$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \left[e^{-|x|}\right]^2 dx = 2\int_0^{\infty} \left[e^{-|x|}\right]^2 dx$$

$$= 2\int_0^{\infty} \left[e^{-x}\right]^2 dx \qquad [\because |x| = x \text{ in } (0, \infty)]$$

$$= 2\int_0^{\infty} e^{-2x} dx = 2\left[\frac{e^{-2x}}{-2}\right]_0^{\infty}$$

$$= -\left[e^{-2x}\right]_0^{\infty} = -[(0)-(1)] = 1 \qquad \dots (2)$$

$$|F(s)|^2 = \frac{2}{\pi}\frac{1}{(1+s^2)^2}$$

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+s^2)^2} ds = \frac{4}{\pi}\int_0^{\infty}\frac{1}{(1+s^2)^2} ds \qquad \dots (3)$$

$$(1) \Rightarrow \frac{4}{\pi}\int_0^{\infty}\frac{1}{(1+s^2)^2} ds = 1 \qquad \text{by (2) \& (3)}$$

$$\int_0^{\infty}\frac{1}{(1+s^2)^2} ds = \frac{\pi}{4}$$

$$\int_0^{\infty}\frac{1}{(1+x^2)^2} dx = \frac{\pi}{4} \qquad [\because s \text{ is a dummy variable}]$$

Example 4.2.c(8) : **If** $f(x) = \begin{cases} \cos x, & |x| < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$**, then find the Fourier transform of f (x) and hence, evaluate** $\int_0^{\infty}\frac{\cos^2 \pi x/2}{(1-x^2)^2} dx$ **using Parseval's identity.**

Solution : We know that,

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx} dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \cos x\, e^{isx} dx \quad \left[\because f(x) = \begin{cases} \cos x, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0, & otherwise \end{cases}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \cos x\,[\cos sx + i\sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \cos sx \cos x\, dx + \frac{i}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \sin sx \cos x\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^{\pi/2} \cos sx \cos x\, dx + \frac{i}{\sqrt{2\pi}}(0)$$

Since, $\cos sx \cos x$ is an even function in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\sin sx \cos x$ is an odd function in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$= \frac{2}{\sqrt{2\pi}}\int_0^{\pi/2}\left[\frac{\cos(s+1)x + \cos(s-1)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\pi/2}[\cos(s+1)x + \cos(s-1)x]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\sin(s+1)\pi/2}{s+1} + \frac{\sin(s-1)\pi/2}{s-1}\right) - (0+0)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin\left(s\frac{\pi}{2}+\frac{\pi}{2}\right)}{s+1} + \frac{\sin\left(\frac{s\pi}{2}-\frac{\pi}{2}\right)}{s-1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{s\pi}{2}\right)}{s+1} - \frac{\sin\left(\frac{\pi}{2}-\frac{s\pi}{2}\right)}{s-1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\cos\frac{s\pi}{2}}{s+1} - \frac{\cos\frac{s\pi}{2}}{s-1}\right] = \frac{\cos\frac{s\pi}{2}}{\sqrt{2\pi}}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{\cos\frac{s\pi}{2}}{\sqrt{2}\,\pi}\left[\frac{s-1-s-1}{s^2-1}\right]$$

$$= \frac{\cos\frac{s\pi}{2}}{\sqrt{2}\,\pi}\left[\frac{-2}{s^2-1}\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos\frac{s\pi}{2}}{1-s^2}\right]$$

By Parseval's identity,

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx \qquad \text{... (1)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = 2\int_0^{\pi/2} \cos^2 x\, dx$$

$$= 2\int_0^{\pi/2} \frac{1+\cos 2x}{2}\, dx = \int_0^{\pi/2} (1+\cos 2x)\, dx$$

$$= \left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \left(\frac{\pi}{2}+0\right) - (0+0)$$

$$= \frac{\pi}{2} \qquad \text{... (2)}$$

$$|F(s)|^2 = \frac{2}{\pi}\,\frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2} \qquad \text{... (3)}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{2}{\pi}\,\frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2}\, ds = \frac{\pi}{2} \quad \text{by (2) \& (3)}$$

$$\Rightarrow \frac{4}{\pi}\int_0^{\infty} \frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2}\, ds = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos^2\frac{s\pi}{2}}{(1-s^2)^2}\, ds = \frac{\pi^2}{8}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos^2\frac{\pi x}{2}}{(1-x^2)^2}\, dx = \frac{\pi^2}{8} \qquad [\because s \text{ is a dummy variable}]$$

Example 4.2.c(9) : Verify convolution theorem for $f(x) = g(x) = e^{-x^2}$.

[A.U M/J 2013] [A.U N/D 2015 R-13]

> **Definition : Convolution theorem for Fourier transforms.** The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.
>
> $$F[f(x) * g(x)] = F\{f(x)\}\; F\{g(x)\}$$

Given : $f(x) = g(x) = e^{-x^2}$

R.H.S. $= F[f(x)]\; F[g(x)]$

We know that, $\boxed{F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx}$

$$F[e^{-x^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2+isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(x+\frac{is}{2}\right)^2+\frac{s^2}{4}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(x+\frac{is}{2}\right)^2} e^{-s^2/4}\, dx$$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-\left(x+\frac{is}{2}\right)^2} dx$$

put $t = x + \dfrac{is}{2}$ | $x \to -\infty \Rightarrow t \to -\infty$

$dt = dx$ | $x \to \infty \Rightarrow t \to \infty$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-t^2} dt$$

Standard integral formula $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

$$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}} e^{-s^2/4}$$

$$F[f(x)] = \frac{1}{\sqrt{2}} e^{-s^2/4}$$

$$F[g(x)] = F[f(x)] = \frac{1}{\sqrt{2}} e^{-s^2/4}$$

$$\therefore F[f(x)]\,.\,F[g(x)] = \left(\frac{1}{\sqrt{2}} e^{-s^2/4}\right)\left(\frac{1}{\sqrt{2}} e^{-s^2/4}\right)$$

$$= \frac{1}{2} e^{-s^2/2} \quad \ldots (1)$$

L.H.S :

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \quad \text{by convolution definition}$$

$$e^{-x^2} * e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2} e^{-(x-u)^2} du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-u)^2-u^2} du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left[\left(u-\frac{x}{2}\right)^2+\frac{x^2}{4}\right]} du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} e^{\frac{-x^2}{2}} du$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} du$$

put $t = u - \dfrac{x}{2}$ | $u \to -\infty \Rightarrow t \to -\infty$

$dt = du$ | $u \to \infty \Rightarrow t \to \infty$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2t^2} dt$$

Put $y = \sqrt{2}\, t$

$dy = \sqrt{2}\, dt$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2} \frac{dy}{\sqrt{2}}$$

$$= \frac{e^{-x^2/2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{e^{-x^2/2}}{2\sqrt{\pi}}\sqrt{\pi} \quad \text{by standard integral formula.}$$

$$= \frac{1}{2} e^{-x^2/2}$$

$$F[f(x) * g(x)] = F\left[\frac{1}{2}e^{-x^2/2}\right]$$

$$= \frac{1}{2} F[e^{-x^2/2}]$$

We know that, $e^{-x^2/2}$ is self reciprocal under Fourier transform.

$$\therefore \quad F[e^{-x^2/2}] = e^{-s^2/2}$$

$$\therefore F[f(x) * g(x)] = \frac{1}{2}e^{-s^2/2} \quad \text{... (2)}$$

$$\therefore (1) = (2)$$

Hence, convolution theorem is verified.

4.3 FOURIER SINE & COSINE TRANSFORMS :

4.3.a. FOURIER COSINE TRANSFORM :

The infinite Fourier cosine transform of $f(x)$ is defined by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

The inverse Fourier cosine transform $F_c[f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds$$

4.3.b INVERSION FORMULA FOR FOURIER COSINE TRANSFORM

Let $F_c(s)$ denote the F.C.T of $f(x)$. Then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

Proof : By the definition of F.C.T,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

Here, $f(x)$ is defined for all $x \geq 0$

Now define $g(x)$ by $g(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ f(-x), & \text{if } x < 0 \end{cases}$

Clearly,

$g(-x) = g(x)$ for all x and hence g is an even function.

To prove that, the Fourier transform of $g(x)$ is the F.C.T of $f(x)$.

$$F[g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) [\cos sx + i \sin sx] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos sx \, dx + \frac{1}{\sqrt{2\pi}} i \int_{-\infty}^{\infty} g(x) \sin sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty g(x) \cos sx \, dx$$

[$\because$ $g(x) \cos sx$ is an even function

$g(x) \sin sx$ is an odd function]

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos sx \, dx$$

$$= F_c[g(x)]$$

$$= F_c[f(x)] \quad [\because g(x) = f(x) \text{ for all } x \geq 0]$$

Hence, by inversion formula for F.T, we have

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\, [\cos sx - i \sin sx]\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_{0}^{\infty} F_c(s) \cos sx\, ds \quad [\because F_c(s) \text{ is an even function}]$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos sx\, ds$$

Since, $g(x) = f(x)$ for all $x \geq 0$, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos sx\, ds$$

4.3.c FOURIER SINE TRANSFORM :

The infinite Fourier sine transform of $f(x)$ is defined by

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

The inverse Fourier sine transform of $F_s[f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s[f(x)] \sin sx\, ds$$

4.3.d INVERSION FORMULA FOR FOURIER SINE TRANSFORM

Let $F_s(s)$ denote the F.S.T of $f(x)$. Then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin sx\, ds$$

Proof : By the definition of F.S.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

Here, $f(x)$ is defined for all $x \geq 0$

Now, we define $g(x)$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \geq 0 \\ -f(x), & \text{if } x < 0 \end{cases}$$

Clearly, $g(x)$ is an odd function

$$F[g(x)] = F_s[f(x)] \quad \text{[by the above 5.8)}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin sx\, ds$$

4.3.e Properties of Fourier sine transform and Fourier cosine transform

1. Linear property

(i) $F_s[af(x) + bg(x)] = a\, F_s[f(x) + b\, F_s[g(x)]$

(ii) $F_c[af(x) + bg(x)] = a\, F_c[f(x)] + b\, F_c[g(x)]$

Proof : (i) We know that,

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx}$$

$$F_s[af(x) + bg(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} [af(x) + bg(x)] \sin sx\, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx + b\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(x) \sin sx\, dx$$

$$= a\, F_s[f(x)] + b\, F_s[g(x)]$$

(ii) We know that,

$$\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx}$$

$$F_c\,[af(x)+bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty [af(x)+bg(x)]\cos sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx\,dx$$

$$= a\,F_c\,[f(x)] + b\,F_c\,[g(x)]$$

2. Modulation property :

(i) $F_s\,[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(ii) $F_s\,[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(iii) $F_c\,[f(x)\sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

(iv) $F_c\,[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

Proof : **[A.U. N/D 2008] [A.U Tvli N/D 2009]**

$$F_s\,[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\sin ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\cos(s-a)x - \cos(s+a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)\,dx\right]$$

$$= \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

(ii) $$F_s\,[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\cos ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\cos(s+a)x + \cos(s-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

(iii) $$F_c\,[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\sin(a+s)x + \sin(a-s)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a+s)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a-s)x\,dx\right]$$

$$= \frac{1}{2}[F_s(a+s) + F_s(a-s)]$$

(iv) $$F_c\,[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\cos ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\cos(s+a)x + \cos(s-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

3. $F_s[f(ax)] = \frac{1}{a} F_s\left[\frac{s}{a}\right]$. **[Change of scale property]**

[A.U N/D 2016 R-13]

Proof : $F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \sin sx \, dx$

put $ax = t$, $a\,dx = dt$ | $x \to 0 \Rightarrow t \to 0$, $x \to \infty \Rightarrow t \to \infty$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a}$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\left(\frac{s}{a} t\right) dt$$

$$= \frac{1}{a} F_s\left[\frac{s}{a}\right]$$

Similarly, $F_c[f(ax)] = \frac{1}{a} F_c\left[\frac{s}{a}\right]$

4. $F_s[f'(x)] = -s F_c(s)$, if $f(x) \to 0$ as $x \to \infty$.

[Transform of derivative]

Proof : $F_s[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \, d[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\sin sx f(x)\right)_0^\infty - s \int_0^\infty f(x) \cos sx \, dx \right]$$

$$= -s \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \quad [\text{Assuming } f(x) \to 0 \text{ as } x \to \infty]$$

$$= -s F_c(s)$$

5. $F_c[f'(x)] = -\sqrt{\frac{2}{\pi}} f(0) + s F_s(s)$ if $f(x) \to 0$ as $x \to \infty$.

[Transform of derivative]

Proof : $F_c[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \, d[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \left[[\cos sx f(x)]_0^\infty + s \int_0^\infty f(x) \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} [(0 - f(0)] + s F_s(s) \quad \text{Assuming } f(x) \to 0 \text{ as } x \to \infty$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + s F_s(s)$$

6. $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$ **[Derivatives of transform]**

[A.U CBT A/M 2011] [A.U A/M 2015 R-08]

Proof : We know that,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

Differentiating both sides w.r.to 's', we get

$$\frac{d}{ds} F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s} (\cos sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) (-x \sin sx) \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, x \sin sx\, dx$$

$$= -F_s\,[x f(x)]$$

$$\text{(i.e.,)}\ F_s\,[x f(x)] = -\frac{d}{ds} F_c\,[f(x)]$$

7. $F_c\,[x f(x)] = \frac{d}{ds} F_s(s)$

[A.U A/M 2015 R-08]

Proof : We know that,

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin s x\, dx$$

Differentiating both sides w.r.to 's', we get

$$\frac{d}{ds} F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\, \frac{d}{ds} \int_0^\infty f(x) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s} (\sin s x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\,(x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, x \cos sx\, dx$$

$$= F_c\,[x f(x)]$$

$$\text{i.e.,}\ F_c\,[x f(x)] = \frac{d}{ds} F_s\,[f(x)]$$

III(a). Problems based on Fourier Cosine Transform

Formula :

$$F_c(s) = F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

Example 4.3.a(1) : Find the Fourier cosine transform of

$$f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \geq a \end{cases}$$

Solution :

We know that, $\boxed{F_c(s) = F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos sx \cos x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{1}{2} \int_0^a [\cos(s+1)x + \cos(s-1)x]\, dx$$

$$\left[\because \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right) - (0+0)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right]$$

provided $s \neq 1$; $s \neq -1$

Example 4.3.a(2) : Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$ and hence, find $F_c\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$.

[A.U CBT A/M 2011]
[A.U N/D 2015 R-13]

Solution : We know that,

$$\boxed{F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx}$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx$$

$$\text{(i.e.,) } F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx$$

$$\frac{d}{ds} F_c(s) = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left[\frac{e^{-ax}}{x} \cos sx\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} (-x \sin sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty -e^{-ax} \sin sx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

Formula :

$$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

Here, $b = s$

$$\frac{d}{ds} F_c(s) = -\sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2}\right]$$

By integrating, we get

$$F(s) = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2 + a^2} ds = -\sqrt{\frac{2}{\pi}} \frac{1}{2} \int \frac{2s}{s^2 + a^2} ds$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \log(s^2 + a^2)$$

$$\text{(i.e.,) } F_c\left[\frac{e^{-ax}}{x}\right] = -\frac{1}{\sqrt{2\pi}} \log(s^2 + a^2)$$

$$\text{similarly, } F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}} \log(s^2 + b^2)$$

$$\text{Now, } F_c\left[\frac{e^{-ax} - e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{-1}{\sqrt{2\pi}} \log(s^2 + a^2) + \frac{1}{\sqrt{2\pi}} \log(s^2 + b^2)$$

$$= \frac{1}{\sqrt{2\pi}} [\log(s^2 + b^2) - \log(s^2 + a^2)]$$

$$= \frac{1}{\sqrt{2\pi}} \log\left[\frac{s^2 + b^2}{s^2 + a^2}\right]$$

Example 4.3.a(3): Find the Fourier cosine transform of e^{-ax}, $a > 0$.

[A.U. April 2001] [A.U N/D 2014 R-08] [A.U N/D 2015 R-8] [A.U. M/J 2016 R-13] [A.U N/D 2016 R-8]

Solution :

We know that, $\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx}$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2}\right]$$

Formula :

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$$

Example 4.3.a(4) : Find the Fourier cosine transform of the function $3e^{-5x} + 5e^{-2x}$.

Solution : Let $f(x) = 3e^{-5x} + 5e^{-2x}$

We know that, $\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$

$$F_c\,[3e^{-5x} + 5e^{-2x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty [3e^{-5x} + 5e^{-2x}] \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty 3e^{-5x} \cos sx\, dx + \int_0^\infty 5e^{-2x} \cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\; 3 \int_0^\infty e^{-5x} \cos sx\, dx + \sqrt{\frac{2}{\pi}}\; 5 \int_0^\infty e^{-2x} \cos sx\, dx$$

$$\boxed{\text{We know that, } \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}}$$

$$= 3\sqrt{\frac{2}{\pi}} \left[\frac{5}{5^2+s^2}\right] + \sqrt{\frac{2}{\pi}}\; 5 \left[\frac{2}{2^2+s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{15}{s^2+25} + \frac{10}{s^2+4}\right]$$

Example 4.3.a(5): **Find the Fourier cosine transform of**

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^a 1 . \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0\right] = \sqrt{\frac{2}{\pi}}\; \frac{\sin sa}{s}$$

Example 4.3.a(6): **Find the Fourier cosine transform of f (x) = x.**

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[x] = \sqrt{\frac{2}{\pi}} \int_0^\infty x \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \int_0^\infty x\, e^{-isx}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[x \frac{e^{-isx}}{(-is)} - (1) \frac{e^{-isx}}{(-is)^2}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[-x \frac{e^{-isx}}{is} + \frac{e^{-isx}}{s^2}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[(0) - \left(0 + \frac{1}{s^2}\right)\right] \qquad [\because e^{-\infty} = 0]$$

$$= \sqrt{\frac{2}{\pi}}\; \text{R.P.} \left[\frac{-1}{s^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\; \frac{1}{s^2}$$

Example 4.3.a(7): **Find the Fourier cosine transform of e^{-ax} cos ax.**

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\,[e^{-ax} \cos ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos ax \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \cos ax \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \left[\frac{\cos(s+a)x + \cos(s-a)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty e^{-ax} \cos(s+a)x \, dx + \int_0^\infty e^{-ax} \cos(s-a)x \, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a}{a^2+(s+a)^2} + \frac{a}{a^2+(s-a)^2}\right]$$

Formula : $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{1}{(s+a)^2+a^2} + \frac{1}{(s-a)^2+a^2}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{1}{s^2+2as+2a^2} + \frac{1}{s^2-2as+2a^2}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{1}{(s^2+2a^2)+2as} + \frac{1}{(s^2+2a^2)-2as}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{s^2+2a^2-2as+(s^2+2a^2)+2as}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{a}{\sqrt{2\pi}} \left[\frac{2(s^2+2a^2)}{s^4+4a^4+4s^2a^2-4a^2s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \, a \left[\frac{s^2+2a^2}{s^4+4a^4}\right]$$

Example 4.3.a(8): **Show that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform.**

[A.U M/J 2007] [A.U CBE Dec. 2009]

Solution :

We know that,

$$\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx}$$

$$F_c[e^{-x^2/2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx$$

[$\because e^{-x^2/2} \cos sx$ is an even function]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx$$

$$= \text{R.P} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} e^{isx} \, dx$$

$$= \text{R.P } F[e^{-x^2/2}]$$

$$= \text{R.P } e^{-s^2/2} \quad \text{by (1)}$$ See example 4.2a(4)

$$F_c[e^{-x^2/2}] = e^{-s^2/2}$$

Hence, $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier cosine transform.

Example 4.3.a(9): Find the Fourier cosine transform of $e^{-ax}\sin ax$.

Solution : We know that,

$$\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx}$$

$$F_c\,[e^{-ax}\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\sin ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^\infty e^{-ax}\,[\sin(s+a)x - \sin(s-a)x]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty e^{-ax}\sin(s+a)x\,dx - \int_0^\infty e^{-ax}\sin(s-a)x\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+a}{a^2+(s+a)^2} - \frac{s-a}{a^2+(s-a)^2}\right]$$

$$\boxed{\because \text{Formula} : \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+a}{s^2+2as+2a^2} - \frac{s-a}{s^2-2as+2a^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s^3-2as^2+2sa^2+as^2-2a^2s+2a^3-s^3-2as^2-2sa^2+as^2+2a^2s+2a^3}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{4a^3-2as^2}{s^4+4a^4+4s^2a^2-4a^2s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2a\,[2a^2-s^2]}{s^4+4a^4} = \sqrt{\frac{2}{\pi}}\;a\left[\frac{2a^2-s^2}{s^4+4a^4}\right]$$

Example 4.3.a(10): Evaluate $F_c\,[x^{n-1}]$ if $0 < x < 1$.

[A.U A/M 2015 R-13]

Deduce that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

[A.U M/J 2012]

Solution : $F_c\,[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx\,dx$

We know that, $\Gamma(n) = \int_0^\infty e^{-y}y^{n-1}\,dy\,, \quad n > 0$

put $y = ax$, we get $\int_0^\infty e^{-ax}x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}\,, \quad a > 0$

Let $a = is$

$$\therefore \int_0^\infty e^{-isx}x^{n-1}\,dx = \frac{\Gamma(n)}{(is)^n}$$

$$\int_0^\infty (\cos sx - i\sin sx)\,x^{n-1}\,dx = \frac{\Gamma(n)\,i^{-n}}{s^n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$

Equating real parts, we get

$$\int_0^\infty x^{n-1}\cos sx\,dx = \frac{\Gamma(n)}{s^n}\cos\left(\frac{n\pi}{2}\right)$$

Using this in (1) we get

$$F_c\,[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\cos\left(\frac{n\pi}{2}\right)$$

put $n = \frac{1}{2}$, we get

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\ \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{\sqrt{\pi}}{\sqrt{s}}\ \frac{1}{\sqrt{2}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

> **Note :** Equating imaginary part, we get
> $$F_s\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\ \frac{\Gamma(n)}{s^n} \sin\left(\frac{n\pi}{2}\right)$$

Example 4.3.a(11): Find the Fourier cosine transform of

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

[A.U. N/D 2006]
[A.U CBT N/D 2011]

Solution : We know that,

$$\boxed{F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx\, dx + \int_1^2 (2-x) \cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{\left[x\frac{\sin sx}{s} - (1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1 + \left[(2-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_1^2\right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{\left[x\frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1 + \left[(2-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_1^2\right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{\left[\left(\frac{1 \sin s}{s} + \frac{\cos s}{s^2}\right) - \left(\frac{1}{s^2}\right)\right] + \left[\left(0 - \frac{\cos 2s}{s^2}\right) - \left(\frac{\sin s}{s} - \frac{\cos s}{s^2}\right)\right]\right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\cos s - \cos 2s - 1}{s^2}\right]$$

Example 4.3.a(12): Find the Fourier cosine transform of $\frac{1}{a^2 + x^2}$.

Solution :

We know that,

$$\boxed{F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$$

$$F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} e^{-s}\right]$$

$$F_c\left[\frac{1}{a^2+x^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2a} e^{-as}\right]$$

$$= \sqrt{\frac{\pi}{2}}\ \frac{1}{a} e^{-as}$$

Example 4.3.a(13): Find the Fourier cosine transform of $e^{-a^2x^2}$.

[A.U M/J 2006, A.U CBT A/M 2011] [A.U Tvli N/D 2011]
[A.U N/D 2012] [A.U A/M 2017 R-13]

Solution : We know that,

$$\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx}$$

$$F_c[e^{-a^2x^2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty e^{-a^2x^2} \cos sx \, dx$$

[$\because e^{-a^2x^2} \cos sx$ is an even function]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-a^2x^2} \cos sx \, dx$$

$$= \text{R.P.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-a^2x^2} e^{isx} \, dx$$

$$= \text{R.P } F[e^{-a^2x^2}]$$

$$= \text{R.P } \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2} \quad \text{by (1) See example 4.2a(10)}$$

$$F_c[e^{-a^2x^2}] = \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2}$$

III. (b) Problems based on Fourier cosine transform and its inversion formula.

Formula :

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

Example 4.3.b(1): Solve the integral equation

$\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$ **and also show that**

$\int_0^\infty \frac{\cos \lambda x}{1+x^2} dx = \frac{\pi}{2} e^{-\lambda}$ **[A.U. Dec. 1996] [A.U N/D 2015 R-08]**

[A.U N/D 2016 R-13]

Solution : Given : $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$

$$\int_0^\infty f(x) \cos sx \, dx = e^{-s} \quad [\text{take } \lambda = s]$$

Multiplying by $\sqrt{\frac{2}{\pi}}$ on both sides, we get

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} e^{-s}$$

by formula, $F_c[f(x)] = \sqrt{\frac{2}{\pi}} e^{-s}$ ... (1)

by Fourier cosine inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-s} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty e^{-s} \cos sx \, ds$$

$$f(x) = \frac{2}{\pi}\left[\frac{1}{1+x^2}\right]$$

[$\because$ Formula : $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

Here $a = 1$, $b = x$]

$$F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{2}{\pi} \left[\frac{1}{1+x^2}\right] \cos sx\, dx$$

$$\sqrt{\frac{2}{\pi}}\, e^{-s} = \frac{2}{\pi} \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\frac{1}{1+x^2}\right] \cos sx\, dx$$

$$e^{-s} = \frac{2}{\pi} \int_0^\infty \left[\frac{1}{1+x^2}\right] \cos sx\, dx$$

$$\int_0^\infty \left[\frac{1}{1+x^2}\right] \cos sx\, dx = \frac{\pi}{2} e^{-s}$$

$$\Rightarrow \int_0^\infty \left[\frac{1}{1+x^2}\right] \cos \lambda x\, dx = \frac{\pi}{2} e^{-\lambda} \quad \text{[Replace } s \text{ by } \lambda]$$

Example 4.3.b(2): Solve the integral equation

$$\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$$

Hence, evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$.

[A.U.April, 2001]

Solution :

Given that, $\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$

$$\int_0^\infty f(x) \cos sx\, dx = \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases} \quad [\text{take } \lambda = s\,]$$

Multiply by $\sqrt{\frac{2}{\pi}}$ on both sides, we have

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}$$

by Fourier cosine formula, $F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}$

$$F_c\,[f(x)] = \begin{cases} \sqrt{\frac{2}{\pi}}\,(1-s), & 0 \le s \le 1 \\ 0, & s > 1 \end{cases} \quad \ldots (1)$$

We know that, Fourier cosine inversion formula is

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c\,[f(x)] \cos sx\, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}}\,(1-s) \cos sx\, ds$$

$$= \frac{2}{\pi} \int_0^1 (1-s) \cos sx\, ds$$

$$= \frac{2}{\pi} \left[(1-s)\frac{\sin sx}{x} - (-1)\left(\frac{-\cos sx}{x^2}\right)\right]_{s=0}^{s=1}$$

$$= \frac{2}{\pi} \left[(1-s)\frac{\sin sx}{x} - \frac{\cos sx}{x^2}\right]_{s=0}^{s=1}$$

$$= \frac{2}{\pi} \left[\left(0 - \frac{\cos x}{x^2}\right) - \left(0 - \frac{1}{x^2}\right)\right]$$

$$= \frac{2}{\pi} \left[\frac{-\cos x}{x^2} + \frac{1}{x^2}\right]$$

$$f(x) = \frac{2}{\pi} \left[\frac{1-\cos x}{x^2}\right]$$

We know that,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{2}{\pi} \left[\frac{1-\cos x}{x^2}\right] \cos sx \, dx \quad \ldots (2)$$

From (1) and (2), we have

$$\sqrt{\frac{2}{\pi}} \frac{2}{\pi} \int_0^\infty \frac{1-\cos x}{x^2} \cos sx \, dx = \begin{cases} \sqrt{\frac{2}{\pi}} (1-s), & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}$$

Now, put $s \to 0$, we get

$$\sqrt{\frac{2}{\pi}} \frac{2}{\pi} \int_0^\infty \frac{1-\cos x}{x^2} dx = \sqrt{\frac{2}{\pi}}$$

$$\frac{2}{\pi} \int_0^\infty \frac{1-\cos x}{x^2} dx = 1$$

$$\int_0^\infty \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{2\sin^2 x/2}{x^2} dx = \frac{\pi}{2}$$

put $t = \frac{x}{2}$	$x \to 0 \Rightarrow t \to 0$
$dt = \frac{1}{2} dx$	$x \to \infty \Rightarrow t \to \infty$

$$\int_0^\infty \frac{2\sin^2 t}{(2t)^2} 2\, dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

Example 4.3.b(3): Find the Fourier cosine transform of $e^{-|x|}$ and deduce that $\int_0^\infty \frac{\cos xt}{1+t^2} dt = \frac{\pi}{2} e^{-|x|}$.

Solution : We know that, $\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx}$

$$F_c[e^{-|x|}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-|x|} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx \, dx \quad [\because \text{ in the interval } (0, \infty), e^{-|x|} = e^{-x}]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2+1}\right]$$

$\because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$
Here, $a = 1$, $b = s$

Now, using Fourier cosine inversion formula

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2+1}\right] \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{s^2+1} ds$$

$$\text{(i.e.,)} \int_0^\infty \frac{\cos sx}{s^2+1} ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-|x|}$$

$$\text{(i.e.,)} \int_0^\infty \frac{\cos xt}{1+t^2} dt = \frac{\pi}{2} e^{-|x|} \quad (\because s \text{ is a dummy variable})$$

Example 4.3.b(4): Find the Fourier cosine transform of e^{-ax}, $a > 0$ and deduce that $\int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a} e^{-ax}$.

Solution : We know that, $\boxed{F_c\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx}$

$$F_c\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right] \qquad \because \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$$

Applying the inversion formula, we have

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c\,[e^{-ax}] \cos sx\, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx\, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds$$

$$\text{(i.e.,)} \int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} e^{-ax},\; a > 0$$

III. (c) Problems Based on Fourier Sine Transform. [F.S.T]

Formula :

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

Example 4.3.c(1): Find the Fourier sine transform of

$$\mathbf{f(x) = e^{-x} \cos x.}$$

Solution : We know that,

$$\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$$

$$F_s\,[e^{-x} \cos x] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos x \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \cos x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \left[\frac{\sin (s+1)x + \sin (s-1)x}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty e^{-x} \sin (s+1)x\, dx + \int_0^\infty e^{-x} \sin (s-1)x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s+1}{(s+1)^2+1} + \frac{s-1}{(s-1)^2+1}\right]$$

$$\left[\because \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s+1}{s^2+2s+2} + \frac{s-1}{s^2-2s+2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{s^3-2s^2+2s+s^2-2s+2+s^3+2s^2+2s-s^2-2s-2}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2s^3}{s^4+4+4s^2-4s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2s^3}{s^2+4} = \frac{1}{\sqrt{2\pi}} \frac{2s^3}{s^2+4}$$

Example 4.3.c(2): Find the Fourier sine transform of

$$\mathbf{f(x) = \begin{cases} \sin x, & 0 \le x < a \\ 0, & x > a \end{cases}}$$

Solution : We know that,

$$\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^a \sin x \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \sin sx \sin x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{\cos(s-1)x - \cos(s+1)x}{2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^a \cos(s-1)x\, dx - \int_0^a \cos(s+1)x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin(s-1)x}{s-1}\right)_0^a - \left(\frac{\sin(s+1)x}{s+1}\right)_0^a\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]$$

where $s \neq 1$ and $s \neq -1$

Example 4.3.c(3): Find the Fourier sine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

[A.U N/D 2010]

Solution : We know that, **[A.U N/D 2016 R-8]**

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx\, dx + \int_1^2 (2-x) \sin sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[x\left(\frac{-\cos sx}{s}\right) - (1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^1\right.$$
$$\left. + \left[(2-x)\left(\frac{-\cos sx}{s}\right) - (-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\left(-x\frac{\cos sx}{s} + \frac{\sin sx}{s^2}\right)\right]_0^1 + \left[\left(-(2-x)\frac{\cos sx}{s} - \frac{\sin sx}{s^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\left(\frac{-\cos s}{s} + \frac{\sin s}{s^2}\right) - (-0+0)\right]\right.$$
$$\left. + \left[\left(-0 - \frac{\sin 2s}{s^2}\right) - \left(\frac{-\cos s}{s} - \frac{\sin s}{s^2}\right)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s - \sin 2s}{s^2}\right]$$

Example 4.3.c(4): Find the Fourier sine transform of $\frac{1}{x}$.

[A.U. CBT Dec. 2008] [A.U N/D 2009] [A.U A/M 2015 R-13]

Solution : We know that, **[A.U N/D 2016 R-13, A/M 2017 R-13]**

$$\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx}$$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx\, dx$$

Let $sx = \theta$	$x \to 0 \Rightarrow \theta \to 0$
$s\,dx = d\theta$	$x \to \infty \Rightarrow \theta \to \infty$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{s}{\theta}\right) \sin\theta \frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2}\right] = \sqrt{\frac{\pi}{2}} \qquad \left[\because \int_0^\infty \frac{\sin\theta}{\theta}\, d\theta = \frac{\pi}{2}\right]$$

Example 4.3.c(5): Find the Fourier sine transform of $3e^{-5x} + 5e^{-2x}$.

Solution : We know that, $\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$

$$F_s\,[3e^{-5x} + 5e^{-2x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty (3e^{-5x} + 5e^{-2x}) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty 3e^{-5x} \sin sx \, dx + \int_0^\infty 5e^{-2x} \sin sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[3\int_0^\infty e^{-5x} \sin sx \, dx + 5\int_0^\infty e^{-2x} \sin sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[3\left[\frac{s}{s^2+25}\right] + 5\left[\frac{s}{s^2+4}\right]\right]$$

$$\left[\because \text{Formula : } \int e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\, s \left[\frac{3}{s^2+25} + \frac{5}{s^2+4}\right]$$

Example 4.3.c(6): Find the Fourier sine transforms of $f(x) = e^{-ax}$.

[A.U. N/D 2007] [A.U N/D 2009 (Trichy)] [A.U N/D 2014 R-08]

Solution : We know that, $\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

Formula :

$$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$$

Example 4.3.c(7): Find the Fourier sine transform of the function $f(x) = \frac{e^{-ax}}{x}$ and hence find $F_s\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$

Solution : We know that,

[A.U. N/D 2006] [A.U. T. Chennai N/D 2011] [A.U N/D 2016 R-13]

$$\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

Diff. w.r.to s on both sides,

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x} \sin sx\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \frac{e^{-ax}}{x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \qquad \left[\because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right]$$

Here $b = s$

Integrating w.r.to 's', we get

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2+a^2}\, ds$$

Note : Put $a = 0$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}\frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}$$

$$= \sqrt{\frac{2}{\pi}}\; a \cdot \frac{1}{a} \cdot \tan^{-1}\frac{s}{a} = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{s}{a}$$

Similarly, $F_s\left[\frac{e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{b}$

$$\therefore F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_s\left[\frac{e^{-ax}}{x}\right] - F_s\left[\frac{e^{-bx}}{x}\right]$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{a} - \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{b}$$

$$= \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\frac{s}{a} - \tan^{-1}\frac{s}{b}\right]$$

Example 4.3.c(8): Find the Fourier sine transform of x^{n-1}. Deduce that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform.

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx}$

$$F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \qquad \left[\because \int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}\right]$$

Taking $n = \frac{1}{2}$, we get

$$F_s\left[x^{\frac{1}{2}-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma\frac{1}{2}}{s^{1/2}}\sin\frac{\pi}{4}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

Hence, $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform.

Example 4.3.c(9): Find the Fourier sine transform of $\frac{x}{a^2+x^2}$.

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{a^2+x^2}\sin sx\,dx}$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\frac{\pi}{2}e^{-as} = \sqrt{\frac{\pi}{2}}e^{-as}$$

III.(d) Problems based on Fourier sine transform and its inversion formula.

Formula :

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds$$

Example 4.3.d(1): Find Fourier sine transform of e^{-ax}, $a > 0$ and deduce that $\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,dx = \frac{\pi}{2}e^{-ax}$.

[A.U. M/J 2007, A.U Tvli N/D 2009, A.U N/D 2010]

[A.U M/J 2016R-13]

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx}$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right] \qquad \left[\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right]$$

Applying the inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2} \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, ds$$

$$\therefore \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-ax}, \; a > 0$$

Example 4.3.d(2): Find the Fourier sine transform of e^{-x}, Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$, $m > 0$.

Solution : We know that, $\boxed{F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx}$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{1+s^2}\right]$$

Formula : $\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

By inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[e^{-x}] \sin sx \, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{1+s^2}\right) \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{1+s^2} ds$$

$$\int_0^\infty \frac{s \sin sx}{1+s^2} ds = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-x}$$

Changing 'x' to 'm' and 's' to 'x', we get

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

Example 4.3.d(3): Find f(x) if its sine transform is $\frac{e^{-sa}}{s}$. Hence, find $F_s^{-1}\left[\frac{1}{s}\right]$. **[A.U N/D 2016 R-13]**

[A.U Trichy N/D 2010, Tvli M/J 2011][A.U N/D 2013]

Solution : Given $F_s[f(x)] = \frac{e^{-sa}}{s}$

By inversion formula, we get

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx \, ds}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-sa}}{s} \sin sx \, ds$$

$$\frac{d}{dx} f(x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty \frac{e^{-sa}}{s} \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial x}\left[\frac{e^{-sa}}{s} \sin sx\right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\frac{e^{-sa}}{s} s \cos sx\right] ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-sa} \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2}$$

Formula : $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

Integrating w.r.to x, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2 + x^2}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)$$

Put $a = 0$, we get

$$F_s^{-1}\left[\frac{1}{s}\right] = \sqrt{\frac{2}{\pi}}\ \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

Example 4.3.d(4): **Solve the integral equation**

$$\int_0^{\infty} f(x) \sin sx\, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$ **[A.U M/J 2014]**

[A.U N/D 2015 R-8]

Solution : Let $F_s[f(x)] = \int_0^{\infty} f(x) \sin sx\, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$

By inversion formula, we get

$$\boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} F_s[f(x)] \sin sx\, ds}$$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 \sin sx\, ds + \int_1^2 2 \sin sx\, ds\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos sx}{x}\right)_0^1 + 2\left(\frac{-\cos sx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[\left[\left(-\frac{\cos x}{x}\right) - \left(-\frac{1}{x}\right)\right] + 2\left[\left(\frac{-\cos 2x}{x}\right) - \left(\frac{-\cos x}{x}\right)\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$= \frac{2}{\pi}\left[\frac{1 - 2\cos 2x + \cos x}{x}\right]$$

$$= \frac{2 + 2\cos x - 4\cos 2x}{\pi x},\ x > 0$$

III.(e) Problems based on properties of F.C.T AND F.S.T.

Example 4.3.e(1): **(i) Find the Fourier cosine transform of $\frac{1}{1+x^2}$**

(ii) Find the Fourier sine transform of $\frac{x}{1+x^2}$.

[A.U N/D 2016 R-8]

Solution : We know that, $\boxed{F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx}$

$$F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\cos sx}{1+x^2}\, dx \quad \text{... (A)}$$

$$\text{Let } I = \int_0^{\infty} \frac{\cos sx}{1+x^2}\, dx \quad \text{... (1)}$$

$$\frac{dI}{ds} = \frac{d}{ds} \int_0^{\infty} \frac{\cos sx}{1+x^2}\, dx$$

$$= \int_0^{\infty} \frac{\partial}{\partial s}\left(\frac{\cos sx}{1+x^2}\right) dx = \int_0^{\infty} \frac{1}{1+x^2} \frac{\partial}{\partial s}[\cos sx]\, dx$$

$$\frac{dI}{ds} = \int_0^{\infty} \frac{-x \sin sx}{1+x^2}\, dx \quad \text{... (2)}$$

$$= -\int_0^{\infty} \frac{x^2 \sin sx}{x(1+x^2)}\, dx = -\int_0^{\infty} \frac{[(1+x^2) - 1] \sin sx}{x(1+x^2)}\, dx$$

$$= -\int_0^{\infty} \frac{\sin sx}{x}\, dx + \int_0^{\infty} \frac{\sin sx}{x(1+x^2)}\, dx$$

$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^{\infty} \frac{\sin sx}{x(1+x^2)}\, dx \quad \text{... (3)}$$

$$\left[\because \int_0^{\infty} \frac{\sin sx}{x}\, dx = \frac{\pi}{2}\right]$$

$$\frac{d^2 I}{ds^2} = 0 + \frac{d}{ds}\left[\int_0^{\infty} \frac{\sin sx}{x(1+x^2)}\, dx\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[\frac{\sin sx}{x(1+x^2)}\right]dx = \int_0^\infty \frac{1}{x(1+x^2)}\frac{\partial}{\partial s}[\sin sx]\,dx$$

Now, $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}dx = \int_0^\infty \frac{\cos sx}{1+x^2}dx = I$$

$$\frac{d^2I}{ds^2} - I = 0$$

$$(D^2-1)I = 0$$

A.E is $m^2 - 1 = 0$

$$m = 1, -1$$

$$\therefore I = Ae^{-s} + Be^{s} \qquad \text{... (4)}$$

$$\frac{dI}{ds} = -Ae^{-s} + Be^{s} \qquad \text{... (5)}$$

$$(4) \Rightarrow Ae^{-s} + Be^{s} = \int_0^\infty \frac{\cos sx}{1+x^2}dx \qquad \text{by (1)}$$

put $s = 0$, $$A + B = \int_0^\infty \frac{1}{1+x^2}dx = [\tan^{-1}x]_0^\infty$$

$$A + B = \frac{\pi}{2} \qquad \text{... (6)}$$

$$(5) \Rightarrow -Ae^{-s} + Be^{s} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx \qquad \text{by (3)}$$

put $s = 0$ $$-A + B = -\frac{\pi}{2} \qquad \text{... (7)}$$

Solving (6) & (7), we get

$$A = \frac{\pi}{2},\ B = 0$$

Substitute in (4), we get

$$I = \frac{\pi}{2}e^{-s} \qquad \text{... (8)}$$

$$F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}e^{-s}\right] = \sqrt{\frac{\pi}{2}}\,e^{-s} \qquad \text{by (A)}$$

(ii) $$F_s\left[\frac{x}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin sx}{1+x^2}dx$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{dI}{ds} \quad \text{by (2)}$$

$$= -\sqrt{\frac{2}{\pi}}\left[-\frac{\pi}{2}e^{-s}\right] \quad \text{by (8)}$$

$$= \sqrt{\frac{\pi}{2}}\,e^{-s}$$

Example 4.3.e(2): **Find the Fourier sine and cosine transformations of xe^{-ax}.**

Solution : (i) We know that, $\boxed{F_s[xf(x)] = \frac{-d}{ds}F_c[f(x)]}$

$$F_s[xe^{-ax}] = -\frac{d}{ds}F_c[e^{-ax}] \qquad [\because f(x) = e^{-ax}]$$

$$= \frac{-d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right] \qquad \left[\because F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\,a\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right]$$

$$= -a\sqrt{\frac{2}{\pi}}\left[\frac{-2s}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{2as}{(s^2+a^2)^2}$$

(ii) We know that, $\boxed{F_c[xf(x)] = \frac{-d}{ds}F_s[f(x)]}$

Solution : $$F_c[xe^{-ax}] = -\frac{d}{ds}F_s[e^{-ax}]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{a^2-s^2}{(s^2+a^2)^2} = \sqrt{\frac{2}{\pi}}\,\frac{s^2-a^2}{(s^2+a^2)^2}$$

Example 4.3.e(3): Find Fourier cosine transform of $e^{-a^2x^2}$ and hence find $F_s\,[xe^{-a^2x^2}]$. **[A.U. N/D 2006]**

Solution : See Example 4.3 a(13), Page No. 4.97

$$F_c\,[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}}\,e^{-s^2/4a^2}$$

Now, $F_s\,[xe^{-a^2x^2}] = \dfrac{-d}{ds}\,F_c\,[e^{-a^2x^2}]$

$$= \frac{-d}{ds}\left[\frac{1}{a\sqrt{2}}\,e^{-s^2/4a^2}\right]$$

$$= -\frac{1}{a\sqrt{2}}\,e^{-s^2/4a^2}\left[\frac{-2s}{4a^2}\right]$$

$$= \frac{s}{2\sqrt{2}\,a^3}\,e^{-s^2/4a^2}$$

Example 4.3.e(4): Find the Fourier sine transform of e^{-ax} and hence find the Fourier cosine transform of xe^{-ax}.

Solution : We know that, **[A.U N/D 2011]**

$$\boxed{F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx}$$

$$F_s\,[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right] \qquad \left[\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right]$$

To find : $F_c\,[xe^{-ax}]$

$$F_c\,[xe^{-ax}] = \frac{d}{ds}\,F_s\,[e^{-ax}]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

III (f) Problems based on Parseval's identity in F.S.T and F.C.T

Example 4.3.f(1): Evaluate : $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

[A.U. April, 2001] [A.U. N/D 2005, M/J 2006, A.U Trichy N/D 2010]

[A.U.T Tvli N/D 2011] [A.U N/D 2014 R-13] [A.U N/D 2015 R-8]

Solution :

Parseval's identity is

$$\boxed{\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_c\,[f(x)]\,F_c\,[g(x)]\,ds} \quad \dots (1)$$

Let $f(x) = e^{-ax}$	Let $g(x) = e^{-bx}$
$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$	$F_s[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin sx\, dx$
$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$	$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx\, dx$
$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2}\right]$	$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + b^2}$

$$f(x)\, g(x) = e^{-ax} e^{-bx} = e^{-(a+b)x}$$

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty e^{-(a+b)x}\, dx = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= 0 - \left[\frac{1}{-(a+b)}\right] = \frac{1}{a+b}$$

$$F_s[f(x)]\, F_s[g(x)] = \frac{2}{\pi} \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

$$\therefore (1) \Rightarrow \frac{1}{a+b} = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2(a+b)}$$

$$\Rightarrow \int_0^\infty \frac{\lambda^2}{(\lambda^2+a^2)(\lambda^2+b^2)}\, ds = \frac{\pi}{2(a+b)}$$ [$\because$ s is a dummy variable]

Note : x^2 in Nr and a & b in Dr. use F.S.T formula

Note : Evaluate : $\int_0^\infty \frac{x^2\, dx}{(x^2+4)(x^2+9)}$ **[A.U A/M 2017 R-8]**

Solution : Step 1 : Prove that : $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2(a+b)}$

Step 2 : Here, $a = 2$, $b = 3$

$$\therefore \int_0^\infty \frac{x^2\, dx}{(x^2+4)(x^2+9)} = \frac{\pi}{2(2+3)} = \frac{\pi}{10}$$

Example 4.3.f(3) : Using transform methods, evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

[A.U Tvli M/J 2011] [A.U M/J 2013, N/D 2013] [A.U M/J 2014]

Solution : Parseval's identity is

$$\boxed{\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c[f(x)]|^2\, ds} \quad \ldots (1)$$

Let $f(x) = e^{-ax}$	$[f(x)]^2 = (e^{-ax})^2 = e^{-2ax}$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$	$\lvert F_c[f(x)]\rvert^2 = \frac{2}{\pi} \frac{a^2}{(s^2+a^2)^2}$
$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$	
$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$	

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty e^{-2ax}\, dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = (0) - \left(\frac{1}{-2a}\right) = \frac{1}{2a}$$

$$\therefore (1) \Rightarrow \frac{1}{2a} = \frac{2a^2}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)^2}\, ds$$

$$\Rightarrow \int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$ [$\because$ s is a dummy variable]

$f(x) = 1, \ \lvert x \rvert < a$	$g(x) = e^{-a\lvert x \rvert}, \ a > 0$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$	$F_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos sx \, dx$
$= \sqrt{\frac{2}{\pi}} \int_0^a \cos sx \, dx$	$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$
$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$	$[\because \lvert x \rvert = x \text{ in } (0, \infty)]$
$= \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$	$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2 + a^2}\right)$

$$f(x)\, g(x) = (1)\, e^{-ax} = e^{-ax} \text{ in } (0, a)$$

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^a e^{-ax}\, dx = \left[\frac{e^{-ax}}{-a}\right]_0^a = \left(\frac{e^{-a^2}}{-a}\right) - \left(\frac{1}{-a}\right)$$

$$= \frac{1}{a}\left[1 - e^{-a^2}\right] \quad \ldots (1)$$

$$F_c[f(x)]\, F_c[g(x)] = \frac{2}{\pi} \frac{a \sin sa}{s(s^2 + a^2)}$$

$$(1) \Rightarrow \frac{1}{a}[1 - e^{-a^2}] = \frac{2a}{\pi} \int_0^\infty \frac{\sin sa}{s(s^2 + a^2)}\, ds$$

$$\Rightarrow \int_0^\infty \frac{\sin sa}{s(s^2 + a^2)}\, ds = \frac{\pi}{2a^2}[1 - e^{-a^2}]$$

$$\Rightarrow \int_0^\infty \frac{\sin ax}{x(x^2 + a^2)}\, dx = \frac{\pi}{2a^2}[1 - e^{-a^2}] \quad [\because s \text{ is a dummy variable}]$$

EXERCISE 4.1 [Fourier integral theorem]

1. Using Fourier integral representation, show that

$$\int_0^\infty \frac{\sin \pi\lambda \sin \lambda x}{1 - \lambda^2}\, d\lambda = \begin{cases} \frac{\pi}{2} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

2. Solve the integral equation $\int_0^\infty f(x) \cos sx\, dx = e^{-s}$. Hence deduce that $\int_0^\infty \frac{\cos sx}{1 + s^2}\, ds = \frac{\pi}{2} e^{-x}$ [**Ans.** $f(x) = \frac{2}{\pi} \frac{1}{1 + x^2}$]

3. Solve for $f(x)$ from the integral equation

$$\int_0^\infty f(x) \sin sx\, dx = \begin{cases} 1 & 0 \le s < 1 \\ 2 & 1 \le s < 2 \\ 0 & s \ge 2 \end{cases}$$

[A.U M/J 2014]

[**Ans.** $f(x) = \frac{2}{\pi x}[1 + \cos x - 2\cos 2x]$]

4. Solve the integral equation $\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1 - \lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$.

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$ [**Ans.** $f(x) = \frac{2}{\pi}\left[\frac{1 - \cos x}{x^2}\right]$]

EXERCISE 4.2 [Fourier Transform]

1. Find the Fourier transform of the function.

1. $f(x) = \begin{cases} 1 + \frac{x}{a}, & -a < x < 0 \\ 1 - \frac{x}{a}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$ **Ans.** $\frac{4 \sin^2\left(\frac{as}{2}\right)}{as^2}$

2. $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$ **Ans.** $\frac{1 + e^{is\pi}}{\sqrt{2\pi}\,(1 - s^2)}$

3. $f(x) = x e^{-x}, 0 \le x \le \infty$ Ans. $\frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1-is)^2}\right]$

4. $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2\in}, & |x| \le \in \\ 0, & |x| > \in \end{cases}$ Ans. $\frac{\sin s \in}{s \in}$

5. $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{Otherwise} \end{cases}$

[Ans. $\left(\frac{\cos s + s \sin s - 1}{\sqrt{2\pi}\, s^2}\right) + i \left(\frac{\sin s - s \cos s}{\sqrt{2\pi}\, s^2}\right)$]

6. $f(x) = e^{-|x|}$ and deduce that $\int_0^\infty \left(\frac{\cos xt}{1+t^2}\right) dt = \frac{\pi e^{-|x|}}{2}$

[Ans. $\sqrt{\frac{2}{4}} \left(\frac{1}{1+s^2}\right)$]

7. $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{l} & \text{if } -l < x < l \\ 0 & \text{if Otherwise} \end{cases}$ [Ans. $\frac{\sqrt{2}\,\pi \sin sl}{sl}$]

8. $f(x) = e^{-a|x|}$ and hence deduce that

(i) $\int_0^\infty \frac{\cos sx}{a^2+t^2} dt = \frac{\pi}{2a} e^{-a|x|}$ (ii) $F[x e^{-a|x|}] = \frac{2as}{(s^2+a^2)^2}$

II. If the Fourier transform of $f(x)$ is $\sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$,

find $F\left[F(x)\left(1 + \cos\frac{\pi x}{a}\right)\right]$ Ans. $\sqrt{\frac{2}{\pi}} \sin as \left[\frac{1}{s} + \frac{2a^2 s}{\pi^2 - a^2 s^2}\right]$

EXERCISE 4.3 [FCT and FST]

I. 1. Find the Fourier cosine transform of e^{-4x}. Deduce that

$\int_0^\infty \frac{\cos 2x}{x^2+16} dx = \frac{\pi}{8} e^{-8}$ and $\int_0^\infty \frac{s \sin 2x}{x^2+16} dx = \frac{\pi}{2} e^{-8}$

[Ans. $F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}} \left[\frac{4}{s^2+16}\right]$]

2. Find the Fourier sine transform of $x e^{-x^2/2}$ [Ans. $s e^{-s^2/2}$]

3. Find $f(x)$ if its cosine transform is

$F_c(s) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left(a - \frac{s}{2}\right) & \text{if } 0 < s < 2a \\ 0 & \text{if } s \ge 2a \end{cases}$ [Ans. $\frac{\sin^2 ax}{\pi x^2}$]

II. Find the Fourier Cosine Transform of

1. $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & x > 1 \end{cases}$ Ans. $F_c(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin s}{s}\right)$

2. $f(x) = \frac{e^{-ax} - e^{-bx}}{x}$ Ans. $\frac{1}{\sqrt{2\pi}} \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

3. $f(x) = e^{-4x}$ and hence deduce that

(i) $\int_0^\infty \frac{\cos 2x}{x^2+16} dx = \frac{\pi}{8} e^{-8}$ (ii) $\int_0^\infty \frac{x \sin 2x}{x^2+16} dx = \frac{\pi}{2} e^{-8}$

Ans. $F(s) = \sqrt{\frac{2}{\pi}} \frac{4}{s^2+16}$

III. Find the Fourier Sine transform of

1. $\frac{x}{x^2+a^2}$ Ans. $\sqrt{\frac{\pi}{2}} e^{-as}$

2. $f(x) = \begin{cases} 0 & \text{if } 0 < x < a \\ x & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases}$

Ans. $\sqrt{\frac{2}{\pi}} \left[\frac{a \cos sa - b \cos sb}{s} + \frac{\sin sb - \sin sa}{s^2}\right]$

3. $f(x) = e^{-|x|}$ Ans. $\sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1}\right)$

4. $f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \ge a \end{cases}$

Ans. $\frac{1}{\sqrt{2\pi}} \left[\frac{2s}{s^2-1} - \left(\frac{\cos(s+1)a}{s+1} + \frac{\cos(s-1)a}{s-1}\right)\right]$

IV. Find the Fourier cosine and sine transforms of

1. $5e^{-2x} + 2e^{-5x}$

Ans. $F_c(s) = 10\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+4} + \frac{1}{s^2+25}\right]$

$F_s(s) = \sqrt{\frac{2}{\pi}}\left[\frac{5s}{s^2+4} + \frac{2s}{s^2+25}\right]$

2. $\cosh x - \sinh x$

Ans. $F_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+1}\right)$

$F_c(s) = \left(\frac{1}{s^2+1}\right)$

******

IMPORTANT QUESTIONS

Find the F.T of $f(x)$ and show that the following by using inversion and parseval's identity

	Function	F.T	F.T and its inversion	Parseval's identity
1.	$f(x)=\begin{cases}1, & \lvert x\rvert < a\\ 0, & \lvert x\rvert \geq a\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa}{s}\right)$	$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$
2.	$f(x)=\begin{cases}1, & \lvert x\rvert < 1\\ 0, & \lvert x\rvert \geq 1\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin s}{s}\right)$	$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$
3.	$f(x)=\begin{cases}1, & \lvert x\rvert < 2\\ 0, & \lvert x\rvert \geq 2\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin 2s}{s}\right)$	$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$
4.	$f(x)=\begin{cases}1-x^2, & \lvert x\rvert < 1\\ 0, & \lvert x\rvert > 1\end{cases}$	$F(s)=\frac{4}{\sqrt{2\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$	$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$	$\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}\,dx = \frac{\pi}{15}$
5.	$f(x)=\begin{cases}1-\lvert x\rvert, & \lvert x\rvert < 1\\ 0, & \lvert x\rvert > 1\end{cases}$	$F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$	$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

PART-A QUESTIONS AND ANSWERS

1. **State Fourier integral theorem.**

[A.U. Ap. 1996, A/M 2005, A/M. 2008, N/D 2008, AU Tvli. N/D 2010] [A.U Tvli M/J 2011, A.U CBT N/D 2011] [A.U M/J 2016 R-13]

See Page 4.1

2. **Show that $f(x) = 1, 0 < x < \infty$ cannot be represented by a Fourier integral.** **[A.U. April/May 2003] [A.U M/J 2014]**

Sol : $\int_0^\infty |f(x)|\,dx = \int_0^\infty 1\,dx = [x]_0^\infty = \infty$

and this value tends to ∞ as $x \to \infty$

i.e., $\int_0^\infty |f(x)|\,dx$ is not convergent.

Hence $f(x) = 1$ cannot be represented by a Fourier integral.

3. **Define Fourier Transform pair. (OR)**

Define Fourier Transform and its inverse transform.

[A.U March 1996] [A.U. A/M. 2001, N/D 2007] [A.U. N/D 2010] [A.U Trichy N/D 2010, CBT N/D 2010,] [A.U. N/D 2011, A.U.T. Tvli. N/D 2011, A.U.T Chennai N/D 2011]

See Page 4.25

4. **What is the Fourier cosine transform of a function ?**

See Page 4.78 **[A.U. O/N 1996]**

5. **Find the Fourier cosine transform of** $f(x) = \begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \geq a \end{cases}$

See Page 4.87

6. **Find the Fourier cosine transform of** e^{-ax}, $a > 0$

See Page 4.89

7. **Find Fourier Cosine transform of e^{-x}** **[A.U. Nov./Dec. 2004] [A.U.T CBT N/D 2011]**

Solution : We know that $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\,dx$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{1+s^2}\right]$$

Formula : $\int_0^\infty e^{-ax} \cos bx\,dx = \frac{a}{a^2+b^2}$

8. **Find the Fourier sine transform of e^{-3x}.**

[Anna University, Nov/Dec., 1996] [A.U M/J 2013]

Solution : $F_s[e^{-3x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-3x} \sin sx\,dx$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+3^2}\right]$$

Formula : $F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\,dx$

9. **Find the Fourier Sine-transform of $3e^{-2x}$**

[Anna University, March, 1996]

Solution : Let $f(x) = 3e^{-2x}$

We know that,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty 3e^{-2x} \sin sx\,dx$$

$$= 3\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \sin sx\,dx$$

$$= 3\sqrt{\frac{2}{\pi}} \left[\frac{e^{-2x}}{4+s^2}(-2\sin sx - s\cos sx)\right]_0^\infty$$

$$= 3\sqrt{\frac{2}{\pi}}\left[[0]-\left[\frac{1}{4+s^2}(-s)\right]\right] = 3\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+4}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{3s}{s^2+4}\right]$$

10. Find the Fourier Sine transform of $\frac{1}{x}$

See Page 4.107 [A.U. A/M 2005, A.U.T Chennai N/D 2011] [A.U M/J 2014] [A.U A/M 2017 R-13]

11. Define Fourier sine transform and its inversion formula.

See Page 4.80 [A.U. April/May, 2004 PTMA]

12. Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$ and hence deduce that $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-a}$.

See Page 4.112 [A.U, March, 1998, 1999 & 2000]

13. If Fourier Transform of $f(x) = F(s)$, then what is Fourier Transform of $f(ax)$? [A.U. N/D 1996] [A.U. M/J 2006]

See Page 4.27

14. If F denotes the Fourier Transform operator, then show that

$$F\left\{x^k f^{(m)}(x)\right\} = (-1)^{k+m}\frac{d^k}{ds^k}s^m F\left\{f(x)\right\}$$

[Anna University, March, 1996]

Solution :

Property : $F[x^n f(x)] = (-i)^n \frac{d^n F(s)}{ds^n}$

Proof : We have, $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$

Differentiating both sides n times w.r.t. 's', we get

$$\frac{d^n}{ds^n}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,(ix)^n\,e^{isx}\,dx$$

$$= \frac{i^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,x^n\,dx = i^n\,F[x^n f(x)]$$

$$\text{Hence, } F[x^n f(x)] = \frac{1}{i^n}\frac{d^n F(s)}{ds^n} = (-i)^n\frac{d^n F(s)}{ds^n}$$

Similarly, we can prove

(i) $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$,

(ii) $F_c[x f(x)] = \frac{d}{ds}[F_s(s)]$

Similarly, we get

$$F\left\{x^k f^{(m)}(x)\right\} = (-1)^{k+m}\frac{d^k}{ds^k}s^m F\left\{f(x)\right\}$$

15. If Fourier transform of $f(x)$ is $F(s)$, prove that the Fourier transform of $f(x)\cos ax$ is $\frac{1}{2}[F(s-a)+F(s+a)]$

See Page 4.29 [A.U, April, 2001]

16. Prove that $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a)+F_c(s-a)]$ where F_c denotes the Fourier cosine transform $f(x)$.

See Page 4.82 [A.U April, May 2001]

17. If $F(s)$ is the Fourier transform of $f(x)$, then show that the Fourier transform of $e^{iax} f(x)$ is $F(s+a)$.

See Page 4.28 [A.U April, 1996] [A.U CBT N/D 2010]

18. **Given that $e^{-x^2/2}$ is self reciprocal under Fourier cosine transform, find (i) Fourier sine transform of $xe^{-x^2/2}$ and (ii) Fourier cosine transform of $x^2 e^{-x^2/2}$**

[Anna University, Dec, 1996]

Solution : Given $F_c\left[e^{-x^2/2}\right] = e^{-s^2/2}$

$$F_s\left[xe^{-x^2/2}\right] = -\frac{d}{ds} F_c\left[xe^{-x^2/2}\right]$$

$$= -\frac{d}{ds}\left[e^{-s^2/2}\right] = -e^{-s^2/2}[-s] = se^{-s^2/2}$$

$$F_c\left[x^2 e^{-x^2/2}\right] = \frac{d}{ds} F_s\left[xe^{-x^2/2}\right]$$

$$= \frac{d}{ds}\left[se^{-s^2/2}\right] = \left[se^{-s^2/2}(-s) + e^{-s^2/2}\right]$$

$$= -s^2 e^{-s^2/2} + e^{-s^2/2} = (1-s^2)\,e^{-s^2/2}$$

19. **If F (s) is the Fourier transform of f (x), then find the Fourier transform of f (x − a).**

[A.U. N/D 2005, N/D 2006, A.U. CBT Dec. 2008, N/D 2005]
[A.U.T Tvli. N/D 2011]

See Page 4.28

20. **State the convolution theorem for Fourier transforms.**

[A.U. April/May 2003, May 2000 PT] [A.U. A/M 2008]

Solution : Convolution theorem (or) Faltung theorem :

If $F(s)$ and $G(s)$ are the Fourier transform of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform.

$$F\,[f(x) * g(x)] = F(S)\,G(S)$$

$$= F\,[f(x)]\;F\,[g(x)]$$

21. **State the Fourier transform of the derivatives of a function.**

[A.U. N/D 2005]

See Page 4.30, Q.No. 6. (i) and (ii)

22. **Find the Fourier sine transform of $f(x) = e^{-x}$.** **[A.U. M/J 2006]**

Solution : We know that $F_s\,[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$

$$F_s\,[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{1+s^2}\right] \quad \left[\because \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}\right]$$

23. **Give a function which is self reciprocal under Fourier sine and cosine transforms.**

[A.U. CBT Dec. 2008]

Solution : $\frac{1}{\sqrt{x}}$

24. **State the modulation theorem in Fourier Transform.**

See Page 4.29 **[A.U CBT Dec. 2008]**

25. **State the Parseval's identity on Fourier Transform.**

See Page 4.34 **[A.U CBT Dec. 2008] [A.U CBT N/D 2010]**
[A.U N/D 2011]

26. **Define self reciprocal with respect to Fourier transform.**

[A.U N/D 2013]

See Page No. 4.38