4

FOURIER TRANSFORMS

Definition. If $f(x)$ is defined in (a, b), the integral transform of $f(x)$ with the Kernel $K(s, x)$ is defined by

$$F(s) = \bar{f}(s) = \int_a^b f(x)K(s,x)\,dx, \text{ if the integral exists.}$$

Here, $K(s, x)$ is called the Kernel of the transform while a, b are fixed limits. If a, b are finite, the transform is finite and if a, b are infinite, it is an infinite transform.

If $K(s, x) = e^{-sx}$ for $x \geq 0$

$= 0$ for $x < 0$

the above infinite transform becomes the well known Laplace transform.

If $$F(s) = \int_a^b f(x)K(s,x)\,dx \qquad \text{... (1)}$$

it may be possible to get $f(x)$ as

$$f(x) = \int_c^d F(s)H(s,x)\,ds \qquad \text{... (2)}$$

In such case, (2) is called the inversion formula for (1). In (1), if $F(s)$ is known while $f(x)$ is unknown, it may be regarded as an integral equation.

From the general integral transform definition, we can get various integral transforms by properly defining the kernal.

Some of the well known integral transforms are listed below together with the corresponding inversion formula.

1. Infinite Fourier Transform (complex form)

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds \quad \text{(inversion formula)}$$

2. Infinite Fourier Cosine Transform

$$F_c(s) = \bar{f}_C(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos sx\,dx$$

3. Infinite Fourier Sine Transform

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \ \sin sx \ ds$$

4. Laplace Transform

$$F(s) = \int_0^\infty e^{-sx} f(x) dx$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{sx} ds$$

5. Hankel Transform

$$F_n(s) = \int_0^\infty x f(x) J_n(sx) dx$$

$$f(x) = \int_0^\infty s F_n(s) J_n(sx) ds$$

Where $J_n(x)$ is the Bessel function of the first kind of order n.

6. Mellin Transform

$$F(s) = \int_0^\infty f(x) x^{s-1} dx$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) x^{-s} ds$$

Theorem 1. *If f(x) satisfies Dirichlet's conditions in (–1, 1), then the complex Fourier series of f(x) is*

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} \quad \text{where} \qquad \ldots (1)$$

$$C_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}} dx \qquad \ldots (2)$$

(For proof, refer to chapter 'Fourier series', Page 92, Note 3)

Therem 2. (FOURIER INTEGRAL THEOREM)

If f(x) is a piece-wise continuously differentiable and absolutely integrable in (– ∞, ∞), then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i(x-t)^s} dt\, ds.$$

Proof. In theorem 1, substituting the value of C_n from (2) in (1), and changing the variable of integration in (2) as t, we get,

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2l} \int_{-l}^{l} f(t) e^{\frac{in\pi(x-t)}{l} dt} \right)$$

Let $\quad \dfrac{\pi}{l} = \delta s. \ (-1 \le x \le 1)$

$$\therefore \qquad f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2l} \int_{-l}^{l} f(t) e^{i(x-t)n\delta s} dt \right)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \delta s \left(\int_{-l}^{l} f(t) e^{i(x-t)n\delta s} dt \right) \text{ since } l = \frac{\pi}{\delta s}$$

Let $l \to \infty$, so that the range $(-l, l)$ becomes $(-\infty, \infty)$. As $l \to \infty$, $\delta s \to 0$. Changing the summation to definite integral, we get,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i(x-t)^s} dt\, ds.$$

Note 1. Using Euler's theorem, $e^{i\theta} = \cos\theta + i \sin\theta$, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \; [\cos (x-t)s + i \sin (x-t)] \; dt\, ds.$$

Equating real and imaginary parts on both sides ($f(x)$ real) we get,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \; \cos (x-t)\, s \; dt\, ds$$

and $$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \; \sin (x-t) \; dt\, ds = 0.$$

Complex Fourier Transform (Infinite)

Let $f(x)$ be a function defined in $(-\infty, \infty)$ and be piece-wise continuous in each finite partial interval and absolutely integrablein $(-\infty, \infty)$. Then the complex Fourier Transform of $f(x)$ is defined by

$$\overline{f}(s) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ixx} dx.$$

Theorem 3. Inversion Theorem for complex Fourier Transform

If f(x) satisfies the Dirichlet's conditions in every finite interval (–l, l) and if it is absolutely integrable in the range, and if

F(s)denotes the complex Fourier transform of f(x), then at every point of continuity of f(x), we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Proof. From Fourier integral theorem

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i(x-t)s} dt\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixs} \left(\int_{-\infty}^{\infty} f(t) e^{-its} dt \right) ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixv} \left(\int_{-\infty}^{\infty} f(t) e^{itv} dt \right) dv \quad \text{putting } s = -v$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixv} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{itv} dt \right) dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixv} (F(v) dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{isx} ds \text{ changing } v \text{ as } s.$$

Note 1. The parameter s is taken as p by some authors.

Note 2. Some authors define the Complex Fourier transforms in different forms, changing the Kernels of the transform. Various forms are listed below.

1. $$\overline{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \overline{f}(s) ds$$

2. $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} F(s) ds$$

3. $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(x) dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} F(s) ds$$

Properties of Fourier Transforms

Theorem 1. *Fourier transform is linear.*

i.e., $F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$ *where F stands for Fourier transform.*

$$F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(x) + bg(x)) e^{isx} dx$$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx + b \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= aF[f(x)] + bF[g(x)].$$

Theorem 2. Shifting theorem

If $F\{f(x)\} = F(s)$, then $F\{f(x - a)\} = e^{isa} F(s)$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i(a+i)s} dt \quad \text{putting } x - a = t$$

$$= e^{ias} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its} dt$$

$$= e^{ias} F(s).$$

Theorem 3. Change of scale property

If $F\{(f, x)\} = F(s)$, then $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$ where $a \neq 0$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)^t} f(t) \frac{dt}{a} \quad \text{putting } ax = t \text{ and } a > 0.$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) \quad \text{if } a > 0.$$

$$F\{f(ax) = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} ei\left(\frac{s}{a}\right)t\, f(t) \frac{dt}{a} \quad \text{if } a < 0$$

$$= -\frac{1}{a} F\left(\frac{s}{a}\right) \quad \text{if } a < 0$$

Hence $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$

Theorem 4. $F\{e^{iax} f(x)\} = F(s + a)\}$ ***(BR 1995 Ap.)***

$$F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)^x} f(x) dx$$

$$= F(s + a).$$

Theorem 5. Modulation theorem

If $F\{(f(x)\} = F(s)$, then

$$F\{f(x) \cos ax\} = \frac{1}{2} [F(s - a) + F(s - a)]$$

Proof:

$$F\{(f(x) \cos ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \cdot \frac{e^{ias} + e^{-iax}}{2} dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\,dx\right]$$

$$= \frac{1}{2}\,[F(s+a) + F(s-a)]$$

Theorem 6. *If $F\{f(x)\} = F(s)$, then $f\{x^n f(x)\} = (-i)n\ \dfrac{d^n}{ds^n}F(s)$*

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{usx} f(x)\,dx$$

Differentiating w.r.t s both sides, n times.

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (ix)^n e^{isx} f(x)\,dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{isx} f(x)dx$$

$$= (i)^n\, \mathrm{F}\{x^n f(x)\}$$

$$F\{x^n f(x)\} = (-i)^n\ \frac{d^n}{ds^n}\{F(s)\}$$

Theorem 7. *$F\{f'(x)\} = -is\,F(s)$ if $f(x)\ \ 0$ as $x \to \pm\infty$.*

$$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f'(x)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} d\{f(x)\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left\{e^{isx} f(x)\right\}_{-\infty}^{\infty} - \mathrm{is}\int_{-\infty}^{\infty} f(x)e^{isx}dx\right]$$

$$= -is\,F(s) \text{ if } f(x) \to 0 \text{ as } x \to \pm\infty$$

Cor.: $F\{f^n(x)\} = (-is)^n F(s)$ if $f, f', f'', \ldots\ldots f^{(n-1)} \to 0$ as $x \to \pm\infty$.

Theorem 8. $F\left\{\int_a^x f(x)\,dx\right\} = \dfrac{F(s)}{(-is)}$

Let $\qquad \phi(x) = \int_a^x f(x)\,dx$

Then, $\qquad \phi'(x) = f(x)$

$$F\{\phi'(x)\} = (-is)\ \bar{\phi}(s)$$

$$= (-is)\ F(\phi(x))$$

$$= (-is)\ F\int_a^x f(x)\,dx$$

$$F\left(\int_a^x f(x)\,dx\right) = \frac{1}{(-is)}F\{\phi'(x)\}$$

$$= \frac{1}{(-is)}F(f(x)) = \frac{F(s)}{(-is)}$$

Example 1. *Find the Complex Fourier transform of*

$f(x) = x$ for $|x| \leq a$

$= 0$ for $|x| > a$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} xe^{isx}\,dx, \text{ the other integrals vanish.}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{xe^{isx}}{is} - \frac{e^{isx}}{(is)^2}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{is}\left\{ae^{isa} + ae^{-isa}\right\} + \frac{1}{s^2}\left\{e^{isa} - e^{-isa}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2ai}{s}\cos sa + \frac{2i}{s^2}(\sin sa)\right]$$

$$= \frac{2i}{s^2}\cdot\frac{1}{\sqrt{2\pi}}[\sin sa - as\cos sa]$$

Example 2. *Find the Fourier transform (complex) of*

$$f(x) = e^{ikx},\ a < x < b$$
$$= 0,\ x < a \text{ and } x > b$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{ikx}\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(k+s)x}}{i\,(k+s)}\right]_a^b$$

$$= \frac{i}{\sqrt{2\pi}\,(k+s)}\left[e^{i\,(k+s)a} - e^{i(k+s)b}\right]$$

Example 3. *Find the Fourier transform (complex) of*

$$f(x) = \begin{cases} 1-x^2 \text{ if } & |x|<1 \\ 0 \quad\ \text{ if } & |x|>1 \end{cases}$$ ***(Anna Ap 2005)***

Hence evaluate $\displaystyle\int_0^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx} (1-x^2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2)\left(\frac{e^{isx}}{is}\right) - (-2x)\left(\frac{e^{isx}}{i^2 s^2}\right) + (-2)\left(\frac{e^{isx}}{t^3 s^3}\right) \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-2}{s^2}\{e^{is} + e^{-is}\} - \frac{2i}{s^3}\{e^{is} - e^{is}\} \right]$$

$$= -\frac{2}{s^3} \frac{1}{\sqrt{2\pi}} [2\ s \cos s - 2 \sin s]$$

$$= \frac{-4}{\sqrt{2\pi}} \left[\frac{s\cos s - \sin s}{s^3} \right]$$

Using inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{-4}{\sqrt{2\pi}} \left(\frac{s\cos s - \sin s}{s^3} \right) e^{-isx} ds \right.$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) e^{-isx} ds$$

$$\int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) (\cos sx - i \sin sx)\, ds$$

$$= -\frac{\pi}{2}(1-x^2) \quad \text{if } |x| < 1$$

$$= 0 \text{ if } |x| > 1$$

Equating real parts,

$$\int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) \cos sx\, ds = -\frac{\pi}{2}(1-x^2) \quad \text{if } |x| < 1$$

$$= 0 \text{ if } |x| > 1$$

Set $x = \dfrac{1}{2}$

$$\therefore \quad \int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) \cos \frac{s}{2} ds = -\frac{\pi}{2}\left(1 - \frac{1}{4}\right) = -\frac{3\pi}{8}$$

$$2\int_{0}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) \cos \frac{s}{2} ds = -\frac{3\pi}{8}$$

$$\therefore \quad \int_{0}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) \cos \frac{s}{2} ds = -\frac{3\pi}{16}$$

Example 4. *Show that the transform of* $e^{-\frac{x^2}{2}}$ *is* $e^{-\frac{s^2}{2}}$ *by finding the Fourier transform of* $e^{-a^2x^2}$*, a > 0.*

$$F\{e^{-a^2x^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha^2x^2} e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} e^{-\frac{s^2}{4a^2}}\,dx$$

$$= e^{-\frac{x^2}{4\alpha^2}} \cdot \frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,dt, \text{ putting } \alpha x - \frac{is}{2\alpha} = t \text{ and } \alpha > 0$$

$$= e^{-\frac{s^2}{4a^2}} \cdot \frac{1}{\alpha\sqrt{2\pi}} \times \sqrt{\pi} \text{ since } \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}a} e^{-\frac{s^2}{4\alpha^2}}$$

Setting $\alpha = \frac{1}{\sqrt{2}}$,

$$F\left\{e^{-\frac{x^2}{2}}\right\} = e^{-\frac{s^2}{2}}$$

That is, $e^{-\frac{x^2}{2}}$ is self reciprocal.

Example 5. Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} 1 \text{ for } |x| < a \\ 0 \text{ for } |x| > a > 0 \end{cases}$$ ***(Anna. Univ. 2002)***

and hence evaluate $\int_{0}^{\infty} \frac{\sin x}{x}\,dx$

and $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\,ds$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ixs}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ixs}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{e^{isx}}{is}\right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{is}\left[e^{ias} - e^{-ias}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$$

Using inversion formula, we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin as}{s}\right) e^{-ixs}\, ds = f(x) = \begin{cases} 1 \text{ for } |x| < a \\ 0 \text{ for } |x| > a \end{cases}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin as}{s}\right) (\cos xs - i \sin xs)\, ds = f(x)$$

Equating real parts,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos xs\, ds = f(x) = \begin{cases} 1 \text{ for } |x| < a \\ 0 \text{ for } |x| > a \end{cases}$$

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin as}{0} \cos xs\, ds = f(x)$$

$$\int_{0}^{\infty} \frac{\sin as}{s} \cos xs\, ds = \frac{\pi}{2} \text{ for } |x| < a$$

$$= 0 \text{ for } |x| > a$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4} \text{ for } |x| = a$$

Setting $x = 0$

$$\int_{0}^{\infty} \frac{\sin as}{s}\, ds = \frac{\pi}{2}$$

$$\therefore \quad \int_{0}^{\infty} \frac{\sin as}{x}\, dx = \frac{\pi}{2}$$

Setting $ax = \theta$, we get,

$$\int_{0}^{\infty} \frac{\sin \theta}{\theta}\, d\theta = \frac{\pi}{2}$$

Convolution Theorem or Faltung Theorem

Def. The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Theorem. *The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transforms.*

That is, $F\, f(x) * (x)\} = F(s).G(s) = F\{f(x)\} \,.\, F\{g(x)\}$

$$F\{f * g\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)\, e^{ixs}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g\,(x-t) dt \right) e^{ixs}\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left(\int_{-\infty}^{\infty} g(x-t) e^{ixs}\, dx \right) dt$$

by changing the order of integration

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, F\{g(x-t)\}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{its}\, G(s)\, dt, \text{ using shifting theorem}$$

$$= G(s). \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{its} dt$$

$= G(s)$. F(s)

$= F(s).G(s)$

By inversion,

F–1 {F (s) G (s)} = f * g = F–1 {F (s) * F–1 {G (s)}

Parseval's identity. If $F(s)$ is the Fourier transform of $f(x)$.

then, $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

Proof. Firstly let us prove

$F\overline{f(-x)\}} = \overline{F(s)}$ where $\overline{F(s)}$ indicates the complex conjugate of $F(s)$

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-v)}\, e^{isv} dv \quad \text{putting } x = -v$$

$$= F\{\overline{f(-v)}\}$$

$$= F\{\overline{f(-x)}\} \text{, changing the dummy variable} \qquad \text{... (1)}$$

By convolution Theorem,

$$F\{f(x)\} * g(x)\} = F(s)\ G(s)$$

$$f * g = F^{-1}\ \{F(s)\ G(s)\}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s) e^{-isx} ds$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} f(t)g(-t)\,dt = \int_{-\infty}^{\infty} F(s)G(s)\,ds \quad \text{... (2)}$$

Since it is true for all $g(t)$, take $g(t) = \overline{f(-t)}$

$$\therefore \quad g(-t) = \overline{f(t)}$$

$G(s) = F\{g(t)\} = F\{\overline{f(-t)}\} = \overline{F(s)}$ using (1). Use this in (2).

$$\int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\,ds$$

$$\therefore \quad \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

Example 6. Using Parseval's identity, prove $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

In example 5, if $f(x) = \begin{cases} 1 \text{ for } |x| < a \\ 0 \text{ for } |x| > a > 0 \end{cases}$

We have proved $F(s) = \sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$.

Using Parseval's identity.

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\therefore \quad \int_{-a}^{a} 1\,dt = \int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin as}{\pi}\right)^2 ds.$$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds.$$

Setting $as = t$, we get

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$$

$$\therefore \quad 2\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$$

$$\therefore \quad \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$$

Example 7. *Find the Fourier transform of f(x) = 1 – | x | if | x | < 1 and hence find the value* $\int_0^\infty \frac{\sin^4 t}{t^4} dt$. *= 0 for | x | > 1* ***(Anna Ap. 2005)***

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1-|x|)e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2\int_0^1 (1-x)\cos sx\,dx \quad \text{since } (1-|x|) \text{ is even}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)$$

Using Parseval's identity.

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)}{s^4} ds = \int_{-1}^{1} (1-|x|^2)dx$$

$$\frac{4}{\pi}\int_0^{\infty} \frac{(1-\cos s)^2}{s^4} ds = 2\int_0^1 (1-x)^2 dx = 2/3$$

$$\frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 s/2}{s^4} ds = 2/3$$

Setting $\frac{s}{2} = x,$ we get

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

Infinite Fourier cosine transform and sine transform:

Infinite Fourier cosine transform

Let $f(x)$ be defined for all $x \geq 0$. Now we shall extend the function $f(x)$ to the negative side of x axis. That is, let the extended function $F(x)$ be defined as

$F(x) = f(x)$ for $x \geq 0$

$= f(-x)$ for $x < 0$ so that $F(x)$ is even in $(-\infty, \infty)$. Now,

$$F\{F(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)[\cos sx + i\sin sx]\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \times 2\int_0^{\infty} F(x) \cos sx\, dx \quad \text{since } F(x) \text{ is even}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \text{ since } F(x) = f(x) \text{ for } x > 0$$

The R.H.S. is defined as the infinite Fourier cosine transform of $f(x)$ denote by $F_c(s)$

$\therefore$ Infinite Fourier cosine transform of $f(x)$ is defined to be

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \quad \text{... (1)}$$

Then by inversion theorem,

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F_c(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F_C(s) \{\cos sx - i \sin sx\} ds$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \int_0^\infty F_c(s) \cos sx \, dx \text{ since } F_c(s) \text{ is even}$$

Since $F(x) = f(x)$ in $(0, \infty)$,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds \quad \text{... (2)}$$

Equation (1) defines the Fourier cosine transform of $f(x)$ and (2) gives the Inversion Theorem for Fourier cosine transform.

Infinite Fourier Sine Transform

Let $f(x)$ be defined for $x \geq 0$. We extend $f(x)$ to the negative side of x axis.

Thus, let,

$$= -f(-x) \text{ for } x < 0$$

so that $F(x)$ is odd in $(-\infty, \infty)$

$$F\{F(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(x) \, [\cos sx + i \sin sx] dx$$

$$= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^\infty F(x) \sin sx \, dx \text{ since } F(x) \text{ is odd}$$

$$= i\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Now, we define Fourier Sine transform of $f(x)$ as

$F_s(s)$ = Imaginary point of $F\ \{F(x)\}$

$$F_s\ \{f(x)\} = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \quad \text{... (3)}$$

By inversion theorm,

$$F(x) = F^{-1}\ \{iF_s\ (s)\}$$

$$= i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_s(s)e^{isx}ds$$

$$f\,(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin sx\,ds \qquad \text{... (4)}$$

since $F_s(s)$ is odd function of s and $F(x) = f(x)$ in $(0, \infty)$

Equation (3) defines infinite Fourier sine transform of $f(x)$ and (4) gives the inversion theorem for Fourier sine transform.

Hence the transform pairs are

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx;\ f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos sx\,ds$$

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx;\ f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin sx\,ds$$

Properties regarding cosine and sine transforms

1. Cosine and sine transforms are linear.

   That is,

   $F_c\{af(x) + bg\ (x) = aF_c\ \{f(x)\} + bF_c\ [g(x)\}$

   The proof is obvious.

2. $F_s\ [af\,(x) + bg(x)] = aF_s\ \{f(x)\} + bF_s\ \{g(x)\}$

3. $F_s\ [f(x)\sin ax] = \frac{1}{2}\,[F_c(s-a) - F_c\ (s+a)]$

4. $F_s\,[\,f(x)\cos ax] = \frac{1}{2}\ [F_s\ (s+a) + F_s\ (s-a)]$

5. $F_c\ [f(x)\sin ax] = \frac{1}{2}\ [F_s\ (a+s) + Fs\ (a-s)]$

6. $F_e\ [f(x)\cos ax] = \frac{1}{2}\ [F_c\ (s+a) + F_c\ (s-a)]$

Proof. $F_s\ [\,f(x)]\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \sin sx\ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot[\cos\,(s-a)x - \cos\,(a+s)x]\ dx$$

$$= \frac{1}{2}\ [F_c(s-a) - F_c\ (s+a)]$$

$$F_s\ [f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\sin sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x).\,[\sin (s + a)x + \sin (s - a)]dx$$

$$= \frac{1}{2}[F_s(s+a)+F_s(s-a)]$$

Similarly, we can prove the results (5) and (6).

7. $F_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{s}{a}\right)$

8. $F_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{s}{a}\right)$

Hint. put $ax = u$ and proceed.

Identities

If $F_c(s)$, $G_c(s)$ are the Fourier cosine transforms and $F_s(s)$, $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

1. $\int_0^\infty f(x)g(x)\,dx = \int_0^\infty f_c(s)G_c(s)ds$

2. $\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s(s)G_s(s)\,ds$

3. $\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |f_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds.$

Proof. $\int_0^\infty F_s(s)G_s(s)ds = \int_0^\infty F_s(s)\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\sin sx\,dx\right\}ds$

$$= \int_0^\infty g(x)\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds\right\}dx$$

$$= \int_0^\infty g(x)f(x)\,dx$$

Similarly, we can prove the first identity and the third follows by setting $g(x) = f(x)$.

Example 8. *Find Fourier cosine and sine transforms of e^{-ax}, $a > 0$ and hence deduce the inversion formula.*

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2},\ \text{if } a > 0 \qquad \text{... (1)}$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2} \text{ if } a > 0 \quad \text{... (2)}$$

By inversion formula of (1).

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2} \cdot \sin sx \right) ds$$

$$\therefore \quad \int_0^\infty \frac{s}{a^2+s^2} \sin sx\, ds = \frac{\pi}{2} e^{-ax}, \ a > 0$$

Changing the variables,

$$\int_0^\infty \frac{x \sin \alpha x}{a^2+x^2} dx = \frac{\pi}{2} e^{-\alpha a}, a > 0 \quad \text{... (3)}$$

Again, by inversion formula of (2)

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2} \right) \cos sx\, ds$$

$$\therefore \quad \int_0^\infty \frac{\cos sx}{a^2+s^2} ds = \frac{\pi}{2a} e^{-ax}$$

Changing the variables,

$$\int_0^\infty \frac{\cos \alpha x}{a^2+x^2} dx = \frac{\pi}{2a} e^{-\alpha a}, a > 0 \quad \text{... (4)}$$

Example 9. *Find Fourier sine transform of* $\frac{x}{a^2+x^2}$ *and Fourier cosine transform of* $\frac{1}{a^2+x^2}$.

$$F_s\left(\frac{x}{a^2+x^2} \right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{a^2+x^2} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} e^{-as} \right] \text{ using (3) of Example 8.}$$

$$= \sqrt{\frac{\pi}{2}} e^{-as}$$

$$F_c\left(\frac{1}{a^2+x^2} \right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{a^2+x^2} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2a} e^{-as} \right] \text{ using (4) of Example 8.}$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{a} e^{-as}.$$

Example 10. *Using Parseval's identity evaluate*

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2} \quad \text{and} \quad \int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx \quad \text{if } a > 0.$$

By example 8. If $f(x) = e^{-ax}$, $F_s(s) = \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$

and $$F_c(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

Using Parseval's identity.

$$\int_0^\infty \frac{2}{\pi} \frac{s^2}{(a^2+s^2)^2} ds = \int_0^\infty e^{-2ax} dx$$

$$= \left(\frac{e^{-2ax}}{-2a} \right)_0^\infty$$

$$= \frac{1}{2a}$$

$$\therefore \quad \int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx = \frac{\pi}{4a} \cdot \text{ if } a > 0$$

Also $$\int_0^\infty \frac{2}{\pi} \frac{a^2}{(a^2+s^2)^2} ds = \int_0^\infty e^{-2ax} dx = \frac{1}{2a}$$

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3} \text{ if } a > 0$$

Example 11. *Evaluate* $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ *using transform methods.*

Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \text{ by example 8}$$

similary,

$$G_c(s) = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$$

$\therefore$ Using, $$\int_0^\infty F_c(s) G_c(s) ds = \int_0^\infty f(x) g(x) dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}ds = \int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{1}{a+b}$$

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}, \text{ if } a,\, b > 0.$$

Example 12. *Find Fourier cosine transform of*

$$f(x) = \begin{cases} \cos x \text{ in } 0 < x < a \\ 0 \qquad\qquad x \geq a \end{cases}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a \cos x\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^a [\cos(s+1)x+\cos(s-1)x]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1}+\frac{\sin(s-1)a}{s-1}\right] \text{ if } s \neq 1, -1$$

Example 13. *Find Fourier sine transform of* $\frac{1}{x}$ ***(Anna Ap. 2005)***

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta}{\theta}d\theta, \text{ putting } sx = \theta$$

$$= \sqrt{\frac{2}{\pi}}\times\frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

Example 14. *Find Fourier sine and cosine transform of* x^{n-1}.

We know, $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, n > 0$

$$= \int_0^\infty e^{-ax}(ax)^{n-1}a\,dx \text{ setting } x \text{ as (as } ax,\, a > 0)$$

$$\therefore \quad \int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}\cdot n > 0, a > 0$$

We can prove the above result even if a is complex.

Setting $$a = is,$$

$$\int_0^\infty e^{-isx} x^{n-1} dx = \frac{\Gamma(n)}{(is)^n}$$

$$= \frac{(-i)^a \Gamma(n)}{s^n}$$

$$= \frac{e^{-\frac{\pi}{2}ni}}{s^n}\Gamma(n) \text{since} - i = 3^{-\frac{\pi}{2}i}$$

Equating real and imaginary parts on both sides, we get

$$\int_0^\infty x^{n-1}\cos sx\,dx = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2}$$

$$\int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}$$

$$\therefore \quad F_c(x^{n-1}) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2}$$

$$F_s(x^{n-1}) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}$$

Taking $n = 1/2$.

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}} \text{since}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$$

Note: $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

Example 15. *Show that*

(i) $Fs[xf(x)] = -\frac{d}{ds}F_c(s)$

(ii) $Fc[xf(x)] = \frac{d}{ds}F_s(s)$ and hence find Fourier cosine and sine transform of xe^{-ax}.

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$\frac{d}{ds}F_c(s) = -\sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\sin sx\,dx$$

$$= -F_s\{xf(x)\}$$

Similarly,

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx \, dx$$

$$= F_c \{xf(x)\}$$

$$F_c(xe^{-ax}) = \frac{d}{ds} F_s(e^{-ax})$$

$$= \frac{d}{ds}\left(\sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}\right) \text{ using example 8}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(a^2+s^2)^2}$$

$$F_s(xe^{-ax}) = -\frac{d}{ds} F_c(e^{-ax})$$

$$= -\frac{d}{ds}\left(\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}\right) \text{ (Refer example 8)}$$

$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(a^2+s^2)^2}$$

Example 16. *Find Fourier cosine transform of* $e^{-a^2x^2}$ *and hence evaluate, Fourier sine transform of* $x\,e^{-a^2x^2}$

$$F_c(e^{-a^2x^2}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} . \cos sx \, dx$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2+isx} dx$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \text{ (Refer example 4)}$$

$$F_s\left(xe^{-a^2x^2}\right) = -\frac{d}{ds} F_c(e^{-a^2x^2})$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \frac{s}{2a^2}$$

$$= \frac{s}{2a^3\sqrt{2}} e^{-\frac{s}{4a^2}}$$

Example 17. *Solve for f(x) from the integral equation*

$$\int_0^\infty f(x)\cos \alpha x\, dx = e^{-\alpha}$$

Proof. Multiplying by $\sqrt{\frac{2}{\pi}}$, we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos \alpha x = \sqrt{\frac{2}{\pi}}\, e^{-\alpha}$$

$$F_c\,\{f(x)\} = \sqrt{\frac{2}{\pi}}\, e^{-\alpha},\ \alpha \text{ parameter}$$

$$\therefore \qquad f(x) = F_c^{-1}\left(\sqrt{\frac{2}{\pi}}\, e^{-\alpha}\right)$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\alpha}\cos \alpha x\, d\alpha$$

$$= \frac{2}{\pi}\cdot\frac{1}{1+x^2} \text{ on integration}$$

Example 18. *Solve for f(x) from the integral equationz*

$$\int_0^\infty f(x)\sin sx\, dx = \begin{cases} 1 \text{ for } 0 \le s < 1 \\ 2 \text{ for } 1 \le s < 2 \\ 0 \text{ for } 2 \ge 2 \end{cases}$$

Poof. Multiplying by $\sqrt{\frac{2}{\pi}}$, both sides,

$$F_s\,(f(x)) = \begin{cases} \sqrt{\frac{2}{\pi}} \text{ for } 0 \le s < 1 \\ 2\sqrt{\frac{2}{\pi}} \text{ for } 1 \le s < 2 \\ 0 \text{ for } s \ge 2 \end{cases}$$

$$\therefore \qquad f(x) = F_s^{-1}\ \text{(R.H.S)}$$

$$= \frac{2}{\pi}\int_0^1 \sin sx\, ds + \frac{4}{\pi}\int_1^2 \sin sx\, ds$$

$$= \frac{2}{\pi}\left(\frac{1-\cos x}{x}\right) + \frac{4}{\pi}\left(\frac{\cos x - \cos 2x}{x}\right)$$

$$f(x) = \frac{2}{\pi x}(1+\cos x - 2\cos 2x)$$

Example 19. *Find the Fourier cosine transform of e^{-x^2} (second method).*

$$I = F_c(e^{-x^2}) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x^2 \cos sx\, dx}$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}}\int_0^{\infty} xe^{-x^2} \sin sx\, dx$$

$$= +\sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin sx\, d\left(\frac{e^{-x^2}}{2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{e^{-x^2}}{2}\sin sx\right)_0^{\infty} - \frac{1}{2}\int_0^{\infty} se^{-x^2}\cos sx\, dx\right]$$

$$= -\frac{s}{2}I$$

$\therefore$ $$\frac{dI}{I} = -\frac{s}{2}ds$$

$\therefore$ $$\log I = -\frac{s^2}{4} + \log c$$

$$I = ce^{-\frac{s^2}{4}} \qquad \text{... (1)}$$

when $$s = 0,\ I = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x^2}dx = \frac{1}{\sqrt{2}}$$

Using in (1),

$$\frac{1}{\sqrt{2}} = c$$

$\therefore$ $$I = F_c(e^{-x^2}) = \frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}$$

Example 20. *Find the complex Fourier transform of dirac delta function $\delta(t-a)$.*

$$F\{\delta(t-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist}\delta(t-a)dt$$

$$= \frac{1}{\sqrt{2\pi}}\underset{h\to 0}{\text{Lt}}\int_a^{a+h}\frac{1}{h}e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}}\underset{h\to 0}{\text{Lt}}\frac{1}{h}\left(\frac{e^{ist}}{is}\right)_a^{a+h}$$

$$= \frac{1}{\sqrt{2\pi}}\underset{h\to 0}{\text{Lt}}\, e^{isa}\left(\frac{e^{ish}-1}{ish}\right)$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \text{ since } \underset{\theta \to 0}{\text{Lt}} \frac{e^{\theta} - 1}{\theta} = 1$$

Note. Dirac delta function $\delta(t - a)$ is defined as

$$\delta(t - a) = \underset{h \to 0}{\text{Lt}} \; I(h, t - a) \text{ where}$$

$$I(h, t - a) = \frac{1}{h} \text{ for } a < t < a + h$$

$$= 0 \text{ for } t < a \text{ and } t > a + h$$

Example 21. *Find the function if its sine transform is* $\dfrac{e^{-as}}{s}$.

Let $\quad F_s(f(x)) = \dfrac{e^{-as}}{s}$

Then, $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{s} \sin sx \, ds \qquad \text{... (1)}$$

$$\therefore \quad \frac{df}{dx} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-as} \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + x^2}$$

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \cdot a \int \frac{dx}{a^2 + x^2}$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a} + c \qquad \text{... (2)}$$

At $x = 0$, $f(0)$ using (1)

Using this in (2), $c = 0$

Hence, $$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a}.$$

setting $a = 0$.

$$F_s^{-1}\left(\frac{1}{s}\right) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

Example 22. *Prove (i)* $F\{x^n f(x)\} = (-i)^n \dfrac{d^n F(s)}{ds^n}$ *and (ii)* $F\{f^{n}(x)\} = (-is)^n F(s)$ *(iii) Hence solve for* $f(x)$ *if*

$$\int_{-\infty}^{\infty} f(t) e^{-|x-t|} \, dt = \phi(x) \text{ where } \phi(x) \text{ is known.}$$

Proof. (*i*)
$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$\frac{d^n}{ds^n}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (ix)^n e^{isx} f(x)\,dx$$

$$(-i)^n \frac{d^n}{ds^n}F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{isx} f(x)\,dx$$

$$= F\{x^n f(x)\}$$

(*ii*) Similarly,

$$F\{f^n(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} \frac{d^n}{dx^n} f(x)\cdot dx$$

$$= (is)^n F(s),$$

Using integration by parts successively and making assuptions that $f, f^1, ..., f^{(n-1)} \to 0$ as x $\to \pm\infty$.

(*iii*) $$\frac{1}{\sqrt{2\pi}}\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-|x-t|}\,dt,$$ from the given the equation

$$= f(x) * e^{-|x|}$$

By convolution theorem,

$$\frac{1}{\sqrt{2\pi}}\bar{\phi}(s) = F(s)\,.\,\sqrt{\frac{2}{\pi}}\frac{1}{1+s^2}$$

$$F(s) = \frac{1}{2}(1+s^2)\bar{\phi}(s)$$

$$= \frac{1}{2}[\bar{\phi}(s) - (-is)^2\bar{\phi}(s)]$$

$\therefore$ $$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\phi''(x)$$ using the result derived in (*ii*).

Exercises 4 (a)

1. Show that the Fourier transform of

$$f(x) = a - |x| \text{ for } |x| < a$$
$$= 0 \text{ for } |x| > a > 0$$

is $\sqrt{\frac{2}{\pi}}\frac{1-\cos as}{s^2}$. Hence show that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

2. Show that the Fourier transform of

$$f(x) = 0 \text{ for } x < \alpha$$
$$= 1 \text{ for } \alpha < x < \beta$$
$$= 0 \text{ for } x > \beta$$

is $\frac{1}{\sqrt{2\pi}}\left(\frac{e^{i\beta s}-e^{i\alpha s}}{is}\right)$

3. Show that the Fourier transform of

$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2a}, \text{ for } |x| \le a \\ 0 \quad \text{for } |x| > a \end{cases} \text{ is } \frac{\sin sa}{sa}$$

4. Find the Fourier and cosine transforms of e^{-x} and hence using the inversion formulae, show that $\int_0^\infty \frac{x \sin \alpha x}{1+x^2}dx = \frac{\pi}{2}e^{-\alpha} = \int_0^\infty \frac{\cos \alpha x}{1+x^2}dx$

5. Show that the Fourier sine transform of

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

is $\frac{2\sqrt{2}}{\sqrt{\pi}}$ $\sin s\,(1 - \cos s) / s^2$. **(1987 Ap. Barathadasan)**

6. Find the Fourier sine and cosine transform of $\cosh x - \sinh x$ (same as question 4).

7. Find the Fourier sine and cosine transform of $ae^{-\alpha x} + be^{-\beta x}$, $\alpha, \beta > 0$.

8. Find the Fourier transform of $f(x) = \begin{cases} 1 \text{ for } |x| \le a \\ 0 \text{ for } |x| > a \end{cases}$

9. Show that the Fourier cosine transform of $\frac{1}{1+x^2}$ is $\sqrt{\frac{\pi}{2}}e^{-s}$

10. Show that the Fourier sine transform of $\frac{x}{1+x^2}$ is $\sqrt{\frac{\pi}{2}}e^{-s}$

11. Show that the Fourier transform of $e^{-\frac{-x^2}{2}}$ is self-reciprocal.

12. Find Fourier transform of $e^{-|x|}$ if $a > 0$.

13. Find Fourier transform of $\frac{1}{\sqrt{|x|}}$.

Fourier Transform of Derivatives

We have already seen (refer Example 22) that,

$F\{f^n(x)\} = (-is)^n F(s)$

(*i*) $\therefore\ F\left(\frac{\partial^2 u}{\partial x^2}\right) = (-is)^2 F\{u(x)\} = -s^2\,\overline{u}$ where $\overline{u}$ is Fourier transform of u w.r.t. x.

(*ii*) $F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}f(0)+sF_s(s)$

$$\text{L.H.S.} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cdot\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\, d\{f(x)\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\{f(x)\cos sx\}_0^\infty + s\int_0^\infty f(x)\sin sx\,dx\right]$$

$$= sF_s(s) - \sqrt{\frac{2}{\pi}}f(0) \text{ assuming } f(x) \to 0 \text{ } \alpha\sigma \text{ } x \to \infty.$$

(*iii*) $F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\, d\,[f(x)]$

$$= \sqrt{\frac{2}{\pi}}\left[(f(x)\sin sx)_0^\infty - s\int_0^\infty f(x)\cos sx\,dx\right]$$

$$= -\,sF_c\,(s)$$

(*iv*) $F_c(F''(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\, d[f'(x)]$

$$= \sqrt{\frac{2}{\pi}}\left[\{f'(x)\cos sx\}_0^\infty + s\int_0^\infty f'(x)\sin sx\,dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + sF_s\{f'(x)\}$$

$$= -s^2\, F_c(s) - \sqrt{\frac{2}{\pi}f'}\,(0)$$

assuming $f(x), f'(x) \to 0$ as x $\to \infty$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty \sin sx\, d[f'(x)]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(f'(x)\sin sx\right)_0^\infty - s\int_0^\infty f'(x)\cos sx\,dx\right]$$

$$= -\,sF_c\,\{f'(x)\}$$

$$= -s\left[sF_s(s) - \sqrt{\frac{2}{\pi}}f(0)\right]$$

$$= -s^2\,F_s(\text{s}) + \sqrt{\frac{2}{\pi}}sf(0)$$

assuming $f(x), f'(x) \to 0$ as $x \to \infty$.

Relationship between Fourier and Laplace Transforms

Consider

$$f(t) = \begin{cases} e^{-xt} g(t) \text{ for } t > 0 \\ 0 \qquad \text{ for } t < 0 \end{cases} \qquad \text{... (1)}$$

Then the Fourier transform of $f(t)$ is given by

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ist} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(is-x)t} g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-pt} g(t) dt \text{ where } p = x - is$$

$$= \frac{1}{\sqrt{2\pi}} L\{g(t)\}$$

$\therefore$ Fourier transform of $f(t) = \frac{1}{\sqrt{2\pi}} \times$ Laplace transform of $g(t)$ defined by (1).

APPLICATIONS TO BOUNDARY VALUE PROBLEMS

Example 1. *Solve the diffusion equation* $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0$ *with the conditions,* $u(x, 0) = f(x)$, *and* $\frac{\partial u}{\partial x}, u$ *tend to zero as x tend to* $\pm\infty$.

Proof. Here, $u = u(x, t), -\infty < x < \infty, t > 0$

Defining Fourier transform of $u(x, t)$ as

$\bar{u}(s, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{isx} dx,$ take Fourier transform of the given differential equation.

$$KF\left\{\frac{\partial^2 u}{\partial x^2}\right\} = F\left\{\frac{\partial u}{\partial t}\right\}$$

i.e.,
$$K(-s^2 \bar{u}(s,t)) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{\partial u}{\partial t} e^{isx} dx$$

$$= \frac{d}{dt}\{F(u)\}$$

$$= \frac{d\bar{u}}{dt}$$

$\therefore \quad \dfrac{d\overline{u}}{dt} + s^2 \cdot k\overline{u} = 0$ where $\overline{u}$ = Fourier transform of $\overline{u}$ ... (1)

Solving (1), $\overline{u}\,(s, t) = ce^{-s^2kt}$

Since $u(x, 0) = f(x)$, taking transform w.r.t. x.

$$\overline{u}\,(s, o) = F(s)$$

Using (3) in (2), we get, $c = F(s)$

$$\overline{u}\,(s, t) = F(s)\ e^{-s^2kt}$$

Taking inverse transform,

$$u(x, t) = F^{-1}\{F(s)e^{-s^2kt}\}$$

$$= f(x) * F^{-1}\ (e^{s^2kt})$$

$$= f(x) * \frac{e^{\frac{x^2}{4kt}}}{\sqrt{2kt}}$$

(since $F^{-1}\left(e^{-\frac{s^2}{4a^2}}\right) = \sqrt{2}ae^{-a^2x^2}$ by example 4)

$$= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\theta)e^{-\frac{(x-\theta)^2}{4kt}}\,d\theta$$

Putting, $\dfrac{x-\theta}{2\sqrt{kt}} = \phi,$ we get $\theta = x - 2\sqrt{kt}\phi$

$$\therefore \quad u(x, t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x - 2\sqrt{kt}\,\phi)e^{-\phi^2}\,d\phi$$

Example 2. *Solve* $\dfrac{\partial^2 u}{\partial t^2} = \alpha^2\dfrac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty,\ t \geq 0$ *with conditions* $u(x, 0) = f(x)$, $\dfrac{\partial u}{\partial t}(x, 0) = g(x)$ *and assuming* $u, \dfrac{\partial u}{\partial x} \to 0$ *as* $x \to \pm\infty$.

Taking Fourier transform on both sides of the differential equation,

$\dfrac{d^2\overline{u}}{dt^2} = \alpha^2(-s^2\overline{u})$ where $\overline{u}$ is Fourier transform of u with respect to x.

$$\frac{d^2\overline{u}}{dt^2} + \alpha^2 s^2\overline{u} = 0$$

Auxiliary equation is $m^2 + \alpha^2 s^2 = 0$

$$m = \pm\, i\alpha s$$

$$\therefore \quad \overline{u}\,(s, t) = Ae^{i\alpha st} + Be - i\alpha st \quad \text{... (1)}$$

Since $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$,

$\bar{u}(s, 0) = F(s)$ and $\frac{d\bar{u}}{dt}(s,o) = G(s)$ on taking transforms.

Using these conditions in (1).

$$\bar{u}(s, 0) = A + B = F(s) \quad \text{... (2)}$$

$$\frac{d\bar{u}}{dt}(s,0) = i\alpha s(A - B) = G(s) \quad \text{... (3)}$$

Solving
$$A = \frac{1}{2}\left[F(s)+\frac{G(s)}{i\alpha s}\right]$$

$$B = \frac{1}{2}\left[F(s)-\frac{G(s)}{i\alpha s}\right]$$

Using these values in (1),

$$\bar{u}(s, t) = \frac{1}{2}\left[F(s)+\frac{G(s)}{i\alpha s}\right]e^{i\alpha st} + \frac{1}{2}\left[F(s)-\frac{G(s)}{i\alpha s}\right]e^{i\alpha st} \quad \text{... (4)}$$

By inversion theorem, (4) reduce to,

$$u(x, t) = \frac{1}{2}\left[f(x-\alpha t)-\frac{1}{\alpha}\int_{\alpha}^{x-\alpha t} g(\theta)\,d\theta\right] + \frac{1}{2}\left[f(x+\alpha t)+\frac{1}{\alpha}\int_{\alpha}^{x+\alpha t} g(\theta)d\theta\right]$$

Using the result

$$F\left(\int_{\alpha}^{x} f(t)dt\right) = \frac{F(s)}{(-is)}$$

Boundary Value Problems using sine and cosine Transforms

Example 3. *Solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *for $x > 0$, $t > 0$ given that*

(*i*) $u(0, t) = 0$ for $t > 0$

(*ii*) $u(x, 0) = \begin{cases} 1 \text{ for } 0 < x < 1 \\ 0 \text{ for } x \geq 1 \end{cases}$ and

(*iii*) $u(x, t)$ is bounded.

(**Note.** If u at $x = 0$ is given, take Fourier sine transform and if $\frac{\partial u}{\partial x}$ at $x = 0$ is given, take Fourier cosine transform.)

In this problem, u at $x = 0$ is given. Therefore, take fourier (infinite) sine transform.

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\,dx$$

i.e., $\dfrac{d\overline{u}}{dt} = -\ s^2\overline{u} + \sqrt{\dfrac{2}{\pi}}\ su\,(0, t)$ where $\overline{u}$ stands for Fourier sine transform of u.

Using boundary condition (*i*), we get

$$\frac{d\overline{u}}{dt} + s^2\overline{u} = 0$$

Solving

$$\overline{u}(s,t) = ce^{-s^2 t} \qquad \text{... }(iv)$$

Now,
$$u(x, 0) = \begin{cases} 1 \text{ for } 0 < x < 1 \\ 0 \text{ for } x \geq 1 \end{cases}$$

Taking Fourier sine transform of this initial condition,

$$\overline{u}\,(s, 0) = \sqrt{\frac{2}{\pi}}\int_0^1 \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \qquad \text{... }(v)$$

Using (*v*) in (*iv*)

$$c = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)$$

Hence (*iv*) reduces to,

$$\overline{u}\,(s, t) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)\cdot e^{-s^2 t}$$

By iunversion theorem, this becomes,

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos s}{s}\right)e^{-s^2 t}\cdot \sin sx\, ds.$$

Example 4. *Solve* $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2}$ *for* $0 \leq x < \infty$, $t > 0$ *given the conditions*

(i) $u(x, 0) = 0$ *for* $x \geq 0$

(ii) $\dfrac{\partial u}{\partial x}(0,t) = -\ a$ *constant*

(iii) $u(x, t)$ *is bounded.*

In this problem, $\dfrac{\partial u}{\partial x}$ at $x = 0$ is given. Hence, take Fourier cosine transform on both sides of the given equation.

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(K\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{d\bar{u}}{dt} = K\left(-s^2\bar{u}\sqrt{\frac{2}{\pi}}\cdot\frac{\partial u}{\partial x}(0,t)\right)$$

$$= -ks^2\bar{u} = \sqrt{\frac{2}{\pi}}ka \text{ using condition } (ii)$$

$$\frac{d\bar{u}}{dt} + ks^2\bar{u} = \sqrt{\frac{2}{\pi}}ka$$

This is linear in $\bar{u}$. Therefore, solving

$$\bar{u}e^{ks^2t} = \int\sqrt{\frac{2}{\pi}}kae^{ks^2t}dt$$

$$= \sqrt{\frac{2}{\pi}}ka\frac{e^{ks^2}t}{ks^2} + c$$

$$\bar{u}\,(s,\,t) = \sqrt{\frac{2}{\pi}}\frac{a}{s^2} + ce^{-ks^2t} \qquad \text{... } (iv)$$

Since $u(x, 0) = 0$ for $x \geq 0$.

$$\bar{u}\,(s,\,0) = 0 \text{ for } x \geq 0,$$

$$\bar{u}\,(s,0) = 0.$$

Using this in (*iv*), we get

$$\bar{u}\,(s,0) = c + \sqrt{\frac{2}{\pi}}\frac{a}{s^2} = 0$$

$$\therefore \qquad c = -\sqrt{\frac{2}{\pi}\frac{a}{s^2}}$$

Substituting this in (*iv*)

$$\bar{u}\,(s,\,t) = \sqrt{\frac{2}{\pi}}\frac{a}{s^2}\left(1 - e^{-k^st}\right)$$

By inversion theorem,

$$u(x,\,t) = \frac{2}{\pi}\cdot a\int_0^\infty \frac{1 - e^{ks^2t}}{s^2}\cos sx\,ds.$$

Example 5. *Solve* $\frac{\partial u}{dt} = k\frac{\partial^2 u}{\partial x^2}$ *for* $x \geq 0$, $t \geq 0$ *under the given conditions* $u = u_0$ *at* $x = 0$, $t > 0$ *with initial condition* $u(x, 0) = 0$, $x \geq 0$

Taking Fourier sine transforms

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{d}{du}\overline{u} = k\left[-s^2\overline{u}+\sqrt{\frac{2}{\pi}}su(0,t)\right]$$

$= -ks^2\ \overline{u}\ +\ \sqrt{\frac{2}{\pi}}ksu_0$ where $\overline{u}$ is the Fourier sine transform of u.

$$\frac{d\overline{u}}{dt}+ks^2\overline{u} = \sqrt{\frac{2}{\pi}}ksu_0$$

This is linear in $\overline{u}$.

$$\therefore \qquad \overline{u}e^{ks^2t} = \sqrt{\frac{2}{\pi}}ku_0\int s\,e^{kx^2t}\,dt$$

$$= \sqrt{\frac{2}{\pi}}\frac{u_0}{s}e^{ks^2t}+c \qquad \text{... (1)}$$

Since, $u(x, 0) = 0$, $\overline{u}\,(s, 0) = 0$. Using this in (1)

$$0 = \sqrt{\frac{2}{\pi}}\frac{u_0}{s}+c$$

$$\therefore \qquad c = -\sqrt{\frac{2}{\pi}}\frac{u_0}{s}$$

$$e^{ks^2t}\overline{u} = \sqrt{\frac{2}{\pi}}\frac{u_0}{s}\left(e^{ks^2t}-1\right)$$

$$\therefore \qquad \overline{u} = \sqrt{\frac{2}{\pi}}\frac{u_0}{s}\left(1-e^{ks^2t}\right)$$

By inversion theorem,

$$u(x, t) = \frac{2u_0}{\pi}\int_0^\infty\left(\frac{1-e^{-ks^2t}}{s}\right)\sin sx\,ds.$$

Exercises 4(b)

1. Show that the solution of $\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2}$.

subject to $u(0, t) = 0$, for $t > 0$ and $u(x, 0) = e^{-x}$ for $x > 0$ and $u(x, t)$ is bounded, is $\frac{2}{\pi}\int_0^\infty\frac{se^{-2s^2t}}{1+s^2}\sin sx\,ds$

2. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ if

(*i*) $\frac{\partial u}{\partial t}(0,t)\ = 0$ for $t > 0$.

(*ii*) $u(x, 0) = \begin{cases} x \text{ for } 0\le x\le 1 \\ 0 \text{ } \mathit{for}\ x>1\end{cases}$

(*iii*) and $u(x, t)$ is bounded for $x > 0$, $t > 0$.

$$\left[\textbf{Ans.}\, u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right) e^{-s^2 t} \cos sx\, ds\right]$$

FINITE FOURIER TRANSFORMS

Let $f(x)$ denotes a function which is sectionally continous over the range $(0, l)$. Then the **finite Fourier sine transform** of $f(x)$ on this interval is defined as

$$F_s(p) = \bar{f}_s(p) = \int_0^l f(x) \sin \frac{p\pi x}{l} dx$$

where p is an integer (instead of s, we take p as a parameter).

Inversion formula for sine transform

It $f_s(p) = F_s(p)$ is the finite Fourier sine transform of $f(x)$ in $(0, t)$ then the inversion formula for sine transform is

$$f(x) = \frac{2}{l}\sum_{p=1}^\infty \bar{f}_s(p) \sin \frac{p\pi x}{l}$$

Proof. For the given function $f(x)$ in $(0, l)$, if we find the half range Fourier sine series, we get,

$$f(x) = \sum_{n=1}^\infty b_n \sin \frac{n\pi x}{l} \qquad \text{... (1)}$$

where $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$\therefore$ $$b_p = \frac{2}{l}\int_0^l f(x) \sin \frac{p\pi x}{l} dx$$

$$= \frac{2}{l}\bar{f}_s(p) \text{ by definition}$$

Substituting in (1),

$\therefore$ $$f(x) = \frac{2}{l}\sum_{p=1}^\infty \bar{f}_s(p) \sin \frac{p\pi x}{l}$$

Finite Fourier Cosine Transform

Let $f(x)$ denote a sectionally contionous function in $(0, l)$.

Then the Finite Fourier cosine transform of $f(x)$ over $(0, l)$ is defined as

$$f_c(p) = \bar{f}_c(p) = \int_0^l f(x) \cos \frac{p\pi x}{l} dx \text{ where } p \text{ is an integer.}$$

Inversion Formula for Cosine Transform

If $\bar{f}_s(p)$ is the finite Fourier cosine transform of $f(x)$ in $(0, l)$, then the inversion formula for cosine transform is

$$f(x) = \frac{1}{l}\overline{f}_c(0) + \frac{2}{l}\sum_{p=l}^{\infty}\overline{f}_c(p)\cos\frac{p\pi x}{l}$$

where $$\overline{f}_c(0) = \int_0^l f(x)\,dx.$$

Proof. If we find half Fourier cosine series for $f(x)$ in $(0, l)$, we obtain,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{... (2) where}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$\therefore$ $$a_p = \frac{2}{l}\overline{f}_c(p)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\overline{f}_c(0).$$

Substituting in (2), we get,

$$f(x) = \frac{1}{l}\overline{f}_c(0) + \frac{2}{l}\sum_{p=1}^{\infty}\overline{f}_c(p)\cos\frac{p\pi x}{l}$$

Example 1. *Find the finite Fourier sine and cosine transforms of*

(i) $f(x) = 1$ *in* $(0, \pi)$

(ii) $f(x) = x$ *in* $(0, l)$

(iii) $f(x) = x^2$ *in* $(0, l)$

(iv) $f(x) = 1$ *in* $0, < x < \pi/2$

$= -1$ *in* $\pi/2 < x < \pi$

(v) $f(x) = x^3$ *in* $(0, l)$

(vi) $f(x) = e^{ax}$ *in* $(0, l)$

(i) $$\overline{f}_s(p) = Fs(1) = \int_0^{\pi} 1\cdot\sin\frac{p\pi x}{\pi}\,dx$$

$$= \left(-\frac{\cos px}{p}\right)_0^{\pi}$$

$$= \frac{1-\cos p\pi}{p} \quad \text{if } p \neq 0$$

$$\overline{f}_c(p) = \int_0^{\pi} 1\cdot\cos px\,dx$$

$$= \left(\frac{\sin px}{p}\right)_0^\pi = \frac{1}{p}(0-0) = 0$$

(*ii*) $\bar{f}_s(p) = F_s(p) = \int_0^l x \sin\frac{p\pi x}{l}\,dx$

$$= \left[(x)\left(\frac{-\cos\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (1)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{-1}{p\pi}(1\cos p\pi)$$

$$= \frac{-l^2}{p\pi}(-1)^p \quad \text{if } p \neq 0$$

$\bar{f}_c(p) = F_c(x) = \int_0^1 x\cos\frac{p\pi x}{l}\,dx$

$$= \left[(x)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (1)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{l^2}{p^2\pi^2}[(-1)^p - 1] \quad \text{if } p \neq 0$$

(*iii*) $\bar{f}_s(p) = F_s(x^2) = \int_0^l x^2 \sin\frac{p\pi x}{l}\,dx$

$$= \left[(x^2)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (2x)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right)\right] + (2)\left(\frac{\cos\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right)\Bigg]_0^l$$

$$= \frac{-l^3}{p\pi}[(-1)^p] + \frac{2l^3}{p^3\pi^3}[(-1)^p - 1] \quad \text{if } p \neq 0$$

$\bar{f}_c(p) = \int_0^l (x^2)\cos\frac{p\pi x}{l}\,dx$

$$= \left[(x^2)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (2x)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right) + (2)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2l^3}{p^2\pi^2}[(-1)^p] \text{ if } p \neq 0$$

(*iv*) $$F_s\{f(x)\} = \int_0^{\pi/2} \sin px\,dx + \int_{\pi/2}^{\pi} (-1)\sin px\,dx$$

$$= \left(-\frac{\cos px}{p}\right)_0^{\pi/2} + \left(\frac{\cos px}{p}\right)_{2x}^{\pi}$$

$$= -\frac{1}{p}\left(\cos\frac{p\pi}{2} - 1\right) + \frac{1}{p}(\cos p\pi - \cos p\pi/2)$$

$$= \frac{1}{p}\left(\cos p\pi - -2\cos\frac{p\pi}{2} + 1\right) \text{ if } p \neq 0$$

$$F_c(f(x)) = \int_0^{\pi/2} \cos px\,dx - \int_{\pi/2}^{\pi} \cos px\,dx$$

$$= \left(\frac{\sin px}{p}\right)_0^{\pi/2} - \left(\frac{\sin px}{p}\right)_{\pi/2}^{\pi} = \frac{2}{p}\sin\frac{p\pi}{2} \text{ if } p \neq 0$$

(*v*) $$F_s(x^3) = \int_0^1 x^3 \sin\frac{p\pi x}{l}\,dx$$

$$= \left[(x^3)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (3x^2)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right)\right] + (6x)\left(\frac{\cos\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right) - (6)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^4\pi^4}{l^4}}\right)\Bigg]_0^l$$

$$= -\frac{l^4}{p\pi}\left[(-1)^p\right] + \frac{6l^4}{p^3\pi^3}(-1)^n \quad \text{if } p \neq 0$$

$$F_c(x^3) = \int_0^l x^3 \cos\frac{p\pi x}{l}\,dx$$

$$= \left[(x^3)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (3x^2)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right) + 6(x)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right) - (6)\left(\frac{\cos\frac{p\pi x}{l}}{\frac{p^4\pi^4}{l^4}}\right)\right]_0^l$$

$$= \frac{3l^4}{\pi^2 p^2}(-1)^p - \frac{6l^4}{p^4\pi^4}\left[(-1)^p - 1\right] \text{ if } p \neq 0$$

(*vi*) $F_s\,(e^{ax}) = \int_0^l e^{ax} \sin\frac{p\pi x}{l}\,dx$

$$= \left\{\frac{e^{ax}}{a^2 + \frac{p^2\pi^2}{l^2}}\left[a\sin\frac{p\pi x}{l} - \frac{p\pi}{l}\cos\frac{p\pi x}{l}\right]\right\}_0^l$$

$$= \frac{e^{ax}}{a^2 + \frac{p^2a^2}{l^2}};\left(-\frac{p\pi}{l}(-1)^p\right) + \frac{1}{a^2 + \frac{p^2\pi^2}{l^2}}\left(\frac{p\pi}{l}\right)$$

$$Fc(e^{ax}) = \left\{\frac{e^{ax}}{a^2 + \frac{p^2\pi^2}{l^2}}\left[a\cos\frac{p\pi x}{l} + \frac{p\pi}{l}\sin\frac{p\pi x}{l}\right]\right\}_0^l$$

$$= \frac{e^{al}}{a^2 + \frac{p^2\pi^2}{l^2}}a(-1)^p - \frac{1}{a^2 + \frac{p^2\pi^2}{l^2}}(a)$$

Note. In all the above problems, if $p = 0$, do the integration separately

Example 2. *Find f(x) if its finite Fourier sine transform is* $\frac{2\pi}{p^3}(-1)^{p-1}$ *for p = 1, 2, ..., 0 < x < π.*

By inversion Theorem,

$$f(x) = \frac{2}{\pi}\sum_{p=1}^{\infty}\frac{2\pi}{p^3}(-1)^{p-1}\sin px$$

$$= 4\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p^3}\sin px$$

Example 3. *Find f(x) if its finitie Fourier sine transform is given by*

(*i*) $F_s(p) = \dfrac{1-\cos p\pi}{p^2\pi^2}$ for p = 1, 2, 3, ... and $0 < x < \pi$

(*ii*) $F_s(p) = \dfrac{16(-1)^{p-1}}{p^3}$ for $p = 1, 2, 3,$ ..... where $0 < x < 8$

(*iii*) $F_s(p) = \dfrac{\cos\frac{2\pi p}{3}}{(2p+1)^2}$ for $p = 1, 2, 3,$ ..... and $0 < x < 1$.

Solution. By inversion theorem

$$(i)\ f(x) = \frac{2}{\pi}\sum_{p=1}^{\infty}\left(\frac{1-\cos p\pi}{p^2\pi^2}\right)\cdot \sin px$$

$$= \frac{2}{\pi^3}\sum_{p=1}^{\infty}\left(\frac{1-\cos p\pi}{p^2}\right)\cdot \sin px$$

$$(ii)\ f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(p)\sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{2}{8}\sum_{p=1}^{\infty}\frac{16(-1)p^{-1}}{p^3}\sin\left(\frac{p\pi x}{8}\right) \text{ since } l = 8$$

$$= 4\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p^3}\sin\left(\frac{p\pi x}{8}\right)$$

$$(iii)\ f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(p)\sin\left(\frac{p\pi x}{l}\right)$$

$$= 2\sum_{p=1}^{\infty}\frac{\cos\left(\frac{2\pi p}{3}\right)}{(2p+1)^2}\ \sin(p\pi x) \text{ since } l = 1$$

Example 4. *Find f(x) if its finite Fourier cosine transform is*

$$(i)\ F_c(p) = \frac{1}{2p}\sin\left(\frac{p\pi}{2}\right) \text{ for p} = 1, 2, 3.....$$

$$= \frac{\pi}{4} \text{ for } p = 0 \text{ given } 0 < x < 2\pi$$

$$(ii)\ F_c(p) = \frac{6\sin\frac{p\pi}{2} - \cos p\pi}{(2p+1)\pi} \text{ for } p = 1, 2, 3, ...$$

$$= \frac{2}{\pi} \text{ for } p = 0 \text{ given } 0 < x < 4$$

$$(iii)\ F_c(p) = \frac{\cos\left(\frac{2p\pi}{3}\right)}{(2p+1)^2} \text{ for p} = 1, 2, 3, ...$$

$$= 1 \text{ for } p = 0 \text{ given } 0 < x < 1$$

Solution: By inversion theorem,

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{p=1}^{\infty}F_c(p)\cdot\cos\frac{p\pi x}{l}.$$

(i) Here $F_c(0) = \pi/4$ and $l = 2\pi$

$$\therefore \qquad f(x) = \frac{1}{2\pi}\left(\frac{\pi}{4}\right)+\frac{2}{2\pi}\sum_{p=1}^{\infty}\frac{1}{2p}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{p\pi x}{2\pi}\right)$$

$$= \frac{1}{8}+\frac{1}{2\pi}\sum_{p=1}^{\infty}\frac{1}{p}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{px}{2}\right).$$

(*ii*) Here, $F_c(0) = \dfrac{2}{\pi}$ and $l = 4$

$$f(x) = \frac{1}{4}\left(\frac{2}{\pi}\right)+\frac{2}{4}\sum_{p=1}^{\infty}\frac{\left(6\sin\frac{p\pi}{2}-\cos p\pi\right)}{(2p+1)\pi}\cos\left(\frac{p\pi x}{4}\right)$$

$$= \frac{1}{2\pi}+\frac{1}{2\pi}\sum_{p=1}^{\infty}\frac{\left(6\sin\frac{p\pi}{2}-\cos p\pi\right)}{(2p+1)}\cdot\cos\left(\frac{p\pi x}{4}\right)$$

(*iii*) Here $F_c(0) = 1$, $l = 1$

$$\therefore \qquad f(x) = \frac{1}{1}+\frac{2}{1}\sum_{p=1}^{\infty}\frac{1}{(2p+1)^2}\cos\left(\frac{2p\pi}{3}\right)\cdot\cos(p\pi x)$$

$$= 1+2\sum_{p=1}^{\infty}\frac{\cos\left(\frac{2p\pi}{3}\right)}{(2p+1)^2}\cos(p\pi x)$$

EXAMPLE 5. *Find the finite Fourier sine transform of in f(x) = 1 (0, π). Use the inversion theorem and find Fourier sine series for f(x) = 1 in (0, π) Hence prove*

(*i*) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.... = \pi/4$ (*ii*) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.... = \pi^2/8$

Solution. $F_s(1) = \int_0^{\pi} 1\cdot\sin\left(\frac{p\pi x}{\pi}\right)dx$

$$\bar{f}_s(p) = \frac{1-\cos p\pi}{p} \text{ if } p \neq 0$$

By inversion theorem,

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty}F_s(p)\sin\frac{p\pi x}{l}$$

$$1 = \frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1-(-1)^p}{p}\cdot\sin px \quad \text{since } l = \pi$$

$$1 = \frac{4}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+.....\right]$$

This is the half range Fourier sine series for $f(x) = 1$ in $(0, \pi)$ getting $x = \pi/2$.

$$\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....\right] = 1$$

$\therefore \qquad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+..... = \pi/4$

In the half Fourier sine series $l_n = \frac{4}{\pi}\cdot\frac{1}{n}$ for n odd

By using Parseval's Theorem.

$$\text{(range)}\left[\frac{1}{2}\sum b_n^2\right] = \int_0^\pi (1)^2\,dx$$

$$\pi\left[\frac{1}{2}\cdot\frac{16}{\pi^2}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2}\right] = \pi$$

i.e., $$\sum_{n=1,3,5}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$$

Example 6. Find f(x) if its finite Fourier cosine transform is $\frac{2l^3}{p^2\pi^2}(-1)^p$ for p = 1, 2, 3, ...

and is $\frac{l^3}{3}$ for $p = 0$; $0 < x < l$.

Proof: By inversion formula,

$$f(x) = \frac{1}{l}\bar{f}_c(0)+\frac{2}{l}\sum_{p=1}^{\infty}\bar{f}_c(p)\cos\left(\frac{p\pi x}{l}\right)$$

$$= \frac{1}{l}\cdot\frac{l^3}{3}+\frac{2}{l}\sum_{p=1}^{\infty}\frac{2l^3(-1)^p}{p^2\pi^2}\cos\frac{p\pi x}{l}$$

$$= \frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{p=1}^{\infty}\frac{(-1)^p}{p^2}\cos\left(\frac{p\pi x}{l}\right)$$

Exercise 4 (c)

Find the finite Fourier sine and cosine transforms of

1. $f(x) = 2x$ in $(0, 4)$ 2. $f(x) = x$ in $(0, \pi)$ **(BR 1995 April)**

3. $f(x) = \cos ax$ in $(0, \pi)$ 4. $f(x) = 1 - \frac{x}{\pi}$ in $(0, \pi)$

5. $f(x) = \begin{cases} x \text{ in } (0, \pi/2) \\ \pi - x \text{ in } (\pi/2, \pi) \end{cases}$ 6. $f(x) = e^{-ax}$ in $(0, l)$

7. Find finite Fourier cosine transform of $\left(1-\frac{x}{\pi}\right)^2$, $0 < x < \pi$

8. Find $f(x)$ if $\bar{f}_c(p) = \dfrac{\sin\left(\frac{p\pi}{2}\right)}{2p}$, $p = 1, 2, 3.....$

and $= \dfrac{\pi}{4}$ if $p = 0$ given $0 < x < 2\pi$.

9. Find finite Fourier cosine and sine transform of $f(x) = \dfrac{\pi}{3} - x + \dfrac{x^2}{2\pi}$. **(BR 1995 April)**

Finite Fourier sine and cosine transform of derivatives

Using the definition and the integration by parts, we can easily prove the following results. For $0 \le x \le 1$.

$$F_s\,\{f''(x)\} = -\frac{p^2\pi^2}{l^2}\bar{f}_s(p) + \frac{p\pi}{l}\left[f(0) - (-1)^p f(l)\right]$$

$$F_c\{f''(x)\} = -\frac{p^2\pi^2}{l^2}\bar{f}_c(p) + f'(l)(-1)^p - f'(0)$$

$$F_s\,\{f'(x)\} = -\frac{p\pi}{l}\bar{f}_c(p)$$

$$F_c\,\{f'(x)\} = f(l)\,(-1)^p - f(0) + \frac{p\pi}{l}\bar{f}_s(p)$$

Proof: (*i*) $F_s\,\{f'(x)\} = \displaystyle\int_0^l f'(x)\sin\frac{p\pi x}{l}dx$

$$= \int_0^l \sin\frac{p\pi x}{l}\cdot d[f(x)]$$

$$= \left(f(x)\sin\frac{p\pi x}{l}\right)_0^l - \int_0^l f(x)\cdot\cos\frac{p\pi x}{l}\cdot\frac{p\pi}{l}dx$$

$$= -\frac{p\pi}{l}\bar{f}_c(p)$$

(*ii*) $F_c\,\{f'(x)\} = \displaystyle\int_0^l f'(x)\cos\frac{p\pi x}{l}dx$

$$= \left(f(x)\cos\frac{p\pi x}{l}\right)_0^l - \int_0^l f(x)\cdot\frac{p\pi}{l}\sin\frac{p\pi x}{l}dx$$

$$= (-1)^p f(l) - f(0) + \frac{p\pi}{l}\bar{f}_s(p)$$

(*iii*) $F_s\,\{f''(x)\} = \displaystyle\int_0^l \sin\frac{p\pi x}{l}d[f'(x)]$

$$= \left(f'(x)\sin\frac{p\pi x}{l}\right)_0^l - \frac{p\pi}{l}\int_0^l f'(x)\cos\frac{p\pi x}{l}dx$$

$$= -\frac{p\pi}{l}\left[(-1)^p f(l) - f(0) + \frac{p\pi}{l}\bar{f}_s(p)\right]$$

$$= -\frac{p^2\pi^2}{l^2}\bar{f}_s(p) + \frac{p\pi}{l}\left[f(0) - (-1)^n f(1)\right]$$

$$(iv)\ Fc\{f''(x)\} = \int_0^l \cos\frac{p\pi x}{l}\, d[f'(x)]$$

$$= \left[f'(x)\cos\frac{p\pi x}{l}\right]_0^l + \frac{p\pi}{l}\int_0^l f'(x)\sin\frac{p\pi x}{l}\, dx$$

$$= (-1)^p f'(l) - f'(0) + \frac{p\pi}{l}\left[\frac{p\pi}{l}\bar{f}_c(p)\right]$$

$$= -\frac{p^2\pi^2}{l^2}\bar{f}_c(p) + f'(1)(-1)^n - f'(0)$$

Note. If $u = u(x, t)$, then

$$F_s\left[\frac{\partial u}{\partial x}\right] = -\frac{p\pi}{l}F_c(u)$$

$$F_c\left[\frac{\partial u}{\partial x}\right] = \frac{p\pi}{l}\ F_s(u) - u(0, t) + (-1)^p\, u(l, t)$$

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = -\frac{p^2\pi^2}{l^2}F_s(u) + \frac{p\pi}{l}[u(0,t) - (-1)^p u(l,t)]$$

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\frac{p^2\pi^2}{l^2}F_c(u) + \frac{\partial u}{\partial x}(l,t)\cos p\pi - \frac{\partial u}{\partial x}(0,t)$$

Example 1. *Using finite Fourier transform, solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ given } u(0, t) = 0 \text{ and } u(4, t) = 0$$

and $u(x,0) = 2x$ where $0 < x < 4$, $t > 0$

Proof. Since $u(0, t)$ is given, take finite fourier sine transform.

$$\int_0^4 \frac{\partial u}{\partial t}\sin\frac{p\pi x}{4}dx = \int_0^4 \frac{\partial^2 u}{\partial x^2}\sin\frac{p\pi x}{4}dx$$

$$\frac{d}{dt}\bar{u}_s = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= -\frac{p^2\pi^2}{16}\bar{u}_s + \frac{p\pi}{4}[u(0,t) - (-1)^p u(4,t)]$$

$$= -\frac{p^2\pi^2}{16}\bar{u}_s \text{ using } u(0, t) = 0,\ u(4, t) = 0$$

$$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{p^2\pi^2}{16}dt$$

Integrating

$$\log\ \bar{u}_s = \frac{p^2\pi^2}{16}t + c$$

$$\bar{u}_s = Ae^{-\frac{x^2p^2}{16}t}$$

since $u(x, 0) = 2x$

$$\bar{u}_s\ (p, 0) = \int_0^4 (2x)\sin\left(\frac{p\pi x}{4}\right)dx$$

$$= -\frac{32}{p\pi}\cos p\pi \qquad \text{... (2)}$$

Using (2) in (1),

$$\bar{u}_s = (p, 0) = A = -\frac{32}{p\pi}\cos p\pi.$$

Substituting in (1),

$$\therefore \qquad \bar{u}_s = -\frac{32}{n\pi}(-1)^n e^{-\frac{p^2\pi^2}{16}t}$$

By inversion theorem,

$$u(x, t) = \frac{2}{4}\sum_{p=1}^{\infty}\frac{32}{p\pi}(-1)^{p+1}e^{-\frac{p^2\pi^2}{16}t}\sin\left(\frac{p\pi x}{4}\right)$$

Example 2. *Solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$*,* $0 < x < 6,\ t > 0$*, given* $\frac{\partial u}{\partial x}(0, t) = 0$*,* $\frac{\partial u}{\partial x}(6, t) = 0$ *and* $u(x, 0) = 2x$

Proof. Since $\frac{\partial u}{\partial x}$ (0, t) is given, use finite Fourier cosine transform.

$$\int_0^6 \frac{\partial u}{\partial t}\cos\frac{p\pi x}{6}dx = \int_0^6 \frac{\partial^2 u}{\partial x^2}\cos\frac{p\pi x}{6}dx$$

$$\frac{d}{dt}\bar{u}_c = -\frac{p^2\pi^2}{36}\bar{u}_c + \frac{\partial u}{\partial x}(6, t)\cos p\pi - \frac{\partial u}{\partial x}(0, t)$$

$$= -\frac{p^2\pi^2}{36}\bar{u}_c$$

$$\frac{d\bar{u}_c}{\bar{u}_c} = -\frac{p^2\pi^2}{36}dt$$

$$\log\ \overline{u}_c = -\frac{p^2\pi^2}{36}t + c$$

$$\overline{u}_c = Ae^{-\frac{p^2\pi^2}{36}t} \qquad \text{... (1)}$$

$$u(x,\ 0) = 2x.$$

$\therefore$ At $t = 0$

$$\overline{u}_c\,(p,\ 0) = \int_0^6 (2x)\cos\frac{p\pi x}{6}dx$$

$$= \frac{72}{p^2\pi^2}(\cos p\pi - 1) \qquad \text{... (2)}$$

Using this in (1),

$$\overline{u}_c\,(p,\ 0) = A = \frac{72}{p^2\pi^2}(\cos p\pi - 1)$$

Substituting in (1),

$$\overline{u}_c\,(p,\ t) = \frac{72}{p^2\pi^2}(\cos p\pi - 1)e^{-\frac{p^2\pi^2}{36}t}$$

By inversion theorem,

$$u(x,\ t) = \frac{1}{l}\overline{f}_c(0) + \frac{2}{l}\sum_{p=1}^{\infty}\overline{f}_c(p)\cos\left(\frac{p\pi x}{l}\right)$$

$$= \frac{1}{6}\int_0^6 (2x)\,dx + \frac{2}{6}\sum_{p=1}^{\infty}\frac{72}{p^2\pi^2}\,(\cos p\pi - 1)e^{-\frac{p^2\pi^2}{36}t}\cdot\cos\left(\frac{p\pi x}{6}\right)$$

$$= 6 + \frac{24}{x^2}\sum_{p=1}^{\infty}\frac{(\cos p\pi - 1)}{p^2}e^{-\frac{p^2\pi^2}{36}t}\cos\left(\frac{p\pi x}{6}\right).$$

Example 3. *Solve* $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, *$0 < x < 4$, $t > 0$ given $u(0, t) = 0$, $u(4, t) = 0$; $u(x, 0) = 3 \sin \pi x - 2 \sin 5\pi x$.*

Proof. Since $u(0, t)$ is given, take finite Fourier sine transform. The equation becomes (as in example 1)

$$\frac{d}{dt}\overline{u}_c = 2\left[-\frac{p^2\pi^2}{16}\overline{u}_s + \frac{p\pi}{4}\{u(0,t) - (-1)^p u(4,t)\}\right]$$

$$= -\frac{p^2\pi^2}{8}\overline{u},$$

Solving we get,

$$\bar{u}_s = Ae^{-\frac{p^2\pi^2}{8}t}$$

$$u(x, 0) = 3 \sin \pi x - 2 \sin 5\pi x$$

Taking sine transform,

$$\bar{u}_s(p, 0) = \int_0^4 (3\sin \pi x - 2\sin 5\pi x)\sin \frac{p\pi x}{4}\, dx$$

$$= 0 \text{ if } p \neq 4,\ p \neq 20.$$

If $\quad p = 4,\ \bar{u}_s(4, 0) = 6$

If $\quad p = 20,\ \bar{u}_s(20, 0) = -4$

$$u(x, t) = \frac{2}{4}\sum_{p=1}^{\infty} \bar{u}_s(p, t)\sin\left(\frac{p\pi x}{4}\right)$$

$$= \frac{1}{2}\left[6e^{-\frac{p^2\pi^2}{8}t}\sin \pi x - 4e^{-\frac{p^2\pi^2}{8}t}\sin 5\pi x\right]$$

where p in the first term is 4 and p in the second term is 20

$$= 3e^{2\pi^2 t}\sin \pi x - 2e^{-50\pi^2 t}\sin 5\pi x.$$

Example 4. *Solve* $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, *$0 < x < \pi$, $t > 0$ given $u(0, t) = u(x, t) = 0$ for $t > 0$ and $u(x, 0)$ $= \sin^3 x$.*

Solution. Since $u(0, t)$ and $u(\pi, t)$ are given, take finite Fourier sine transforms on both sides of the given equation with respect to x in $(0, \pi)$.

$$\int_0^\pi \frac{\partial u}{\partial t}\sin \frac{p\pi x}{\pi}\, dx = \int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin \frac{p\pi x}{\pi}\, dx$$

$$\frac{d}{dt}\bar{u}_s = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= -p^2\bar{u}_s + \text{p}[\text{u}(0, t) - (-1)^p u(\pi, t)]$$

$$= -p^2\bar{u}_s \text{ using } u(0, t) = u(\pi, t) = 0.$$

$$\frac{d\bar{u}_s}{\bar{u}_s} = -p^2 dt$$

Integrating

$$\log(\bar{u}_s) = -p^2 t + k$$

$$\therefore \quad \bar{u}_s(p, t) = ce^{-pt} \qquad \text{... (1)}$$

Since, $u(x, 0) = \sin^3 x$

$$\bar{u}_s(p, 0) = \int_0^\pi \sin^3 x \sin px\, dx$$

$$= \int_0^\pi \left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)\sin px\, dx$$

$$= 0 \text{ for } p \neq 1, p \neq 3$$

when $p = 1$, $\bar{u}_s(1, 0) = \int_0^\pi \sin^4 x dx$

$$= 2\int_0^{\pi./2} \sin^4 x\, dx$$

$$= 2\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2\right)$$

$$= \frac{3}{8}\pi \qquad \text{... (2)}$$

when $p = 3$, $\bar{u}_s(3, 0) = \int_0^\pi \left(-\frac{1}{4}\sin^2 3x\right)dx$

$$= -\frac{1}{4}\int_0^\pi \frac{1-\cos 6x}{2}dx$$

$$= -\frac{1}{8}\left[x - \frac{\sin 6x}{6}\right]_0^\pi$$

$$= -\pi/8 \qquad \text{... (3)}$$

Using (2) and (3) in (1),

when $p = 1$, $\bar{u}_s(1, 0) = C = \dfrac{3\pi}{8}$

when $p = 3$, $\bar{u}_s\ (3, 0) = C = -\pi/8$

For all other values of p, $c = 0$

By inversion formula, $u(x, t) = \dfrac{2}{\pi}\displaystyle\sum_{p=1}^{\infty} \bar{u}_s(p, t)\sin nx$

i.e.,
$$u(x, t) = \frac{2}{\pi}\left[\frac{3\pi}{8}e^{-t}\sin x - \frac{\pi}{8}e^{-9t}\sin 3x\right]$$

$$= \frac{3}{4}3^{-t}\sin x - \frac{1}{4}e^{-9t}\sin 3x$$

Example 5. *Solve* $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, *$0 < x < 10$ given $u(0, t) = u(10, t) = 0$ for $t > 0$ and $u(x, 0) = 10x - x^2$ for $0 < x < 10$.*

Solution. As the previous problem, take finite Fourier sine transform on both sides of the heat equation with respect to x in (0, 10).

$$\therefore \qquad \frac{d}{dt}\bar{u}_s = -\frac{p^2\pi^2}{100}\bar{u}_s + \frac{p\pi}{10}[u(0, t) - (-1)^p u(10, t)]$$

$$= -\frac{p^2\pi^2}{100}\overline{u}, \text{ using } u(0, t) = u(10, t) = 0$$

Solving for $\overline{u}_s$, we get

$$\overline{u}_s = C_e \frac{-p^2\pi^2}{100} t \qquad \text{... (1)}$$

Since $u(x, 0) = 10x - x^2$,

$$\overline{u}_s(p, 0) = \int_0^{10} (10x - x^2)\sin\frac{p\pi x}{10}\,dx$$

$$= \left\{(10x - x^2)\left[\frac{-\cos\frac{p\pi x}{10}}{\frac{p\pi}{10}}\right] - (10 - 2x)\left[-\frac{\sin\frac{p\pi x}{10}}{\frac{p^2\pi^2}{100}}\right] + (-2)\left[\frac{\frac{\cos p\pi x}{10}}{\frac{p^3\pi^3}{1000}}\right]\right\}_0^{10}$$

$$= -\frac{2000}{D^3\pi^3}[(-1)^p - 1]$$

$$= \begin{cases} \frac{4000}{p^3\pi^3} \text{ if } p \text{ is odd} \\ 0 \quad \text{ if } p \text{ is even} \end{cases}$$

Using (1), $\overline{u}_s(p, 0) = C$.. using this again in (1), C is eliminated. Hence, taking inversion formula, use (1)

$$u(x, t) = \frac{2}{10}\sum \overline{u}_s \cdot \sin\frac{p\pi x}{10}$$

$$= \frac{800}{\pi^3}\sum_{p=1,3,5}^{\infty} \frac{1}{p^3}\sin\frac{p\pi x}{10} e^{\frac{-p^2p^2}{100}t}$$

Exercises 4 (d)

1. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 6$, $t > 0$ given that

$$u(0, t) = u(6, t) \text{ and } u(x, 0) = \begin{Bmatrix} 1 \text{ for } 0 < x < 3 \\ 0 \text{ for } 3 < x < 6 \end{Bmatrix}$$

$$\left[\textbf{Ans. } u(x,t) = \frac{2}{\pi}\sum_{p=1}^{\pi}\left(\frac{1 - \cos\frac{p\pi}{2}}{p}\right) e^{-\frac{p^2\pi^2 t}{36}} \sin\left(\frac{p\pi x}{6}\right)\right]$$

2. Solve $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ subject to conditions $v(0, t) = 1$, $v(\pi, t) = 3$

$v(x, 0) = 1$ for $0 < x < \pi$, $t > 0$. $\left[\textbf{Ans. } v(x, t) = \frac{4}{\pi}\sum_{p=1}^{\infty}\frac{\cos p\pi}{p}e^{-p^2 t}\sin px + 1 + \frac{2x}{\pi}.\right]$

3. Solve $\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$ given

$\theta(0, t) = 0$, $\theta(\pi, t) = 0$, $\theta(x, 0) = 2x$ for $0 < x < \pi$, $t > 0$ [Refer example 3]

4. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $0 < x < l$, $t > 0$ given $\frac{\partial u}{\partial x}$ $(0, t) = 0$ $\frac{\partial u}{\partial x}(1, t) = 0$ for $t > 0$ and $u(x,$

$0) = 1x - x^2$ for $0 < x < l$ $\left[\textbf{Ans. } u(x,t) = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum\frac{1}{p^2}\cos\frac{2p\pi x}{l}e^{\frac{-4p^2\pi^2 t}{l^2}}\right]$

5. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ for $0 < x < 1$, $t > 0$ given $u(0, t) = u$ (l, t) = 0 for t > 0 and $u(x, 0)$

$= \sin 3\pi x + \sin \pi x$. $\left[\textbf{Ans. } u(x,t) = e^{-2\pi^2 t}\sin\pi x + e^{-18\pi^2 t}\sin 3\pi x\right]$

6. Solve $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ subject to

$u(0, t) = u$ $(1, t)0$ for $t > 0$ and $u(x, 0) = x$ for $0 \le x \le l/2$

$= l - x$ for $l/2 \le x = < 1$

$$\left[\textbf{Ans. } u(x,t) = \frac{4l}{\pi^2}\sum\frac{1}{p^2}\sin\frac{p\pi}{2}\sin\frac{p\pi x}{l}e^{\frac{-\alpha^2 p^2\pi^2 t}{t^2}}\right]$$

SHORT ANSWER QUESTIONS

1. Define Periodic function.

2. If $f(x)$ is expressed in Fourier series of period 2π in $(c, c + 2\pi)$ write down the Fourier series and formulae for Euler's Coefficients.

3. If $f(x)$ is expressed in Fourier Series of period $2l$ in $(c, c + 2l)$ write down the Fourier series and forumulae for Euler's Coefficients.

4. If $f(x)$ is expressed in half range Fourier cosine series or sine series in $(0, l)$ of periodicity $2l$, write down two series and the corresponding integrals for coefficients.

5. State Dirichlet's conditions.

6. Draw the graph of $f(x) = |x|$

7. Define $|x|$ in $(-\infty, \infty)$

8. Find half range Fourier Cosine series or sine series in $(0, \pi)$ of periodicity 2π if $f(x) = k$.

9. Define odd and even functions.

10. State any property you know regarding $\int_{-l}^{l} f(x)\,dx$ if $f(x)$ is odd or even.