Chapter 1

# Partial Differential Equations

## 1.1 INTRODUCTION

Partial differential equations are found in problems involving wave phenomena, heat conduction in homogeneous solids and potential theory. As an equation containing ordinary differential coefficients is called an ordinary differential equation, an equation containing partial differential coefficients is called a partial differential equation. Partial derivatives come into being only when there is a dependent variable which is a function of two or more independent variables. Hence in a partial differential equation, there will be one dependent variable and two or more independent variables. However we will mostly deal with partial differential equations containing only two independent variables. In what follows, $z$ will be taken as the dependent variable and $x$ and $y$ the independent variables so that $z = f(x, y)$. We will use the following standard notations to denote the partial derivatives:

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

The *order* of a partial differential equation is that of the highest order derivative occurring in it.

## 1.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Though our main interest is to solve partial differential equations, it will be advantageous if we know how partial differential equations are formed. Knowledge of the formation of partial differential equations will help us to distinguish between two kinds of solutions of the equation. Partial differential equations can be formed by eliminating either arbitrary constants or arbitrary functions from functional relations satisfied by the dependent and independent variables. When we form partial differential equations the following points may be considered for proper procedure and checking.

1. If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the process of elimination results in a partial differential equation of the first order.

**Note** ✍

*In the formation of ordinary differential equations, the order of the equation is equal to the number of constants eliminated.*

2. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the process of elimination will lead to a partial differential equation of second or higher orders.
3. If the partial differential equation is formed by eliminating arbitrary functions, the order of the equation will be, in general, equal to the number of arbitrary functions eliminated.

## 1.3 ELIMINATION OF ARBITRARY CONSTANTS

By way of verifying point 3 of Section 1.2, let us consider the functional relation among

$$x, y, z, \text{ i.e. } f(x, y, z, a, b) = 0 \tag{1}$$

where $a$ and $b$ are arbitrary constants to be eliminated.

Differentiating (1) partially with respect to $x$ and $y$, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0, \text{ i.e. } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot p = 0 \tag{2}$$

and
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0, \text{ i.e. } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot q = 0 \tag{3}$$

Equations (2) and (3) will contain $a$ and $b$.

If we eliminate $a$ and $b$ from equations (1), (2) and (3), we get partial differential equation (involving $p$ and $q$) of the first order. This justifies point 1 of Section 1.2.

## 1.4 ELIMINATION OF ARBITRARY FUNCTIONS

By way of verifying point 3 of Section 1.2 above, let us consider the relation

$$f(u, v) = 0 \tag{1}$$

where $u$ and $v$ are functions of $x$, $y$, $z$ and $f$ is an arbitrary function to be eliminated. Differentiating (1) partially with respect to $x$,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \tag{2}$$

[since $u$ and $v$ are functions of $x$, $y$, $z$ and $z$ is in turn, a function of $x$, $y$]

Differentiating (2) partially with respect to $y$,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \tag{3}$$

Instead of eliminating $f$, let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3).

From (2) and (3), we get

$$\frac{u_x + u_z p}{u_y + u_z q} = \frac{v_x + v_z p}{v_y + v_z q}, \text{ where } u_x = \frac{\partial u}{\partial x}, \text{ etc.}$$

i.e. $$u_x v_y + u_x v_z q + u_z v_y p = u_y v_x + u_y v_z p + u_z v_x q$$

i.e. $$(u_y v_z - u_z v_y) p + (u_z v_x - u_x v_z) q = (u_x v_y - u_y v_x) \tag{4}$$

i.e. $Pp + Qq = R$, say, where $P$, $Q$ and $R$ are functions of $x$, $y$, $z$.

Now equation (4) is a partial differential equation of order 1.

This justifies point 3 of Section 1.2.

## Note ✍

1. *To verify point 3 of Section 1.2, we could have taken a functional relation containing a function of one argument, but we have shown that the order of the partial differential equation formed depends only on the number of arbitrary functions eliminated and not on the number of arguments of the function.*
2. *The equation (4) is called Lagrange's linear equation, whose solution will be discussed later.*

### Worked Examples 1(a)

### *Example 1*

Form the partial differential equation by eliminating the arbitrary constants $a$ and $b$ from the following.

(i) $\log z = a \log x + \sqrt{1 - a^2} \log y + b$

(ii) $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$

(i) $$\log z = a \log x + \sqrt{1 - a^2} \log y + b \tag{1}$$

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we get

$$\frac{1}{z} p = a/x \tag{2}$$

and $$\frac{1}{z} q = \frac{\sqrt{1 - a^2}}{y} \tag{3}$$

If we ignore (1), $b$ is eliminated.

From (2), $a = \dfrac{px}{z}$ and using this in (3), we get

$$\frac{1}{z^2}q^2 = \frac{1}{y^2}\left\{1 - \frac{p^2x^2}{z^2}\right\}$$

i.e.
$$\frac{p^2x^2}{z^2} + \frac{q^2y^2}{z^2} = 1$$

or
$$p^2x^2 + q^2y^2 = z^2$$

(ii)
$$(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha \tag{1}$$

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we get

$$2(x-a) = 2zp\cot^2\alpha \tag{2}$$

and
$$2(y-b) = 2zq\cot^2\alpha \tag{3}$$

Using (2) and (3) in (1), we have

$$z^2(p^2+q^2)\cot^4\alpha = z^2\cot^2\alpha$$

i.e.
$$p^2 + q^2 = \tan^2\alpha$$

### *Example 2*

Form the partial differential equation by eliminating the arbitrary constants $a$ and $b$ from the following.

(i) $\sqrt{1+a^2}\log\left(z + \sqrt{z^2-1}\right) = x + ay + b$

(ii) $z = \dfrac{1}{2}x\sqrt{x^2+a^2} + \dfrac{1}{2}y\sqrt{y^2-a^2} + \dfrac{a^2}{2}\log\left\{\dfrac{x+\sqrt{x^2+a^2}}{y+\sqrt{y^2-a^2}}\right\} + b$

(i)
$$\sqrt{1+a^2}\log\left(z + \sqrt{z^2-1}\right) = x + ay + b \tag{1}$$

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we get

$$\sqrt{1+a^2}\cdot\frac{1}{z+\sqrt{z^2-1}}\cdot\left\{1 + \frac{z}{\sqrt{z^2-1}}\right\}p = 1$$

i.e.
$$\sqrt{1+a^2}\cdot p/\sqrt{z^2-1} = 1 \tag{2}$$

and
$$\sqrt{1+a^2}\cdot\frac{1}{z+\sqrt{z^2-1}}\cdot\left\{1 + \frac{z}{\sqrt{z^2-1}}\right\}q = a$$

i.e.
$$\sqrt{1+a^2}\cdot\frac{q}{\sqrt{z^2-1}} = a \tag{3}$$

From (2) and (3), we get $$\frac{p}{q} = \frac{1}{a} \quad (4)$$

Using (4) in (2), we get

$$\sqrt{1 + \frac{q^2}{p^2}}.p = \sqrt{z^2 - 1}$$

i.e. $$\sqrt{p^2 + q^2} = \sqrt{z^2 - 1} \quad \text{or} \quad p^2 + q^2 + 1 = z^2$$

(*ii*) $$z = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}y\sqrt{y^2 - a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2 + a^2})$$

$$-\frac{a^2}{2}\log(y + \sqrt{y^2 - a^2}) + b \quad (1)$$

Differentiating (1) partially with respect to $x$,

$$p = \frac{1}{2}\left\{x.\frac{x}{\sqrt{x^2 + a^2}} + \sqrt{x^2 + a^2}\right\} + \frac{a^2}{2}.\frac{1}{x + \sqrt{x^2 + a^2}}\left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$

$$= \frac{1}{2}\left[\frac{2x^2 + a^2}{\sqrt{x^2 + a^2}} + \frac{a^2}{\sqrt{x^2 + a^2}}\right] = \sqrt{x^2 + a^2} \quad (2)$$

Similarly, differentiating (1) partially with respect to $y$, we get

$$q = \sqrt{y^2 - a^2} \quad (3)$$

From (2) and (3),

$$p^2 - x^2 = y^2 - q^2$$

i.e. $$p^2 + q^2 = x^2 + y^2$$

### *Example 3*

Form a partial differential equation by eliminating the arbitrary constants $a$, $b$, $c$ from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

We note that the number of constants is more than the number of independent variables. Hence the order of the resulting equation will be more than 1.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}p = 0 \quad (2)$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2}q = 0 \quad (3)$$

Differentiating (2) partially with respect to $x$,

$$\frac{1}{a^2}+\frac{1}{c^2}(zr+p^2) \tag{4}$$

where $$r=\frac{\partial^2 z}{\partial x^2}$$

From (2), $$-\frac{c^2}{a^2}=\frac{zp}{x} \tag{5}$$

From (4), $$\frac{-c^2}{a^2}=zr+p^2 \tag{6}$$

From (5) and (6), we get

$xz\dfrac{\partial^2 z}{\partial x^2}+x\left(\dfrac{\partial z}{\partial x}\right)^2=z\dfrac{\partial z}{\partial x}$ which is the required partial differential equation. This is not the only way of eliminating $a$, $b$ and $c$. Had we differentiated (2) partially with respect to $y$, we would have got

$$\frac{2}{c^2}\{zs+pq\}=0, \text{ where } s=\frac{\partial^2 z}{\partial x\partial y}$$

i.e. $$z\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial z}{\partial y}=0$$

which is also a partial differential equation corresponding to (1).

If we differentiate (3) partially with respect to $y$ and eliminate $b$ and $c$, we will get yet another partial differential equation, namely

$$yz\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0$$

### *Example 4*

Find the partial differential equation of the family of planes, the sum of whose $x$, $y$, $z$ intercepts is unity.

The equation of a plane which cuts off intercepts $a$, $b$, $c$ on the coordinate axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \tag{1}$$

If sum of the intercepts is unity, $a+b+c=1$ or

$$c=1-a-b \tag{2}$$

Using (2) in (1), we get the equation of a plane, the sum of whose $x$, $y$, $z$ - intercepts is unity as

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{1-a-b}=1$$

or $$b(1-a-b)x+a(1-a-b)y+abz=ab(1-a-b) \tag{3}$$

If $a$ and $b$ are treated as arbitrary constants, (3) represents the family of planes having the given property. Differentiating (3) partially with respect to $x$ and then with respect to $y$, we have

$$b(1-a-b)+abp=0 \text{ or } 1-a-b=-ap \tag{4}$$

and

$$a(1-a-b)+abq=0 \text{ or } 1-a-b=-bq \tag{5}$$

From (4) and (5), we get

$$ap=bq \text{ or } \frac{a}{q}=\frac{b}{p}=k \tag{6}$$

Using (6) in (4), $1-k(p+q)=-kpq$

i.e.

$$k=\frac{1}{p+q-pq}$$

$$\therefore \quad a=\frac{q}{p+q-pq},\ b=\frac{p}{p+q-pq} \text{ and } 1-a-b=\frac{-pq}{p+q-pq}$$

Using these values in (3), we have

$$-k^2p^2qx-k^2pq^2y+k^2pqz=-k^3p^2q^2$$

i.e.

$$-px-qy+z=-kpq$$

or $z=px+qy-\dfrac{pq}{p+q-pq}$, which is the required partial differential equation.

### *Example 5*

Find the differential equation of all planes which are at a constant distance $k$ from the origin.

The equation of a plane which is at a distance $k$ from the origin is

$$x\cos\alpha+y\cos\beta+z\cos\nu=k$$

where $\cos\alpha$, $\cos\beta$, $\cos\nu$ are the direction cosines of a normal to the plane.

Taking $\cos\alpha=a$, $\cos\beta=b$ and $\cos\nu=c$ and noting that $a^2+b^2+c^2=1$, the equation of the plane can be assumed as

$$ax+by+\sqrt{1-a^2-b^2}\,z=k \tag{1}$$

If $a$ and $b$ are treated as arbitrary constants, equation (1) represents all planes having the given property.

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we have

$$a+\sqrt{1-a^2-b^2}\,p=0 \tag{2}$$

and

$$b+\sqrt{1-a^2-b^2}\,q=0 \tag{3}$$

From (2) and (3), $$\frac{a}{p} = \frac{b}{q} = -\sqrt{1-a^2-b^2} = \lambda, \text{ say}$$

$\therefore$ $$a = \lambda p, b = \lambda q \text{ and } \sqrt{1-\lambda^2(p^2+q^2)} = -\lambda$$

i.e. $$1 - \lambda^2(p^2+q^2) = \lambda^2$$

$\therefore$ $$\lambda^2 = \frac{1}{1+p^2+q^2} \text{ or } \lambda = -\frac{1}{\sqrt{1+p^2+q^2}}$$

$$(\because \lambda \text{ is negative, as } \lambda = -\sqrt{1-a^2-b^2})$$

Using these values in (1), we get

$$\lambda px + \lambda qy - \lambda z = k$$

i.e. $$z = px + qy - \frac{k}{\lambda} \text{ or}$$

$z = px + qy + k\sqrt{1+p^2+q^2}$, which is the required partial differential equation.

### *Example 6*

Find the differential equation of all spheres of the same radius $c$ having their centres on the $yoz$-plane.

The equation of a sphere having its centre at $(0, a, b)$, that lies on the $yoz$-plane and having its radius equal to $c$ is

$$x^2 + (y-a)^2 + (z-b)^2 = c^2 \quad (1)$$

If $a$ and $b$ are treated as arbitrary constants, (1) represents the family of spheres having the given property.

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we have

$$2x + 2(z-b)p = 0 \quad (2)$$

and $$2(y-a) + 2(z-b)q = 0 \quad (3)$$

From (2), $$z - b = -\frac{x}{p} \quad (4)$$

Using (4) in (3), $$y - a = \frac{qx}{p} \quad (5)$$

Using (4) and (5) in (1), we get

$$x^2 + \frac{q^2x^2}{p^2} + \frac{x^2}{p^2} = c^2$$

i.e. $(1 + p^2 + q^2)x^2 = c^2p^2$, which is the required partial differential equation.

### *Example 7*

Find the differential equation of all spheres whose centres lie on the $x$-axis.
The equation of any sphere whose centre is $(a, 0, 0)$ (that lies on the $x$-axis) and whose radius is $b$ is

$$(x-a)^2 + y^2 + z^2 = b^2 \tag{1}$$

If $a$ and $b$ are treated as arbitrary constants, (1) represents the family of spheres having the given property.

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we have

$$2(x-a) + 2zp = 0 \tag{2}$$

$$2y + 2zq = 0 \tag{3}$$

The required equation is provided by (3).

i.e. $$\text{it is } z\frac{\partial z}{\partial y} + y = 0$$

### *Example 8*

Find the differential equation of all spheres whose radii are the same.
The equation of all spheres with equal radius can be taken as

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \tag{1}$$

where $a$, $b$, $c$ are arbitrary constants and $R$ is a given constant.

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we have

$$(x-a) + (z-c)p = 0 \tag{2}$$

and $$(y-b) + (z-c)q = 0 \tag{3}$$

Differentiating (2) and (3) with respect to $x$ and $y$ respectively, we get

$$1 + (z-c)r + p^2 = 0 \tag{4}$$

and $$1 + (z-c)t + q^2 = 0 \tag{5}$$

Eliminating $(z-c)$ from (4) and (5), we have

$$\frac{r}{t} = \frac{1+p^2}{1+q^2}$$

i.e. $$r(1+q^2) = t(1+p^2), \text{ where } r = \frac{\partial^2 z}{\partial x^2} \text{ and } t = \frac{\partial^2 z}{\partial y^2}.$$

## Note ✍

*The answer is not unique. We can get different partial differential equations.*

### *Example 9*

Form the partial differential equation by eliminating the arbitrary function '$f$' from

(*i*) $z = e^{ay} f(x + by)$; and

(*ii*) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

(i) $$z = e^{ay}.f(x + by)$$

i.e. $$e^{-ay} z = f(x + by) \tag{1}$$

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we get

$$e^{-ay} p = f'(u) \cdot 1 \tag{2}$$

$$e^{-ay} q - ae^{-ay} z = f'(u) b \tag{3}$$

where $u = x + by$
Eliminating $f'(u)$ from (2) and (3), we get

$$\frac{q - az}{p} = b$$

i.e. $$q = az + bp$$

(*ii*) $$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

i.e. $$z - y^2 = 2f\left(\frac{1}{x} + \log y\right) \tag{1}$$

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we get

$$p = 2f'(u).\left(\frac{-1}{x^2}\right) \tag{2}$$

and $$q - 2y = 2f'(u).\left(\frac{1}{y}\right) \tag{3}$$

where $u = \frac{1}{x} + \log y$
Dividing (2) by (3), we have

$$\frac{p}{q - 2y} = \frac{-y}{x^2}$$

i.e. $$px^2 + qy = 2y^2$$

which is the required partial differential equation.

### *Example 10*

Form the partial differential equation by eliminating the arbitrary function '$f$' from

(*i*) $xy + yz + zx = f\left(\dfrac{z}{x+y}\right)$ and

(*ii*) $f(z - xy, x^2 + y^2) = 0$

(*i*)
$$xy + yz + zx = f\left(\frac{z}{x+y}\right) \tag{1}$$

Differentiating (1) partially with respect to $x$ and then with respect to $y$, we have

$$y + yp + xp + z = f'(u)\left\{\frac{(x+y)p - z}{(x+y)^2}\right\} \tag{2}$$

and
$$x + yq + z + xq = f'(u)\left\{\frac{(x+y)q - z}{(x+y)^2}\right\} \tag{3}$$

Dividing (2) by (3), we have

$$\frac{(y+z)+(x+y)p}{(z+x)+(x+y)q} = \frac{(x+y)p - z}{(x+y)q - z}$$

i.e. $(x+y)(z+x)p - z(z+x) - z(x+y)q$

$$= (x+y)(y+z)q - z(y+z) - z(x+y)p$$

i.e.
$$(x+y)(x+2z)p - (x+y)(y+2z)q = z(x-y)$$

which is a Lagrange linear equation.

(*ii*)
$$f(z - xy, x^2 + y^2) = 0 \tag{1}$$

i.e.
$$f(u.v) = 0$$

If we assume that $u$ can be expressed as a single-valued function of $v$, (1) can be rewritten as

$$z - xy = \phi(x^2 + y^2) \tag{2}$$

where $\phi$ is an arbitrary function.

Differentiating (2) partially with respect to $x$ and then with respect to $y$, we have

$$p - y = \phi'(u).2x \tag{3}$$

and
$$q - x = \phi'(u).2y \tag{4}$$

Eliminating $\phi'(u)$ from (3) and (4), we get

$$\frac{p-y}{q-x} = \frac{x}{y} \text{ or } yp - xq = y^2 - x^2$$

## Note ✍

*Without assuming that $u = \phi(v)$, we can eliminate 'f' and form the equation alternatively as given in the following example.*

### *Example 11*

Form the partial differential equation by eliminating '$f$' from

(*i*) $f(z - xy, x^2 + y^2) = 0$ and
(*ii*) $f(x^2 + y^2 + z^2, ax + by + cz) = 0$

(i)
$$f(z - xy, x^2 + y^2) = 0 \tag{1}$$

By putting $z - xy = u$ and $x^2 + y^2 = v$, (1) becomes

$$f(u, v) = 0 \tag{2}$$

Differentiating (2) partially with respect to $x$ and then with respect to $y$, we have

$$\frac{\partial f}{\partial u}.(p - y) + \frac{\partial f}{\partial v}(2x) = 0 \tag{3}$$

and
$$\frac{\partial f}{\partial u}(q - x) + \frac{\partial f}{\partial v}(2y) = 0 \tag{4}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} p - y & 2x \\ q - x & 2y \end{vmatrix} = 0$$

i.e.
$$2y(p - y) - 2x(q - x) = 0$$

or
$$yp - xq = y^2 - x^2$$

(*ii*)
$$f(x^2 + y^2 + z^2, ax + by + cz) = 0 \tag{1}$$

Putting $u = x^2 + y^2 + z^2$ and $v = ax + by + cz$, (1) becomes

$$f(u, v) = 0 \tag{2}$$

Differentiating (2) partially with respect to $x$ and then with respect to $y$, we have

$$\frac{\partial f}{\partial u}(2x + 2zp) + \frac{\partial f}{\partial v}(a + cp) = 0 \tag{3}$$

and
$$\frac{\partial f}{\partial u}(2y + 2zq) + \frac{\partial f}{\partial v}(b + cq) = 0 \tag{4}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} x + zp & a + cp \\ y + zq & b + cq \end{vmatrix} = 0$$

i.e.
$$(x + zp)(b + cq) = (y + zq)(a + cp)$$

i.e.
$$(cy - bz)p + (az - cx)q = bx - ay$$

## *Example 12*

Form the partial differential equation by eliminating the arbitrary functions $f$ and $g$ from $z = f(2x + y) + g(3x - y)$

$$z = f(2x + y) + g(3x - y) \tag{1}$$

Differentiating (1) partially with respect to $x$,

$$p = f'(u).2 + g'(v).3 \tag{2}$$

where $u = 2x + y$ and $v = 3x - y$
Differentiating (1) partially with respect to $y$,

$$q = f'(u).1 + g'(v)(-1) \tag{3}$$

Differentiating (2) partially with respect to $x$ and then with respect to $y$,

$$r = f''(u).4 + g''(v).9 \tag{4}$$

and

$$s = f''(u).2 + g''(v).(-3) \tag{5}$$

Differentiating (3) partially with respect to $y$,

$$t = f''(u).1 + g''(v).1 \tag{6}$$

Eliminating $f''(u)$ and $g''(v)$ from (4), (5) and (6) using determinants, we have

$$\begin{vmatrix} 4 & 9 & r \\ 2 & -3 & s \\ 1 & 1 & t \end{vmatrix} = 0$$

i.e.

$$5r + 5s - 30t = 0$$

or

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

## *Example 13*

Form the differential equation by eliminating the arbitrary functions $f$ and $\phi$ from $z = f(ax + by) + \phi(cx + dy)$.

$$z = f(u) + \phi(v) \tag{1}$$

where $u = ax + by$ and $v = cx + dy$

Differentiating partially with respect to $x$ and $y$,

$$p = f'(u) \cdot a + \phi'(v) \cdot c \tag{2}$$

$$q = f'(u) \cdot b + \phi'(v)d \tag{3}$$

$$r = f''(u) \cdot a^2 + \phi''(v) \cdot c^2 \tag{4}$$

$$s = f''(u) \cdot ab + \phi''(v) \cdot cd \tag{5}$$

$$t = f''(u) \cdot b^2 + \phi''(v) \cdot d^2 \tag{6}$$

Eliminating $f''(u)$ and $\phi''(v)$ from (4),(5),(6), we have

$$\begin{vmatrix} r & a^2 & c^2 \\ s & ab & cd \\ t & b^2 & d^2 \end{vmatrix} = 0$$

i.e. $$(abd^2 - b^2cd)r - (a^2d^2 - b^2c^2)s + (a^2cd - abc^2)t = 0$$

i.e. $$bd(ad - bc)r - (ad + bc)(ad - bc)s + ac(ad - bc)t = 0$$

i.e. $$bd\frac{\partial^2 z}{\partial x^2} - (ad + bc)\frac{\partial^2 z}{\partial x \partial y} + ac\frac{\partial^2 z}{\partial y^2} = 0.$$

### *Example 14*

Form the differential equation by eliminating $f$ and $g$ from $z = xf(ax + by) + g(ax + by)$.

$$z = x \cdot f(u) + g(u) \tag{1}$$

where $u = ax + by$.
Differentiating partially with respect to $x$ and $y$,

$$p = xf'(u) \cdot a + f(u) + g'(u) \cdot a \tag{2}$$

$$q = xf'(u) \cdot b + g'(u) \cdot b \tag{3}$$

$$r = x \cdot f''(u)a^2 + f'(u) \cdot 2a + g''(u) \cdot a^2 \tag{4}$$

$$s = xf''(u)ab + f'(u)b + g''(u)ab \tag{5}$$

$$t = xf''(u)b^2 + g''(u) \cdot b^2 \tag{6}$$

$[(4) \times b - (5) \times 2a]$ gives

$$br - 2as = -a^2b[xf''(u) + g''(u)] \tag{7}$$

$$= -a^2b \times \frac{1}{b^2}t, \quad \text{from (6)}$$

i.e. $$b^2\frac{\partial^2 z}{\partial x^2} - 2ab\frac{\partial^2 z}{\partial x \partial y} + a^2\frac{\partial^2 z}{\partial y^2} = 0$$

### *Example 15*

Form the differential equation by eliminating the arbitrary functions $f$ and $g$ from

$$z = f(x+iy) + (x+iy)g(x-iy), \quad \text{where} \quad i = \sqrt{-1} \quad \text{and} \quad x+iy \neq z$$

$$z = f(u) + (x+iy)g(v) \tag{1}$$

where $u = x+iy$ and $v = x-iy$.

Differentiating partially with respect to $x$ and $y$,

$$p = f'(u)\cdot 1 + (x+iy)g'(v)\cdot 1 + g(v) \tag{2}$$

$$q = f'(u)\cdot i + (x+iy)g'(v)(-i) + g(v)\cdot i \tag{3}$$

$$r = f''(u)\cdot 1 + (x+iy)g''(v)\cdot 1 + 2g'(v)\cdot 1 \tag{4}$$

$$s = f''(u)\cdot i + (x+iy)g''(v)(-i) \tag{5}$$

$$t = f''(u)(-1) + (x+iy)g''(v)\cdot(-1) + 2g'(v) \tag{6}$$

Adding (4) and (6), we get

$$r + t = 4g'(v) \tag{7}$$

From (2) and (3), we get

$$p + iq = 2(x+iy)g'(v) \tag{8}$$

Eliminating $g'(v)$ from (7) and (8), we get

$$r + t = 2\frac{(p+iq)}{x+iy}$$

i.e $$(x+iy)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 2\left(\frac{\partial z}{\partial x} + i\frac{\partial z}{\partial y}\right)$$

## Note ✍

*Equation (5), giving the value of s, is not at all used.*

### *Example 16*

If $u = f(x^2+y) + \phi(x^2-y)$, show that $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{x}\dfrac{\partial u}{\partial x} - 4x^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

$$u = f(v) + \phi(w) \tag{1}$$

where $v = x^2 + y$ and $w = x^2 - y$.

Differentiating partially with respect to $x$ and $y$,

$$\frac{\partial u}{\partial x} = f'(v) \cdot 2x + \phi'(w) \cdot 2x \tag{2}$$

$$\frac{\partial u}{\partial y} = f'(v) \cdot 1 + \phi'(w) \cdot (-1) \tag{3}$$

$$\frac{\partial^2 u}{dx^2} = f'(v) \cdot 2 + f''(v) \cdot 4x^2 + \phi'(w) \cdot 2 + \phi''(w) \cdot 4x^2 \tag{4}$$

$$\frac{\partial^2 u}{\partial x \partial y} = f''(v) \cdot 2x + \phi''(w) \cdot (-2x) \tag{5}$$

$$\frac{\partial^2 u}{\partial y^2} = f''(v) \cdot 1 + \phi''(w) \cdot 1 \tag{6}$$

Eq. (4) can be rewritten as

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= 2\{f'(v) + \phi'(w)\} + 4x^2\{f''(v) + \phi''(w)\} \\ &= 2 \times \frac{1}{2x}\frac{\partial u}{\partial x} + 4x^2 \cdot \frac{\partial^2 u}{\partial y^2}, \qquad \text{from (2) and (6)}\end{aligned}$$

i.e.

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\frac{\partial u}{\partial x} - 4x^2\frac{\partial^2 u}{\partial y^2} = 0$$

### *Example 17*

Form the differential equation by eliminating $f$ and $\phi$ from $z = f(x+y) \cdot \phi(x-y)$.

$$z = f(u) \cdot \phi(v) \tag{1}$$

where $u = x + y$ and $v = x - y$.

Differentiating partially with respect to $x$ and $y$, we get

$$p = f(u) \cdot \phi'(v) + f'(u) \cdot \phi(v) \tag{2}$$

$$q = f(u)\phi'(v)(-1) + f'(u)\phi(v) \tag{3}$$

$$r = f(u)\phi''(v) + 2f'(u)\phi'(v) + f''(u) \cdot \phi(v) \tag{4}$$

$$s = f(u)\phi''(v)(-1) + f''(u) \cdot \phi(v) \tag{5}$$

$$t = f(u) \cdot \phi''(v) - 2f'(u)\phi'(v) + f''(u)\phi(v) \tag{6}$$

Subtracting (5) from (3), we get

$$r - t = 4f'(u) \cdot \phi'(v) \tag{7}$$

From (1) and (2), we get

$$\begin{aligned} p^2 - q^2 &= 4f(u)\cdot\phi(u)\cdot f'(u)\cdot\phi'(v) \\ &= z(r-t) \text{ from (1) and (7)} \end{aligned}$$

i.e. $$z\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

### *Example 18*

Form the differential equation by eliminating $f$ and $\phi$ from $z = xf(y/x) + y\phi(x)$.

$$z = xf(u) + y\phi(x) \tag{1}$$

where $u = \frac{y}{x}$.
Differentiating partially with respect to $x$ and $y$, we get

$$p = xf'(u)\cdot\left(-\frac{y}{x^2}\right) + f(u) + y\phi'(x)$$

i.e. $$p = -\frac{y}{x}\cdot f'(u) + f(u) + y\phi'(x) \tag{2}$$

$$q = x\cdot f'(u)\cdot\frac{1}{x} + \phi(x)$$

i.e. $$q = f'(u) + \phi(x) \tag{3}$$

$$r = -\frac{y}{x}\cdot f''(u)\left(-\frac{y}{x^2}\right) + y\phi''(x)$$

i.e. $$r = \frac{y^2}{x^3}f''(u) + y\phi''(x) \tag{4}$$

$$s = -\frac{y}{x^2}f''(u) + \phi'(x) \tag{5}$$

$$t = \frac{1}{x}f''(u) \tag{6}$$

Eliminating $f''(u)$ from (5) and (6), we get

$$s + \frac{y}{x}t = \phi'(x) \tag{7}$$

From (2) and (3), we get

$$px + qy = \{xf(u) + y\phi(x)\} + xy\phi'(x)$$

i.e. $$px + qy = z + xy\phi'(x) \tag{8}$$

Eliminating $\phi'(x)$ from (7) and (8), we get

$$xys + y^2t = px + qy - z$$

i.e. $$xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$

### *Example 19*

Form the differential equation by eliminating $f$ and $\phi$ from $z = f(y) + \phi(x+y+z)$

$$z = f(y) + \phi(u) \tag{1}$$

where $u = x + y + z$.
Differentiating partially with respect to $x$ and $y$, we get

$$p = \phi'(u)(1+p) \tag{2}$$

$$q = f'(y) + \phi'(u)(1+q) \tag{3}$$

$$r = \phi'(u)\cdot r + \phi''(u)\cdot(1+p)^2 \tag{4}$$

$$s = \phi'(u)\cdot s + \phi''(u)(1+p)(1+q) \tag{5}$$

$$t = f''(y) + \phi'(u)t + \phi'(u)(1+q)^2 \tag{6}$$

From (4), $$r\{1 - \phi'(u)\} = (1+p)^2\phi''(u) \tag{7}$$

From (5), $$s\{1 - \phi'(u)\} = (1+p)(1+q)\phi''(u) \tag{8}$$

Dividing (7) by (8), we get

$$\frac{r}{s} = \frac{1+p}{1+q}$$

i.e. $$\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial x}\right)\frac{\partial^2 z}{\partial x^2}$$

### *Example 20*

Form the differential equation by eliminating the arbitrary function $\phi$ from

$$z = \frac{1}{x}\phi(y-x) + \phi'(y-x).$$

## Note ✍

*Though $\phi'$ is the derivative of $\phi$, we should not assume that only one function is to be eliminated. We have to eliminate two functions $\phi$ and $\phi'$ and hence the resulting partial differential equation will be of order 2.*

$$z = \frac{1}{x}\,\phi(u) + \phi'(u) \tag{1}$$

where $u = y - x$
Differentiating partially with respect to $x$ and $y$, we get

$$p = \frac{1}{x}\phi'(u)\cdot(-1) - \frac{1}{x^2}\phi(u) + \phi''(u)(-1) \tag{2}$$

$$q = \frac{1}{x}\phi'(u)\cdot 1 + \phi''(u)\cdot 1 \tag{3}$$

$$r = \frac{1}{x}\phi''(u)\cdot 1 + \frac{2}{x^2}\phi'(u) + \frac{2}{x^3}\phi(u) + \phi'''(u)\cdot 1 \tag{4}$$

$$s = -\frac{1}{x^2}\phi'(u) - \frac{1}{x}\phi''(u) + \phi'''(u)(-1) \qquad (5)$$

$$t = \frac{1}{x}\phi''(u)\cdot 1 + \phi'''(u)\cdot 1 \qquad (6)$$

From (4) and (6), we get

$$\begin{aligned} r - t &= \frac{2}{x^2}\phi'(u) + \frac{2}{x^3}\phi(u) \\ &= \frac{2}{x^2}\left\{\frac{1}{x}\phi(u) + \phi'(u)\right\} \\ &= \frac{2}{x^2}z \end{aligned}$$

i.e.
$$x^2\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = 2z$$

## Exercise 1(a)

### Part A (Short-Answer Questions)

1. Write down the form of the P.D.E. (partial differential equation), obtained by eliminating '$f$' from $f(u, v) = 0$.

Form the P.D.E.s by eliminating the arbitrary constants $a$ and $b$ from the following relations:

2. $z = (x + a)(y + b)$
3. $z = (x^2 + a^2)(y^2 + b^2)$
4. $z = ax + by + ab$
5. $z = ax + by + a^2 + b^2$
6. $z = ax^3 + by^3$
7. $z = a(x + y) + b$
8. $ax^2 + by^2 + z^2 = 1$
9. $(x - a)^2 + (y - b)^2 = z^2$.

Form the P.D.E.s by eliminating the arbitrary functions from the following relations.

10. $z = f(x^2 + y^2)$
11. $z = \phi(x^3 - y^3)$
12. $z = f(bx - ay)$

13. $z = \phi(xy)$

14. $z = f\left(\frac{y}{x}\right)$

15. $z = f(x) + \phi(y)$

16. $z = f(x) + \phi(y) + axy$

17. $z = f(y) + x\phi(y)$

18. $z = yf(x) + \phi(x)$

19. $z = xf(y) + \phi(y) - \sin x$

20. $z = yf(x) + \phi(x) - \cos y$

**Part B**

21. Form the P.D.E. by eliminating $a$ and $b$ from $z = xy + y\sqrt{x^2 - a^2} + b$.

22. Form the P.D.E. by eliminating $a$ and $b$ from $z = ax - \frac{a}{a+1}y + b$.

23. Form the P.D.E. by eliminating $a$ and $b$ from $4z(1 + a^2) = (x + ay + b)^2$.

24. Form the P.D.E. by eliminating $a$ and $b$ from $z^2 + \left\{z\sqrt{z^2 - 4a^2} - 4a^2 \log\left(z + \sqrt{z^2 - 4a^2}\right)\right\} = 4(x + ay + b)$.

25. Form the P.D.E. by eliminating $a$ and $b$ from $3z = ax^3 + 2\sqrt{a-1}\,y^{3/2} + b$.

26. Find the P.D.E. of all planes which cut off equal intercepts on the $x$ and $y$ axes.

27. Find the P.D.E. of all planes passing through the origin.

28. Find the P.D.E. of all spheres whose centres lie on the $z$-axis.

29. Find the P.D.E. of all spheres of radius $c$ having their centres on the $xoy$-plane.

30. Find the P.D.E. of all spheres of radius $c$ having their centres on the $zox$-plane.

31. Form the P.D.E. by eliminating the arbitrary function '$f$' from
(a) $z = f\left(\frac{xy}{z}\right)$; (b) $z = f(x^2 + y^2 + z^2)$

32. Form the P.D.E. by eliminating the arbitrary function $f$ from
(a) $xyz = f(x + y + z)$; (b) $\frac{xy}{z} = f(x^2 - y + z)$

33. Form the P.D.E. by eliminating '$\phi$' from
(a) $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$; (b) $\phi(x^3 - y^3, x^2 - z^2) = 0$

34. Form the P.D.E. by eliminating '$\phi$' from
(a) $\phi\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0$; (b) $\phi\left(x^2 - y^2 - 2z, \frac{y}{zx}\right) = 0$

35. Form the P.D.E. by eliminating '$\phi$' from

$$(a)\ \phi\left(\frac{x-y}{y-z},\ xy+yz+zx\right)=0; \quad (b)\ \phi\left(\frac{x+y+z}{z}, x^2-y^2\right)=0$$

Form the P.D.E.s by eliminating the arbitrary functions from the following relations.

36. $z = f(x+iy) + g(x-iy)$, where $i=\sqrt{-1}$ and $x+iy \neq z$.
37. $z = f(2y+3x) + g(y-3x)$.
38. $z = f_1(y-x) + f_2(y+x) + f_3(y+2x)$.
39. $z = xf(2x+3y) + g(2x+3y)$
40. $z = f(x+y) + yg(x+y)$
41. $z = (x-iy)f(x+iy) + g(x-iy)$, where $i=\sqrt{-1}$ and $x+iy \neq z$.
42. $z = f(\sqrt{x}+y) + g(\sqrt{x}-y)$
43. $z = f(x)\cdot\phi(y)$
44. $z = yf(x) + x\phi(y)$
45. $z = f(x+y+z) + \phi(x-y)$.

## 1.5 SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

The relation between the independent variables and the dependent variable (containing arbitrary constants or functions) from which a partial differential equation is formed is called the *primitive* or *solution* of the P.D.E.

In other words, a *solution* of a P.D.E. is a relation between the independent and the dependent variables, which satisfies the P.D.E. Solution of a P.D.E. is also called *integral* of the P.D.E.

As was seen in Section 1.2, the primitive of a P.D.E. may contain arbitrary constants or arbitrary functions. Accordingly, we have two types of solutions for a P.D.E.

A solution of a P.D.E. which contains as many arbitrary constants as the number of independent variables is called *the complete solution* or *complete integral* of the equation.

A solution of a P.D.E. which contains as many arbitrary functions as the order of the equation is called the *general solution* or *general integral* of the equation.

Both these types of solutions can be obtained for the same P.D.E. For example, the equation $z = px + qy$ is obtained when we eliminate the arbitrary constants $a$ and $b$ from $z = ax + by$ or the arbitrary function '$f$' from $z = x\cdot f\left(\frac{y}{x}\right)$.

Thus $z = ax + by$ is the complete solution and $z = x\cdot f\left(\frac{y}{x}\right)$ is the general solution of the P.D.E. $z = px + qy$.

The complete solution $z = ax + by$ can be rewritten as $z = x\left\{a + b\left(\frac{y}{x}\right)\right\}$. Comparing this with the general solution $z = xf\left(\frac{y}{x}\right)$, we note that $a + b\left(\frac{y}{x}\right)$ is a particular case of $f(y/x)$. Hence the general solution of a P.D.E. is more general than the complete solution. Thus when the solution of a P.D.E. is required, we should try to give the general solution. However there are certain P.D.E.s for which methods are not available for finding the general solutions directly, but methods are available for finding the complete solutions only in other cases. In such cases, we indicate the procedure for finding the general solution from the complete solution as explained in Section 1.6.

## 1.6 PROCEDURE TO FIND GENERAL SOLUTION

Let
$$F(x, y, z, p, q) = 0 \tag{1}$$
be a first order P.D.E. Let its complete solution be
$$\phi(x, y, z, a, b) = 0 \tag{2}$$
where $a$ and $b$ are arbitrary constants.

Let $b = f(a)$ [or $a = g(b)$], where '$f$' is an arbitrary function.

Then (2) becomes
$$\phi[x, y, z, a, f(a)] = 0 \tag{3}$$

Differentiating (2) partially with respect to $a$, we get
$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot f'(a) = 0 \tag{4}$$

Theoretically, it is possible to eliminate '$a$' between (3) and (4).

This eliminant, which contains the arbitrary function '$f$', is the general solution of (1).

A solution obtained by giving particular values to the arbitrary constants in the complete solution or to the arbitrary functions in the general solution is called *a particular solution* or *particular integral* of the P.D.E.

Thus for the P.D.E. $z = px + qy$, for which the complete solution is $z = ax + by$ and the general solution is $z = x \cdot f\left(\frac{y}{x}\right)$, the following are particular solutions.

(i) $z = 2x + 3y$
(ii) $z = 3x - 4y$
(iii) $z = x \cdot e^{\frac{y}{x}}$
(iv) $z = x \sin\left(\frac{y}{x}\right)$

There is yet another type of solution of a P.D.E., called the *singular solution* or *singular integral*. Geometrically the singular solution of a P.D.E. represents the envelope of the family of surfaces represented by the complete solution of that P.D.E. The singular solution will neither contain arbitrary constants nor arbitrary functions but at the same time cannot be obtained as particular case of the complete or general solution.

## 1.7 PROCEDURE TO FIND SINGULAR SOLUTION

Let $$F(x, y, z, p, q) = 0 \tag{1}$$

be a first order P.D.E.

Let its complete solution be

$$\phi(x, y, z, a, b) = 0 \tag{2}$$

Differentiating (2) partially with respect to $a$ and then $b$, we have

$$\frac{\partial \phi}{\partial a} = 0 \tag{3}$$

and $$\frac{\partial \phi}{\partial b} = 0 \tag{4}$$

The eliminant of $a$ and $b$ from equations (2), (3) and (4), if it exists, is the singular solution of the P.D.E. (1).

As pointed out earlier, P.D.E.s can be divided into two categories — one for which methods are readily available only for finding complete solutions and the other for which methods are available for finding general solutions. First order non-linear equations that belong to the first category will be discussed in Section 1.8.

## 1.8 COMPLETE SOLUTIONS OF FIRST ORDER NON-LINEAR P.D.E.S

A P.D.E., the partial derivatives occurring in which are of the first degree, is said to be *linear*; otherwise it is said to be *non-linear*.

First order non-linear P.D.E.s, for which complete solutions can be found out, are divided into four standard types. Some first-order non-linear P.D.E.s, which do not fall under any of the four standard types, can be transformed into one or the other of the standard types by suitable changes of variables. We shall discuss below the special methods of finding the complete solutions for these types of equations.

### *Type I*

*Equations of the form* $f(p, q) = 0$, i.e. the P.D.E.s that contain $p$ and $q$ only explicitly.

For equations of this type, it is known that a solution will be of the following form,

$$z = ax + by + c \tag{1}$$

But this solution contains three arbitrary constants, whereas the number of independent variables is two. Hence if we can reduce the number of arbitrary constants in (1) by one, it becomes the complete solution of the equation $f(p, q) = 0$. Now from (1), $p = a$ and $q = b$. If (1) is to be a solution of $f(p, q) = 0$, the values of $p$ and $q$ obtained from (1) should satisfy the given equation.

i.e. $$f(a, b) = 0$$

Solving this, we can get $b = \phi(a)$, where $\phi$ is a known function. Using this value of $b$ in (1), the complete solutiion of the given P.D.E. is

$$z = ax + \phi(a)y + c \tag{2}$$

The general solution can be obtained from (2) by the method given earlier.

To find the singular solution, we have to eliminate $a$ and $c$ from

$$z = ax + \phi(a)y + c, \quad x + \phi'(a)y = 0 \quad \text{and} \quad 1 = 0$$

of which the last equation is absurd. Hence there is no singular solution for equations of type I.

### *Type II*

*Clairaut's type*, i.e. the P.D.E.s of the form

$$z = px + qy + f(p, q) \tag{1}$$

For equations of this type also, it is known that a solution will be of the form

$$z = ax + by + c \tag{2}$$

If we can reduce the number of arbitrary constants in (2) by one, it becomes the complete solution of (1).

From (2) we get $p = a$ and $q = b$.

As before, $$z = ax + by + f(a, b) \tag{3}$$

From (2) and (3), we get $c = f(a, b)$

Thus the complete solution of (1) is given by (3).

## Note

***Without going through all these formalities, we can quickly write down the complete solution of a clairaut's type of P.D.E. by simply replacing p and q by a and b in it respectively.***

The general and singular solutions of (1) can be found out by the usual methods. For clairaut's type of equations, singular solutions will normally exist.

### *Type III*

*Equations not containing x and y explicitly*, i.e. equations of the form

$$f(z, p, q) = 0 \tag{1}$$

For equations of this type, it is known that a solution will be of the form

$$z = \phi(x + ay) \tag{2}$$

where '$a$' is an arbitrary constant and $\phi$ is a specific function to be found out.

Putting $x + ay = u$, (2) becomes $z = \phi(u)$ or $z(u)$

$$\therefore \qquad p = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$\text{and} \qquad q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

If (2) is to be a solution of (1), the values of $p$ and $q$ obtained should satisfy (1).

$$\text{i.e. } f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \tag{3}$$

From (3), we can get

$$\frac{dz}{du} = \psi(z, a) \tag{4}$$

Now (4) is an ordinary differential equation, which can be solved by the variable separable method.

The solution of (4), which will be of the form $g(z, a) = u + b$ or $g(z, a) = x + ay + b$, is the complete solution of (1).

The general and singular solutions of (1) can be found out by the usual methods.

### *Type IV*

*Equations of the form*

$$f(x, p) = g(y, q) \tag{1}$$

that is equations which do not contain $z$ explicitly and in which terms containing $p$ and $x$ can be separated from those containing $q$ and $y$.

To find the complete solution of (1), we assume that $f(x, p) = g(y, q) = a$, where '$a$' is an arbitrary constant.

Solving $f(x, p) = a$, we can get $p = \phi(x, a)$ and solving $g(y, q) = a$, we can get $q = \psi(y, a)$.

Now

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \text{ or } p dx + q dy$$

$$\text{i.e.} \qquad dz = \phi(x, a)dx + \psi(y, a)dy$$

Integrating with respect to the concerned variables, we get

$$z = \int \phi(x, a)dx + \int \psi(y, a)dy + b \tag{2}$$

The complete solution of (1) is given by (2), which contains two arbitrary constants $a$ and $b$.

The general and singular solutions of (1) are found out by the usual methods.

## 1.9 EQUATIONS REDUCIBLE TO STANDARD TYPES — TRANSFORMATION

### *Type A*

Equations of the form $f(x^m p, y^n q) = 0$ or $f(x^m p, y^n q, z) = 0$,
where $m$ and $n$ are constants, each not equal to 1.
We make the transformations $x^{1-m} = X$ and $y^{1-n} = Y$.

Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = (1-m)x^{-m}P$, where $P \equiv \frac{\partial z}{\partial X}$ and

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = (1-n)y^{-n}Q$, where $Q \equiv \frac{\partial z}{\partial Y}$.

Therefore the equation $f(x^m p, y^n q) = 0$ reduces to $f\{(1-m)P, (1-n)Q\} = 0$, which is a type I equation.

The equation $f(x^m p, y^n q, z) = 0$ reduces to $f\{(1-m)P, (1-n)Q, z\} = 0$, which is a type III equation.

### *Type B*

Equations of the form $f(px, qy) = 0$ or $f(px, qy, z) = 0$

## Note

*These equations correspond to $m = 1$ and $n = 1$ of the type A equations.*

The required transformations are

$$\log x = X \text{ and } \log y = Y$$

In this case, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x}$ or $px = P$ and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{1}{y}$ or $qy = Q$, where $P \equiv \frac{\partial z}{\partial X}$ and $Q \equiv \frac{\partial z}{\partial Y}$.

Therefore the equation $f(px, qy) = 0$ reduces to $f(P, Q) = 0$, which is a type I equation.

The equation $f(px, qy, z) = 0$ reduces to $f(P, Q, z) = 0$, which is a type III equation.

### *Type C*

Equations of the form $f(z^k p, z^k q) = 0$ or $f(z^k p, z^k q, x, y) = 0$, where $k$ is a constant $\neq -1$.

We make the transformation $Z = z^{k+1}$

Then $$P = \frac{\partial Z}{\partial x} = (k+1)z^k p \text{ and}$$

$$Q = \frac{\partial Z}{\partial y} = (k+1)z^k q$$

Therefore the equation $f(z^k p, z^k q) = 0$ reduces to $f\left(\dfrac{P}{k+1}, \dfrac{Q}{k+1}\right) = 0$, which is a type I equation and the equation $f(z^k p, z^k q, x, y) = 0$ reduces to $f\left(\dfrac{P}{k+1}, \dfrac{Q}{k+1}, x, y\right) = 0$, which may be a type IV equation.

### *Type D*

Equations of the form $f\left(\dfrac{p}{z}, \dfrac{q}{z}\right) = 0$ or $f(p/z, q/z, x, y) = 0$, which correspond to $k = -1$ of type C equations.

The required transformation is $Z = \log z$

Then $P = \dfrac{\partial Z}{\partial x} = \dfrac{1}{z}p$ and $Q = \dfrac{\partial Z}{\partial y} = \dfrac{1}{z}q$

Therefore the equations $f(p/z, q/z) = 0$ and $f(p/z, q/z, x, y) = 0$ reduce respectively to type I and type IV equations.

### *Type E*

Equations of the form $f\left(x^m z^k p, y^n z^k q\right) = 0$ where $m, n \neq 1$; $k \neq -1$

We make the transformations

$$X = x^{1-m}, Y = y^{1-n} \text{ and } Z = z^{k+1}$$

Then

$$P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz} \cdot \frac{\partial z}{\partial x} \cdot \frac{dx}{dX}$$
$$= (k+1)z^k p \cdot \frac{x^m}{1-m}$$

and

$$Q = (k+1)z^k q \frac{y^n}{1-n}$$

$\therefore$ The given equation reduces to

$$f\left\{\left(\frac{1-m}{k+1}\right)P, \left(\frac{1-n}{k+1}\right)Q\right\} = 0$$

which is a type I equation.

### *Type F*

Equations of the form $f\left(\dfrac{px}{z}, \dfrac{qy}{z}\right) = 0$

By putting $X = \log x$, $Y = \log y$ and $Z = \log z$ the equation reduces to $f(P, Q) = 0$, where $P = \dfrac{\partial Z}{\partial X}$ and $Q = \dfrac{\partial Z}{\partial Y}$.

## Worked Examples 1(b)

### *Example 1*

Solve the equation $pq + p + q = 0$.

This equation contains only $p$ and $q$ explicitly.

$\therefore$ Let a solution of the equation be

$$z = ax + by + c \tag{1}$$

From (1), we get $p = a$ and $q = b$.

Since (1) is a solution of the given equation,

$$ab + a + b = 0$$

$\therefore$
$$b = -\frac{a}{a+1} \tag{2}$$

Using (2) in (1), the required complete solution of the equation

$$z = ax - \frac{a}{a+1}y + c \tag{3}$$

To find the general solution, we put $c = f(a)$ in (3), where '$f$' is an arbitrary function.

i.e.
$$z = ax - \frac{a}{a+1}y + f(a) \tag{4}$$

Differentiating (4) partially with respect to $a$, we get

$$x - \frac{1}{(a+1)^2}y + f'(a) = 0 \tag{5}$$

Eliminating $a$ between (4) and (5), we get the required general solution.

To find the singular solution, we have to differentiate (3) partially with respect to $a$ and $c$.

When we differentiate (3) partially with respect to $c$, we get $0 = 1$, which is absurd.

Hence, no singular solution exists for the given equation.

### *Example 2*

Solve the equation $p^2 + q^2 = 4pq$.

$$p^2 + q^2 - 4pq = 0 \tag{1}$$

As (1) contains only $p$ and $q$, a solution of (1) will be of the form

$$z = ax + by + c \tag{2}$$

From (2), we get $p = a$ and $q = b$.

Since (2) is a solution of (1),

$$a^2 + b^2 - 4ab = 0$$

Solving for $b$, we get

$$b = \frac{4a \pm \sqrt{16a^2 - 4a^2}}{2}$$
$$= (2 \pm \sqrt{3})a$$

Using in (2), the complete solution of (1) is

$$z = ax + (2 \pm \sqrt{3})ay + c \tag{3}$$

There is no singular solution for (1), as in Example 1.

To get the general solution, we put $c = f(a)$ in (3), which becomes

$$z = ax + (2 \pm \sqrt{3})ay + f(a) \tag{4}$$

where $f$ is an arbitrary function.

Differentiating (4) partially with respect to $a$, we get

$$0 = x + (2 \pm \sqrt{3})y + f'(a) \tag{5}$$

The eliminant of '$a$' between (4) and (5) gives the general solution of (1).

### *Example 3*

Solve the equation $x^4 p^2 - yzq - z^2 = 0$.

As it is, the equation

$$x^4 p^2 - yzq - z^2 = 0 \tag{1}$$

does not belong to any of the four standard types.

Rewriting Eq. (1), we get

$$\left(\frac{x^2 p}{z}\right)^2 - \left(\frac{yq}{z}\right) = 1 \tag{2}$$

As L.H.S. of (2) is a function of $\frac{x^2 p}{z}$ and $\frac{yq}{z}$, we make the transformations

$X = x^{-1}$, $Y = \log y$ and $Z = \log z$

(by the transformation rules for type A and type F equations)

Then $$p = \frac{\partial z}{\partial x} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial X} \cdot \frac{dX}{dx} = zP\left(-\frac{1}{x^2}\right)$$

$\therefore$ $$\frac{x^2 p}{z} = -P$$

and $$q = \frac{\partial z}{\partial y} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial Y} \cdot \frac{dY}{dy} = zQ.\frac{1}{y}$$

$\therefore$ $$\frac{yq}{z} = Q$$

Equation (2) becomes

$$P^2 - Q = 1 \tag{3}$$

Equation 3 contains only $P$ and $Q$ explicitly.

Therefore a solution of (3) will be of the form

$$Z = aX + bY + c \tag{4}$$

$\therefore$ $P = a$ and $Q = b$, obtained from (4), satisfy Eq. 3.

$$\therefore \quad a^2 - b = 1$$

$$\therefore \quad b = a^2 - 1$$

$\therefore$ The complete solution of (3) is

$$Z = aX + (a^2 - 1)Y + c$$

$\therefore$ The complete solution of (1) is

$$\log z = \frac{a}{x} + (a^2 - 1)\log y + c$$

Singular solution does not exist and general solution is found out as usual.

### *Example 4*

Solve the equation $z^2\left(\dfrac{p^2}{x^2} + \dfrac{q^2}{y^2}\right) = 1$.

The given equation does not belong to any of the four standard types.

It can be rewritten as

$$(x^{-1}zp)^2 + (y^{-1}zq)^2 = 1 \tag{1}$$

which is of the form $(x^m z^k p)^2 + (y^n z^k q)^2 = 1$ [Refer to type E equations]

$\therefore$ We make the transformations

$$X = x^{1-m},\ Y = y^{1-n} \text{ and } Z = z^{k+1}$$

i.e. $$X = x^2,\ Y = y^2 \text{ and } Z = z^2$$

Then

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dZ}\cdot\frac{\partial Z}{\partial X}\cdot\frac{dX}{dx} = \frac{1}{2z}\cdot P\cdot 2x$$

$$\therefore \quad P = x^{-1}zp$$

Similarly, $Q = y^{-1}zq$.

Using these in (1), it becomes

$$P^2 + Q^2 = 1 \tag{2}$$

As (2) contains only $P$ and $Q$ explicitly, a solution of the equation will be of the form

$$Z = aX + bY + c \tag{3}$$

$\therefore$ $P = a$ and $Q = b$, obtained from (3), satisfy Eq. 2.

i.e. $$a^2 + b^2 = 1.$$

$\therefore$ $$b = \pm\sqrt{1-a^2}$$

$\therefore$ The complete solution of (2) is

$$Z = aX \pm \sqrt{1-a^2}Y + c$$

$\therefore$ The complete solution of (1) is

$$z^2 = ax^2 \pm \sqrt{1-a^2}y^2 + c$$

Singular solution does not exist and general solution is found out as usual.

### *Example 5*

Solve the equation $pq\,xy = z^2$.

The equation

$$pq\,xy = z^2 \tag{1}$$

does not belong to any of the four standard types.

Rewriting (1),

$$\left(\frac{px}{z}\right)\left(\frac{qy}{z}\right) = 1 \tag{2}$$

As (2) contains $\frac{px}{z}$ and $\frac{qy}{z}$, we make the substitutions $X = \log x$, $Y = \log y$ and $Z = \log z$ [Refer to type F equations]

Then $$P = \frac{\partial z}{\partial x} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial X} \cdot \frac{dX}{dx} = z \cdot P \cdot \frac{1}{x}$$

i.e. $$\frac{px}{z} = P$$

Similarly $$\frac{qy}{z} = Q$$

Using these in (2), it becomes

$$PQ = 1 \tag{3}$$

which contains only $P$ and $Q$ explicitly. A solution of (3) is of the form

$$Z = aX + bY + c \tag{4}$$

$\therefore$ $P = a$ and $Q = b$, obtained from 4, satisfy (3)

i.e. $$ab = 1 \quad \text{or} \quad b = \frac{1}{a}$$

$\therefore$ The complete solution of (3) is $Z = aX + \frac{1}{a}Y + c$

$\therefore$ The complete solution of (1) is

$$\log z = a \log x + \frac{1}{a} \log y + c \tag{5}$$

General solution of (1) is obtained as usual.

## Note ✍

*To find the singular solution of (1), we should not use the complete solution of (3). We should use only that of (1) given in (5).*

If we put $c = \log k$, (5) becomes

$$\log z = \log (x^a y^{1/a} k)$$

i.e. $$z = x^a y^{1/a} k \tag{6}$$

Differentiating (6) partially with respect to $a$,

$$\log x - \frac{1}{a^2} \log y = 0 \tag{7}$$

Differentiating (6) partially with respect to $k$,

$$0 = x^a y^{1/a} \tag{8}$$

Eliminating $a$ and $k$ from (6), (7) and (8), that is using (8) in (6), the singular solution of equation (1) is $z = 0$.

### *Example 6*

Solve the equation $z^4 q^2 - z^2 p = 1$.

The equation can be solved directly, as it contains $p, q$ and $z$ only explicitly. However we shall transform it into a simpler equation and solve it.

The equation can be rewritten as

$$(z^2 q)^2 - (z^2 p) = 1 \tag{1}$$

which contains $z^2 p$ and $z^2 q$.

Hence we make the transformation $Z = z^3$ [Refer to type C equations]

$$\therefore \qquad P = \frac{\partial Z}{\partial x} = 3z^2 p$$

$$\text{i.e.} \qquad z^2 p = \frac{P}{3}$$

Similarly $z^2 q = \dfrac{Q}{3}$

Using these values in (1), we get

$$Q^2 - 3P = 9 \tag{2}$$

As (2) is an equation containing $P$ and $Q$ only, a solution of (2) will be of the form

$$Z = ax + by + c \tag{3}$$

Now $P = a$ and $Q = b$, obtained from (3) satisfy Eq. 2.

$$\therefore \qquad b^2 - 3a = 9$$

$$\text{i.e.} \qquad b = \pm\sqrt{3a + 9}$$

$\therefore$ Complete solution of (2) is $Z = ax \pm \sqrt{3a+9}y + c$, i.e. complete solution of (1) is $z^3 = ax \pm \sqrt{3a+9}y + c$. Singular solution does not exist. General solution is found out as usual.

## *Example 7*

Solve the equation $z = px + qy + p^2 + pq + q^2$

The given equation

$$z = px + qy + (p^2 + pq + q^2) \tag{1}$$

is a Clairaut's type equation.

$\therefore$ The complete solution of (1) is

$$z = ax + by + a^2 + ab + b^2 \tag{2}$$

[got by replacing $p$ and $q$ in (1) by $a$ and $b$]

Let us now find the singular solution of (1).

Differentiating (2) partially with respect to $a$ and then $b$, we get

$$x + 2a + b = 0 \tag{3}$$

$$\text{and} \qquad y + a + 2b = 0 \tag{4}$$

The eliminant of $a$ and $b$ from (2), (3) and (4) is the required singular solution.

Solving (3) and (4) for $a$ and $b$, we get

$$a = \frac{1}{3}(y - 2x) \quad \text{and} \quad b = \frac{1}{3}(x - 2y)$$

Using these values in (2), the singular solution is

$$z = \frac{x}{3}(y-2x) + \frac{y}{3}(x-2y) + \frac{1}{9}(y-2x)^2$$

$$+\frac{1}{9}(y-2x)(x-2y) + \frac{1}{9}(x-2y)^2$$

i.e. $$9z = 3x(y-2x) + 3y(x-2y)$$

$$+(y-2x)^2 + (y-2x)(x-2y) + (x-2y)^2$$

i.e. $$3z + x^2 - xy + y^2 = 0$$

General solution of (1) is found out as usual.

***Example 8***

Solve the equation $z = px + qy + \left(\frac{q}{p} - p\right)$.

The given equation

$$z = px + qy + \left(\frac{q}{p} - p\right) \tag{1}$$

is a Clairaut's type equation.

$\therefore$ The complete solution of (1) is

$$z = ax + by + \frac{b}{a} - a \tag{2}$$

The general solution of (1) is found out as usual.

To find the singular solution of (1), we differentiate (2) partially with respect to $a$ and then $b$.

We get

$$0 = x - b/a^2 - 1 \tag{3}$$

and $$0 = y + 1/a \tag{4}$$

Using $a = -\frac{1}{y}$ got from (4) in (3), we get

$$x - by^2 - 1 = 0$$

i.e. $$b = \frac{x-1}{y^2}$$

Using these values of $a$ and $b$ in (2), we get

$$z = -x/y + \frac{x-1}{y} - \left(\frac{x-1}{y}\right) + \frac{1}{y}$$

i.e. $yz = 1 - x$, which is the singular solution of (1).

## *Example 9*

Solve the equation $Z = px + qy + c\sqrt{1 + p^2 + q^2}$.
The given equation

$$z = px + qy + c\sqrt{1 + p^2 + q^2} \qquad (1)$$

is a Clairaut's type equation.
$\therefore$ Its complete solution is

$$z = ax + by + c\sqrt{1 + a^2 + b^2} \qquad (2)$$

where $a$ and $b$ are arbitrary constants and $c$ is a given constant.

The general solution of (1) is found out from (2) as usual.

To find the singular solution of (1), we differentiate (2) partially with respect to $a$ and then $b$.

$$0 = x + \frac{ca}{\sqrt{1 + a^2 + b^2}} \qquad (3)$$

and
$$0 = y + \frac{cb}{\sqrt{1 + a^2 + b^2}} \qquad (4)$$

From (3) and (4), we get $\frac{a}{b} = \frac{x}{y}$ or $\frac{a}{x} = \frac{b}{y} = k$, say

$\therefore$
$$a = kx \text{ and } b = ky$$

Using these values in (3), we have

$$\frac{kc}{\sqrt{1 + k^2(x^2 + y^2)}} = -1$$

since $k$ is negative.

i.e.
$$1 + k^2(x^2 + y^2) = k^2c^2$$

or
$$k^2(c^2 - x^2 - y^2) = 1$$

i.e.
$$k = -\frac{1}{\sqrt{c^2 - x^2 - y^2}}$$

$\therefore$
$$a = -\frac{x}{\sqrt{c^2 - x^2 - y^2}}, \quad b = -\frac{y}{\sqrt{c^2 - x^2 - y^2}}$$

and
$$\sqrt{1 + a^2 + b^2} = \frac{c}{\sqrt{c^2 - x^2 - y^2}}$$

Using these values in (2), the singular solution of (1) is got as

$$z = -\frac{x^2}{\sqrt{c^2 - x^2 - y^2}} - \frac{y^2}{\sqrt{c^2 - x^2 - y^2}} + \frac{c^2}{\sqrt{c^2 - x^2 - y^2}}$$

i.e.
$$z = \sqrt{c^2 - x^2 - y^2} \quad \text{or}$$

$$x^2 + y^2 + z^2 = c^2$$

### *Example 10*

Solve the equation $(pq - p - q)(z - px - qy) = pq$.

Rewriting the given equation as

$$z = px + qy + \frac{pq}{pq - p - q} \tag{1}$$

we identify it as a Clairaut's type equation.

Hence its complete solution is

$$z = ax + by + \frac{ab}{ab - a - b} \tag{2}$$

The general solution of (1) is found out as usual from (2).

Let us now find the singular solution of (1).

Differentiating (2) partially with respect to $a$ and then $b$, we get

$$0 = x + \frac{(ab - a - b)b - ab(b - 1)}{(ab - a - b)^2}$$

i.e.

$$0 = x - \frac{b^2}{(ab - a - b)^2} \tag{3}$$

and similarly

$$0 = y - \frac{a^2}{(ab - a - b)^2} \tag{4}$$

From (3) and (4), we get $\dfrac{a^2}{b^2} = y/x$ or

$$\frac{a}{\sqrt{y}} = \frac{b}{\sqrt{x}} = k, \text{ say}$$

$\therefore$

$$a = k\sqrt{y} \text{ and } b = k\sqrt{x}$$

Using these values in (3), we get

$$k^2 x - (k^2\sqrt{xy} - k\sqrt{y} - k\sqrt{x})^2 x = 0$$

i.e.

$$(k\sqrt{xy} - \sqrt{x} - \sqrt{y}) = 1$$

$\therefore$

$$k = \frac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{xy}}$$

Hence $a = \dfrac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{x}}$ and $b = \dfrac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{y}}$

Also

$$\frac{ab}{ab - a - b} = \frac{1}{1 - 1/b - 1/a} = \frac{1}{1 - \dfrac{\sqrt{y}}{1 + \sqrt{x} + \sqrt{y}} - \dfrac{\sqrt{x}}{1 + \sqrt{x} + \sqrt{y}}}$$

$$= 1 + \sqrt{x} + \sqrt{y}$$

Using these values in (2), the singular solution of (1) is

$$z = \sqrt{x}(1 + \sqrt{x} + \sqrt{y}) + \sqrt{y}(1 + \sqrt{x} + \sqrt{y}) + (1 + \sqrt{x} + \sqrt{y})$$

i.e. $z = (1 + \sqrt{x} + \sqrt{y})^2$.

***Example 11***

Transform the equation $4xyz = pq + 2px^2y + 2qxy^2$ by means of the substitutions $X = x^2$ and $Y = y^2$ and hence solve it.

$$P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{p}{2x}$$

and similarly $$Q = \frac{q}{2y}$$

Rewriting the given equation, we have

$$4z = \frac{pq}{xy} + 2px + 2qy \tag{1}$$

Using the transformations in (1), it becomes

$$4z = 4PQ + 4PX + 4QY$$

i.e. $$z = PX + QY + PQ \tag{2}$$

which is a Clairaut's type of equation.

The complete solution of (2) is

$$z = aX + bY + ab \tag{3}$$

Therefore the complete solution (1) is

$$z = ax^2 + by^2 + ab \tag{4}$$

The general solution of (1) is obtained from (4) as usual.

The singular solution of (1) is obtained as follows.

Differentiating (4) partially with, respect to $a$ and then $b$, we get

$$0 = x^2 + b \tag{5}$$

and $$0 = y^2 + a \tag{6}$$

From (5) and (6), $a = -y^2$ and $b = -x^2$. Using these values in (4), the singular solution of (1) is

$$z = -x^2y^2 - x^2y^2 + x^2y^2$$

i.e. $$z + x^2y^2 = 0$$

***Example 12***

Solve the equation $z^2(p^2 + q^2 + 1) = c^2$, where $c$ is a constant.

The given equation

$$z^2(p^2 + q^2 + 1) = c^2 \tag{1}$$

does not contain $x$ and $y$ explicitly.

Therefore (1) has a solution of the form

$$z = y(u) = z(x + ay) \tag{2}$$

where $z(u) = z(x + ay)$ is a function of $(x + ay)$, where $a$ is an arbitrary constant.

From (2), we have $p = \dfrac{dz}{du}$ and $q = \dfrac{dz}{du} \cdot a$

Since (2) is a solution of (1), we get

$$z^2 \left\{ \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 + 1 \right\} = c^2$$

i.e. $$(1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{c^2}{z^2} - 1$$

i.e. $$\sqrt{1 + a^2}\frac{dz}{du} = \frac{\sqrt{c^2 - z^2}}{z}$$

i.e. $$\sqrt{1 + a^2}\frac{zdz}{\sqrt{c^2 - z^2}} = du \qquad (3)$$

Integrating (3), the complete solution of (1) is

$$-\frac{1}{2}\sqrt{1 + a^2}\int \frac{-2zdz}{\sqrt{c^2 - z^2}} = u + b$$

i.e. $$-\sqrt{1 + a^2}\sqrt{c^2 - z^2} = x + ay + b \text{ or}$$

$$(1 + a^2)(c^2 - z^2) = (x + ay + b)^2 \qquad (4)$$

The general and singular solutions of (1) are found out from (4) as usual.

### *Example 13*

Solve the equation $p(1 - q^2) = q(1 - z)$

The given equation

$$p(1 - q^2) = q(1 - z) \qquad (1)$$

does not contain $x$ and $y$ explicitly.

Therefore (1) has a solution of the form

$$z = z(u) = z(x + ay) \qquad (2)$$

where $a$ is an arbitrary constant.

From (2), $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$

Since (2) is a solution of (1), we get

$$\frac{dz}{du}\left\{1 - a^2\left(\frac{dz}{du}\right)^2\right\} = a\frac{dz}{du}(1 - z)$$

i.e. $$\frac{dz}{du}\left[1 - a^2\left(\frac{dz}{du}\right)^2 - a + az\right] = 0$$

As z is not a constant, $\frac{dz}{du} \neq 0$

$$\therefore \qquad 1 - a^2\left(\frac{dz}{du}\right)^2 - a + az = 0$$

i.e.
$$a^2\left(\frac{dz}{du}\right)^2 = az + 1 - a$$

or
$$a\frac{dz}{du} = \sqrt{az + 1 - a} \qquad (3)$$

Solving (3), we get

$$a\int \frac{dz}{\sqrt{az + 1 - a}} = u + b$$

i.e.
$$2\sqrt{az + 1 - a} = x + ay + b \quad \text{or}$$

$$4(az + 1 - a) = (x + ay + b)^2 \qquad (4)$$

which is the complete solution of (1).

The general and singular solutions of (1) are found out from (4) as usual.

### *Example 14*

Solve the equation $\qquad 9pqz^4 = 4(1 + z^3).$

The given equation

$$9pqz^4 = 4(1 + z^3) \qquad (1)$$

does not contain $x$ and $y$ explicitly.

Therefore (1) has got a solution of the form

$$z = z(u) = z(x + ay) \qquad (2)$$

where $a$ is an arbitrary constant.

From (2), $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

Since (2) is a solution of (1), we get

$$9a\left(\frac{dz}{du}\right)^2 z^4 = 4(1 + z^3)$$

i.e.
$$3\sqrt{a}z^2\frac{dz}{du} = 2\sqrt{1 + z^3} \qquad (3)$$

Solving (3), we get

$$\frac{\sqrt{a}}{2}\int \frac{3z^2 dz}{\sqrt{1 + z^3}} = u + b$$

i.e.
$$\sqrt{a} \cdot \sqrt{1 + z^3} = x + ay + b \qquad \text{or}$$

$$a(1 + z^3) = (x + ay + b)^2 \qquad (4)$$

which is the complete solution of (1).

The general and singular solutions of (1) are found out from (4) as usual.

### *Example 15*

Solve the equation $\dfrac{x^2}{p} + \dfrac{y^2}{q} = z$.

The given equation does not belong to any of the standard types.

It can be rewritten as

$$\frac{1}{px^{-2}} + \frac{1}{qy^{-2}} = z \tag{1}$$

As equation (1) contains $px^{-2}$ and $qy^{-2}$, we make the substitutions $X = x^3$ and $Y = y^3$. [Refer to type A equations]

Then $P = \dfrac{\partial z}{\partial X} = p \cdot \dfrac{1}{3x^2}$ or $px^{-2} = 3P$ and similarly $qy^{-2} = 3Q$.

Then (1) becomes

$$\frac{1}{P} + \frac{1}{Q} = 3Z \tag{2}$$

As (2) does not contain $X$ and $Y$ explicitly, it has a solution of the form

$$z = z(u) = z(X + aY) \tag{3}$$

From (3), $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

Since (3) is a solution of (2), we get

$$\frac{dz}{du}(1 + a) = 3az\left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du}\left(3az\frac{dz}{du} - a - 1\right) = 0$$

$$\text{As} \quad \frac{dz}{du} \neq 0, \qquad 3az\frac{dz}{du} = a + 1 \tag{4}$$

Solving (4), $\displaystyle\int 3az\, dz = (a + 1)u + b$

i.e. $\quad \frac{3}{2}az^2 = (a + 1)(X + aY) + b$

which is the complete solution of equation (2).

$\therefore$ The complete solution of equation (1) is

$$\frac{3}{2}az^2 = (a + 1)(x^3 + ay^3) + b$$

where $a$ and $b$ are arbitrary constants.

The general and singular solutions are found out as usual.

### *Example 16*

Solve the equation

$$p^2 + x^2y^2q^2 = x^2z^2$$

The given equation does not belong to any of the standard types.
Rewriting it, we have

$$\left(x^{-1}p\right)^2 + (yq)^2 = z^2 \tag{1}$$

As equation (1) contains $x^{-1}p$ and $yq$, we make the transformations $X = x^2$ and $Y = \log y$ [Refer to type A and type B equations]

$\therefore$ $$\frac{\partial z}{\partial X} = p \cdot \frac{1}{2x} \quad \text{and} \quad Q = \frac{\partial z}{\partial Y} = qy$$

i.e. $$x^{-1}p = 2P \quad \text{and} \quad yq = Q$$

Using these values in (1), it becomes

$$4P^2 + Q^2 = z^2 \tag{2}$$

As (2) does not contain $X$ and $Y$ explicitly, it has got a solution of the form

$$z = z(u) = z(X + aY) \tag{3}$$

From (3), we have

$$P = \frac{dz}{du} \text{ and } Q = a\frac{dz}{du}$$

Using these values in (2), we get

$$\left(\frac{dz}{du}\right)^2 (4 + a^2) = z^2$$

i.e. $$\sqrt{a^2+4}\ \frac{dz}{du} = z \tag{4}$$

Solving (4), we get

$$\sqrt{a^2+4}\log z = X + aY + b$$

which is the complete solution of (2).

$\therefore$ The complete solution of (1) is

$$\sqrt{a^2+4}\log z = x^2 + a\log y + b$$

where $a$ and $b$ are arbitrary constants.

The general and singular solutions are found out as usual.

### *Example 17*

Solve the equation

$$x^2p^2 + xpq = z^2$$

The given equation can be rewritten as

$$(xp)^2 + (xp)q = z^2 \tag{1}$$

Putting $X = \log x$, we get $P = \dfrac{\partial z}{\partial X} = px$

Using this in (1), it becomes

$$P^2 + Pq = z^2 \tag{2}$$

As Eq. 2 does not contain $X$ and $y$ explicitly, it has a solution of the form

$$z = z(u) = z(X + ay) \tag{3}$$

From (3),

$$P = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Using these values in (2), we have

$$\left(\frac{dz}{du}\right)^2 + a\left(\frac{dz}{du}\right)^2 = z^2$$

i.e.

$$\sqrt{1+a}\,\frac{dz}{du} = z \tag{4}$$

Solving (4), we get $\sqrt{1+a}\log z = X + ay + b$, which is the complete solution of (2).

$\therefore$ The complete solution of (1) is

$$\sqrt{1+a}\log z = \log x + ay + b$$

The general and singular solutions are found out as usual.

### *Example 18*

Solve the equation

$$q^2y^2 = z(z - px)$$

As the given equation contains $px$ and $qy$, we make the following substitutions.

$$X = \log x \quad \text{and} \quad Y = \log y$$

$$\therefore \quad P = \frac{\partial z}{\partial X} = px \quad \text{and} \quad Q = \frac{\partial z}{\partial Y} = qy$$

Using these in the given equation, it becomes

$$Q^2 = z(z-P) \quad \text{or} \quad Pz + Q^2 = z^2 \tag{1}$$

As Eq. (1) does not contain $X$ and $Y$ explicitly, it has a solution of the form

$$z = z(u) = z(X + aY) \tag{2}$$

From (2),

$$P = \frac{dz}{du} \quad \text{and} \quad Q = a\frac{dz}{du}$$

Using these values in (1), it becomes

$$z\frac{dz}{du} + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

or

$$a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{du} - z^2 = 0 \tag{3}$$

Solving (3) for $\frac{dz}{du}$, we get

$$\begin{aligned}\frac{dz}{du} &= \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2} \\ &= \frac{(-1 \pm \sqrt{1+4a^2})z}{2a^2}\end{aligned}$$

Solving this equation, we get

$$2a^2 \int \frac{dz}{z} = (-1 \pm \sqrt{1+4a^2})u + b$$

i.e.

$$2a^2 \log z = (-1 \pm \sqrt{1+4a^2})(X + aY) + b$$

which is the complete solution of (1).

$\therefore$ The complete solution of the given equation is

$$2a^2 \log z = (-1 \pm \sqrt{1+4a^2})(\log x + a \log y) + b$$

The general and singular solutions are found out as usual.

### *Example 19*

Solve the equation

$$\sqrt{p} + \sqrt{q} = x + y$$

The given equation does not contain $z$ explicitly and is variable separable.

That is the equation can be rewritten as

$$\sqrt{p} - x = y - \sqrt{q} = a, \text{ say} \tag{1}$$

$$\therefore \quad p = (x + a)^2 \quad \text{and} \quad q = (y - a)^2$$

Now $$dz = p dx + q dy$$

$$= (x + a)^2 dx + (y - a)^2 dy \tag{2}$$

Integrating both sides with respect to the concerned variables, we get

$$z = \frac{(x + a)^3}{3} + \frac{(y - a)^3}{3} + b \tag{3}$$

where $a$ and $b$ are arbitrary constants. Equation (3) is the complete solution of the given equation.

General solution is found out as usual. Singular solution does not exist.

### *Example 20*

Solve the equation

$$yp = 2xy + \log q$$

The given equation, which does not contain $z$, can be rewritten as

$$p - 2x = \frac{1}{y} \log q = a, \text{ say} \tag{1}$$

$$\therefore \quad p = 2x + a \quad \text{and} \quad q = e^{ay}$$

Now $$dz = p dx + q dy$$

i.e. $$dz = (2x + a) dx + e^{ay} dy \tag{2}$$

Integrating (2), we get

$$z = x^2 + ax + \frac{1}{a} e^{ay} + b \tag{3}$$

where $a$ and $b$ are arbitrary constants.

Equation (3) is the complete solution of the given equation.

General solution is found out as usual.

Singular solution does not exist.

### *Example 21*

Solve the equation

$$p^2(1 + x^2)y = qx^2$$

The given equation, which does not contain $z$, can be rewritten as

$$p^2 \frac{(1 + x^2)}{x^2} = \frac{q}{y} = a, \text{ say} \tag{1}$$

$$p = \frac{\sqrt{a} \cdot x}{\sqrt{1+x^2}} \quad \text{and} \quad q = ay$$

$$dz = p dx + q dy$$

$$= \sqrt{a} \cdot \frac{x}{\sqrt{1+x^2}} dx + ay dy \tag{2}$$

Integrating (2), we get the complete solution of the given equation as

$$z = \sqrt{a(1+x^2)} + \frac{ay^2}{2} + b \tag{3}$$

where $a$ and $b$ are arbitrary constants.

From (3), we get the general solution as usual. Singular solution does not exist.

### *Example 22*

Solve the equation $\qquad z^2(p^2 + q^2) = x + y$

The given equation

$$z^2(p^2 + q^2) = x + y \tag{1}$$

does not belong to any of the standard types.

Equation (1) can be rewritten as

$$(zp)^2 + (zq)^2 = x + y$$

Since the equation contains $zp$ and $zq$, we make the substitution $Z = z^2$

$$\therefore \qquad P = \frac{\partial Z}{\partial x} = 2zp \quad \text{and} \quad Q = \frac{\partial Z}{\partial y} = 2zq$$

Using these in (1), it becomes

$$P^2 + Q^2 = 4x + 4y \tag{2}$$

which does not contain $Z$ explicitly.

Rewriting (2), we get

$$P^2 - 4x = 4y - Q^2 = 4a, \text{ say} \tag{3}$$

$$\therefore \qquad P = 2\sqrt{x+a} \quad \text{and} \quad Q = 2\sqrt{y-a}$$

$$dZ = P dx + Q dy$$

$$= 2\sqrt{x+a}\, dx + 2\sqrt{y-a}\, dy$$

Integrating, we get

$$Z = \frac{4}{3}(x+a)^{3/2} + \frac{4}{3}(y-a)^{3/2} + b$$

i.e. $$z^2 = \frac{4}{3} \cdot (x+a)^{3/2} + \frac{4}{3} \cdot (y-a)^{3/2} + b$$

which is the complete solution of (1).

General solution is found out as usual.

Singular solution does not exist.

***Example 23***

Solve the equation

$$p^2 + q^2 = z^2(x^2 + y^2)$$

The given equation does not belong to any of the standard types.
It can be rewritten as

$$(z^{-1}p)^2 + (z^{-1}q)^2 = x^2 + y^2 \tag{1}$$

As the Eq. (1) contains $z^{-1}p$ and $z^{-1}q$, we make the substitution $Z = \log z$

$$\therefore \qquad P = \frac{p}{z} \quad \text{and} \quad Q = \frac{q}{z}$$

Using these values in (1), it becomes

$$P^2 + Q^2 = x^2 + y^2 \tag{2}$$

As Eq. 2 does not contain $Z$ explicitly, we rewrite it as

$$P^2 - x^2 = y^2 - Q^2 = a^2, \text{ say} \tag{3}$$

From (3),

$$P = \sqrt{x^2 + a^2} \quad \text{and} \quad Q = \sqrt{y^2 - a^2}$$

$$\begin{aligned} dZ &= P dx + Q dy \\ &= \sqrt{x^2 + a^2}dx + \sqrt{y^2 - a^2}dy \end{aligned}$$

Integrating, we get

$$Z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}(y/a) + b$$

$\therefore$ The complete solution of (1) is

$$\log z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}(y/a) + b$$

where $a$ and $b$ are arbitrary constants.
General solution is found out as usual.
Singular solution does not exist.

***Example 24***

Solve the equation $(x + pz)^2 + (y + qz)^2 = 1$.

The given equation does not belong to any of the standard types.

But the equation contains $pz$ and $qz$.

Therefore we make the substitution $Z = z^2$.

Then $P = \dfrac{\partial Z}{\partial x} = 2zp$ and $Q = 2zq$.

Using these values in the given equation, it becomes

$$\left(x + \frac{P}{2}\right)^2 + \left(y + \frac{Q}{2}\right)^2 = 1 \tag{1}$$

Equation (1) does not contain $Z$ explicitly. Rewriting (1), we have

$$\left(x + \frac{P}{2}\right)^2 = 1 - \left(y + \frac{Q}{2}\right)^2 = a^2, \text{ say} \tag{2}$$

From (2), $x + \dfrac{P}{2} = a$ or $P = 2(a - x)$ and $y + \dfrac{Q}{2} = \sqrt{1 - a^2}$ or $Q = 2(\sqrt{1 - a^2} - y)$

Now $dZ = P dx + Q dy$

$$= 2(a - x)dx + 2(\sqrt{1 - a^2} - y)dy \tag{3}$$

Integrating (3) and replacing $Z$ by $z^2$, the complete solution of the given equation is

$$z^2 = -(a - x)^2 + 2\sqrt{1 - a^2}y - y^2 + b$$

General solution is found out as usual. Singular solution does not exist.

### *Example 25*

Solve the equation $pz^2 \sin^2 x + qz^2 \cos^2 y = 1$. The given equation does not belong to any of the standard types.

The given equation contains $(z^2 p)$ and $(z^2 q)$.

Therefore we make the substitution $Z = z^3$

$$\therefore \quad P = \frac{\partial Z}{\partial x} = 3z^2 p \text{ and } Q = 3z^2 q$$

Using these values in the given equation, it becomes

$$\frac{P}{3} \sin^2 x + \frac{Q}{3} \cos^2 y = 1 \tag{1}$$

Equation (1) does not contain $Z$ explicitly. Rewriting (1), we have

$$\frac{P}{3} \sin^2 x = 1 - \frac{Q}{3} \cos^2 y = a, \text{ say} \tag{2}$$

From (2), $P = 3a \operatorname{cosec}^2 x$ and $Q = 3(1 - a) \sec^2 y$

Now $dZ = Pdx + Qdy$

$$= 3a \operatorname{cosec}^2 x dx + 3(1 - a) \sec^2 y dy \tag{3}$$

Integrating (3) and replacing $Z$ by $z^3$, the complete solution of the given equation is

$$z^3 = -3a \cot x + 3(1 - a) \tan y + b$$

General solution is found out as usual. Singular solution does not exist.

## Exercise 1(b)

### Part A (Short-Answer Questions)

1. Define complete solution and general solution of a P.D.E.
2. How will you find the general solution of a P.D.E. from its complete solution?
3. What is the geometrical significance of the singular solution of a P.D.E.?
4. How will you find the singular solution of a P.D.E. from its complete solution?
5. Find the complete solution of the P.D.E. $q = f(p)$.
6. Find the complete solution of the P.D.E. $z = px + qy + f(p, q)$.

Find the complete solution of the following P.D.E.s.

7. $pq = k$
8. $p = e^q$
9. $p^2 + q^2 = 2$
10. $p + q = z$
11. $p^2 = qz$
12. $pq = z$
13. $pq = xy$
14. $px = qy$
15. $pe^y = qe^x$
16. Rewrite the equation $pqz = p^2(qx + p^2) + q^2(py + q^2)$ as a Clairaut's equation and hence write down its complete solution.

### Part B

17. Solve the equation (a) $\sqrt{p} + \sqrt{q} = 1$; (b) $p^2 + q^2 = k^2$. Find the singular solutions, if they exist.
18. Solve the equation $3p^2 - 2q^2 = 4pq$. Find the singular solution, if it exists.

19. Solve the equation $p^2 - 2pq + 3q = 5$. Find the singular solution, if it exists.

Convert the following equations into equations of the form $f(p, q) = 0$ and hence solve them.

20. $p^2x^2 + q^2y^2 = z^2$
21. $p^2x + q^2y = z$
22. $px^2 + qy^2 = z^2$
23. $z^2(p^2 - q^2) = 1$
24. $2x^4p^2 - yzq - 3z^2 = 0$
25. $(y - x)(qy - px) = (p - q)^2$ [**Hint**: Put $x + y = X$ and $xy = Y$]

Find the singular solutions of the following partial differential equations.

26. $z = px + qy - 2\sqrt{pq}$
27. $\dfrac{z}{pq} = \dfrac{x}{q} + \dfrac{y}{p} - \sqrt{pq}$
28. $z = px + qy + p^2q^2$
29. $(p + q)(z - px - qy) = 1$
30. $z = px + qy + p^2 - q^2$
31. $z = px + qy + \sqrt{p^2 + q^2}$
32. $(1 - x)p + (2 - y)q = 3 - z$

Solve the following equations.

33. $p^2 + q^2 = z$
34. $1 + p^2 + q^2 = z^2$
35. (a) $pz = 1 + q^2$; (b) $qz = 1 + p^2$
36. $p(1 + q^2) = q(z - a)$
37. $9(p^2z + q^2) = 4$

Convert the following equations into equations of the form $f(p, q, z) = 0$ and hence solve them.

38. $\dfrac{p}{x^2} + \dfrac{q}{y^2} = z$
39. $(p^2x^2 + q^2)z^2 = 1$
40. $p^2x^4 + y^2zq = 2z^2$

Solve the following equations.

41. $q = px + p^2$

42. $yp + xq + pq = 0$
43. $yp - x^2q^2 = x^2y$
44. $q(p - \sin x) = \cos y$

Convert the following equations into equations of the form $f(p, q, x, y) = 0$ and hence solve them.

45. $(p^2 - q^2)z = x - y$
46. $(p^2 + q^2)z^2 = x^2 + y^2$
47. $p^2 + x^2y^2q^2 = x^2z^2$
48. $4z^2q^2 = y - x + 2zp$
49. $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$ [**Hint**: Put $x + y = X$ and $x - y = Y$]
50. $(p^2 + q^2)(x^2 + y^2) = 1$ [**Hint**: Put $x = r\cos\theta$ and $y = r\sin\theta$]

## 1.10 GENERAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations, for which the general solution can be obtained directly, can be divided into the following three categories

1. Equations that can be solved by direct (partial) integration. For example, consider the equation

$$\frac{\partial z}{\partial x} = a \tag{1}$$

If $z$ were a function of $x$ only, direct integration with respect to $x$ will give the solution as

$$z = ax + b \tag{2}$$

If (2) is to be the general solution of (1), $b$ need not be a constant, but it may be an arbitrary function of $y$, say $f(y)$. Then (2) becomes

$$z = ax + f(y) \tag{3}$$

When we differentiate (3) partially with respect to $x$, we get Eq. (1). As (3) contains an arbitrary function, it is the general solution.
Thus when we get the solution of an equation by partial integration with respect to $x$ [or $y$], we should take an arbitrary function of $y$ [or $x$] in the place of arbitrary constants taken when ordinary integration is performed.
Equations, in which the dependent variable occurs only in the partial derivatives, can be solved by this partial integration method.

2. Lagrange's linear equation of the first order, which will be discussed in Section 1.11.
3. Linear partial differential equations of higher order with constant coefficients, which will be discussed in Section 1.12.

## 1.11 LAGRANGE'S LINEAR EQUATION

A linear partial differential equation of the first order, which is of the form $Pp+Qq = R$ where $P$, $Q$, $R$ are functions of $x$, $y$, $z$, is called *Lagrange's linear equation*.
We have already shown that the elimination of the arbitrary function '$f$' from $f(u, v) = 0$ leads to Lagrange's linear equation.

### General solution of Lagrange's linear equation

The general solution of the equation $Pp + Qq = R$ is $f(u, v) = 0$, where '$f$' is an arbitrary function and $u(x, y, z) = a$ and $v(x, y, z) = b$ are independent solutions of the simultaneous differential equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

***Proof***

$$f(u, v) = 0 \tag{1}$$

Differentiating (1) partially with respect to $x$ and then $y$, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \tag{2}$$

and

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \tag{3}$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (2) and (3), we get

$$\frac{u_x + u_z p}{u_y + u_z q} = \frac{v_x + v_z p}{v_y + v_z q}$$

i.e.

$$(u_y v_z - u_z v_y)p + (u_z v_x - u_x v_z)q = u_x v_y - u_y v_x \tag{4}$$

Taking $P = u_y v_z - u_z v_y$, $Q = u_z v_x - u_x v_z$ and $R = u_x v_y - u_y v_x$, Eq. (4) takes the form

$$Pp + Qq = R \tag{5}$$

Since the primitive of equation (5) is equation (1), that contains an arbitrary function '$f$', we conclude that $f(u, v) = 0$ is the general solution of the Lagrange's linear equation (5).

Now consider $u = a$ and $v = b$

$\therefore$ $du = 0$ and $dv = 0$

i.e.

$$u_x dx + u_y dy + u_z dz = 0 \tag{6}$$

and

$$v_x dx + v_y dy + v_z dz = 0 \tag{7}$$

Solving (6) and (7) for d$x$, d$y$, d$z$, we get

$$\frac{dx}{u_y v_z - u_z v_y} = \frac{dy}{u_z v_x - u_x v_z} = \frac{dz}{u_x v_y - u_y v_x}$$

i.e. $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (8)$$

When we eliminate $a$ and $b$ from $u = a$ and $v = b$, we get the simultaneous equations (8). In other words, the solutions of equations (8) are $u = a$ and $v = b$.

Therefore the general solution of $Pp + Qq = R$ is $f(u, v) = 0$, where $u = a$ and $v = b$ are independent solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

### *Working rule to solve $Pp + Qq = R$*

(i) To solve $Pp + Qq = R$, we form the corresponding subsidiary simultaneous equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

(ii) Solving these equations, we get two independent solutions $u = a$ and $v = b$.

(iii) Then the required general solution is $f(u, v) = 0$ or $u = \phi(v)$ or $v = \psi(u)$.

## 1.12 SOLUTION OF THE SIMULTANEOUS EQUATIONS $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

### Method of grouping

By grouping any two of three ratios, it may be possible to get an ordinary differential equation containing only two variables, eventhough $P$; $Q$; $R$ are, in general, functions of $x$, $y$, $z$. By solving this equation, we can get a solution of the simultaneous equations. By this method, we may be able to get two independent solutions, by using different groupings.

### Method of multipliers

If we can find a set of three quantities $l$, $m$, $n$, which may be constants or functions of the variables $x$, $y$, $z$, such that $l\text{P}+m\text{Q}+n\text{R} = 0$, then a solution of the simultaneous equations is found out as follows.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

Since $l\text{P} + m\text{Q} + n\text{R} = 0$, $l\,dx + m\,dy + n\,dz = 0$. If $l\,dx + m\,dy + n\,dz$ is an exact differential of some function $u(x, y, z)$, then we get $du = 0$. Integrating this, we get $u = a$, which is a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Similarly, if we can find another set of independent multipliers $l'$, $m'$, $n'$, we can get another independent solution $v = b$.

## Note ✍

1. *We may use the method of grouping to get one solution and the method of multipliers to get the other solution of* $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. *The subsidiary equations are called Lagrange's subsidiary simultaneous equations.*
3. *The multipliers* $l, m, n$ *are called Lagrange multipliers.*

### Worked Examples 1(c)

#### *Example 1*

Solve the equations (i) $\frac{\partial^2 z}{\partial x^2} = xy$; (ii) $\frac{\partial^2 z}{\partial y^2} = \sin xy$

(i)
$$\frac{\partial^2 z}{\partial x^2} = xy \tag{1}$$

Integrating both sides of (1) partially with respect to $x$ (i.e. treating $y$ as a constant),

$$\frac{\partial z}{\partial x} = y\frac{x^2}{2} + \phi(y) \tag{2}$$

Integrating (2) partially with respect to $x$,

$$z = \frac{x^3}{6}y + f(y) + x \cdot \phi(y) \tag{3}$$

where $f(y)$ and $\phi(y)$ are arbitrary functions. Equation (3) is the required general solution of (1).

(ii)
$$\frac{\partial^2 z}{\partial y^2} = \sin xy \tag{4}$$

Integrating (4) partially with respect to $y$,

$$\frac{\partial z}{\partial y} = -\frac{1}{x}\cos xy + \phi(x) \tag{5}$$

Integrating (5) partially with respect to $y$,

$$z = -\frac{1}{x^2}\sin xy + f(x) + y \cdot \phi(x) \tag{6}$$

where $f(x)$ and $\phi(x)$ are arbitrary functions. Equation (6) is the required general solution of (4).

### *Example 2*

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$, if $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$. Also show that $u \to \sin x$, when $t \to \infty$.

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x \tag{1}$$

Integrating (1) partially with respect to $x$,

$$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t) \tag{2}$$

When $x = 0$, $\dfrac{\partial u}{\partial t} = 0$. (given)
Using this in (2), we get $f(t) = 0$.

$\therefore$ Equation (2) becomes $\dfrac{\partial u}{\partial t} = e^{-t}\sin x$ (3)

Integrating (3) partially with respect to $t$, we get

$$u = -e^{-t}\sin x + g(x) \tag{4}$$

Using the given condition, namely, $u = 0$ when $t = 0$, in (4), we get

$$0 = -\sin x + g(x) \text{ or } g(x) = \sin x$$

Using this value in (4), the required particular solution of (1) is $u = \sin x(1 - e^{-t})$.

$$\text{Now } \lim_{t\to\infty}(u) = \sin x\left[\lim_{t\to\infty}(1 - e^{-t})\right]$$
$$= \sin x$$

That is when $t \to \infty$, $u \to \sin x$.

### *Example 3*

Solve the equation $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, given that $z = e^y$ and $\dfrac{\partial z}{\partial x} = 1$ when $x = 0$.

$$\frac{\partial^2 z}{\partial x^2} + z = 0 \tag{1}$$

If $z$ were a function of $x$ alone, the equation (1) would have been the ordinary differential equation

$$\frac{d^2 z}{dx^2} + z = 0, \text{ i.e. } (D^2 + 1)z = 0 \tag{2}$$

The auxiliary equation of (2) is $m^2 + 1 = 0$. Its roots are $\pm i$. Hence the solution of (2) is

$$z = A\cos x + B\sin x \tag{3}$$

Solution (3) can be assumed to be obtained by integrating (2) ordinarily with respect to $x$.

If we replace $A$ and $B$ in (3) by arbitrary functions of $y$, the solution can be assumed to have been obtained by integrating (1) partially with respect to $x$.

Thus the general solution of (1) is

$$z = f(y) \cdot \cos x + g(y) \cdot \sin x \tag{4}$$

From (4),
$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \tag{5}$$

Using the condition that $z = e^y$ when $x = 0$ in (4), we get

$$f(y) = e^y \tag{6}$$

Using the condition that $\frac{\partial z}{\partial x} = 1$ when $x = 0$ in (5),

$$g(y) = 1 \tag{7}$$

Using (6) and (7) in (4), the required solution of (1) is $z = e^y \cos x + \sin x$.

### *Example 4*

Solve the equations $\frac{\partial z}{\partial x} = 3x - y$ and $\frac{\partial z}{\partial y} = -x + \cos y$ simultaneously.

$$\frac{\partial z}{\partial x} = 3x - y \tag{1}$$

$$\frac{\partial z}{\partial y} = -x + \cos y \tag{2}$$

Integrating (1) partially with respect to $x$,

$$z = \frac{3x^2}{2} - yx + f(y) \tag{3}$$

Differentiating (3) partially with respect to $y$,

$$\frac{\partial z}{\partial y} = -x + f'(y) \tag{4}$$

Comparing (2) and (4), we get $f'(y) = \cos y$

$$\therefore \quad f(y) = \sin y + c \tag{5}$$

$\therefore$ The required solution is

$z = \frac{3}{2}x^2 - xy + \sin y + c$, where $c$ is an arbitrary constant.

### *Example 5*

By changing the independent variables by the transformations $u = x - y$ and $v = x + y$, show that the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ can be transformed as $\frac{\partial^2 z}{\partial v^2} = 0$ and hence solve it.

$$u = x - y \text{ and } v = x + y$$

$$\therefore \qquad x = \frac{u+v}{2} \text{ and } y = \frac{v-u}{2}$$

If we express $x$ and $y$ in $z$ in terms of $u$ and $v$, $z$ becomes a function of $u$ and $v$.

$$z_x = \frac{\partial z}{\partial x} = z_u \cdot u_x + z_v \cdot v_x, \text{ where } z_u = \frac{\partial z}{\partial u} \text{ and } u_x = \frac{\partial u}{\partial x}, \text{ etc.}$$

$$= z_u + z_v$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y = -z_u + z_v$$

$$z_{xx} = (z_{uu} + z_{uv}) + (z_{vu} + z_{vv}) = z_{uu} + 2z_{uv} + z_{vv}$$

$$z_{xy} = (-z_{uu} + z_{uv}) + (-z_{vu} + z_{vv}) = -z_{uu} + z_{vv}$$

$$z_{yy} = z_{uu} - z_{uv} + (-z_{vu} + z_{vv}) = z_{uu} - 2z_{uv} + z_{vv}$$

Using these values in the given equation $z_{xx}+2z_{xy}+z_{yy} = 0$, it becomes $4z_{vv} = 0$.

i.e.
$$\frac{\partial^2 z}{\partial v^2} = 0 \qquad (1)$$

Integrating (1) partially with respect to $v$,

$$\frac{\partial z}{\partial v} = g(u) \qquad (2)$$

Integrating (2) partially with respect to $v$,

$$z = v \cdot g(u) + f(u) \qquad (3)$$

$\therefore$ The solution of the given equation is

$$z = f(x-y) + (x+y)g(x-y)$$

### *Example 6*

By changing the independent variables by the transformations $u = x$ and $v = \frac{y}{x}$, transform the equation $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$ and hence solve it.

When $u = x$ and $v = y/x$, $\quad x = u$ and $y = uv$.

$\therefore \qquad z$, which is a function of $x$ and $y$, can also treated as a function of $u$ and $v$.

$$z_x = z_u \cdot u_x + z_v \cdot v_x = z_u - \frac{y}{x^2}z_v$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y = \frac{1}{x} \cdot z_v$$

$$z_{xx} = z_{uu} + z_{uv}\left(-\frac{y}{x^2}\right) + \frac{2y}{x^3}z_v - \frac{y}{x^2}\left[z_{vu} + z_{vv}\left(\frac{-y}{x^2}\right)\right]$$

$$z_{xy} = z_v \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x}\left[z_{vu} + z_{vv}\left(-\frac{y}{x^2}\right)\right]; \; z_{yy} = \frac{1}{x}\left[z_{vv} \cdot \frac{1}{x}\right]$$

Using these values in the given equation, it becomes,

$$\left(x^2 z_{uu} - yz_{uv} + \frac{2y}{x}z_v - yz_{uv} + \frac{y^2}{x^2}z_{vv}\right)$$
$$+\left(-\frac{2y}{x}z_v + 2yz_{uv} - \frac{2y^2}{x^2}z_{vv}\right) + \left(\frac{y^2}{x^2}z_{vv}\right) = 0$$

i.e. $$x^2 z_{uu} = 0 \quad \text{or} \quad z_{uu} = 0 \tag{1}$$

Integrating (1) partially with respect to $u$,

$$z_u = \phi(v) \tag{2}$$

Integrating (2) partially with respect to $u$,

$$z = f(v) + u \cdot \phi(v) \tag{3}$$

$\therefore$ Solution of the given equation is

$$z = f(y/x) + x \cdot \phi(y/x)$$

***Example 7***

Transform the partial differential equation $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$ to the form $\frac{\partial^2 z}{\partial u \partial v} = 0$ by using the substitutions $u = x + \alpha y$ and $v = x + \beta y$, where $\alpha$ and $\beta$ are appropriate constants and hence solve the given equation.

Clearly $z$, which is a function of $x$ and $y$, can also be treated as a function of u and $v$.

$$z_x = z_u + z_v; \quad z_y = \alpha z_u + \beta z_v$$

$$z_{xx} = z_{uu} + 2z_{uv} + z_{vv}; \quad z_{xy} = z_{uu} \cdot \alpha + z_{uv} \cdot \beta$$

$$+ z_{vu} \cdot \alpha + z_{vv} \cdot \beta \text{ or } \alpha z_{uu} + (\alpha + \beta)z_{uv} + \beta z_{vv} \cdot$$

$$z_{yy} = \alpha(z_{uu} \cdot \alpha + z_{uv} \cdot \beta) + \beta(z_{vu} \cdot \alpha + z_{vv} \cdot \beta)$$

$$= \alpha^2 z_{uu} + 2\alpha\beta z_{uv} + \beta^2 z_{vv}.$$

Using these values in the given equation, it becomes

$$(z_{uu} + 2z_{uv} + z_{vv}) - 5\left[\alpha z_{uu} + (\alpha + \beta)z_{uv} + \beta z_{vv}\right]$$
$$+ 6\left[\alpha^2 z_{uu} + 2\alpha\beta z_{uv} + \beta^2 z_{vv}\right] = 0$$

i.e. $(6\alpha^2 - 5\alpha + 1)z_{uu} + [2 - 5(\alpha + \beta) + 12\alpha\beta]\, z_{uv} + (6\beta^2 - 5\beta + 1)z_{vv} = 0$ (1)

Since (1) has to reduce to the form $z_{uv} = 0$, coefficient of $z_{uu} = 0 =$ coefficient of $z_{vv}$.

i.e. $$6\alpha^2 - 5\alpha + 1 = 0 \text{ and } 6\beta^2 - 5\beta + 1 = 0$$

i.e. $$\alpha = \frac{1}{2}, \frac{1}{3} \quad \text{and } \beta = \frac{1}{2}, \frac{1}{3}$$

If we choose equal values for $\alpha$ and $\beta$, coefficient of $z_{uv}$ also becomes zero. Hence we choose $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$.

For these values of $\alpha$ and $\beta$, equation (1) becomes

$$-\frac{1}{6}z_{uv} = 0 \text{ or } \frac{\partial^2 z}{\partial u \partial v} = 0 \tag{2}$$

Integrating (2) partially with respect to u,

$$\frac{\partial z}{\partial v} = \phi(v) \tag{3}$$

Integrating (3) partially with respect to $v$,

$$z = \int \phi(v)\mathrm{d}v + f(u)$$

i.e. $$z = f(u) + g(v)$$

$\therefore$ The solution of the given equation is

$$z = f\left(x + \frac{1}{2}y\right) + g\left(x + \frac{1}{3}y\right)$$

$$\text{or } z = f(y + 2x) + g(y + 3x)$$

### *Example 8*

Solve the equation $x^2p + y^2q + z^2 = 0$.
The given equation

$$x^2p + y^2q = -z^2 \tag{1}$$

is a Lagrange's linear equation with $P = x^2$, $Q = y^2$ and $R = -z^2$
The subsidiary equations are

$$\frac{\mathrm{d}x}{x^2} = \frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}z}{-z^2}$$

Taking the first two ratios, we get an ordinary differential equation in $x$ and $y$, namely, $\frac{\mathrm{d}x}{x^2} = \frac{\mathrm{d}y}{y^2}$.

Integrating, we get $-\frac{1}{x} = -\frac{1}{y} - a$

i.e.
$$\frac{1}{x} - \frac{1}{y} = a \qquad (2)$$

Taking the last two ratios, we get the equation $\frac{dy}{y^2} = \frac{-dz}{z^2}$

$$\frac{dy}{y^2} = \frac{-dz}{z^2}$$

Integrating, we get $\frac{-1}{y} = \frac{1}{z} - b$

Solving,

$$\frac{1}{y} + \frac{1}{z} = b \qquad (3)$$

$\therefore$ The general solution of the given equation is $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0$, where '$f$' is an arbitrary function.

### *Example 9*

Solve the equation $y^2p - xyq = x(z - 2y)$.

The given equation is a Lagrange's linear equation with $P = y^2$, $Q = -xy$, $R = x(z - 2y)$. The subsidiary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking the first two ratios, we get

$$\frac{dx}{y} = \frac{dy}{-x} \text{ or } -x\,dx = y\,dy$$

Integrating, we get $\quad \frac{x^2}{2} + \frac{y^2}{2} = \frac{a}{2}$ or $x^2 + y^2 = a \qquad (1)$

From the subsidiary equations, we have

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} = \frac{z\,dy + y\,dz}{-2xy^2}$$

From the first and last ratios, we get

$$\frac{dx}{1} = \frac{d(yz)}{-2x} \quad \text{or} \quad -2x\,dx = d(yz)$$

Integrating, we get $\quad x^2 + yz = b \qquad (2)$

From (1) and (2) the general solution of the given equation is $f(x^2 + y^2, x^2 + yz) = 0$.

***Example 10***

Solve the equation $(p - q)z = z^2 + (x + y)$. This is a Lagrange's linear equation with $P = z$, $Q = -z$ and $R = z^2 + (x + y)$.

The subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)}$$

From the first two ratios, we get $dx = -dy$

Integrating, we get $$x + y = a^2 \tag{1}$$

## Note ✍

*Neither the method of grouping nor the method of multipliers can be used to get the second solution.*

We make use of solution (1), i.e. we put $x + y = a^2$ in the third ratio.

From the first and third ratios, we get

$$\frac{dx}{z} = \frac{dz}{z^2 + a^2} \text{ or } 2dx = \frac{2zdz}{z^2 + a^2}$$

Integrating, we get $2x = \log (z^2 + a^2) + b$. Now using the value of $a^2$ from (1), the second solution is

$$2x - \log (z^2 + x + y) = b \tag{2}$$

From (1) and (2), the general solution of the given equation is

$$f[x + y, 2x - \log (x + y + z^2)] = 0$$

***Example 11***

Solve the equation $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

This is a Lagrange's linear equation with $P = (z^2 - 2yz - y^2)$, $Q = xy + zx$ and $R = xy - zx$.

The subsidiary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y + z)} = \frac{dz}{x(y - z)}$$

From the last two ratios, we have

$$(y - z)dy = (y + z)dz$$

i.e. $$ydy - (zdy + ydz) - zdz = 0$$

i.e. $$ydy - d(yz) - zdz = 0$$

Integrating, we get

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = \frac{a}{2} \text{ or}$$

$$y^2 - 2yz - z^2 = a \qquad (1)$$

Using the multipliers, $x, y, z$, each of the above ratios $= \dfrac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{0}$

$$\therefore \qquad x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = 0$$

Integrating, we get $\qquad x^2 + y^2 + z^2 = b \qquad (2)$

Therefore the general solution of the given equation is $f(y^2 - 2yz - z^2, x^2 + y^2 + z^2) = 0$.

### *Example 12*

Solve the equation $(x - 2z)p + (2z - y)q = y - x$. This is a Lagrange's linear equation with $P = x - 2z$, $Q = 2z - y$ and $R = y - x$.

The subsidiary equations are

$$\frac{\mathrm{d}x}{x - 2z} = \frac{\mathrm{d}y}{2z - y} = \frac{\mathrm{d}z}{y - x} \qquad (1)$$

Using the multipliers 1, 1, 1, each ratio in (1) $= \dfrac{\mathrm{d}x + \mathrm{d}y + \mathrm{d}z}{0}$

$$\therefore \qquad \mathrm{d}x + \mathrm{d}y + \mathrm{d}z = 0$$

Integrating, we get, $\qquad x + y + z = a \qquad (2)$

Using the multipliers $y, x, 2z$, each ratio in (1) $= \dfrac{y\mathrm{d}x + x\mathrm{d}y + 2z\mathrm{d}z}{0}$

$$\therefore \qquad \mathrm{d}(xy) + 2z\mathrm{d}z = 0$$

Integrating, we get $\qquad xy + z^2 = b \qquad (3)$

Therefore the general solution of the given equation is $f(x + y + z,\ xy + z^2) = 0$

### *Example 13*

Solve the equation $(x^2 - y^2 - z^2)p + 2xyq = 2zx$. This is a Lagrange's linear equation with $P = x^2 - y^2 - z^2$, $\quad Q = 2xy$, $\quad R = 2zx$.

The subsidiary equations are

$$\frac{\mathrm{d}x}{x^2 - y^2 - z^2} = \frac{\mathrm{d}y}{2xy} = \frac{\mathrm{d}z}{2zx} \qquad (1)$$

Taking the last two ratios, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log a$

i.e. $$\frac{y}{z} = a \tag{2}$$

Using the multipliers $x, y, z,$ each of the ratios in (1) $= \dfrac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$ (3)

Taking the last ratio in (1) and the ratio in (3),

$$\frac{dz}{2zx} = \frac{\frac{1}{2}d(x^2 + y^2 + z^2)}{x(x^2 + y^2 + z^2)}$$

i.e. $$\frac{dz}{z} = \frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$$

Integrating, we get $\log b + \log z = \log (x^2 + y^2 + z^2)$

i.e. $$\frac{x^2 + y^2 + z^2}{z} = b \tag{4}$$

Therefore the general solution of the given equation is $f\left(\dfrac{y}{z}, \dfrac{x^2 + y^2 + z^2}{z}\right) = 0.$

### *Example 14*

Solve the equation $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$.
This is a Lagrange's linear equation with $P = x^2(y - z)$, $Q = y^2(z - x)$, $R = z^2(x - y)$.

The subsidiary equations are

$$\frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)} \tag{1}$$

Using the multipliers $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$, each of the ratios in (1) $= \dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

$$\therefore \quad \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, we get $-\dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = -a$

or $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a \tag{2}$$

Using the multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$, each of the ratios in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$$\therefore \qquad \frac{1}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y + \frac{1}{z}\mathrm{d}z = 0$$

Integrating, we get $\log x + \log y + \log z = \log b$

or $$xyz = b \tag{3}$$

Therefore the general solution of the given equation is $f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \quad xyz\right) = 0$.

### *Example 15*

Solve the equation $(mz - ny)p + (nx - lz)q = ly - mx$. Hence write down the solution of the equation $(2z - y)p + (x + z)q + 2x + y = 0$.
The equation $(mz - ny)p + (nx - lz)q = ly - mx$
is a Lagrange's linear equation with $P = mz - ny$, $Q = nx - lz$, $R = ly - mx$.
The subsidiary equations are

$$\frac{\mathrm{d}x}{mz - ny} = \frac{\mathrm{d}y}{nx - lz} = \frac{\mathrm{d}z}{ly - mx} \tag{1}$$

Using the two sets of multipliers $l, m, n$ and $x, y, z$, each of the above ratios in (1)

$$= \frac{l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z}{0} \quad \text{and also} \quad = \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{0}$$

$$\therefore \qquad l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0 \quad \text{and} \quad x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = 0$$

Integrating both the equations, we get

$$lx + my + nz = a \text{ and } x^2 + y^2 + z^2 = b$$

Therefore the general solution of the given equation is $f(lx + my + nz, x^2 + y^2 + z^2) = 0$.
Comparing the equation

$$(2z - y)p + (x + z)q = -2x - y \tag{2}$$

with the previous equation (1), we get $l = -1$, $m = 2$, $n = 1$.
Therefore the solution of equation (2) is

$$f(-x + 2y + z, x^2 + y^2 + z^2) = 0$$

### *Example 16*

Solve the equation $(y + z)p + (z + x)q = x + y$.
This is a Lagrange's linear equation with $P = y + z$, $Q = z + x$ and $R = x + y$.

The subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \tag{1}$$

Each of the ratios in (1) is equal to

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(z-x)}{-(z-x)} \tag{2}$$

Taking the first two ratios in (2), we get

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, we get $\log(x-y) = \log(y-z) + \log a$

i.e.
$$\frac{x-y}{y-z} = a \tag{3}$$

## Note ✍

*Taking the last two ratios in (2) and integrating, we get another solution, namely*

$$\frac{z-x}{y-z} = b \tag{4}$$

*But solution (4) is not independent of solution (3), since* $-\left(1+\frac{x-y}{y-z}\right) = -(1+a)$, *i.e.* $\frac{z-x}{y-z} = b$.

*Hence we should use solution, (3) or (4) only to write down the general solution of the given equation.*

Now each of the ratios in (1) is also equal to

$$\frac{d(x+y+z)}{2(x+y+z)} \tag{5}$$

Taking the first ratio in (2) and the ratio (5), we have $\frac{d(x+y+z)}{(x+y+z)} = -\frac{2d(x-y)}{x-y}$

Integrating, we get $\log(x+y+z) = -2\log(x-y) + \log c$

i.e.
$$(x-y)^2(x+y+z) = c \tag{6}$$

Therefore the general solution of the given equation is $f\left\{\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right\} = 0$

### *Example 17*

Solve the equation $x(y^2+z^2)p + y(z^2+x^2)q = z(y^2-x^2)$.

This is a Lagrange's linear equation with $P = x(y^2+z^2)$, $Q = y(z^2+x^2)$ and $R = z(y^2-x^2)$.

The subsidiary equations are

$$\frac{dx}{x(y^2+z^2)} = \frac{dy}{y(z^2+x^2)} = \frac{dz}{z(y^2-x^2)} \tag{1}$$

Using the multipliers $-\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of the ratios in (1) $= \dfrac{-\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

Integrating, we get $-\log x + \log y + \log z = \log a$

i.e.
$$\frac{yz}{x} = a \tag{2}$$

Using the multipliers $x, -y, z$, each of the ratios in (1) $= \dfrac{xdx - ydy + zdz}{0}$

$$\therefore \quad xdx - ydy + zdz = 0$$

Integrating, we get
$$x^2 - y^2 + z^2 = b \tag{3}$$

Therefore the general solution of the given equation is $f\left(\frac{yz}{x},\ x^2 - y^2 + z^2\right) = 0$

### *Example 18*

Find the integral surface of the equation $px + qy = z$, passing through $x + y = 1$ and $x^2 + y^2 + z^2 = 4$.

The general solution or integral of the Lagrange's linear equation

$$px + qy = z \tag{1}$$

represents a surface. This surface is called the integral surface of the equation.

Now the particular integral surface passing through the circle given by (2) and (3) is required.

$$x + y = 1 \tag{2}$$

$$x^2 + y^2 + z^2 = 4 \tag{3}$$

First let us find the general integral surface of equation (1).

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \tag{4}$$

Two independendent solutions of (4) are easily found as

$$\frac{x}{y} = a \tag{5}$$

and
$$\frac{y}{z} = b \tag{5$'$}$$

Therefore the general integral surface of (1) is

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0 \tag{6}$$

Instead of finding the particular value of '$f$' that satisfies (2) and (3), we proceed alternatively as follows.

We eliminate $x$, $y$, $z$ from (2), (3), (5) and (5)$'$ and get a relation satisfied by $a$ and $b$, which are then replaced by their equivalents, namely, $\frac{x}{y}$ and $\frac{y}{z}$ respectively.

Using (5)$'$ in (3), $$x^2 + y^2 + \frac{y^2}{b^2} = 4 \tag{7}$$

Using (5) in (2) and (7), we have

$$x\left(1 + \frac{1}{a}\right) = 1 \tag{8}$$

and $$x^2\left(1 + \frac{1}{a^2} + \frac{1}{a^2b^2}\right) = 4 \tag{9}$$

Eliminating $x$ between (8) and (9), we get

$$\frac{(a^2b^2 + b^2 + 1)}{b^2(a+1)^2} = 4 \tag{10}$$

Substituting for $a$ and $b$ from (5) and (6) in (10), we get $\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1 = 4\frac{y^2}{z^2}\left(\frac{x+y}{y}\right)^2$.

viz., $x^2 + y^2 + z^2 = 4(x+y)^2$, which is the equation of the required integral surface.

### *Example 19*

Show that the integral surface of the equation $2y(z-3)p + (2x-z)q = y(2x-3)$ that passes through the circle $x^2 + y^2 = 2x, z = 0$ is $x^2 + y^2 - z^2 - 2x + 4z = 0$.

The subsidiary equations of the given Lagrange's equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \tag{1}$$

Taking the first and last ratios in (1), we have

$$\frac{dx}{2z-6} = \frac{dz}{2x-3} \text{ or } (2x-3)dx = (2z-6)dz$$

Integrating, we get $$x^2 - z^2 - 3x + 6z = a \tag{2}$$

Using the multipliers $1, 2y, -2$, each ratio in (1) $= \dfrac{dx + 2ydy - 2dz}{0}$

$\therefore \qquad dx + 2y\,dy - 2dz = 0$

Integrating, we get $$x + y^2 - 2z = b \tag{3}$$

The required surface has to pass through

$$x^2 + y^2 = 2x \quad \text{and} \tag{4}$$

$$z = 0 \tag{5}$$

Using (5) in (2) and (3), we get

$$x^2 - 3x = a \tag{6}$$

and $$x + y^2 = b \tag{7}$$

From (6) and (7), we get

$$x^2 + y^2 - 2x = a + b \tag{8}$$

Using (4) in (8), we have

$$a + b = 0 \tag{9}$$

Substituting for $a$ and $b$ from (2) and (3) in (9), we get $x^2 + y^2 - z^2 - 2x + 4z = 0$, which is the equation of the required integral surface.

### *Example 20*

Show that the integral surface of the partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0$, $z = 1$ is $x^2 + y^2 + 2xyz - 2z + 2 = 0$.

The subsidiary equations of the given Lagrange's equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{(x^2 - y^2)z} \tag{1}$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of the ratios in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore \qquad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$xyz = a \tag{2}$$

Using the multipliers $x, y, -1$, each of the ratios in (1) $= \dfrac{x\,dx + y\,dy - dz}{0}$

$\therefore \qquad x\,dx + y\,dy - dz = 0$

Integrating, we get

$$x^2 + y^2 - 2z = b \tag{3}$$

The required surface has to pass through

$$x + y = 0 \tag{4}$$

and $$z = 1 \tag{5}$$

Using (4) and (5) in (2) and (3), we have

$$-x^2 = a \tag{6}$$

and $$2x^2 - 2 = b \tag{7}$$

Eliminating $x$ between (6) and (7), we get

$$2a + b + 2 = 0 \tag{8}$$

Substituting for $a$ and $b$ from (2) and (3) in (8), we get $2xyz + x^2 + y^2 - 2z + 2 = 0$ or $x^2 + y^2 + 2xyz - 2z + 2 = 0$, which is the equation of the required surface.

## Exercise 1(c)

### Part A (Short-Answer Questions)

Solve the following equations.

1. $\dfrac{\partial^2 z}{\partial x^2} = 0$
2. $\dfrac{\partial^2 z}{\partial y^2} = 0$
3. $\dfrac{\partial^2 z}{\partial x \partial y} = 0$
4. $\dfrac{\partial^2 z}{\partial x^2} = e^{x+y}$
5. $\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + 3y)$
6. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{xy}$
7. $\dfrac{\partial^2 z}{\partial x^2} = \sin y$

8. $\dfrac{\partial^2 z}{\partial y^2} = \cos y$

9. $\dfrac{\partial^2 z}{\partial x \partial y} = k$

10. $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

11. Give the working rule to solve the Lagrange's linear equation.

Find the general solutions of the following Lagrange's linear equations.

12. $pyz + qzx = xy$
13. $yq - xp = z$
14. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$
15. $p \tan x + q \tan y = \tan z$
16. $px^2 + qy^2 = z^2$

## Part B

17. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $\dfrac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$ when $y$ is an odd multiple of $\dfrac{\pi}{2}$.

18. Solve the equation $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$, given that $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$.

19. Solve the equation $\dfrac{\partial^2 z}{\partial y^2} = z$, given that $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$ when $y = 0$.

20. Solve the equations $p = 6x + 3y$, $\quad q = 3x - 4y$ simultaneously.

21. Solve the equation $x\dfrac{\partial z}{\partial x} = 2x + y + 3z$.

22. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} + 18xy^2 + \sin(2x - y) = 0$.

23. Solve the equation $\dfrac{\partial^2 z}{\partial y^2} - 5\dfrac{\partial z}{\partial y} + 6z = 12y$.

24. Solve the equations $\dfrac{\partial^2 z}{\partial x^2} = 0$, $\quad \dfrac{\partial^2 z}{\partial y^2} = 0$ simultaneously.

25. By changing the independent variables by the transformations $u = x + at$, $v = x - at$, show that the equation $\dfrac{\partial^2 z}{\partial t^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$ gets transformed into the equation $\dfrac{\partial^2 z}{\partial u \partial v} = 0$. Hence obtain the general solution of the equation.

26. By changing the independent variables by the transformations $z = x + iy$, $z^* = x - iy$, where $i = \sqrt{-1}$, show that the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ gets transformed into the equation $\dfrac{\partial^2 u}{\partial z \partial z^*} = 0$. Hence obtain the general solution of the equation.

27. Use the transformations $x = u + v$, $y = u - v$ to change the equation $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$ as $\dfrac{\partial^2 z}{\partial u \partial v} = 0$ and hence solve it.

28. Find the solution of the equation $y^2 \dfrac{\partial^2 z}{\partial x^2} - 2xy \dfrac{\partial^2 z}{\partial x \partial y} + x^2 \dfrac{\partial^2 z}{\partial y^2} = \dfrac{y^2}{x}\dfrac{\partial z}{\partial x} + \dfrac{x^2}{y}\dfrac{\partial z}{\partial y}$, by transforming it to a simpler form using the substitutions $u = x^2 + y^2$, $v = x^2 - y^2$.

29. Reduce the equation $4y^3 z_{xx} - yz_{yy} + z_y = 0$ to a simpler form by using the transformations $u = y^2 + x$ and $v = y^2 - x$ and hence solve it.

Find the general solutions of the following linear partial differential equations.

30. (i) $p \cot x + q \cot y = \cot z$
(ii) $(a - x)p + (b - y)q = (c - z)$

31. $\dfrac{y^2 z}{x} p + xzq = y^2$

32. (i) $x^2 p + y^2 q = (x + y)z$; (ii) $x^2 p - y^2 q = (x - y)z$

33. $(y^2 + z^2)p - xyq + xz = 0$

34. $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$

35. $p - q = \log(x + y)$

36. $z(xp - yq) = y^2 - x^2$

37. (i) $(y - z)p + (z - x)q = x - y$; (ii) $(y - z)p + (x - y)q = z - x$

38. (i) $x(y - z)p + y(z - x)q = z(x - y)$
(ii) $\dfrac{y - z}{yz} p + \dfrac{z - x}{zx} \cdot q = \dfrac{x - y}{xy}$

39. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

40. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. [See example (16)]

41. (i) $(y + z)p - (x + z)q = x - y$ (ii) $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

42. $(y^3 x - 2x^4)p + (2y^4 - x^3 y)q = (x^3 - y^3)z$.

43. Find the integral surface of the equation $px + qy = z$, that passes through the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 2$.

44. Find the integral surface of the equation $yp + xq + 1 = z$, that passes through the curve $z = x^2 + y + 1$ and $y = 2x$.

45. Show that the integral surface of the equation $(x^2-a^2)p+(xy-az\tan\alpha)q = xz - ay\cot\alpha$, that passes through the curve $x^2 + y^2 = a^2, z = 0$ is $x^2 + y^2 - a^2 = z^2\tan^2\alpha$.

## 1.13 LINEAR P.D.E.'S OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

Linear partial differential equations of higher order with constant coefficients may be divided into two categories as given below.

(i) Equations in which the partial derivatives occurring are all of the same order (of course, with degree 1 each) and the coefficients are constants. Such equations are called *homogeneous linear P.D.E.s* with constant coefficients.
(ii) Equations in which the partial derivatives occurring are not of the same order and the coefficients are constants are called *non-homogeneous linear P.D.E.s* with constant coefficients.

For example,

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y} \text{ and}$$
$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} - 4\frac{\partial^3 z}{\partial x \partial y^2} + 12\frac{\partial^3 z}{\partial y^3} = x + 2y$$

are equations of the first category.

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = x^2 + y^2 \text{ and}$$
$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = \cos(x + 2y)$$

are equations of the second category.

*The standard form* of a homogeneous linear partial differential equation of the $n^{\text{th}}$ order with constant coefficients is

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots + a_n\frac{\partial^n z}{\partial y^n} = R(x, y) \tag{1}$$

where $a$'s are constants.

If we use the operators $D \equiv \dfrac{\partial}{\partial x}$ and $D' \equiv \dfrac{\partial}{\partial y}$, we can symbolically write equation (1) as

$$\left(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots + a_n D'^n\right) z = R(x, y) \tag{2}$$

i.e.

$$f(D, D')z = R(x, y) \tag{3}$$

where $f(D, D')$ is a homogeneous polynomial of the $n^{\text{th}}$ degree in $D$ and $D'$.

The method of solving (3) is similar to that of solving ordinary linear differential equations with constant coefficients.

The general solution of (3) is of the form $z$ = (Complementary function) + (Particular integral), where the complementary function (C.F.) is the R.H.S. of the general solution of $f(D, D')z = 0$ and the particular integral (P.I.) is given symbolically by $\frac{1}{f(D, D')}R(x, y)$.

**Complementary function of** $f(D, D')z = R(x, y)$

C.F. of the solution of $f(D, D')z = R(x, y)$ is the R.H.S. of the solution of

$$f(D, D')z = 0 \tag{1}$$

Let us assume that

$$z = \phi(y + mx) \tag{2}$$

is a solution of equation (1), where $\phi$ is an arbitrary function.

Differentiating (2) partially with respect to $x$ and then with respect to $y$, we have

$$Dz = \frac{\partial z}{\partial x} = m\phi'(y + mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2\phi''(y + mx)$$

$$\vdots$$

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n\phi^{(n)}(y + mx)$$

Similarly, $$D'^n_z = \frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y + mx) \text{ and}$$

$$D^{n-r} D'^r_y = \frac{\partial^n z}{\partial x^{n-r}\partial y^r} = m^{n-r}\phi^{(n)}(y + mx)$$

Since (2) is a solution of (1), we have

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n)\phi^{(n)}(y + mx) = 0 \tag{3}$$

Since $\phi$ is arbitrary, $\phi^{(n)}(y + mx) \not\equiv 0$

$\therefore$ (3) reduces to $a_0 m^n + a_1 m^{n-1} + \ldots + a_n = 0$ or $f(m, 1) = 0$ (4)

Thus $z = \phi(y + mx)$ will be a solution of (1), if $m$ satisfies the algebraic equation (4) or $m$ is a root of equation (4), which we get by replacing $D$ by $m$ and $D'$ by 1 in the equation $f(D, D')z = 0$ and by dropping $z$ from it.

The equation $f(m, 1) = 0$ is called *the auxiliary equation*, which is an algebraic equation of the $n^{\text{th}}$ degree in $m$ and hence will have $n$ roots.

### *Case(i)*

The roots of (4) are distinct (real or complex).

Let them be $m_1, m_2, \dots m_n$.

The solutions of (1) corresponding to these roots are $z = \phi_1(y+m_1x)$, $z = \phi_2(y+m_2x), \dots, z = \phi_n(y+m_nx)$. The general solution of (1) is given by a linear combination of these solutions.

That is the general solution of (1) is given by

$$z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_n(y+m_nx)$$

$\therefore$ C.F. of the solution of $f(D, D')z = R(x, y)$ is $\phi_1(y+m_1x)+\phi_2(y+m_2x)+\dots+\phi_n(y+m_nx)$, where $\phi_r$'s are arbitrary functions.

### *Case(ii)*

Two of the roots of (4) are equal and others are distinct.

Let them be $m_1, m_1, m_3, m_4, \dots, m_n$.

## Note

*If we apply the rule arrived at in Case (i), the solution of (1) will be* $z = [\phi_1(y+m_1x) + \phi_2(y+m_1x)] + \phi_3(y+m_3x) + \dots + \phi_n(y+m_nx)$, *i.e.* $z = \phi(y+m_1x) + \phi_3(y+m_3x) + \dots + \phi_n(y+m_nx)$, *which contains only* $(n-1)$ *arbitrary functions. Hence it cannot be the general solution of Equation (1).*

Then $f(m, 1) \equiv a_0(m-m_1)^2(m-m_3)\dots(m-m_n)$

$$\therefore \quad f(D, D') \equiv a_0(D-m_1D')^2(D-m_3D')\dots(D-m_nD')$$

Hence solution of (1) will be a combination of the solutions of the component equations

$$(D-m_1D')^2z = 0, (D-m_3D')z = 0, \dots, (D-m_nD')z = 0$$

Consider $(D-m_rD')z = 0$, i.e. $p - m_rq = 0$, which is a Lagrange's linear equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_r} = \frac{dz}{0}$$

Solving, we get $y + m_rx = a$ and $z = b$.

$\therefore$ General solution of $(D-m_rD')z = 0$ is $f_r(y+m_rx, z) = 0$ or $z = \phi_r(y+m_rx)$.

Now consider $\quad (D-m_1D')^2z = 0 \quad (5)$

Let $\quad (D-m_1D')z = u \quad (6)$

$\therefore$ becomes $(D-m_1D')u = 0 \quad (7)$

The solution of (7) is $u = \phi_1(y + m_1x)$. Using this value of $u$ in (6), it becomes

$$(D - m_1D')z = \phi_1(y + m_1x)$$

or
$$p - m_1q = \phi_1(y + m_1x) \tag{8}$$

which is a Lagrange's equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi_1(y + m_1x)}$$

Solving, we get $y + m_1x = a$ and $z - x\phi_1(y + m_1x) = b$

$\therefore$ The solution of Eq. (8) and hence Eq. (5) is

$$f\,[y + m_1x, z - x \cdot \phi_1(y + m_1x)] = 0$$

or
$$z - x \cdot \phi_1(y + m_1x) = \phi_2(y + m_1x)$$

or
$$z = x \cdot \phi_1(y + m_1x) + \phi_2(y + m_1x)$$

$\therefore$ General solution of equation (1) is

$$z = x\phi_1(y + m_1x) + \phi_2(y + m_1x) + \phi_3(y + m_3x) + \ldots + \phi_n(y + m_nx)$$

$\therefore$ C.F. of the solution of $f(D, D')z = R(x, y)$ is

$$x\phi_1(y + m_1x) + \phi_2(y + m_1x) + \phi_3(y + m_3x) + \ldots + \phi_n(y + m_nx)$$

***Case(iii)***

'$r$' of the roots of Eq. (4) are equal and others distinct.

i.e.
$$m_1 = m_2 = m_3 = \ldots = m_r$$

Proceeding as in Case (ii), we can show that the part of the C.F. of the solution of $f(D, D')z = R(x, y)$ is

$$\phi_1(y + m_1x) + x\phi_2(y + m_1x) + x^2\phi_3(y + m_1x) + \ldots + x^{r-1}\phi_r(y + m_1x)$$

*The Particular Integral* of the solution of $f(D, D')z = R(x, y)$.

As in the case of ordinary differential equations, there are formulas/methods for finding particular integrals (P.I.) of the solution of homogeneous (and also non-homogeneous) linear P.D.E.s with constant coefficients. The formulas/methods are given below without proof.

1. $\dfrac{1}{f(D, D')}e^{ax+by} = \dfrac{1}{f(a, b)}e^{ax+by}$, if $f(a, b) \neq 0$

1(a). If $f(a, b) = 0$, $(D - \frac{a}{b}D')$ or its power will be a factor of $f(D, D')$. In this case we factorise $f(D, D')$ and proceed as in ordinary differential equations and use the following results.

$$\frac{1}{\left(D-\frac{a}{b}D'\right)}e^{ax+by} = xe^{ax+by}; \quad \frac{1}{\left(D-\frac{a}{b}D'\right)^2}e^{ax+by} =$$

$$\frac{x^2}{2!}e^{ax+by}, \ldots, \quad \frac{1}{\left(D-\frac{a}{b}D'\right)^r}e^{ax+by} = \frac{x^r}{r!}e^{ax+by}$$

The above results can be derived by using Lagrange's linear equation method.

For example, let $\dfrac{1}{D-\frac{a}{b}D'}e^{ax+by} = z.$

i.e. $p - \dfrac{a}{b}q = e^{ax+by}$

The subsidiary equations are

$$\frac{\mathrm{d}x}{1} = \frac{b\mathrm{d}y}{-a} = \frac{\mathrm{d}z}{e^{ax+by}}$$

The solutions of these equations are $ax+by = c_1$ and $z = xe^{c_1}$ or $z = xe^{ax+by}$.

2. $\dfrac{1}{f(D^2, DD', D'^2)} \begin{matrix}\sin\\ \cos\end{matrix} (ax+by)$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \begin{matrix}\sin\\ \cos\end{matrix} (ax+by)$$

provided $f(a^2, -ab, -b^2) \neq 0$.

2(a).

If $f(-a^2, -ab, -b^2) = 0$, then $\left(D^2 - \dfrac{a^2}{b^2}D'^2\right)$ will be a factor of $f(D^2, DD', D'^2)$. In this case, we proceed as in ordinary differential equations and use the results

$$\frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\sin(ax+by) = -\frac{x}{2a}\cos(ax+by) \text{ and}$$

$$\frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\cos(ax+by) = \frac{x}{2a}\sin(ax+by)$$

which may be verified by the reader.

3. $\dfrac{1}{f(D, D')}x^m y^n = [f(D, D')]^{-1}x^m y^n$ where $[f(D, D')]^{-1}$ is to be expanded in series of powers of $D$ and $D'$.

4. $\dfrac{1}{f(D, D')} e^{ax+by} F(x, y) = e^{ax+by} \cdot \dfrac{1}{f(D+a, D'+b)} F(x, y)$.

5. $\dfrac{1}{D - mD'} F(x, y) = \left[ \int F(x, a - x) dx \right]_{a \to y+mx}$

This result can be derived by assuming that $\dfrac{1}{D - mD'} F(x, y) = z$ and solving for $z$ by using Lagrange's linear equation method.

## 1.14 COMPLEMENTARY FUNCTION FOR A NON-HOMOGENEOUS LINEAR EQUATION

Let the non-homogeneous linear equation be $f(D, D') = 0$.
We resolve $f(D, D')$ into linear factors of the form $(D - aD' - b)$.
The C.F. is the linear combination or simply the sum of (the R.H.S. functions of) the solutions of the component equations $(D - a_r D' - b_r)z = 0$.

Now let us consider the equation $(D - aD' - b)z = 0$

i.e. $p - aq = bz$, which is a Lagrange's linear equation

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-a} = \frac{dz}{bz}$$

One solution of these equations $y + ax = c_1$. The other solution is $\log z = bx + \log c_2$

$$\text{or} \quad z = c_2 e^{bx}$$

$\therefore$ The general solution of the equation is

$$\phi\left(\frac{z}{e^{bx}}, y + ax\right) = 0 \quad \text{or} \quad z = e^{bx} f(y + ax)$$

## Note ✍

*The rules / methods for finding P.I.s are the same as those for homogeneous linear equations.*

## 1.15 SOLUTION OF P.D.E.S BY THE METHOD OF SEPARATION OF VARIABLES

In the next few chapters on applications of partial differential equations, we will have to solve *boundary value problems*, i.e. partial differential equations that satisfy certain given conditions called boundary conditions.

When solving a boundary value problem, if we first find the general solution of the concerned partial differential equation, it will be very difficult to find particular values of the arbitrary functions involved in the general solution that satisfy the boundary conditions. Hence in such situations, we try to find particular solutions of the partial differential equation that satisfy the boundary conditions and then combine them to get the solution of the boundary value problem.

A simple but powerful method of obtaining such particular solution is the *method of separation of variables.* In this method of solving a P.D.E. with $z$ as the dependent variable and $x$ and $y$ as independent variables, the solution is assumed to be of the form $z = f(x) \cdot g(y)$, where $f$ is a function of $x$ alone and $g$ is a function of $y$ alone.

This assumption makes the solution of the P.D.E. depend on solutions of ordinary differential equations.

The variable separable solution of a P.D.E. is called a particular solution, as it can be verified to be a particular form of the general solution of the P.D.E.

For example, consider the equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \tag{1}$$

A variable separable solution of (1) can be obtained as

$$z = (ae^{px} + be^{-px})(ce^{\text{pat}} + de^{-\text{pat}}) \tag{2}$$

where $a, b, c, d, p$ are constants.

(2) can be rewritten as

$$z = \left\{ac\ e^{p(x+at)} + bd\ e^{-p(x+at)}\right\} + \left\{ad\ e^{p(x-at)} + bc\ e^{-p(x-at)}\right\} \tag{3}$$

(3) is a particular case of

$$z = f(x + at) + \phi(x - at)$$

which is the general solution of (1) [see Problem 25 in Exercise 1(c)].

## Worked Examples 1(d)

### *Example 1*

Solve the equation

$$(D^3 + 2D^2 D' - DD'^2 - 2D'^3)z = 0$$

The auxiliary equation (got by replacing $D$ by $m$ and $D'$ by 1 in the given P.D.E.) is

$$m^3 + 2m^2 - m - 2 = 0$$

i.e. $\quad m^2(m+2) - (m+2) = 0$

i.e. $\quad (m-1)(m+1)(m+2) = 0$

$\therefore \quad m = 1, -1, -2$

$\therefore$ General solution of the given equation is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y - 2x)$$

## Note ✍

*There is no particular integral in the general solution, since the R.H.S. member of the given P.D.E. is zero.*

### *Example 2*

Solve the equation

$$(D^3 - D^2D' - 8DD'^2 + 12D'^3)z = 0$$

The auxiliary equation is $m^3 - m^2 - 8m + 12 = 0$

$m = 2$ is a root of the auxiliary equation.

It is $(m-2)(m^2+m-6) = 0$ or $(m-2)(m-2)(m+3) = 0$

$$\therefore \qquad m = 2, 2, -3$$

$\therefore$ The general solution of the given equation is

$$z = xf_1(y+2x) + f_2(y+2x) + f_3(y-3x)$$

### *Example 3*

Solve the equation $\qquad (D^2 - 3DD' + 2D'^2)z = 2\cosh(3x+4y)$

The auxiliary equation is $\qquad m^2 - 3m + 2 = 0$

i.e. $\qquad (m-1)(m-2) = 0$

$\therefore \qquad m = 1, 2$

$\therefore$ The C.F. of the given P.D.E. = $f_1(y+x) + f_2(y+2x)$

$$\text{P.I.} = \frac{1}{D^2 - 3DD' + 2D'^2} 2\cosh(3x+4y)$$

$$= \frac{1}{D^2 - 3DD' + 2D'^2}\left[e^{3x+4y} + e^{-(3x+4y)}\right]$$

$$= \frac{1}{3^2 - 3.3.4 + 2.4^2} e^{3x+4y} + \frac{1}{(-3)^2 - 3(-3)(-4) + 2(-4)^2} e^{-(3x+4y)}$$

$$= \frac{1}{5}\left[e^{3x+4y} + e^{-(3x+4y)}\right]$$

$$= \frac{2}{5}\cosh(3x+4y)$$

$\therefore$ The general solution of the given equation is $z = f_1(y+x) + f_2(y+2x) + \frac{2}{5}\cosh(3x+4y)$.

### *Example 4*

Solve the equation

$$(9D^2 + 6DD' + D'^2)z = (e^x + e^{-2y})^2$$

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \quad \text{i.e.} \quad (3m+1)^2 = 0$$

$$\therefore \qquad m = -1/3, -1/3$$

$$\therefore \qquad \text{C.F.} = xf_1(y - \tfrac{1}{3}\cdot x) + f_2(y - \tfrac{1}{3}\cdot x) \quad \text{or}$$

$$xf_1(3y - x) + f_2(3y - x)$$

$$\text{P.I.} = \frac{1}{9D^2 + 6DD' + D'^2}(e^x + e^{-2y})^2$$

$$= \frac{1}{9D^2 + 6DD' + D'^2}\left(e^{2x} + e^{-4y} + 2e^{x-2y}\right)$$

$$= \frac{1}{(3D+D')^2}e^{2x} + \frac{1}{(3D+D')^2}e^{-4y} + 2\cdot\frac{1}{(3D+D')^2}e^{x-2y}$$

$$= \frac{1}{36}e^{2x} + \frac{1}{16}e^{-4y} + 2e^{x-2y}$$

$\therefore$ The general solution of the given equation is

$$z = xf_1(3y-x) + f_2(3y-x) + \frac{1}{36}e^{2x} + \frac{1}{16}e^{-4y} + 2e^{x-2y}$$

### *Example 5*

Solve the equation

$$(D^3 - 3DD'^2 + 2D'^3)z = e^{2x-y} + e^{x+y}$$

The auxiliary equation is $m^3 - 3m + 2 = 0$

i.e. $\qquad (m-1)(m^2+m-2) = 0$

i.e. $\qquad (m-1)^2(m+2) = 0$

$$\therefore \qquad m = 1, 1, -2$$

$$\therefore \qquad \text{C.F.} = xf_1(y+x) + f_2(y+x) + f_3(y-2x)$$

$$\text{P.I.} = \frac{1}{D^3 - 3DD'^2 + 2D'^3}(e^{2x-y} + e^{x+y})$$
$$= \frac{1}{(D+2D')(D-D')^2}e^{2x-y} + \frac{1}{(D-D')^2(D+2D')}e^{x+y}$$
$$= \frac{1}{9}\cdot\frac{1}{D+2D'}e^{2x-y} + \frac{1}{9}\cdot\frac{1}{(D-D')^2}e^{x+y}$$
$$= \frac{1}{9}\left[xe^{2x-y} + \frac{x^2}{2}e^{x+y}\right]$$

$\therefore$ The general solution of the given equation is

$$z = xf_1(y+x) + f_2(y+x) + f_3(y-2x) + \frac{x}{9}e^{2x-y} + \frac{x^2}{18}e^{x+y}$$

### *Example 6*

Solve the equation

$$(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = (1 + e^{2x+y})^2$$

The auxiliary equation is $m^3 - 6m^2 + 12m - 8 = 0$

i.e. $\quad (m-2)^3 = 0$

$\therefore \quad m = 2, 2, 2$

$\therefore \quad \text{C.F.} = x^2 f_1(y+2x) + x\cdot f_2(y+2x) + f_3(y+2x)$

$$\text{P.I.} = \frac{1}{(D-2D')^3}(1+e^{2x+y})^2$$
$$= \frac{1}{(D-2D')^3}(1) + 2\cdot\frac{1}{(D-2D')^3}e^{2x+y} + \frac{1}{(D-2D')^3}e^{4x+2y}$$
$$= \frac{x^3}{3!} + 2\frac{x^3}{3!}e^{2x+y} + \frac{x^3}{3!}e^{4x+2y}$$
$$\left[\text{since } \frac{1}{(D-2D')^3}(1) = \frac{1}{(D-2D')^3}e^{0.x+0.y} \text{ and } \frac{1}{\left(D-\frac{a}{b}D'\right)^3}e^{ax+by} = \frac{x^3}{3!}e^{ax+by}\right]$$
$$= \frac{x^3}{6}(1+e^{2x+y})^2$$

$\therefore$ The general solution of the given equation is

$$z = x^2 f_1(y+2x) + xf_2(y+2x) + f_3(y+2x) + \frac{x^3}{6}(1+e^{2x+y})^2.$$

### *Example 7*

Solve the equation $(D^2 + 2DD' + D'^2)z = x^2y + e^{x-y}$

The auxiliary equation is $m^2 + 2m + 1 = 0$ or $(m+1)^2 = 0 \quad \therefore \quad m = -1, -1$

$\therefore \quad \text{C.F.} = xf_1(y-x) + f_2(y-x)$

$$
\begin{aligned}
(\text{P.I.})_1 &= \frac{1}{(D+D')^2}x^2y \\
&= \frac{1}{D^2\left(1+\dfrac{D'}{D}\right)^2}x^2y \\
&= \frac{1}{D^2}\left(1+\frac{D'}{D}\right)^{-2}(x^2y) \\
&= \frac{1}{D^2}\left(1-\frac{2D'}{D}+3\frac{D'^2}{D^2}\right)(x^2y) \\
&= \frac{1}{D^2}\left\{x^2y-\frac{2}{D}x^2\right\} \\
&= y\cdot\frac{1}{D^2}(x^2)-2\cdot\frac{1}{D^3}(x^2) \\
&= y\cdot\frac{x^4}{3.4}-2\cdot\frac{x^5}{3.4.5} \\
&= \frac{x^4y}{12}-\frac{x^5}{30} \\
(\text{P.I.})_2 &= \frac{1}{(D+D')^2}e^{x-y}=\frac{x^2}{2!}e^{x-y}
\end{aligned}
$$

$\therefore$ The general solution is

$$
z = xf_1(y-x)+f_2(y-x)+\frac{x^4y}{12}-\frac{x^5}{30}+\frac{x^2}{2}e^{x-y}
$$

### *Example 8*

Solve the equation

$$
(D^3-7DD'^2-6D'^3)z = x^2+xy^2+y^3
$$

The auxiliary equation is

$$
m^3-7m-6=0, \quad \text{i.e.} \quad (m+1)(m^2-m-6)=0
$$

i.e.

$$
(m+1)(m+2)(m-3)=0
$$

$\therefore$ $m=-1,-2,3$

$\therefore$ C.F. $= f_1(y-x)+f_2(y-2x)+f_3(y+3x)$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^3-7DD'^2-6D'^3}(x^2+xy^2+y^3) \\
&= \frac{1}{D^3}\left\{1-\frac{\left(7DD'^2+6D'^3\right)}{D^3}\right\}^{-1}(x^2+xy^2+y^3) \\
&= \frac{1}{D^3}\left[1+\frac{D'^2}{D^3}(7D+6D')+\cdots\right](x^2+xy^2+y^3)
\end{aligned}
$$

$$= \left[\frac{1}{D^3} + \frac{1}{D^6}(7DD'^2 + 6D'^3)\right](x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}\{7D \cdot (2x + 6y) + 36\}$$

$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}(50)$$

$$= \frac{x^5}{3.4.5} + y^2 \cdot \frac{x^4}{2.3.4} + y^3 \cdot \frac{x^3}{1.2.3} + 50 \cdot \frac{x^3}{1.2.3}$$

$$= \frac{1}{60}x^5 + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3$$

$\therefore$ The general solution is

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{x^5}{60} + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3$$

### *Example 9*

Solve the equation

$$(D^2 + 4DD' - 5D'^2)z = xy + \sin(2x + 3y)$$

The auxiliary equation is $m^2 + 4m - 5 = 0$

i.e. $(m + 5)(m - 1) = 0$

$\therefore$ $m = -5, 1$

$\therefore$ $\text{C.F.} = \phi_1(y - 5x) + \phi_2(y + x)$

$$(\text{P.I.})_1 = \frac{1}{D^2 + 4DD' - 5D'^2}(xy)$$

$$= \frac{1}{D^2\left\{1 + \frac{D'}{D^2}(4D - 5D')\right\}}(xy)$$

$$= \frac{1}{D^2}\left\{1 + \frac{D'}{D^2}(4D - 5D')\right\}^{-1}(xy)$$

$$= \frac{1}{D^2}\left\{1 - \frac{D'}{D^2}(4D - 5D') + \ldots\right\}(xy)$$

$$= \frac{1}{D^2}(xy) - \frac{1}{D^4} \cdot 4D'(xy)$$

$$= \frac{x^3y}{6} - \frac{1}{D^4}(4x)$$

$$= \frac{1}{6}x^3y - \frac{1}{30}x^5$$

$$(\text{P.I})_2 = \frac{1}{D^2 + 4DD' - 5D'^2} \sin(2x + 3y)$$
$$= \frac{1}{-2^2 + 4 \cdot (-2.3) - 5(-3^2)} \sin(2x + 3y)$$
$$= \frac{1}{17} \sin(2x + 3y)$$

$\therefore$ General solution is

$$z = \phi_1(y - 5x) + \phi_2(y + x) + \frac{1}{6}x^3 y - \frac{1}{30}x^5 + \frac{1}{17} \sin(2x + 3y)$$

### *Example 10*

Solve the equation

$$(D^2 + D'^2)z = \sin 2x \sin 3y + 2\sin^2(x + y)$$

The auxiliary equation is $m^2 + 1 = 0$

i.e. $\quad m = \pm i$

$\therefore \quad \text{C.F.} = \phi_1(y + ix) + \phi_2(y - ix)$

$$(\text{P.I.})_1 = \frac{1}{D^2 + D'^2} \sin 2x \sin 3y$$
$$= \frac{1}{D^2 + D'^2} \cdot \frac{1}{2}\left\{\cos(2x - 3y) - \cos(2x + 3y)\right\}$$
$$= \frac{1}{2}\left[\frac{1}{-4-9}\cos(2x - 3y) - \frac{1}{-4-9}\cos(2x + 3y)\right]$$
$$= -\frac{1}{13} \cdot \frac{1}{2}\{\cos(2x - 3y) - \cos(2x + 3y)\}\Big]$$
$$= -\frac{1}{13} \sin 2x \sin 3y$$

$$(\text{P.I.})_2 = \frac{1}{D^2 + D'^2} 2\sin^2(x + y)$$
$$= \frac{1}{D^2 + D'^2}\left\{1 - \cos(2x + 2y)\right\}$$
$$= \frac{1}{D^2}\left(1 + \frac{D'^2}{D^2}\right)^{-1}(1) - \frac{1}{D^2 + D'^2}\cos(2x + 2y)$$
$$= \frac{1}{D^2}(1) - \frac{1}{-4-4}\cos(2x + 2y)$$
$$= \frac{x^2}{2} + \frac{1}{8}\cos(2x + 2y)$$

$\therefore$ General solution is

$$z = \phi_1(y + ix) + \phi_2(y - ix) - \frac{1}{13}\sin 2x \sin 3y + \frac{x^2}{2} + \frac{1}{8}\cos(2x + 2y)$$

### *Example 11*

Solve the equation

$$(16D^4 - D'^4)z = \cos(x + 2y)$$

The auxiliary equation is $16m^4 - 1 = 0$

i.e. $\quad (m^2 - 1/4)(m^2 + 1/4) = 0$

$\therefore \quad m = \pm 1/2, \pm i/2$

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + f_2\left(y - \frac{1}{2}x\right) + f_3\left(y + \frac{ix}{2}\right) + f_4\left(y - \frac{ix}{2}\right)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{16D^4 - D'^4}\cos(x + 2y) \\
&= \frac{1}{(4D^2 - D'^2)(4D^2 + D'^2)}\cos(x + 2y) \\
&= \frac{1}{(4D^2 - D'^2)} \cdot \frac{1}{4(-1) + (-4)}\cos(x + 2y) \\
&= -\frac{1}{8} \cdot \frac{1}{4D^2 - D'^2}\cos(x + 2y) \\
&= -\frac{1}{32} \cdot \frac{1}{D^2 - \frac{1}{4}D'^2}\cos(x + 2y)
\end{aligned}$$

$$= -\frac{1}{32} \cdot \frac{x}{2}\sin(x + 2y) \quad \left[\because \frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\cos(ax + by) = \frac{x}{2a}\sin(ax + by)\right]$$

$$= -\frac{1}{64}x\sin(x + 2y)$$

$\therefore$ General solution is

$$z = f_1\left(y + \frac{x}{2}\right) + f_2\left(y - \frac{x}{2}\right) + f_3\left(y + \frac{ix}{2}\right) + f_4\left(y - \frac{ix}{2}\right) - \frac{1}{64}x\sin(x + 2y)$$

### *Example 12*

Solve the equation

$$(D^3 + D^2D' - 4DD'^2 - 4D'^3)z = \cos(2x + y)$$

The auxiliary equation is $m^3 + m^2 - 4m - 4 = 0$

i.e. $\quad m^2(m + 1) - 4(m + 1) = 0$

i.e. $\quad (m + 1)(m + 2)(m - 2) = 0$

$\therefore \quad m = -1, -2, 2$

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D^2-4D'^2)(D+D')}\cos(2x+y) \\
&= \frac{1}{(D^2-4D'^2)} \cdot \frac{(D-D')}{D^2-D'^2}\cos(2x-y) \\
&= \frac{1}{(D^2-4D'^2)} \cdot \frac{1}{-4-(-1)}(D-D')\cos(2x-y) \\
&= -\frac{1}{3} \cdot \frac{1}{D^2-4D'^2}\{-2\sin(2x-y) - \sin(2x-y)\} \\
&= \frac{1}{D^2-4D'^2}\sin(2x-y) \\
&= -\frac{x}{4}\cos(2x-y) \left[\because \frac{1}{D^2-\frac{a^2}{b^2}D'^2}\sin(ax+by) = -\frac{x}{2a}\cos(ax+by)\right]
\end{aligned}$$

$\therefore$ General solution is

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x) - \frac{x}{4}\cos(2x-y)$$

### *Example 13*

Solve the equation

$$(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+y}$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$\therefore \quad m = 1, 1$

$\therefore \quad$ C.F. $= xf_1(y+x) + f_2(y+x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-D')^2}e^{x+y}(x^2y^2) \\
&= e^{x+y}\frac{1}{\{(D+1)-(D'+1)\}^2}x^2y^2 \\
&= e^{x+y}\frac{1}{(D-D')^2}x^2y^2 \\
&= e^{x+y}\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}(x^2y^2) \\
&= e^{x+y}\frac{1}{D^2}\left(1+\frac{2D'}{D}+3\frac{D'^2}{D^2}\right)(x^2y^2) \\
&= e^{x+y}\frac{1}{D^2}\left\{x^2y^2 + \frac{2}{D}(2x^2y) + \frac{3}{D^2}(2x^2)\right\}
\end{aligned}$$

$$= e^{x+y}\left[y^2 \cdot \frac{1}{D^2}(x^2) + 4y \cdot \frac{1}{D^3}(x^2) + 6 \cdot \frac{1}{D^4}(x^2)\right]$$

$$= \left(\frac{1}{12}x^4y^2 + \frac{1}{15}x^5y + \frac{1}{60}x^6\right)e^{x+y}$$

$\therefore$ General solution is

$$z = xf_1(y+x) + f_2(y+x) + \left(\frac{1}{12}y^2 + \frac{1}{15}xy + \frac{1}{60}x^2\right)x^4e^{x+y}$$

## *Example 14*

Solve the equation

$$(D^2 - D'^2)z = e^{x-y}\sin(2x+3y)$$

The auxiliary equation is $m^2 - 1 = 0$

$\therefore \quad m = \pm 1$

$\therefore$ C.F. $= f_1(y+x) + f_2(y-x)$

$$\text{P.I.} = \frac{1}{D^2 - D'^2}e^{x-y}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{(D+1)^2 - (D'-1)^2}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D^2 - D'^2 + 2(D+D')}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{2(D+D')+5}\sin(2x+3y)$$

$$= e^{x-y}\frac{2(D+D')-5}{4(D+D')^2-25}\sin(2x+3y)$$

$$= e^{x-y}\{2(D+D')-5\}\cdot\frac{1}{4(D^2+2DD'+D'^2)-25}\sin(2x+3y)$$

$$= e^{x-y}\{2(D+D')-5\}\cdot\left(-\frac{1}{125}\right)\sin(2x+3y)$$

$$= -\frac{1}{125}e^{x-y}\{4\cos(2x+3y) + 6\cos(2x+3y) - 5\sin(2x+3y)\}$$

$$= \frac{1}{25}e^{x-y}\{\sin(2x+3y) - 2\cos(2x+3y)\}$$

$\therefore$ General solution is

$$z = f_1(y+x) + f_2(y-x) + \frac{1}{25}e^{x-y}\{\sin(2x+3y) - 2\cos(2x+3y)\}$$

## *Example 15*

Solve the equation

$$(D^2 - 5\,DD' + 6\,D'^2)\,z = y\sin x$$

The auxiliary equation is $m^2 - 5m + 6 = 0$

i.e. $(m-2)(m-3)=0$

$\therefore$ $m=2, 3$

$\therefore$ $\text{C.F.} = \phi_1(y+2x)+\phi_2(y+3x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-2D')(D-3D')}y\sin x \\
&= \frac{1}{D-2D'}\left[\int (a-3x)\sin x\,dx\right]_{a\to y+3x} \\
&= \frac{1}{D-2D'}[(a-3x)(-\cos x)+3(-\sin x)]_{a\to y+3x} \\
&= \frac{1}{D-2D'}[-y\cos x-3\sin x] \\
&= -\left\{\int [(a-2x)\cos x+3\sin x]\,dx\right\}_{a\to y+2x} \\
&= -\Big[(a-2x)\sin x+2(-\cos x)-3\cos x\Big]_{a\to y+2x} \\
&= 5\cos x - y\sin x
\end{aligned}$$

$\therefore$ General solution is

$$z = \phi_1(y+2x)+\phi_2(y+3x)+5\cos x - y\sin x$$

### *Example 16*

Solve the equation

$$(4D^2-4DD'+D'^2)z = 16\log(x+2y)$$

The auxiliary equation is $4m^2-4m+1=0$

i.e. $(2m-1)^2=0$

$\therefore$ $m=1/2, 1/2$

$$\therefore\quad \text{C.F.} = xf_1\left(y+\frac{1}{2}x\right)+f_2\left(y+\frac{1}{2}x\right) \quad \text{or}$$

$$xf_1(2y+x)+f_2(2y+x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(2D-D')^2}16\log(x+2y) \\
&= 4\cdot\frac{1}{(D-1/2D')}\cdot\frac{1}{D-1/2D'}\log(x+2y) \\
&= 4\cdot\frac{1}{D-1/2D'}\left\{\int \log\left[x+2\left(a-\frac{1}{2}x\right)\right]dx\right\}_{a\to y+\frac{1}{2}x}
\end{aligned}$$

$$= 4 \cdot \frac{1}{D - 1/2D'}\left[\int \log(2a)\mathrm{d}x\right]_{a \to y + \frac{1}{2}x}$$

$$= 4 \cdot \frac{1}{D - 1/2D'}\{x \log(x + 2y)\}$$

$$= 4\left[\int x \log\left\{x + 2\left(a - \frac{1}{2}x\right)\right\}\mathrm{d}x\right]_{a \to y + \frac{1}{2}x}$$

$$= 4\left[\int x \log(2a)\mathrm{d}x\right]_{a \to y + \frac{1}{2}x}$$

$$= 2x^2(\log 2a)_{a \to y + \frac{1}{2}x}$$

$$= 2x^2 \log(x + 2y)$$

$\therefore$ General solution is

$$z = xf_1(x + 2y) + f_2(x + 2y) + 2x^2 \log(x + 2y)$$

### *Example 17*

Solve the equation

$$(D^2 + 2DD' + D'^2 - 2D - 2D')z = \cosh(x - y)$$

The given equation is a non-homogeneous linear equation

$$\begin{aligned} D^2 + 2DD' + D'^2 - 2D - 2D' &\equiv (D + D')^2 - 2(D + D') \\ &= (D + D')(D + D' - 2) \end{aligned}$$

$\therefore$ The given equation is

$$(D + D')(D + D' - 2)z = \cosh(x - y)$$

$\therefore$ C.F. $= f_1(y - x) + e^{2x} \cdot f_2(y - x)$ [$\because$ the part of C.F. corresponding to $(D - aD' - b)z = 0$ is $e^{bx} f(y + ax)$]

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D + D')(D + D' - 2)} \frac{1}{2}\{e^{x-y} + e^{-x+y}\} \\ &= \frac{1}{2} \cdot \frac{1}{D + D'} \cdot \frac{-1}{2}(e^{x-y} + e^{-x+y}) \\ &= -\frac{1}{4} \cdot (xe^{x-y} + xe^{-x+y}) \\ &= -\frac{x}{2}\cosh(x - y) \end{aligned}$$

$\therefore$ General solution is

$$z = f_1(y - x) + e^{2x} f_2(y - x) - \frac{x}{2}\cosh(x - y)$$

### *Example 18*

Solve the equation $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y} + xy$

$$D^2 - D'^2 - 3D + 3D' \equiv (D + D')(D - D') - 3(D - D')$$
$$= (D - D')(D + D' - 3)$$

$\therefore$ The given equation is

$$(D - D')(D + D' - 3)z = e^{x+2y} + xy$$

$\therefore$

$$\text{C.F.} = f_1(y + x) + e^{3x} f_2(y - x)$$

$$(\text{P.I.})_1 = \frac{1}{(D - D')(D + D' - 3)} e^{x+2y}$$
$$= \frac{1}{(D + D' - 3)} \cdot (-1) e^{x+2y}$$
$$= -x e^{x+2y}$$

$$(\text{P.I.})_2 = \frac{1}{(D - D')(D + D' - 3)} xy$$
$$= -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left\{1 - \frac{D + D'}{3}\right\}^{-1} xy$$
$$= \frac{1}{3D}\left(1 + \frac{D'}{D}\right)\left\{1 + \frac{1}{3}(D + D') + \frac{1}{9}(D + D')^2 + \frac{1}{27}(D + D')^3 + \ldots\right\} xy$$
$$= -\frac{1}{3}\left(\frac{1}{D} + \frac{D'}{D^2}\right)\left\{1 + \frac{1}{3}D + \frac{1}{3}D' + \frac{1}{9}D^2 + \frac{2}{9}DD' + \frac{1}{27}D^3 + \frac{1}{9}D^2D'\right\}(xy)$$
$$= -\frac{1}{3}\left[\frac{1}{D} + \frac{1}{3} + \frac{1}{3}\frac{D'}{D} + \frac{1}{9}D + \frac{2}{9}D' + \frac{1}{9}DD' + \frac{D'}{D^2} + \frac{1}{3}\frac{D'}{D} + \frac{1}{9}D'\right.$$
$$\left. + \frac{1}{27}DD'\right](xy)$$
$$= -\frac{1}{3}\left[\frac{D'}{D^2} + \frac{2}{3}\frac{D'}{D} + \frac{1}{D} + \frac{1}{3} + \frac{1}{9}D + \frac{1}{3}D' + \frac{4}{27}DD'\right] xy$$
$$= -\frac{1}{3}\left[\frac{x^3}{6} + \frac{x^2}{3} + \frac{x^2 y}{2} + \frac{1}{3}xy + \frac{1}{9}y + \frac{1}{3}x + \frac{4}{27}\right]$$

$\therefore$ General solution is

$$z = \text{C.F} + (\text{P.I.})_1 + (\text{P.I.})_2$$

### *Example 19*

Solve the equation $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = x + y + \sin(2x + y)$

$$D^2 - 3DD' + 2D'^2 + 2D - 2D' \equiv (D - D')(D - 2D') + 2(D - D')$$
$$= (D - D')(D - 2D' + 2)$$

$\therefore$ The given equation is

$$(D - D')(D - 2D' + 2)z = (x + y) + \sin(2x + y)$$

$$\text{C.F.} = f_1(y + x) + e^{-2x} f_2(y + 2x)$$

$$(\text{P.I.})_1 = \frac{1}{(D - D')(D - 2D' + 2)}(x + y)$$

$$= \frac{1}{2D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 + \frac{D - 2D'}{2}\right)^{-1}(x + y)$$

$$= \frac{1}{2}\left(\frac{1}{D} + \frac{D'}{D^2}\right)\left\{1 - \frac{1}{2}(D - 2D') + \frac{1}{4}(D - 2D')^2 + \cdots\right\}(x + y)$$

$$= \frac{1}{2}\left[\frac{1}{D} + \frac{D'}{D^2}\right]\left[1 - \frac{1}{2}D + D' + \frac{1}{4}D^2 - DD'\right](x + y)$$

$$= \frac{1}{2}\left[\frac{1}{D} - \frac{1}{2} + \frac{D'}{D} + \frac{1}{4}D - D' + \frac{D'}{D^2} - \frac{1}{2}\frac{D'}{D} + \frac{1}{4}D'\right](x + y)$$

$$= \frac{1}{2}\left(\frac{D'}{D^2} + \frac{1}{2}\cdot\frac{D'}{D} + \frac{1}{D} - \frac{1}{2} + \frac{1}{4}D - \frac{3}{4}D'\right)(x + y)$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x}{2} + \frac{x^2}{2} + xy - \frac{1}{2}y - \frac{1}{2}x + 1/4 - 3/4\right]$$

$$= \frac{1}{2}x^2 + \frac{1}{2}xy - \frac{1}{4}y - 1/4$$

$$(\text{P. I.})_2 = \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'}\sin(2x + y)$$

$$= \frac{1}{-4 + 6 - 2 + 2(D - D')}\sin(2x + y)$$

$$= \frac{(D + D')}{2(D^2 - D'^2)}\sin(2x + y)$$

$$= \frac{-1}{6}\{2\cos(2x + y) + \cos(2x + y)\}$$

$$= \frac{-1}{2}\cos(2x + y)$$

$\therefore$ General solution is

$$z = f_1(y + x) + e^{-2x} f_2(y + 2x) + \frac{1}{2}x^2 + \frac{1}{2}xy - \frac{1}{4}y - \frac{1}{4} - \frac{1}{2}\cos(2x + y)$$

### *Example 20*

Solve the equation $(D^2 - DD' + D' - 1)z = e^{2x+3y} + \cos^2(x + 2y)$

$$D^2 - DD' + D' - 1 \equiv (D^2 - 1) - D'(D - 1)$$
$$= (D - 1)(D - D' + 1)$$

$\therefore$ The given equation is

$$(D-1)(D-D'+1)z = e^{2x+3y} + \cos^2(x+2y)$$

$$\therefore \qquad \text{C.F.} = e^x f_1(y) + e^{-x} f_2(y+x)$$

$$\begin{aligned}
(\text{P.I.})_1 &= \frac{1}{(D-1)(D-D'+1)} e^{2x+3y} \\
&= \frac{1}{(2-1)(D-D'+1)} e^{2x+3y} \\
&= x e^{2x+3y}
\end{aligned}$$

$$\begin{aligned}
(\text{P.I.})_2 &= \frac{1}{(D-1)(D-D'+1)} \frac{1}{2}\{1+\cos(2x+4y)\} \\
&= \frac{1}{2}(-1) + \frac{1}{2}\cdot\frac{1}{(D^2-DD'+D'-1)}\cos(2x+4y) \\
&= -1/2 + \frac{1}{2}\cdot\frac{1}{-4+8+D'-1}\cos(2x+4y) \\
&= -1/2 + 1/2.\frac{D'-3}{(D'^2-9)}\cos(2x+4y) \\
&= \frac{-1}{2} - \frac{1}{50}\{-4\sin(2x+4y) - 3\cos(2x+4y)\} \\
&= \frac{1}{50}\{4\sin(2x+4y) + 3\cos(2x+4y)\} - 1/2
\end{aligned}$$

$\therefore$ General solution is

$$z = e^x f_1(y) + e^{-x} f_2(y+x) + xe^{2x+3y} - 1/2 + \frac{1}{50}\{4\sin(2x+4y) + 3\cos(2x+4y)\}$$

### *Example 21*

Solve the equation $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y + ye^x$

$$\begin{aligned}
2D^2 - DD' - D'^2 + 6D + 3D' &\equiv (2D+D')(D-D') + 3(2D+D') \\
&= (2D+D')(D-D'+3)
\end{aligned}$$

$\therefore$ The given equation is

$$(2D+D')(D-D'+3)z = xe^y + ye^x$$

$$\therefore \qquad \text{C.F.} = f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_2(y+x)$$

or $\qquad f_1(2y-x) + e^{-3x}\cdot f_2(y+x)$

$$(\text{P.I.})_1 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'}(xe^y)$$

$$= e^y \cdot \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)}(x)$$

$$= e^y \cdot \frac{1}{2 + 5D + D' + 2D^2 - DD' - D'^2}(x)$$

$$= \frac{e^y}{2}\{1 + \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)\}^{-1}(x)$$

$$= \frac{e^y}{2}\{1 - \frac{5}{2} \cdot D\}(x)$$

$$= \frac{1}{4}(2x - 5)e^y$$

$$(\text{P.I.})_2 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'}(ye^x)$$

$$= e^x \cdot \frac{1}{2(D+1)^2 - (D+1)D' - D'^2 + 6(D+1) + 3D'}(y)$$

$$= e^x \cdot \frac{1}{8 + 10D + 2D' + 2D^2 - DD' - D'^2}(y)$$

$$= \frac{e^x}{8}\left\{1 + \frac{1}{8}(10D + 2D' + 2D^2 - DD' - D'^2)\right\}^{-1}(y)$$

$$= \frac{e^x}{8}\left\{1 - \frac{1}{4}D'\right\}(y)$$

$$= \frac{1}{32}(4y - 1)e^x$$

$\therefore$ General solution is

$$z = f_1(2y - x) + e^{-3x} f_2(y + x) + \frac{1}{4}(2x - 5)e^y + \frac{1}{32}(4y - 1)e^x$$

### *Example 22*

Solve the equation $2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$, by the method of separation of variables.

Let $z = X(x) \cdot Y(y)$ be a solution of

$$2xz_x - 3yz_y = 0 \tag{1}$$

Then $z_x = X'Y$ and $z_y = XY'$, where $X' = \frac{dX}{dx}$ and $Y' = \frac{dY}{dy}$ satisfy Eq. (1).

i.e. $$2xX'Y - 3yXY' = 0$$

i.e. $$2x\frac{X'}{X} = 3y\frac{Y'}{Y}$$

L.H.S. is a function of $x$ alone and R.H.S. is a function of $y$ alone. They are equal for all values of $x$ and $y$. This is possible only if each is a constant.

$\therefore$ $$2x\frac{X'}{X} = 3y\frac{Y'}{Y} = k$$

i.e. $$2\frac{X'}{X} = \frac{k}{x} \tag{2}$$

and $$\frac{3Y'}{Y} = \frac{k}{y} \tag{3}$$

Integrating both sides of (2) with respect to $x$,

$$2\log X = k\log x + \log A$$

i.e. $$X^2 = Ax^k \text{ or } X = ax^{k/2} \tag{4}$$

Similarly, from (3), $Y = by^{k/3}$

$\therefore$ Required solution of (1) is

$$z = abx^{k/2}y^{k/3} \text{ or } z = cx^{k/2}y^{k/3}$$

### *Example 23*

Solve the equation $\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + z$, by the method of separation of variables, given that

$$z(x, 0) = 6e^{-3x}$$

Let

$$z = X(x)\cdot Y(y) \tag{1}$$

be a solution of

$$z_x = 2z_y + z \tag{2}$$

Then $z_x = X'Y$ and $z_y = XY'$ satisfy equation (2).

i.e. $$X'Y = 2XY' + XY$$

Dividing throughout by $XY$, we get

$$\frac{X'}{X} = 2\frac{Y'}{Y} + 1 = k$$

[$\because$ the L.H.S. is a function of $x$ alone and the R.H.S. is a function of $y$ alone]

$\therefore$ $$\frac{X'}{X} = k \tag{3}$$

and $$\frac{Y'}{Y} = \frac{k-1}{2} \tag{4}$$

Integrating (3) and (4) with respect to $x$ and $y$ respectively, we get

$$\log X = kx + \log A \text{ and } \log Y = \left(\frac{k-1}{2}\right) y + \log B$$

i.e. $$X = Ae^{kx} \text{ and} Y = Be^{\left(\frac{k-1}{2}\right)y}$$

$\therefore$ Required solution is

$$z = ce^{kx} \cdot e^{\left(\frac{k-1}{2}\right)y} \tag{5}$$

Given that $z(x, 0) = 6e^{-3x}$

$\therefore$ $ce^{kx} = 6e^{-3x}$

$\therefore$ $c = 6$ and $k = -3$

Using these values in (5), the required solution is $z = 6e^{-(3x+2y)}$.

## *Example 24*

Solve the equation $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$, by the method of separation of variables.

Let $$z = X(x).Y(y) \tag{1}$$

be a solution of the equation

$$z_{xx} - 2z_x + z_y = 0 \tag{2}$$

Then $z_x = X'Y$, $z_{xx} = X''Y$ and $z_y = XY'$ satisfy (2).

i.e. $$X''Y - 2X'Y + XY' = 0$$
Dividing throughout by $XY$, we get

$$\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0$$

i.e. $$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = k$$

i.e. $$X'' - 2X' - kX = 0 \tag{3}$$

and $$Y' + kY = 0 \tag{4}$$

i.e. $$(D^2 - 2D - k)X = 0 \tag{5}$$

where $D \equiv \dfrac{\mathrm{d}}{\mathrm{d}x}$ and

$$\frac{Y'}{Y} = -k \tag{6}$$

A.E. of (5) is $m^2 - 2m - k = 0$

$$\therefore \qquad m = \frac{2 \pm \sqrt{4+4k}}{2} \text{ or } 1 \pm \sqrt{k+1}$$

$\therefore$ Solution of (5) is

$$X = Ae^{(1+\sqrt{k+1})x} + Be^{(1-\sqrt{k+1})x}$$

Solution of (6) is

$$Y = ce^{-ky}$$

Using these values in (1), the required solution is

$$z = \{Ae^{(1+\sqrt{k+1})x} + Be^{(1-\sqrt{k+1})x}\}ce^{-ky}$$

or $$z = \{c_1e^{(1+\sqrt{k+1})x} + c_2e^{(1-\sqrt{k+1})x}\}e^{-ky}$$

### *Example 25*

Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 5u$, by the method of separation of variables, given that $u = 0$ and $\dfrac{\partial u}{\partial x} = e^{-3y}$ when $x = 0$ and for all values of $y$.

Let $$u(x, y) = X(x).Y(y) \tag{1}$$

be a solution of

$$u_{xx} = u_y + 5u \tag{2}$$

Then $u_{xx} = X''Y$ and $u_y = XY'$ satisfy (2)

i.e. $$X''Y = XY' + 5XY$$

Dividing throughout by $XY$, we get

$$\frac{X''}{X} = \frac{Y'}{Y} + 5 = k$$

$$X'' - kX = 0 \tag{3}$$

and
$$\frac{Y'}{Y} = k - 5 \tag{4}$$

Assuming that $k$ is positive, the solutions of (3) and (4) are

$$X = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

and
$$Y = ce^{(k-5)y}$$

Using these values in (1), the required solution is

$$u(x, y) = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x})e^{(k-5)y} \tag{5}$$

Given: $u = 0$ when $x = 0$ and for all $y$

$\therefore$
$$(C_1 + C_2)e^{(k-5)y} = 0$$

i.e.
$$C_1 + C_2 = 0 \tag{6}$$

Differentiating (5) partially with respect to $x$, we have

$$\frac{\partial u}{\partial x} = \sqrt{k}(C_1 e^{\sqrt{k}x} - C_2 e^{-\sqrt{k}x})e^{(k-5)y} \tag{7}$$

Given: $\dfrac{\partial u}{\partial x} = e^{-3y}$, when $x = 0$ and for all $y$.

$\therefore$
$$\sqrt{k}(C_1 - C_2)e^{(k-5)y} = e^{-3y}$$

$\therefore$
$$\sqrt{k}(C_1 - C_2) = 1 \tag{8}$$

and
$$k - 5 = -3 \tag{9}$$

Solving (6), (8) and (9), we get

$$k = 2, C_1 = \frac{1}{2\sqrt{2}} \quad \text{and} \quad C_2 = -\frac{1}{2\sqrt{2}}$$

Using these values in (5), the required solution is

$$u(x, y) = \frac{1}{\sqrt{2}} \sinh x\sqrt{2} \cdot e^{-3y}.$$

## Exercise 1(d)

### Part A (Short-Answer Questions)

Solve the following equations.

1. $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = 0$
2. $(D - D')^3 z = 0$

3. $(D^2 + D'^2)^2 z = 0$
4. $(D^3 + 4D^2D' - 5DD'^2)z = 0$
5. $(2D^2D' - 5DD'^2 - 3D'^3)z = 0$
6. $(D + D' - 1)(D - D' + 1)z = 0$
7. $D(D - 2D' + 3)z = 0$
8. $D'(D + 3D' - 2)z = 0$
9. $(D + D')(D - D' - 1)z = 0$
10. $(D - D')(D + D' + 1)z = 0$

Find the particular integrals of the following equations.

11. $(D^2 + 2DD' + D'^2)z = e^{x-y}$
12. $(D^2 - DD' - 2D'^2)z = \sin(3x + 4y)$
13. $(D^2 - 4D'^2)z = \sin(2x + y)$
14. $\{(D - 1)^2 - D'^2\}z = e^{x+y}$
15. $(D^2 - D'^2 + D)z = \cos(x + y)$

Solve the following partial differential equations by the method of separation of variables.

16. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, given that $u(x, 0) = 4e^{-x}$
17. $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$
18. $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given that $z(x, 0) = 4e^{-3x}$
19. $x^2\frac{\partial z}{\partial y} + y^3\frac{\partial z}{\partial x} = 0$
20. $\frac{\partial u}{\partial y} = 2\frac{\partial^2 u}{\partial x^2}$

**Part B**

Solve the following partial differential equations.

21. $(D^2 + 3DD' - 4D'^2)z = (e^{2x} - e^{-y})^3$
22. $(D^3 - 7DD'^2 - 6D'^3)z = \sinh(2x - 3y)$
23. $(D^2 - 7DD' + 12D'^2)z = (e^{3x} + e^{4x})e^y$
24. $(D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$
25. $(D^3 + 2D^2D')z = e^{2x} + 3x^2y$
26. $(D^2 - 3DD' + 2D'^2)z = e^{2x+3y} + \sin(x - 2y)$

27. $(D^2 - 6DD' + 9D'^2)z = x^2y^2 + \cos(3x + y)$
28. $(D^2 - DD')z = \cos x \cos 2y$
29. $(8D^3 - 4D^2D' - 18DD'^2 + 9D'^3)z = \sin(3x + 2y)$
30. $(D^2 - 3DD' + 2D'^2)z = (2 + 4x)e^{x+2y}$
31. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$
32. $(D^2 + DD' - 6D'^2)z = y \cos x$
33. $(D^2 + D'^2)z = \dfrac{8}{x^2 + y^2}$
34. $D(D^2 + 4DD' + 3D'^2 - 3D - 5D' + 2)z = e^x + e^y$
35. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = \cosh(2x + y)$
36. $(D^2 - DD' + D)z = x^2 + y^2$
37. $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$
38. $(D^2 - D'^2 - 2D + 1)z = xy + e^{2x+3y}$
39. $(D^2 + DD' + D' - 1)z = \sinh(3x - 2y)$
40. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \cos 2x \cos y$
41. $(2DD' + D'^2 - 3D')z = 4\sin^3(x + 2y)$
42. $(D^2 - D'^2 + D + 3D' - 2)z = xe^x + ye^y$
43. Solve equation $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, by the method of separation of variables, given that $u(0, y) = 3e^{-y} - e^{-5y}$ [**Hint**: Assume the R.H.S. of the solution as the sum of two terms of the form $Ce^{\frac{kx}{4}+(3-k)y}$ with different values for $c$ and $k$]
44. Solve equation $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$, by the method of separation of variables, given that $z = 0$ and $\dfrac{\partial z}{\partial x} = 4e^{-3y} + 6e^{-8y}$ when $x = 0$.
45. Solve the equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y} + 2z$, by the method of separation of variables, given that $z = 0$ and $\dfrac{\partial z}{\partial x} = 1 + e^{-3y}$ when $x = 0$.

# Answers

**Exercise 1(a)**

2. $pq = z$

3. $pq = 4xyz$

4. $z = px + qy + pq$

5. $z = px + qy + p^2 + q^2$

6. $px + qy = 3z$

7. $p = q$

8. $px + qy = z - \dfrac{1}{z}$

9. $p^2 + q^2 = 1$

10. $py = qx$

11. $py^2 + qx^2 = 0$

12. $ap + bq = 0$

13. $px = qy$

14. $px + qy = 0$

15. $s = 0$

16. $s = a$

17. $r = 0$

18. $t = 0$

19. $r = \sin x$

20. $t = \cos y$

21. $px + qy = pq$

22. $pq = p + q$

23. $p^2 + q^2 = z$

24. $pz = 1 + q^2$

25. $yp - x^2q^2 = x^2y$

26. $p = q$

27. $z = px + qy$

28. $py = qx$

29. $z^2(p^2 + q^2 + 1) = c^2$

30. $(p^2 + q^2 + 1)y^2 = c^2q^2$

31. (a) $px = qy$; (b) $py = qx$

32. (a) $x(y - z)p + y(z - x)q = z(x - y)$;
(b) $x(y - z)p + y(z + 2x^2)q = z(y + 2x^2)$

33. (a) $px^2 + qy^2 = z^2$
(b) $y^2zp + x^2zq = xy^2$

34. (a) $(y^2 + z^2)p - xyq + xz = 0$;
(b) $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$

35. (a) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$;
(b) $yp + xq = z$

36. $r + t = 0$

37. $2r + 3s - 9t = 0$

38. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{(\partial x)^2 \partial y} - \dfrac{\partial^3 z}{\partial x(\partial y)^2} + 2\dfrac{\partial^3 z}{\partial y^3} = 0.$

39. $9r - 12s + 4t = 0.$

40. $r - 2s + t = 0.$

41. $(x - iy)(r - t) = 2(p - iq).$

42. $4xr - t + 2p = 0.$

43. $zs = pq.$

44. $xys = px + qy - z.$

45. $(1 + q)r + (q - p)s - (1 + p)t = 0.$

**Exercise 1(b)**

7. $z = ax + \dfrac{k}{a}y + b.$

8. $z = ax + y\log a + b.$

9. $z = ax \pm \sqrt{2 - a^2}\,y + b.$

10. $(1 + a)\log z = x + ay + b.$

11. $\log z = a(x + ay) + b.$

12. $4az = (x + ay + b)^2.$

13. $z = a\dfrac{x^2}{2} + \dfrac{y^2}{2a} + b.$

14. $z = a\log(xy) + b.$

15. $z = a(e^x + e^y) + b.$

16. $z = ax + by + \dfrac{a^3}{b} + \dfrac{b^3}{a}.$

17. (a) C.S. is $z = ax + (1 - \sqrt{a})^2 y$; No singular solution (S. S.).
(b) C.S. is $z = ax \pm \sqrt{k^2 - a^2}\,y + b$; No S.S.

18. C.S. is $z = ax + \dfrac{1}{2}(-2 \pm \sqrt{10})ay + b$; No S.S.

19. C.S. is $z = ax + \left(\dfrac{5 - a^2}{3 - 2a}\right) y + b$; No S.S.

20. $\log z = a \log x \pm \sqrt{1 - a^2} \log y + b$.

21. $\sqrt{z} = a\sqrt{x} \pm \sqrt{1 - a^2}\sqrt{y} + b$.

22. $\dfrac{1}{z} = \dfrac{a}{x} + \dfrac{(1 - a)}{y} + b$.

23. $z^2 = ax \pm \sqrt{a^2 - 4} \cdot y + b$.

24. $\log z = \dfrac{a}{x} + (2a^2 - 3) \log y + b$.

25. $z = a^2(x + y) + axy + b$.

26. $xy = 1$.

27. $729z^4 = 1024xy$.

28. $16z^3 + 27x^2y^2 = 0$.

29. $z^4 = 16xy$.

30. $4z = y^2 - x^2$.

31. $x^2 + y^2 = 1$.

32. $z = 3$.

33. $4(1 + a^2)z = (x + ay + b)^2$.

34. $\sqrt{1 + a^2} \log(z + \sqrt{z^2 - 1}) = x + ay + b$.

35. (a) $z^2 \pm z\sqrt{z^2 - 4a^2} - 4a^2 \log(z + \sqrt{z^2 - 4a^2}) = 4(x + ay + b)$

(b) $az^2 \mp z\sqrt{a^2z^2 - 4} \pm \frac{4}{a} \log(az + \sqrt{a^2z^2 - 4}) = 4(x + ay + b)$

36. $4(bz - ab - 1) = (x + by + c)^2$.

37. $(z + a^2)^3 = (x + ay + b)^2$.

38. $3(1 + a) \log z = x^3 + ay^3 + b$.

39. $\sqrt{a^2 + 1}z^2 = 2\ (\log x + ay + b)$.

40. $2 \log z = (a \pm \sqrt{a^2 + 8}) \left(\frac{1}{x} + \frac{a}{y} + b\right)$.

41. $4z = -x^2 \pm \left\{x\sqrt{x^2 + 4a^2} + 4a^2 \log\ (x + \sqrt{x^2 + 4a^2}) + 4(a^2y + b)\right\}$.

42. $2z = ax^2 - \frac{a}{a+1}y^2 + b$.

43. $3z = ax^3 + 2\sqrt{a - 1}y^{3/2} + b$.

44. $z = ax - \cos x + \frac{1}{a} \sin y + b$.

45. $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$.

46. $z^2 = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} + y\sqrt{y^2 - a^2} - a^2 \cosh^{-1} \frac{y}{a} + b$.

47. $\log z = \frac{\sqrt{a}x^2}{2} + \sqrt{1 - a} \log y + b$.

48. $z^2 = x^2 + ax + \frac{2}{3}(y + a)^{3/2} + b$.

49. $z = \sqrt{a(x + y)} + \sqrt{(1 - a)(x - y)} + b$.

50. $z = \frac{a}{2} \log\ (x^2 + y^2) + \sqrt{1 - a^2} \tan^{-1} \left(\frac{y}{x}\right) + b$.

## Exercise 1(c)

1. $z = xf(y) + \phi(y)$.

2. $z = yf(x) + \phi(x)$.

3. $z = f(x) + \phi(y)$.

4. $z = xf(y) + \phi(y) + e^{x+y}$.

5. $z = yf(x) + \phi(x) - \frac{1}{9}\cos(2x + 3y)$.

6. $z = f(x) + \phi(y) + \log x \cdot \log y$.

7. $z = xf(y) + \phi(y) + \frac{x^2}{2}\sin y$.

8. $z = yf(x) + \phi(x) - \cos y$.

9. $z = f(x) + \phi(y) + kxy$.

10. $z = f(x) + \phi(y) + \frac{xy}{3}\left(x^2 + y^2\right)$

12. $f(x^2 - y^2, y^2 - z^2) = 0$

13. $f\left(xy, \frac{y}{z}\right) = 0$.

14. $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$.

15. $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$.

16. $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$.

17. $z = (1 + \cos x)\cos y$.

18. $z = c\cosh ax + \sinh ax \sin y$.

19. $z = e^y \cosh x + e^{-y}\sinh x$.

20. $z = 3x^2 + 3xy - 2y^2 + c$

21. $z = x^3 f(y) - x - y/3$.

22. $z = f(x) + \phi(y) - 3x^2y^3 - \frac{1}{2}\sin(2x - y)$

23. $z = e^{2y} f(x) + e^{3y} g(x) + 2y + 5/3$.

24. $z = Axy + Bx + Cy + D$.

25. $z = f(x + at) + \phi(x - at)$.

26. $z = f(x + iy) + \phi(x - iy)$.

27. $z = f(x + y) + \phi(x - y)$.

28. $z = (x^2 - y^2) f(x^2 + y^2) + \phi(x^2 + y^2)$.

29. $z = f(y^2 + x) + \phi(y^2 - x)$.

30. (i) $f\left(\dfrac{\sec x}{\sec y}, \dfrac{\sec y}{\sec z}\right) = 0$;
(ii) $f\left(\dfrac{a - x}{b - y}, \dfrac{b - y}{c - z}\right) = 0$.

31. $f(x^3 - y^3, x^2 - z^2) = 0$.

32. (i) $f\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{x - y}{z}\right) = 0$;
(ii) $f\left(\dfrac{1}{x} + \dfrac{1}{y}, \dfrac{x + y}{z}\right) = 0$.

33. $f(y/z, x^2 + y^2 + z^2) = 0$.

34. $f\left(\dfrac{y}{z}, \dfrac{x^2 + y^2 + z^2}{z}\right) = 0$.

35. $f\left[x \log(x + y) - z, x + y\right] = 0$.

36. $f(xy, x^2 + y^2 + z^2) = 0$.

37. (i) $f(x + y + z, x^2 + y^2 + z^2) = 0$;
(ii) $f(x + y + z, x^2 + 2yz) = 0$.

38. (i) $f(x + y + z, xyz) = 0$;
(ii) $f(x + y + z, xyz) = 0$.

39. $f(xyz, x^2 + y^2 + z^2) = 0.$

40. $f\left(\dfrac{x-y}{y-z}, xy + yz + zx\right) = 0.$

41. (i) $f(x + y + z, x^2 + y^2 - z^2) = 0;$
(ii) $f(2x + 3y + 4z, x^2 + y^2 + z^2) = 0.$

42. $f\left(x^3y^3z, \dfrac{x}{y^2} + \dfrac{y}{x^2}\right) = 0.$

43. $xy + yz + zx = 0.$

44. $3(2x + 2y - 3z + 3)^2 = (y - x)(x + y)^3.$

**Exercise 1(d)**

1. $z = f_1(y - 2x) + f_2(y + 2x) + f_3(y + 3x).$

2. $z = f_1(y + x) + xf_2(y + x) + x^2 f_3(y + x).$

3. $z = f_1(y + ix) + xf_2(y + ix) + f_3(y - ix) + xf_4(y - ix)$

4. $z = f_1(y) + f_2(y - 5x) + f_3(y + x).$

5. $z = f_1(x) + f_2\left(y - \dfrac{x}{2}\right) + f_3(y + 3x).$

6. $z = e^x f_1(y - x) + e^{-x} f_2(y + x).$

7. $z = f_1(y) + e^{-3x} f_2(y + 2x).$

8. $z = f_1(x) + e^{2x} f_2(y - 3x).$

9. $z = f_1(y - x) + e^x f_2(y + x).$

10. $z = f_1(y + x) + e^{-x} f_2(y - x).$

11. $\dfrac{x^2}{2} e^{x-y}.$

12. $\dfrac{1}{35} \sin(3x + 4y).$

13. $-\dfrac{x}{4}\cos(2x+y)$.

14. $-e^{x+y}$.

15. $\sin(x+y)$.

16. $u = 4e^{-x+\frac{3}{2}y}$.

17. $u = 8e^{-12x-3y}$.

18. $z = 4e^{-3x+t}$

19. $z = ce^{k(3y^4-4x^3)}$.

20. $u = e^{2ky}\left(Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}\right)$.

21. $z = f_1(y+x) + f_2(y-4x) + \dfrac{1}{36}e^{6x} - \dfrac{3}{5}xe^{4x-y} - \dfrac{1}{8}e^{2x-2y} - \dfrac{1}{36}e^{-3y}$.

22. $z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) + \dfrac{1}{44}\cosh(2x-3y)$.

23. $z = f_1(y+3x) + f_2(y+4x) + x\left(e^{4x+y} - e^{3x+y}\right)$.

24. $z = f_1(y-x) + xf_2(y-x) + \dfrac{1}{4}\cdot\left(x^4 - 2x^3y + 2x^2y^2\right)$.

25. $z = f_1(y) + xf_2(y) + f_3(y+2x) + \dfrac{1}{4}xe^{2x} + \dfrac{x^5}{60}\left(y + \dfrac{x}{3}\right)$

26. $z = f_1(y+x) + f_2(y+2x) + \dfrac{1}{4}e^{2x+3y} - \dfrac{1}{15}\sin(x-2y)$.

27. $z = f_1(y+3x) + xf_2(y+3x) + \dfrac{x^4}{60}(9x^2 + 12xy + 5y^2) + \dfrac{x^2}{2}\cos(3x+y)$.

28. $z = f_1(y) + f_2(y+x) + \dfrac{1}{2}\cos(x+2y) - \dfrac{1}{6}\cos(x-2y)$.

29. $z = f_1(2y+x) + f_2(2y+3x) + f_3(2y-3x) - \dfrac{x}{96}\sin(3x+2y)$

30. $z = f_1(y+x) + f_2(y+2x) + \dfrac{2}{9}e^{x+2y}(11+6x)$.

31. $z = f_1(y-x) + xf_2(y-x) + f_3(y+x) + \frac{e^x}{25}(2\sin 2y + \cos 2y)$.

32. $z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x$.

33. $z = f_1(y+ix) + f_2(y-ix) + \frac{1}{2}\left[\log(x^2+y^2)\right]^2 + 2\left(\tan^{-1}\frac{y}{x}\right)^2$.

34. $z = f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x) - xe^x + xye^y$.

35. $z = e^x f_1(y+x) + e^{2x} f_2(y+x) - \frac{x}{2}e^{2x+y} + \frac{1}{12}e^{-2x-y}$.

36. $z = f_1(y) + e^{-x} f_2(y+x) + \frac{x^3}{3} + xy^2 - x^2 + 2xy + 4x$.

37. $z = e^x f_1(y-x) + e^{3x} f_2(y-2x) + (x+2y+6)$

38. $z = e^x f_1(y+x) + e^x f_2(y-x) + (x+2)y - \frac{1}{8}e^{2x+3y}$.

39. $z = e^{-x} f_1(y) + e^x f_2(y-x) + \frac{1}{8}xe^{3x-2y} - \frac{1}{8}e^{2y-3x}$.

40. $z = f_1(y-x) + e^{-2x} f_2(y+2x) + \frac{1}{12}\sin(2x+y) + \frac{1}{4}\sin(2x-y) - \frac{1}{2}\cos(2x-y)$.

41. $z = f_1(x) + e^{3x/2} \cdot f_2\left(y - \frac{x}{2}\right) + \frac{3}{50}\{3\cos(x+2y) - 4\sin(x+2y)\}$
$+ \frac{1}{306}\{4\sin(3x+6y) - \cos(3x+6y)\}$.

42. $z = e^x f_1(y-x) + e^{-2x} f_2(y+x) + \frac{e^x}{54}(9x^2 - 6x + 2) +$
$e^y\left(xy - \frac{x^2}{2} - y - 3\right)$.

43. $u = 3e^{x-y} - e^{2x-5y}$.

44. $z = \left(e^{3x} - e^{-x}\right)e^{-3y} + \left(e^{4x} - e^{-2x}\right)e^{-8y}$.

45. $z = \frac{1}{\sqrt{2}}\sinh x\sqrt{2} + e^{-3y}\sin x$.