Learning Outcomes - Unit 3

When you have completed this unit you will be able to

- ***Find the smallest positive root of the polynomial and transcendental equations by Newton Raphson method.***
- ***Solving Simultaneous Linear Algebraic equations by direct and indirect methods.***
- ***Understand the meaning on interpolation.***
- ***Use the Gregory-Newton interpolation formula with forward and backward differences for equally spaced domain points.***

- ***Use Lagrange's formula when the domain points are not equally spaced.***
- ***Evaluate a derivative at a value using an appropriate numerical method.***
- ***Calculate a definite integral using an appropriate numerical method.***
- ***Workout numerical differentiation and integration whenever and wherever routine methods are not applicable.***
- ***Applications based on Numerical Differentiation & Integration.***

➢***Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as iteration process and is repeated till the result is obtained to a desired degree of accuracy.***

POLYNOMIAL EQUATION

Let $f_n(x)$ be a polynomial in x. Then $f_n(x) = 0$ is known as polynomial equation.

TRANSCENDENTAL EQUATION

If a function f(x) contains some other functions such as trigonometric, logarithmic, exponential etc., then $f(x) = 0$ is called transcendental equation.

To solve the polynomial equations and transcendental equations, we can use the following intermediate value theorem to locate the initial approximation.

If $f(x)$ is continuous in the interval (a, b) and if $f(a)$ and $f(b)$ are of opposite signs, then the equation $f(x) = 0$ has at least one root between $x = a$ and $x = b$.

i.e., $a < Root < b$

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NEWTON RAPHSON METHOD OR METHOD OF TANGENTS (for algebraic and transcendental equations)

- ***Newton's method is the best known procedure for finding the roots of an equation.***
- ***It is applicable to the solution of all types of equations, i.e., algebraic and transcendental and also useful for finding complex roots.***

GAUSS JORDAN METHOD (for Simultaneous Linear Algebraic Equations)

The method is based on the idea of reducing the given system of equations $\mathbf{Ax} = \mathbf{b}$, to a diagonal system of equations $\mathbf{Ix} = \mathbf{d}$, where $\mathbf{I}$ is the identity matrix, using elementary row operations. We know that the solutions of both the systems are identical. This reduced system gives the solution vector $\mathbf{x}$. This reduction is equivalent to finding the solution as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

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GAUSS SEIDEL METHOD (for Simultaneous Linear Algebraic Equations)

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations,

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z &= d_1 \\
a_2 x + b_2 y + c_2 z &= d_2 \qquad \ldots(1) \\
a_3 x + b_3 y + c_3 z &= d_3
\end{aligned}
$$

Let us assume

$$
\begin{aligned}
|a_1| &> |b_1| + |c_1| \\
|b_2| &> |a_2| + |c_2| \\
|c_3| &> |a_3| + |b_3|
\end{aligned}
$$

Then, iterative method can be used for the system (1). Solve for x, y, z (whose coefficients are the larger values) in terms of the other variables. That is,

$$
\begin{aligned}
x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\
y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \qquad \ldots(2) \\
z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y)
\end{aligned}
$$

We start with the initial values $y^{(0)}, z^{(0)}$ for y and z and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)}$ and $y^{(1)}$, use $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

Note 1. For all systems of equations, this method will not work (since convergence is not assured). It converges only for special systems of equations.

2. Iteration method is self-correcting method. That is, any error made in computation, is corrected in the subsequent iterations.

3. The iteration is stopped when the values of x, y, z start repeating with the required degree of accuracy.

Finite Differences:

Let $y = f(x)$ be the given function of x .

Let $y_0, y_1, y_2, \ldots, y_n$ be the values of y corresponding to the values $x_0, x_1, x_2, \ldots, x_n$.

- The values of y are called ***entries*** and the values of x are called ***arguments***.
- Usually the arguments $x_0, x_1, x_2, \ldots, x_n$ are in general not equally spaced.
- If we subtract each value of y from the proceeding value (except y_0), we get $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, etc., The results obtained are known as first differences of y and it is denoted by Δ.

Here Δ denotes an operation called forward difference operator.

$$\Delta y_0 = y_1 - y_0 ;$$

$$\Delta y_1 = y_2 - y_1 ;$$

$$\Delta y_2 = y_3 - y_2 ;$$

$$\ldots,$$

$$\Delta y_{n-1} = y_n - y_{n-1}.$$

For the purpose of our practical work, let us assume that the arguments are equally spaced. The arguments $x_0, x_1, x_2, \ldots, x_n$ can be taken as $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$.

Here 'h' is called interval of differencing.

Forward difference operator :

Forward difference operator Δ is defined as

$\Delta f(x) = f(x + h) - f(x)$.

Hence $\Delta f(x_0) = f(x_0 + h) - f(x_0)$ i.e., $\Delta y_0 = y_1 - y_0$

$\Delta f(x_1) = f(x_1 + h) - f(x_1)$,i.e., $\Delta y_1 = y_2 - y_1$

Similarly $\Delta y_2 = y_3 - y_2$

$$\Delta y_3 = y_4 - y_3$$

$$\Delta y_4 = y_5 - y_4$$

and so on.

In general, $\Delta y_{n-1} = y_n - y_{n-1}$

Second forward difference

Second forward difference is defined as

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x + h) - f(x))$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1$$

$$\Delta^2 y_2 = \Delta(\Delta y_2) = \Delta(y_3 - y_2) = \Delta y_3 - \Delta y_2$$

and so on.

In general,

$$\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta(y_n - y_{n-1}) = \Delta y_n - \Delta y_{n-1}$$

Third Forward difference

$$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(\Delta y_1 - \Delta y_0) = \Delta^2 y_1 - \Delta^2 y_0$$

$$\Delta^3 y_1 = \Delta(\Delta^2 y_1) = \Delta(\Delta y_2 - \Delta y_1) = \Delta^2 y_2 - \Delta^2 y_1$$

$$\Delta^3 y_2 = \Delta(\Delta^2 y_2) = \Delta(\Delta y_3 - \Delta y_2) = \Delta^2 y_3 - \Delta^2 y_2$$

and so on.

In general,

$$\Delta^3 y_{n-1} = \Delta(\Delta^2 y_{n-1}) = \Delta(\Delta y_n - \Delta y_{n-1}) = \Delta^2 y_n - \Delta^2 y_{n-1}$$

Backward difference operator ∇

Backward difference operator ∇ is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x_1) = f(x_1) - f(x_1 - h), i.e., \nabla y_1 = y_1 - y_0$$

$$\nabla f(x_2) = f(x_2) - f(x_2 - h), i.e., \nabla y_2 = y_2 - y_1$$

$$\nabla f(x_3) = f(x_3) - f(x_3 - h), i.e., \nabla y_3 = y_3 - y_2$$

Similarly $\nabla y_4 = y_4 - y_3$

In general

$$\nabla f(x_n) = f(x_n) - f(x_n - h), i.e., \nabla y_n = y_n - y_{n-1}$$

Interpolation

- The process of computing the intermediate values of y, from the given set of tabular values of the function is known as Interpolation.

- Let $y = f(x)$ be the given function of x. Here x is known as argument and y is known as entry.

- In general, the arguments are equally spaced. Assume that the arguments $x_0, x_1, \ldots, x_n$ can be taken as $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$.

- Here 'h' is known as interval of differencing.

- y_0 means $f(x_0)$; y_1 means $f(x_1)$ ……………, and y_n is $f(x_n)$.

Given the tabular values of the function y = f(x)

x	x_0	x_1	x_2	x_3	..	..	..	..	x_n
y	y_0	y_1	y_2	y_3	..	..	..	..	y_n

- If we require to compute y_i i.e., $y(x = x_i)$ from the given set of tabular values where $x_0 < x_i < x_n$ is known as interpolation.
- Similarly, if we compute the values of y, outside the given interval is known as extrapolation.
- But in general, the word interpolation is used for both the cases.

NEWTON RAPHSON METHOD

Let x_0 be an approximate root of the equation $f(x) = 0$.

If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

$\therefore$ expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ a closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula*.

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In general, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, 3, \ldots\ldots$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula*.

GEOMETRICAL INTERPRETATION OF NEWTON RAPHSON METHOD

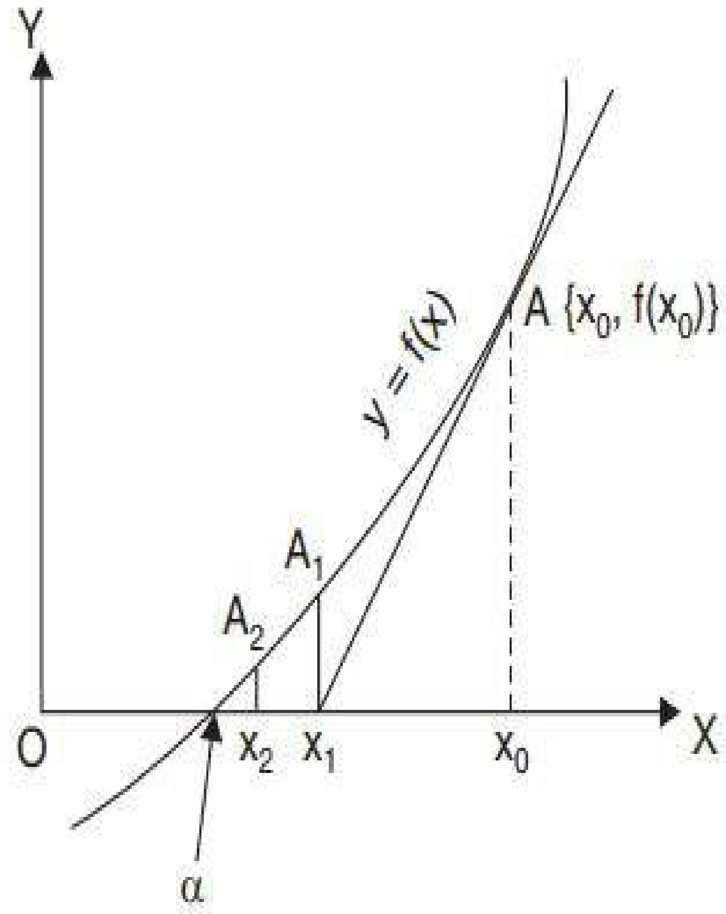
Let x_0 be a point near the root α of equation $f(x) = 0$, then tangent at A $\{x_0, f(x_0)\}$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

It cuts x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

which is I approximation to root α. If A_1 corresponds to x_1 on the curve, then tangent at A_1 will cut x-axis at x_2, nearer to α and is therefore II approximation to root α.

Repeating this process, we approach the root α quite rapidly. Hence, the method consists in replacing the part of the curve between A and x-axis by the means of the tangent to the curve at A_0.



Note:

The rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root is called the **rate of convergence.**

Working Rule to solve the equation by Newton Raphson method

Step 1: Find an interval (a, b) in which the root of f(x) = 0 lies, using intermediate value theorem, where a < b.

Step 2: Out of f(a) and f(b), choose which is nearer to zero. If f(a) is nearer to zero, then a is an initial approximation x_0 of the given equation.

Step 3: Apply Newton Raphson iterative formula to find out better approximate root.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where n = 0, 1, 2, ...,

Step 4: Repeat the process to get successive approximation.

Remarks:

1. This method if useful in cases of large values of the derivative of f(x), i.e., when the graph of f(x) while crossing the x-axis is nearly vertical.
2. If the derivative of f(x) is zero, this method fails.
3. This method is also used to obtain complex roots.
4. Order of convergence of this method is ***two***.
5. Condition for convergence of Newton's method is

$$|f(x)f''(x)| < |f'(x)|^2$$

6. The error at any stage is proportional to the square of the error in the previous stage.

Problems

1. Find the smallest positive root of $3x - \cos x - 1 = 0$ by Newton Raphson method.

Solution: $f(x) = 3x - \cos x - 1 = 0$.

$f'(x) = 3 + \sin x = 0$.

$f(0) = -2 = -ve$

$f(1) = 1.459697694 = +ve$.

Since $f(0)$ and $f(1)$ are of opposite signs, there is atleast real root must lie between 0 and 1.

Let $x_0 = 0.5$

Newton-Raphson iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

$$= x_n - \frac{[3x_n - \cos x_n - 1]}{[3 + \sin x_n]}$$

$$x_1 = x_0 - \frac{(3x_0 - \cos x_0 - 1)}{(3 + \sin x_0)} = 0.608518649$$

$$x_2 = x_1 - \frac{3x_1 - \cos x_1 - 1}{3 + \sin x_1} = 0.60710187 8$$

$$x_3 = x_2 - \frac{3x_2 - \cos x_2 - 1}{3 + \sin x_2} = 0.607101648$$

$$x_4 = x_3 - \frac{3x_3 - \cos x_3 - 1}{3 + \sin x_3} = 0.607101648$$

Root of $3x - \cos x - 1 = 0$ is 0.607101648.

2. Find the negative root of $x^3 - 4x + 9 = 0$ by Newton Raphson method.

Solution: Let $f(x) = x^3 - 4x + 9 = 0$.

Then $f(-x) = (-x)^3 - 4(-x) + 9 = 0$.

$= -x^3 + 4x + 9 = 0$.

Take $\phi(x) = -x^3 + 4x + 9 = 0$.

$\phi(0) = 9 = +ve$

$\phi(1) = 12 = +ve$

$\phi(2) = 9 = +ve$

$\phi(3) = -6 = -ve.$

$\therefore$ Root of $\phi(x) = 0$ lies between 2 & 3.

Let $x_0 = 2.5$

Newton - Raphson iterative formula is

$$x_{n+1} = x_n - \frac{\phi(x_n)}{\phi'(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

$$= x_n - \frac{(-x_n^3 + 4x_n + 9)}{(-3x_n^2 + 4)}$$

$$x_1 = x_0 - \frac{(-x_0^3 + 4x_0 + 9)}{(-3x_0^2 + 4)} = 2.728813559$$

$$x_2 = x_1 - \frac{(-x_1^3 + 4x_1 + 9)}{(-3x_1^2 + 4)} = 2.706749049$$

$$x_3 = x_2 - \frac{(-x_2^3 + 4x_2 + 9)}{(-3x_2^2 + 4)} = 2.706527977$$

$$x_4 = x_3 - \frac{(-x_3^2 + 4x_3 + 9)}{(-3x_3^2 + 4)} = 2.706527955$$

$$x_5 = x_4 - \frac{(-x_4^2 + 4x_4 + 9)}{(-3x_4^2 + 4)} = 2.706527955$$

Positive root of $\phi(x) = 0$ is 2.706527955

Negative root of $f(x) = 0$ is -2.706527955

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3. Find an iterative formula to find the reciprocal of a number N by Newton Raphson method and hence obtain $\frac{1}{19}$.

Solution: Let $x = \frac{1}{N}$. Then $N = \frac{1}{x} \Rightarrow N - \frac{1}{x} = 0$.

Assume $f(x) = N - \frac{1}{x} = 0$; $f'(x) = \frac{1}{x^2}$

Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

$f(x) = x - \frac{1}{N}$

$f'(x) = 1$

$$x_{n+1} = \cancel{x_n} - \frac{\left(\cancel{x_n} - \frac{1}{N}\right)}{1}$$

$= \frac{1}{N}$

i.e., $x_{n+1} = x_n - \dfrac{\left(N - \frac{1}{x_n}\right)}{\left(\frac{1}{x_n^2}\right)}$

$= x_n - x_n^2\left(N - \frac{1}{x_n}\right)$

$= x_n - Nx_n^2 + x_n$

$= 2x_n - Nx_n^2$

$$x_{n+1} = x_n(2 - Nx_n), \quad n = 0, 1, 2, \ldots$$

To find $\frac{1}{19}$: Here $N = 19$.

$$x_{n+1} = x_n(2 - 19x_n), \quad n = 0, 1, 2, \ldots$$

Take $x_0 = 0.05$ $\because \frac{1}{20} = 0.05$

$$x_1 = x_0(2 - 19x_0) = 0.0525$$

$$x_2 = x_1(2 - 19x_1) = 0.052631\,25$$

$$x_3 = x_2(2 - 19x_2) = 0.052631578$$

$$x_4 = x_3(2 - 19x_3) = 0.052631578$$

$$\therefore \frac{1}{19} \simeq 0.052631578$$

Practice Problems

1. Using Newton Raphson method, solve the following:

(i) $2x^3 - 3x - 6 = 0$. (1.783769)

(ii) $4x - e^x = 0$ (find the root between 2 and 3). (2.1533)

(iii) $x \log_{10} x - 1.2 = 0$. (2.7405)

(iv) $x^x = 100$ (find the root between 3 and 4) (3.597)

(v) $x \tan x = 1.28$ (0.988)

(vi) $xe^x = \cos x$ (0.5177)

2. Find an iterative formula to find $\sqrt{N}$ $(N>0)$ and hence find $\sqrt{5}$.

3. Find the iterative formula to find cube root of a positive number N and hence find $(24)^{1/3}$.

4. Find the iterative formula to find p^{th} root of a positive number N.

2. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right), \quad n = 0, 1, 2, \ldots$

3. $x_{n+1} = \frac{1}{2}\left(2x_n + \frac{N}{x_n^2}\right), \quad n = 0, 1, 2, \ldots$

4. $x_{n+1} = \frac{(p-1)x_n^p + N}{p\, x_n^{p-1}}, \quad n = 0, 1, 2, \ldots$

Gauss Jordan Method (Direct Method).

(for Simultaneous Linear Algebraic Equations)

Consider

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \text{---} (1)$$

$n \rightarrow$ equations

$n \rightarrow$ variables $(x_1, x_2, \ldots, x_n)$

This system is equivalent to $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \rightarrow \text{coefficient matrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \rightarrow \text{unknown matrix.}$$

$$\& \; B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \rightarrow \text{constant matrix.}$$

The values of $x_1, x_2, \ldots, x_n$ which satisfy (1) is called solution of the system.

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Direct Methods: Methods which produce exact solution after certain number of steps are called direct methods. It involves certain amount of fixed computation.

Gauss Jordan Method

In Gauss Jordan method, we reduce the coefficient matrix as a diagonal matrix and the solution is obtained directly.

Problems

1. Solve $x+2y+z=3$

$2x+3y+3z=10$

$3x-y+2z=13$ by Gauss Jordan method.

$x = 2$
$y = -1$
$z = 3$

Solution: The given system is equivalent to $AX = B$ where $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}$; $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; $B = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$

Augmented matrix (A, B) is

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 7 & 1 & -4 \end{array}\right) \qquad \begin{array}{l} R_2 \to 2R_1 - R_2 \\ R_3 \to 3R_1 - R_3 \end{array}$$

$$R_2 \to (2 \quad 4 \quad 2 \quad 6) - (2 \quad 3 \quad 3 \quad 10)$$
$$= (0 \quad 1 \quad -1 \quad -4)$$

$$R_3 \to (3 \quad 6 \quad 3 \quad 9) - (3 \quad -1 \quad 2 \quad 13)$$
$$= (0 \quad 7 \quad 1 \quad -4)$$

$$= \left(\begin{array}{ccc|c} -1 & 0 & -3 & -11 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -8 & -24 \end{array}\right) \qquad \begin{array}{l} R_1 \to 2R_2 - R_1 \\ R_3 \to 7R_2 - R_3 \end{array}$$

$$(A, B) = \left(\begin{array}{ccc|c} -1 & 0 & -3 & -11 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad R_3 \to \frac{R_3}{(-8)}$$

$$= \left(\begin{array}{ccc|c} -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$$\left.\begin{array}{l} -x + 0 \cdot y + 0 \cdot z = -2 \\ 0 \cdot x + 1 \cdot y + 0 \cdot z = -1 \\ 0 \cdot x + 0 \cdot y + 1 \cdot z = 3 \end{array}\right\} \Rightarrow \begin{array}{l} x = 2 \\ y = -1 \\ z = 3 \end{array}$$

Practice Problems

Solve the following by Gauss Jordan Method.

1. $x+y+z+w=2$

   $2x-y+2z-w=-5$

   $3x+2y+3z+4w=7$

   $x-2y-3z+2w=5$

   $x=0$; $y=1$

   $z=-1$; $w=2$

2. $3.15x-1.96y+3.85z=12.95$

   $2.13x+5.12y-2.89z=-8.61$

   $5.92x+3.05y+2.15z=6.88$

   $x=1.7089$

   $y=-1.8005$

   $z=1.0488$

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Gauss Seidel Method (for simultaneous linear algebraic equations)

↳ Indirect or Iterative method.

Consider the system
$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let us assume
$$\left.\begin{aligned} |a_1| &> |b_1| + |c_1| \\ |b_2| &> |a_2| + |c_2| \\ |c_3| &> |a_3| + |b_3|. \end{aligned}\right\}$$

The given system can be solved by Gauss Seidel Method.

This condition is sufficient but not necessary.

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Note: 1. Iteration method is a self-correcting method. That is, any error made in computation is corrected in the subsequent iterations.

2. The iteration process is stopped when the values of unknowns start repeating with the required degree of accuracy.

Gauss seidel Method

Consider
$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Assume that $|a_1| > |b_1| + |c_1|$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

Initial values: $y^{(0)} = 0$; $z^{(0)} = 0$

Iteration 1:

$$x^{(1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(0)}\right)$$

$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right)$$

Iteration 2:

$$x^{(2)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(1)} - c_1 z^{(1)}\right)$$

$$y^{(2)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(2)} - c_2 z^{(1)}\right)$$

$$z^{(2)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(2)} - b_3 y^{(2)}\right)$$

and so on.

If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterative values of x, y, z, then the $(r+1)^{th}$ iterative values are given by

$$x^{(r+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(r)} - c_1 z^{(r)}\right)$$

$$y^{(r+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)}\right)$$

$$z^{(r+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)}\right)$$

The process of iteration is continued until the convergence is assured.

Problem

1. Solve the following equations by Gauss Seidel method.

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

Solution: Since the diagonal elements in the coefficient matrix are not dominant, we rearrange the equations as follows such that the elements in the coefficient matrix are dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Here $|28| > |4| + |-1|$

$|17| > |2| + |4|$

$|10| > |1| + |3|$.

$$x = \frac{1}{28}(32 - 4y + z)$$
$$y = \frac{1}{17}(35 - 2x - 4z)$$
$$z = \frac{1}{10}(24 - x - 3y)$$

$x = 0.993594143$

$y = 1.506977808$

$z = 1.848547243$

Gauss Seidel Method

Initial values are $y^{(0)} = 0$; $z^{(0)} = 0$.

Iteration 1: $x^{(1)} = \frac{1}{28}\left[32 - 4(0) + 0\right] = 1.142857$

$$y^{(1)} = \frac{1}{17}\left[35 - 2(1.142857) - 4(0)\right]$$

$$= 1.924370 .$$

$$z^{(1)} = \frac{1}{10}\left[24 - 1.142857 - 3(1.924370)\right]$$

$$= 1.708403 .$$

Iteration 2: $x^{(2)} = \frac{1}{28}\left[32 - 4(1.924370) + 1.708403\right]$

$= 0.928962$

$y^{(2)} = \frac{1}{17}\left[35 - 2(0.928962) - 4(1.708403)\right]$

$= 1.547557$

$z^{(2)} = \frac{1}{10}\left[24 - 0.928962 - 3(1.547557)\right]$

$= 1.842837$

Initial values: $y = 0 \,;\, z = 0$.

$x \to A$

$y \to B$

$z \to C$

$$x = \frac{1}{28}(32 - 4\overset{B}{y} + \overset{C}{z}) \qquad y = \frac{1}{17}(35 - 2\overset{A}{x} - 4\overset{C}{z}) \qquad z = \frac{1}{10}(24 - \overset{A}{x} - 3\overset{B}{y})$$

Iteration Number	x	y	z
1	1.142857	1.924370	1.708403
2	0.928962	1.547557	1.842837
3	0.987593	1.509027	1.848532
4	0.993301	1.507016	1.848565
5	0.993589	1.506974	1.848549
6	0.993595	1.506977	1.848547

$x = 0.9936 \,;\, y = 1.5070 \,;\, z = 1.8485$

Practice Problems

Solve the following by Gauss Seidel Method.

1)
$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

2)
$$8x + y + z = 8$$
$$2x + 4y + z = 4$$
$$x + 3y + 3z = 5$$

3)
$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = -3$$

Interpolation

- The process of computing the intermediate values of y, from the given set of tabular values of the function is known as Interpolation.

- Let $y = f(x)$ be the given function of x. Here x is known as argument and y is known as entry.

- In general, the arguments are equally spaced. Assume that the arguments $x_0, x_1, \ldots, x_n$ can be taken as $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$.

- Here 'h' is known as interval of differencing.

- y_0 means $f(x_0)$; y_1 means $f(x_1)$ ………….., and y_n is $f(x_n)$.

Given the tabular values of the function $y = f(x)$

x	x_0	x_1	x_2	x_3	..	..	..	..	x_n
y	y_0	y_1	y_2	y_3	..	..	..	..	y_n

- If we require to compute y_i i.e., $y(x = x_i)$ from the given set of tabular values where $x_0 < x_i < x_n$ is known as interpolation.
- Similarly, if we compute the values of y, outside the given interval is known as extrapolation.
- But in general, the word interpolation is used for both the cases.

We know that $\Delta f(x) = f(x + h) - f(x)$

Also $\Delta y_0 = y_1 - y_0$;

$\Delta y_1 = y_2 - y_1$;

$\Delta y_2 = y_3 - y_2$ and so on

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$;

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$;

$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$ and so on

and $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$;

$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$;

$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$ and so on

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
$(= x_0 + h)$		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
$(= x_0 + 2h)$		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
$= (x_0 + 3h)$		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
$= (x_0 + 4h)$		Δy_4				
x_5	y_5					
$= (x_0 + 5h)$						

Here, the first entry y_0 is called leading term and Δy_0, $\Delta^2 y_0$, ... are called leading differences.

Also we know that

$$\nabla y_1 = y_1 - y_0 = \Delta y_0$$

$$\nabla y_2 = y_2 - y_1 = \Delta y_1$$

$$\nabla y_3 = y_3 - y_2$$ and so on.

$$\nabla^2 y_1 = \nabla y_1 - \nabla y_0 \qquad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$$ and so on.

$$\nabla^3 y_1 = \nabla^2 y_1 - \nabla^2 y_0$$

$$\nabla^3 y_2 = \nabla^2 y_2 - \nabla^2 y_1$$

$$\nabla^3 y_{\cancel{2}3} = \nabla^2 y_{\cancel{2}3} - \nabla^2 y_{\cancel{1}2}$$ and so on.

x	y	∇y / Δy	$\nabla^2 y$ / $\Delta^2 y$	$\nabla^3 y$ / $\Delta^3 y$
0	y_0 4	Δy_0 3	$\Delta^2 y_0$ -2	$\Delta^3 y_0$ 3 $\nabla^3 y_n$
1	7	1	1 $\nabla^2 y_n$	
2	8	2 ∇y_n		
3	10 y_n			

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1 $(= x_0 + h)$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2 $(= x_0 + 2h)$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3 $(= x_0 + 3h)$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4 $(= x_0 + 4h)$	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5 $(= x_0 + 5h)$	y_5					

- ***Forward difference table*** ***is used to interpolate the values of y nearer to the*** ***beginning value of the table****. In calculation, we use the* ***uppermost values*** ***in the forward difference table.***

- ***Backward difference table*** ***is used to interpolate the values of y nearer to the*** ***end value of the table****. In calculation, we use the* ***lowermost values*** ***in the backward difference table.***

Newton's forward difference formula for equal intervals

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

where $u = \dfrac{x - x_0}{h}$

- This formula is used to interpolate (or extrapolate) the values of y nearer to beginning value of the table.
- u lies between 0 and 1.

Newton's backward interpolation formula for equal intervals is

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots\dots\dots\dots\dots\dots$$

$$where \quad v = \frac{x - x_n}{h}$$

<u>Note 1:</u> This formula is used to interpolate the values of y nearer to the end of set tabular values.

2. v lies between -1 and 0.

Problem

Find the values of y at $x = 21$ and $x = 28$ from the following data:

x :	20	23	26	29
y :	0.3420	0.3907	0.4384	0.4848

Solution: Here $h = 3$.

Here $x_0 = 20$ & $x_n = 29$

$$u = \frac{x - x_0}{h} = \frac{21 - 20}{3} = \frac{1}{3} = 0.3333$$

(i) To find y at $x = 21$

Since $x = 21$ is nearer to the beginning of the table, we use Newton's forward interpolation formula.

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420 (y_0)			
		0.0487 (Δy_0)		
23	0.3907		−0.0010 ($\Delta^2 y_0$)	
		0.0477		−0.0003 ($\Delta^3 y_0$, $\nabla^3 y_n$)
26	0.4384		−0.0013 ($\nabla^2 y_n$)	
		0.0464 (∇y_n)		
29	0.4848 (y_n)			

Newton's forward interpolation formula is

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots$$

$$\therefore y(21) = 0.3420 + (0.3333)(0.0487)$$

$$+ \frac{(0.3333)(0.3333-1)}{2!}(-0.0010)$$

$$+ \frac{(0.3333)(0.3333-1)(0.3333-2)}{3!}(-0.0003)$$

$$= 0.3420 + 0.016232 + 0.0001 - 0.000018$$

$$= 0.3583 \text{ app.}$$

(ii) To find y at $x = 28$

Since $x = 28$ is nearer to the end of tabular values, we use Newton's backward interpolation formula.

Newton's backward difference interpolation formula is

$$y(x) = y_n + v \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \cdots$$

where $v = \dfrac{x - x_n}{h}$

i.e., $v = \dfrac{28 - 29}{3} = \dfrac{-1}{3}$

$$\therefore y(28) = 0.4848 + (-0.3333)(0.0464) + \frac{(-0.3333)(-0.3333+1)}{2!}(-0.0013) + \frac{(-0.3333)(-0.3333+1)(-0.3333+2)}{3!}(-0.0003)$$

$$= 0.4848 - 0.015465 + 0.000144 + 0.0000185$$

$$= 0.4694 \text{ app.}$$

$$\therefore y(21) = 0.3583$$

and $y(28) = 0.4694$

Problems:

1. Find a polynomial of degree two which takes the values

x :	0	1	2	3	4	5	6	7
y :	1	2	4	7	11	16	22	29

Solution: Here $h = 1$.

To find a polynomial to the above data, we can use either Newton's forward interpolation formula or Newton's backward interpolation formula.

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1	1	1	0
1	2	2	1	0
2	4	3	1	0
3	7	4	1	0
4	11	5	1	0
5	16	6	1	
6	22	7		
7	29			

Here $y_0 = 1$

$\Delta y_0 = 1$

$\Delta^2 y_0 = 1$

$\Delta^3 y_0 = 0$

Newton's forward interpolation formula is

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Where $u = \frac{x - x_0}{h} = \frac{x-0}{1} = x$

$$\therefore y(x) = 1 + x(1) + \frac{x(x-1)}{2!}(1)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x}{2}$$

$$= \frac{1}{2}(x^2 + x + 2)$$ which is the required polynomial.

2. The following table gives the marks secured by 100 students in Numerical Analysis subject:

Range of Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of Students	25	35	22	11	7

Use Newton's forward interpolation formula to find (i) the number of students who got more than 55 marks and (ii) the number of students who secured in the range from 36 to 45.

Solution: The given table is re-arranged as follows:

Marks obtained (x)	Number of Students (y)
Less than 40	25
Less than 50	$25+35=60$
Less than 60	$60+22=82$
Less than 70	$82+11=93$
Less than 80	$93+7=100$.

Here $h=10$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	y_0 25				
		Δy_0 35			
50	60		$\Delta^2 y_0$ −13		
		22		$\Delta^3 y_0$ 2	
60	82		−11		$\Delta^4 y_0$ 5
		11		7	
70	93		−4		
		7			
80	100				

Newton's forward interpolation formula is

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \frac{x - x_0}{h}$

(i) To find the number of students who got more than 55 marks.

First let us find y_{55} or $y(55)$, i.e., the number of students who got less than 55 marks.

Now $u = \frac{55 - 40}{10} = 1.5$

$$\therefore y(55) = 25 + (1.5)(35) + \frac{(1.5)(1.5-1)}{2!}(-13)$$

$$+ \frac{(1.5)(1.5-1)(1.5-2)}{3!}(2) + \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{4!}(5)$$

$$= 25 + 52.5 - 4.875 - 0.125 + 0.1171875$$

$$= 72.617 \simeq 73$$

$\therefore$ Number of students who got more than 55 marks $\Big\} = 100 - 73$

$$= 27.$$

(ii) To find the number of students who got marks between 36 and 45.

i.e., To find $y_{45} = y(45)$ = Number of students who got marks < 45

& $y_{36} = y(36)$ = Number of students who got marks < 36.

To find y(45): Here $u = \frac{45-40}{10} = 0.5$

$$y(45) = 25 + (0.5)(35) + \frac{(0.5)(0.5-1)}{2!}(-13)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{3!}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(5)$$

i.e., $y(45) = 25 + 17.5 + 1.625 + 0.125 - 0.195313$

$$= 44.054 \simeq 44$$

Next to find $y(36)$: Here $u = \frac{36-40}{10} = -0.4$

$$y(36) = 25 + (-0.4)(35) + \frac{(-0.4)(-0.4-1)}{2!}(-13)$$

$$+ \frac{(-0.4)(-0.4-1)(-0.4-2)}{3!}(2) + \frac{(-0.4)(-0.4-1)(-0.4-2)(-0.4-3)}{4!}(5)$$

$$= 25 - 14 - 3.64 - 0.448 + 0.952$$

$$= 7.864 \simeq 8$$

Hence the number of students who secured marks in the range from 36 to 45 is

$$y(45) - y(36)$$

$$= 44 - 8$$

$$= 36$$

Homework

1. Find the value of $\sin 52^\circ$ from the following data:

θ :	45°	50°	55°	60°
$\sin\theta$:	0.7071	0.7660	0.8192	0.8660

Here $u = \dfrac{\theta - \theta_0}{h}$

$$= \frac{52^\circ - 45^\circ}{5^\circ} = 1.4$$

2. The following are the number of deaths in four successive ten year age groups. Find the number of deaths at 45 - 50 and 50 - 55.

$\hookrightarrow y(50) - y(45)$ $\hookrightarrow y(55) - y(50)$

Age group :	25 - 35	35 - 45	45 - 55	55 - 65
Deaths :	13229	18139	24225	31496

Hint:

Age upto x :	35	45	55	65
Deaths y :	13229	31368	.....	.....

To find: $y(50)$ by NFIF | NBIF

INTERPOLATION WITH UNEQUAL INTERVALS

In the previous lectures on Interpolation we had the intervals of differencing to be a constant h.

In other words, we had $x_i – x_{i-1} = h$ (constant) ,

for i = 1, 2, ……. , n.

If the values of x's are given at unequal intervals, our Newton's forward, backward formulae and central difference interpolation formulae will not hold good.

It is then desirable to develop interpolation formulae for unequally spaced values of x. We shall study "Lagrange's interpolation formula" in this lecture.

LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ takes the value $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, \ldots, x_n$, then

$y(x) =$

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1$$

$$+ \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

NOTE:

- Lagrange’s interpolation formula for n points is a polynomial of degree n – 1 which is known as Lagrangian polynomial and is very simple to implement on a Computer.
- Lagrange’s interpolation formula is more general in that it is applicable to either equal or unequal intervals and the abscissae $x_0, x_1, x_2, \ldots\ldots.., x_n$ need not be in order.

Problems:

1. The function $y = \sin x$ is tabulated below:

x:	0	$\pi/4$	$\pi/2$
$y=\sin x$:	0	0.70711	1.0

Here $h = \pi/4$

Using Lagrange's interpolation formula, find the value of $\sin\left(\frac{\pi}{6}\right)$.

Solution: From the data

$$x_0 = 0 \quad ; \quad x_1 = \pi/4 \quad ; \quad x_2 = \pi/2$$

$$y_0 = 0 \quad ; \quad y_1 = 0.70711 \quad ; \quad y_2 = 1.0$$

To find: $\sin\left(\frac{\pi}{6}\right)$

Here the function is $y = \sin x$, where $x = \pi/6$.

Lagrange's interpolation formula is

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2$$

$$\therefore y\left(\frac{\pi}{6}\right) = \frac{\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{\left(0-\frac{\pi}{4}\right)\left(0-\frac{\pi}{2}\right)} \cdot (0)$$

$$+\frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}(0.70711)+\frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}(1)$$

i.e, $y\left(\frac{\pi}{6}\right)=\frac{\left(-\frac{\pi}{12}\right)\left(-\frac{2\pi}{6}\right)}{\left(-\frac{\pi}{4}\right)\left(-\frac{\pi}{2}\right)}(0)+\frac{\left(\frac{\pi}{6}\right)\left(-\frac{2\pi}{6}\right)}{\left(\frac{\pi}{4}\right)\left(-\frac{\pi}{4}\right)}(0.70711)$

$$+\frac{\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{12}\right)}{\left(\frac{\pi}{2}\right)\left(\frac{\pi}{4}\right)}(1)$$

$$=0+\frac{16}{18}(0.70711)-\frac{8}{72}$$

$$=0.51743$$

2. Find the unique polynomial $y(x)$ of degree 2 such that

$y(1) = 1$; $y(3) = 27$; $y(4) = 64$.

Solution: Given:

x :	1	3	4
y :	1	27	64

2 1

$x_0 = 1$; $x_1 = 3$; $x_2 = 4$

$y_0 = 1$; $y_1 = 27$; $y_2 = 64$

By Lagrange's interpolation formula

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2$$

$$\text{i.e., } y(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27)$$

$$+ \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$$

$$= \frac{x^2-7x+12}{(-2)(-3)} + \frac{x^2-5x+4}{(2)(-1)}(27) + \frac{x^2-4x+3}{(3)(1)}(64)$$

$$= \frac{x^2-7x+12}{6} - \frac{27}{2}(x^2-5x+4) + \frac{64}{3}(x^2-4x+3)$$

$$= \frac{1}{6}\left[x^2-7x+12-81(x^2-5x+4)+128(x^2-4x+3)\right]$$

$$= 8x^2-19x+12$$

$$\therefore y(x) = 8x^2-19x+12 \rightarrow \text{Lagrangian Polynomial}$$

3. The function $y = f(x)$ is given at the points $(7,3)$, $(8,1)$, $(9,1)$ and $(10,9)$. Find the value of y for $x = 9.5$ using Lagrange's interpolation formula.

Solution: We are given

x :	7	8	9	10
y :	3	1	1	9

Here $h = 1$

To find : Value of y when $x = 9.5$

$x_0 = 7$; $x_1 = 8$; $x_2 = 9$; $x_3 = 10$

$y_0 = 3$; $y_1 = 1$; $y_2 = 1$; $y_3 = 9$

Lagrange's interpolation formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$\therefore y(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)}(3) + \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)}(1)$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)}(1) + \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)}(9)$$

i.e., $$y(9.5) = \frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)}(3) + \frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)}(1)$$

$$+ \frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)}(1) + \frac{(2.5)(1.5)(0.5)}{(3)(2)(1)}(9)$$

$$= \frac{0.375}{2} - \frac{0.625}{2} + \frac{1.875}{2} + \frac{5.625}{2}$$

$$= \frac{0.375 - 0.625 + 1.875 + 5.625}{2}$$

$$= \frac{7.25}{2} = 3.625$$

$\therefore$ Value of y when $x = 9.5$ is 3.625

Practice Problems

$\rightarrow$ 2nd degree polynomial

1. Find the parabola passing through the points (0, 1), (1, 3) and (3, 55) using Lagrange's interpolation formula.

2. Given

x :	0	1	3	4
y :	-6	0	0	6

find $y(2)$.

3. Certain corresponding values of x and $\log_{10} x$ are (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find $\log_{10} 301$.

Answers:
1. $y(x) = 8x^2 - 6x + 1$
2. $y(2) = 0$
3. $\log_{10} 301 = 2.4786$.

INVERSE INTERPOLATION

So far, given a set of values of x and y we were finding the values of y corresponding to some $x = x_k$ (which is not given in the table). Here we treat y as a function of x.

Now the problem is, given some $y = y_r$, we should find the corresponding x. This process of finding x given y is called the inverse interpolation.

In such a case, we will take y as independent variable and x as dependent variable and use Lagrange's interpolation formula.

LAGRANGE'S INVERSE INTERPOLATION FORMULA

$$x(y) = \frac{(y-y_1)(y-y_2)................(y-y_n)}{(y_0-y_1)(y_0-y_2)..............(y_0-y_n)} x_0$$

$$+ \frac{(y-y_0)(y-y_2)................(y-y_n)}{(y_1-y_0)(y_1-y_2)..............(y_1-y_n)} x_1$$

$$+ .... +$$

$$+ \frac{(y-y_0)(y-y_1)................(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)..............(y_n-y_{n-1})} x_n$$

The above formula is called formula of inverse interpolation.

Problem 1: From the data given below, find the value of x when $y = 13.5$

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.8	14.7	17.07	19.91

Solution: Here $x_0 = 93.0$; $x_1 = 96.2$; $x_2 = 100.0$; $x_3 = 104.2$; $x_4 = 108.7$; $y_0 = 11.38$; $y_1 = 12.8$; $y_2 = 14.7$; $y_3 = 17.07$ and $y_4 = 19.91$.

To find: x when $y = 13.5$ by using Lagrange's inverse interpolation formula.

By Lagrange's formula for inverse interpolation,

$$x(y) = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)} \cdot x_0$$

$$+ \frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)} \cdot x_1$$

$$+ \dots + \frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})} \cdot x_n$$

Now $$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} \cdot x_0$$

$$+ \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)} \cdot x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)} \cdot x_2$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)} \cdot x_3$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} \cdot x_4$$

$$x(y) = \frac{(y-12.8)(y-14.7)(y-17.07)(y-19.91)(93)}{(11.38-12.8)(11.38-14.7)(11.38-17.07)(11.38-19.91)}$$

$$+ \frac{(y-11.38)(y-14.7)(y-17.07)(y-19.91)(96.2)}{(12.8-11.38)(12.8-14.7)(12.8-17.07)(12.8-19.91)}$$

$$+ \frac{(y-11.38)(y-12.8)(y-17.07)(y-19.91)(100)}{(14.7-11.38)(14.7-12.8)(14.7-17.07)(14.7-19.91)}$$

$$+ \frac{(y-11.38)(y-12.8)(y-14.7)(y-19.91)(104.2)}{(17.07-11.38)(17.07-12.8)(17.07-14.7)(17.07-19.91)}$$

$$+ \frac{(y-11.38)(y-12.8)(y-14.7)(y-17.07)(108.7)}{(19.91-11.38)(19.91-12.8)(19.91-14.7)(19.91-17.07)}$$

Putting $y = 13.5$ on the right hand side, and simplifying

$$x = -7.8126929 + 68.3721132 + 43.595887$$

$$- 7.2733429 + 0.770084198$$

$$= 97.6557503$$

$\therefore$ Value of x when $y = 13.5$ is 97.6557503

Advantages of Lagrange's interpolation formula

- The formula is simple and easy to remember.
- There is no need to construct the divided difference table.
- The answers for higher order polynomials will be more accurate.
- Lagrange's formula has a better performance at the boundaries which make it more convenient for real time applications.

Disadvantages of Lagrange's Interpolation Formula

- The application of the formula is not speedy.
- It becomes a tedious job to do when the polynomial order increases because the number of points increases and we need to evaluate approximate solutions for each point.
- There is always a chance to commit some error in the calculations.
- The calculation provide no check whether the functional values used in the formula are correct or not.

Practice Problems

1. Find the age corresponding to the annuity value 13.6 from the following table:

Age(x) :	30	35	40	45	50
Annuity Value(y):	15.9	14.9	14.1	13.3	12.5

2. Find the value of x when y(x) = 19 given

x :	0	1	2
y :	0	1	20

Answers

1. $x(y = 13.6) = 43$
2. $x(y = 19) = 2.8$