Pigeon hole principle.

If n pigeon holes are occupied by $n+1$ or more pigeons then atleast one pigeon hole is occupied by more than one pigeon.

Generalized one:

If n pigeon holes are occupied by $k \cdot n+1$ or more pigeons then atleast one pigeon hole is occupied by $k+1$ or more pigeons.

Regular Languages

⇒ All finite languages are Regular.

If the no. of strings in a language is finite then we can have a regular expression by making a union of all the strings & an equivalent DFA will be there.

⇒ If a language is infinite, it can be regular or non regular. Myhill - Nerode theorem can be used to say whether a language is regular, & if regular how many states are required in the minimal DFA to recognize the language.

⇒ Pumping lemma for regular languages can be used to prove that a language is not regular. But reverse is not possible.

Lemma:- For any regular language L there exist an integer n such that for all $x \in L$ & $|x| \geq n$, there exist $u, v, w \in \Sigma^*$ such that

1. $x = uvw$
2. $|uv| \leq n$
3. $|v| \geq 1$
4. $\forall i \geq 0 \; uv^iw \in L$

Furthermore n is no greater than the no. of states of the smallest finite automata accepting L.

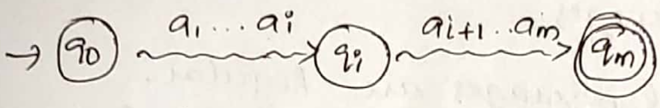
Explanation

If a language is regular, it's accepted by a DFA. $M = (Q, \Sigma, \delta, q_0, F)$ with $Q = \{Q_1 \ldots Q_n\}$ n states

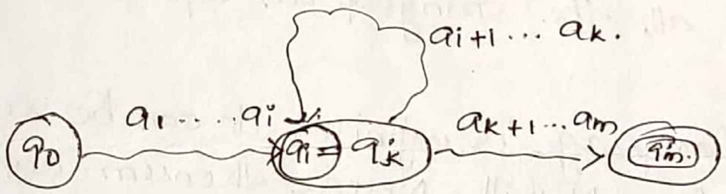
Consider an input of n or more symbols $\in L$.

$a_1 a_2 \ldots a_m \quad m \geq n$.

then. DFA starts transition at q_0. after reading a_1 it enters q_1 ~~then~~ ... after a_i it enters q_i & after a_m it enters q_m & string will get accepted

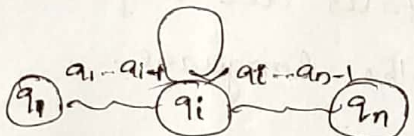


As DFA has n states & $m \geq n$. by pigeon hole principle some q_is & q_js will be equal.



Cond 1: Strings considered should be of length $\geq n$. (no. of states)

Consider:-



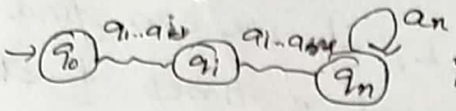
If the string's not entering loop then nothing will be there to be pumped. & ~~if~~ for that length of the string should be $n-1$.

by condition 1: $|x| \geq n$, this condition is avoided.

Cond 2: $|v| \geq 1$:- Atleast one ~~string~~ element should be there in string ~~loop~~ to be pumped./looped.

~~String~~ that string can occur anywhere. starting from q_1 to q_n. / in between $a_1 \ldots a_n$ of ~~am~~. $a_1 \ldots a_m$.

If ~~loop~~ string to be pumped is at last



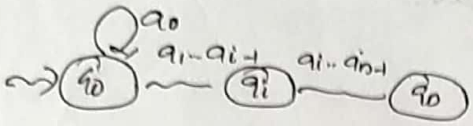
in this case $u = a_1 \dots a_{n-1}$

$v = a_n$

$w = -$

So $|uv| = n$.

If it's at q_0:



$u = -$

$v = a_0$ $\qquad |uv| = 1$

$w = a_i \dots a_{n-1}$

$|uvw| = n$.

So $|uv| \leq n$.

If it's not possible to find a v which can be pumped 0 or more times so that the resulting string is in L. then L is not <u>regular</u>.

~~Example~~:

Q ~~Prove that $a^n b^n$ is not regular using pumping lemma.~~

Pumping lemma. states that for all strings in a regular language we can find a pattern (v) such that it can be pumped 0/more times and all resulting strings will be in the language.

So if we can find a pattern like that then it's possible that the language is regular.

If no such pattern exists then the language is definitely not regular.

Corrolary of lemma is not true:

For a language L, if we can find a pattern v which can be pumped n times & still resulting strings are in the language $\not\Rightarrow$ that the language is regular:

Eg: $a^m b^{n^2}$.

Assume $a^m b^{n^2}$ is regular with pumping length 4.

Consider the string aaabbbb.

$u = aa$

$v = a$

$w = bbbb$.

$uv^iw = \{aabbbb, aaabbbb, aaaabbbb, aaaaabbbb \ldots\}$

Here all elements in uv^iw $\in a^m b^{n^2}$.

We can find a pattern here so we cant prove $a^m b^{n^2}$ is not regular.

But actually $a^m b^{n^2}$ is not regular.

Pumping Lemma.

Q1 Prove that the language $a^n b^n$ where $n \geq 0$ is regular.

Assume $a^n b^n$ is regular & we can design a dfa for it with m states.

Pumping length = m.

By pumping lemma there is a m s.t. if $z \in L \wedge |z| \geq m$

$z = uvw$ s.t. $uv^iw \in L \ \forall i \geq 0$. $|uv| \leq m$

Consider string $a^m b^m$.

$\underbrace{aaa\ldots}_{u}\ \underbrace{|a\ldots|}_{v}\ \underbrace{abbbb\ldots}_{w}$

let $u = a^q$

$v = a^p$ where $p + q + r = m$.

$w = a^r b^m$

$uv^iw = \{ a^q a^r b^m,\ a^q a^p a^r b^m,\ a^q v^{2p} a^r b^m \ldots \}$.

$a^q (a^p)^i b^m \notin L$

Violating pumping lemma & so.

Our Assumption is wrong.

by Contradiction L is not regular.

Q2. $L = a^{n^2} \mid n \geq 1$.

Assume L is regular with minimal DFA having n states

$z = a^{m^2}$ $m \geq 1$.

z can be divided into uvw s.t.

$u = a^x$ $v = a^y$ $w = a^z$ s.t. $y \geq 1$. $x + y + z = m^2$.

By pumping lemma

$\forall i \geq 0$ $uv^iw \in L$.

$uv^iw = \{ a^x a^z,\ a^x a^y a^z,\ a^x a^{2y} a^z \ldots \}$.

$= \{ a^{x+z},\ a^{x+y+z},\ a^{x+2y+z},\ a^{x+3y+z} \ldots \}$.

$a^{\cancel{x+z}}$ $x + y + z = m^2$.

$\cancel{x+y}$ $m^2 - y$ is not always a perfect square.

So our Assumption is wrong.

OR. Consider $z = aaaa : a^{2^2}$

$u = a$

$v = aa$

$w = a$.

$uv^iw : \{aa, aaaa, aaaaaa, aaaaaaaa \ldots\}$

$= \{a^2, a^4, a^6, a^8 \ldots\}$.

a^6 & a^8 is not of the form a^{n^2} & our Assumption is wrong.

So By Contradiction a^{n^2} is not regular

Q3. Prove that $1^P \mid P$ is prime is not regular.

Assume : L is regular with n state DFA.

By PL : $x = uvw$ s. $|uvw| = p$.

$uv^iw \in L \quad \forall i \geq 0$.

But if $i = p+1$.

$|uv^iw| = |uvw| + |v|^p$

$= p + p\cancel{q}v$

$p(1+\cancel{q})v$ which is not prime.

So Assumption is wrong.

By Contradiction uv^iw is not regular.

OR

Let $z = aaa$.

$u = a$

$v = a$ $\quad uv^iw : \{aa, aaa, aaaa \ldots\}$

$w = a$.

where some of them $\notin L$.

& assumption is wrong

So L is not regular.

Q4. Prove ww^R is not regular using pumping lemma.

Assume ww^R is regular & minimal DFA having n states (pumping length = n).

Consider string $0^n110^n \in L$. $|0^n110^n| = 2n+2 \geq n$.

let $u = 0^{n-l}$

$v = 0^l$ $l \geq 1$.

$|uv| \leq n$ ✓

$|v| \geq 1$. ✓

So according to pumping lemma

$\forall i \geq 0$ $uv^iw \in L$.

$uv^iw = \{ \cancel{00}, 0^{n-l}110^n, 0^n110^n, 0^{n+l}110^n \ldots \}$

$\notin L$.

So our assumption is wrong.

By contradiction L is not regular.

~~Q5~~

ww^R is not regular

But wxw^R $x \in \Sigma^+$ is regular.

$\equiv$ All strings starting & ending with same symbol.

xww^Ry. $x, y \in \Sigma^*$ is also regular

$\equiv$ All strings having 00/11 as substring.

$\left.\begin{array}{l} xww^R \\ ww^R \end{array}\right\}$ & are not regular.

wxw^R with a restriction on x like $|x| \leq 5$ is not regular.

xww^Ry with restriction on x/y is also not regular.

Q5. $0^x 1^y$ ~~x~~ $x \leq y$

Assume $0^x 1^y$ is regular with DFA having p states.

Consider string $0^p 1^{p+1} \in L$

$u = 0^{p-1}$

$v = 0$ $\quad |uv| \leq p$ ✓

$w = 1^{p+1}$ $\quad |v| \geq 1$ ✓

According to pumping lemma. $\forall i \geq 0$ $uv^iw \in L$.

$uv^iw = \{0^{p-1}1^{p+1} \quad 0^p 1^{p+1} \quad 0^{p+1}1^{p+1} \quad 0^{p+2}1^{p+1} \ldots\}$

When $0^{p+2}1^{p+1} \quad 0^{p+3}1^{p+1} \ldots \notin L$ so our Assumption is wrong. ∴

By contradiction L is not regular.

Q6. $L = \{a^n b^j : n \leq j^2\}$

Assume $a^n b^j$ $n \leq j^2$ is regular with minimal DFA having p states.

Consider the string $a^p b^{\sqrt{p}} \in L$ $\quad |a^p b^{\sqrt{p}}| > p$.

According to pumping lemma there exist a u, v, w such that

$|uv| \leq p$, $|v| \geq 1$ $\forall i \geq 0$ $uv^iw \in L$.

$u = a^{p-1}$

$v = a^p$

$w = b^{\sqrt{p}}$

$uv^iw = \{a^{p-1}b^{\sqrt{p}}, a^p b^{\sqrt{p}}, a^{p+1}b^{\sqrt{p}}, \ldots\}$

~~$a^{p+1} b^{\sqrt{p}} \notin L$ as $p+1 \nleq (\sqrt{p})$~~

at $i = p^2$ string is

$a^{p-1} a^{p^2} b^{\sqrt{p}}$

$= a^{p^2+p-1} b^{\sqrt{p}}$

$p^2+p+1 \nleq p^2$ $\notin L$ so

L is not regular.

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