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Numerical Differentiation

The process of finding derivatives of the function from the given set of tabular values is known as numerical differentiation.

Newton's forward difference formula to get the derivative.

Newton's forward interpolation formula is given by

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$$y(x) = y_0 + u\Delta y_0 + \frac{\overset{u^2-u}{u(u-1)}}{2!}\Delta^2 y_0 + \frac{\overset{u^3-3u^2+2u}{u(u-1)(u-2)}}{3!}\Delta^3 y_0$$

$$+ \frac{\overset{u^4-6u^3+11u^2-6u}{u(u-1)(u-2)(u-3)}}{4!}\Delta^4 y_0 + \cdots$$

where $u = \frac{x - x_0}{h}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{dy}{du}\cdot\frac{1}{h} \quad \text{since } \frac{du}{dx} = \frac{1}{h}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right] \quad \text{— (1)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right] \quad -(2)$$

Similarly,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{du}\left(\frac{d^2y}{dx^2}\right)\frac{du}{dx} = \frac{d}{du}\left(\frac{d^2y}{dx^2}\right)\frac{1}{h}$$

$$= \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{(u-9)}{6}\Delta^4 y_0 + \cdots\cdots\right] \quad -(3)$$

When $x = x_0$, $u = \frac{x - x_0}{h} = \frac{x_0 - x_0}{h} = 0$

In particular,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \quad \text{---} (4)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \left(\frac{d^2y}{dx^2}\right)_{u=0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \quad \text{---} (5)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \left(\frac{d^3y}{dx^3}\right)_{u=0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \quad \text{---} (6)$$

Equations (1), (2) & (3) give the first three derivatives at a general point x. Equations (4), (5) & (6) give the first three derivatives at an initial point x_0

Newton's backward difference formula to get the derivative.

Newton's backward interpolation formula is

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

where $v = \dfrac{x - x_n}{h}$

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$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2v+1)}{2!}\nabla^2 y_n + \frac{3v^2+6v+2}{3!}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right] \quad \text{--- (1)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{12v^2+36v+22}{24}\nabla^4 y_n + \cdots\right] \quad \text{--- (2)}$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{24v+36}{24}\nabla^4 y_n + \cdots\right] \quad \text{--- (3)}$$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \quad \text{— (4)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \left(\frac{d^2y}{dx^2}\right)_{v=0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \quad \text{— (5)}$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \left(\frac{d^3y}{dx^3}\right)_{v=0} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \quad \text{— (6)}$$

(1), (2) & (2) $\rightarrow$ derivatives at any point x.

(4), (5) & (6) $\rightarrow$ " at an end point x_n.

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Problem

1. The population of a town is given below. Find the rate of growth of population in 1931, 1941, 1961 and 1971.

Year (x):	1931	1941	1951	1961	1971
Population in 000's (y):	40.62	60.8	79.95	103.56	132.65

Solution: To find $\frac{dy}{dx}$ at $x = 1931$ & 1941, we use Newton's forward difference formula and to find $\frac{dy}{dx}$ at $x = 1961$ & 1971, we use Newton's backward difference formula.

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 1931$	y_0 40.62				
		Δy_0 20.18			
1941	60.8		$\Delta^2 y_0$ −1.03		
		19.15		$\Delta^3 y_0$ 5.49	
1951	79.95		4.46		$\Delta^4 y_0$ −4.47 $\nabla^4 y_n$
		23.61		1.02 $\nabla^3 y_n$	
1961	103.56		5.48 $\nabla^2 y_n$		
		29.09 ∇y_n			
$x_n = 1971$	132.65 y_n				

WKT $u = \frac{x - x_0}{h}$

When $x = 1931$, $u = \frac{1931 - 1931}{10} = 0$.

$$\left(\frac{dy}{dx}\right)_{x=1931} = \left(\frac{dy}{dx}\right)_{u=0}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{10}\left[20.18 - \frac{(-1.03)}{2} + \frac{5.49}{3} - \frac{(-4.47)}{4}\right]$$

$$= 2.36425$$

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When $x = 1941$, $u = \frac{1941 - 1931}{10} = 1$.

$$\left(\frac{dy}{dx}\right)_{x=x} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1941} = \left(\frac{dy}{dx}\right)_{u=1} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 - \frac{1}{6}\Delta^3 y_0 + \frac{2}{24}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{10}\left[20.18 - \frac{(-1.03)}{2} - \frac{5.49}{6} + \frac{2}{24}(-4.47)\right] = 1.83775$$

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WKT $v = \dfrac{x - x_n}{h}$

When $x = 1971$, $v = \dfrac{1971 - 1971}{10} = 0$.

$$\left(\frac{dy}{dx}\right)_{x=1971} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{10}\left[29.09 + \frac{5.48}{2} + \frac{1.02}{3} + \frac{(-4.47)}{4}\right]$$

$$= 3.10525$$

When $x = 1961$, $v = \dfrac{1961 - 1971}{10} = -1$

$$\left(\frac{dy}{dx}\right)_{x=x} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2!}\nabla^2 y_n + \frac{3v^2+6v+2}{3!}\nabla^3 y_n + \frac{(4v^3+18v^2+22v+6)}{4!}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1961} = \left(\frac{dy}{dx}\right)_{v=-1}$$

$$= \frac{1}{h}\left[\nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{6}\nabla^3 y_n - \frac{2}{24}\nabla^4 y_n\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1961} = \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) - \frac{1}{6}(1.02) - \frac{2}{24}(-4.47)\right]$$

$$= 2.65525$$

Practice Problems

1. Find the first and second derivative of the function tabulated below at $x = 3$.

x :	3·0	3·2	3·4	3·6	3·8	4·0
$f(x)$:	− 14	− 10·032	− 5·296	− 0·256	6·672	14

2. Find the first three derivatives of the function at $x = 1\cdot5$ from the table below.

x :	1·5	2·0	2·5	3·0	3·5	4·0
y :	3·375	7·0	13·625	24·0	38·875	59·0

3. From the table below find y' and y'' at $x = 1\cdot05$.

x :	1·00	1·05	1·10	1·15	1·20	1·25	1·30
y :	1·00000	1·02470	1·04881	1·07238	1·09544	1·11803	1·14017

4. Find the first and second derivative of $\sqrt{x}$ at $x = 15$ from the table below.

x :	15	17	19	21	23	25
$\sqrt{x}$:	3·873	4·123	4·359	4·583	4·796	5·000

(B.Sc. BR. 1989)

5. The following data give the corresponding values for pressure and specific volume of a superheated steam.

Volume v :	2	4	6	8	10
Pressure p :	105	42·7	25·3	16·7	13·0

Find the rate of change of pressure w.r.t. volume when $v = 2$.

6. Obtain the second derivative of y at $x = 0\cdot96$ from the data.

x :	0·96	0·98	1·00	1·02	1·04
y :	0·7825	0·7739	0·7651	0·7563	0·7473

7. Find the value of cos (1·74) from the following table.

x :	1·7	1·74	1·78	1·82	1·86
$\sin x$:	0·9916	0·9857	0·9781	0·9691	0·9584

Problems

1. A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t (seconds). Calculate the angular velocity and angular acceleration of the rod at $t = 0.6$ seconds.

t :	0	0.2	0.4	0.6	0.8	1.0
θ :	0	0.12	0.49	1.12	2.02	3.20

Solution: angular velocity $= \left(\dfrac{d\theta}{dt}\right)_{t=0.6}$

angular acceleration $= \left(\dfrac{d^2\theta}{dt^2}\right)_{t=0.6}$

Since $t = 0.6$ is towards the end, we will use Newton's backward difference formula.

We know that $v = \frac{t - t_n}{h} = \frac{0.6 - 1.0}{0.2} = -2$

where $t_n = 1.0$ and $h = 0.2$

Now we know that

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2!}\nabla^2 y_n + \frac{3v^2+6v+2}{3!}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{4!}\nabla^4 y_n + \cdots\right]$$

and $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6v+6}{3!}\nabla^3 y_n + \frac{12v^2+36v+22}{4!}\nabla^4 y_n + \dots\right]$

Difference Table

t	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$	$\nabla^5\theta$
0	0					
		0.12				
0.2	0.12		0.25			
		0.37		0.01		
0.4	0.49		0.26		0	
		0.63		0.01		0 $\nabla^5\theta_n$
0.6	1.12		0.27		0 $\nabla^4\theta_n$	
		0.90		0.01 $\nabla^3\theta_n$		
0.8	2.02		0.28 $\nabla^2\theta_n$			
		1.18 $\nabla\theta_n$				
$t_n = 1.0$	3.20 θ_n					

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$$\left(\frac{d\theta}{dt}\right)_{t=0.6} = \left(\frac{d\theta}{dt}\right)_{v=-2}$$

$$= \frac{1}{h}\left[\nabla\theta_n - \frac{3}{2}\nabla^2\theta_n + \frac{2}{6}\nabla^3\theta_n + \cdots\right]$$

$$= \frac{1}{0.2}\left[1.18 - \frac{3}{2}(0.28) + \frac{1}{3}(0.01)\right] = 3.81\ \text{rad}/\text{sec}.$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.6} = \left(\frac{d^2\theta}{dt^2}\right)_{v=-2}$$

$$= \frac{1}{h^2}\left[\nabla^2\theta_n - \nabla^3\theta_n + \cdots\right] = \frac{1}{(0.2)^2}\left[0.28 - 0.01\right]$$

$$= 6.75\ \text{rad}/\text{sec}^2$$

2. Find the value of $\sec 31^\circ$ using the following table.

θ (in deg.)	31°	32°	33°	34°
$y = \tan\theta$	0.6008	0.6249	0.6494	0.6745

Solution: $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$.

Here $\theta_0 = 31^\circ$; $h = 1^\circ$ and $\theta = 31^\circ$

and $u = \frac{\theta - \theta_0}{h} = \frac{31^\circ - 31^\circ}{1^\circ} = \frac{0^\circ}{1^\circ} = 0$.

Difference Table

θ	$y=\tan\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$
$\theta_0 = 31^\circ$	y_0: 0.6008			
		Δy_0: 0.0241		
32°	0.6249		$\Delta^2 y_0$: 0.0004	
		0.0245		$\Delta^3 y_0$: 0.0002
33°	0.6494		0.0006	
		0.0251		
34°	0.6745			

Since $\theta = 31^\circ$ is the initial value and we require $\sec 31^\circ$, we use Newton's forward difference formula to get this derivative.

$$\left(\frac{dy}{d\theta}\right)_{\theta=\theta_0} = \left(\frac{dy}{d\theta}\right)_{\theta=31^\circ} = \left(\frac{dy}{d\theta}\right)_{u=0}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$$

$$= \frac{1}{1^\circ}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{180}{\pi}\left[0.02397\right] = \frac{180 \times 7 \times 0.02397}{22} = 1.3728$$

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$$\left(\frac{dy}{d\theta}\right)_{\theta=31^\circ} = 1.3728 \text{ app.}$$

i.e, $\sec^2 31^\circ = 1.3728$

$\Rightarrow \sec 31^\circ = \underline{1.17}17$ app.

Exact value of $\sec 31^\circ = 1.166633397$.

Numerical differentiation for unequal intervals

If the arguments are not equally spaced, we may find the polynomial by Lagrange's interpolation formula and then differentiate it as many times as required.

Problem: Evaluate y' and y'' at $x = 2$ given

x :	0	1	3	6
y :	18	10	-18	90

Solution: Since the values of x are not equally spaced, we use Lagrange's interpolation formula to find a polynomial to the given data.

$x_0 = 0$; $x_1 = 1$; $x_2 = 3$; $x_3 = 6$

$y_0 = 18$; $y_1 = 10$; $y_2 = -18$; $y_3 = 90$

Lagrange's interpolation formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$y(x) = \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{x(x-3)(x-6)}{1(1-3)(1-6)}(10)$$

$$+ \frac{x(x-1)(x-6)}{3(3-1)(3-6)}(-18) + \frac{x(x-1)(x-3)}{6(6-1)(6-3)}(90)$$

$$= -(x-1)(x^2-9x+18) + x(x^2-9x+18)$$

$$+ x(x^2-7x+6) + x(x^2-4x+3)$$

$$= \cancel{-x^3} + \cancel{9}x^2 - 1\cancel{8}x + x^2 - \cancel{9}x + 18 + x^3 - \cancel{9}x^2 + \cancel{18}x$$

$$\cancel{+x^3} - 7x^2 + \cancel{6}x + x^3 - 4x^2 + \cancel{3}x$$

$$= 2x^3 - 10x^2 + 18$$

$$\therefore \quad y(x) = 2x^3 - 10x^2 + 18$$

$$y'(x) = 6x^2 - 20x$$

$$y''(x) = 12x - 20$$

Hence $y'(2) = 6(2)^2 - 20(2)$

$$= -16$$

& $y''(2) = 12(2) - 20$

$$= 4$$

Problems

1. A curve passes through the points $(0,3)$, $(2,3)$, $(5,11)$ and $(6,27)$. Find the slope of the curve at $x=1$.

2. Find the max. and min. value of y from the table below.

x	0	1	2	3	4	5
y	0	0.25	0	2.25	16	56.25

Ans: Min at 0, 2 is 0

Max. at 1 is 0.25

Lecture Notes - NUMERICAL INTEGRATION

Class : II Year B. Tech (EEE, ECE & EIE)

Engineering Mathematics – IV
Course Code : MAT301R01

By
D. Sarala
SASH / SASTRA

NUMERICAL INTEGRATION

- Approximate calculation of a definite integral using numerical techniques.
- Function values are tabulated at regularly spaced intervals / Function is specified in the given problem over an interval.
- Gives the approximate value of an integral.
- Different from analytical integration.

- *Numerical integration* is the process of evaluating a definite integral of a function from a set of numerical values of a function.

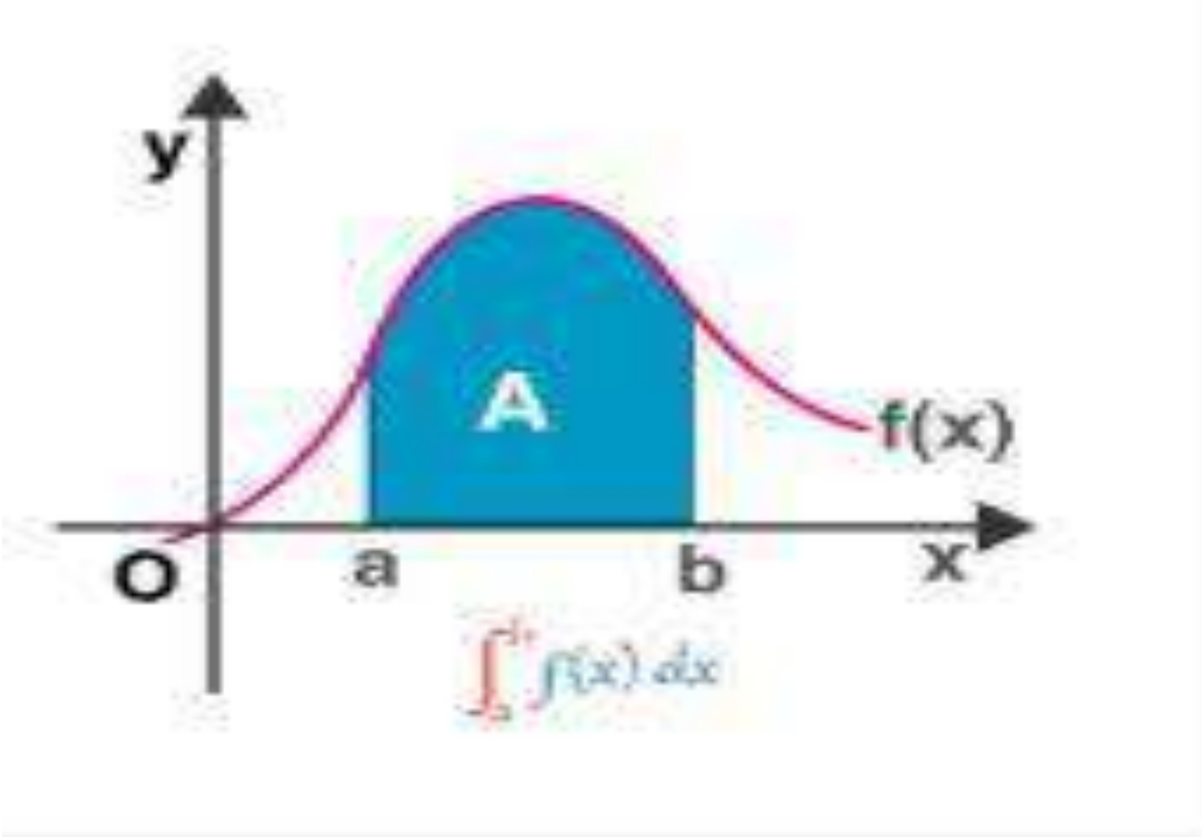
  Alternatively, we may define numerical integration as follows:

- It is the process of computing the value of definite integral

$$\int_a^b y\,dx$$

  when the integrand function y = f(x) is given as discrete set of points (x_i, y_i), i = 0, 1, 2, ……. , n.

Geometrical Interpretation of a definite integral



$$\int_a^b y\,dx \longrightarrow$$ Gives the area between y = f(x), x – axis and the ordinates x = a and x = b.

Numerical Integration – Newton Cotes Integration Formulas

- The Newton-Cotes Integration Formulas are the most common numerical integration schemes.
- They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate.

Newton Cotes Integration Formula

For an explicitly known function $y = f(x)$, let y take values $y_0,\ y_1,\ y_2\ , \dots y_n$ for x taking values $x_0,\ \ x_1,\ x_2\ , \dots x_n$, where x_i's are equispaced.

To evaluate $I = \int_a^b f(x)dx$, divide the interval (a, b) into n equal parts, each of width h, such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Clearly $x_n = x_0 + nh$Then $I = \int_a^b f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx, \quad nh = b - a$

By Newton's forward interpolation formula:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots, \quad x = x_0 + ph$$

$$\therefore I = \int_{x_0}^{x_0+nh} f(x)dx = h\int_0^n f(x_0 + ph)dp$$

$\because x = x_0 + ph \Rightarrow dx = hdp$, also when $x = x_0,\ \ p = 0$

$x = x_0 + nh, p = n$

$$\Rightarrow I \equiv h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right] dp$$

$$\equiv h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$\Rightarrow I \equiv nh\left[y_0 + \frac{n}{2}\Delta y_0 + \left(\frac{n^2}{3} - \frac{n}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^3}{4} - n^2 + n\right)\frac{\Delta^3 y_0}{3!} + \cdots\right] \quad \ldots (1)$$

This is known as Newton's Cote's quadrature formula. Different quadrature formulae are derived by taking $n = 1, 2, 3, \ldots$ in equation (1).

TRAPEZOIDAL RULE (n = 1)

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots\dots + y_{n-1})]$$

SIMPSON'S ONE-THIRD RULE (n = 2)

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \dots\dots\dots) + 4(y_1 + y_3 + \dots\dots\dots)]$$

SIMPSON'S THREE-EIGHTH RULE (n = 3)

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots\dots\dots) + 2(y_3 + y_6 + \dots\dots\dots)]$$

where x_0 = initial value of x

y_0 = initial value of y

x_n = final value of x

y_n = final value of y

h = length of the interval

$= (x_n - x_0) / n$

n = number of intervals

TRAPEZOIDAL RULE

- It is applicable for any number of intervals.
- The error is of order h^2 .
- The accuracy of the integral can be improved by increasing the number of intervals and by decreasing the value of h.
- In this rule y(x) is a linear function of x.
- The Trapezoidal rule is so called because it approximates the integral by the sum of the areas of Trapezoids.

SIMPSON'S ONE-THIRD RULE

- It is applicable for even number of intervals.
- The error is of order h^4 .
- In this rule y(x) is a polynomial of degree 2.
- It uses 3 data points.
- Simpson's one-third rule approximates the area of two adjacent strips by the area under a quadratic parabola.

SIMPSON'S THREE-EIGHTH RULE

- It is applicable for the intervals which is multiple of 3.
- The error is of order h^4 .
- In this rule y(x) is a polynomial of degree 3.
- It uses 4 data points.

Truncation error in Trapezoidal rule:

$$|E| < \frac{(b-a)h^2}{12} M, \text{ if the interval is } (a,b) \text{ and } h = \frac{b-a}{n}$$

$$\text{where } M \text{ is the maximum value of } |y_0''|, |y_1''|, |y_2''|, \ldots\ldots\ldots\ldots$$

Truncation error in Simpson's rule:

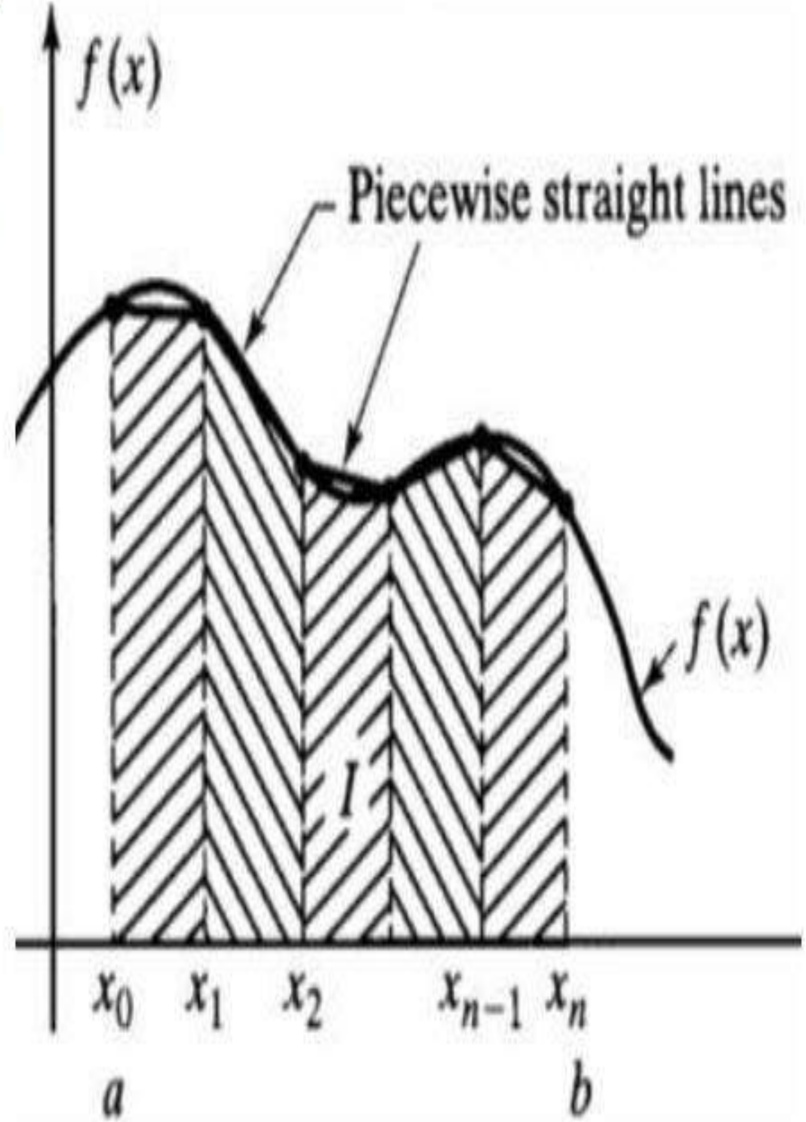
$$|E| < \frac{(b-a)h^4}{180} M, \text{ if the interval is } (a,b) \text{ and } h = \frac{b-a}{2n}$$

$$\text{where } M \text{ is the maximum value of } |y_0''''|, |y_2''''|, |y_4''''|, \ldots\ldots\ldots\ldots$$

Geometrical Significance of Trapezoidal Rule

In trapezoidal rule, the curve $y = f(x)$ is replaced by n piecewise straight lines joining the points (x_0, y_0) and (x_1, y_1) ; (x_1, y_1) and (x_2, y_2); ... ; (x_{n-1}, y_{n-1}) and (x_n, y_n).

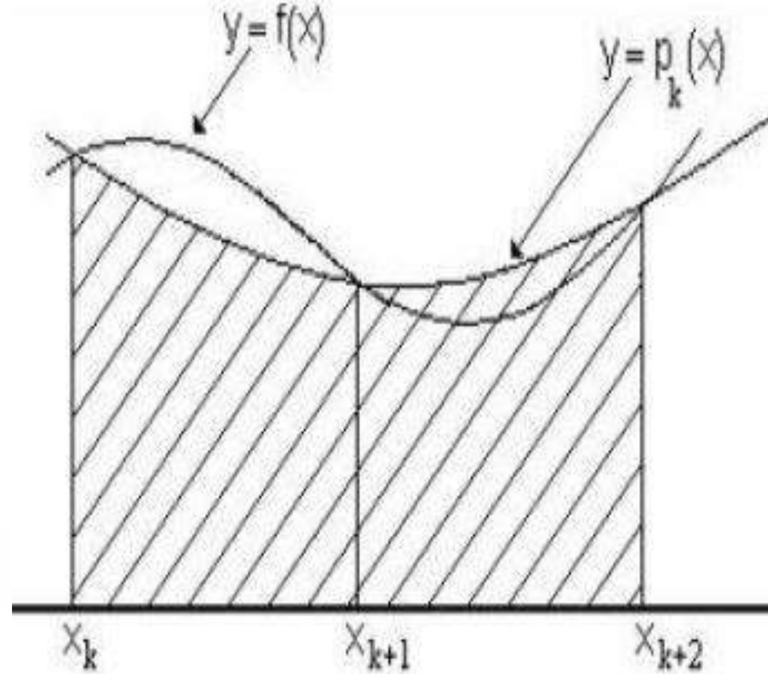
The area under the curve $y = f(x)$, between the ordinates $x = x_0$; $x = x_n$ and above $x-$axis is approximately equal to the sum of areas of n trapezoids obtained within the enclosed region, shown by shaded portion of adjoining figure.



Geometrical Significance of Simpson's One-Third Rule

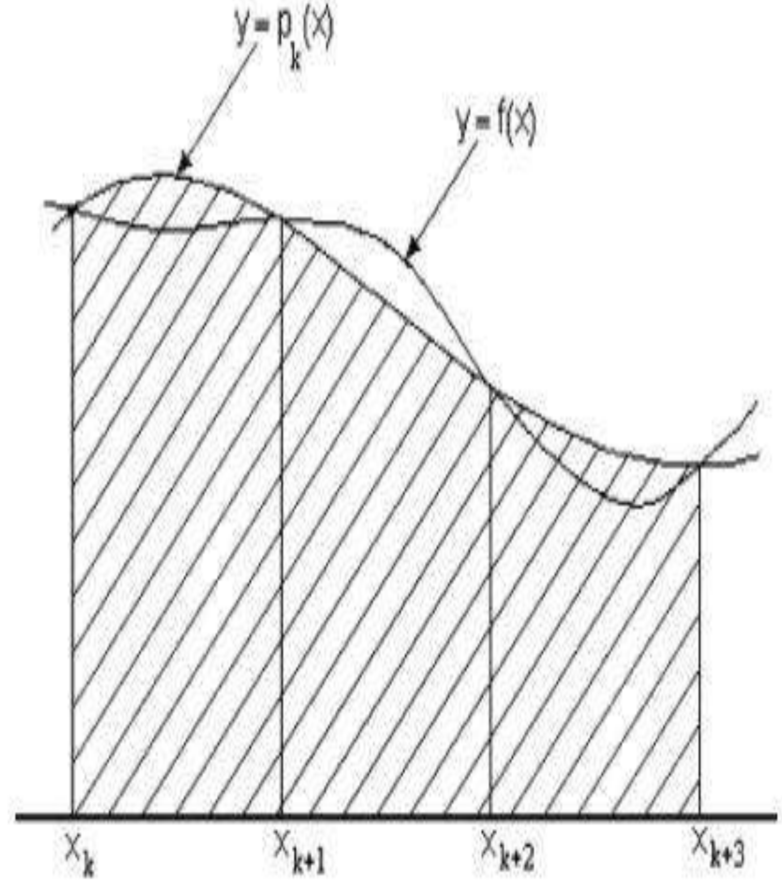
In Simpson's one-third rule, the curve $y = f(x)$ is replaced by arcs of 2^{nd} degree parabolas with vertical axis as shown in given figure.

Simpson's one-third rule requires the given interval to be divided into even number of sub-intervals, since we are finding areas of two strips at a time.



<u>**Geometrical Significance of Simpson's Three-Eighth Rule**</u>

The Simpson's 3/8 rule is similar to the 1/3 rule except that curve $y = f(x)$ is replaced by arcs of 3^{rd} degree polynomial curve, as shown in given figure. It is used when it is required to take 3 segments at a time. Thus number of intervals must be a multiple of 3.



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Problems

1. Evaluate $\int_{4}^{5.2} \log_e x \, dx$ using Trapezoidal and Simpson's rules.

Solution: Here $f(x) = \log_e x$

$x_0 = a = 4$ and $x_n = b = 5.2$

Let us take the number of intervals $= 6 = n$.

Now $h = \dfrac{b-a}{n}$ $\left[= \dfrac{\text{upper Limit} - \text{Lower Limit}}{\text{number of intervals}} \right.$

$= \dfrac{5.2-4}{6} = 0.2$

x :	4	4.2	4.4	4.6	4.8	5.0	5.2
	x_0						x_n
$f(x) = \log_e x$:	1.38629	1.43508	1.48160	1.52605	1.56861	1.60943	1.64865
	y_0	y_1	y_2	y_3	y_4	y_5	$y_6 = y_n$

Trapezoidal rule is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\therefore \int_4^{5.2} \log_e x\,dx = \frac{0.2}{2}\Big[(1.38629+1.64865)$$
$$+2(1.43508+1.48160+1.52605$$
$$+1.56861+1.60943)\Big]$$
$$= 1.827648$$

Simpson's $1/3$rd rule is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\cdots)+4(y_1+y_3+y_5+\cdots)\right]$$

$$\therefore \int_4^{5.2} \log_e x\,dx = \frac{0.2}{3}\left[(1.38629+1.64865) + 2(1.48160+1.56861) + 4(1.43508+1.52605+1.60943)\right]$$

$$= 1.82784$$

Simpson's $3/8^{th}$ rule is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+\cdots)\Big]$$

$$\therefore \int_{4}^{5.2} \log_e x\,dx = \frac{3(0.2)}{8}\Big[(1.38629+1.64865) + 3(1.43505+1.4816+1.56861+1.60943) + 2(1.52605)\Big]$$

$$= 1.82783$$

2. By dividing the range into 10 equal parts, evaluate $\int_0^{\pi} \sin x \, dx$ by Trapezoidal and Simpson's rules. Verify your answer with actual integration.

Solution: Here $f(x) = \sin x$ and $n = 10$.

$$h = \frac{b-a}{n} \quad \text{where } a = 0 \ \& \ b = \pi$$

$$= \pi/10$$

x:	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7
$f(x) = \sin x$:	0	0.3090	0.5878	0.8090	0.9511	1	0.9511	0.8090

x:	$8\pi/10$	$9\pi/10$	π
	y_8	y_9	y_{10}
$f(x)$:	0.5878	0.3090	0

Trapezoidal rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots)\right]$$

$$\therefore \int_0^{\pi} \sin x\,dx = \frac{(\pi/10)}{2}\left[(0+0)+2(0.3090+0.5878+0.8090+0.9511+1+0.9511+0.8090+0.5878+0.3090)\right]$$

$$= 1.9843 \text{ app.}$$

Simpson's $1/3^{rd}$ rule is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+\cdots)\right]$$

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$$\therefore \int_0^{\pi} \sin x \, dx = \frac{(\pi/10)}{3}\Big[(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.309 + 0.809 + 1 + 0.809 + 0.309)\Big]$$

$$= 2.00091 \text{ app.}$$

Since the number of intervals is not a multiple of 3, we cannot use Simpson's $3/8^{th}$ rule.

By actual integration, $\int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi}$

$$= -\left[\cos\pi - \cos 0\right]$$

$$= -(-1-1) = 2$$

3. A curve passes through the points (1, 2), (1.5, 2.4), (2, 2.7), (2.5, 2.8), (3, 3), (3.5, 2.6) and (4, 2.1). Obtain the area bounded by the curve, the x axis, $x = 1$ and $x = 4$. Also find the volume of solid of revolution got by revolving this area about the x axis.

Solution: $\text{Area} = \int_a^b y \, dx$

$$\text{volume} = \pi \int_a^b y^2 \, dx$$

Here $h = 0.5$

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1
	y_0	y_1	y_2	y_3	y_4	y_5	$y_6 = y_n$

By Simpson's one-third rule,

$$\text{Area} = \int_a^b y\,dx = \int_1^4 y\,dx$$

$$= \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+\cdots)\right]$$

$$= \frac{0.5}{3}\left[(2+2.1) + 2(2.7+3) + 4(2.4+2.8+2.6)\right]$$

$$= 7.7833 \text{ Sq. units.}$$

By Simpson's one-third rule,

$$\text{volume} = \pi \int_a^b y^2 \, dx = \pi \int_1^4 y^2 \, dx$$

$$= \pi \cdot \frac{h}{3} \left[(y_0^2 + y_n^2) + 2(y_2^2 + y_4^2 + \cdots) + 4(y_1^2 + y_3^2 + \cdots) \right]$$

$$= \pi \frac{(0.5)}{3} \left[(2^2 + 2.1^2) + 2(2.7^2 + 3^2) + 4(2.4^2 + 2.8^2 + 2.6^2) \right]$$

$$= \pi \times 20.405$$

$$= 64.13 \text{ cubic units}$$

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Practice Problems

1. Evaluate $\int_0^1 \sqrt{\sin x + \cos x}\, dx$ using 7 ordinates.
(y_0 to $y_6 \to 6$ intervals)

2. Compute $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ taking $h = 0.2$ using Trapezoidal and Simpson's rules. Compare your results by actual integration.

3. Find $\int_1^{1.4} e^{-x^2}\, dx$ by dividing the range into 4 equal parts by (i) Trapezoidal rule
(ii) Simpson's $1/3^{rd}$ rule

4. Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2\,dx}{1+x^3}$ by using Simpson's $1/3^{rd}$ rule with $h = 0.25$.

<u>Hint:</u> $f(x) = \frac{x^2}{1+x^3}$; $h = 0.25$

By Simpson's $1/3^{rd}$ rule, $I = \int_0^1 \frac{x^2\,dx}{1+x^3} = ?$ — (1)

By actual integration, $I = \int_0^1 \frac{x^2\,dx}{1+x^3} = \left[\frac{1}{3}\log(1+x^3)\right]_0^1$

$= \log 2^{1/3}$ — (2)

From (1) & (2), $\log 2^{1/3} =$

Problems

1. The velocity v of a particle at distance s from a point on its path is given by the table below:

S in metre	0	10	20	30	40	50	60
V m/sec	47	58	64	65	61	52	38

Estimate the time taken to travel 60 metres by using Simpson's one-third rule.

Solution: we know that $V = \frac{ds}{dt}$

$$\Rightarrow dt = \frac{1}{V} ds$$

$$\Rightarrow t = \int_0^{60} \frac{1}{V} ds = \int_0^{60} y\, ds, \text{ where } y = \frac{1}{V}$$

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S in m :	0	10	20	30	40	50	60
v m/s :	47	58	64	65	61	52	38
$y = \frac{1}{v}$:	$\frac{1}{47}$	$\frac{1}{58}$	$\frac{1}{64}$	$\frac{1}{65}$	$\frac{1}{61}$	$\frac{1}{52}$	$\frac{1}{38}$
	y_0	y_1	y_2	y_3	y_4	y_5	$y_6 = y_n$

Simpson's one third rule is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+\cdots)\right]$$

$$\therefore \int_0^{60} y\,ds = \frac{10}{3}\left[\left(\frac{1}{47}+\frac{1}{38}\right) + 2\left(\frac{1}{64}+\frac{1}{61}\right) + 4\left(\frac{1}{58}+\frac{1}{65}+\frac{1}{52}\right)\right]$$

i.e., $t = 1.06346$ secs.

2. The table below gives the velocity of a moving particle at time t seconds. Find the distance covered by the particle in 12 seconds and also the acceleration at t = 2 seconds.

t :	0	2	4	6	8	10	12
v :	4	6	16	34	60	94	136

Solution: we know that $v = \frac{ds}{dt} \Rightarrow ds = v\,dt \Rightarrow s = \int_0^{12} v\,dt$

and $a = \frac{dv}{dt}$

To find: $s = \int_0^{12} v\,dt$ and $a = \left(\frac{dv}{dt}\right)_{t=2}$

Simpson's one-third rule is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+\cdots)\right]$$

$$\therefore S = \int_0^{12} v\,dt = \frac{2}{3}\left[(4+136)+2(16+60)+4(6+34+94)\right]$$

$$= \underline{552} \text{ metres}$$

Newton's forward difference formula to get the derivative is

$$a = \frac{dv}{dt} = \frac{1}{h}\left[\Delta v_0 + \frac{(2u-1)}{2!}\Delta^2 v_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 v_0 + \cdots\right]$$

where $u = \dfrac{t-t_0}{h}$

$$\left(\frac{dv}{dt}\right)_{t=2} = \left(\frac{dv}{dt}\right)_{u=1} \qquad \text{since } u = \frac{2-0}{2} = 1$$

$$= \frac{1}{h}\left[\Delta v_0 + \frac{1}{2}\Delta^2 v_0 - \frac{1}{6}\Delta^3 v_0 + \cdots\right]$$

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$
0	$v_0 = 4$			
		Δv_0 2		
2	6		$\Delta^2 v_0$ 8	
		10		$\Delta^3 v_0$ 0
4	16		8	
		18		0
6	34		8	
		26		0
8	60		8	
		34		0
10	94		8	
		42		
12	136			

$\therefore$ Accelaration $= \left(\frac{dv}{dt}\right)_{t=2}$

$$= \frac{1}{2}\left[2 + \frac{1}{2}(8)\right]$$

$$= 3\,m/s^2$$

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Romberg's Method

$$I = \int_{x_0}^{x_n} f(x)\,dx = I_2 + \frac{1}{3}(I_2 - I_1)$$

where I_1 and I_2 are the values of I got by two different values of h, by Trapezoidal rule.

Note: By applying Trapezoidal rule many times, every time halving h, we get a sequence of values

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3. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Romberg's method. Hence obtain an approximate value of π.

Solution: Here $f(x) = \frac{1}{1+x^2}$

$$h = \frac{1-0}{2} = 0.5$$

x :	0	0.5	1
$f(x) = \frac{1}{1+x^2}$:	1	0.8	0.5

Trapezoidal rule is $\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{0.5}{2}\left[(1+0.5)+2(0.8)\right] = 0.775$$

$$h = \frac{0.5}{2} = 0.25$$

x	:	0	0.25	0.5	0.75	1
$f(x) = \frac{1}{1+x^2}$	:	1	0.9411	0.8	0.64	0.5

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{0.25}{2}\left[(1+0.5) + 2(0.9411 + 0.8 + 0.64)\right]$$

$$= 0.7827$$

$$h = \frac{0.25}{2} = 0.125$$

x :	0	0.125	0.25	0.375	0.5	0.625	0.75
$f(x)$:	1	0.9846	0.9411	0.8767	0.8	0.7191	0.64

x :	0.875	1
$f(x)$:	0.5663	0.5

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{0.125}{2}\left[(1+0.5)+2(0.9846+0.9411+0.8767+0.8+0.7191+0.64+0.5663)\right]$$

$$= 0.7847$$

The different values got by Trapezoidal rule for different h's are

0.775	0.7827	0.7847
(I_1)	(I_2)	
	(I_1)	(I_2)

Romberg's formula is

$$I = I_2 + \frac{1}{3}(I_2 - I_1) = 0.7827 + \frac{1}{3}(0.7827 - 0.775)$$

$$= 0.7852 \quad (I_1)$$

& $$I = I_2 + \frac{1}{3}(I_2 - I_1) = 0.7847 + \frac{1}{3}(0.7847 - 0.7827)$$

$$= 0.7853 \quad (I_2)$$

To improve the value of I,

$$I = I_2 + \frac{1}{3}(I_2 - I_1) = 0.7853 + \frac{1}{3}(0.7853 - 0.7852)$$

$$= 0.7853$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = 0.7853 \quad \text{---} \ (1)$$

By actual integration, $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4} \quad \text{---} \ (2)$$

From (1) & (2), $\dfrac{\pi}{4} = 0.7853$

$$\Rightarrow \pi \simeq 3.1412$$

Note: RMS value $= \sqrt{\dfrac{\int_a^b y^2\,dx}{b-a}}$

28. The velocity of a train which starts from rest is given by the following table, time being reckoned in minutes from the start and speed in miles per hour.

Minutes :	2	4	6	8	10	12	14	16	18	20
Miles/hour :	10	18	25	29	32	20	11	5	2	0

Find the total distance covered in 20 minutes.

[Hint: It is given that it starts from rest. So, at $t = 0$, $v = 0$. Introduce this idea to get 11 ordinates to use Simpson's rule.]

29. The velocity v of a particle moving in a straight line covers a distance x in time t. They are related as follows:

x :	0	10	20	30	40
v :	45	60	65	54	42

Find time taken to traverse the distance of 40 units. (*MKU BE '73*)

30. Find the distance travelled by the train between 11·50 A.M and 12·30 P.M. from the data given below:

Time :	11·50 A.M.	12·00 noon	12·10 P.M.	12·20 P.M.	12·30 P.M.
Speed in kmph :	48·2	70·0	82·6	85·6	78·4

31. Evaluate $\int_{1\cdot 0}^{1\cdot 3} \sqrt{x}\, dx$ taking $h = 0\cdot 05$ by various methods. What is the error made in using Trapezoidal rule.

32. By using Trapezoidal rule, taking sub-interval lengths as $h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}$, the integral $\int_0^{\pi/2} \sin x\, dx$ was evaluated (for a specific h). The values of the integrals are 0·987116, 0·996785, 0·999196 and 0·999799. Using Romberg's method, improve the result.

33. Find the approximate value of $\int_0^1 \frac{dx}{1+x}$ by Trapezoidal rule taking $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and then use Romberg's method to get more accurate result.

34. Evaluate $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$, using Simpson's rule taking six equal intervals.

35. A curve passes through the points (1, 0·2), (2, 0·7), (3, 1), (4, 1·3), (5, 1·5), (6, 1·7), (7, 1·9), (8, 2·1), (9, 2·3). Find the volume of the solid generated by revolving the area between the curve, the x-axis and $x = 1$, $x = 0$ about the x-axis.

CHAPTER 24

Case Studies: Numerical Integration and Differentiation

The purpose of this chapter is to apply the methods of numerical integration and differentiation discussed in Part Six to practical engineering problems. Two situations are most frequently encountered. In the first case, the function under study can be expressed in analytic form but is too complicated to be readily evaluated using the methods of calculus. Numerical methods are applied to situations of this type by using the analytic expression to generate a table of arguments and function values. In the second case, the function to be evaluated is inherently tabular in nature. This type of function usually represents a series of measurements, observations, or some other empirical information. Data for either case is directly compatible with several schemes discussed in this part of the book.

Section 24.1, which deals with heat calculations from chemical engineering, involves equations. In this application, an analytic function is integrated numerically to determine the heat required to raise the temperature of a material.

Sections 24.2 and 24.3 also involve functions that are available in equation form. Section 24.2, which is taken from civil engineering, uses numerical integration to determine the total wind force acting on the mast of a racing sailboat. Section 24.3 determines the root-mean-square current for an electric circuit. This example is used to demonstrate the utility of Romberg integration and Gauss quadrature.

Section 24.4 focuses on the analysis of tabular information to determine the work required to move a block. Although this application has a direct connection with mechanical engineering, it is germane to all other areas of engineering. Among other things, we use this example to illustrate the integration of unequally spaced data.

24.1 INTEGRATION TO DETERMINE THE TOTAL QUANTITY OF HEAT (CHEMICAL/BIO ENGINEERING)

Background. Heat calculations are employed routinely in chemical and bio engineering as well as in many other fields of engineering. This application provides a simple but useful example of such computations.

One problem that is often encountered is the determination of the quantity of heat required to raise the temperature of a material. The characteristic that is needed to carry out this computation is the heat capacity c. This parameter represents the quantity of heat required to raise a unit mass by a unit temperature. If c is constant over the range of temperatures being examined, the required heat ΔH (in calories) can be calculated by

$$\Delta H = mc\ \Delta T \tag{24.1}$$

where c has units of cal/(g · °C), m = mass (g), and ΔT = change in temperature (°C). For example, the amount of heat required to raise 20 g of water from 5 to 10°C is equal to

$$\Delta H = 20(1)(10 - 5) = 100 \text{ cal}$$

where the heat capacity of water is approximately 1 cal/(g · °C). Such a computation is adequate when ΔT is small. However, for large ranges of temperature, the heat capacity is not constant and, in fact, varies as a function of temperature. For example, the heat capacity of a material could increase with temperature according to a relationship such as

$$c(T) = 0.132 + 1.56 \times 10^{-4} T + 2.64 \times 10^{-7} T^2 \tag{24.2}$$

In this instance you are asked to compute the heat required to raise 1000 g of this material from −100 to 200°C.

Solution. Equation (PT6.4) provides a way to calculate the average value of $c(T)$:

$$\bar{c}(T) = \frac{\int_{T_1}^{T_2} c(T)\, dT}{T_2 - T_1} \tag{24.3}$$

which can be substituted into Eq. (24.1) to yield

$$\Delta H = m \int_{T_1}^{T_2} c(T)\, dT \tag{24.4}$$

where $\Delta T = T_2 - T_1$. Now because, for the present case, $c(T)$ is a simple quadratic, ΔH can be determined analytically. Equation (24.2) is substituted into Eq. (24.4) and the result integrated to yield an exact value of ΔH = 42,732 cal. It is useful and instructive to compare this result with the numerical methods developed in Chap. 21. To accomplish this, it is necessary to generate a table of values of c for various values of T:

T, °C	*c*, cal/(*g* · °C)
−100	0.11904
−50	0.12486
0	0.13200
50	0.14046
100	0.15024
150	0.16134
200	0.17376

These points can be used in conjunction with a six-segment Simpson's 1/3 rule to compute an integral estimate of 42,732. This result can be substituted into Eq. (24.4) to yield a value of ΔH = 42,732 cal, which agrees exactly with the analytical solution. This exact agreement would occur no matter how many segments were used. This is to be expected because c is a quadratic function and Simpson's rule is exact for polynomials of the third order or less (see Sec. 21.2.1).

TABLE 24.1 Results using the trapezoidal rule with various step sizes.

Step Size, °C	ΔH	ε_t (%)
300	96,048	125
150	43,029	0.7
100	42,864	0.3
50	42,765	0.07
25	42,740	0.018
10	42,733.3	<0.01
5	42,732.3	<0.01
1	42,732.01	<0.01
0.05	42,732.00003	<0.01

The results using the trapezoidal rule are listed in Table 24.1. It is seen that the trapezoidal rule is also capable of estimating the total heat very accurately. However, a small step ($< 10°C$) is required for five-place accuracy. The same calculation can also be implemented with software. For example, MATLAB yields

```
>> m=1000;
>> DH=m*quad(@(T) 0.132+1.56e-4*T+2.64e-7*T.^2,-100,200)

DH =
  42732
```

24.3 ROOT-MEAN-SQUARE CURRENT BY NUMERICAL INTEGRATION (ELECTRICAL ENGINEERING)

Background. The average value of an oscillating electric current over one period may be zero. For example, suppose that the current is described by a simple sinusoid: $i(t) = \sin(2\pi t/T)$, where T is the period. The average value of this function can be determined

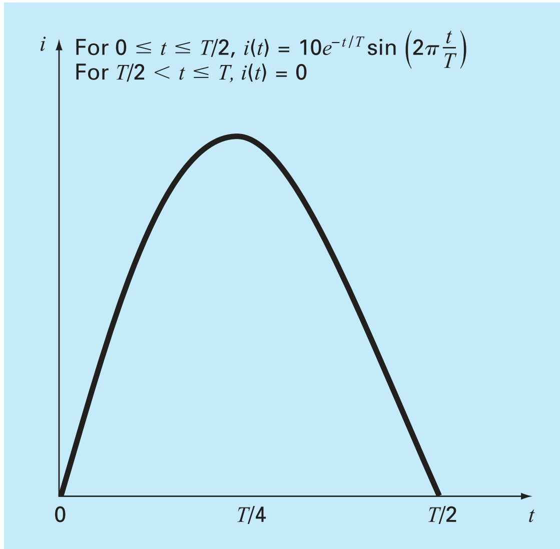

FIGURE 24.3
A periodically varying electric current.

by the following equation:

$$i = \frac{\int_0^T \sin\left(\frac{2\pi t}{T}\right) dt}{T - 0} = \frac{-\cos(2\pi) + \cos 0}{T} = 0$$

Despite the fact that the net result is zero, such current is capable of performing work and generating heat. Therefore, electrical engineers often characterize such current by

$$I_{\text{RMS}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\, dt} \tag{24.11}$$

where $i(t)$ = the instantaneous current. Calculate the RMS or root-mean-square current of the waveform shown in Fig. 24.3 using the trapezoidal rule, Simpson's 1/3 rule, Romberg integration, and Gauss quadrature for $T = 1$ s.

Solution. Integral estimates for various applications of the trapezoidal rule and Simpson's 1/3 rule are listed in Table 24.4. Notice that Simpson's rule is more accurate than the trapezoidal rule.

The exact value for the integral is 15.41261. This result is obtained using a 128-segment trapezoidal rule or a 32-segment Simpson's rule. The same estimate is also determined using Romberg integration (Fig. 24.4).

In addition, Gauss quadrature can be used to make the same estimate. The determination of the root-mean-square current involves the evaluation of the integral ($T = 1$)

$$I = \int_0^{1/2} (10e^{-t} \sin 2\pi t)^2\, dt \tag{24.12}$$

TABLE 24.4 Values for the integral calculated using various numerical schemes. The percent relative error ε_t is based on a true value of 15.41261.

Technique	Segments	Integral	ε_t (%)
Trapezoidal rule	1	0.0	100
	2	15.16327	1.62
	4	15.40143	0.0725
	8	15.41196	4.21×10^{-3}
	16	15.41257	2.59×10^{-4}
	32	15.41261	1.62×10^{-5}
	64	15.41261	1.30×10^{-6}
	128	15.41261	0
Simpson's 1/3 rule	2	20.21769	−31.2
	4	15.48082	−0.443
	8	15.41547	−0.0186
	16	15.41277	1.06×10^{-3}
	32	15.41261	0

FIGURE 24.4
Result of using Romberg integration to estimate the RMS current.

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$
0	20.21769	15.16503	15.41502	15.41261	15.41261
15.16327	15.48082	15.41111	15.41262	15.41261	
15.40143	15.41547	15.41225	15.41261		
15.41196	15.41277	15.41261			
15.41257	15.41262				
15.41261					

First, a change in variable is performed by applying Eqs. (22.23) and (22.24) to yield

$$t = \frac{1}{4} + \frac{1}{4}t_d \qquad dt = \frac{1}{4}\,dt_d$$

These relationships can be substituted into Eq. (24.12) to yield

$$I = \int_{-1}^{1} \left[10e^{-[1/4+(1/4)t_d]} \sin 2\pi\left(\frac{1}{4} + \frac{1}{4}t_d\right)\right]^2 \frac{1}{4}\,dt \tag{24.13}$$

For the two-point Gauss-Legendre formula, this function is evaluated at $t_d = -1/\sqrt{3}$ and $1/\sqrt{3}$, with the results being 7.684096 and 4.313728, respectively. These values can be substituted into Eq. (22.17) to yield an integral estimate of 11.99782, which represents an error of $\varepsilon_t = 22.1\%$.

The three-point formula is (Table 22.1)

$$I = 0.5555556(1.237449) + 0.8888889(15.16327) + 0.5555556(2.684915)$$
$$= 15.65755 \qquad |\varepsilon_t| = 1.6\%$$

The results of using the higher-point formulas are summarized in Table 24.5.

TABLE 24.5 Results of using various-point Gauss quadrature formulas to approximate the integral.

Points	Estimate	ε_t (%)
2	11.9978243	22.1
3	15.6575502	−1.59
4	15.4058023	4.42×10^{-2}
5	15.4126391	-2.01×10^{-4}
6	15.4126109	-1.82×10^{-5}

The integral estimate of 15.41261 can be substituted into Eq. (24.12) to compute an I_{RMS} of 3.925890 A. This result could then be employed to guide other aspects of the design and operation of the circuit.

$$\text{RMS value} = \sqrt{\frac{\int_a^b y^2\,dx}{b-a}}$$

To find : $\quad I = \int_0^{1/2} \left(10e^{-t}\sin 2\pi t\right)^2 dt$ when $T = 1$

where $i(t) = 10e^{-t/T}\sin\left(2\pi\frac{t}{T}\right), \quad 0 \le t \le T/2$

$= 0, \quad T/2 < t \le T$

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Trapezoidal rule

t :	0	0.1	0.2	0.3	0.4	0
$[i(t)]^2$:	0	28.2865	60.631	49.6405	15.5239	0

where $i(t) = 10e^{-t}\sin 2\pi t$

Trapezoidal rule is

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+\cdots+y_{n-1})\right]$$

$$\therefore \int_0^{0.5} [i(t)]^2\,dt = \frac{0.1}{2}\left[(0+0)+2(28.2865+60.631+49.6405+15.5239)\right]$$

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i.e., $\int_0^{0.5} [i(t)]^2 dt = 15.40819$

$\therefore I_{RMS} = \sqrt{15.40819} = 3.9253$

If you wish to use Simpson's 1/3rd rule, take $n = 6$.

$h = \frac{b-a}{n} = \frac{0.5-0}{6} = 0.0833$

t:	0	0.0833	0.1666	0.2499	0.3332	0.4165	0.4998
$[i(t)]^2$:	0	21.1481	53.721	60.6652	38.5538	10.908	0

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Simpson's $1/3^{rd}$ rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[(y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+\cdots)\Big]$$

$$\therefore \int_0^{0.5} (i(t))^2\,dt = \frac{0.0833}{3}\Big[(0+0)+2(53.721+38.5538) + 4(21.1481+60.6652+10.9080)\Big]$$

$$= 15.4226$$

$$\therefore I_{RMS} = \sqrt{15.4226} = 3.9272$$

$i(t) = 10^{-t/T} \sin\left(2\pi \frac{t}{T}\right)$

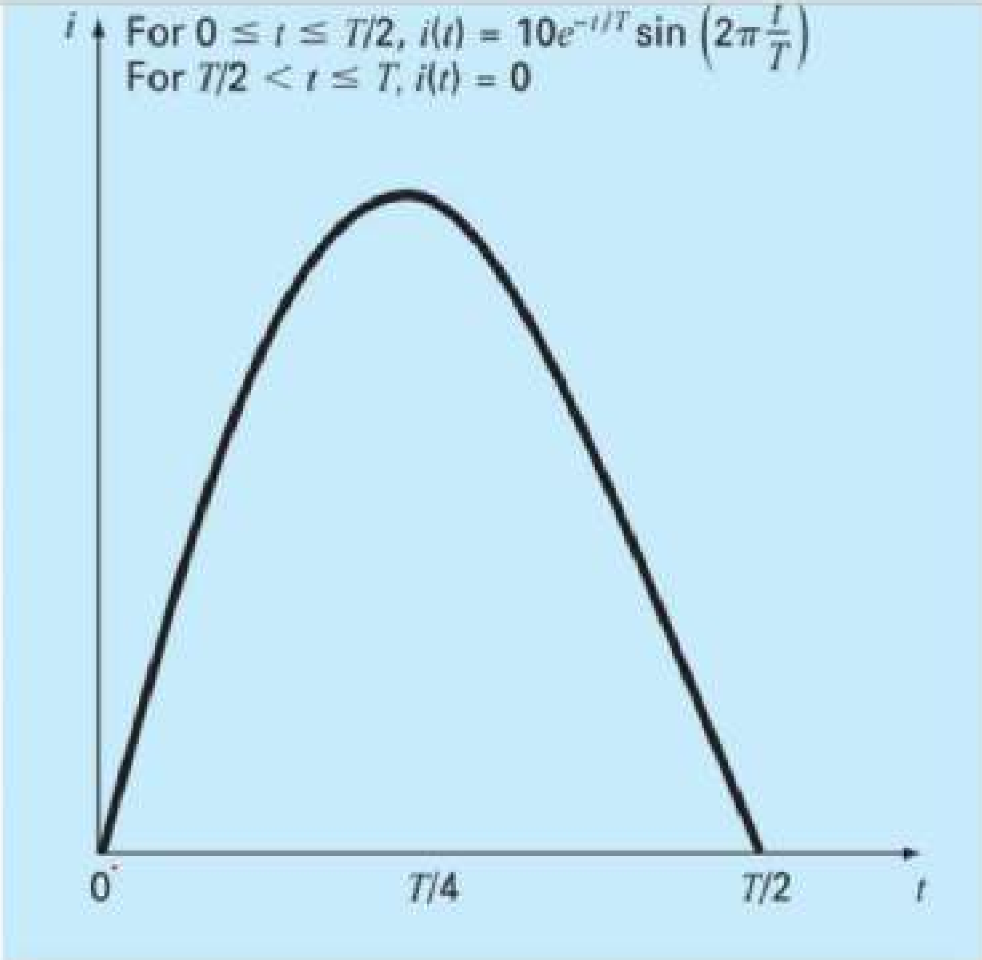

FIGURE 24.3
A periodically varying electric current.

Integration to determine the total quantity of heat

Chemical and bio Engg useful

determination of quantity of heat required to raise the temperature of the material.

heat capacity $\underline{\underline{C}}$. to raise a unit mass by a unit temp.

If c is constant, ΔH (in calories)

$$\boxed{\Delta H = mc\Delta T} \longrightarrow (I)$$

c – cal / (g °c)

m = mass

ΔT - change in temp

The amount of heat required to raise 20 g of water from 5 to 10°c is equal to

[heat capacity of the water is approximately 1 cal/(g°c)

$$\boxed{\Delta H = mc\Delta T}$$

$= 20 * 1 * (10-5)$

$= 20 * 5 = 100$ cal

ΔT is small.

Varies as fun. of temp.

For example the heat capacity of a material could increase with temp according to a relationship such as

$$c(T) = 0.132 + 1.56 * 10^{-4} T + 2.64 * 10^{-7} T^2$$

To compute the heat required to raise 1000g of this material from -100 to $200^\circ c$

Sol) To find ΔH

Average value of $C(T)$ is to be calculated by

$$\bar{C}(T) = \frac{\int_{T_1}^{T_2} C(T)\,dT}{T_2 - T_1} \longrightarrow \text{(II)}$$

WKT $\Delta H = mC\Delta T$

$$\boxed{\Delta T = T_2 - T_1}$$

sub (II) in ΔH

$$\Delta H = m \frac{\int_{T_1}^{T_2} C(T)\,dT}{\cancel{T_2 - T_1}} \cdot \cancel{\Delta T}$$

$$\Delta H = m \int_{T_1}^{T_2} C(T)\, dT$$

$$= 1000 \int_{-100}^{200} [0.132 + 1.56 * 10^{-4} T + 2.64 * 10^{-7} T^2]\, dT$$

Consider $\int_{-100}^{200} [0.132 + 1.56 * 10^{-4} T + 2.64 * 10^{-7} T^2]\, dT$

Simpson's 1/3 Rule $n = 6$

$$\boxed{h = \frac{b-a}{n}}$$

$$h = \frac{200-(-100)}{6} = \frac{200+100}{6} = \frac{300}{6} = 50$$

	T_0	T_1	T_2	T_3	T_4	T_5	T_6
T	-100	-50	0	50	100	150	200
$C(T)$	0.11904	0.12486	0.132	0.14046	0.15024	0.16134	0.17376
	C_0	C_1	C_2	C_3	C_4	C_5	C_6

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\cdots)+4(y_1+y_3+\cdots)\right]$$

$$\therefore \int_{-100}^{200} C(T)\,dT = \frac{h}{3}\left[(C_0+C_6)+2(C_2+C_4)+4(C_1+C_3+C_5)\right]$$

$$= \frac{50}{3}\left[(0.11904+0.17376)+2(0.132+0.15024) + 4(0.12486+0.14046+0.16134)\right]$$

$$= 42.732$$

$$\therefore \Delta H = m\int_{-100}^{200} C(T)\,dT = 1000 * 42.732$$

$$= 42{,}732 \text{ cal}$$

Actual

$$\int_{-100}^{200} C(T)\,dT = \left[0.132\,T + 1.56\times10^{-4}\,\frac{T^2}{2} + 2.64\times10^{-7}\,\frac{T^3}{3}\right]_{-100}^{200}$$

Root mean square current by Numerical Integration

Def Root Mean square (RMS) value AC current is defined as the steady or DC current which when flowing through a circuit for a given time period produces the same heat as produced by the AC current flowing through the same circuit for the same time period. RMS value is also known as effective value or virtual value of AC current.

To calculate RMS value, we need to first calculate the average value of square of AC current/voltage for one time period. Then we find the square root of the calculated average value. This gives the RMS value.

Since the average value of any fun^{n} $f(x)$ having time period T is given by $\frac{1}{T}\int_0^T f(x)\,dx$. $\therefore$, the average value of square of $f(x) = \frac{1}{T}\int_0^T [f(x)]^2 dx$

$$\boxed{\therefore \text{ RMS value} = \sqrt{\frac{1}{T}\int_0^T [f(x)]^2\,dx}}$$

average value of an oscillating electric current over one period may be zero. Ex simple sinusoid ; $i(t) = \sin\left(\frac{2\pi t}{T}\right)$

where T is the period.

$$\text{Average value} = \frac{1}{T}\int_0^T [i(t)]\,dt$$

$$= \frac{1}{T}\int_0^T \sin\left(\frac{2\pi t}{T}\right) dt = \frac{1}{T}\left[-\frac{\cos\left(\frac{2\pi t}{T}\right)}{\frac{2\pi}{T}}\right]_0^T$$

$$= -\frac{1}{T}\cdot\frac{T}{2\pi}\left[\cos\frac{2\pi t}{T}\right]_0^T$$

$$= -\frac{1}{2\pi}\left[\cos 2\pi - \cos 0\right] = -\frac{1}{2\pi}\left[(-1)^2 - 1\right]$$

$$= -\frac{1}{2\pi}\left[1-1\right] = 0$$

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T (i(t))^2 dt}$$

$i(t)$ – the instantaneous current.

Calculate the RMS value of the wave form as show in fig using trapezoidal & simpson's rule for T = 1 s

Sol

$$i(t) = 10 e^{-t/T} \sin\left(\frac{2\pi t}{T}\right) \quad ; 0 < t < T/2$$

$$= 0 \quad ; \text{otherwise.}$$

Given $T = 1\,s$

$$\therefore\ i(t) = 10e^{-t}\sin(2\pi t) \quad ; \quad 0 < t < \tfrac{1}{2}$$

$$= 0 \quad ; \quad \text{Otherwise}$$

$$I_{MS} = \frac{1}{T}\int_0^T [i(t)]^2\,dt$$

$$= \int_0^1 [i(t)]^2\,dt = \int_0^{1/2} [10e^{-t}\sin(2\pi t)]^2\,dt$$

$(a, b) = (0, 0.5)$

Trapezoidal rule $h = 0.1$

t	0	0.1	0.2	0.3	0.4	0.5
$[i(t)]^2$	0	28.2865	60.6310	49.6405	15.5239	0

radian:

$t=0 \quad (i(t))^2 = [10 e^{-0} \sin(2\pi * 0)]^2 = 0$

$t=0.1 \quad [i(t)]^2 = [10 e^{-0.1} \sin(0.2\pi)]^2 = 28.2865$

$t=0.2 \quad [i(t)]^2 = [10 e^{-0.2} \sin(0.4\pi)]^2 = 60.6310$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$\int_0^{0.5} [i(t)]^2\,dt = \frac{0.1}{2}\left[(0+0)+2(28.2865+60.6310+49.6405+15.5239)\right]$$

$$= 15.40819$$

$$\therefore \text{RMS current} = \sqrt{15.40819} =$$

Simpson's Rule $n=6$

$$h = \frac{b-a}{n} = \frac{0.5-0}{6} = 0.0833$$

$n=8$
$h=0.0625$

t	0	0.0833	0.1666	0.2499	0.3332	0.4165
$[i(t)]^2$	0	21.1481	53.7210	60.6652	38.5538	10.9080

$$(i(t)) = 10e^{-t}\sin 2\pi t$$

0.4998
0

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+\cdots)\right]$$

$$\int_0^{0.5}[i(t)]^2\,dt = \frac{0.0833}{3}\Big[(0+0)+2(53.7210+38.5538)$$

$$+4(21.1481+60.6652+10.9080)\Big]$$

$$= 15.4226$$

$\therefore$ RMS current $= \sqrt{15.4226} =$