Permutations & Combinations

Learning Objectives

- ✓ Fundamental Principles Counting.
- ✓ Arrangement - Linear & Geometrical.
- ✓ Selection, Grouping and Distribution.

Two Fundamental Principles of Counting

Event A $\rightarrow$ can occur in m ways.

Event B $\rightarrow$ can occur in n ways.

Sum Rule – (A or B) $\rightarrow$ can occur in $m + n$ ways.

Product Rule – A and B $\rightarrow$ can occur in $m \times n$ ways.

Definition

Permutation - Arrangement in a definite order of number of objects taken some or all at a time. Order of things is very important.

E.g.- Passwords, Arranging people, Digits, Alphabets, Colours etc.

Combination – A selection that can be formed by taking some or all a finite set of things or objects. Order is not important.

E.g.- Selection of Teams, Subjects, Menu, Items etc.

Basic Formulae

${}^{n}P_{r} \rightarrow$ Number of arrangements of 'n' different objects taken 'r' at a time

$${}^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

${}^{n}C_{r} \rightarrow$ Number of selections of 'n' different objects taken 'r' at a time.

$${}^{n}C_{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!} = \frac{{}^{n}P_{r}}{r!} = {}^{n}C_{n-r}.$$

Permutations under constraints

1. The total number of arrangements of 'n' things taken 'r' at a time, in which a particular thing always occurs = ${}^{n-1}P_{r-1}$

2. The total number of arrangements of 'n' things taken 'r' at a time, in which a particular thing never occurs = ${}^{n-1}P_{r}$

3. The number of permutations of 'n' different things taking 'r' at a time when each thing may be repeated any number of times in any permutations is given by $n \times n \times n \ \ldots . n, r \ times. \ i.e. n^r$ ways.

Permutations under constraints

4. The number of arrangements when things are not all different such as arrangement of 'n' things, when 'p' of them are of one kind, 'q' of another kind, 'r' is still of another kind and so on, the total number of permutations is given by $\dfrac{n!}{(p!\,q!\,r\,!.......)}$.

Combinations under constraints

1. Number of combinations of n different things taken 'r' at a time in which 'p' particular things will always occur is ${}^{n-p}C_{r-p}$

2. Number of combinations of n dissimilar things taken 'r' at a time in which 'p' particular things will never occur is ${}^{n-p}C_r$

3. The number of ways in which (m + n) things can be divided into two groups containing m & n things respectively ${}^{m+n}C_m = {}^{m+n}C_n.$

Combinations under constraints

4. The total number of ways of dividing n identical items among r persons, each of whom can receive 0, 1, 2, or more items ($\leq$ n) is $^{n+r-1}C_{r-1}$

5. Number of diagonals in n sided polygon= $^{n}C_2 - n$

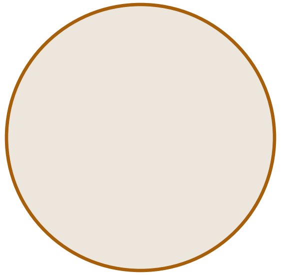
6. Sum of all possible combinations of n distinct things $= 2^n$ i.e. $^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_n$

Geometrical Arrangements

Circular Arrangements

- Number of circular permutations of n things all taken at a time = (n – 1)!
- Number of circular permutations of n different things taking r at a time is = $n_{Cr} \times (r-1)! = \frac{{}^{n}P_{r}}{r}$ considering clockwise and anticlockwise as distinct different.
- If we consider clockwise and anticlockwise as same, then it is $\frac{(n-1)!}{2}$ & $\frac{{}^{n}P_{r}}{2r}$ respectively.

  E.g. Necklace or garland etc.



Geometrical Arrangements

Arrangement around a regular polygon

- If N people are to arranged around a Ksided regular polygon, such that each side of that polygon contains same number of people, then the number of arrangements will be $\frac{N}{K} \times (N-1)! = \frac{N!}{K}$
- If the polygon is not regular, then it is $N!$
- For rectangular table, it is $\frac{N!}{2}$, 2 signifies the degree of symmetry of a rectangle.

Example 1

How many four-digit numbers can be formed from the digits 2, 3, 5, 6, 7 & 9 which are divisible by 4 and none of the digits is repeated?

A. 96 B. 88 C. 80 D. 144

Solution: Option A

The last two digits of the number must be 32, 36, 52, 56, 72, 76, 92, 96.

_ _ _ _

$4 \times 3 \times 8 = 96.$

Example 2

How many 4-digit numbers divisible by 5 can be formed with the digits 0, 1, 2, 3, 4, 5, 6 and 6?

(Repetition not allowed, means only 6 can be used 2 times).

A. 220 B. 249 C. 216 D. 288

Solution:

÷5, 4-digit number _ _ _ _ using 0, 1, 2, 3, 4, 5, 6 & 6.

Case I – 0/1 six

_ _ _ <u>0</u> ($6 \times 5 \times 4 \times 1 = 120$)

_ _ _ <u>5</u> ($5 \times 5 \times 4 \times 1 = 100$).

So 220 numbers are there with either 1 or 0 six.

Example 2 Cont.

Case II – 2 sixes

_ _ _ _0 (5C1×3)

_ _ _ _5 (5C1×3 – 1).

So 15+14 = 29 numbers with 2 sixes.

Total 220 + 29 = 249.

Option B.

Example 3

In how many ways can the letters of the word "OPTICAL" be rearranged such that vowels are always together?

A. 120 B. 720 C. 2140 D. 5360

Solution: Option B.

Consider the vowels $\boxed{\text{OIA}}$ as a single entity.

So arranging PTCL + $\boxed{\text{OIA}}$ $= (4+1)! \times 3! = 720$.

Example 4

In how many ways can the letters of the word "OPTICAL" be arranged such that vowels are never together?

A. 720 B. 7! – 720 C. 2140 D. 1440

Solution: Option D

First place the consonants, and then place the vowels in between them.

_ P _ T _ C _ L _.

So PTCL can be arranged in 4! & 3 vowels in the 5 gaps in $^5P_3$ ways.

Hence $24 \times 5 \times 4 \times 3 = 1440$.

Example 5

There are 20 people among whom 2 are sisters. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the 2 sisters.

A. 18! B. $2! \times 19!$ C. 19! D. $2! \times 18!$

Solution:

Select 1 guy to sit between the 2 sisters, 18C1 ways.

Now 17 guys + $\boxed{S1\_S2}$ = 18, can be arranged in a circular table in (18-1)! = 17! ways.

Also the sisters can exchange their positions, 2! ways.

Hence $2 \times 18 \times 17! = 2 \times 18!$ ways.

Option D.

Example 6

If you jumble and arrange the word LABOUR in all possible ways and arrange all the words so formed as in a dictionary. What will be the rank of the word LABOUR?

Solution:

The alphabetical order is A B L O R U.

A _ _ _ _ _ = 5! = 120,

B _ _ _ _ _ = 5! = 120,

L A B O R U

L A B O U R.

So the rank is 120 + 120 + 1 + 1 = 242.

Example 7
Distribution problem

In how many ways can 5 Chocolates be distributed among 3 Kids such that

i. The Chocolates are different.
ii. The Chocolates are identical.
iii. The Chocolates are different, but given to triplets (identical children)
iv. The Chocolates are identical, and given to triplets (identical children)

(Each children is eligible to get all 5 chocolates)

Example 7 (i)

In how many ways can 5 different Chocolates be distributed among 3 Kids.

Chocolate 1 can be given to Kid_1, Kid_2 or Kid_3, 3 ways.

$C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5$

$K_1 \quad K_2 \quad K_3$

Similarly each chocolates can be distributed in 3 ways.

Hence $3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$ ways.

Example 7 (i) - Aliter

5 Different chocolates to 3 kids

$(5, 0, 0) \rightarrow 5C_5 \times 3C_1 = 1 \times 3 = 3$ ways.

$(4, 1, 0) \rightarrow 5C_4 \times 3C_1 \times 1C_1 \times 2C_1 = 5 \times 3 \times 1 \times 2 = 30$ ways.

$(3, 2, 0) \rightarrow 5C_3 \times 3C_1 \times 2C_2 \times 2C_1 = 10 \times 3 \times 1 \times 2 = 60$ ways.

$(3, 1, 1) \rightarrow 5C_3 \times 3C_1 \times \frac{2C_1 \times 2C_1}{2} = 10 \times 3 \times \frac{2 \times 2}{2} = 60$ ways.

$(2, 2, 1) \rightarrow 5C_2 \times 3C_1 \times \frac{3C_2 \times 2C_1}{2} = 10 \times 3 \times \frac{6 \times 2}{2} = 90$ ways.

Hence total 243 ways.

Example 7(ii)

In how many ways can 5 identical Chocolates be distributed among 3 Kids.

C C C C C $\longrightarrow$ K_1 K_2 K_3

Suppose all 5 chocolates for K_3, and nil for K_1, K_2 can be expressed as | | C C C C C

Similarly 1 for K_1, and 1 for K_2, 3 for K_3 can be expressed as C | C | C C C

Likewise, distribution for "2 1 2" is as C C | C | C C

So, it is simply arranging 7 items, in which 5 are one kind and 2 are another, in

$\frac{7!}{5!2!} = 21$ ways ($^7C_2$).

Example 7(ii) - Aliter

In how many ways can 5 identical Chocolates be distributed among 3 Kids.

$(5, 0, 0) \rightarrow \frac{3!}{2!} = 3$ ways.

$(4, 1, 0) \rightarrow 3! = 6$ ways.

$(3, 2, 0) \rightarrow 3! = 6$ ways.

$(3, 1, 1) \rightarrow \rightarrow \frac{3!}{2!} = 3$ ways.

$(2, 2, 1) \rightarrow \rightarrow \frac{3!}{2!} = 3$ ways.

Hence total 21 ways

Example 7(iii)

5 Different chocolates to Triplets

$(5, 0, 0) \rightarrow 5C_5 \times 1 = 1$ way.

$(4, 1, 0) \rightarrow 5C_4 \times 1C_1 = 5$ ways.

$(3, 2, 0) \rightarrow 5C_3 \times 2C_2 = 10$ ways.

$(3, 1, 1) \rightarrow 5C_3 \times \frac{2C_1}{2} = 10$ ways.

$(2, 2, 1) \rightarrow 5C_2 \times \frac{3C_2}{2} = 15$ ways.

Hence total 41 ways.

Example 8

The number of non-negative integral solutions of the equation

$a + b + c + d = 20$ is

A. 1208 B. 4024 C. 1140 D. 1771

Solution:

The total number of ways of dividing n identical items among r persons, each of whom can receive 0, 1, 2, or more items ($\leq$ n) is ${}^{n+r-1}C_{r-1}$.

Here $n = 20$, $r = 4$.

${}^{20+4-1}C_{4-1} = {}^{23}C_3 = 1771$.

Option D.

Example 9-Envelope problem

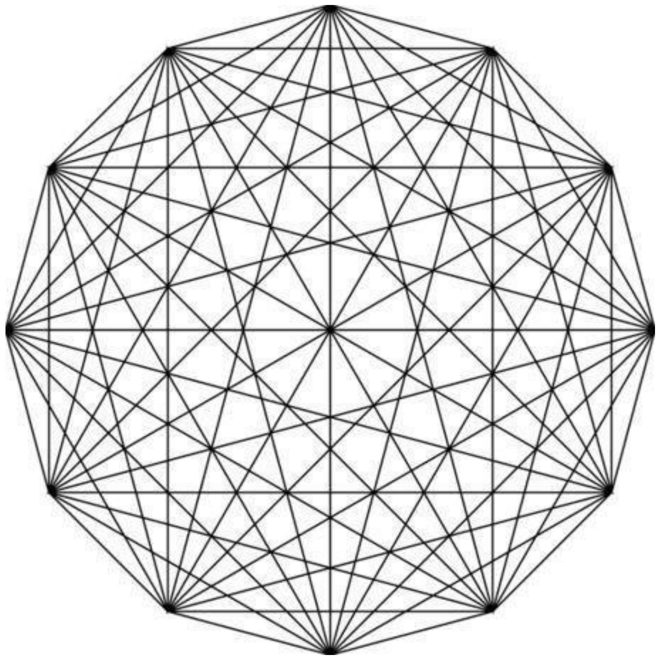
There are 4 different letters and 4 addressed envelopes. In how many ways can the letters be put in the envelopes so that at least one letter goes to the correct address?

Solution: Total – No letter goes to right envelope = at least one letter goes right envelope

✓	×	**Ways**
4	0	1
3	1	0
2	2	6
1	3	8
0	4	9.

Hence 24 – 9 = 15

Example 10-Diagonal problem



A polygon has 54 diagonals. Find the number of sides.

A. 10 B. 14 C. 12 D. 9

Solution:

Number of diagonals in n sided polygon= ${}^{n}C_{2} - n.$

${}^{n}C_{2} - n = 54$

Solve for n,

$n^2 - 3n - 108 = 0,$

$(n - 12)(n + 9) = 0$

n = 12.

Thank you

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