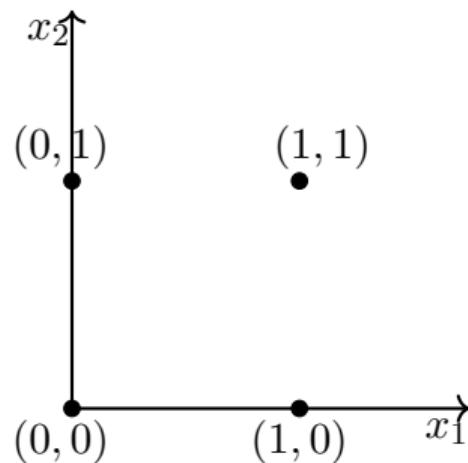
$$x_1 + x_2 = \sum_{i=1}^2 x_i \geq 2$$

$$y \in \{0, 1\}$$

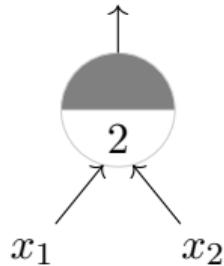


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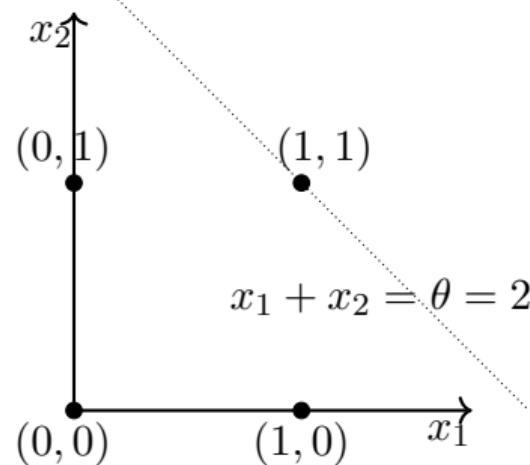


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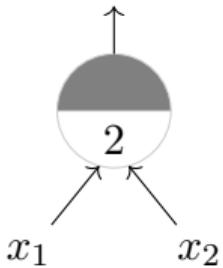


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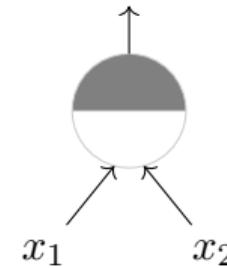
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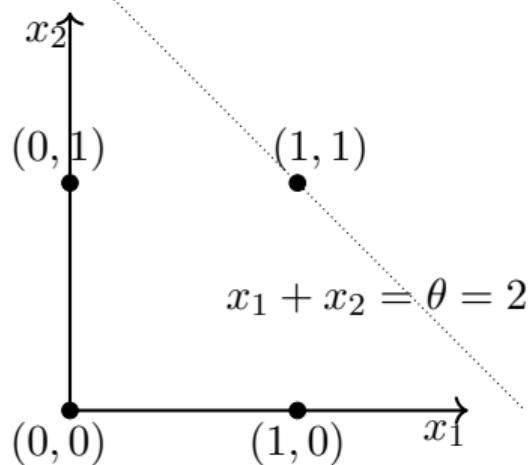
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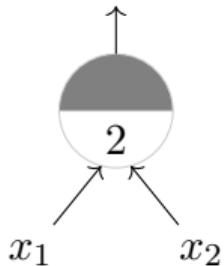
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Tautology (always ON)



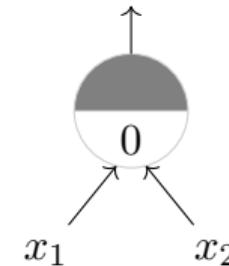
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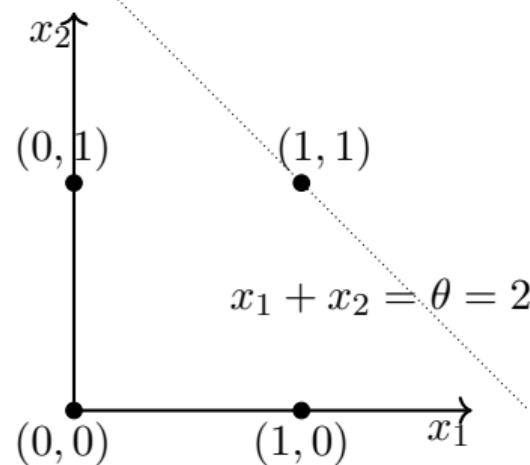
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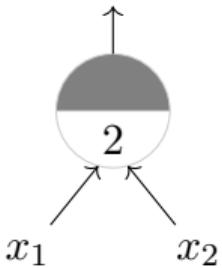
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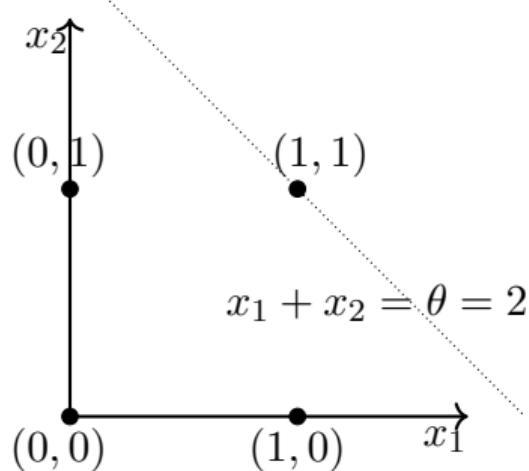


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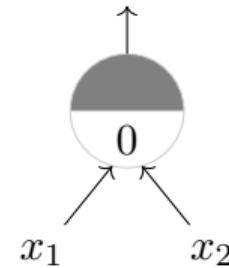


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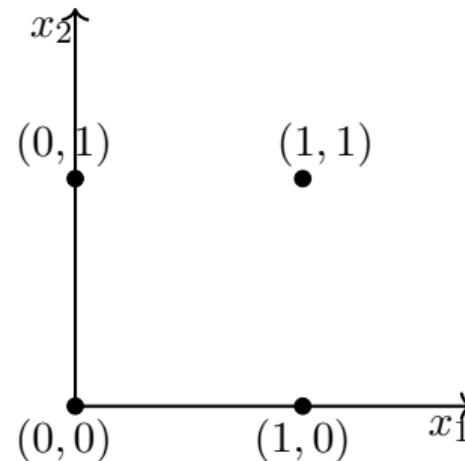
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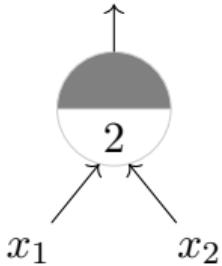
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Tautology (always ON)

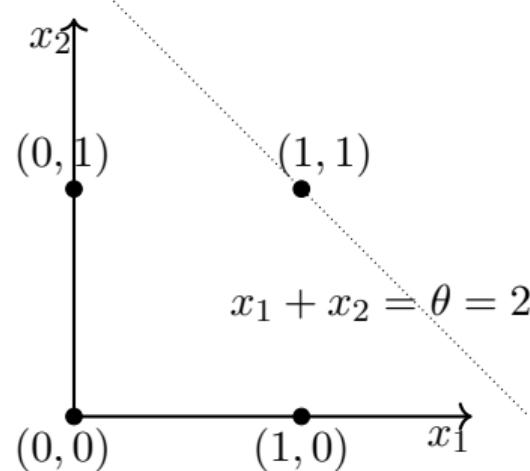


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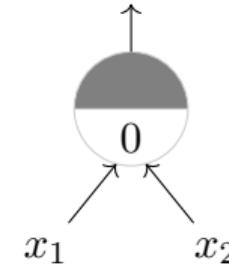


AND function

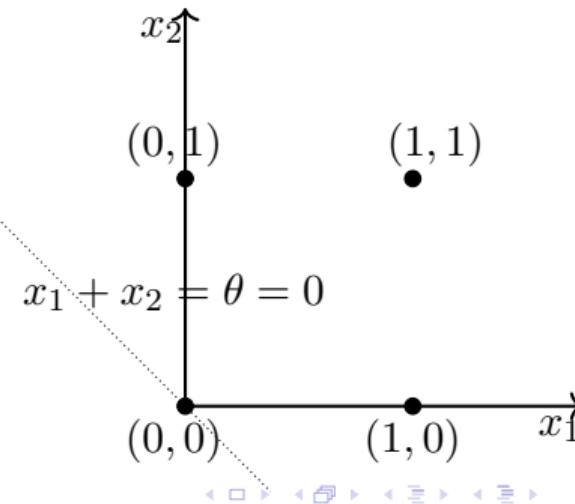
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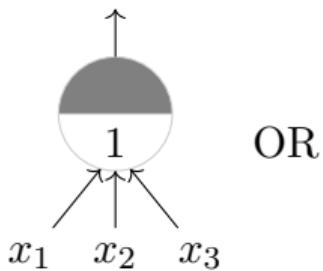
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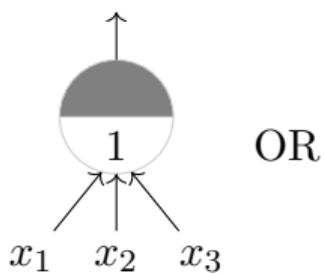


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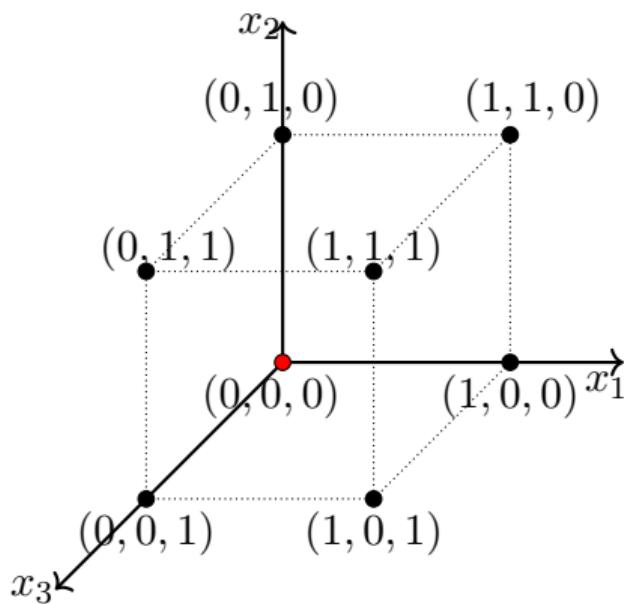


- What if we have more than 2 inputs?

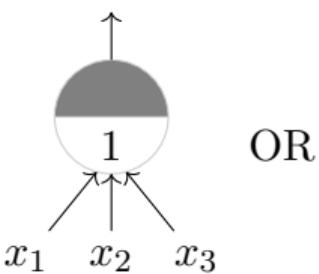
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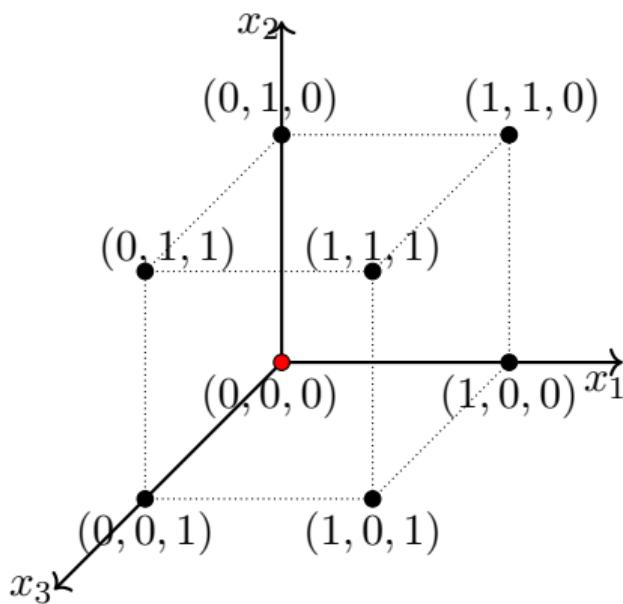


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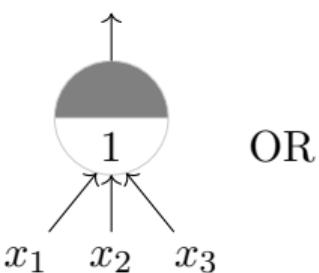


OR

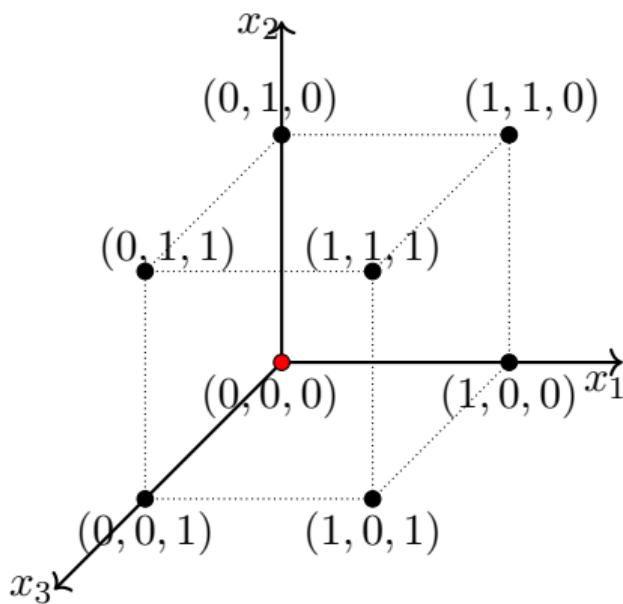
- What if we have more than 2 inputs?
- Well, instead of a line we will have a plane



$$y \in \{0, 1\}$$

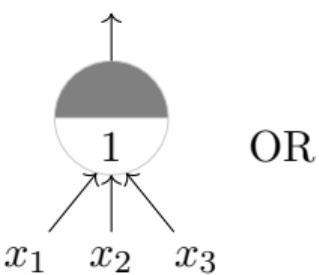


OR



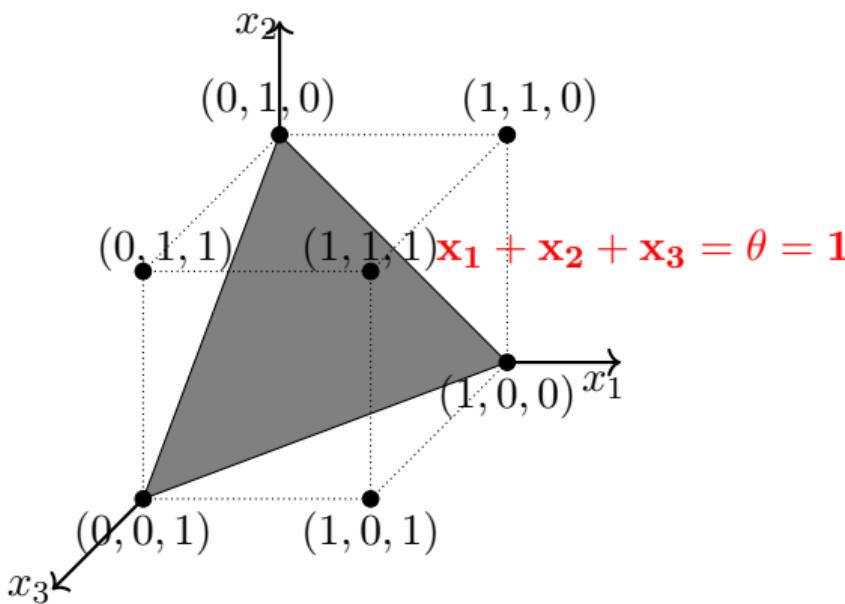
- What if we have more than 2 inputs?
- Well, instead of a line we will have a plane
- For the OR function, we want a plane such that the point $(0,0,0)$ lies on one side and the remaining 7 points lie on the other side of the plane

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The story so far ...

- A single McCulloch Pitts Neuron can be used to represent boolean functions which are linearly separable

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- A single McCulloch Pitts Neuron can be used to represent boolean functions which are linearly separable
- Linear separability (for boolean functions) : There exists a line (plane) such that all inputs which produce a 1 lie on one side of the line (plane) and all inputs which produce a 0 lie on other side of the line (plane)

Module 2.3: Perceptron

The story ahead ...

- What about non-boolean (say, real) inputs ?

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- What about non-boolean (say, real) inputs ?
- Do we always need to hand code the threshold ?

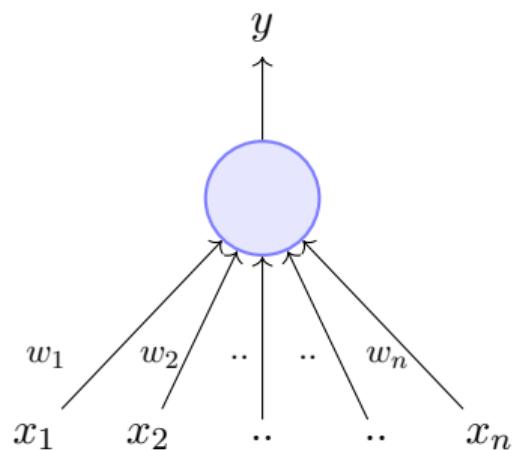
The story ahead ...

- What about non-boolean (say, real) inputs ?
- Do we always need to hand code the threshold ?
- Are all inputs equal ? What if we want to assign more weight (importance) to some inputs ?

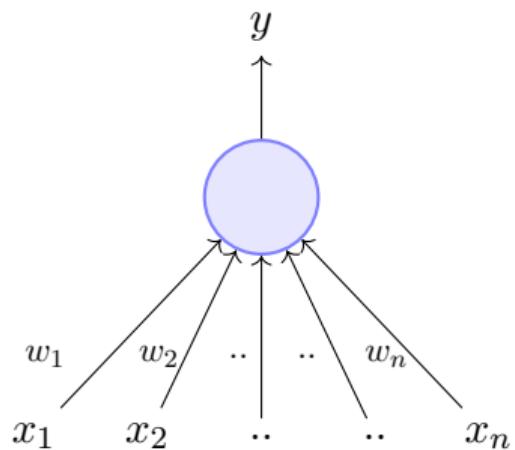
The story ahead ...

- What about non-boolean (say, real) inputs ?
- Do we always need to hand code the threshold ?
- Are all inputs equal ? What if we want to assign more weight (importance) to some inputs ?
- What about functions which are not linearly separable ?

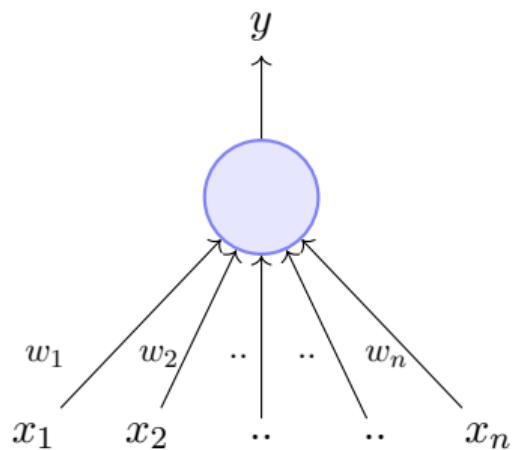
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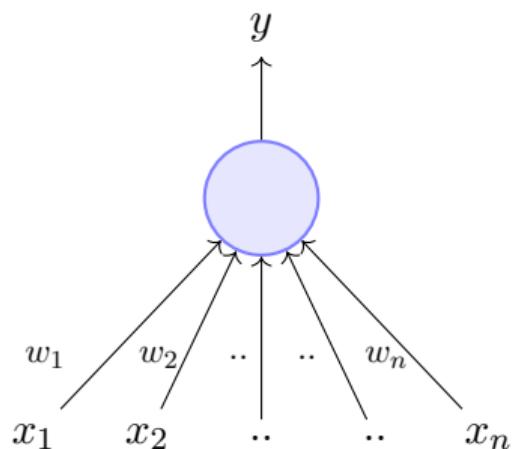
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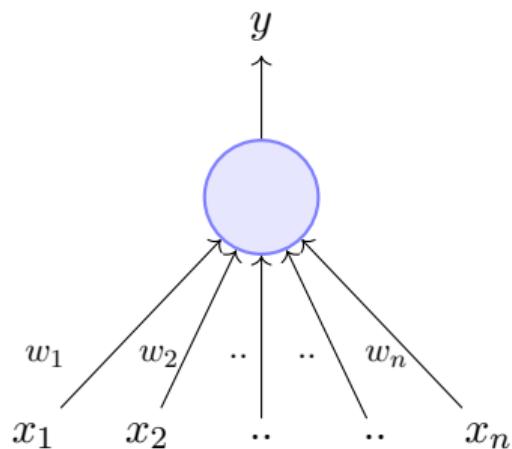
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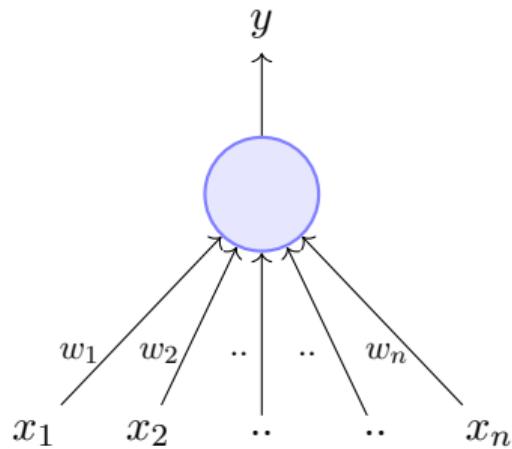
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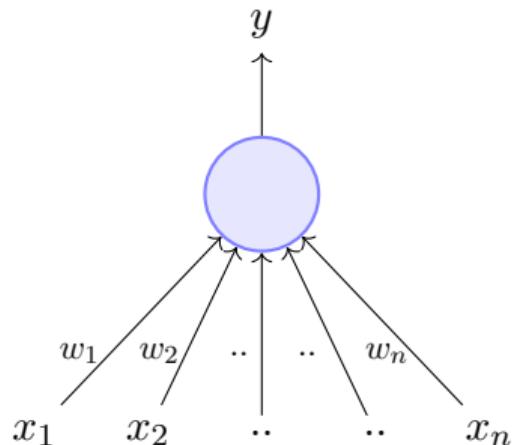


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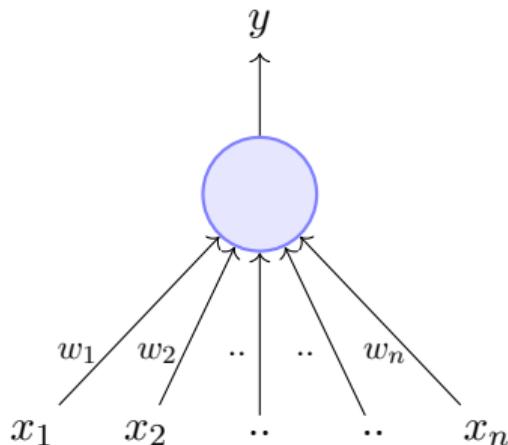


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- Refined and carefully analyzed by Minsky and Papert (1969) - their model is referred to as the **perceptron** model here



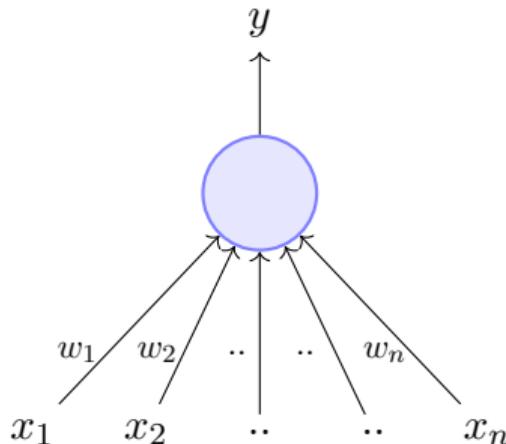


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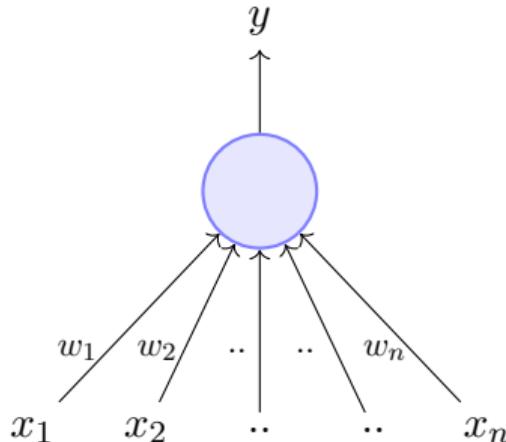
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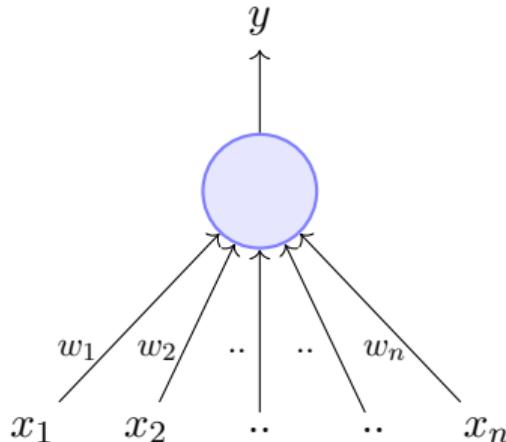


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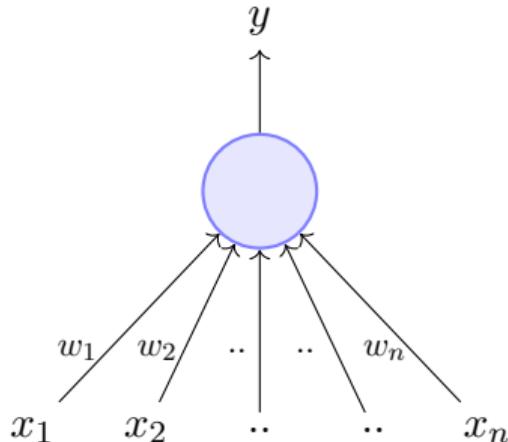
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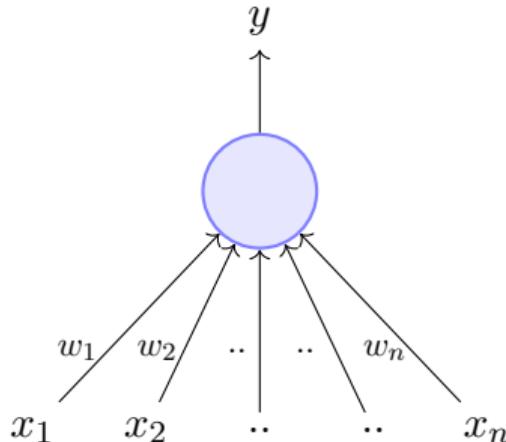
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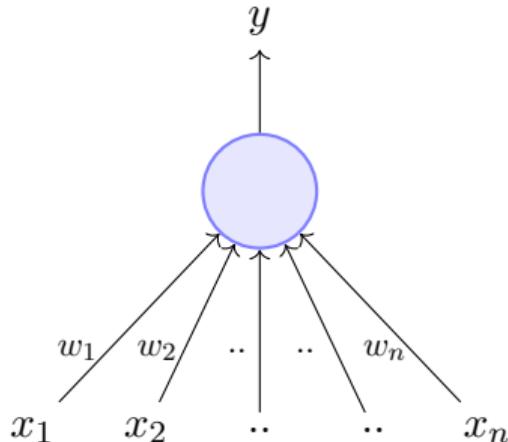
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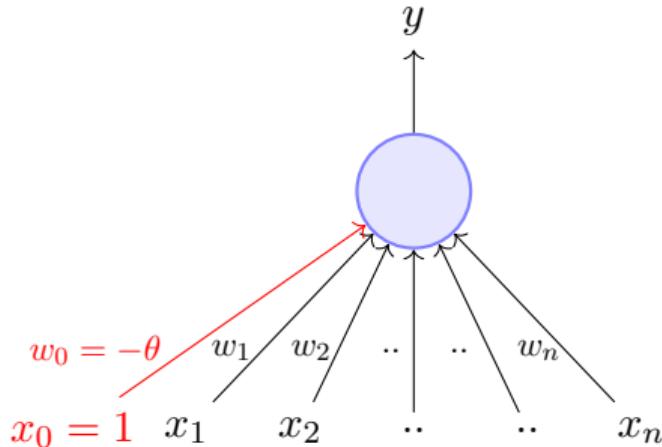
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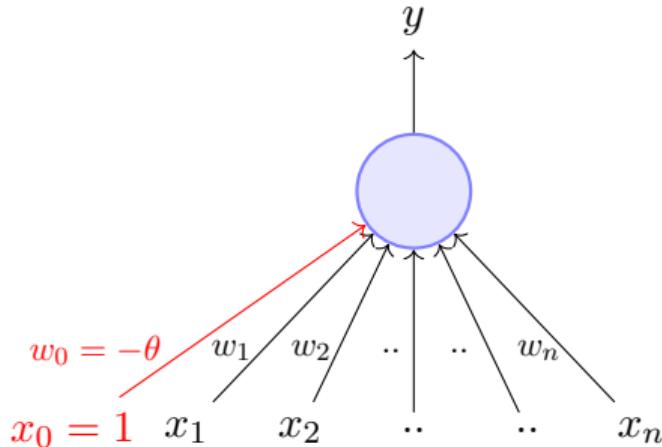
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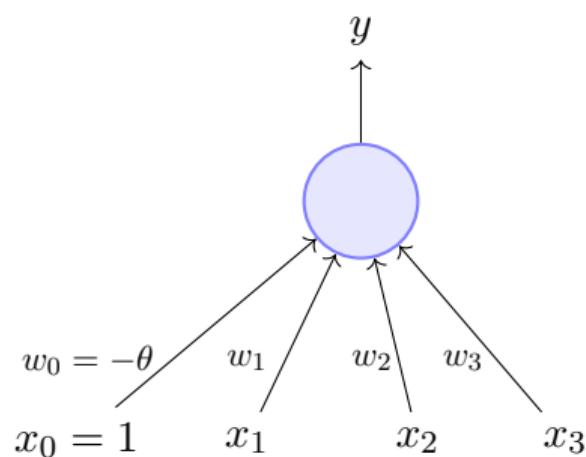
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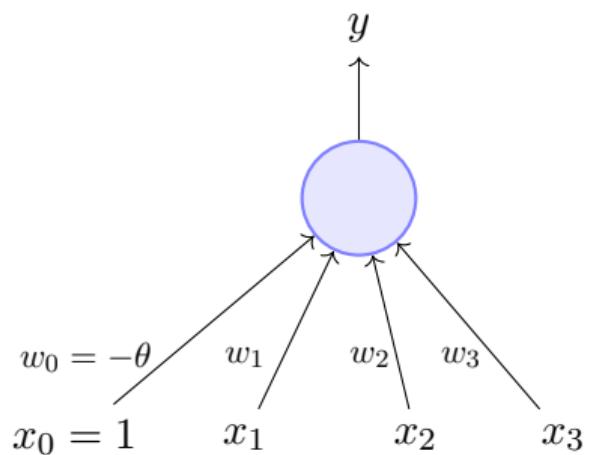
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We will now try to answer the following questions:

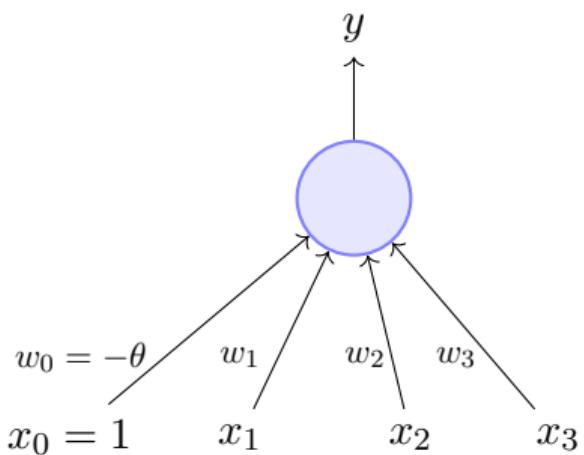
- Why are we trying to implement boolean functions?
- Why do we need weights ?
- Why is $w_0 = -\theta$ called the bias ?

- Consider the task of predicting whether we would like a movie or not





- Consider the task of predicting whether we would like a movie or not
- Suppose, we base our decision on 3 inputs (binary, for simplicity)

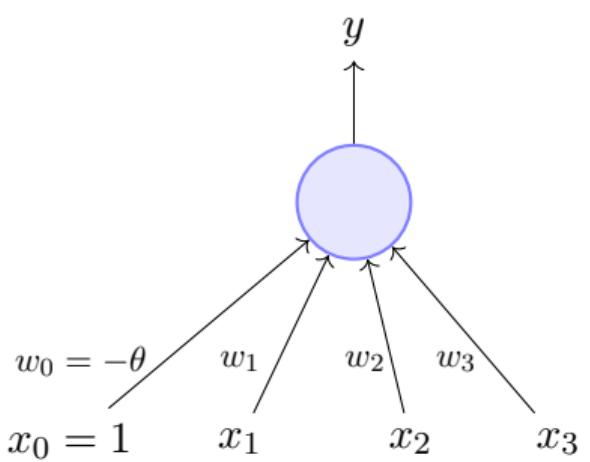


- Consider the task of predicting whether we would like a movie or not
- Suppose, we base our decision on 3 inputs (binary, for simplicity)
- Based on our past viewing experience (**data**), we may give a high weight to *isDirectorNolan* as compared to the other inputs

$x_1 = \text{isActorDamon}$

$x_2 = \text{isGenreThriller}$

$x_3 = \text{isDirectorNolan}$

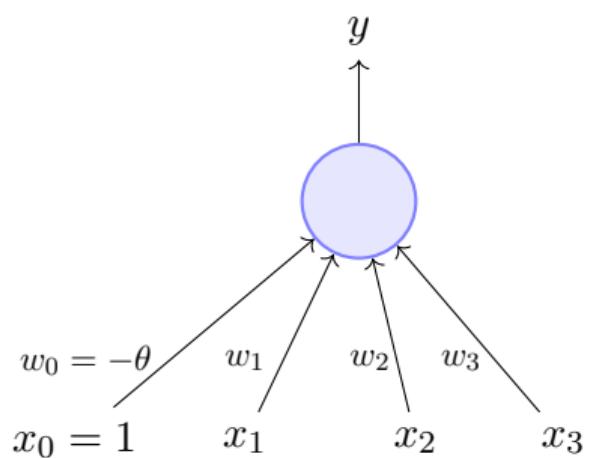


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- Suppose, we base our decision on 3 inputs (binary, for simplicity)
- Based on our past viewing experience (**data**), we may give a high weight to *isDirectorNolan* as compared to the other inputs
- Specifically, even if the actor is not *Matt Damon* and the genre is not *thriller* we would still want to cross the threshold θ by assigning a high weight to *isDirectorNolan*

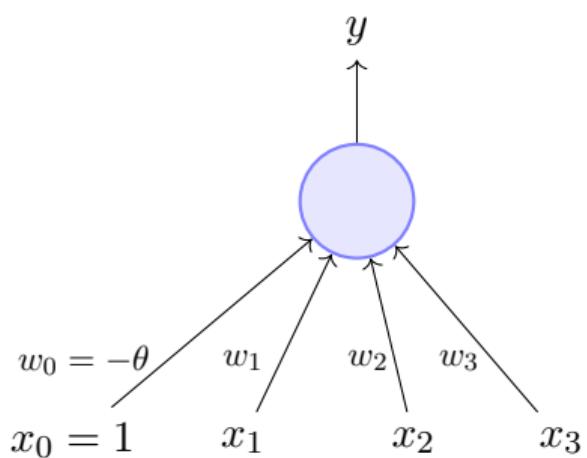


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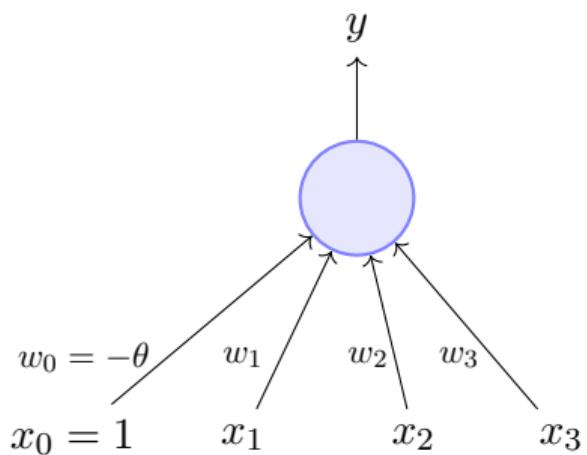


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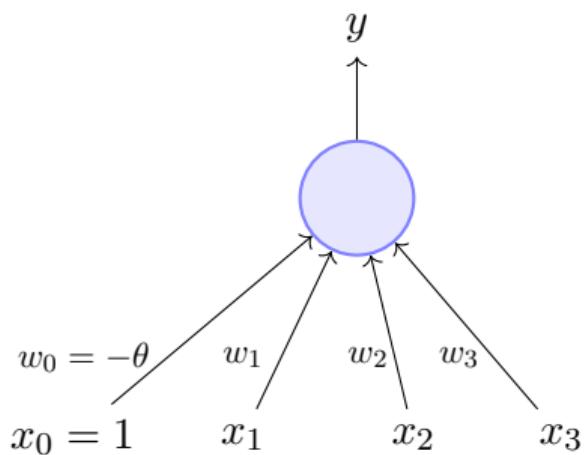


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- The weights (w_1, w_2, \dots, w_n) and the bias (w_0) will depend on the data (viewer history in this case)

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What kind of functions can be implemented using the perceptron? Any difference from McCulloch Pitts neurons?

McCulloch Pitts Neuron

(assuming no inhibitory inputs)

$$y = 1 \quad if \sum_{i=0}^n x_i \geq 0$$

$$= 0 \quad if \sum_{i=0}^n x_i < 0$$

Perceptron

$$y = 1 \quad if \sum_{i=0}^n \textcolor{red}{w_i} * x_i \geq 0$$

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- In other words, a single perceptron can only be used to implement linearly separable functions
- Then what is the difference? The weights (including threshold) can be learned and the inputs can be real valued
- We will first revisit some boolean functions and then see the perceptron learning algorithm (for learning weights)

x_1	x_2	OR
0	0	

x_1	x_2	OR
0	0	0

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i$

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
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x_1	x_2	OR	
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- One possible solution to this set of inequalities is $w_0 = -1, w_1 = 1.1, w_2 = 1.1$ (and various other solutions are possible)

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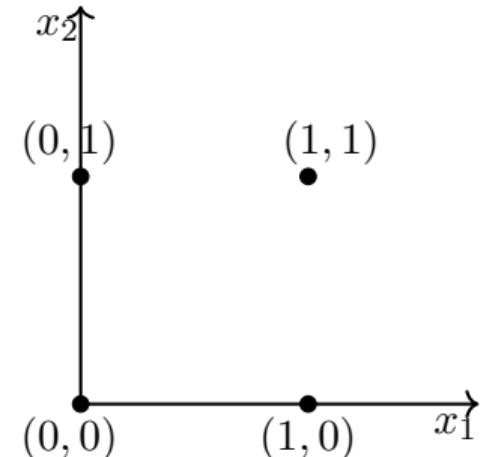
$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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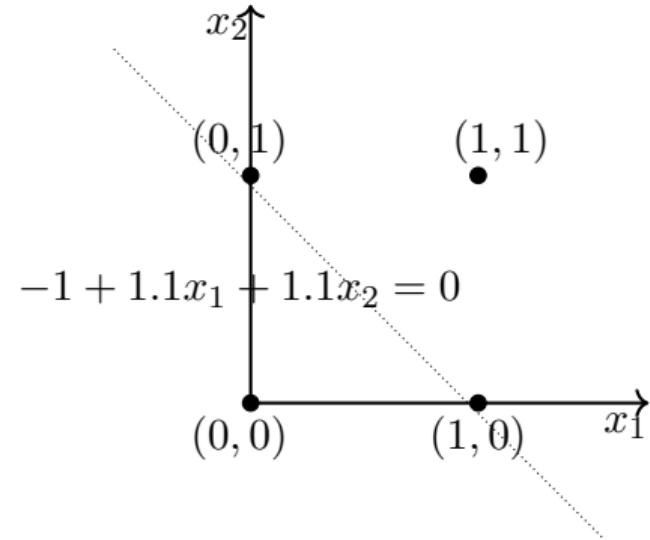
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1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
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1	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$



$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

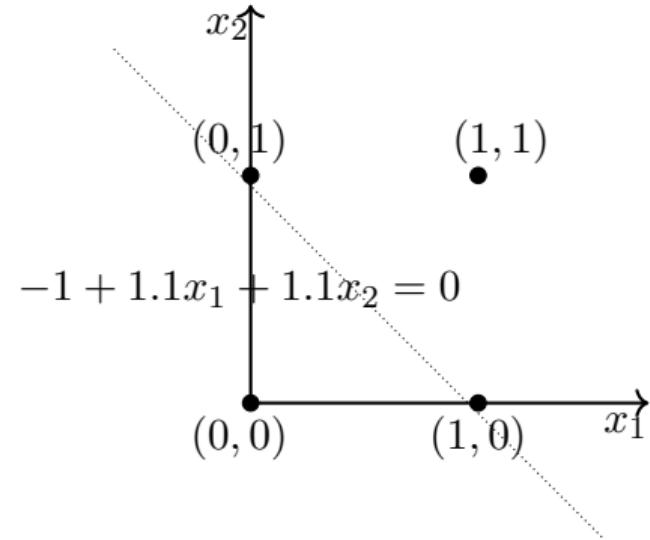
$$w_0 + w_1 \cdot 0 + w_2 \cdot 1 \geq 0 \implies w_2 \geq -w_0$$

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$$w_0 + w_1 \cdot 1 + w_2 \cdot 1 \geq 0 \implies w_1 + w_2 \geq -w_0$$

- One possible solution to this set of inequalities is $w_0 = -1, w_1 = 1.1, w_2 = 1.1$ (and various other solutions are possible)

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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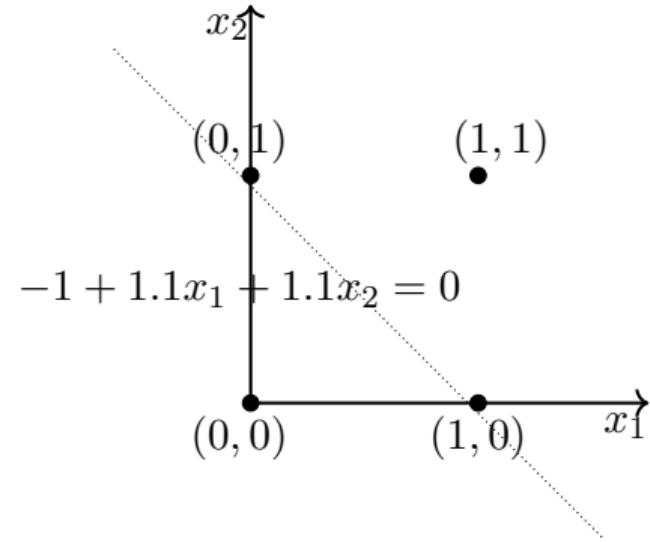
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- Note that we can come up with a similar set of inequalities and find the value of θ for a McCulloch Pitts neuron also

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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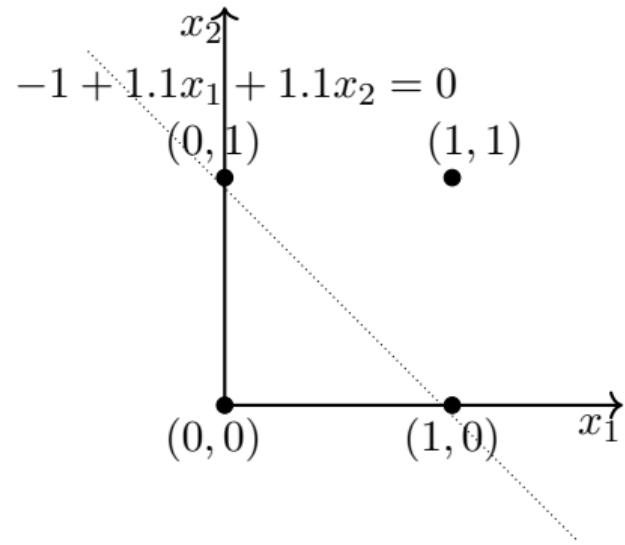
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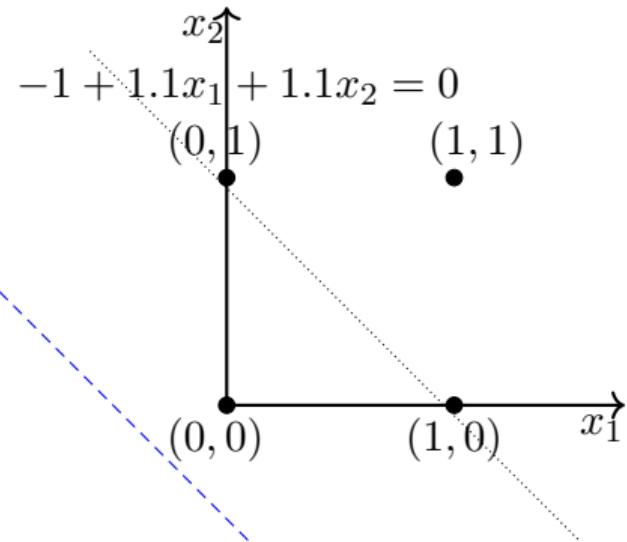
- Note that we can come up with a similar set of inequalities and find the value of θ for a McCulloch Pitts neuron also (Try it!)

Module 2.4: Errors and Error Surfaces

- Let us fix the threshold ($-w_0 = 1$) and try different values of w_1, w_2

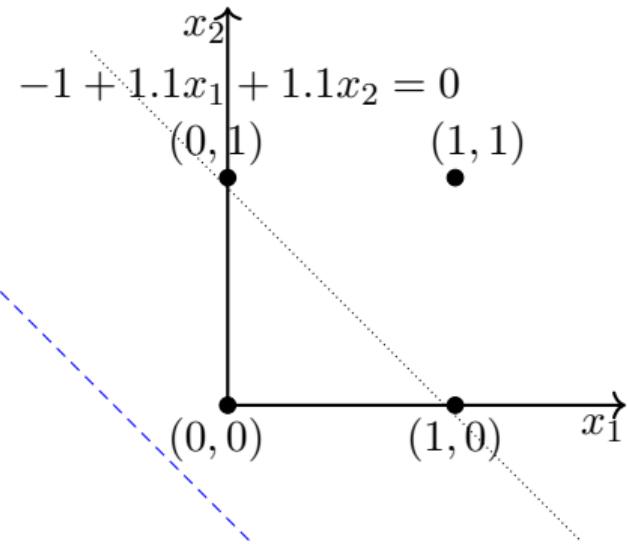


- Let us fix the threshold ($-w_0 = 1$) and try different values of w_1, w_2
- Say, $w_1 = -1, w_2 = -1$



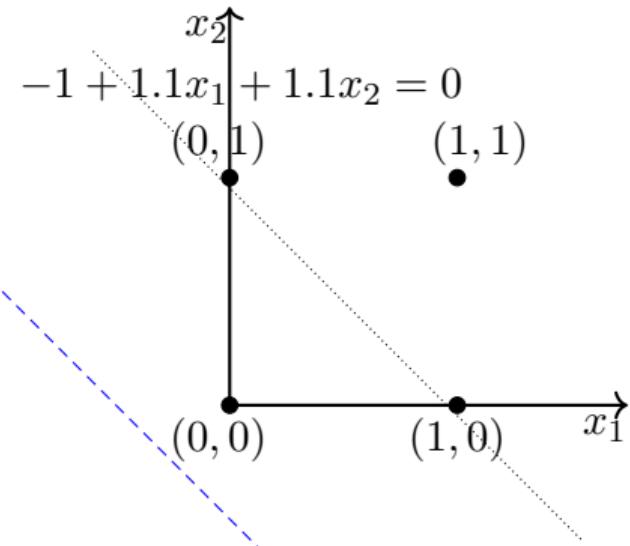
$$-1 + (-1)x_1 + (-1)x_2 = 0$$

- Let us fix the threshold ($-w_0 = 1$) and try different values of w_1, w_2
- Say, $w_1 = -1, w_2 = -1$
- What is wrong with this line?



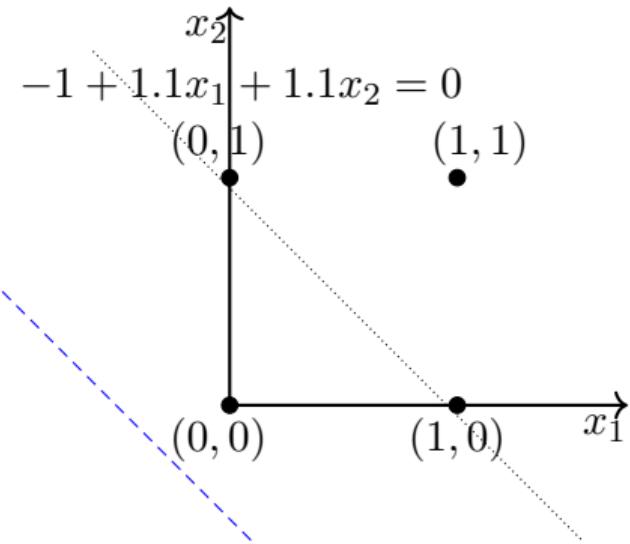
$$-1 + (-1)x_1 + (-1)x_2 = 0$$

- Let us fix the threshold ($-w_0 = 1$) and try different values of w_1, w_2
- Say, $w_1 = -1, w_2 = -1$
- What is wrong with this line? We make an error on 1 out of the 4 inputs



$$-1 + (-1)x_1 + (-1)x_2 = 0$$

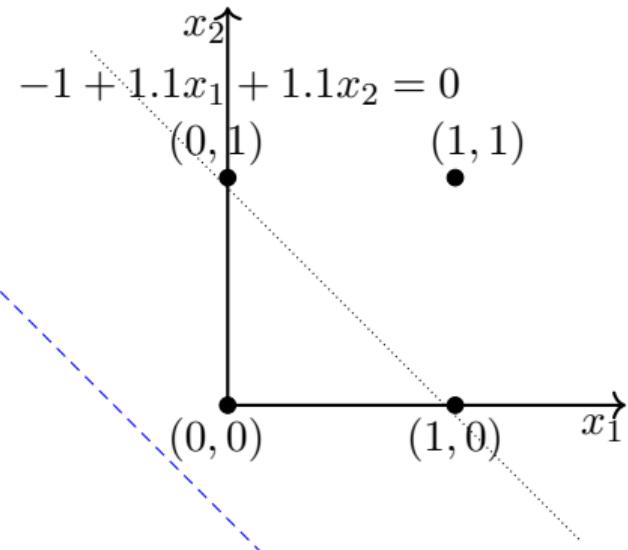
- Let us fix the threshold ($-w_0 = 1$) and try different values of w_1, w_2
- Say, $w_1 = -1, w_2 = -1$
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- Lets try some more values of w_1, w_2 and note how many errors we make



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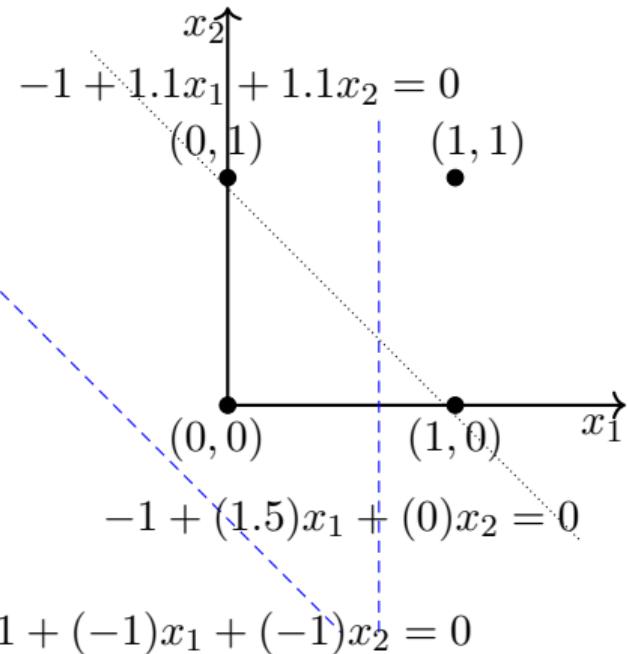
w_1	w_2	errors
-1	-1	3



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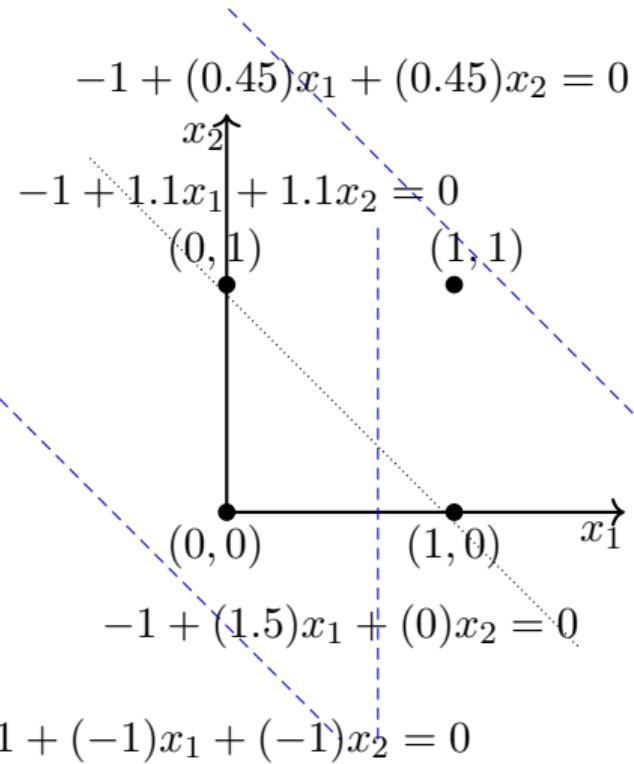
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w_1	w_2	errors
-1	-1	3
1.5	0	1



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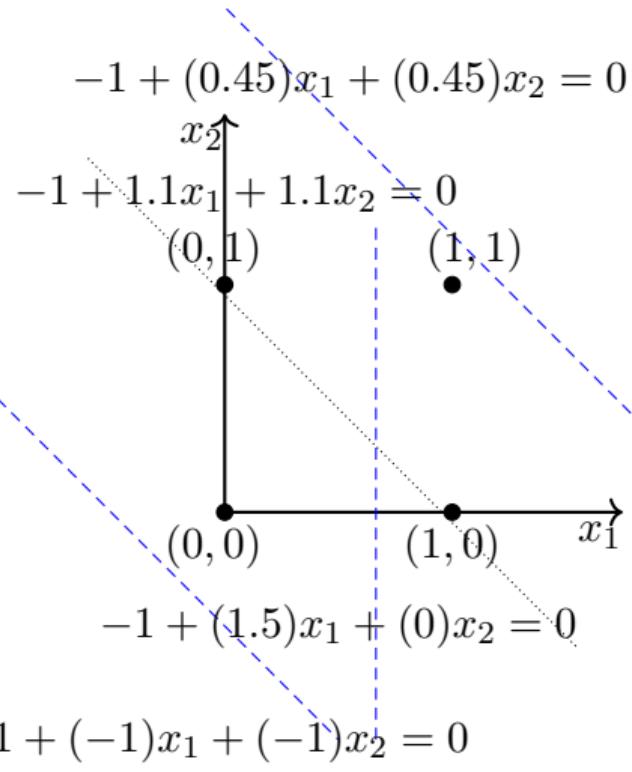
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0.45	0.45	3



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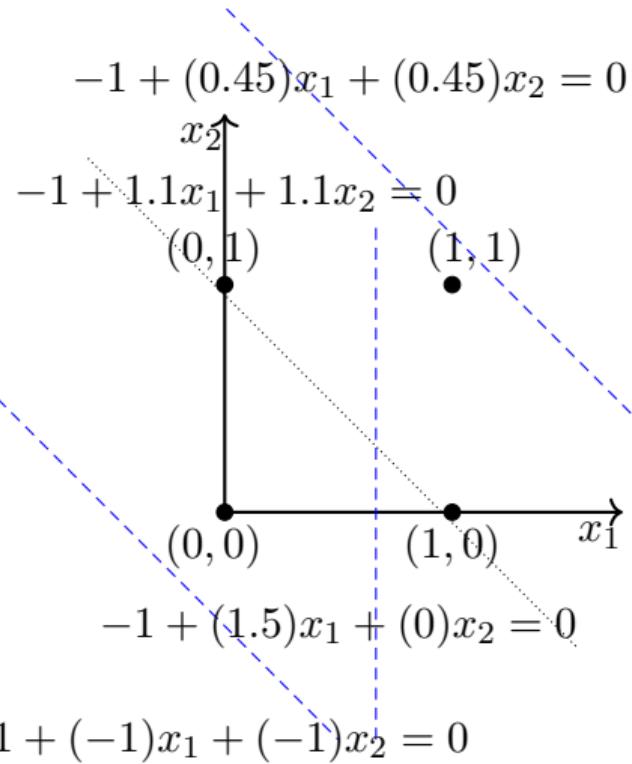
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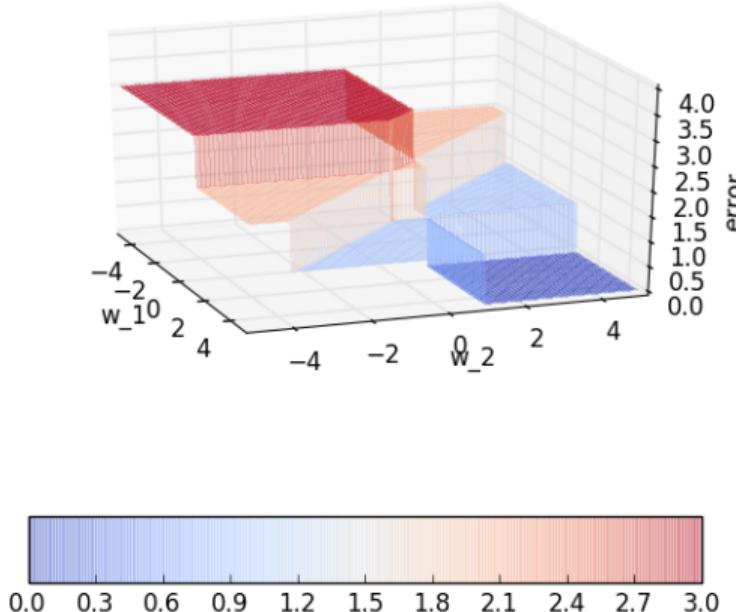
- We are interested in those values of w_0, w_1, w_2 which result in 0 error
- Let us plot the error surface corresponding to different values of w_0, w_1, w_2



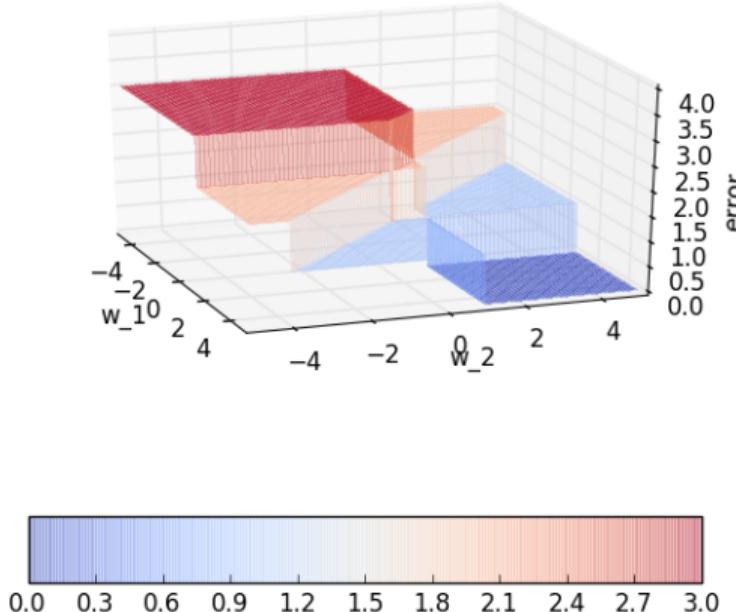
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- For the OR function, an error occurs if $(x_1, x_2) = (0, 0)$ but $-w_0 + w_1 * x_1 + w_2 * x_2 \geq 0$ or if $(x_1, x_2) \neq (0, 0)$ but $-w_0 + w_1 * x_1 + w_2 * x_2 < 0$



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- We are interested in finding an algorithm which finds the values of w_1, w_2 which minimize this error

Module 2.5: Perceptron Learning Algorithm

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- Apart from implementing boolean functions (which does not look very interesting) what can a perceptron be used for ?
- Our interest lies in the use of perceptron as a binary classifier. Let us see what this means...

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$x_1 = \text{isActorDamon}$

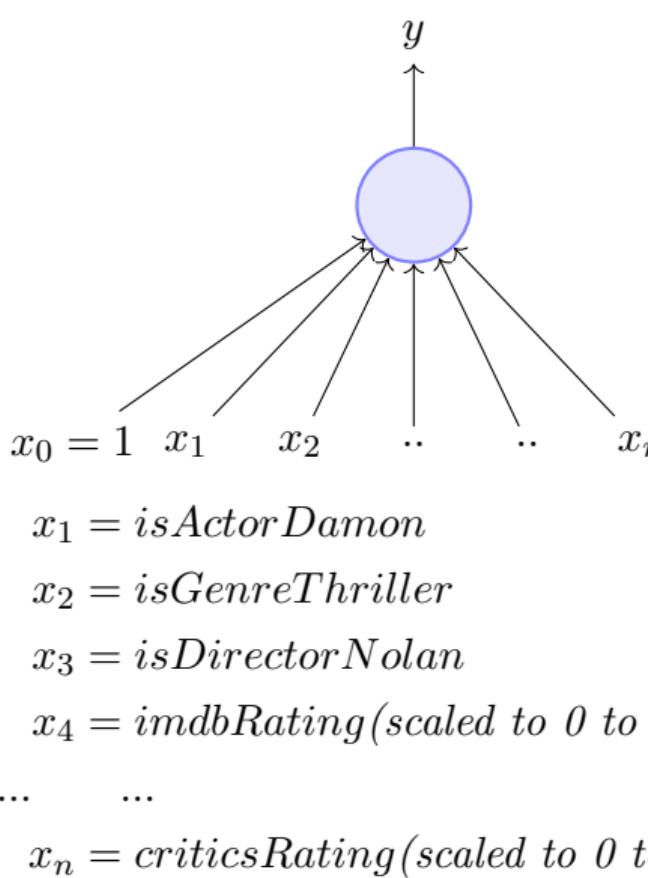
$x_2 = \text{isGenreThriller}$

$x_3 = \text{isDirectorNolan}$

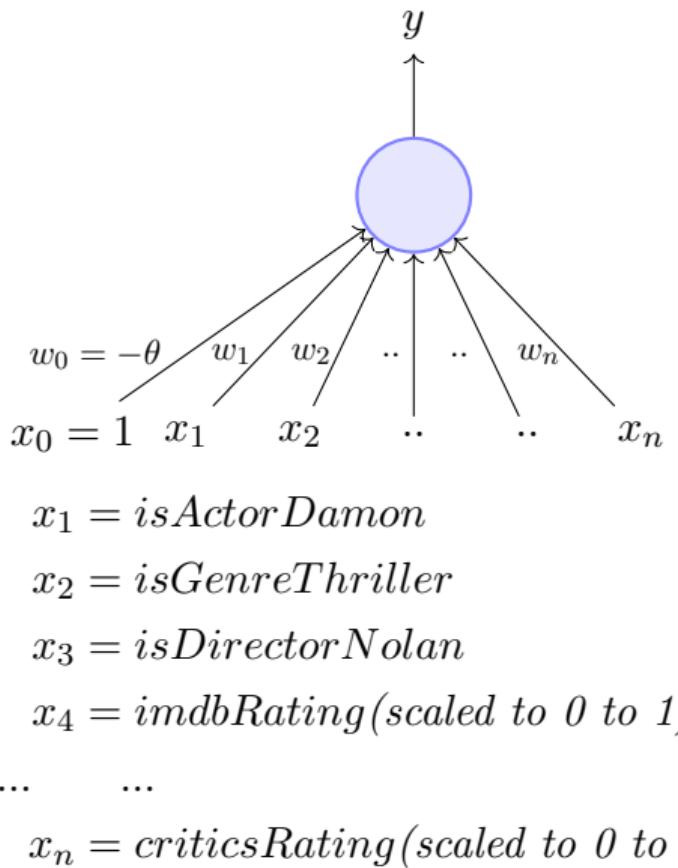
$x_4 = \text{imdbRating}(\text{scaled to 0 to 1})$

... ...

$x_n = \text{criticsRating}(\text{scaled to 0 to 1})$



- Let us reconsider our problem of deciding whether to watch a movie or not
- Suppose we are given a list of m movies and a label (class) associated with each movie indicating whether the user liked this movie or not : binary decision
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- We will assume that the data is linearly separable and we want a perceptron to learn how to make this decision
- In other words, we want the perceptron to find the equation of this separating plane (or find the values of $w_0, w_1, w_2, \dots, w_m$)

Algorithm: Perceptron Learning Algorithm

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|

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Pick random $\mathbf{x} \in P \cup N$;

if $\mathbf{x} \in P$ and $\sum_{i=0}^n w_i * x_i < 0$ **then**

$\mathbf{w} = \mathbf{w} + \mathbf{x}$;

end

end

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Algorithm: Perceptron Learning Algorithm

```
P ← inputs with label 1;  
N ← inputs with label 0;  
Initialize w randomly;  
while !convergence do  
    Pick random x ∈ P ∪ N ;  
    if x ∈ P and  $\sum_{i=0}^n w_i * x_i < 0$  then  
        | w = w + x ;  
    end  
    if x ∈ N and  $\sum_{i=0}^n w_i * x_i \geq 0$  then  
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- Why would this work ?
- To understand why this works we will have to get into a bit of Linear Algebra and a bit of geometry...

- Consider two vectors \mathbf{w} and \mathbf{x}

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$$\mathbf{w} = [w_0, w_1, w_2, \dots, w_n]$$

$$\mathbf{x} = [1, x_1, x_2, \dots, x_n]$$

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$$\mathbf{x} = [1, x_1, x_2, \dots, x_n]$$

$$\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = \sum_{i=0}^n w_i * x_i$$

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- Every point (\mathbf{x}) on this line satisfies the equation $\mathbf{w}^T \mathbf{x} = 0$
- What can you tell about the angle (α) between \mathbf{w} and any point (\mathbf{x}) which lies on this line ?
- The angle is 90° ($\because \cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|} = 0$)

- Consider two vectors \mathbf{w} and \mathbf{x}

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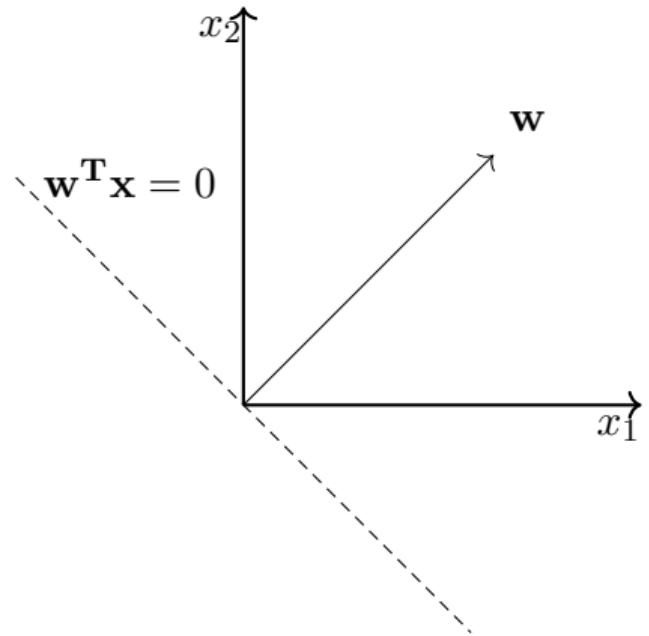
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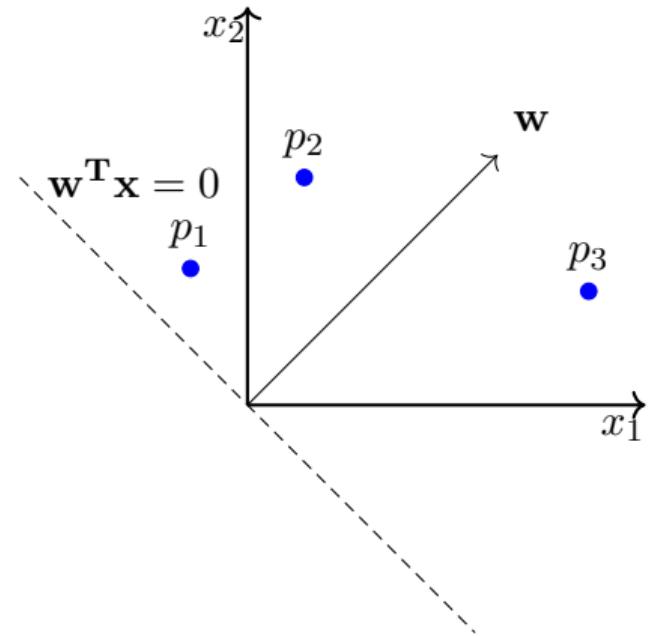
$$y = 1 \quad if \quad \mathbf{w}^T \mathbf{x} \geq 0$$

$$= 0 \quad if \quad \mathbf{w}^T \mathbf{x} < 0$$

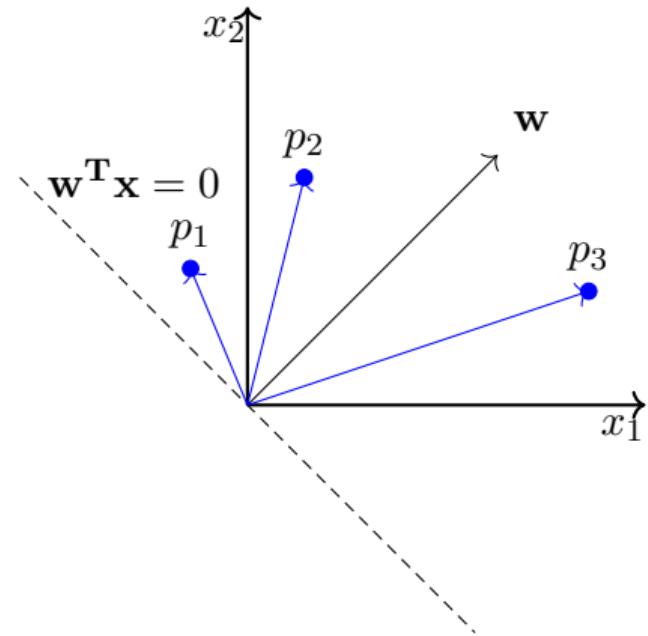
- We are interested in finding the line $\mathbf{w}^T \mathbf{x} = 0$ which divides the input space into two halves
- Every point (\mathbf{x}) on this line satisfies the equation $\mathbf{w}^T \mathbf{x} = 0$
- What can you tell about the angle (α) between \mathbf{w} and any point (\mathbf{x}) which lies on this line ?
- The angle is 90° ($\because \cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|} = 0$)
- Since the vector \mathbf{w} is perpendicular to every point on the line it is actually perpendicular to the line itself



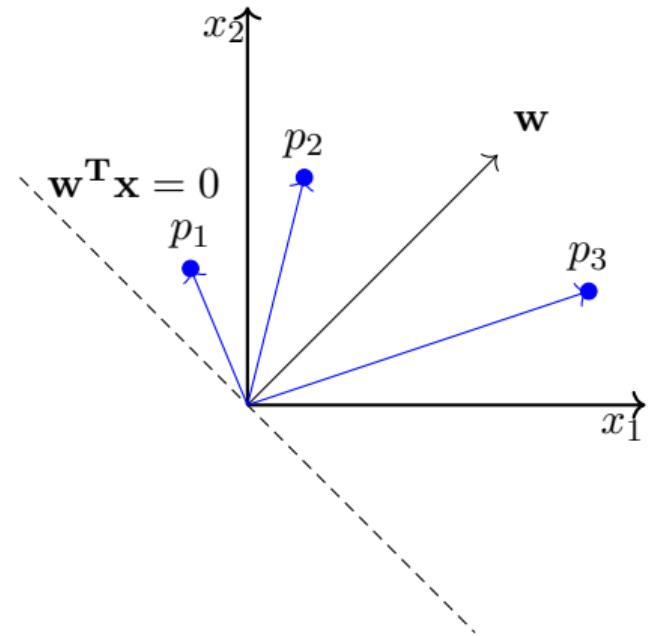
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)



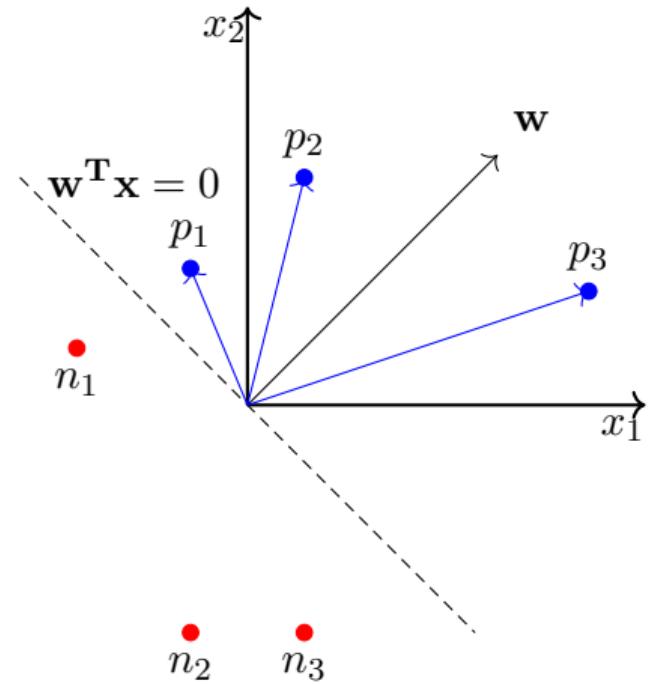
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)
- What will be the angle between any such vector and \mathbf{w} ?



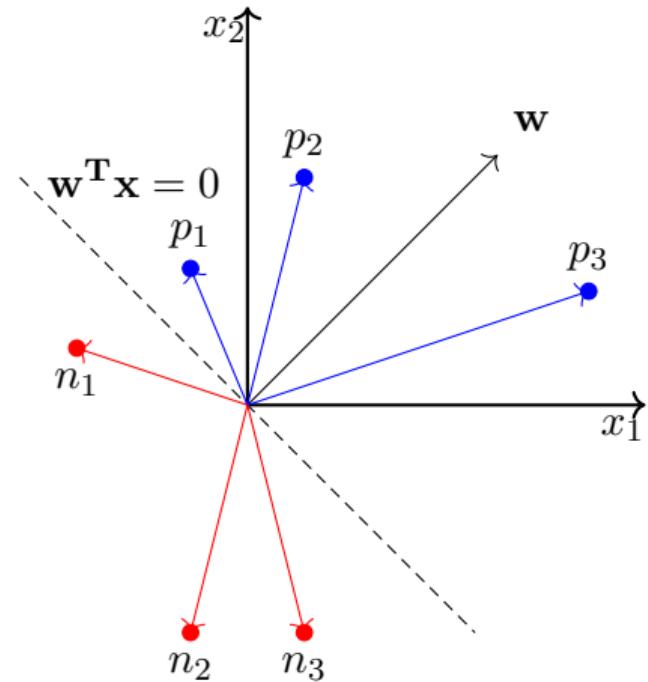
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)
- What will be the angle between any such vector and \mathbf{w} ? Obviously, less than 90°



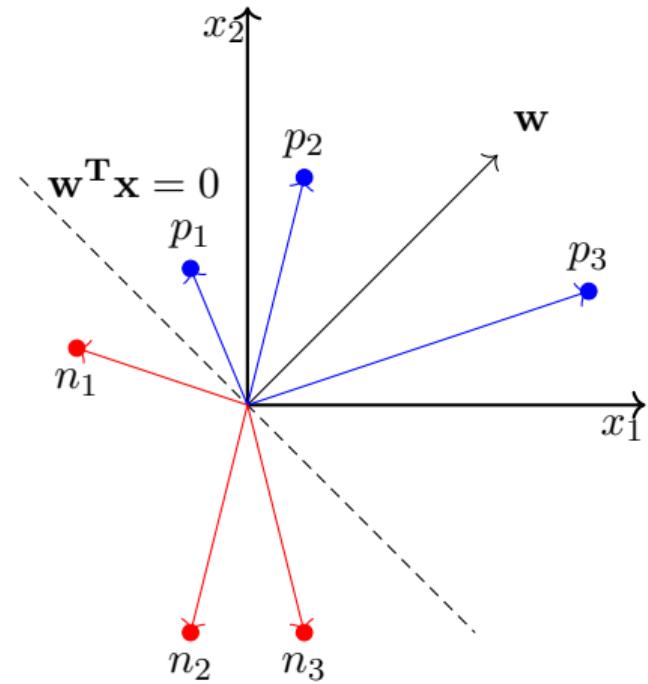
- Consider some points (vectors) which lie in the positive half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} \geq 0$)
- What will be the angle between any such vector and \mathbf{w} ? Obviously, less than 90°
- What about points (vectors) which lie in the negative half space of this line (*i.e.*, $\mathbf{w}^T \mathbf{x} < 0$)



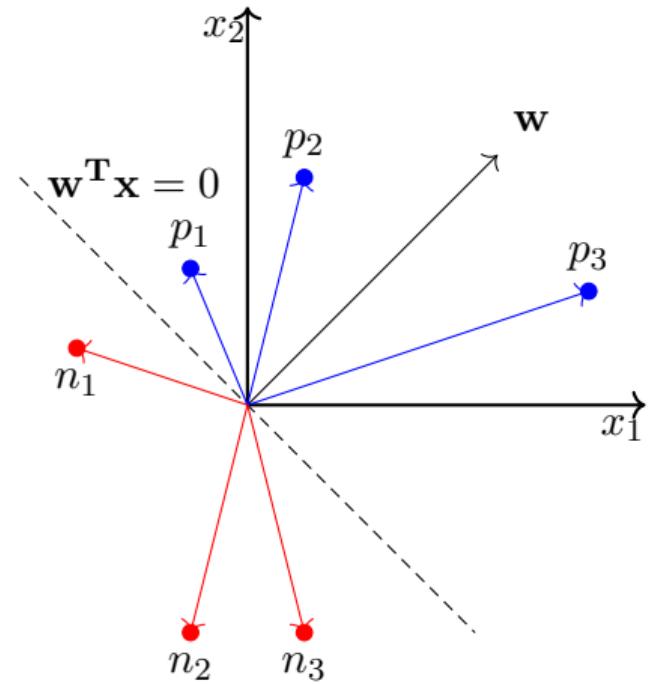
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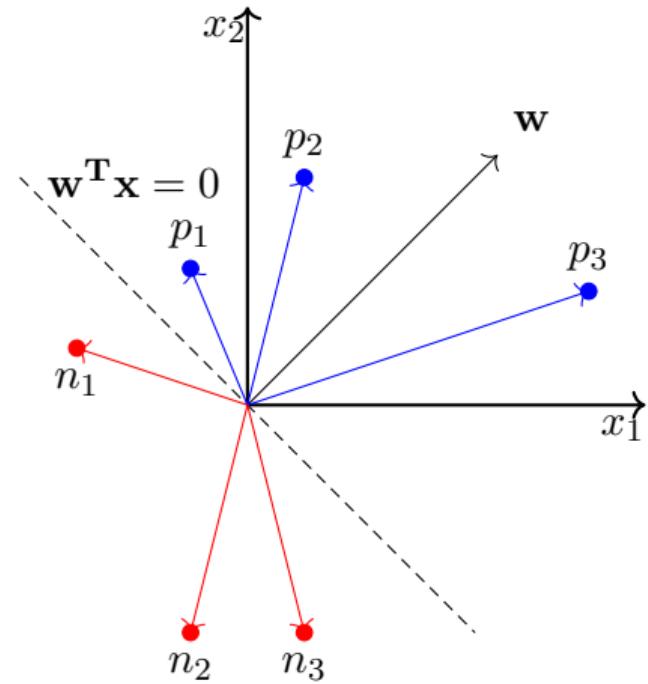
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- Of course, this also follows from the formula ($\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$)



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- Of course, this also follows from the formula ($\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$)
- Keeping this picture in mind let us revisit the algorithm



Algorithm: Perceptron Learning Algorithm

```
P ← inputs with label 1;  
N ← inputs with label 0;  
Initialize w randomly;  
while !convergence do  
    Pick random x ∈ P ∪ N ;  
    if x ∈ P and w.x < 0 then  
        | w = w + x ;  
    end  
    if x ∈ N and w.x ≥ 0 then  
        | w = w - x ;  
    end  
end  
//the algorithm converges when all the  
inputs are classified correctly
```

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

Algorithm: Perceptron Learning Algorithm

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- For $\mathbf{x} \in P$ if $\mathbf{w} \cdot \mathbf{x} < 0$ then it means that the angle (α) between this \mathbf{x} and the current \mathbf{w} is greater than 90°

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

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- For $\mathbf{x} \in P$ if $\mathbf{w} \cdot \mathbf{x} < 0$ then it means that the angle (α) between this \mathbf{x} and the current \mathbf{w} is greater than 90° (but we want α to be less than 90°)

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

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- What happens to the new angle (α_{new}) when $\mathbf{w}_{new} = \mathbf{w} + \mathbf{x}$

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

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- What happens to the new angle (α_{new}) when $\mathbf{w}_{new} = \mathbf{w} + \mathbf{x}$

$$\cos(\alpha_{new}) \propto \mathbf{w}_{new}^T \mathbf{x}$$

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

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$$\cos(\alpha_{new}) > \cos\alpha$$

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$$\cos(\alpha_{new}) > \cos\alpha$$

- Thus α_{new} will be less than α and this is exactly what we want

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- What happens to the new angle (α_{new}) when $\mathbf{w}_{new} = \mathbf{w} - \mathbf{x}$

$$\cos(\alpha_{new}) \propto \mathbf{w}_{new}^T \mathbf{x}$$
$$\propto (\mathbf{w} - \mathbf{x})^T \mathbf{x}$$

$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

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$$\begin{aligned} \cos(\alpha_{new}) &\propto \mathbf{w}_{new}^T \mathbf{x} \\ &\propto (\mathbf{w} - \mathbf{x})^T \mathbf{x} \\ &\propto \mathbf{w}^T \mathbf{x} - \mathbf{x}^T \mathbf{x} \\ &\propto \cos\alpha - \mathbf{x}^T \mathbf{x} \end{aligned}$$
$$\cos(\alpha_{new}) < \cos\alpha$$

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        | w = w + x ;  
    end  
    if x ∈ N and w.x ≥ 0 then  
        | w = w - x ;  
    end  
end  
//the algorithm converges when all the  
inputs are classified correctly
```

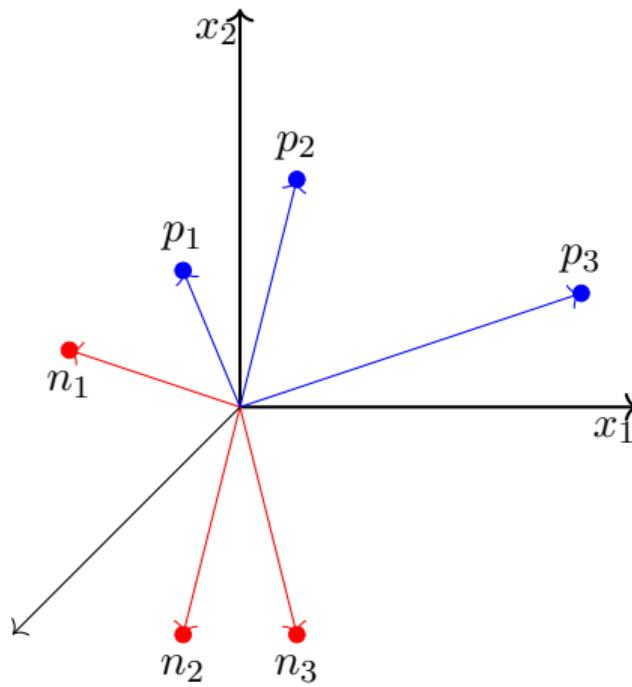
$$\cos\alpha = \frac{\mathbf{w}^T \mathbf{x}}{\|\mathbf{w}\| \|\mathbf{x}\|}$$

- For $\mathbf{x} \in N$ if $\mathbf{w} \cdot \mathbf{x} \geq 0$ then it means that the angle (α) between this \mathbf{x} and the current \mathbf{w} is less than 90° (but we want α to be greater than 90°)
- What happens to the new angle (α_{new}) when $\mathbf{w}_{new} = \mathbf{w} - \mathbf{x}$

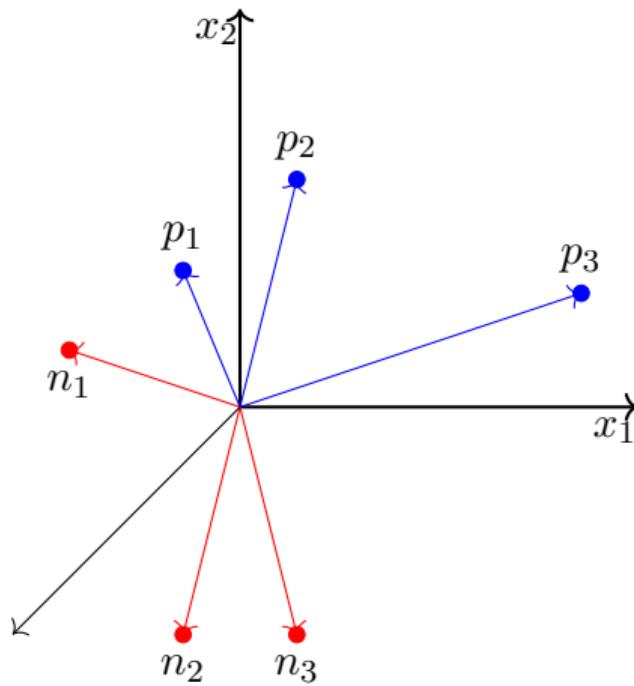
$$\begin{aligned} \cos(\alpha_{new}) &\propto \mathbf{w}_{new}^T \mathbf{x} \\ &\propto (\mathbf{w} - \mathbf{x})^T \mathbf{x} \\ &\propto \mathbf{w}^T \mathbf{x} - \mathbf{x}^T \mathbf{x} \\ &\propto \cos\alpha - \mathbf{x}^T \mathbf{x} \end{aligned}$$
$$\cos(\alpha_{new}) < \cos\alpha$$

- Thus α_{new} will be greater than α and this is exactly what we want

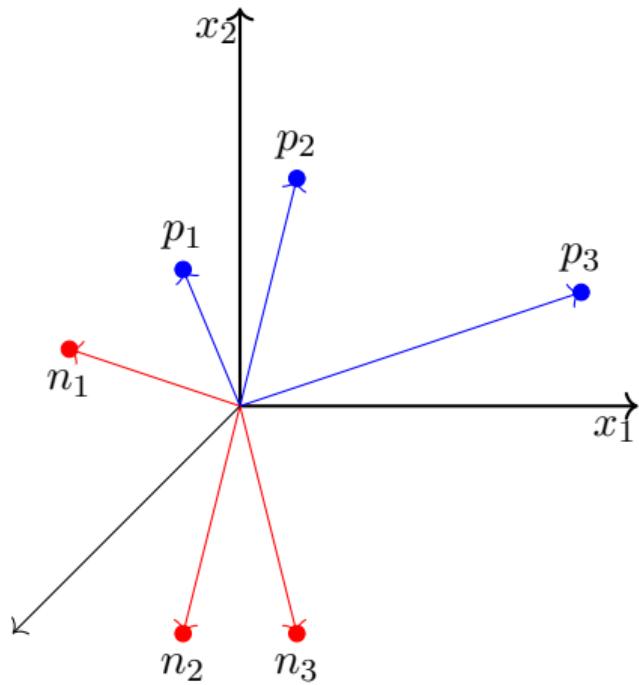
- We will now see this algorithm in action for a toy dataset



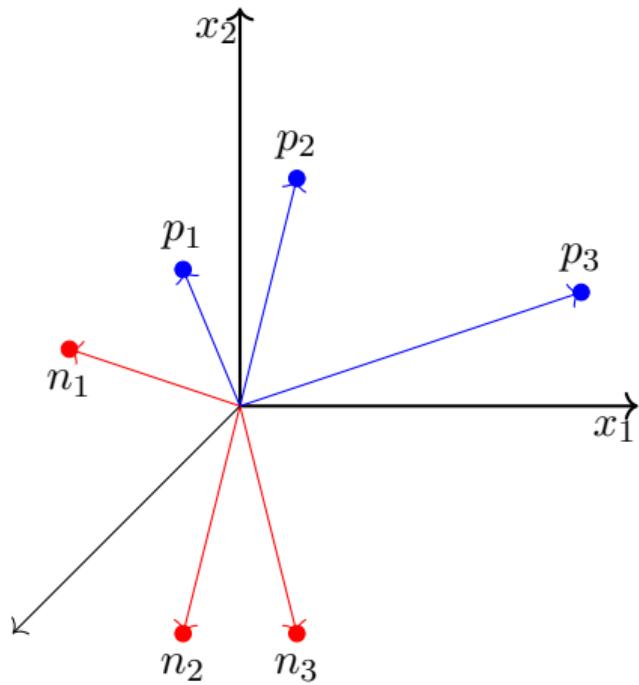
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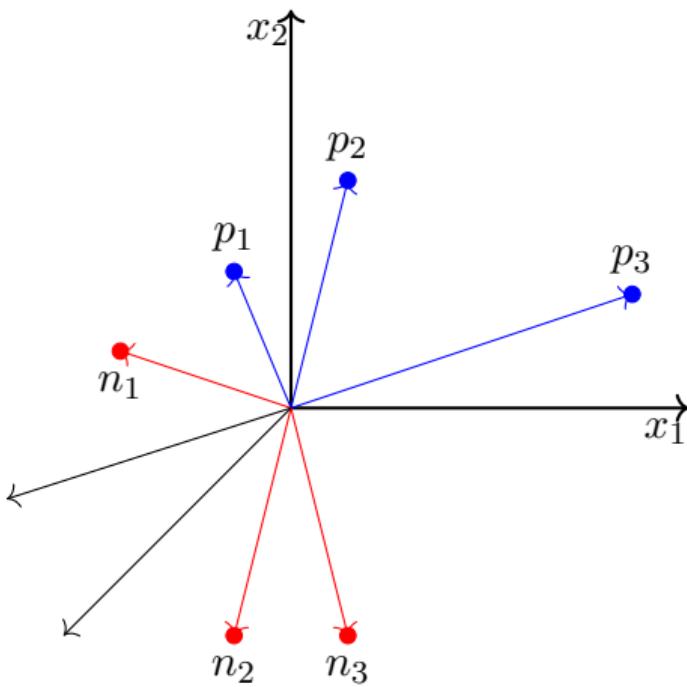
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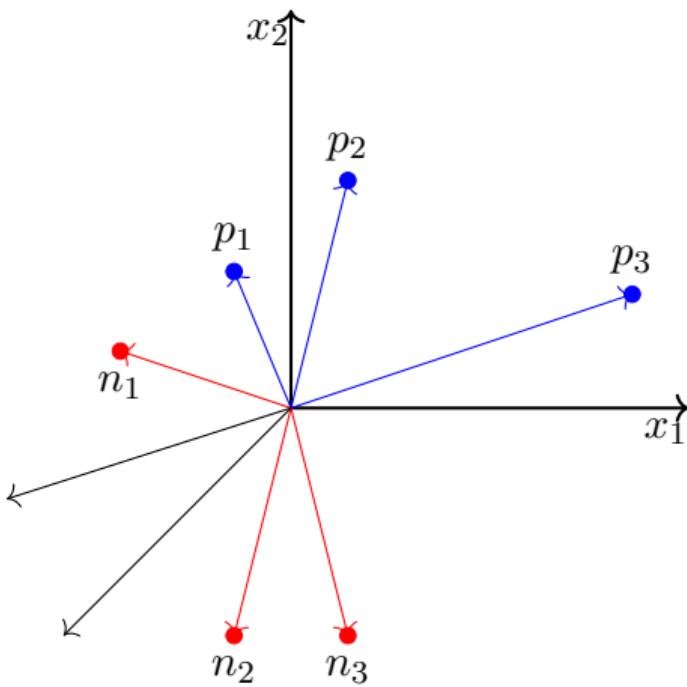
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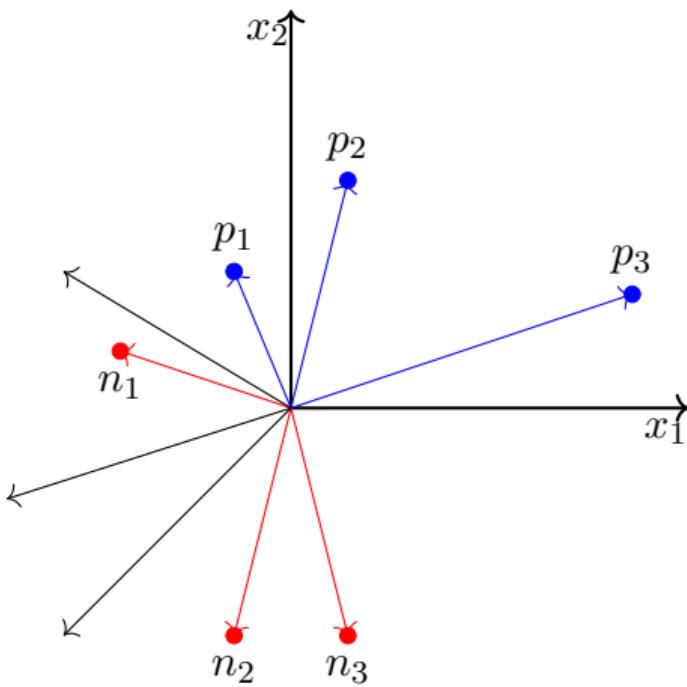
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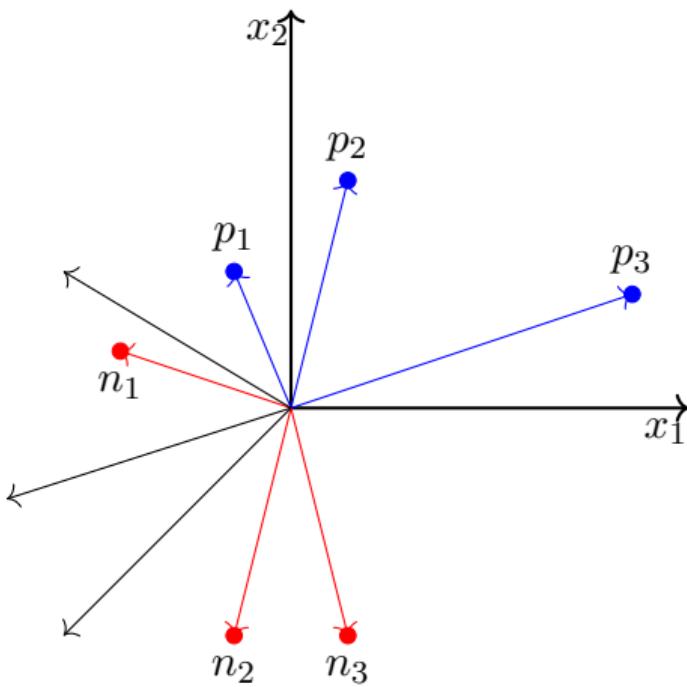
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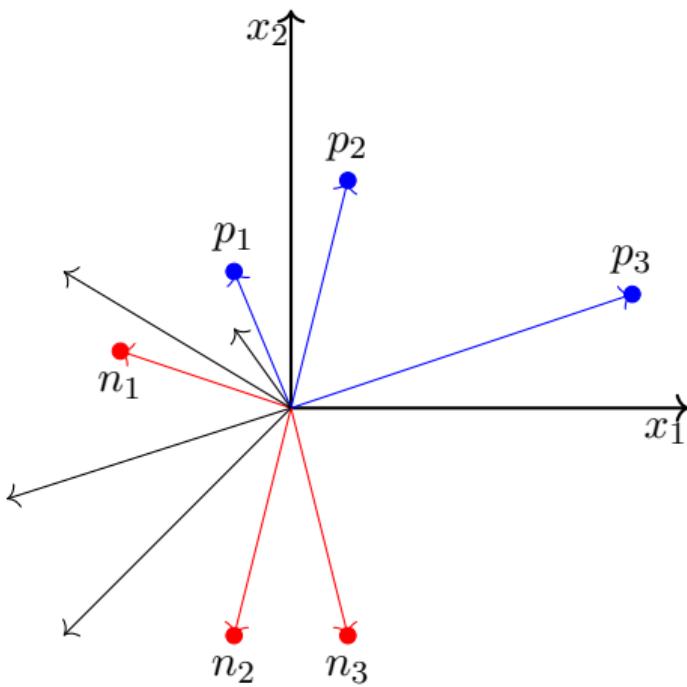
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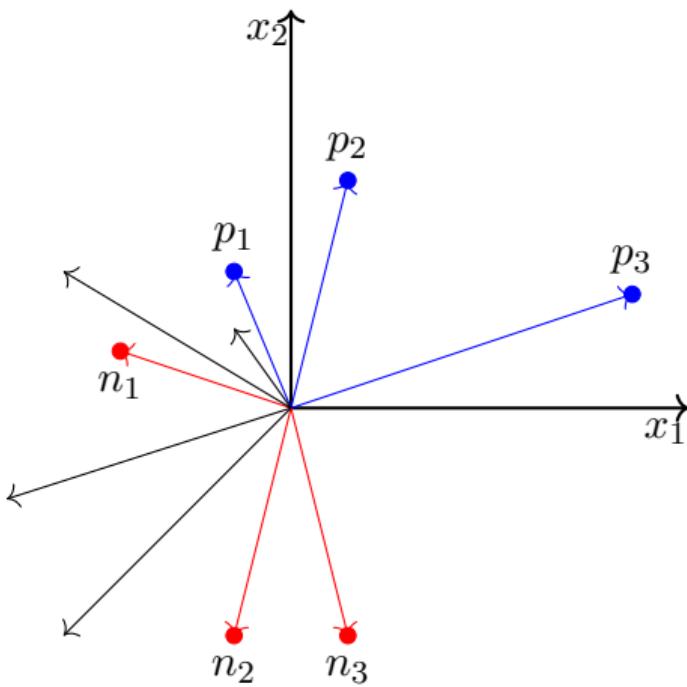
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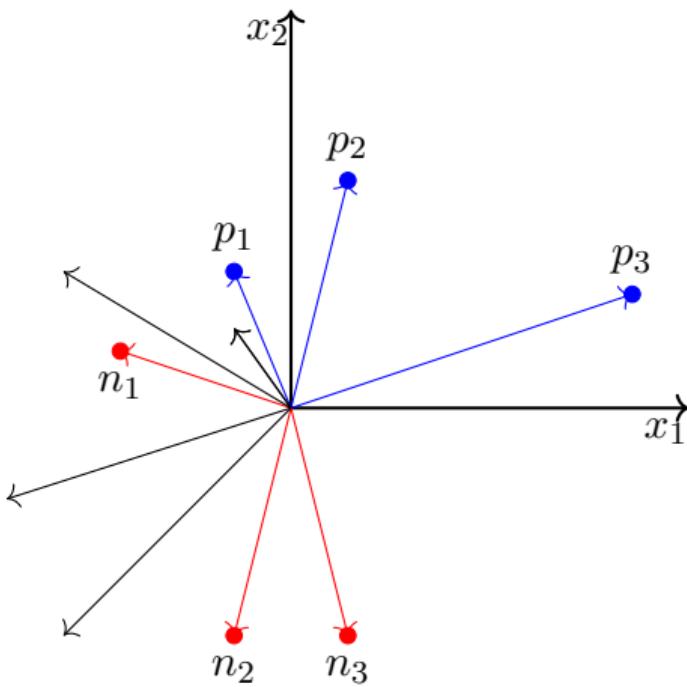
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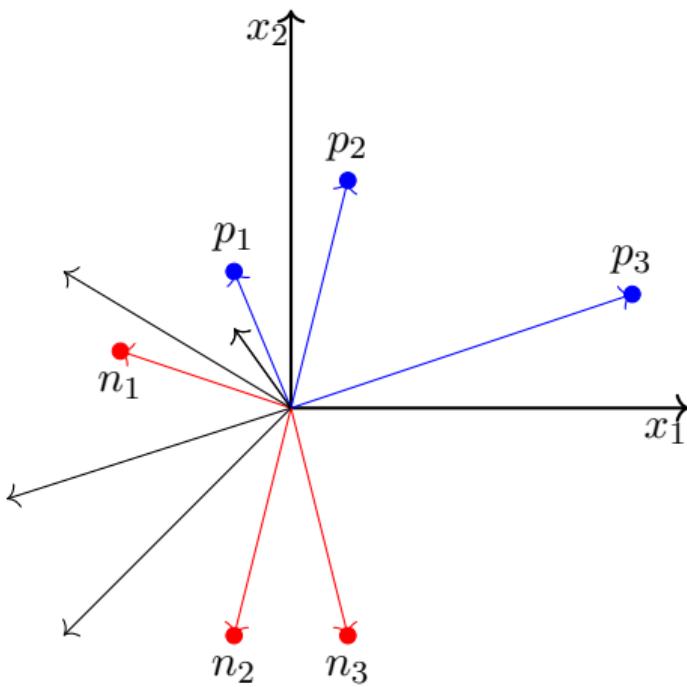
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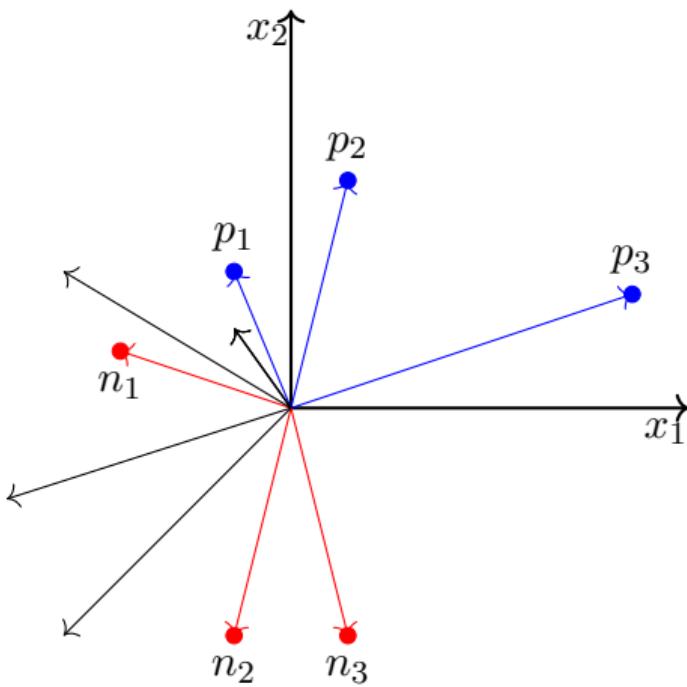
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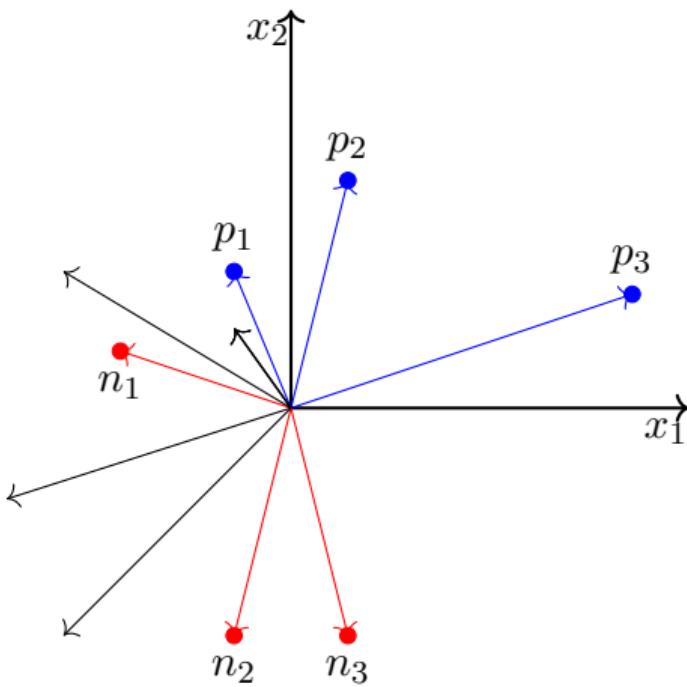
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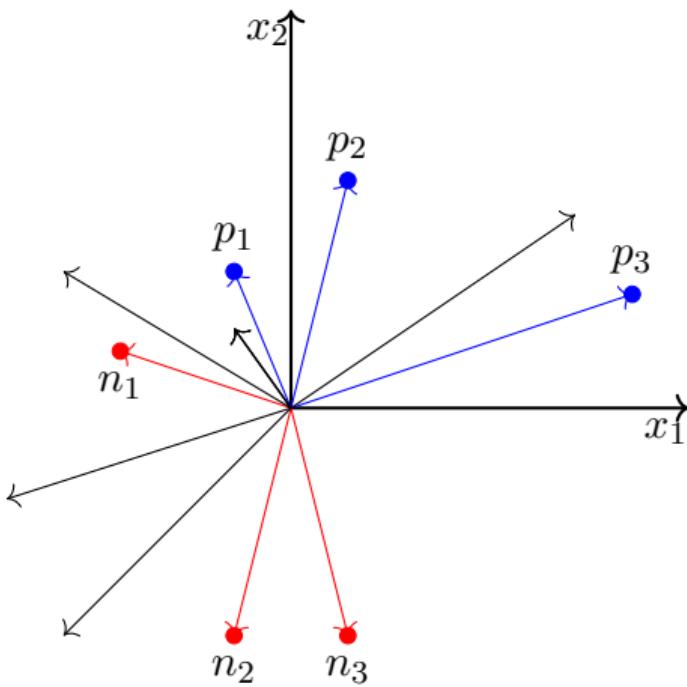
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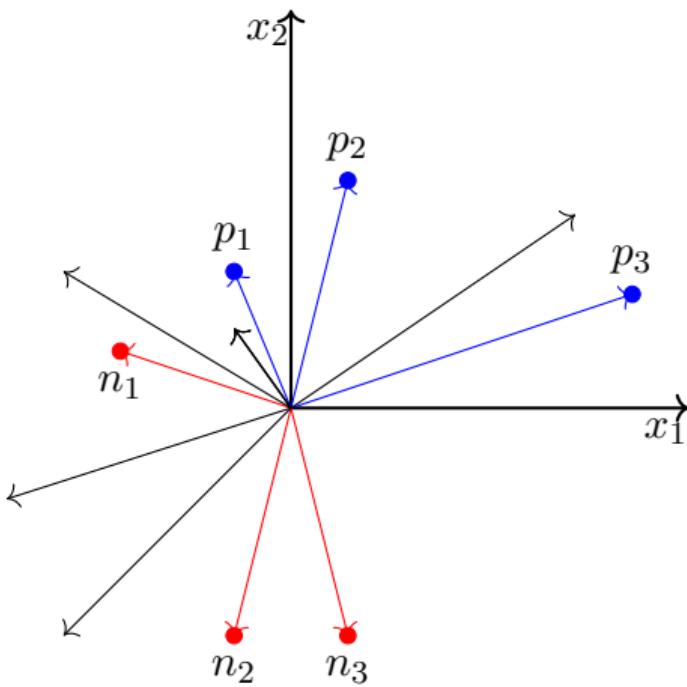
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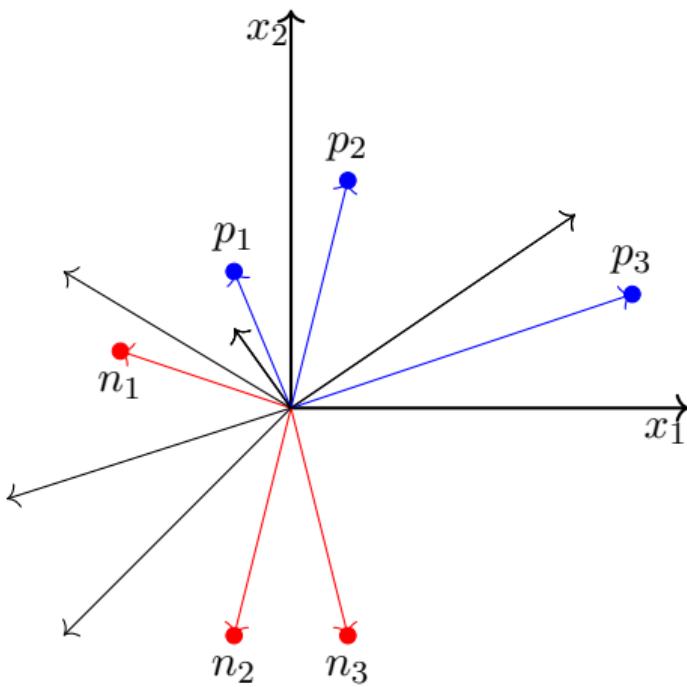
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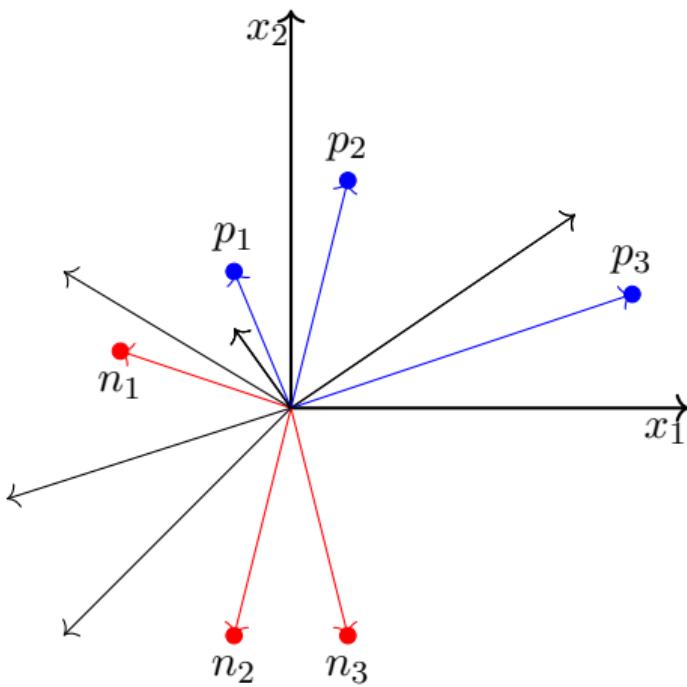
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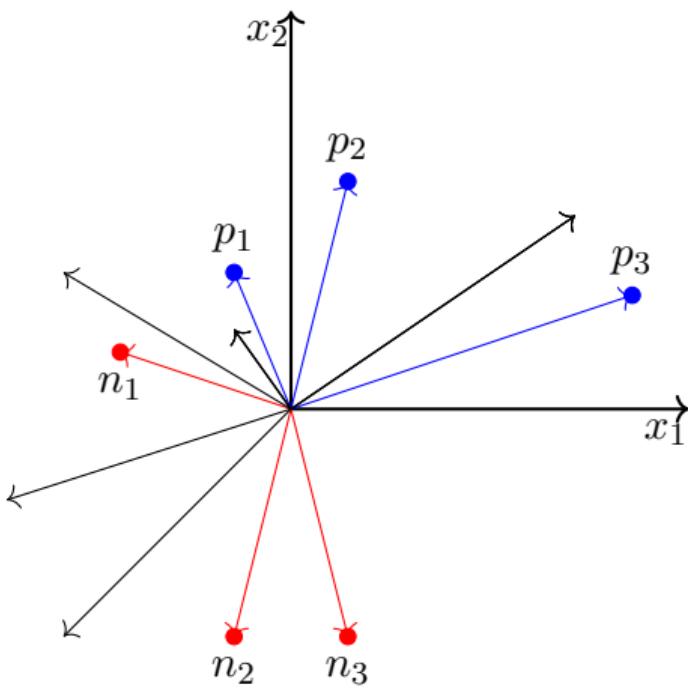
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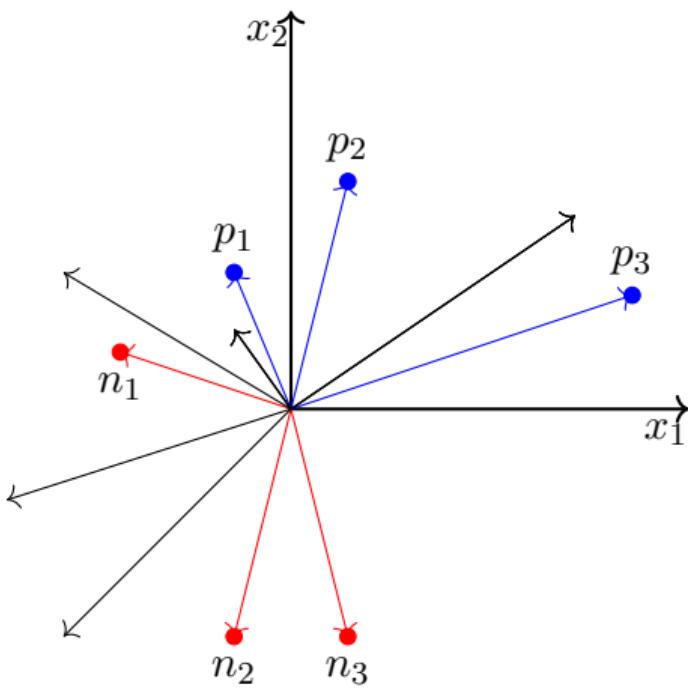
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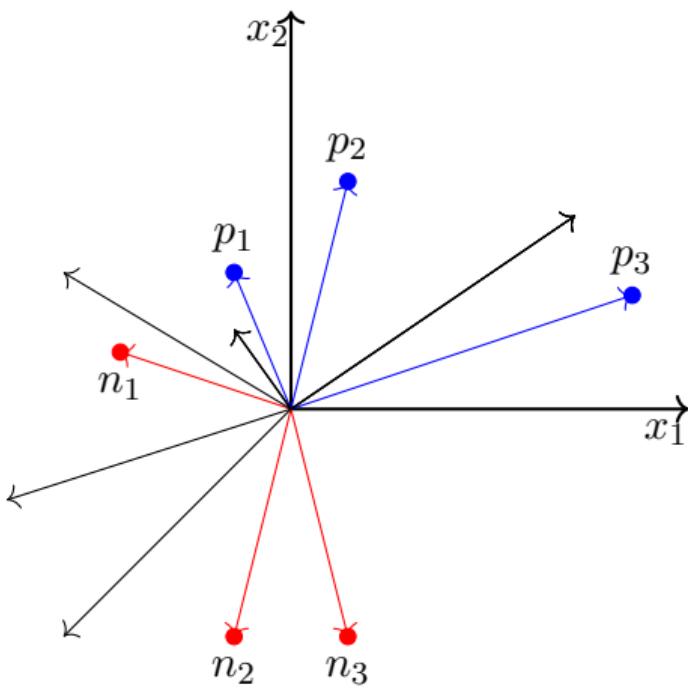
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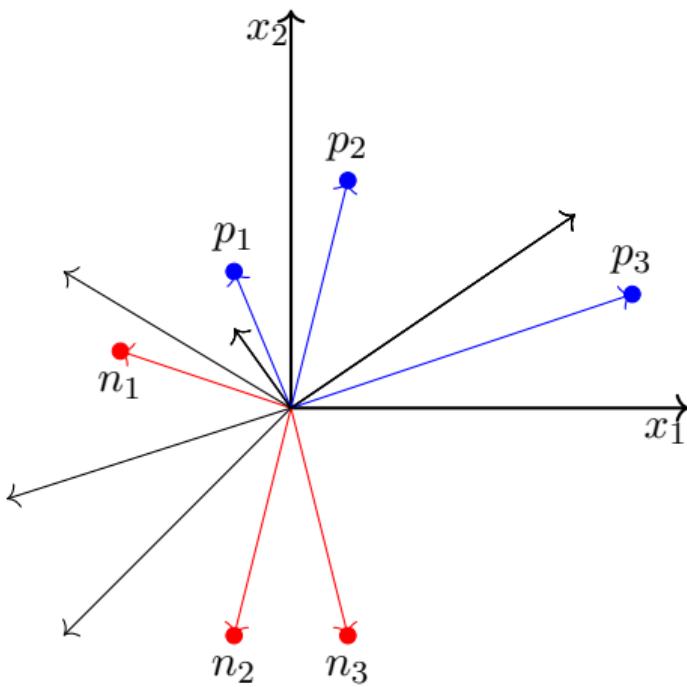
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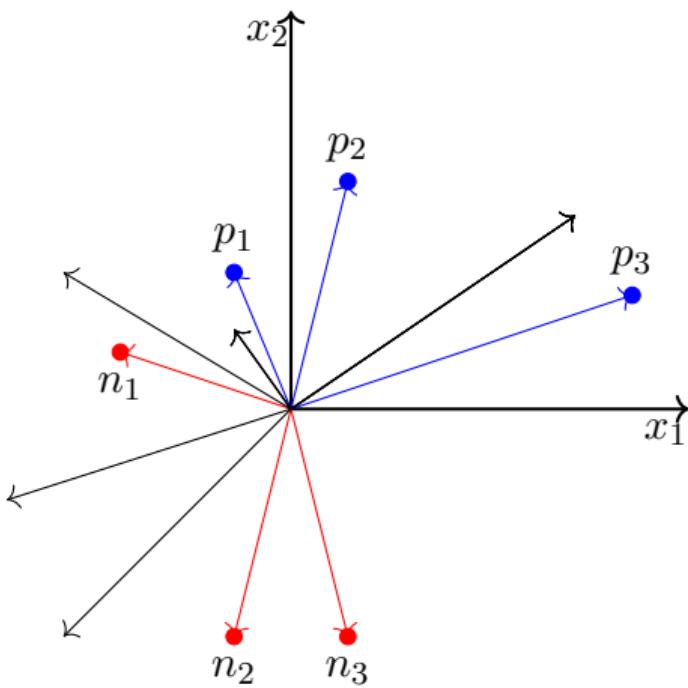
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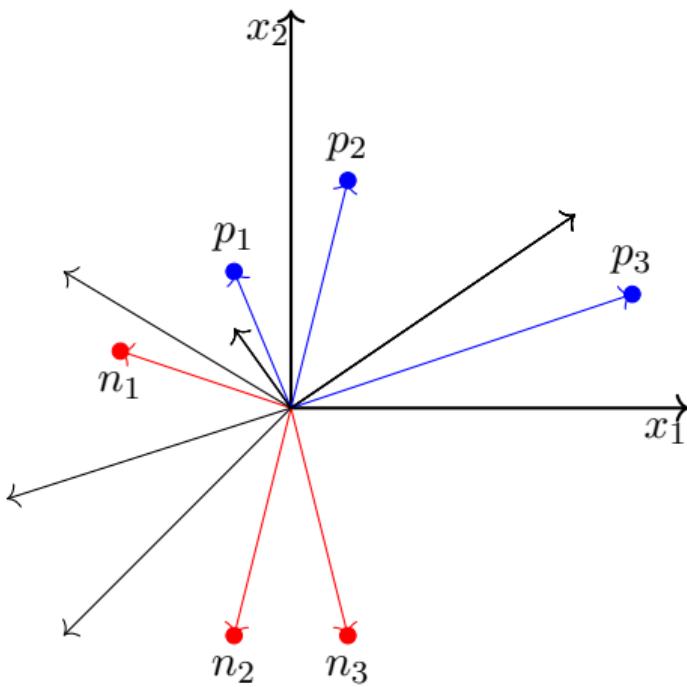
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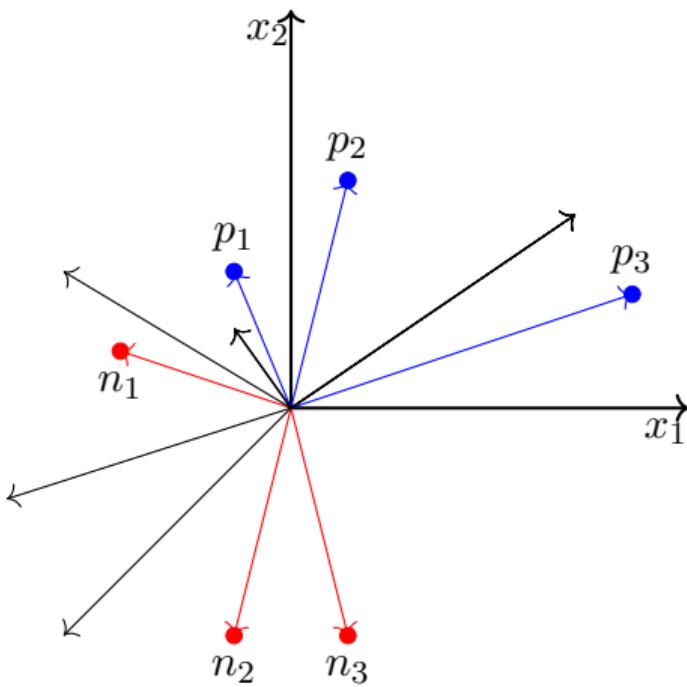
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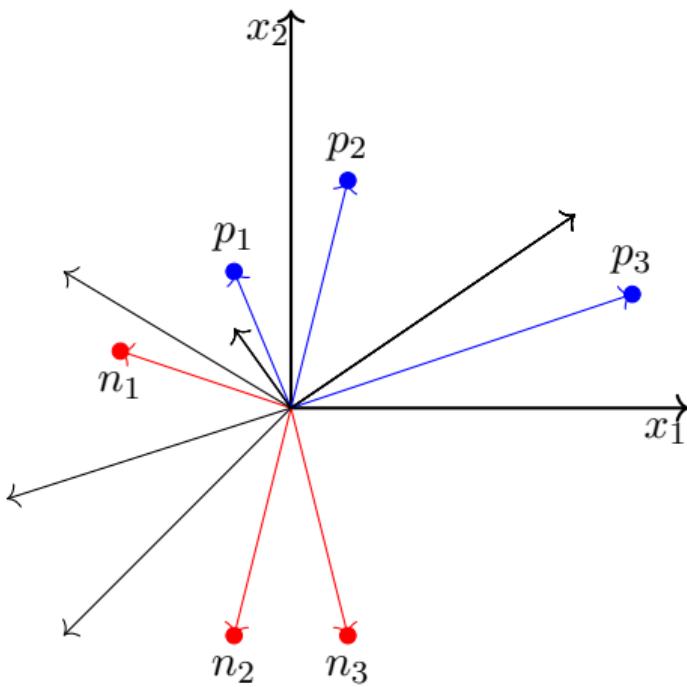
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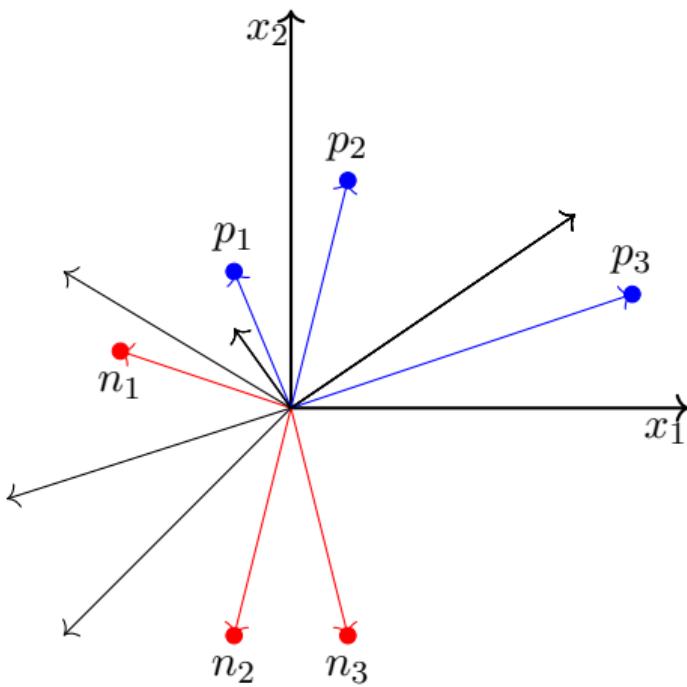
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- We now run the algorithm by randomly going over the points
- The algorithm has converged

Module 2.6: Proof of Convergence

- Now that we have some faith and intuition about why the algorithm works, we will see a more formal proof of convergence ...

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Proof: On the next slide

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        else  $\leftarrow$   $N^-$  ;
    end if
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- Let w^* be the normalized solution vector (we know one exists as the data is linearly separable)

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Proof (continued:)

So far we have, $w^T \cdot p_i \leq 0$ (and hence we made the correction)

$$\cos\beta = \frac{w^* \cdot w_{t+1}}{\|w_{t+1}\|} \quad (\text{by definition})$$

Numerator $\geq w^* \cdot w_0 + k\delta$ (proved by induction)

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- $\cos\beta$ thus grows proportional to \sqrt{k}
- As k (number of corrections) increases $\cos\beta$ can become arbitrarily large
- But since $\cos\beta \leq 1$, k must be bounded by a maximum number
- Thus, there can only be a finite number of corrections (k) to w and the algorithm will converge!

Coming back to our questions ...

- What about non-boolean (say, real) inputs?
- Do we always need to hand code the threshold?
- Are all inputs equal? What if we want to assign more weight (importance) to some inputs?
- What about functions which are not linearly separable ?

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- Are all inputs equal? What if we want to assign more weight (importance) to some inputs? A perceptron allows weights to be assigned to inputs
- What about functions which are not linearly separable ? Not possible with a single perceptron but we will see how to handle this ..

Module 2.7: Linearly Separable Boolean Functions

- So what do we do about functions which are not linearly separable ?

- So what do we do about functions which are not linearly separable ?
- Let us see one such simple boolean function first ?

x_1	x_2	XOR	
0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^2 w_i x_i \geq 0$
1	1	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$

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1	1	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$

$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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1	1	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$

$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

$$w_0 + w_1 \cdot 0 + w_2 \cdot 1 \geq 0 \implies w_2 \geq -w_0$$

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0	0	0	$w_0 + \sum_{i=1}^2 w_i x_i < 0$
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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

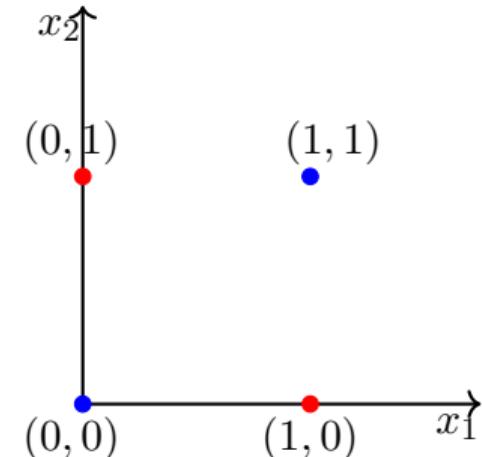
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- The fourth condition contradicts conditions 2 and 3

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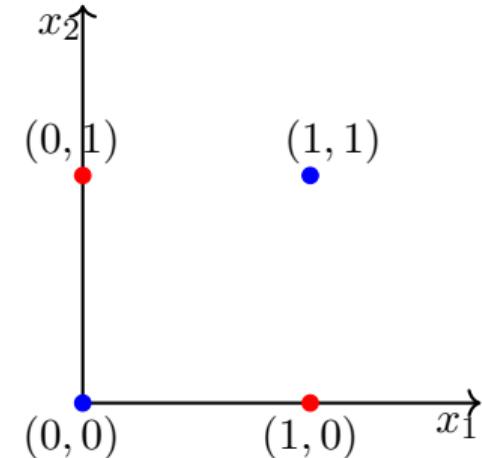
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- Hence we cannot have a solution to this set of inequalities

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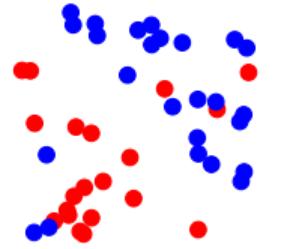
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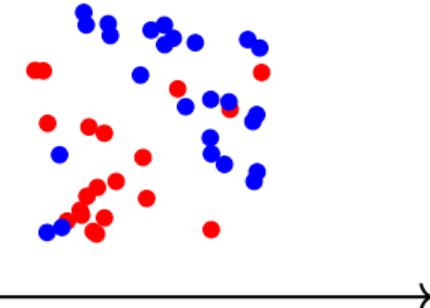
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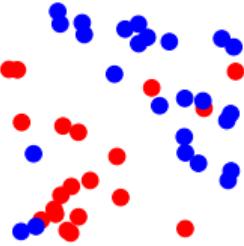
- And indeed you can see that it is impossible to draw a line which separates the red points from the blue points



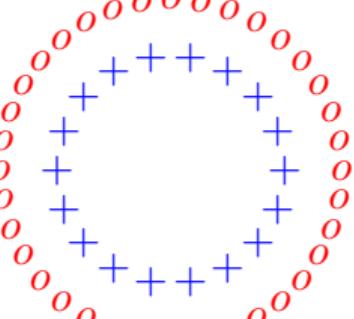
- Most real world data is not linearly separable and will always contain some outliers

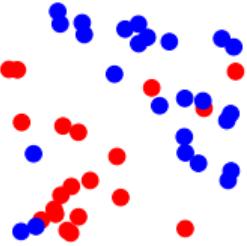


- Most real world data is not linearly separable and will always contain some outliers
- In fact, sometimes there may not be any outliers but still the data may not be linearly separable

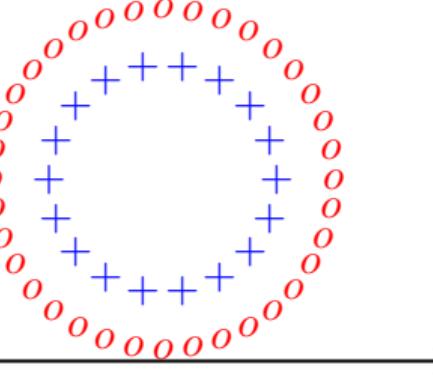


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- Most real world data is not linearly separable and will always contain some outliers
- In fact, sometimes there may not be any outliers but still the data may not be linearly separable
- We need computational units (models) which can deal with such data
- While a single perceptron cannot deal with such data, we will show that a network of perceptrons can indeed deal with such data



- Before seeing how a network of perceptrons can deal with linearly inseparable data, we will discuss boolean functions in some more detail ...

- How many boolean functions can you design from 2 inputs ?

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- Let us begin with some easy ones which you already know ..

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x_1	x_2
0	0
0	1
1	0
1	1

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x_1	x_2	f_1
0	0	0
0	1	0
1	0	0
1	1	0

- How many boolean functions can you design from 2 inputs ?
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x_1	x_2	f_1	f_{16}
0	0	0	1
0	1	0	1
1	0	0	1
1	1	0	1

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x_1	x_2	f_1	f_2	f_{16}
0	0	0	0	1
0	1	0	0	1
1	0	0	0	1
1	1	0	1	1

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x_1	x_2	f_1	f_2	f_8	f_{16}
0	0	0	0	0	1
0	1	0	0	1	1
1	0	0	0	1	1
1	1	0	1	1	1

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0	1	0	0	0	1	1
1	0	0	0	1	1	1
1	1	0	1	0	1	1

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1	0	0	0	1	1	0	1	1
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0	1	0	0	0	0	1	1	1	1	0	1
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0	1	0	0	0	0	1	1	1	1	0	0	1
1	0	0	0	1	1	0	0	1	1	0	0	1
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0	1	0	0	0	0	1	1	1	1	0	0	0	1
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0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
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0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	1
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0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1

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0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

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x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ? 2^{2^n}

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ? 2^{2^n}
- How many of these 2^{2^n} functions are not linearly separable ?

- How many boolean functions can you design from 2 inputs ?
- Let us begin with some easy ones which you already know ..

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	

- Of these, how many are linearly separable ? (turns out all except XOR and !XOR - feel free to verify)
- In general, how many boolean functions can you have for n inputs ? 2^{2^n}
- How many of these 2^{2^n} functions are not linearly separable ? For the time being, it suffices to know that at least some of these may not be linearly inseparable (I encourage you to figure out the exact answer :-))

Module 2.8: Representation Power of a Network of Perceptrons

- We will now see how to implement **any** boolean function using a network of perceptrons ...

- For this discussion, we will assume True = +1 and False = -1

- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons



x_1

x_2

- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



x_1

x_2

- For this discussion, we will assume True = +1 and False = -1
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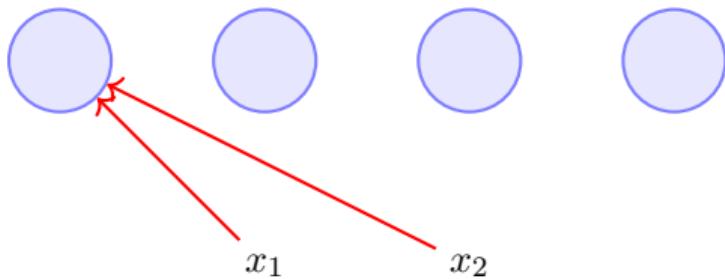
x_1

x_2

red edge indicates $w = -1$

blue edge indicates $w = +1$

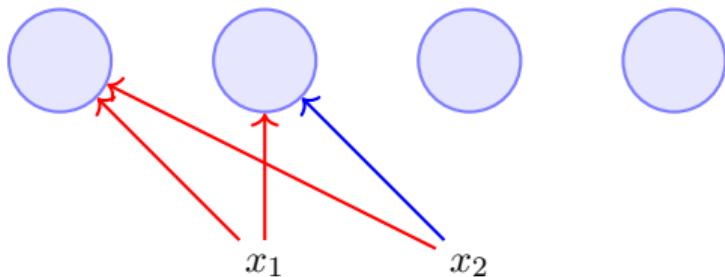
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



red edge indicates $w = -1$

blue edge indicates $w = +1$

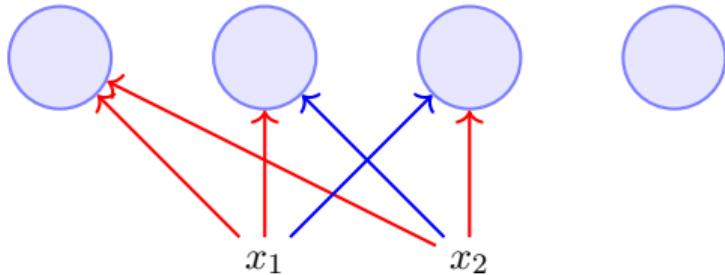
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



red edge indicates $w = -1$

blue edge indicates $w = +1$

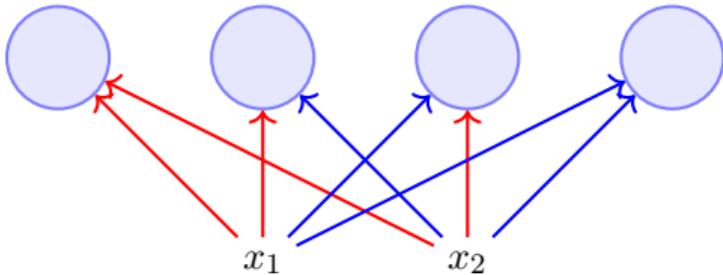
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



red edge indicates $w = -1$

blue edge indicates $w = +1$

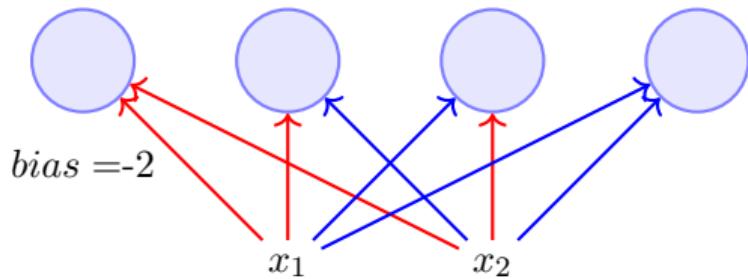
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights



red edge indicates $w = -1$

blue edge indicates $w = +1$

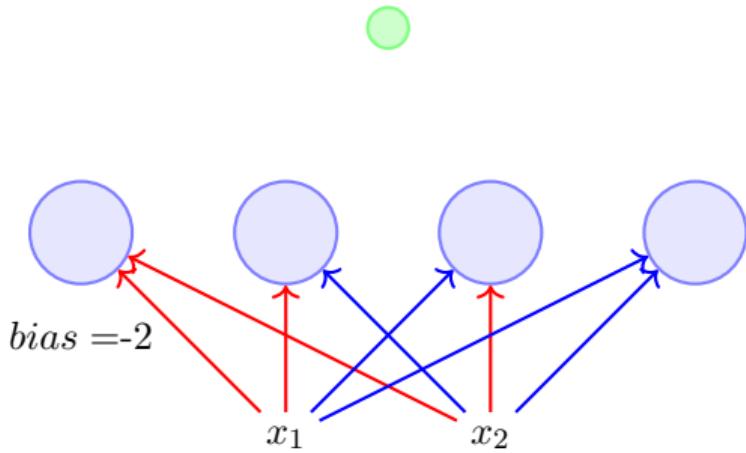
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias (w_0) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is ≥ 2)



red edge indicates $w = -1$

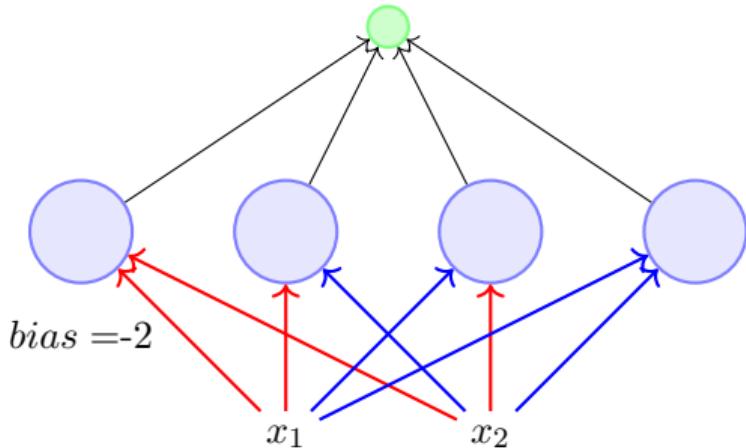
blue edge indicates $w = +1$

- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias (w_0) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is ≥ 2)
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)



red edge indicates $w = -1$
 blue edge indicates $w = +1$

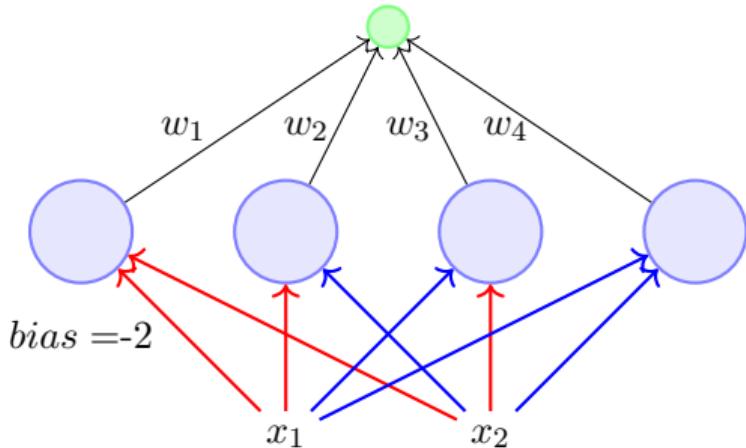
- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
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red edge indicates $w = -1$

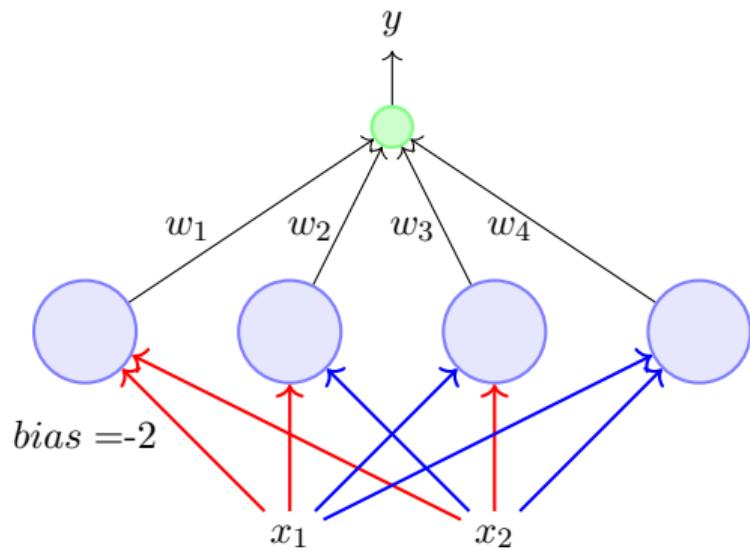
blue edge indicates $w = +1$

- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias (w_0) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is ≥ 2)
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)



red edge indicates $w = -1$

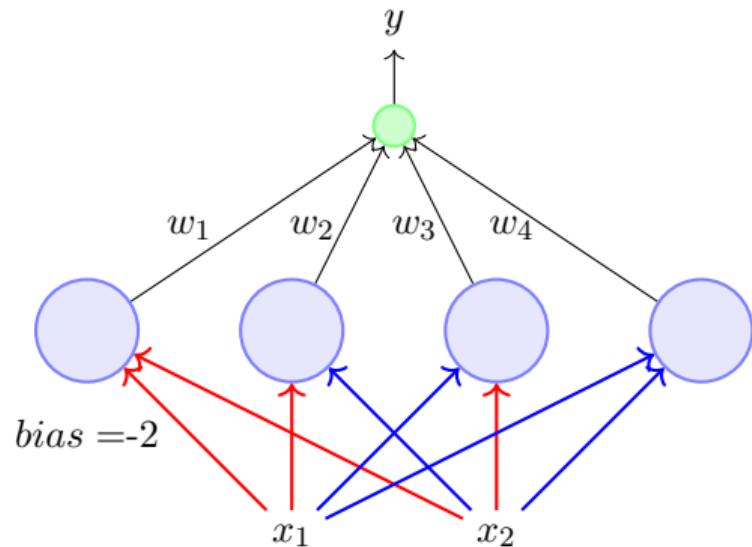
blue edge indicates $w = +1$



- For this discussion, we will assume True = +1 and False = -1
- We consider 2 inputs and 4 perceptrons
- Each input is connected to all the 4 perceptrons with specific weights
- The bias (w_0) of each perceptron is -2 (i.e., each perceptron will fire only if the weighted sum of its input is ≥ 2)
- Each of these perceptrons is connected to an output perceptron by weights (which need to be learned)
- The output of this perceptron (y) is the output of this network

Terminology:

- This network contains 3 layers

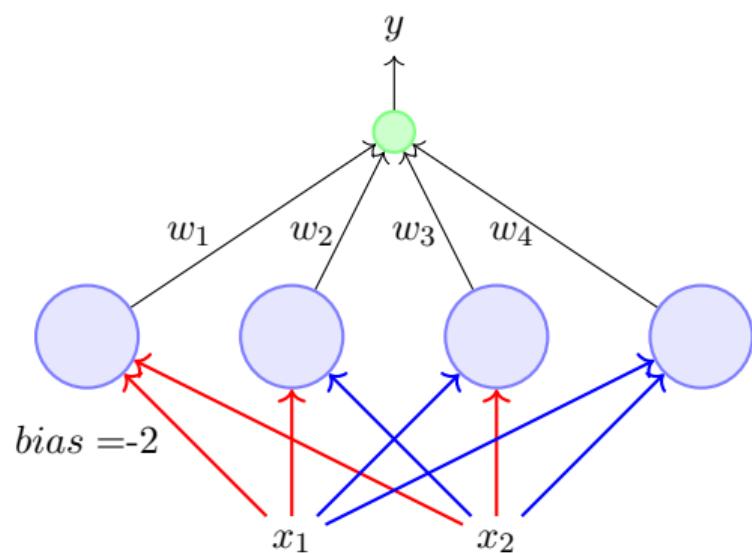


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**

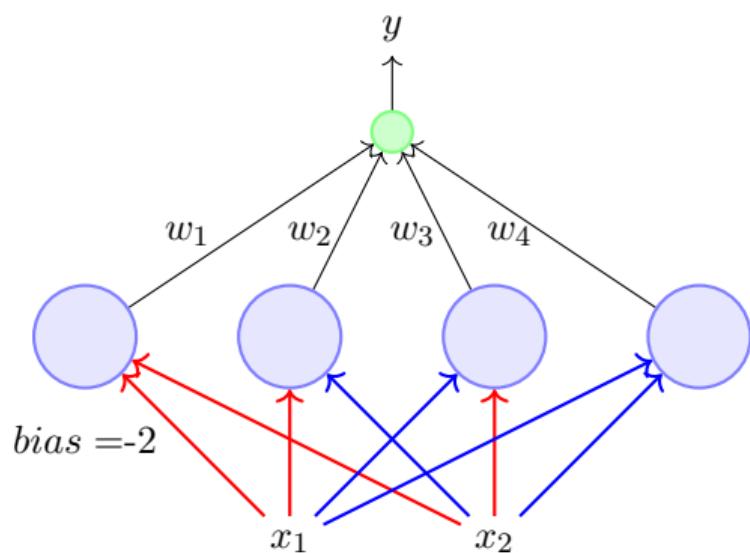


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**

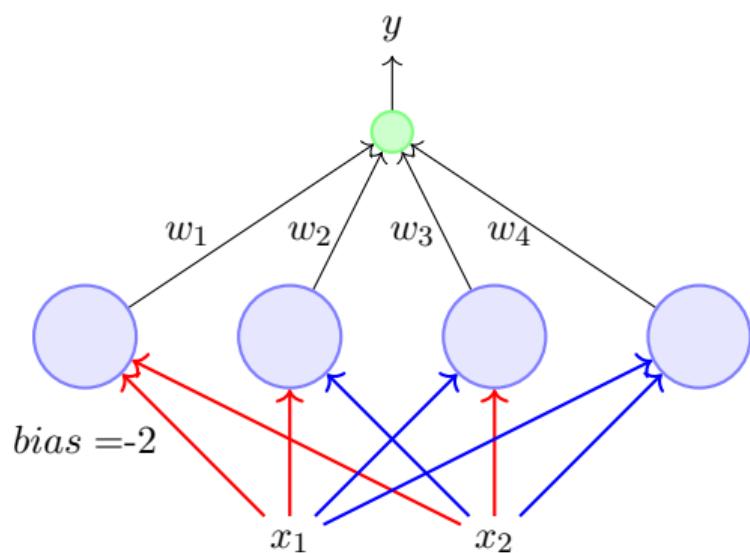


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**
- The final layer containing one output neuron is called the **output layer**

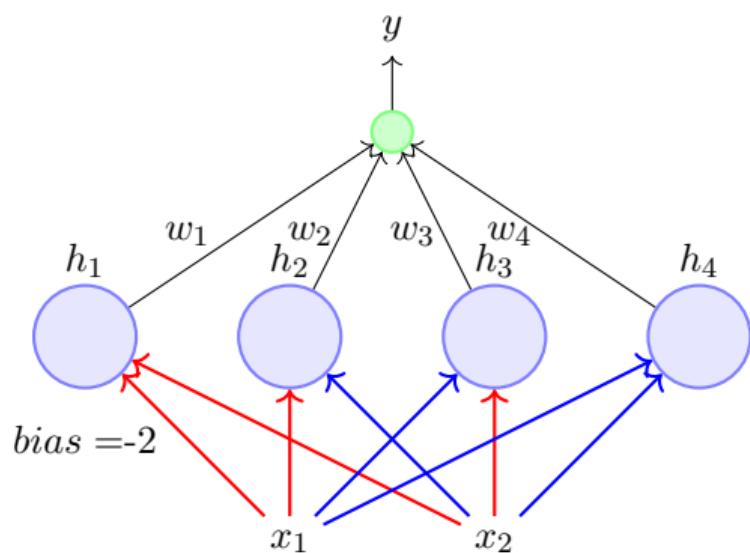


red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

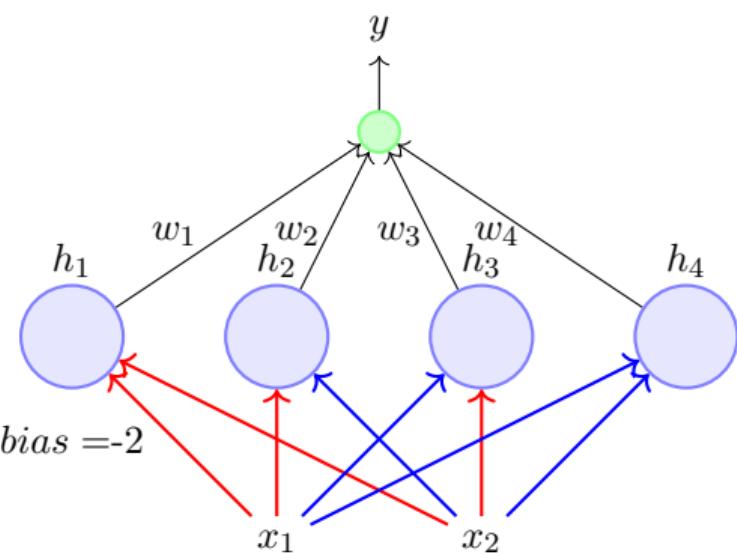
- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**
- The final layer containing one output neuron is called the **output layer**
- The outputs of the 4 perceptrons in the hidden layer are denoted by h_1, h_2, h_3, h_4



red edge indicates $w = -1$

blue edge indicates $w = +1$

Terminology:

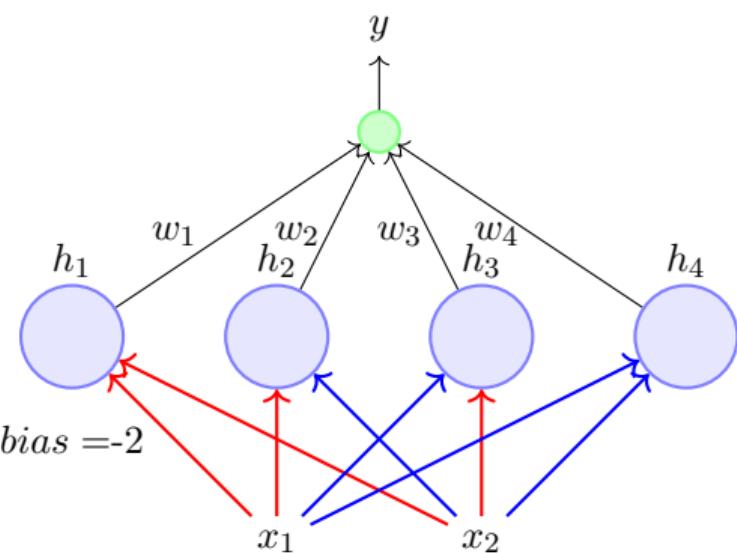


red edge indicates $w = -1$

blue edge indicates $w = +1$

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**
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- The outputs of the 4 perceptrons in the hidden layer are denoted by h_1, h_2, h_3, h_4
- The red and blue edges are called layer 1 weights

Terminology:

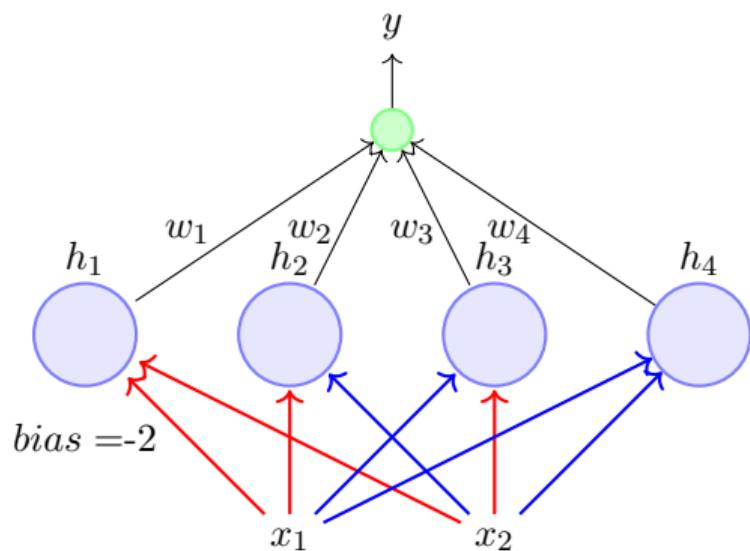


red edge indicates $w = -1$

blue edge indicates $w = +1$

- This network contains 3 layers
- The layer containing the inputs (x_1, x_2) is called the **input layer**
- The middle layer containing the 4 perceptrons is called the **hidden layer**
- The final layer containing one output neuron is called the **output layer**
- The outputs of the 4 perceptrons in the hidden layer are denoted by h_1, h_2, h_3, h_4
- The red and blue edges are called layer 1 weights
- w_1, w_2, w_3, w_4 are called layer 2 weights

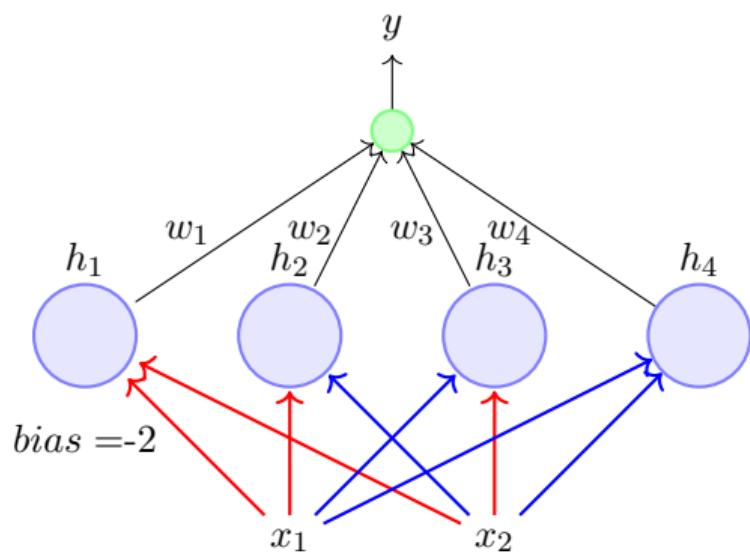
- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !



red edge indicates $w = -1$

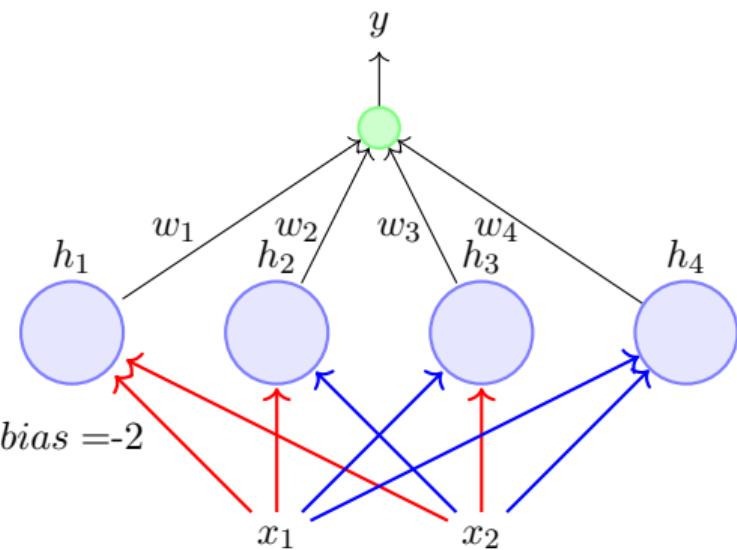
blue edge indicates $w = +1$

- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !
- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network



red edge indicates $w = -1$

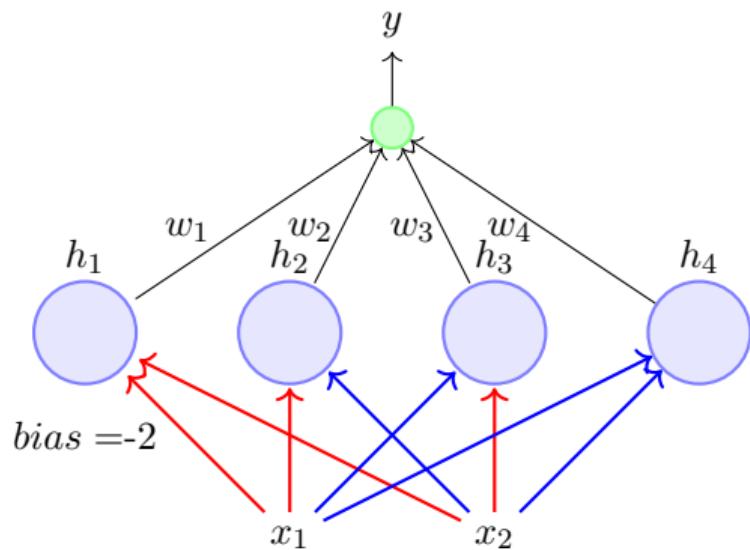
blue edge indicates $w = +1$



red edge indicates $w = -1$

blue edge indicates $w = +1$

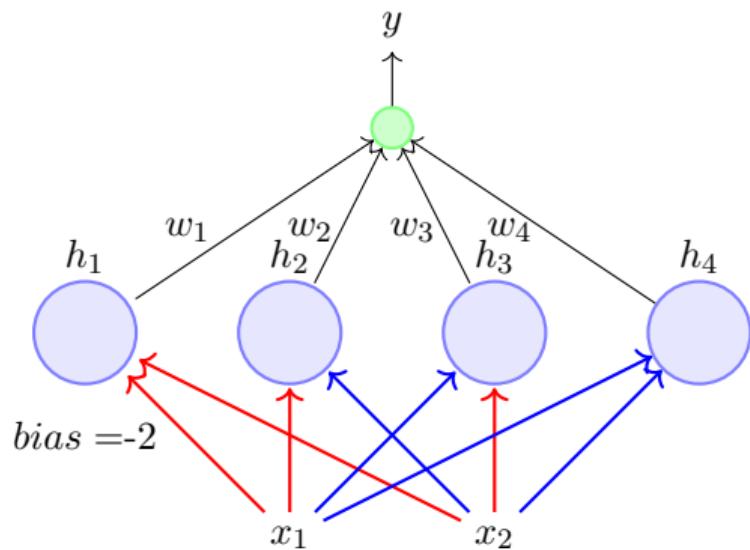
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- Astonishing claim!



red edge indicates $w = -1$

blue edge indicates $w = +1$

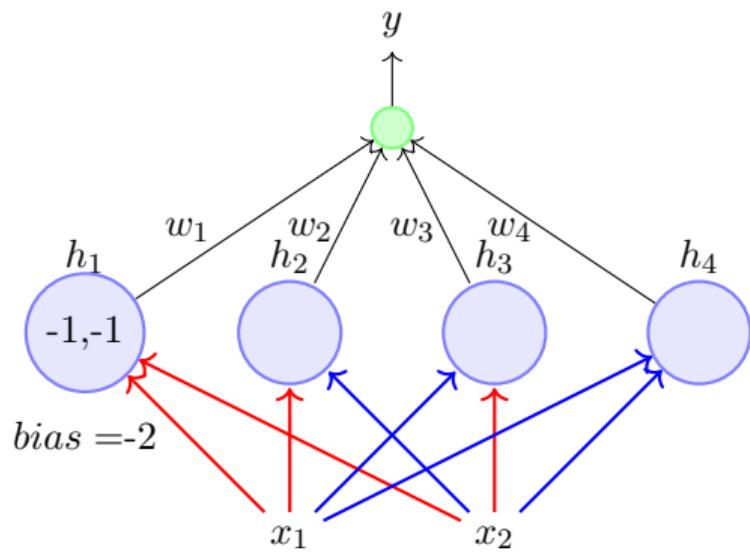
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- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on



red edge indicates $w = -1$

blue edge indicates $w = +1$

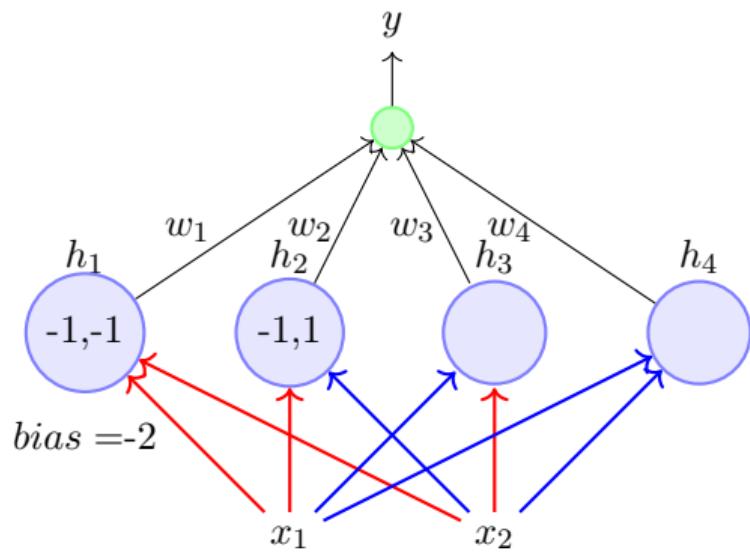
- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !
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- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)



red edge indicates $w = -1$

blue edge indicates $w = +1$

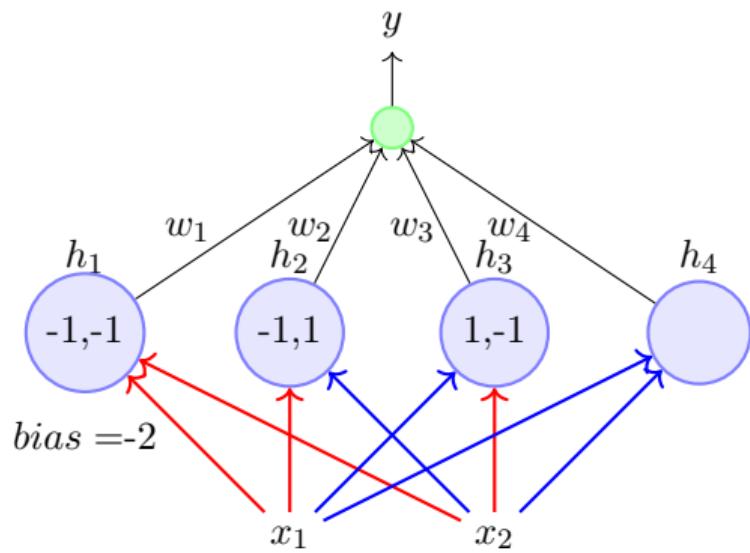
- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !
- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the first perceptron fires for $\{-1,-1\}$



red edge indicates $w = -1$

blue edge indicates $w = +1$

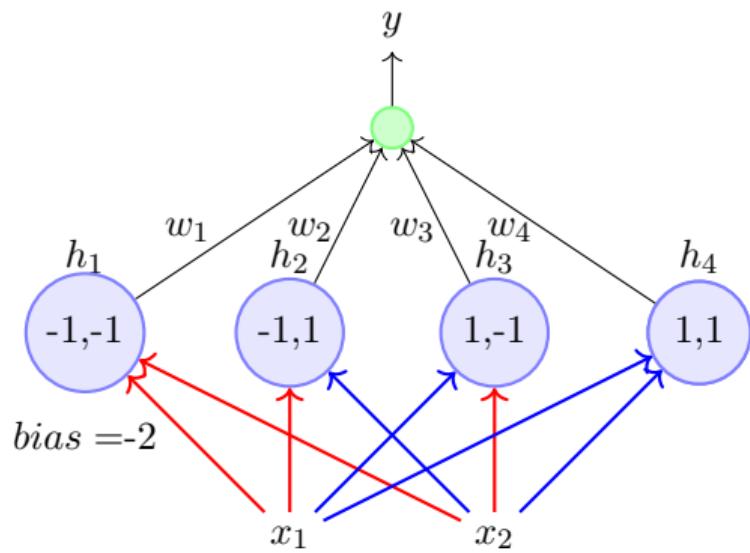
- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !
- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the second perceptron fires for $\{-1, 1\}$



red edge indicates $w = -1$

blue edge indicates $w = +1$

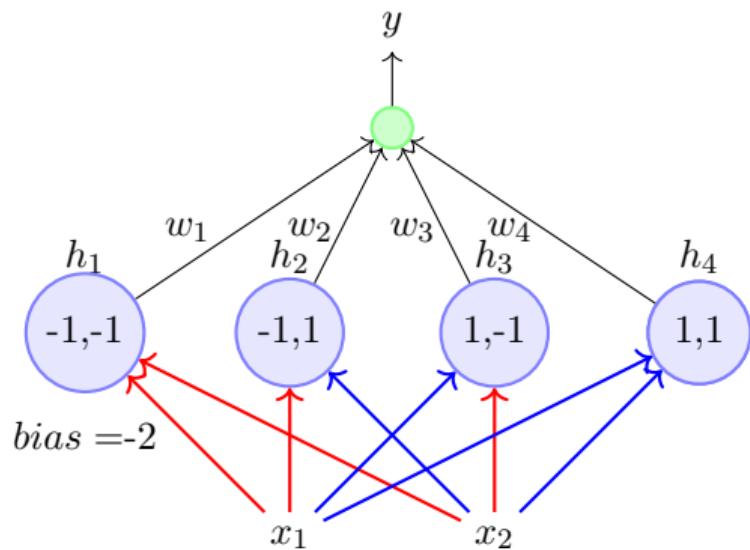
- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !
- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the third perceptron fires for $\{1,-1\}$



red edge indicates $w = -1$

blue edge indicates $w = +1$

- We claim that this network can be used to implement **any** boolean function (linearly separable or not) !
- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- the fourth perceptron fires for $\{1,1\}$

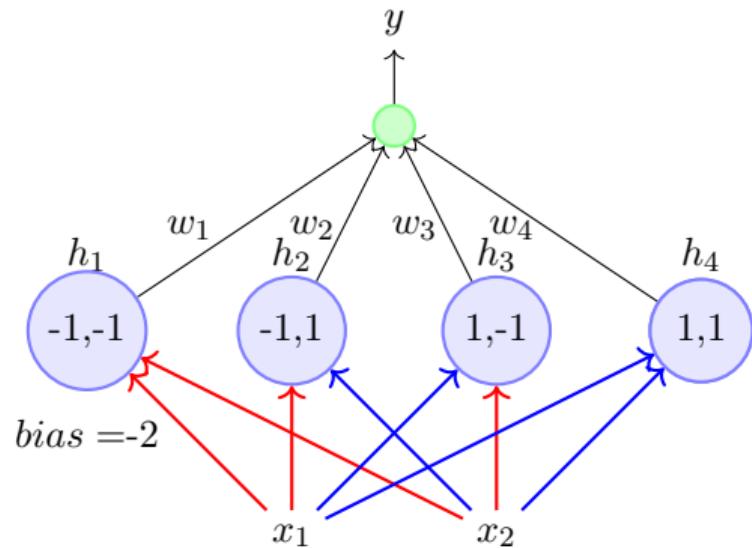


red edge indicates $w = -1$

blue edge indicates $w = +1$

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- In other words, we can find w_1, w_2, w_3, w_4 such that the truth table of any boolean function can be represented by this network
- Astonishing claim! Well, not really, if you understand what is going on
- Each perceptron in the middle layer fires only for a specific input (and no two perceptrons fire for the same input)
- Let us see why this network works by taking an example of the XOR function

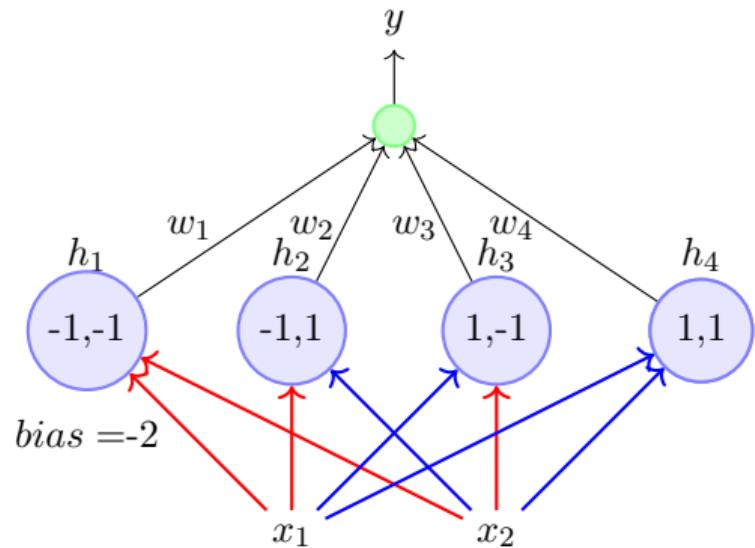
- Let w_0 be the bias output of the neuron (i.e., it will fire if $\sum_{i=1}^4 w_i h_i \geq w_0$)



red edge indicates $w = -1$

blue edge indicates $w = +1$

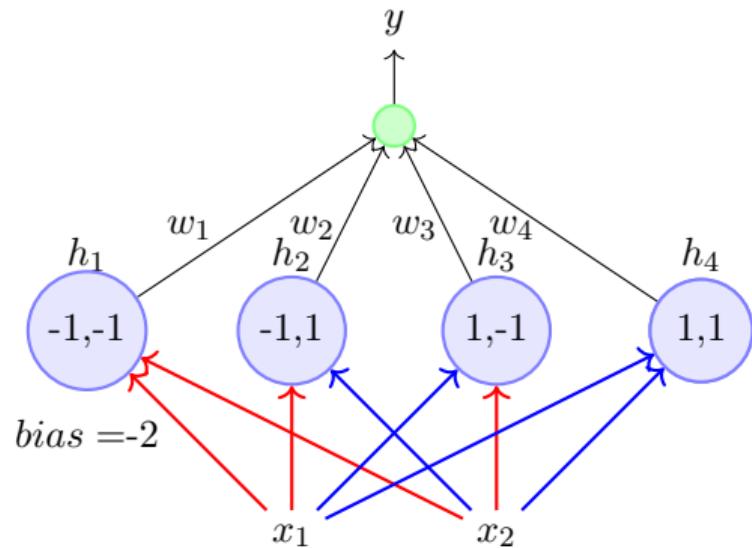
- Let w_0 be the bias output of the neuron (i.e., it will fire if $\sum_{i=1}^4 w_i h_i \geq w_0$)



red edge indicates $w = -1$
blue edge indicates $w = +1$

x_1	x_2	XOR	h_1	h_2	h_3	h_4	$\sum_{i=1}^4 w_i h_i$
0	0	0	1	0	0	0	w_1

- Let w_0 be the bias output of the neuron (i.e., it will fire if $\sum_{i=1}^4 w_i h_i \geq w_0$)

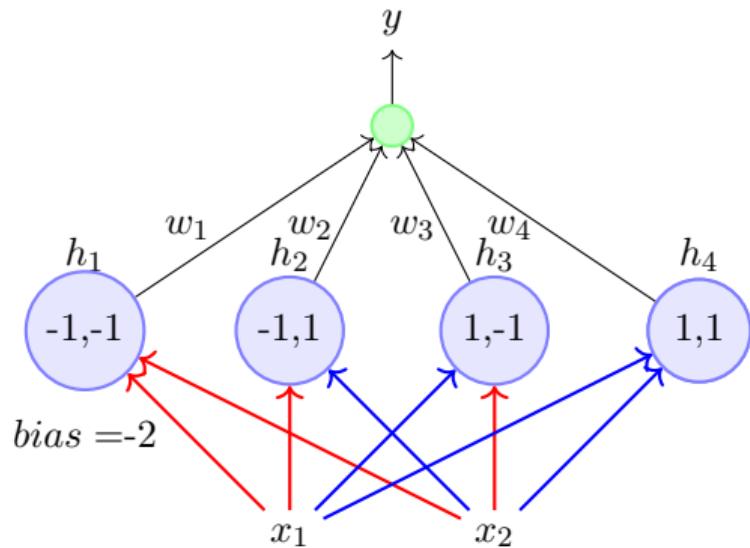


red edge indicates $w = -1$

blue edge indicates $w = +1$

x_1	x_2	XOR	h_1	h_2	h_3	h_4	$\sum_{i=1}^4 w_i h_i$
0	0	0	1	0	0	0	w_1
0	1	1	0	1	0	0	w_2

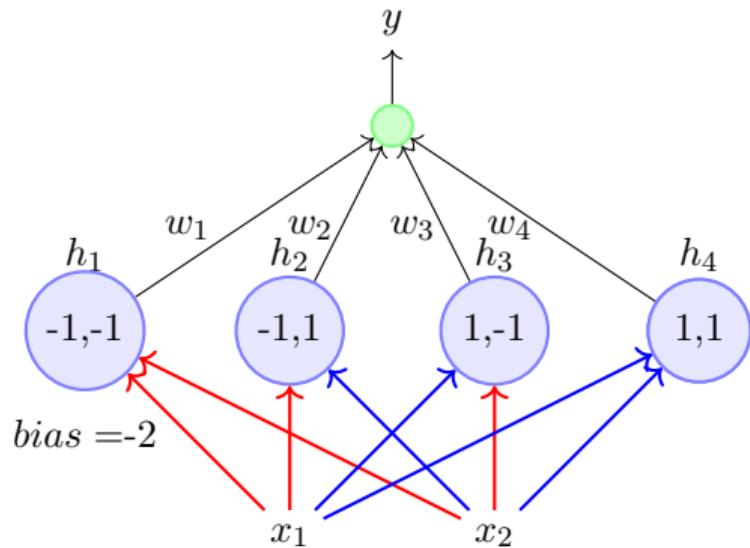
- Let w_0 be the bias output of the neuron (i.e., it will fire if $\sum_{i=1}^4 w_i h_i \geq w_0$)



red edge indicates $w = -1$
 blue edge indicates $w = +1$

x_1	x_2	XOR	h_1	h_2	h_3	h_4	$\sum_{i=1}^4 w_i h_i$
0	0	0	1	0	0	0	w_1
0	1	1	0	1	0	0	w_2
1	0	1	0	0	1	0	w_3

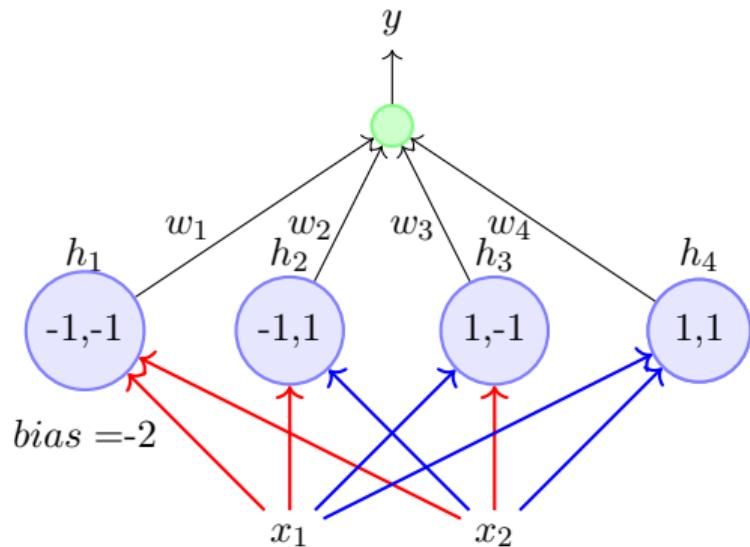
- Let w_0 be the bias output of the neuron (i.e., it will fire if $\sum_{i=1}^4 w_i h_i \geq w_0$)



red edge indicates $w = -1$
 blue edge indicates $w = +1$

x_1	x_2	XOR	h_1	h_2	h_3	h_4	$\sum_{i=1}^4 w_i h_i$
0	0	0	1	0	0	0	w_1
0	1	1	0	1	0	0	w_2
1	0	1	0	0	1	0	w_3
1	1	0	0	0	0	1	w_4

- Let w_0 be the bias output of the neuron (i.e., it will fire if $\sum_{i=1}^4 w_i h_i \geq w_0$)

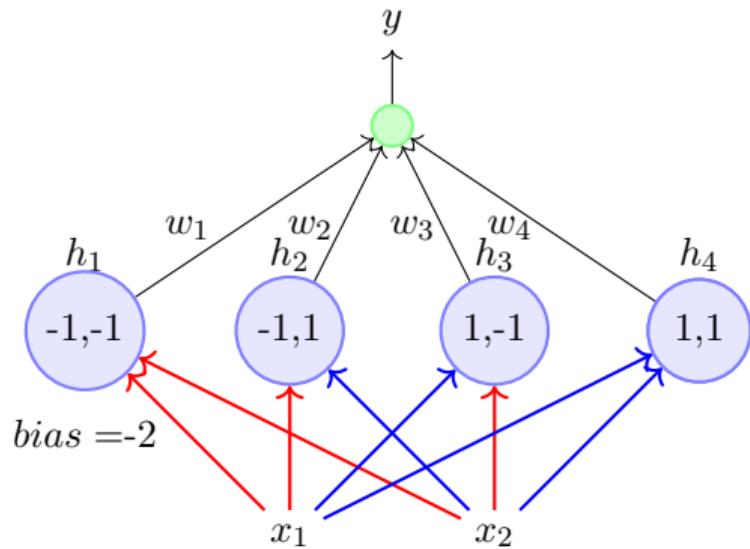


red edge indicates $w = -1$
 blue edge indicates $w = +1$

x_1	x_2	XOR	h_1	h_2	h_3	h_4	$\sum_{i=1}^4 w_i h_i$
0	0	0	1	0	0	0	w_1
0	1	1	0	1	0	0	w_2
1	0	1	0	0	1	0	w_3
1	1	0	0	0	0	1	w_4

- This results in the following four conditions to implement XOR: $w_1 < w_0, w_2 \geq w_0, w_3 \geq w_0, w_4 < w_0$

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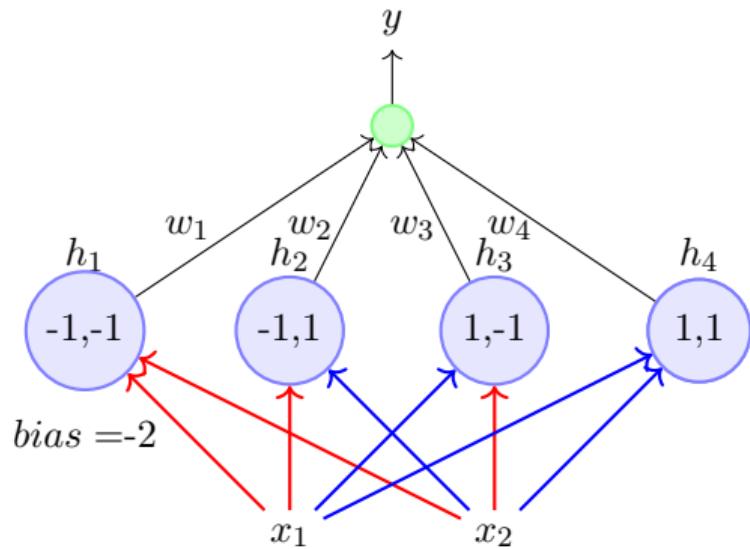


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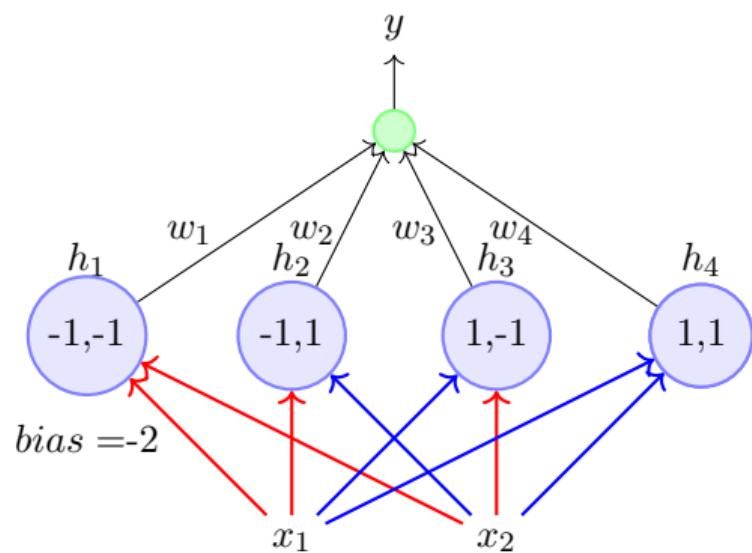


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- Unlike before, there are no contradictions now and the system of inequalities can be satisfied
- Essentially each w_i is now responsible for one of the 4 possible inputs and can be adjusted to get the desired output for that input

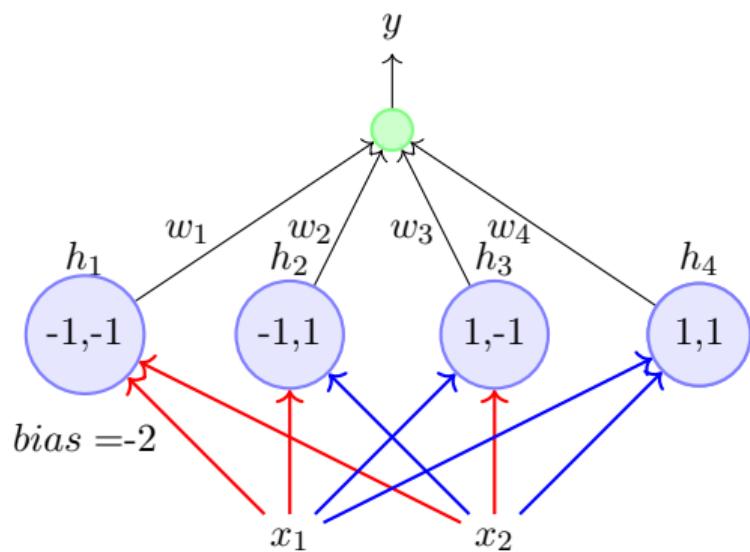
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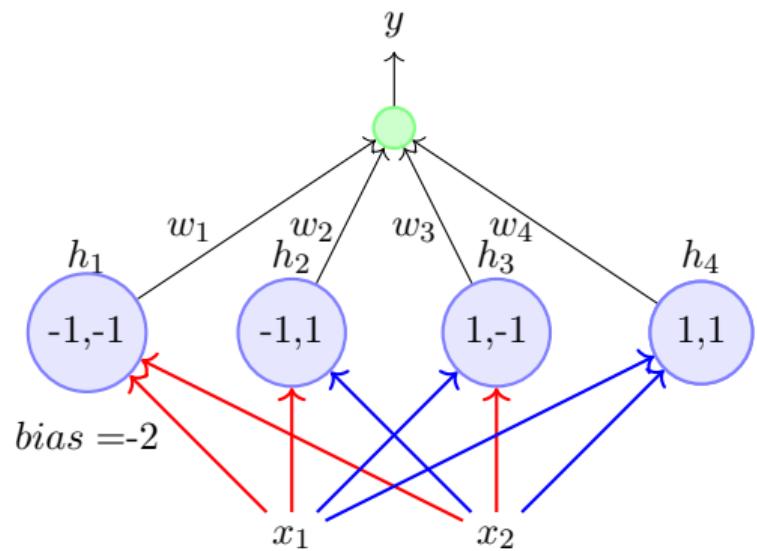
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- It should be clear that the same network can be used to represent the remaining 15 boolean functions also
- Each boolean function will result in a different set of non-contradicting inequalities which can be satisfied by appropriately setting w_1, w_2, w_3, w_4



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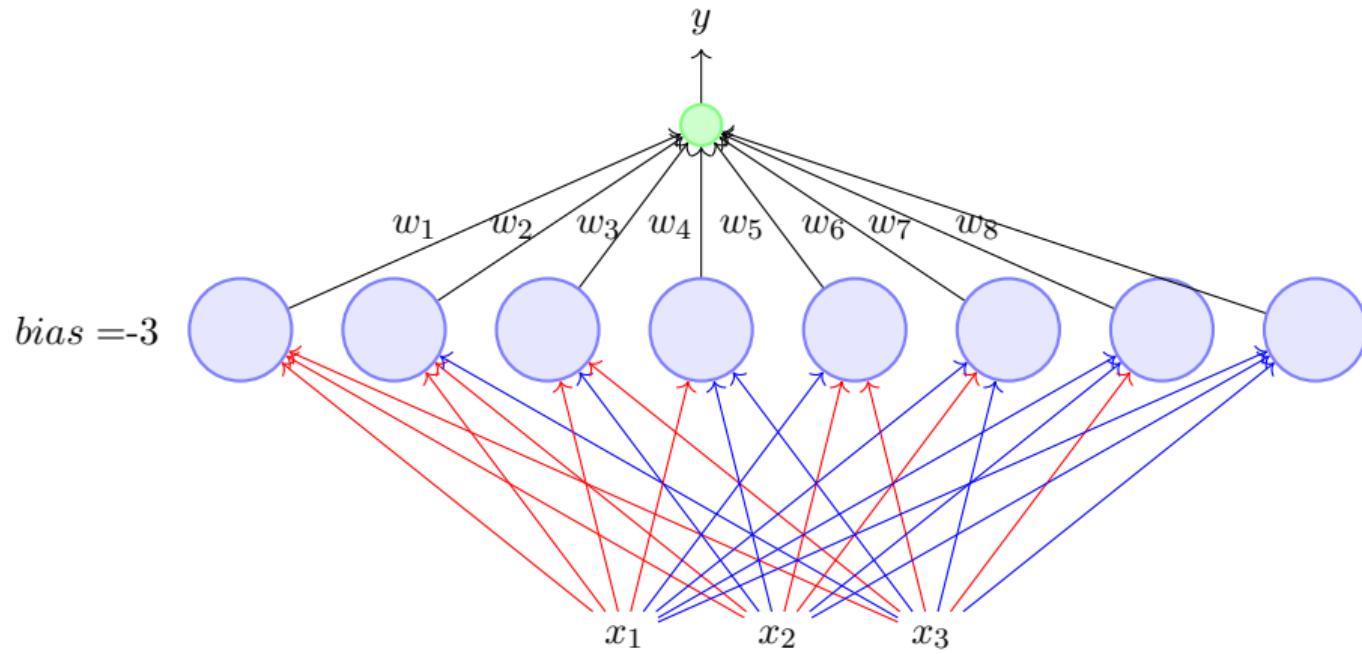
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- Each boolean function will result in a different set of non-contradicting inequalities which can be satisfied by appropriately setting w_1, w_2, w_3, w_4
- Try it!

- What if we have more than 3 inputs ?

- Again each of the 8 perceptrons will fire only for one of the 8 inputs
- Each of the 8 weights in the second layer is responsible for one of the 8 inputs and can be adjusted to produce the desired output for that input



- What if we have n inputs ?

Theorem

Any boolean function of n inputs can be represented exactly by a network of perceptrons containing 1 hidden layer with 2^n perceptrons and one output layer containing 1 perceptron

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Catch: As n increases the number of perceptrons in the hidden layers obviously increases exponentially

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- How does this help us with our original problem: which was to predict whether we like a movie or not? Let us see!

- We are given this data about our past movie experience

$$\begin{array}{ll}
 p_1 & \left[\begin{array}{ccccc} x_{11} & x_{12} & \dots & x_{1n} & y_1 = 1 \end{array} \right] \\
 p_2 & \left[\begin{array}{ccccc} x_{21} & x_{22} & \dots & x_{2n} & y_2 = 1 \end{array} \right] \\
 \vdots & \left[\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right] \\
 n_1 & \left[\begin{array}{ccccc} x_{k1} & x_{k2} & \dots & x_{kn} & y_i = 0 \end{array} \right] \\
 n_2 & \left[\begin{array}{ccccc} x_{j1} & x_{j2} & \dots & x_{jn} & y_j = 0 \end{array} \right] \\
 \vdots & \left[\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right]
 \end{array}$$

- We are given this data about our past movie experience
- For each movie, we are given the values of the various factors (x_1, x_2, \dots, x_n) that we base our decision on and we are also given the value of y (like/dislike)

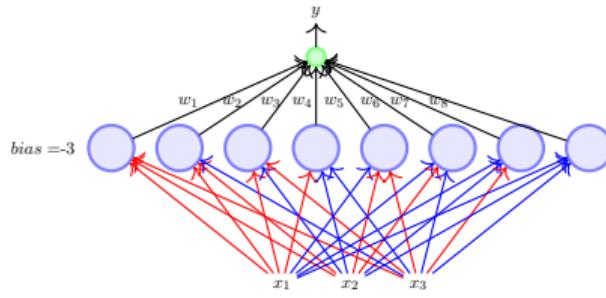
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- p_i 's are the points for which the output was 1 and n_i 's are the points for which it was 0
- The data may or may not be linearly separable
- The proof that we just saw tells us that it is possible to have a network of perceptrons and learn the weights in this network such that for any given p_i or n_j the output of the network will be the same as y_i or y_j (i.e., we can separate the positive and the negative points)

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- Networks of the form that we just saw (containing, an input, output and one or more hidden layers) are called Multilayer Perceptrons (MLP, in short)

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- More appropriate terminology would be “Multilayered Network of Perceptrons” but MLP is the more commonly used name
- The theorem that we just saw gives us the representation power of a MLP with a single hidden layer
- Specifically, it tells us that a MLP with a single hidden layer can represent **any** boolean function