Unit-IV

FOURIER TRANSFORMS

Statement of Fourier integral theorem - Fourier transform pair - Fourier sine and cosine transforms - Properties - Transforms of simple functions - Convolution theorem - Parseval's identity.

4.0 INTRODUCTION

Integral transforms are used in the solution of partial differential equations. The choice of a particular transform to be used for the solution of a differential equation depends upon the nature of the boundary conditions of the equation and the facility with which the transform F(s) can be converted to give $f(x)$.

4.1 STATEMENT OF FOURIER INTEGRAL THEOREM

4.1.a Fourier integral theorem (without proof)

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{is(t-x)}\, dt\, ds \quad \dots (1)$$

Equation (1) can be re-written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left[\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \right] ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \right] ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx}\, F(s)\, ds, \text{ where}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

i.e., $F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$ [$\because$ t is a dummy variable]

or equivalently,

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda$$

This is known as Fourier integral theorem or Fourier integral formula.

(OR)

Let us assume the following conditions on a function $f(x)$

1. $f(x)$ is piece-wise continuous in any finite interval (a, b)
2. $\int_{-\infty}^{\infty} |f(x)|\, dx$ is convergent.

Then the Fourier integral theorem states that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda$$

The double integral in the right hand side is known as a Fourier integral expansion of $f(x)$

(OR)

If $f(x)$ is a function defined in $(-l, l)$ satisfying Dirichlet's conditions, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda$$

The double integral in the right hand side is known as Fourier integral to represent $f(x)$

Note : We assume the following conditions on $f(x)$

(i) $f(x)$ is defined as single-valued except at finite points in $(-l, l)$

(ii) $f(x)$ is periodic outside $(-l, l)$ with period $2l$.

(iii) $f(x)$ and $f'(x)$ are sectionally continuous in $(-l, l)$

(iv) $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges

i.e., $f(x)$ is absolutely integrable in $(-\infty, \infty)$

I.(a) Problems based on Fourier integral theorem :

Example 4.1.a(1) : Show that $f(x) = 1, 0 < x < \infty$ cannot be represented by a Fourier integral.

Solution : $\int_{0}^{\infty} |f(x)|\, dx = \int_{0}^{\infty} 1\, dx = [x]_0^{\infty} = \infty - 0 = \infty$

i.e., $\int_{0}^{\infty} |f(x)|\, dx$ is not convergent.

Hence, $f(x) = 1$ cannot be represented by a Fourier integral.

Example 4.1.a(2) : If $f(x) = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$, show that $f(x) = \int_{0}^{\infty} \frac{\cos \lambda x \sin \lambda}{\lambda}\, d\lambda$

Hence show that $\int_{0}^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{is(t-x)}\, dt\, ds$$

i.e., $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$, where ... (1)

$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$ ... (2)

Given : $f(x) = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$ i.e., $f(x) = \begin{cases} \frac{\pi}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$(2) \Rightarrow F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{\pi}{2} e^{isx}\, dx$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \int_{-1}^{1} e^{isx}\, dx$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \int_{-1}^{1} \left[\cos sx + i \sin sx\right] dx$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \left[\int_{-1}^{1} \cos sx\, dx + i \int_{-1}^{1} \sin sx\, dx\right]$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left(\frac{1}{2}\right) \left[2 \int_{0}^{1} \cos sx\, dx + i\,(0)\right]$$

$\boxed{\begin{array}{l} \because \cos sx \text{ is an even function in } (-1, 1) \\ \sin sx \text{ is an odd function in } (-1, 1) \end{array}}$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left[\frac{\sin sx}{s}\right]_{x=0}^{x=1}$$

$$= \left(\sqrt{\frac{\pi}{2}}\right)\left[\frac{\sin s}{s} - 0\right]$$

$$= \sqrt{\frac{\pi}{2}}\ \frac{\sin s}{s} \qquad \text{... (3)}$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}}\ \frac{\sin s}{s}\, e^{-isx}\, ds$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin s}{s} \left[\cos sx - i \sin sx\right] ds$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin s}{s} \cos sx\, ds - \frac{i}{2} \int_{-\infty}^{\infty} \frac{\sin s}{s} \sin sx\, ds$$

$$= \left(\frac{1}{2}\right)\left[(2) \int_{0}^{\infty} \frac{\sin s}{s} \cos sx\, ds\right] - \frac{i}{2}[0]$$

$\boxed{\begin{array}{l} \because \frac{\sin s}{s} \cos sx \text{ is an even function in } (-\infty, \infty) \\ \frac{\sin s}{s} \sin sx \text{ is an odd function in } (-\infty, \infty) \end{array}}$

$$= \int_{0}^{\infty} \frac{\sin s}{s} \cos sx\, ds$$

$$= \int_{0}^{\infty} \frac{\sin \lambda}{\lambda} \cos \lambda x\, d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\text{i.e., } f(x) = \int_{0}^{\infty} \frac{\cos \lambda x \sin \lambda}{\lambda}\, d\lambda \qquad \text{... (4)}$$

Put $x = 0$, (we cannot assign any value to the dummy variable 'λ')

$$\left.\begin{array}{l} f(x) \\ \text{at } x = 0 \end{array}\right\} = \frac{\pi}{2}, \qquad [\because x = 0 \text{ is a point of continuity in } -1 < x < 1]$$

$$\therefore (4) \Rightarrow \frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda \qquad \text{i.e., } \int_{0}^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$$

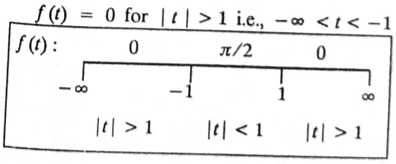
Aliter :

We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t - x)\, dt\, d\lambda \qquad \text{... (1)}$$

Here, $f(t) = \frac{\pi}{2}$ for $|t| < 1$ is $-1 < t < 1$

$f(t) = 0$ for $|t| > 1$ i.e., $-\infty < t < -1$ and $1 < t < \infty$



$$(1) \Rightarrow f(x) = \frac{1}{\pi} \int_0^\infty \int_{-1}^{1} \frac{\pi}{2} \cos \lambda (t - x)\, dt\, d\lambda$$

$$= \frac{1}{\pi} \frac{\pi}{2} \int_0^\infty \int_{-1}^{1} \cos \lambda (t - x)\, dt\, d\lambda$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{\sin \lambda (t - x)}{\lambda}\right]_{-1}^{1} d\lambda$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{\sin \lambda (1 - x)}{\lambda} - \frac{\sin \lambda (-1 - x)}{\lambda}\right] d\lambda$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{\sin \lambda (1 - x)}{\lambda} + \frac{\sin \lambda (1 + x)}{\lambda}\right] d\lambda$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{1}{2} \int_0^\infty \frac{\sin \lambda (1 - x) + \sin \lambda (1 + x)}{\lambda}\, d\lambda$$

$$= \frac{1}{2} \int_0^\infty \frac{\sin (\lambda - \lambda x) + \sin (\lambda + \lambda x)}{\lambda}\, d\lambda$$

$$= \frac{1}{2} \int_0^\infty \frac{2 \sin \lambda \cos \lambda x}{\lambda}\, d\lambda \quad [\because \sin(A+B) + \sin(A-B) = 2 \sin A \cos B]$$

$$= \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda$$

i.e., $\displaystyle\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = f(x)$

$$= \frac{\pi}{2} \text{ for } |x| < 1$$

$$= 0 \text{ for } |x| > 1$$

Putting $x = 0$, we get $\displaystyle\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$

Example 4.1.a(3) : Express the function $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ **as a Fourier integral. Hence evaluate** $\displaystyle\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda$ **and find the value of** $\displaystyle\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda$.

[A.U. April, 2001] [A.U N/D 2015 R-8]

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{is(t-x)}\, dt\, ds$$

i.e., $\displaystyle f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$, where ... (1)

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \quad \text{... (2)}$$

Given : $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ i.e., $f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$$(2) \Rightarrow F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} [\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^1 \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because$ $\cos sx$ is an even function in $(-1, 1)$

$\sin sx$ is an odd function in $(-1, 1)$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_{x=0}^{x=1}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} - 0\right] \qquad [\because \sin 0 = 0]$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\, \frac{\sin s}{s} \qquad \ldots (3)$$

$$(1) \Rightarrow \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin s}{s}\, e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} [\cos sx - i \sin sx]\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} \sin sx\, ds$$

$$= \frac{1}{\pi} \left[2 \int_0^{\infty} \frac{\sin s}{s} \cos sx\, ds\right] - \frac{i}{\pi} [0]$$

$\because \frac{\sin s}{s} \cos sx$ is an even function in $(-\infty, \infty)$

$\frac{\sin s}{s} \sin sx$ is an odd function in $(-\infty, \infty)$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin s}{s} \cos sx\, ds$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda}{\lambda} \cos \lambda x\, d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\text{i.e., } \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} f(x) \qquad \ldots (4)$$

$$\text{Hence, } \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = \begin{cases} \frac{\pi}{2}, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Put $x = 0$, (we cannot assign any value to the dummy variable 'λ')

$$\left.\begin{matrix} f(x) \\ \text{at } x = 0 \end{matrix}\right\} = 1 \qquad [\because x = 0 \text{ is a point of continuity in } -1 \le x \le 1]$$

$$\therefore (4) \Rightarrow \quad \int_0^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$$

Example 4.1.a(4) : Find the Fourier integral of the function

$$\mathbf{f(t)} = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}$$

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{is(t-x)}\, dt\, ds$$

$$\text{i.e., } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds, \text{ where} \qquad \ldots (1)$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad \ldots (2)$$

$$\text{Given : } f(t) = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases} \quad \text{i.e., } f(t) = \begin{cases} e^{at}, & -\infty < t < 0 \\ e^{-at}, & 0 < t < \infty \end{cases}$$

$$\text{i.e., } f(x) = \begin{cases} e^{ax}, & -\infty < x < 0 \\ e^{-ax}, & 0 < x < \infty \end{cases}$$

$$(2) \Rightarrow F(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{ax}\, e^{isx}\, dx + \int_0^{\infty} e^{-ax}\, e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(a+is)x}\, dx + \int_0^{\infty} e^{-(a-is)x}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{e^{(a+is)x}}{a+is}\right)_{-\infty}^{0} + \left(\frac{e^{-(a-is)x}}{-(a-is)}\right)_0^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} - 0 + 0 - \left(\frac{1}{-(a-is)}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} + \frac{1}{a-is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a-is+a+is}{a^2+s^2}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2a}{a^2+s^2}\right]$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+s^2}\,e^{-isx}\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+s^2}[\cos sx - i\sin sx]\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+s^2}\cos sx\,ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+s^2}\sin sx\,ds$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{a}{a^2+s^2}\cos sx\,ds - \frac{i}{\pi}(0)$$

$\because \frac{a}{a^2+s^2}\cos sx$ is an even function in $(-\infty, \infty)$ &

$\frac{a}{a^2+s^2}\sin sx$ is an odd function in $(-\infty, \infty)$

$$= \frac{2a}{\pi}\int_{0}^{\infty}\frac{\cos sx}{a^2+s^2}\,ds$$

$$f(x) = \frac{2a}{\pi}\int_{0}^{\infty}\frac{\cos\lambda x}{a^2+\lambda^2}\,d\lambda \quad [\because s \text{ is a dummy variable}]$$

i.e., $$\int_{0}^{\infty}\frac{\cos\lambda x}{a^2+\lambda^2}\,d\lambda = \frac{\pi}{2a}f(x)$$

$$= \begin{cases}\frac{\pi}{2a}e^{at}, & t<0\\ \frac{\pi}{2a}e^{-at}, & t>0\end{cases}$$

Example 4.1.a(5) : Find the Fourier integral of the function

$$f(x) = \begin{cases}0, & x<0\\ 1/2, & x=0\\ e^{-x}, & x>0\end{cases}$$

Verify the representation directly at the point $x = 0$.

[A.U N/D 2010, M/J 2012]

Solution : We know that, the Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{is(t-x)}\,dt\,ds$$

i.e., $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)\,e^{-isx}\,ds, \text{ where} \qquad \dots (1)$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{isx}\,dx \qquad \dots (2)$$

Given : $f(x) = \begin{cases}0, & x<0\\ ½, & x=0\\ e^{-x}, & x>0\end{cases}$ i.e., $f(x) = \begin{cases}0, & -\infty<x<0\\ ½, & x=0\\ e^{-x}, & 0<x<\infty\end{cases}$

$$(2) \Rightarrow F(s) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-x}\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-(1-is)x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(1-is)x}}{-(1-is)}\right]_{x=0}^{x=\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \left(\frac{1}{-(1-is)}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-is}\right] = \frac{1}{\sqrt{2\pi}}\,\frac{1+is}{1+s^2} \qquad \dots (3)$$

$$(1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,\frac{1+is}{1+s^2}\,e^{-isx}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+s^2}e^{-isx} + i\,\frac{s}{1+s^2}e^{-isx}\right] ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+s^2}(\cos sx - i\sin sx)\, ds$$

$$+ \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{s}{1+s^2}(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+s^2}\cos sx\, ds - \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+s^2}\sin sx\, ds$$

$$+ \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{s}{1+s^2}\cos sx\, ds + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{s}{1+s^2}\sin sx\, ds$$

$$= \frac{1}{2\pi}\left[2\int_{0}^{\infty}\frac{1}{1+s^2}\cos sx\, ds\right] - \frac{i}{2\pi}[0]$$

$$+ \frac{i}{2\pi}[0] + \frac{1}{2\pi}\left[2\int_{0}^{\infty}\frac{s}{1+s^2}\sin sx\, ds\right]$$

$\because$ $\frac{1}{1+s^2}\cos sx$ is an even function in $(-\infty, \infty)$,

$\frac{s}{1+s^2}\sin sx$ is an even function in $(-\infty, \infty)$,

$\frac{1}{1+s^2}\sin sx$ is an odd function in $(-\infty, \infty)$ &

$\frac{s}{1+s^2}\cos sx$ is an odd function in $(-\infty, \infty)$

$$= \frac{1}{\pi}\left[\int_{0}^{\infty}\frac{1}{1+s^2}\cos sx\, ds + \int_{0}^{\infty}\frac{s}{1+s^2}\sin sx\, ds\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos sx + s\sin sx}{1+s^2}\, ds$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda \qquad \ldots (4)$$

[$\because$ s is a dummy variable]

Verification :

Put $x = 0$ in (4), we get

$$f(0) = \frac{1}{\pi}\int_{0}^{\infty}\frac{1}{1+\lambda^2}\, d\lambda = \frac{1}{\pi}\left[\tan^{-1}\lambda\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}$$

The value of the given function at $x = 0$ is $\frac{1}{2}$.

Hence, verified.

4.1.b Complex form of the Fourier integrals.

The Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos[\lambda(t-x)]\, dt\, d\lambda \qquad \ldots (1)$$

because $\cos[\lambda(t-x)]$ is an even function of λ.

Also since $\sin[\lambda(t-x)]$ is an odd function of λ.

$$\text{We have, } \int_{-\infty}^{\infty} f(t)\sin[\lambda(t-x)]\, d\lambda = 0$$

$$\text{i.e., } \int_{-\infty}^{\infty} f(t)\, i\sin[\lambda(t-x)]\, d\lambda = 0$$

$$(1) \Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,[\cos\lambda(t-x) + i\sin\lambda(t-x)]\, dt\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\, d\lambda$$

which is the complex form of the Fourier integral.

FOURIER SINE AND COSINE INTEGRALS

Fourier sine and cosine integrals

(i) $$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin st \sin sx\, dt\, ds$$

[Fourier sine integral]

(ii) $$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos st \cos sx\, dt\, ds$$

[Fourier cosine integral]

Proof : We know that, the Fourier integral theorem is

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos s(t-x)\, dt\, ds$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos(st - sx)\, dt\, ds$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)[\cos st\cos sx + \sin st \sin sx]\, dt\, ds$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos st\cos sx\, dt\, ds$$

$$+ \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\sin st\sin sx\, dt\, ds \quad \text{... (1)}$$

Case (i) : When $f(t)$ is odd,

$f(t)\cos st$ is an odd function in $(-\infty, \infty)$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{\pi}(0) + \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin st\sin sx\, dt\, ds$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin st\sin sx\, dt\, ds \quad \text{.... (2)}$$

Case (ii) : When $f(t)$ is even,

$f(t)\sin st$ is an odd function in $(-\infty, \infty)$

$$\therefore (1) \Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos st\cos sx\, dt\, ds + \frac{1}{\pi}(0)$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos st\cos sx\, dt\, ds \quad \text{... (3)}$$

Note I :

Equation (2) can be re-written as

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx\left[\int_0^\infty f(t)\sin st\, dt\right] ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\, dt\right] ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\, F_s(s)\, ds, \text{ where}$$

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\, dt$$

i.e., $$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

[$\because$ t is a dummy variable]

Note II :

Equation (3) can be re-written as

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos sx\left[\int_0^\infty f(t)\cos st\, dt\right] ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\, dt\right] ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\, F_c[s]\, ds, \text{ where}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt$$

i.e., $$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

[$\because$ t is a dummy variable]

I.(b) Problems based on Fourier cosine and Fourier sine integrals

Example 4.1.b(1) : Find Fourier cosine integral of the function e^{-ax}. Hence deduce the value of the integral $\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda$

Solution : We know that, the Fourier Cosine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds$$

i.e., $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds, \text{ where} \quad \dots (1)$$

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \quad \dots (2)$$

Given : $f(x) = e^{-ax}$

$$(2) \Rightarrow F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

$$\left[\text{Formula : } \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right]$$

i.e., $$F_c[s] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx \, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2+a^2} ds$$

$$\Rightarrow \int_0^\infty \frac{\cos sx}{s^2+a^2} ds = \frac{\pi}{2a} f(x)$$

i.e., $$\int_0^\infty \frac{\cos \lambda x}{\lambda^2+a^2} d\lambda = \frac{\pi}{2a} f(x) \quad [\because s \text{ is a dummy variable}]$$

$$= \frac{\pi}{2a} e^{-ax}$$

Put $a = 1$, we get

$$\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$$

Example 4.1.b(2) : Using Fourier integral formula, show that

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos \lambda x \, d\lambda$$

Solution : We know that, the Fourier Cosine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds$$

i.e., $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds, \text{ where} \quad \dots (1)$$

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \quad \dots (2)$$

Here, $f(x) = e^{-x} \cos x$

(2) $\Rightarrow$ $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos x \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \left[\frac{\cos(s+1)x + \cos(s-1)x}{2}\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-x}\cos(s+1)x}{2} dx + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-x}\cos(s-1)x}{2} dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+(s+1)^2}\right] + \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+(s-1)^2}\right]$$

$$\left[\because \text{Formula} : \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+(s+1)^2} + \frac{1}{1+(s-1)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{1+s^2+1+2s} + \frac{1}{1+s^2+1-2s}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{1}{s^2+2+2s} + \frac{1}{s^2+2-2s}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{s^2+2-2s+s^2+2+2s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \left[\frac{2(s^2+2)}{(s^2+2)^2-4s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s^2+2}{s^4+4+4s^2-4s^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{s^2+2}{s^4+4}\right]$$

(1) $\Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s^2+2}{s^4+4}\right) \cos sx \, ds$

$$= \frac{2}{\pi} \int_0^\infty \frac{s^2+2}{s^4+4} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos \lambda x \, d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\Rightarrow e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos \lambda x \, d\lambda$$

Example 4.1.b(3) : Express $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$ as a Fourier sine integral and hence evaluate $\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin(x\lambda) \, d\lambda$

Solution :

We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx \, dt \, ds$$

i.e., $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$, where ... (1)

$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$ ... (2)

Given : $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

(2) $\Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s}\right]_{x=0}^{x=\pi}$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{\cos sx}{s}\right]_{x=0}^{x=\pi}$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\cos \pi s}{s} - \frac{1}{s}\right]$$

i.e., $$F_s(s) = \sqrt{\frac{2}{\pi}}\,\frac{1}{s}\,[1 - \cos \pi s] \qquad \ldots (3)$$

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{1}{s}(1 - \cos \pi s)\sin sx\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1-\cos \pi s}{s}\sin sx\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1-\cos \pi \lambda}{\lambda}\sin \lambda x\,d\lambda$$

[∵ s is a dummy variable]

$$\Rightarrow \int_0^\infty \frac{1-\cos \pi\lambda}{\lambda}\sin \lambda x\,d\lambda = \frac{\pi}{2}f(x) \qquad \ldots (4)$$

$$= \begin{cases}\left(\frac{\pi}{2}\right)(1), & 0 < x < \pi \\ \left(\frac{\pi}{2}\right)(0), & x > \pi\end{cases} = \begin{cases}\frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi\end{cases}$$

Note : If $x = \pi$ is a finite point of discontinuity of $f(x)$, then

$$\left.\begin{array}{r} f(x) \\ at\ x = \pi \end{array}\right\} = \frac{1+0}{2} = \frac{1}{2}$$

$$\therefore (4) \Rightarrow \int_0^\infty \frac{1-\cos \pi\lambda}{\lambda}\sin \pi\lambda\,d\lambda = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

Example 4.1.b(4) : Using the Fourier integral representation show that $\int_0^\infty \frac{\omega \sin x\omega}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-x}$ $(x > 0)$

Solution :

We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin st\,\sin sx\,dt\,ds$$

i.e., $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds, \text{ where} \qquad \ldots (1)$$

$$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx \qquad \ldots (2)$$

Here, $f(x) = e^{-x}$, $x > 0$

$$(2) \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+1}\right]$$

[∵ Formula : $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$]

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+1}\sin sx\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\omega}{\omega^2+1}\sin \omega x\,d\omega$$ [∵ s is a dummy variable]

$$\Rightarrow \int_0^\infty \frac{\omega}{1+\omega^2}\sin \omega x\,d\omega = \frac{\pi}{2}f(x)$$

$$= \frac{\pi}{2}e^{-x}, \quad x > 0$$

Example 4.1.b(5) : Using Fourier integral formula, prove that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda,\ a, b > 0$$

Solution: We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin st\,\sin sx\,dt\,ds$$

i.e., $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx\, ds$, where ... (1)

$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$ ... (2)

Here, $f(x) = e^{-ax} - e^{-bx}, \quad x > 0$, and $a, b > 0$

$$(2) \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty (e^{-ax} - e^{-bx}) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2}\right] - \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + b^2}\right]$$

$$\left[\because \text{Formula} : \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\, s \left[\frac{1}{s^2 + a^2} - \frac{1}{s^2 + b^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\, s \left[\frac{s^2 + b^2 - s^2 - a^2}{(s^2 + a^2)(s^2 + b^2)}\right]$$

$$= \sqrt{\frac{2}{\pi}}\, s \left[\frac{b^2 - a^2}{(s^2 + a^2)(s^2 + b^2)}\right]$$

$$(1) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\, s \left[\frac{b^2 - a^2}{(s^2 + a^2)(s^2 + b^2)}\right] \sin sx\, ds$$

$$= \frac{2}{\pi}(b^2 - a^2) \int_0^\infty \frac{s}{(s^2 + a^2)(s^2 + b^2)} \sin sx\, ds$$

$$= \frac{2(b^2 - a^2)}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)}\, d\lambda,$$

$[\because s$ is a dummy variable$]$

$$\Rightarrow \frac{2(b^2 - a^2)}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)}\, d\lambda = f(x)$$

$$= e^{-ax} - e^{-bx},\ x > 0 \text{ and } a, b > 0$$

Example 4.1.b(6) : Applying the Fourier sine integral formula to the function $f(t) = \sin x$ **when** $0 < x \leq \pi$

$= 0$ **when** $x > \pi$

show that $\int_0^\infty \frac{\sin \lambda x \ \sin \pi\lambda}{1 - \lambda^2}\, d\lambda = \frac{\pi}{2} \sin x$ **if** $0 < x < \pi$

$= 0$ **if** $x > \pi$

Solution: We know that, the Fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \ \sin sx\, dt\, ds$$

i.e., $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx\, ds$, where ... (1)

$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$ ... (2)

Given : $f(x) = \begin{cases} \sin x, & 0 < x \leq \pi \\ 0, & x > \pi \end{cases}$

$$(2) \Rightarrow F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin sx \sin x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \frac{\cos(s-1)x - \cos(s+1)x}{2}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right) \int_0^\pi \left[\cos(s-1)x - \cos(s+1)x\right] dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_{x=0}^{x=\pi}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{\sin(s-1)\pi}{(s-1)} - \frac{\sin(s+1)\pi}{s+1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{\sin s\pi \,.\, \cos\pi - \cos s\pi \sin\pi}{s-1}\right.$$

$$\left. - \frac{\sin s\pi \cos\pi + \cos s\pi \sin\pi}{s+1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\left[\frac{-\sin s\pi}{s-1} + \frac{\sin s\pi}{s+1}\right] \qquad [\because \sin\pi = 0]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\sin s\pi\left[\frac{-1}{s-1} + \frac{1}{s+1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\sin s\pi\left[\frac{-s-1+s-1}{s^2-1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\sin s\pi\left[\frac{-2}{s^2-1}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{\sin s\pi}{s^2-1}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin s\pi}{1-s^2}$$

$$(1) \Rightarrow \quad f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\sin s\pi}{1-s^2}\right)\sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin s\pi}{1-s^2}\sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\pi}{1-\lambda^2}\sin\lambda x\, d\lambda \qquad [\because s \text{ is a dummy variable}]$$

$$\Rightarrow \int_0^\infty \frac{\sin\lambda\pi \sin\lambda x}{1-\lambda^2}\, d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}\sin x, & 0 < x \le \pi \\ 0, & x > \pi \end{cases}$$

4.2 FOURIER TRANSFORM PAIR :

4.2.a. Fourier Transform : [Complex Fourier Transform]

Definition : The complex (or infinite) Fourier Transform

[A.U N/D 2016 R-8]

The complex (or infinite) Fourier Transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad \dots (1)$$

Then the function $f(x)$ is the inverse Fourier Transform of $F(s)$ and is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \qquad \dots (2)$$

The above (1) & (2) are jointly called Fourier transform pair.

OTHER FORMATS OF FOURIER TRANSFORM PAIR		
(1)	$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$	$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds$
(2)	$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$
(3)	$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds$
(4)	$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$	$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$
(5)	$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$	$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds$

Note : Whatever definitions or format we use, there will be a difference in constant factor while finding $F(s) = F[f(x)]$. But this will be adjusted while expressing $f(x)$ as a Fourier integral.

For example, $\int_0^{\infty} \frac{\sin t}{t} dt$ or $\int_0^{\infty} \frac{\sin \lambda}{\lambda} d\lambda$ is equal to $\frac{\pi}{2}$, whatever definitions or format we use.

4.2.b. INVERSION FORMULA FOR FOURIER TRANSFORM

Let $f(x)$ be a function satisfying Dirichlet's conditions in every finite interval $(-l, l)$. Let F(s) denote the Fourier transform of $f(x)$. Then at every point of continuity of $f(x)$, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

Proof : By Fourier integral theorem,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i(x-t)s}\, dt\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixs} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-its}\, dt\right] ds$$

put $s = -\omega$,

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-ix\omega} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{it\omega}\, dt\right] (-d\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega}\, F(s)\, d\omega \text{ (by definition of F.T)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-ixs}\, ds \qquad [\because \omega \text{ is a dummy variable}]$$

PROPERTIES - TRANSFORMS OF SIMPLE FUNCTIONS

4.2.c. PROPERTIES OF FOURIER TRANSFORMS :

1. **Linear property**

F [a f(x) + b g(x)] = a F [f(x)] + b F [g(x)]

where a and b are real numbers. **[A.U N/D 2015 R-8]**

Proof : $F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]\, e^{isx}\, dx$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx$$

$$= a\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right] + b\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx\right]$$

$$= a\, F[f(x)] + b\, F[g(x)]$$

2. **Change of scale property**

For any non-zero real a, $F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right]$.

[A.U Tvli. N/D 2010] [A.U M/J 2013] [A.U N/D 2015 R-13]

Proof : We know that, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $t = ax$, $x \to -\infty \Rightarrow t \to -\infty$, if $a > 0$

$dt = a\,dx$, $x \to \infty \Rightarrow t \to \infty$, if $a > 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist/a} \frac{dt}{a} = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\, dt$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s/a)x}\, dx \quad [\because t \text{ is a dummy variable}]$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right] \qquad \text{... (1)}$$

Similarly if $a < 0$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

put $t = -ax$, $x \to -\infty \Rightarrow t \to \infty$

$dt = -a\,dx$, $x \to \infty \Rightarrow t \to -\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{(is/a)t} \frac{dt}{-a}$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{(is/a)t} \frac{dt}{a} = \frac{-1}{a} F\left[\frac{s}{a}\right] \quad \dots (2)$$

Combining (1) & (2), we get

$$F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right], \quad a \neq 0$$

3. Shifting property [A.U. May, 2000, April, 1999, April/May 2001]

(i) $F[f(x-a)] = e^{ias} F(s)$ (ii) $F[e^{iax} f(x)] = F[s+a]$

[A.U. N/D 2006, A/M 2006, M/J 2007, CBT Dec. 2008] [A.U CBT N/D 2010] [A.U N/D 2013] [A.U N/D 2014 R-08, 13] [A.U A/M 2015 R-13, A.U A/M 2017 R-13]

Proof : (i) We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

put $t = x - a$ | $x \to -\infty \Rightarrow t \to -\infty$

$dt = dx$ | $x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist} e^{isa}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = e^{ias} F[f(t)]$$

$$= e^{isa} F[s]$$

(ii) We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$= F[s+a]$$

4. Modulation Property :

Modulation theorem :

If F (s) is the Fourier transform of f (x), then

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

[A.U CBT Dec. 2008] [A.U N/D 2014 R-2013]

Proof : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] e^{isx}\, dx$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{i(s+a)x} + e^{i(s-a)x}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}\,[F(s+a) + F(s-a)]$$

5. $F[x^n f(x)] = (-i)^n \dfrac{d^n F(s)}{ds^n}$. **[A.U Tvli M/J 2011]**

Proof : We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Differentiating both sides n times w.r.to s, we get

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}}\, \frac{d^n}{ds^n}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n}\,[f(x)\, e^{isx}]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,(ix)^n\, e^{isx}\, dx$$

$$= i^n\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, x^n\, e^{isx}\, dx$$

$$= (i)^n\, F[x^n f(x)]$$

Hence, $F[x^n f(x)] = \dfrac{1}{(i)^n}\dfrac{d^n}{ds^n} F(s) = (-i)^n \dfrac{d^n}{ds^n} F(s)$

6. (i) $F[f'(x)] = -is\, F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

(ii) $F[f^{(n)}(x)] = (-i)^n s^n F(s)$ if $f(x), f'(x), \ldots f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$.

Proof : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\, d[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\left[[e^{isx} f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, e^{isx}\,(is)\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[(0-0) - is\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right]$$

$$= (-is)\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad [\because f(x) \to 0 \text{ as } x \to \pm\infty]$$

$$= -is\, F(s)$$

Similarly, $F[f^{(n)}(x)] = (-is)^n F(s)$ if $f, f', f'', \ldots f^{(n-1)} \to 0$ as $x \to \pm\infty$.

7. $F\left[\int_a^x f(x)\, dx\right] = \dfrac{F(s)}{(-is)}$

Proof : Let $\phi(x) = \int_a^x f(x)\, dx$

then $\phi'(x) = f(x)$

$$F[\phi'(x)] = (-is)\, F[\phi(x)] \text{ by property (6)}$$

$$= (-is)\, F\left[\int_a^x f(x)\, dx\right]$$

$$F\left[\int_a^x f(x)\, dx\right] = \frac{1}{-is}\, F[\phi'(x)] = \frac{1}{-is}\, F[f(x)]$$

8. $F\,[\overline{f(x)}] = \overline{F(-s)}$

Proof : We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\therefore F(-s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

Taking complex conjugate on both sides

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)}\; e^{isx}\, dx = F[\overline{f(x)}]$$

9. $F[f(-x)] = F(-s)$

Proof : We know that,

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(-x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-x)\, e^{isx}\, dx$$

put $-x = t \qquad x \to -\infty \Rightarrow t \to \infty$

$-dx = dt \qquad x \to \infty \Rightarrow t \to -\infty$

$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(t)\, e^{-ist}\,(-dt)$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\int_{\infty}^{-\infty} f(t)\, e^{-ist}\, dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

$$= F(-s)$$

Note : $F[\overline{f(-x)}] = \overline{F(s)}$

4.2.d. CONVOLUTION THEOREM - PARSEVAL'S IDENTITY

Definition : Convolution

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

Convolution Theorem :

[A.U Trichy N/D 2009, CBT A/M 2011, N/D 2011, A/M 2012]

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

(i.e.,) $F[f(x) * g(x)] = F(s)\, G(s) = F[f(x)]\, F[g(x)]$

Proof : We know that, $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [f(x) * g(x)]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right] e^{isx}\, dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dt\, dx$$

by changing the order of integration, we get

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dx\, dt$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\; F[g(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, G(s)\, dt$$

by shifting property $F[f(x-a)] = e^{ias} F(s)$

$$= G(s) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]$$

$$= G(s)\, F(s) = F(s)\, G(s)$$

Note : $F^{-1}[F(s)\, G(s)] = f(x) * g(x)$

$$= F^{-1}[F(s)] * F^{-1}[G(s)]$$

PARSEVAL'S IDENTITY :

If F (s) is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

[A.U. CBT Dec. 2008, A.U N/D 2010, A.U CBT N/D 2010, M/J 2012]

Proof : By convolution theorem,

$$F[f(x) * g(x)] = F(s).\, G(s)$$

$$f(x) * g(x) = F^{-1}[F(s).\, G(s)]$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{-isx}\, ds \quad \dots (1)$$

Put $x = 0$, we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, ds \quad \dots (2)$$

$g(-t) = \overline{f(t)}$ there it follows that

$G(s) = \overline{F(s)}$

$$(2) \Rightarrow \int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt = \int_{-\infty}^{\infty} F(s)\, \overline{F(s)}\, ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

Note : In the same way, we can prove Parseval's identity for Fourier sine and cosine transforms.

If $F_s[f(s)] = F_s[s]$ and $F_c[g(x)] = F_c(s)$ then

(i) $\int_0^{\infty} |f(x)|^2\, dx = \int_0^{\infty} |F_s(s)|^2\, ds$ and

(ii) $\int_0^{\infty} |g(x)|^2\, dx = \int_0^{\infty} |F_c(s)|^2\, ds.$

II. (a) Problems based on Fourier Transform [Complex Fourier Transform]

Formula : $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

Example 4.2.a(1) : Find the Fourier Transform of

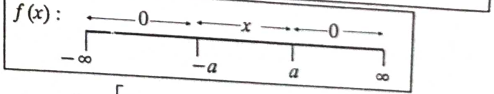
$$f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$$

[A.U. Oct/Nov. 1996, Tvli M/J 2011] [A.U A/M 2007 R-8]

Solution : The given function can be written as

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \text{ and } a < x < \infty \end{cases}$$

We know that, $\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0\, dx + \int_{-a}^{a} x e^{isx}\, dx + \int_{a}^{\infty} 0\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\,[\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} x \sin sx\, dx$$

$x \cos sx$ is an odd function in $(-a, a)$ $\therefore \int_{-a}^{a} x \cos sx \, dx = 0$

$x \sin sx$ is an even function in $(-a, a)$ $\therefore \int_{-a}^{a} x \sin sx \, dx = 2\int_{0}^{a} x \sin sx \, dx$

$$= \frac{1}{\sqrt{2\pi}}(0) + \frac{i}{\sqrt{2\pi}} 2\int_{0}^{a} x \sin sx \, dx$$

$$= \frac{2i}{\sqrt{2\pi}} \left[(x)\left[\frac{-\cos sx}{s}\right] - (1)\left[\frac{-\sin sx}{s^2}\right]\right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}} \left[-x\frac{\cos sx}{s} + \frac{\sin sx}{s^2}\right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}} \left[\left(\frac{-a \cos sa}{s} + \frac{\sin sa}{s^2}\right) - (-0+0)\right]$$

$$= i\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - as \cos sa}{s^2}\right]$$

Example 4.2.a(2) : Find the Fourier Transform of

$$f(x) = \begin{cases} 1 & \text{in } |x| < a \\ 0 & \text{in } |x| \geq a \end{cases}$$

[A.U. Dec. 2005, April, 2004, April 2003] [A.U A/M 2017 R-8]

Solution : The given function can be written as

$$f(x) = \begin{cases} 1 & \text{in } -a < x < a \\ 0 & \text{in } -\infty < x < -a \text{ and } a < x < \infty \end{cases}$$

We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$f(x)$: ←0→ ←1→ ←0→ ; $-\infty$, $-a$, a, ∞

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \sin sx \, dx$$

$\cos sx$ is an even function in $(-a, a)$ $\therefore \int_{-a}^{a} \cos sx \, dx = 2\int_{0}^{a} \cos sx$

$\sin sx$ is an odd function in $(-a, a)$ $\therefore \int_{-a}^{a} \sin sx \, dx = 0$

$$= \frac{1}{\sqrt{2\pi}} 2\int_{0}^{a} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0\right] = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s}\right]$$

Example 4.2.a(3) : Find the Fourier transform of f(x) given by

$$f(x) = \begin{cases} x^2 & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$$

Solution : The given function can be written as

$$f(x) = \begin{cases} x^2 & \text{if } -a \leq x \leq a \\ 0 & \text{if } -\infty < x < -a \text{ and } a < x < \infty \end{cases}$$

We know that, $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$f(x)$: ←0→ ←x^2→ ←0→ ; $-\infty$, $-a$, a, ∞

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 [\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} x^2 \sin sx \, dx$$

> $x^2 \cos sx$ is an even function in $(-a, a)$
>
> $$\therefore \int_{-a}^{a} x^2 \cos sx \, dx = 2\int_{0}^{a} x^2 \cos sx \, dx$$
>
> $x^2 \sin sx$ is an odd function in $(-a, a)$
>
> $$\therefore \int_{-a}^{a} x^2 \sin sx \, dx = 0$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} x^2 \cos sx \, dx + \frac{i}{\sqrt{2\pi}} (0)$$

$$= \sqrt{\frac{2}{\pi}} \left[(x^2) \left[\frac{\sin sx}{s}\right] - (2x) \left[\frac{-\cos sx}{s^2}\right] + (2) \left[\frac{-\sin sx}{s^3}\right] \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[x^2 \frac{\sin sx}{s} + \frac{2x}{s^2} \cos sx - \frac{2}{s^3} \sin sx \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{a^2 \sin sa}{s} + \frac{2a \cos sa}{s^2} - \frac{2}{s^3} \sin sa \right) - (0 + 0 - 0) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2 s^2 \sin sa + 2as \cos sa - 2 \sin sa}{s^3} \right]$$

> **Definition : Self reciprocal :** [A.U N/D 2013]
>
> If a transformation of a function $f(x)$ is equal to $f(s)$ then the function $f(x)$ is called self reciprocal.

Example 4.2.a(4) : Show that the Fourier Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$.

(OR) **[A.U. Dec-1996, May-2000]**
[A.U N/D 2011, A.U CBT N/D 2011] [A.U M/J 2013]
[A.U M/J 2016 R-13]

Show that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transform.

Solution :

Given : $f(x) = e^{-x^2/2}$,

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx} \, dx$$

$$F[e^{-x^2/2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2) + isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}}\right)^2 - isx\right]} dx$$

$a^2 - 2ab$

> We know that,
>
> $$a^2 - 2ab = (a-b)^2 - b^2$$
>
> Here, $a = \frac{x}{\sqrt{2}}$
>
> $$2ab = isx$$
>
> $$2\frac{x}{\sqrt{2}} b = isx$$
>
> $$b = \frac{is}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2 - \left(\frac{is}{\sqrt{2}}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-\left[\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right]^2} dx$$

Put $y = \frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}$ | $x \to -\infty \Rightarrow y \to -\infty$

$dy = \frac{1}{\sqrt{2}} dx$ | $x \to \infty \Rightarrow y \to \infty$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

i.e., $F[e^{-x^2/2}] = e^{-s^2/2}$ ... (1)

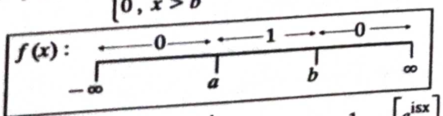
Hence, $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.

Example 4.2.a(5) : Find the Fourier transform of f(x) defined by

$$f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$$

Solution : We know that, $\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx}$

Given : $f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$



$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isb}}{is} - \frac{e^{isa}}{is}\right] = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isb} - e^{isa}}{is}\right]$$

Example 4.2.a(6) : Show that the Fourier transform of

$f(x) = |x|$ for $|x| < a$

$= 0$ for $|x| > a, a > 0$

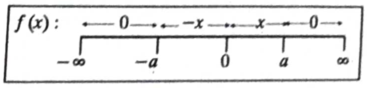
is $\sqrt{\frac{2}{\pi}} \left[\frac{sa \sin sa + \cos sa - 1}{s^2}\right]$

Solution : Given function can be written as

$f(x) = |x|$ for $-a < x < a$

$= 0$ for $|x| > a, a > 0$

(i.e.,) $f(x) = \begin{cases} -x & \text{for } -a < x < 0 \\ +x & \text{for } 0 < x < a \\ 0 & \text{for } |x| > a \end{cases}$



We know that,

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |x| [\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |x| \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} |x| \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^a x \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because |x| \cos sx$ is an even function in $(-a, a)$, $|x| \sin sx$ is an odd function in $(-a, a)$ & $|x| = x$ in $(0, a)$

$$= \sqrt{\frac{2}{\pi}} \int_0^a x \cos sx\, dx = \sqrt{\frac{2}{\pi}} \left[(x)\left[\frac{\sin sx}{s}\right] - (1)\left[\frac{-\cos sx}{s^2}\right]\right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{a \sin sa}{s} + \frac{\cos sa}{s^2}\right) - \left(0 + \frac{1}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s\, a \sin sa + \cos sa - 1}{s^2}\right]$$

Example 4.2.a(7) : Find the (complex) Fourier transform of

$$f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a, x > b \end{cases}$$ **[A.U N/D 2009]**

Solution : We know that, $\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx}\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)}\right]_a^b = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{i(k+s)}\right] \left[e^{i(k+s)x}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{i(k+s)} [e^{i(k+s)b} - e^{i(k+s)a}]$$

$$= \frac{-i}{\sqrt{2\pi}(k+s)} [e^{i(k+s)b} - e^{i(k+s)a}]$$

$$= \frac{i}{\sqrt{2\pi}(k+s)} [e^{i(k+s)a} - e^{i(k+s)b}]$$

Example 4.2.a(8) : Find the Fourier transform of

$$f(x) = \begin{cases} \cos x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution : We know that,

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

Given : $f(x) = \begin{cases} \cos x & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_0^1 \cos x\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{(is)^2 + 1} [is \cos x + \sin x]\right]_0^1$$

$$\boxed{\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \quad \text{Here, } a = is,\ b = 1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{1 - s^2} [is \cos x + \sin x]\right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{e^{is}}{1 - s^2} (is \cos 1 + \sin 1)\right) - \left(\frac{1}{1 - s^2} (is)\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{is}}{1 - s^2} is \cos 1 + \frac{e^{is}}{1 - s^2} \sin 1 - \frac{is}{1 - s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1 - s^2} [e^{is} is \cos 1 + e^{is} \sin 1 - is]$$

Example 4.2.a(9) : Find the Fourier transform of $\frac{1}{\sqrt{|x|}}$

Solution · We know that, **[A.U M/J 2014]**

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} e^{isx}\, dx$$

$$\text{Let } I = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \frac{e^{isx}}{\sqrt{-x}} dx + \int_0^{\infty} \frac{e^{isx}}{\sqrt{x}} dx\right] \quad \ldots (1)$$

Since, $|x| = \begin{cases} -x, & x < 0 \\ x, & x > 0 \end{cases}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{isx}}{\sqrt{-x}} dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{isx}}{\sqrt{x}} dx$$

$$= I_1 + I_2$$

In I_1, put $-x = y$; $-dx = dy$

when $x \to -\infty \Rightarrow y \to \infty$

$x \to 0 \quad \Rightarrow y \to 0$

$$\therefore I_1 = \int_{\infty}^{0} \frac{e^{-isy}}{\sqrt{y}}(-dy) = \int_{0}^{\infty} \frac{e^{-isy}}{\sqrt{y}}\,dy = \int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{x}}\,dx$$

[$\because$ y is a dummy variable]

There is no change in I_2

$$\therefore I = I_1 + I_2$$

$$= \left[\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{x}}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{e^{isx}}{\sqrt{x}}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{e^{isx}+e^{-isx}}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{2\cos sx}{\sqrt{x}}\,dx \quad [\because \cos sx = \frac{e^{isx}+e^{-isx}}{2}]$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \text{R.P. } \frac{e^{-isx}}{\sqrt{x}}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\text{ R.P. }\int_{0}^{\infty} \frac{e^{-isx}}{\sqrt{isx}\,\sqrt{is}}\,d(isx)$$

$$= \sqrt{\frac{2}{\pi}}\text{ R.P. }\frac{1}{\sqrt{is}}\int_{0}^{\infty} \frac{e^{-T}dT}{\sqrt{T}} \text{ where } T = isx$$

$$= \sqrt{\frac{2}{\pi}}\text{ R.P. }\frac{(i)^{-1/2}}{\sqrt{s}}\int_{0}^{\infty} e^{-T}T^{-1/2}\,dT$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{s}}\text{ R.P. }\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]^{-1/2}\Gamma_{1/2}$$

$$[\because i = \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}]$$

$$= \frac{1}{\sqrt{s}}\sqrt{\frac{2}{\pi}}\text{ R.P. }\left[\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right]\sqrt{\pi} \quad [\because \Gamma_{1/2}=\sqrt{\pi}]$$

$$= \frac{1}{\sqrt{s}}\sqrt{\frac{2}{\pi}}\cos\frac{\pi}{4}\sqrt{\pi}$$

$$= \sqrt{\frac{2}{s}}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

$$\therefore F\left[\frac{1}{\sqrt{|x|}}\right] = \frac{1}{\sqrt{s}}$$

Example 4.2.a(10) : Find the Fourier transform of $e^{-a^2x^2}$, $a > 0$, Hence, show that $e^{-x^2/2}$ is self reciprocal under Fourier transform.

[A.U N/D 2014 R-08, 13] [A.U A/M 2015] [A.U N/D 2016 R-13]

Solution : Given : $f(x) = e^{-a^2x^2}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

We know that,
$A^2 - 2AB = (A-B)^2 - B^2$
Here, $A = ax$
$2AB = isx$
$2ax\,B = isx$
$B = \frac{is}{2a}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-[(ax)^2-isx]}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left[(ax)-\left(\frac{is}{2a}\right)\right]^2-\left(\frac{is}{2a}\right)^2\right]}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Put $y = ax - \frac{is}{2a}$ | $x \to -\infty \Rightarrow y \to -\infty$

$dy = a\,dx$ | $x \to \infty \Rightarrow y \to \infty$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-y^2} \frac{1}{a} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a}\right) e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a}\right) e^{-s^2/4a^2} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

$$F[e^{-a^2x^2}] = \frac{1}{\sqrt{2}\,a} e^{-s^2/4a^2} \qquad \ldots (1)$$

Put $a = \frac{1}{\sqrt{2}}$, we get

$$F[f(x)] = e^{-s^2/2}$$

i.e., $F[e^{-x^2/2}] = e^{-s^2/2}$

Hence, $e^{-x^2/2}$ is self reciprocal under Fourier transform.

> **Note :** e^{-x^2} **is not self reciprocal, because, by putting $a = 1$ in (4), we get** $F[e^{-x^2}] = \frac{1}{\sqrt{2}} e^{-s^2/4} \neq e^{-s^2}$

Example 4.2.a(11) : Find the Fourier transform of Dirac delta function $\delta(t - a)$.

Solution : The Dirac delta function is defined as

$$\delta(t-a) = \lim_{h \to 0} I(h, t-a), \text{ where}$$

$$I(h, t-a) = \begin{cases} \frac{1}{h} & \text{for } a < t < a+h \\ 0 & \text{for } t < a \text{ and } t > a+h \end{cases}$$

The Fourier transform of $\delta(t-a)$ is

$$F[\delta(t-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \delta(t-a)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \int_{a}^{a+h} \frac{1}{h} e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \frac{1}{h} \left[\frac{e^{ist}}{is}\right]_a^{a+h}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \frac{1}{h} \left[\frac{e^{is(a+h)} - e^{ias}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \left[\frac{e^{ias}(e^{ish} - 1)}{ish}\right]$$

$$= \frac{e^{ias}}{\sqrt{2\pi}} \lim_{h \to 0} \left[\frac{e^{ish} - 1}{ish}\right]$$

$$= \frac{e^{ias}}{\sqrt{2\pi}} \qquad \left[\text{since } \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta}\right) = 1\right]$$

II.(b) Problems based on Fourier transform and its inversion formula

Formula :

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

Example 4.2.b(1) : Find the Fourier transform of the function f (x) defined by $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$

Hence, prove that $\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$

[A.U. April, 1996, 2000, 2001, M/J 2006] [A.U. CBT Dec. 2008] [A.U. CBT Dec. 2009][A.U. M/J 2016 R-8] [A.U N/D 2016 R4]

Solution : The given function can be written as

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

We know that,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, [\cos sx + i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^1 (1 - x^2) \cos sx\, dx \right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because$ $(1 - x^2) \cos sx$ is an even function in $(-1, 1)$

$(1 - x^2) \sin sx$ is an odd function in $(-1, 1)$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x^2) \cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1 - x^2) \left[\frac{\sin sx}{s}\right] - (-2x) \left[\frac{-\cos sx}{s^2}\right] + (-2) \left[\frac{-\sin sx}{s^3}\right] \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[(1 - x^2) \frac{\sin sx}{s} - 2x \frac{\cos sx}{s^2} + \frac{2 \sin sx}{s^3} \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(0 - \frac{2 \cos s}{s^2} + \frac{2 \sin s}{s^3} \right) - (0 - 0 + 0) \right]$$

$$= \frac{4}{s^3 \sqrt{2\pi}} [\sin s - s \cos s] \quad \ldots (1)$$

By Fourier inversion formula, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{s^3 \sqrt{2\pi}} (\sin s - s \cos s)\, e^{-isx}\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) \cos sx\, ds - i \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) \sin sx\, ds \quad \ldots (2)$$

$$= \frac{2}{\pi} \left[2 \int_0^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) \cos sx\, ds \right] - i \frac{2}{\pi} [0]$$

$$\because \frac{\sin s - s\cos s}{s^3}\cos sx \text{ is an even function in } (-\infty, \infty)$$
$$\frac{\sin s - s\cos s}{s^3}\sin sx \text{ is an odd function in } (-\infty, \infty)$$

i.e., $f(x) = \dfrac{4}{\pi}\displaystyle\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos sx\, ds$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos sx\, ds = \frac{\pi}{4} f(x) \qquad \ldots (3)$$

Put $x = \dfrac{1}{2}$, we get

$$\left.\begin{matrix} f(x) \\ \text{at } x = \frac{1}{2} \end{matrix}\right\} = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

[$\because x = \dfrac{1}{2}$ is a point of continuity is $-1 < x < 1$]

$$\therefore (3) \Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\, ds = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16}$$

Example 4.2.b(2) : Find the Fourier Transform of $f(x) = \begin{cases} a - |x|, & \text{for } |x| < a \\ 0, & \text{for } |x| > a > 0 \end{cases}$ **hence deduce that**

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

[A.U Nov/Dec, 1996] [A.U. CBT Dec. 2008] [A.U Tvli N/D 2011] [A.U A/M 2015 R-08]

Solution : We know that,

$$\boxed{F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

Here, $f(x) = \begin{cases} a - |x|, & \text{for } |x| < a \text{ (i.e.,) } -a < x < a \\ 0, & \text{otherwise} \end{cases}$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} [a - |x|]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} [a - |x|](\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} [a - |x|]\cos sx\, dx + i\int_{-a}^{a} [a - |x|]\sin sx\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a - x)\cos sx\, dx + 0\right]$$

$$\boxed{\begin{array}{l} \text{Since, } [a - |x|]\cos sx \text{ is an even function in } (-a, a) \\ \qquad [a - |x|]\sin sx \text{ is an odd function in } (-a, a)\ \& \\ \qquad a - |x| = a - x \text{ in } (0, a) \end{array}}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a [a - x]\cos sx\, dx \qquad [\because |x| = x \text{ in } (0, a)]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a - x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[(a - x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(0 - \frac{\cos as}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos as}{s^2} + \frac{1}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{s^2}[1 - \cos as]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[2\sin^2\frac{as}{2}\right] \qquad \ldots (1)$$

By inversion formula,

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\, \frac{1}{s^2} \sin^2 \frac{as}{2}\, e^{-isx}\, ds$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{s^2} \sin^2 \frac{as}{2}\, e^{-isx}\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin \frac{as}{2}}{s}\right]^2 e^{-isx}\, ds$$

We have to deduce that $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$ in the above integrand put

$x = 0$ and $a = 2$, we get

$$(1) \Rightarrow f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}\, ds$$

$$= \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2 s}{s^2}\, ds \quad \left[\because \frac{\sin^2 s}{s^2} \text{ is an even function}\right]$$

$$\int_0^{\infty} \frac{\sin^2 s}{s^2}\, ds = \frac{\pi}{4} f(0)$$

$$= \frac{\pi}{4} [2] \qquad \left[\because f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & \text{otherwise} \end{cases}\right.$$

$f(0) = a$ but Here $a = 2$

$$\text{(i.e.,)} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2} \qquad [\because s \text{ is a dummy variable}]$$

Note :

Even function :

If $f(-x) = f(x)$ in $(-l, l)$ then $f(x)$ is an even function.

Odd function :

If $f(-x) = -f(x)$ in $(-l, l)$ then $f(x)$ is an odd function

In the above problem, $a \cos sx$ is an even function,

$a \sin sx$ is an odd function

$|x|$ is an even function,

$|x| \cos sx$ is an even function

$|x| \sin sx$ is an odd function.

Example 4.2.b(3): Find Fourier transform of $e^{-a|x|}$ [AU N/D 2012] and hence deduce that

(i) $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}}\, \frac{2as}{(s^2+a^2)^2}$ [A.U N/D 2016 R-8]

(ii) $\int_0^{\infty} \frac{\cos xt}{a^2+t^2}\, dt = \frac{\pi}{2a} e^{-a|x|}$ [A.U A/M 2015 R-08]

[A.U. Oct.2001, M/J 2007 M/J 2014][A.U N/D 2014 R-08]

Solution : We know that,

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$F[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_0^{\infty} e^{-ax} \cos sx\, dx \qquad [\because |x| = x \text{ in } (0, \infty)]$$

Since, $e^{-a\|x\|}\cos sx$ is an even function in $(-\infty, \infty)$ $\therefore \int_{-\infty}^{\infty} e^{-a\|x\|}\cos sx\,dx = 2\int_0^{\infty} e^{-ax}\cos sx\,dx$
$e^{-a\|x\|}\sin sx$ is an odd function, $(-\infty, \infty)$ $\int_{-\infty}^{\infty} e^{-a\|x\|}\sin sx\,dx = 0$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right] \quad \boxed{\because \text{Formula: } \int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}}$$

Using inversion formula, we get

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\,ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\,a\int_{-\infty}^{\infty}\frac{e^{-isx}}{s^2+a^2}\,ds$$

$$= \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx - i\sin sx}{s^2+a^2}\,ds$$

$$= \frac{a}{\pi}\left[2\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds\right]$$

Since, $\frac{\cos sx}{s^2+a^2}$ is an even function in $(-\infty, \infty)$ $\therefore \int_{-\infty}^{\infty}\frac{\cos sx}{s^2+a^2}\,ds = 2\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds$
$\frac{\sin sx}{s^2+a^2}$ is an odd function in $(-\infty, \infty)$ $\therefore \int_{-\infty}^{\infty}\frac{\sin sx}{s^2+a^2}\,ds = 0$

$$f(x) = \frac{2a}{\pi}\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds$$

$\frac{d}{dx} e$

$$\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds = \frac{\pi}{2a}f(x)$$

$$\int_0^{\infty}\frac{\cos xt}{a^2+t^2}\,dt = \frac{\pi}{2a}e^{-a|x|} \quad [\because s \text{ is a dummy variable}]$$

$$F[xe^{-a|x|}] = -i\frac{d}{ds}F(s)$$

$$= -i\frac{d}{ds}F[e^{-a|x|}]$$

$$= -i\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right]$$

$$= -i\sqrt{\frac{2}{\pi}}\,a\,\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right]$$

$$= -ia\sqrt{\frac{2}{\pi}}\left[\frac{0-2s}{(s^2+a^2)^2}\right]$$

$$= i\sqrt{\frac{2}{\pi}}\ \frac{2as}{(s^2+a^2)^2}$$

Example 4.2.b(4): Find the Fourier transform of $e^{-|x|}$ and hence find the Fourier transform of $e^{-|x|}\cos 2x$.
[A.U CBT A/M 2011] [A.U A/M 2015 R-2008]

Solution : We know that, $\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$

$$F[e^{-|x|}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}\cos sx\, dx + 0$$

> Since, $e^{-|x|}\cos sx$ is an even function in $(-\infty, \infty)$
> $e^{-|x|}\sin sx$ is an odd function in $(-\infty, \infty)$ and
> $|x| = x$ in $(0, \infty)$

$$\therefore F(s) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-x}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+s^2}\right] \qquad \left[\because \int_0^{\infty} e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}\right]$$

To find : $F[e^{-|x|}\cos 2x]$

By Modulation theorem,

$$\boxed{F[f(x)\cos ax] = \frac{1}{2}[F(s-a) + F(s+a)]}$$

$$F[e^{-|x|}\cos 2x] = \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\left[\frac{1}{(s-2)^2+1}\right] + \sqrt{\frac{2}{\pi}}\left[\frac{1}{(s+2)^2+1}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{s^2-4s+5} + \frac{1}{s^2+4s+5}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s^2+4s+5+s^2-4s+5}{(s^2+5)^2-(4s)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\ \frac{2(s^2+5)}{s^4+10s^2+25-16s^2}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+5}{s^4-6s^2+25}\right]$$

II.(c) Problems based on inversion formula, Parseval's identity and Convolution theorem

Formula :

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

Example 4.2.c(1) : Find the Fourier transform of

$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a \end{cases}$ **where a is a positive real number.**

Hence deduce that (i) $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$ **and (ii)** $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

[A.U. March, 1996] [A.U N/D 2007, A/M 2008, M/J 2013]

[A.U A/M 2015 R-13] [A.U M/J 2016 R-8]

Solution :

The given function can be written as $f(x) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_0^a \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}} [0]$$

$\because$ cos sx is an even function in $(-a, a)$ & sin sx is an odd function in $(-a, a)$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos sx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) \quad \text{... (1)}$$

(i) Now, by **Fourier inversion formula,** we have

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) (\cos sx - i \sin sx)\, ds \quad \text{[using (1)]}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \sin sx\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - \frac{i}{\pi}(0)$$

$[\because \left(\frac{\sin as}{s}\right)$ sin sx is an odd function in $(-\infty, \infty)]$

$$= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds$$

$[\because \left(\frac{\sin as}{s}\right)$ cos sx is an even function in $(-\infty, \infty)]$

$$\int_0^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds = \frac{\pi}{2} f(x) \quad \text{... (2)}$$

Put $x = 0 \Rightarrow \left.\begin{matrix} f(x) \\ at\ x = 0 \end{matrix}\right\} = 1$

$$(2) \Rightarrow \int_0^{\infty} \frac{\sin as}{s}\, ds = \frac{\pi}{2} \quad \text{... (3)}$$

put $as = t$	$s \to 0 \Rightarrow t \to 0$
$a\, ds = dt$	$s \to \infty \Rightarrow t \to \infty$
$\Rightarrow ds = \frac{1}{a} dt$	

$$(3) \Rightarrow \int_0^{\infty} \frac{\sin t}{(t/a)} \frac{1}{a} dt = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

(ii) Using Parseval's identity

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds\,,}$$

We get, $\displaystyle\int_{-a}^{a} 1\,dx = \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\,\frac{\sin as}{s}\right)^2 ds$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds = a\pi$$

put $t = as$	$s \to -\infty \Rightarrow t \to -\infty$
$dt = a\,ds$	$s \to \infty \Rightarrow t \to \infty$
$ds = \frac{1}{a}\,dt$	

$$\int_{-\infty}^{\infty}\left(\frac{\sin t}{\frac{t}{a}}\right)^2 \frac{dt}{a} = a\pi$$

$$2\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi \quad \left[\because \left(\frac{\sin t}{t}\right)^2 \text{ is an even function}\right]$$

$$\therefore \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Example 4.2.c(2) : Find the Fourier Transform of

$$f(x) = \begin{cases} 1-|x| & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$ **Hence deduce that**

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}\,,\quad \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

[A.U. April, 2001, May, 2001]
[A.U. A/M 2005, N/D 2005, M/J 2006, N/D 2007, CBT Dec. 2008]
[A.U N/D 2009, A.U Tvli. N/D 2010, A.U CBT N/D 2010]
[A.U N/D 2011, N/D 2012] [CBT N/D 2011] [A.U N/D 2014 R-13]
[A.U N/D 2015 R-13] [A.U N/D 2016 R-13]

Solution : Given $f(x) = \begin{cases} 1-|x|\,, & -1 < x < 1 \\ 0 & , x < -1 \text{ and } x > 1 \end{cases}$

The Fourier transform of $f(x)$ is

$$\boxed{F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\cos sx\,dx + \frac{i}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\sin sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{1}(1-x)\cos sx\,dx + 0$$

[$\because$ $(1-|x|)\sin sx$ is an odd function in $(-1, 1)$]

[$\because$ $(1-|x|)\cos sx$ is an even function in $(-1, 1)$]

[$\because$ $|x| = x$ in $(0, 1)$]

$$= \sqrt{\frac{2}{\pi}}\left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(0 - \frac{\cos s}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right]$$

Put $x = 0$, we get

$$(2) \Rightarrow f(0) = \frac{4}{\pi}\int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\,dt = 1, \text{ when } a = 1$$

$$\Rightarrow \boxed{\int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}}$$

(ii) Using **Parseval's identity,**

$$\boxed{\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx} \quad \ldots (1)$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,ds = \int_{-1}^{1} (1-x^2)^2\,dx \qquad [\text{Here, } a = 1]$$

$$= 2\int_0^1 \left[1 + x^4 - 2x^2\right] dx = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[\left(1 + \frac{1}{5} - \frac{2}{3}\right) - (0 + 0 - 0)\right] = 2\left[\frac{8}{15}\right] = \frac{16}{15} \quad \ldots (2)$$

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \frac{8}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds \qquad [\text{Here, } a = 1]$$

$$= \frac{16}{\pi}\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds \quad \ldots (3)$$

$$(1) \Rightarrow \frac{16}{\pi}\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

i.e., $\displaystyle\int_0^{\infty} \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt = \frac{\pi}{15}$ [$\because$ s is a dummy variable]

Example 4.2.c(4) : Find the Fourier transform of $e^{-a|x|}$ if $a > 0$

Deduce that $\displaystyle\int_0^{\infty} \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}$ if $a > 0$.

Solution : Given : $f(x) = e^{-a|x|}$

See Example 4.2.b(3) in page no. 4.53

$$F(s) = F[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

By Parseval's identity,

If F(S) is the Fourier transform of $f(x)$, then

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds} \quad \ldots (1)$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} \left[e^{-ax}\right]^2 dx = 2\int_0^{\infty} \left[e^{-ax}\right]^2 dx$$

$$= 2\int_0^{\infty} e^{-2ax}\,dx = 2\left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty}$$

$$= -\frac{1}{a}\left[e^{-2ax}\right]_0^{\infty} = \frac{1}{a}[0 - 1] = \frac{1}{a} \quad \ldots (2)$$

$$|F(s)|^2 = \frac{2}{\pi}\,\frac{a^2}{(s^2+a^2)^2} \quad \ldots (3)$$

$$(1) \Rightarrow \frac{1}{a} = \int_{-\infty}^{\infty} \frac{2}{\pi}\,\frac{a^2}{(s^2+a^2)^2}\,ds \qquad \text{by (2) \& (3)}$$

$$= \frac{4a^2}{\pi}\int_0^{\infty} \frac{1}{(s^2+a^2)^2}\,ds$$

$$\int_0^{\infty} \frac{1}{(s^2+a^2)^2}\,ds = \frac{\pi}{4a^3}$$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3} \text{ if } a > 0 \; [\because s \text{ is a dummy variable}]$$

Example 4.2.c(5) : Find the Fourier transform of

$$f(x) = 1 - x^2 \text{ if } |x| < 1$$
$$= 0 \quad \text{if } |x| \geq 1$$

Hence, show that $\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2}\, ds = \frac{3\pi}{16}$

Also show that $\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}\, dx = \frac{\pi}{15}$.

[A.U N/D 2013]

Solution : See Example 4.2.b(1) in page no. 4.48 for problems based on Fourier transform and its inversion formula

$$F(s) = F[f(x)] = \frac{4}{s^3\sqrt{2\pi}}[\sin s - s\cos s]$$

Now, if $F[f(x)] = F(s)$ by **Parseval's identity**

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds \quad \text{... (1)}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-1}^{1} (1-x^2)^2 dx = 2\int_0^1 (1-x^2)^2 dx$$

$$= 2\int_0^1 \left[1 + x^4 - 2x^2\right] dx = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right] = 2\left(\frac{8}{15}\right) = \frac{16}{15}$$

$$|F(s)|^2 = \frac{16}{s^6(2\pi)}[\sin s - s\cos s]^2 = \frac{8}{s^6\pi}[\sin s - s\cos s]^2$$

$$(1) \Rightarrow \quad \frac{16}{15} = \int_{-\infty}^{\infty} \frac{8}{\pi s^6}[\sin s - s\cos s]^2 ds$$

$$= \frac{16}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6} ds$$

$$\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6} ds = \frac{\pi}{15}$$

i.e., $\int_0^\infty \frac{(\sin x - x\cos x)^2}{x^6} dx = \frac{\pi}{15}$ [$\because$ s is a dummy variable]

i.e., $\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15}$ [$\because (a-b)^2 = (b-a)^2$]

Example 4.2.c(6) : Find the Fourier transform of f (x) given by

$$f(x) = 1 \text{ for } |x| < 2$$
$$= 0 \text{ for } |x| > 2$$

and hence, evaluate $\int_0^\infty \frac{\sin x}{x} dx$ **and** $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$.

[A.U. Nov/Dec.2003] [A.U A/M 2017 R-13]

Solution : The given equation can be written as

$$f(x) = 1 \text{ if } -2 < x < 2$$
$$= 0 \text{ otherwise}$$

$$F(s) = F[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} \cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-2}^{2} \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^2 \cos sx\, dx\right] + \frac{i}{\sqrt{2\pi}}[0]$$

[$\because$ $\cos sx$ is an even function in $(-2, 2)$ & $\sin sx$ is an odd function in $(-2, 2)$]

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_{x=0}^{x=2}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin 2s}{s} - 0\right]$$

$$F(s) = F[f(x)] = \sqrt{\frac{2}{\pi}}\frac{\sin 2s}{s}$$

(i) Now, by Fourier inversion formula, we have

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin 2s}{s}(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2s}{s}\cos sx\, ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2s}{s}\sin sx\, dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2s}{s}\cos sx\, dx - \frac{i}{\pi}(0) \quad [\because \frac{\sin 2s}{s}\sin sx \text{ is an odd function in } (-\infty, \infty)]$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin 2s}{s}\cos sx\, dx \quad [\because \frac{\sin 2s}{s}\cos sx \text{ in an even function in } (-\infty, \infty)]$$

$$\text{(i.e.,)} \int_0^{\infty}\left(\frac{\sin 2s}{s}\right)\cos sx\, dx = \frac{\pi}{2} f(x)$$

put $x = 0$, we get $\int_0^{\infty}\frac{\sin 2s}{s}\, ds = \frac{\pi}{2} f(0) = \frac{\pi}{2}$

$$\therefore \int_0^{\infty}\frac{\sin 2s}{s}\, ds = \frac{\pi}{2} \quad [\because f(0) = 1 \text{ in } |x| < 2]$$

Now, put $t = 2s$ $\quad s \to 0 \Rightarrow t \to 0$

$dt = 2\, ds$ $\quad s \to \infty \Rightarrow t \to \infty$

$$\therefore \int_0^{\infty}\frac{\sin t}{(t/2)}\frac{dt}{2} = \frac{\pi}{2}$$

Hence, $\int_0^{\infty}\frac{\sin t}{t}\, dt = \frac{\pi}{2}$. (i.e.,) $\int_0^{\infty}\frac{\sin x}{x}\, dx = \frac{\pi}{2}$ $\quad [\because$ t is a dummy variable$]$

(ii) Using Parseva's identity, $\boxed{\int_{-\infty}^{\infty}|f(x)|^2\, dx = \int_{-\infty}^{\infty}|F(s)|^2\, ds}$ ... (1)

$$\int_{-\infty}^{\infty}|f(x)|^2\, dx = \int_{-2}^{2} 1^2\, dx = \int_{-2}^{2} dx = [x]_{-2}^{2} = 2-(-2) = 4 \quad \text{... (2)}$$

$$|F(s)|^2 = \frac{2}{\pi}\frac{\sin^2 2s}{s^2}$$

$$\int_{-\infty}^{\infty}|F(s)|^2\, ds = \int_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sin^2 2s}{s^2}\, ds = \frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 2s}{s^2}\, ds$$

put	$t = 2s$	$\Rightarrow dt = 2\, ds$
	$s \to 0$	$\Rightarrow t \to 0,\ s \to \infty \Rightarrow t \to \infty$

$$= \frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 t}{(t/2)^2}\frac{1}{2}\, dt = \frac{8}{\pi}\int_0^{\infty}\frac{\sin^2 t}{t^2}\, dt \quad \text{... (3)}$$

$$(1) \Rightarrow 4 = \frac{8}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt \quad \text{by (2) \& (3)}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2} \quad [\because t \text{ is a dummy variable}]$$

Example 4.2.c(7) : Find the Fourier transform of $e^{-|x|}$, using Parseval's identity show that $\int_0^{\infty}\frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$.

Solution : $F[e^{-|x|}] = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+s^2}\right)$ already proved.

By Parseval's identity,

$$\boxed{\int_{-\infty}^{\infty}|F(s)|^2\, ds = \int_{-\infty}^{\infty}|f(x)|^2\, dx} \quad \text{... (1)}$$