Chapter 4

Fourier Transforms

4.1 INTRODUCTION

The development of mathematical representation of periodic phenomena using complex numbers leads to complex form of the Fourier series representation of periodic function. The representation of periodic signals as a linear combination of harmonically related complex exponentials can be extended to develop a representation of aperiodic signals as linear combination of complex exponentials. This leads to Fourier Transforms.

Fourier transform is widely used in the theory of communication engineering, wave propagation and other branches of applied mathematics.

We have already discussed Laplace transforms and its applications for the solution of ordinary differential equations. In this Chapter, we shall discuss three other integral transforms and their applications for the solution of partial differential equations.

The effect of applying an integral transform to a partial differential equation is to reduce the number of independent variables by one, as will be seen in the solution of problems towards the end. The choice of a particular transform is decided by the nature of the boundary conditions and the convenience of inverting the transform function $\bar{f}(s)$ to give $f(x)$.

The three integral transforms, namely, Complex Fourier transform, Fourier Cosine transform and Fourier Sine transform and their inverse transforms are defined by means of a powerful theorem, known as Fourier Integral Theorem, which is given in Section 4.2.

4.2 FOURIER INTEGRAL THEOREM

If $f(x)$ is piecewise continuous, has piecewise continuous derivatives in every finite interval in $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot e^{is(x-t)} \, \mathrm{d}t \, \mathrm{d}s \text{ or equivalently}$$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\{s(x-t)\} \, \mathrm{d}t \, \mathrm{d}s.$$

Proof

When $f(x)$ satisfies the conditions given in the theorem, we can prove that $f(x)$ can be expanded as an infinite series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{1}$$

in $(-l, l)$ however large l may be, where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(t) e^{-in\pi t/l}\, \mathrm{d}t \tag{2}$$

Note

The series (1) is the Complex form of the Fourier series of $f(x)$ in $(-l, l)$.

Putting $s_n = \dfrac{n\pi}{l}$ and inserting (2) in (1), we have

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2l}\int_{-l}^{l} f(t) e^{is_n(x-t)}\, \mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-l}^{l}\left[\sum_{n=-\infty}^{\infty} f(t) e^{is_n(x-t)} \cdot \frac{\pi}{l}\right] \mathrm{d}t$$

on interchanging summation and integration.

$$= \frac{1}{2\pi}\int_{-l}^{l}\left[\sum_{n=-\infty}^{\infty} f(t) e^{is_n(x-t)} \Delta S_n\right] \mathrm{d}t, \text{ since}$$

$$\Delta s_n = s_{n+1} - s_n = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}.$$

Taking limits as $\Delta S_n \longrightarrow 0$ or equivalently $l \longrightarrow \infty$, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t) e^{is(x-t)}\, \mathrm{d}s\right] \mathrm{d}t \tag{3}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) e^{is(x-t)}\, \mathrm{d}t\ \mathrm{d}s \tag{4}$$

[$\because$ the limits of integration are constants]

From (3),
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} [\cos s(x-t) + i \sin s(x-t)]\, \mathrm{d}s\, \mathrm{d}t$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot 2 \int_{0}^{\infty} \cos s(x-t)\, \mathrm{d}s\, \mathrm{d}t$$

[$\because$ $\cos s(x-t)$ is an even function and $\sin s(x-t)$ is an odd function of s in $(-\infty, \infty)$]

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos s(x-t)\, \mathrm{d}t\, \mathrm{d}s \tag{5}$$

[$\because$ the limits of the integration are constants]

Note ✍

1. *In (5), the limits for t are $-\infty$ and ∞ and those for s are 0 and ∞.*
2. *The R.H.S. of (4) is called the Fourier Complex integral or Fourier Complex integral representation of $f(x)$. The R.H.S. of (5) is called the Fourier integral or Fourier integral representation of $f(x)$.*
3. *At a point of discontinuity, the value of the integral in the R.H.S. of (3) or (5) $= \frac{1}{2}\{f(x-0) + f(x+0)\}$.*

From (5), we have

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t)[\cos sx \cos st + \sin sx \cdot \sin st]\, \mathrm{d}t\, \mathrm{d}s$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \cos sx \left(\int_{-\infty}^{\infty} f(t) \cos st\, \mathrm{d}t \right) \mathrm{d}s + \frac{1}{\pi} \int_{0}^{\infty} \sin sx \left(\int_{-\infty}^{\infty} f(t) \sin st\, \mathrm{d}t \right) \mathrm{d}s \tag{6}$$

If $f(x)$ [or $f(t)$] is even,

$f(t) \cos st$ is an even function of t and $f(t) \sin st$ is an odd function of t. Hence, by the property of definite integrals, we get the following from (6),

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(t) \cos sx \cos st\, \mathrm{d}t\, \mathrm{d}s \tag{7}$$

The R.H.S. of (7) is called *the Fourier Cosine integral* of $f(x)$, provided $f(x)$ is even.

If $f(x)$ [or $f(t)$] is odd, $f(t)\cos st$ is an odd function of t and $f(t)\sin st$ is an even function of t.

Hence, by the property of definite integrals, we get the following from (6),

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin sx\,\sin st\ dt\ ds \tag{8}$$

The R.H.S. of (8) is called *the Fourier sine integral* of $f(x)$, provided $f(x)$ is odd.

4.3 FOURIER TRANSFORMS

Definition

$\int_{-\infty}^{\infty} f(x)e^{-isx}\ dx$ is called *the Fourier transform of* $f(x)$ and is denoted by $\bar{f}(s)$ or $F\{f(x)\}$. F is the Fourier transform operator. Thus $\bar{f}(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-isx}\ dx$, where s is used as the transform variable. Sometimes the letter p or ω is used as the transform variable.

From Step (4) of Fourier integral theorem, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left[\int_{-\infty}^{\infty} f(t)e^{-ist}\ dt\right] ds \tag{4$'$}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{ixs}\ ds$$

Now $\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{ixs}\ ds$ is called *the inverse Fourier Transform of* $\bar{f}(s)$ and denoted by $F^{-1}\{\bar{f}(s)\}$.

Thus, once the Fourier transform of $f(x)$ is defined as given above, the inverse Fourier transform of $\bar{f}(s)$ is provided by Fourier Integral theorem.

(4)$'$ may be rewritten as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\ dt\right] ds$$

Accordingly, some authors define $F\{f(x)\}$ and $F^{-1}\{\bar{f}(s)\}$ (Fourier transform pair) as follows:

$$F\{f(x)\} = \bar{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx}\, dx \text{ and}$$

$$F^{-1}\{\bar{f}(s)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s) \cdot e^{ixs}\, ds$$

Definition

$\int_0^\infty f(x) \cos sx \; dx$ is called *Fourier Cosine transform* of $f(x)$ and is denoted by $\bar{f}_C(s)$ or $F_C\{f(x)\}$. F_C is the Fourier cosine transform operator.

From Step (7) of Fourier integral theorem we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos sx \left[\int_0^\infty f(t) \cos st \; dt \right] ds \qquad (7)'$$

$$= \frac{2}{\pi} \int_0^\infty \bar{f}_C(s) \cos xs \; ds$$

$\frac{2}{\pi} \int_0^\infty \bar{f}_C(s) \cos xs \; ds$ is called the *inverse Fourier Cosine transform* of $\bar{f}_C(s)$ and denoted by $F_C^{-1}\{\bar{f}_C(s)\}$. Thus, once the Fourier cosine transform of $f(x)$ is defined, the inverse Fourier cosine transform of $\bar{f}_C(s)$ is provided by the Fourier cosine integral formula.

(7)′ may be rewritten as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \; dt \right] ds$$

Accordingly, some authors define the Fourier cosine transform pair as follows,

$$F_C\{f(x)\} = \bar{f}_C(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \; dx \text{ and}$$

$$F_C^{-1}\{\bar{f}_C(s)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_C(s) \cos xs \; ds$$

Definition

$\int_0^\infty f(x) \sin sx \; dx$ is called *Fourier Sine transform* of $f(x)$ and is denoted by $\bar{f}_S(s)$ or $F_S\{f(x)\}$.

F_S is the Fourier sine transform operator. Thus

$$\bar{f}_S(s) = F_S\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx$$

From Step (8) of Fourier integral theorem, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx \left[\int_0^\infty f(t) \sin st \, dt\right] ds \tag{8$'$}$$

$$= \frac{2}{\pi}\int_0^\infty \bar{f}_S(s) \sin xs \, ds$$

$\frac{2}{\pi}\int_0^\infty \bar{f}_S(s) \sin xs \; ds$ is called the *inverse Fourier sine transform* of $\bar{f}_S(s)$ and denoted by $F_S^{-1}\{\bar{f}_s(s)\}$. Thus once the Fourier sine transform of $f(x)$ is defined, the inverse Fourier sine transform of $\bar{f}_S(s)$ is provided by the Fourier sine integral formula.

(8)$'$ may be rewritten as

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t) \sin st \; dt\right] ds$$

Accordingly, some authors define the Fourier sine transform pair as follows.

$$F_S\{f(x)\} = \bar{f}_S(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin sx \, dx \text{ and}$$

$$F_S^{-1}\{\bar{f}_S(s)\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(s) \sin xs \; ds$$

4.4 ALTERNATIVE FORM OF FOURIER COMPLEX INTEGRAL FORMULA

The Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t) \cos s(x-t) \; dt \; ds$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t) \cos s(t-x) \; dt \; ds$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \left[e^{is(t-x)} + \bar{e}^{is(t-x)} \right] \mathrm{d}t \ \mathrm{d}s$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(t-x)} \ \mathrm{d}t \ \mathrm{d}s + \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \bar{e}^{is(t-x)} \ \mathrm{d}t \ \mathrm{d}s.$$

Putting $s = -s'$ in the second integral, we get

$$f(x) = \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(t-x)} \ \mathrm{d}t \ \mathrm{d}s + \frac{1}{2\pi} \int_{-\infty}^{0} \int_{-\infty}^{\infty} f(t) e^{is'(t-x)} \ \mathrm{d}t \ \mathrm{d}s'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(t-x)} \ \mathrm{d}t \ \mathrm{d}s$$

[on changing s' into s and combining the two integrals] (1)

(1) provides an alternative formula for $f(x)$. Comparing this with the Fourier Complex integral formula derived in (4) of Fourier integral theorem, we note that x and t can be interchanged in the exponential function.

Note ✍

Equation (1) can be re-written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left[\int_{-\infty}^{\infty} f(t) e^{ist} \ \mathrm{d}t \right] \mathrm{d}s$$

Based on this, some authors define Fourier transform pair as follows:

$$\bar{f}(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx} \ \mathrm{d}x \textit{ and}$$

$$F^{-1}\{\bar{f}(s)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s) e^{-ixs} \ \mathrm{d}s$$

We shall follow the definitions which were given earlier.

4.5 RELATIONSHIP BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM

Let $f(t)$ be defined as $f(t) = \begin{cases} e^{-xt}\phi(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$

Then $\quad F\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-iyt}\, dt$

where y is the Fourier transform variable.

$$= \int_{-\infty}^{0} 0 \cdot e^{-iyt}\, dt + \int_{0}^{\infty} e^{-xt}\, \phi(t)e^{-iyt}\, dt$$

$$= \int_{0}^{\infty} e^{-st}\phi(t)\, dt, \text{ where } s = x + iy$$

i.e. $\quad F\{f(t)\} = L\{\phi(t)\}$

Worked Examples 4(a)

Example 1

Find the Fourier integral representation of $f(x)$ defined as

$$f(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1/2, & \text{for } x = 0 \\ e^{-x}, & \text{for } x > 0 \end{cases}$$

Verify the representation at $x = 0$.

Fourier (complex) integral representation is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)e^{-ist} \cdot e^{isx}\, dt\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \left[\int_{-\infty}^{0} + \int_{0}^{\infty} f(t)e^{-ist}\, dt \right] ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \left[\int_{0}^{\infty} e^{-(1+is)t} \cdot dt \right] ds,$$

on using the given values of $f(t)$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \left\{ \frac{e^{-(1+is)t}}{-(1+is)} \right\}_{t=0}^{t=\infty} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \cdot \frac{1}{1+is}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1 - is)}{1 + s^2} (\cos xs + i \sin xs) \, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + s^2} [\{\cos xs + s \sin xs\} + i\{\sin xs - s \cos xs\}] \, ds$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \left(\frac{\cos xs + s \sin xs}{1 + s^2} \right) ds \tag{1}$$

by property of definite integrals, as the real part is even and the imaginary part is odd.

Putting $x = 0$ in the integral representation (1), we get $f(0) = \frac{1}{\pi} \int_{0}^{\infty} \frac{ds}{1 + s^2} = \frac{1}{\pi} [\tan^{-1} s]_0^{\infty} = \frac{1}{2}$. Thus the integral representation (1) holds good for $x = 0$ also.

Example 2

Using Fourier integral formula, prove that

$$e^{-x} \cos x = \frac{2}{\pi} \int_{0}^{\infty} \frac{(\lambda^2 + 2) \cos x\lambda}{\lambda^4 + 4} \, d\lambda.$$

Note ✍

λ has been used instead of 's'

The presence of $\cos x\lambda$ in the integral suggests that the Fourier cosine integral formula for $e^{-x} \cos x$ has been used.

Fourier cosine integral representation is given by

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(t) \cos \lambda t \cos \lambda x \, dt \, d\lambda$$

$$\therefore \; e^{-x} \cos x = \frac{2}{\pi} \int_{0}^{\infty} \cos x\lambda \, d\lambda \left[\int_{0}^{\infty} e^{-t} \cos t \cos \lambda t \, dt \right]$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \cos x\lambda \, d\lambda \cdot \left[\frac{1}{2} \int_{0}^{\infty} e^{-t} \{\cos(\lambda + 1)t + \cos(\lambda - 1)t\} \, dt \right]$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \cos x\lambda \, d\lambda \left[\frac{e^{-t}}{(\lambda + 1)^2 + 1} \{-\cos(\lambda + 1)t + (\lambda + 1) \sin(\lambda + 1)t\} \right.$$

$$\left. + \frac{e^{-t}}{(\lambda - 1)^2 + 1} \{-\cos(\lambda - 1)t + (\lambda - 1) \sin(\lambda - 1)t\} \right]_0^{\infty}$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\frac{1}{(\lambda+1)^2+1} + \frac{1}{(\lambda-1)^2+1}\right\}\cos x\lambda \; d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)\cos x\lambda}{\lambda^4+4}\, d\lambda.$$

Example 3

Using Fourier integral formula, prove that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{u \sin xu}{(u^2+a^2)(u^2+b^2)}\, du (a, b > 0)$$

Note ✍

The letter 'u' has been used instead of 's'

The presence of sin xu in the integral suggests that the Fourier sine integral formula has been used.

Fourier sine integral representation is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin ut \sin xu \; dt \; du$$

$$\therefore \quad e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^\infty \sin x\, u \; du \left[\int_0^\infty (e^{-at} - e^{-bt})\sin ut \; dt\right]$$

$$= \frac{2}{\pi}\int_0^\infty \sin xu \; du \left[\frac{e^{-at}}{a^2+u^2}\{-a\sin ut - u\cos ut\} - \frac{e^{-bt}}{b^2+u^2}\{-b\sin ut - u\cos ut\}\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \sin xu \; du \left[\frac{u}{a^2+u^2} - \frac{u}{b^2+u^2}\right]$$

$$= \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{u \sin xu}{(u^2+a^2)(u^2+b^2)}\, du$$

Example 4

Find the Fourier transform of the unit step function and unit impulse function.

(i) The unit step function is defined as

$$u_a(x) = \begin{cases} 0, & \text{for } x < a \\ 1, & \text{for } x \geq a \end{cases}$$

$$\therefore \qquad F\{u_a(x)\} = \int_a^{\infty} e^{-isx}\, dx = \left[\frac{e^{-isx}}{-is}\right]_a^{\infty} = \frac{1}{is} e^{-ias}$$

In particular $\qquad F\{u_0(x)\} = \frac{1}{is} \quad \text{or} \quad -\frac{i}{s}$

(ii) The unit impulse function or Dirac Delta function $\delta_a(x)$ is defined as $\lim\limits_{\in \to 0} [f(x)]$, where

$$f(x) = \begin{cases} \frac{1}{\in}, & \text{for } a - \frac{\in}{2} \leq x \leq a + \frac{\in}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$\therefore \qquad F\{f(x)\} = \int_{a-\in/2}^{a+\in/2} \frac{1}{\in} e^{-isx}\, dx$$

$$= \frac{1}{\in}\left[\frac{e^{-isx}}{-is}\right]_{a-\in/2}^{a+\in/2}$$

$$= \frac{1}{i \in s}\left\{e^{-is(a-\in/2)} - e^{-is(a+\in/2)}\right\}$$

$$= e^{-ias} \cdot \left[\frac{\sin\left(\frac{\in S}{2}\right)}{\left(\frac{\in S}{2}\right)}\right]$$

$$\therefore \qquad F\{\delta_a(x)\} = \lim_{\in \to 0}\left[e^{-ias} \cdot \left\{\frac{\sin\left(\frac{\in S}{2}\right)}{\frac{\in S}{2}}\right\}\right]$$

$$= e^{-ias}$$

In particular, $\qquad F\{\delta_0(x)\} = 1$

Example 5

Find the Fourier transform of $f(x)$, defined as

$$f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

and hence find the value of $\int_0^{\infty} \frac{\sin x}{x}\, dx$

$$F\{f(x)\} = \int_{-a}^{a} e^{-isx}\, dx = \int_{-a}^{a} (\cos sx - i \sin sx)\, dx$$

$$= 2\int_{0}^{a} \cos sx\, dx, \text{ by the property of definite integrals}$$

$$= \frac{2}{s} \sin as.$$

Taking Fourier inverse transforms,

$$F^{-1}\left\{\frac{2}{s}\sin as\right\} = f(x)$$

i.e. $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{s} \sin\, a\, s\, e^{ixs}\, ds = f(x)$$

i.e. $$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{s} \sin\, a\, s(\cos\, xs + i \sin xs)\, ds = f(x)$$

i.e. $$\frac{2}{\pi}\int_{0}^{\infty} \frac{1}{s} \sin as\, \cos\, xs\, ds = f(x) \left[\because \frac{1}{s} \sin\, as \sin xs \text{ is odd}\right]$$

i.e. $$\int_{0}^{\infty} \frac{1}{s} \sin\, as\, \cos\, xs\, ds = \begin{cases} \pi/2, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

Putting $a = 1$ and $x = 0$,
so that $|0| < 1$, we get

$$\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

Changing the dummy variable s into x, we get

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Example 6

Find the inverse Fourier transform of $\bar{f}(s)$ given by $\bar{f}(s) = \begin{cases} a - |s|, & \text{for } |s| \le a \\ 0, & \text{for } |s| > a \end{cases}$

Hence show that $\displaystyle\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$

$$F^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{ixs}\ ds$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \{a-|s|\}\ (\cos xs + i\sin xs)\ ds$$

$$= \frac{1}{\pi}\int_{0}^{a} (a-s)\ \cos xs\ ds \qquad [\because \{a-|s|\}\sin xs \text{ is odd}]$$

$$= \frac{1}{\pi}\left[(a-s)\frac{\sin xs}{x} - \frac{\cos xs}{x^2}\right]_0^a$$

$$= \frac{1}{\pi x^2}(1-\cos ax)$$

$$= \frac{a^2}{2\pi}\left\{\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right\}^2$$

$\therefore$
$$F\left[\frac{a^2}{2\pi}\left\{\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right\}^2\right] = \bar{f}(s)$$

i.e.
$$\frac{a^2}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right\}^2 e^{-isx}\ dx = \begin{cases} a-|s|, & \text{for } |s|\le a\\ 0, & \text{for } |s|>a\end{cases}$$

Taking $a = 2$ and letting $s \longrightarrow 0$, we get

$$\int_{-\infty}^{\infty}\left\{\frac{\sin x}{x}\right\}^2\ dx = \pi$$

Since the integrand is an even function of x,
we get

$$\int_{0}^{\infty}\left\{\frac{\sin x}{x}\right\}^2\ dx = \frac{\pi}{2}$$

Example 7

Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} 1-x^2, & \text{for } |x|<1\\ 0, & \text{for } |x|>1\end{cases}$$

Hence evaluate $\displaystyle\int_0^{\infty}\left\{\frac{\sin x - x\cos x}{x^3}\right\}\cos\frac{x}{2}\ dx$

$$F\{f(x)\} = \int_{-1}^{1} (1 - x^2) e^{-isx} \, dx$$

$$= 2\int_{0}^{1} (1 - x^2) \cos sx \, dx \quad [\because (1 - x^2) \sin sx \text{ is an odd function of } x]$$

$$= 2\left[(1 - x^2)\frac{\sin sx}{s} - 2x\frac{\cos sx}{s^2} + 2\frac{\sin sx}{s^3}\right]_0^1$$

$$= \frac{4}{s^3}(\sin s - s\cos s)$$

Taking inverse Fourier transform, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(\sin s - s\cos s)(\cos xs + i\sin xs) \, ds = f(x)$$

i.e.
$$\int_{0}^{\infty} \frac{(\sin s - s\cos s)}{s^3} \cos xs \, ds = \pi/4. \begin{cases} 1 - x^2, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Taking $x = \frac{1}{2}$,

we have,
$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\frac{s}{2} \, ds = \frac{3\pi}{16}$$

Replacing the dummy variables s by x, we get the required result.

Example 8

Find the Fourier transform of $e^{-a^2x^2}$. Hence

(i) prove that $e^{-x^2/2}$ is self-reciprocal with respect to Fourier transforms; and
(ii) find the Fourier cosine transform of e^{-x^2}.

$$F\left\{e^{-a^2x^2}\right\} = \int_{-\infty}^{\infty} e^{-a^2x^2} \cdot e^{-isx} \, dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax + \frac{is}{2a}\right)^2} \cdot e^{-\frac{s^2}{4a^2}} \, dx$$

$$= e^{-s^2/4a^2} \cdot \frac{1}{a} \int_{-\infty}^{\infty} e^{-t^2} \, dt, \text{ on putting } ax + \frac{is}{2a} = t$$

$$= \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} \tag{1}$$

(i) Had we assumed the definition of the Fourier transform as

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} \, dx$$

(1) would have become

$$F\left\{e^{-a^2x^2}\right\} = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

Putting $a = \frac{1}{\sqrt{2}}$ in (2), we get

$$F\left\{e^{-x^2/2}\right\} = e^{-s^2/2} \text{ and so } F^{-1}\left\{e^{-s^2/2}\right\} = e^{-x^2/2}$$

i.e. $e^{-x^2/2}$ is self-reciprocal with respect to Fourier transforms.
(ii) From (1), we have

$$\int_{-\infty}^{\infty} e^{-a^2x^2} (\cos sx - i \sin sx) \, dx = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$$

Equating the real parts on both sides, we get

$$\int_{0}^{\infty} e^{-a^2x^2} \cos sx \, dx = \frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}$$

or

$$F_C\left\{e^{-a^2x^2}\right\} = \frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}$$

Example 9

Find the Fourier cosine transform of e^{-ax} and use it to find the Fourier transform of $e^{-a|x|} \cos bx$.

$$F_C(e^{-ax}) = \int_{0}^{\infty} e^{-ax} \cos sx \, dx$$

$$= \left[\frac{e^{-ax}}{s^2 + a^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= \frac{a}{s^2 + a^2}, \quad a > 0 \qquad (1)$$

Now $$F(e^{-a|x|} \cos bx) = \int_{-\infty}^{\infty} e^{-a|x|} \cos bx \cdot e^{-isx} \, dx$$

$$= 2 \int_0^{\infty} e^{-ax} \cos bx \cos sx \, dx,$$

(since the integral of the odd part of the integrand vanishes)

$$= \int_0^{\infty} e^{-ax} \{\cos (s+b)x + \cos (s-b)x\} \, dx$$

$$= F_C\{e^{-ax}\}_{s \to s+b} + F_C\{e^{-ax}\}_{s \to s-b}$$

$$= a \left\{ \frac{1}{(s+b)^2 + a^2} + \frac{1}{(s-b)^2 + a^2} \right\}, \text{ by (1)}$$

Example 10

Find the Fourier cosine transform of $f(x)$ defined as $f(x) = \begin{cases} 1, & \text{for } 0 < x < a \\ 0, & \text{for } x \geq a \end{cases}$

Hence find the inverse Fourier cosine transform of $\left(\frac{\sin as}{s} \right)$. Verify your answer by directly finding $F_C^{-1} \left(\frac{\sin as}{s} \right)$.

$$F_C\{f(x)\} = \int_0^{\infty} f(x) \cos sx \, dx = \int_0^{a} \cos sx \, dx$$

$$= \frac{\sin as}{s}$$

$\therefore$ $$F_C^{-1} \left\{ \frac{\sin as}{s} \right\} = f(x) = \begin{cases} 1, & \text{for } 0 < x < a \\ 0, & \text{for } x \geq a \end{cases}$$

Now $$F_C^{-1} \left(\frac{\sin as}{s} \right) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin as}{s} \cos xs \, ds$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\sin(a+x)s + \sin(a-x)s}{s} \, ds$$

$$= \frac{1}{\pi}\left\{\int_0^\infty \frac{\sin(a+x)s}{s}\,ds + \int_0^\infty \frac{\sin(a-x)s}{s}\,ds\right]$$

$$= \begin{cases} \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right), \text{ if } x < a \\ \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\right), \text{ if } x > a \end{cases} \quad \left(\because \int_0^\infty \frac{\sin ms}{s}\,ds = \frac{\pi}{2}, \text{ when } m > 0\right)$$

$$= \begin{cases} 1, \text{ if } 0 < x < a \\ 0, \text{ if } x \geq a \end{cases}$$

Example 11

Find the Fourier sine transform of $f(x)$ defined as $f(x) = \begin{cases} \sin x, & \text{when } 0 < x < a \\ 0, & \text{when } x > a \end{cases}$

$$F_S\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx$$

$$= \int_0^a \sin x \sin sx\,dx$$

$$= \frac{1}{2}\int_0^a [\cos(s-1)x - \cos(s+1)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]$$

Example 12

Find the Fourier cosine transform of $f(x)$ defined as $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$

$$F_C\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx$$

$$= \int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx$$

$$= \left[x\frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1 + \left[(2-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_1^2$$

$$= \frac{2\cos s}{s^2} - \frac{1}{s^2}(1 + \cos 2s)$$
$$= \frac{2\cos s(1 - \cos s)}{s^2}$$

Example 13

Find the Fourier sine transform of e^{-ax} $(a > 0)$. Hence find $F_S\{xe^{-ax}\}$ and $F_S\left\{\frac{e^{-ax}}{x}\right\}$.

Deduce the value of $\int\limits_0^\infty \frac{\sin sx}{x}\,\mathrm{d}x$.

$$F_S(e^{-ax}) = \int\limits_0^\infty e^{-ax}\sin sx\,\mathrm{d}x$$
$$= \left[\frac{e^{-ax}}{s^2 + a^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$
$$= \frac{s}{s^2 + a^2}$$

i.e. $$\int\limits_0^\infty e^{-ax}\sin sx\,\mathrm{d}x = \frac{s}{s^2 + a^2} \qquad (1)$$

Differentiating both sides of (1) with respect to 'a', we get,

$$\int\limits_0^\infty -xe^{-ax}\sin sx\,\mathrm{d}x = -\frac{2as}{(s^2 + a^2)^2}$$

i.e. $$F_S(xe^{-ax}) = \frac{2as}{(s^2 + a^2)^2}$$

Integrating both sides of (1) with respect to 'a' between a and ∞,

$$\int\limits_0^\infty \left(\frac{e^{-ax}}{-x}\right)_a^\infty \sin sx\,\mathrm{d}x = \left[-\cot^{-1}\left(\frac{a}{s}\right)\right]_a^\infty$$

i.e. $$\int\limits_0^\infty \left(\frac{e^{-ax}}{x}\right)\sin sx\,\mathrm{d}x = \cot^{-1}\left(\frac{a}{s}\right)$$

i.e. $$F_S\left(\frac{e^{-ax}}{x}\right) = \cot^{-1}\left(\frac{a}{s}\right), \quad a > 0 \qquad (2)$$

Taking limits on both sides of (2) as $a \to 0$,

we get $\quad F_S\left(\dfrac{1}{x}\right) = \cot^{-1}(0) = \dfrac{\pi}{2}$

Thus $\quad \displaystyle\int_0^\infty \frac{\sin sx}{x}\, dx = \frac{\pi}{2}, \quad s > 0.$

Example 14

Find the Fourier cosine transform of $\dfrac{1}{x^2 + a^2}$

$$F_C\left\{\frac{1}{x^2+a^2}\right\} = \int_0^\infty \frac{\cos sx}{x^2+a^2}\, dx = I, \text{ say}$$

$$I = \int_0^\infty \frac{\cos sx}{x^2+a^2}\, dx \tag{1}$$

Differentiating both sides of (1) with respect to 's', we get

$$\frac{dI}{ds} = \int_0^\infty -\frac{x}{x^2+a^2} \sin sx\ dx \left[= -F_S\left(\frac{x}{x^2+a^2}\right)\right] \tag{2}$$

$$= \int_0^\infty \frac{a^2 - (x^2+a^2)}{x(x^2+a^2)} \sin sx\, dx$$

$$= a^2 \int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\, dx - \int_0^{\pi/2} \frac{\sin sx}{x}\, dx$$

$$= a^2 \int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\, dx - \pi/2 \tag{3}$$

Differentiating both sides of (3) with respect to 's' again, we get

$$\frac{d^2 I}{ds^2} = a^2 \int_0^\infty \frac{\cos sx}{x^2+a^2}\, dx$$

i.e. $$\frac{d^2 I}{ds^2} - a^2 I = 0 \tag{4}$$

Solving the differential equation (4), we get

$$I = Ae^{as} + Be^{-as} \tag{5}$$

From (1), when $s = 0,\quad I = \left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right)_0^\infty = \frac{\pi}{2a}$

Using this in (5), we have $A + B = \frac{\pi}{2a}$ (6)

From (3), when $s = 0$, $\frac{dI}{ds} = -\frac{\pi}{2}$.

Using this in (5), we have $A - B = -\frac{\pi}{2a}$ (7)

Solving (6) and (7), we get $\quad A = 0$ and $B = \frac{\pi}{2a}$

Using this in (5), we have

$$I = F_C\left\{\frac{1}{x^2 + a^2}\right\} = \frac{\pi}{2a}e^{-as}$$

Also from (2), we have $\quad \frac{dI}{ds} = -F_S\left(\frac{x}{x^2 + a^2}\right) = -\frac{\pi}{2}e^{-as}$

$$\therefore \quad F_S\left(\frac{x}{x^2 + a^2}\right) = \frac{\pi}{2}e^{-as}.$$

Example 15

Find $f(x)$, if its Fourier sine transform is $\frac{s}{s^2 + 1}$.

Given $\quad F_S\{f(x)\} = \frac{s}{s^2 + 1}$

$$\therefore \quad f(x) = F_S^{-1}\left\{\frac{s}{s^2 + 1}\right\} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2 + 1}\sin xs\ ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(s^2 + 1 - 1)}{s(s^2 + 1)}\sin xs\ ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin xs}{s}ds - \frac{2}{\pi}\int_0^\infty \frac{\sin xs}{s(s^2 + 1)}ds$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2} - \frac{2}{\pi}\int_0^\infty \frac{\sin xs}{s(s^2 + 1)}ds \tag{1}$$

Differentiating (1) with respect to x, we get

$$\frac{df}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{\cos xs}{s^2 + 1}ds\left[= -F_C^{-1}\left(\frac{1}{s^2 + 1}\right)\right] \tag{2}$$

and $\quad \frac{d^2 f}{dx^2} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2 + 1}\sin xs\ ds = f$

i.e. $$\frac{d^2 f}{dx^2} - f = 0 \tag{3}$$

Solving (3), we get $f = Ae^{x} + Be^{-x}$ (4)

From (1), when $x = 0$, $f = 1$. Using this in (4)

we have $$A + B = 1 \tag{5}$$

From (2), when $x = 0$, $\frac{df}{dx} = -\frac{2}{\pi}(\tan^{-1} s)_0^{\infty} = -1$. Using this in (4),

we have $$A - B = -1 \tag{6}$$

Solving (5) and (6), we get $A = 0$ and $B = 1$

$$\therefore \qquad f(x) = F_S^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-x}$$

Also from (2), we have $f_c^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-x}$

Example 16

Find the Fourier sine and cosine transforms of x^{n-1}. Hence deduce that $\frac{1}{\sqrt{x}}$ is self-reciprocal under both the transforms. Also find $F\left\{\frac{1}{\sqrt{|x|}}\right\}$.

Consider $$\int_0^{\infty} x^{n-1} e^{-isx}\, dx = \int_0^{\infty} \left(\frac{t}{is}\right)^{n-1} e^{-t}\left(\frac{dt}{is}\right)$$

on putting $isx = t$

$$= \left(\frac{-i}{s}\right)^n \int_0^{\infty} t^{n-1} e^{-t}\, dt$$

$$= \frac{\boxed{(n)}}{s^n} \cdot (e^{-i\pi/2})^n \quad \left[\because e^{-i\pi/2} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i\right]$$

i.e. $$\int_0^{\infty} x^{n-1}(\cos sx - i\sin sx)\, dx = \left\{\frac{\boxed{(n)}}{s^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)\right\} \tag{1}$$

Equating real parts on both sides of (1) we get

$$\int_0^{\infty} x^{n-1} \cos sx\, dx = \frac{\boxed{(n)}}{s^n} \cos\frac{n\pi}{2}$$

i.e. $$F_C(x^{n-1}) = \frac{\boxed{(n)}}{s^n} \cos\frac{n\pi}{2} \tag{2}$$

Similarly, equating imaginary parts on both sides of (1), we get,

$$F_S(x^{n-1}) = \frac{\overline{|(n)}}{s^n} \sin \frac{n\pi}{2} \tag{3}$$

Assuming that $F_C\{f(x)\} = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(x) \cos sx \, dx$ and

$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(x) \sin sx \, dx \text{ and taking } n = \frac{1}{2}$$

we get

$$F_C\left(\frac{1}{\sqrt{x}}\right) = F_S\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}} \frac{\overline{|\left(\frac{1}{2}\right)}}{\sqrt{s}} \cdot \sin \frac{\pi}{4}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

Thus $\frac{1}{\sqrt{x}}$ is self-reciprocal under both Fourier cosine and sine transforms.

Now $F\left\{\frac{1}{\sqrt{|x|}}\right\} = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} e^{-isx} \, dx$

$$= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} (\cos sx - i \sin sx) \, dx$$

$$= 2 \int\limits_0^\infty \frac{1}{\sqrt{x}} \cos sx \, dx \quad \text{[by the property of even and odd functions]}$$

$$= 2 \frac{\overline{|\left(\frac{1}{2}\right)}}{\sqrt{s}} \cos \frac{\pi}{4} \qquad \left[\text{by putting } n = \frac{1}{2} \text{ in (2)}\right]$$

$$= \sqrt{\frac{2\pi}{s}}$$

Example 17

Solve, for $f(x)$, the integral equation

$$\int\limits_0^\infty f(x) \cos sx \, dx = \begin{cases} 1 - s, & \text{for } 0 < s < 1 \\ 0, & \text{for } s > 1 \end{cases}$$

Given

$$F_C\{f(x)\} = \begin{cases} 1 - s, & \text{for } 0 < s < 1 \\ 0, & \text{for } s > 1 \end{cases}$$

$$= \bar{f}_C(s), \text{ say.}$$

$$\therefore \quad f(x) = F_C^{-1}\{\bar{f}_C(s)\}$$
$$= \frac{2}{\pi}\int_0^1 (1-s)\cos xs \, ds$$
$$= \frac{2}{\pi}\left[(1-s)\frac{\sin xs}{x} - \frac{\cos xs}{x^2}\right]_0^1$$
$$= \frac{2}{\pi x^2}(1-\cos x)$$

Example 18

Solve, for $f(x)$, the integral equation

$$\int_0^\infty f(x)\sin xt \, dx = \begin{cases} 1, & \text{for } 0 \le t < 1 \\ 2, & \text{for } 1 \le t < 2 \\ 0, & \text{for } t \ge 2 \end{cases}$$

Note ✍

(Instead of the usual transform variable s, the letter 't' is used)

$$\text{Given} \quad F_S\{f(x)\} = \bar{f}_S(t) = \begin{cases} 1, & \text{for } 0 \le t \le 1 \\ 2, & \text{for } 1 \le t < 2 \\ 0, & \text{for } t \ge 2 \end{cases}$$

$$\therefore \quad f(x) = F_S^{-1}\{\bar{f}_S(t)\}$$
$$= \frac{2}{\pi}\left[\int_0^1 1 \cdot \sin xt \, dt + \int_1^2 2 \cdot \sin xt \, dt\right]$$
$$= \frac{2}{\pi}\left[\left(\frac{-\cos xt}{x}\right)_0^1 + 2\left(\frac{-\cos xt}{x}\right)_1^2\right]$$
$$= \frac{2}{\pi x}[(1-\cos x) + 2(\cos x - \cos 2x)]$$
$$= \frac{2}{\pi x}(1 + \cos x - 2\cos 2x).$$

Exercise 4(a)

Part A (Short Answer Questions)

1. State both the forms of Fourier integral theorem.
2. Write down the Fourier cosine and sine integral representations of $f(x)$.
3. Write down the complex Fourier transform pair.
4. Write down the Fourier cosine transform pair.

5. Write down the Fourier sine transform pair.

6. Solve, for $f(x)$, the equation $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx}\, dx = \phi(s)$.

7. Find $f(x)$, if $\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx \; dx = \phi(s)$.

8. Find $f(x)$, if $\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx \; dx = \phi(s)$.

9. Find $f(x)$, if $\int_{-\infty}^{\infty} f(x)e^{isx}\, dx = \phi(s)$.

10. How are Fourier and Laplace transforms related?
11. Find the Fourier cosine transform of $e^{-ax} (a > 0)$.
12. Find the Fourier sine transform of $e^{-ax} (a > 0)$.
13. Find the Fourier exponential transform of $e^{-a|x|}$, $a > 0$.
14. Find the Fourier cosine transform of $2e^{-5x} + 5e^{-2x}$.
15. Find the Fourier sine transform of $e^{-2x} + 4e^{-3x}$.
16. Find the Fourier complex transform of $f(x)$,

if $\quad f(x) = \begin{cases} k, & \text{in } |x| \leq l \\ 0, & \text{in } |x| > l \end{cases}$

Part B

17. Find the Fourier integral representation of

$$f(x) = \begin{cases} 1, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence evaluate $\int_{0}^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda$.

18. Use the appropriate Fourier integral formula to prove that

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos x\lambda}{\lambda^2 + a^2} d\lambda.$$

19. Find the Fourier sine integral formula for $\frac{\pi}{2} e^{-x}$.
20. Use Fourier integral formula to prove that

$$\int_{0}^{\infty} \left(\frac{1 - \cos \pi\lambda}{\lambda} \right) \sin x\lambda \; d\lambda = \begin{cases} \pi/2, & \text{when } \quad 0 < x < \pi \\ 0, & \text{when } \quad x > \pi \end{cases}$$

21. Use Fourier integral formula to prove that

$$\int_0^\infty \frac{\sin \pi\lambda \sin x\lambda}{1-\lambda^2}\, d\lambda = \begin{cases} \dfrac{\pi}{2}\sin x, & \text{when } 0 \le x \le \pi \\ 0, & \text{when } x > \pi \end{cases}$$

22. Find the Fourier transform of $f(x)$, given by

$$f(x) = \begin{cases} x, & \text{for } |x| \le a \\ 0, & \text{for } |x| > a \end{cases}$$

23. Find the Fourier transform of $f(x)$, given by

$$f(x) = \begin{cases} 1-|x|, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence show that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$.

24. Find the inverse Fourier transform of $\phi(s)$, defined as

$$\phi(s) = \begin{cases} 1+s^2, & \text{for } |s| < 1 \\ 0, & \text{for } |s| > 1 \end{cases}$$

25. Find the inverse Fourier transform of $\phi(s)$, defined as

$$\phi(s) = \begin{cases} 1, & \text{for } |s| < s_0 \\ 0, & \text{for } |s| > s_0 \end{cases}$$

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin mx}{x}\, dx, m > 0$.

26. Find the Fourier cosine transform of e^{-ax}; $a > 0$. Hence find $F_C\{xe^{-ax}\}$ and $F\{|x|e^{-a|x|}\}$.
27. Find the Fourier sine transform of e^{-ax}; $a > 0$. Hence find $F_S\{xe^{-ax}\}$ and $F\{xe^{-a|x|}\}$.
28. Find the Fourier transform of $e^{-ax^2}\cos bx$.
29. Find the inverse transform of e^{-s^2}.
30. Find the inverse Fourier transform of $\dfrac{1}{(1+s^2)^2}$.

(**Hint:** Use contour integration).

31. Find the Fourier sine transform of $f(x)$, given by

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 0, & \text{for } x > 1 \end{cases}$$

32. Find the Fourier sine transform of $f(x)$, if

$$f(x) = \begin{cases} 0, & 0 < x < a \\ x, & a \le x \le b \\ 0, & x > b \end{cases}$$

33. Find the Fourier cosine transform of $f(x)$, if

$$f(x) = \begin{cases} \cos x, & \text{for } 0 < x < a \\ 0, & \text{for } x > a \end{cases}$$

34. Find the Fourier sine transform of $\dfrac{x}{1+x^2}$.

35. Find the Fourier cosine transform of e^{-x^2}, using the definition directly.

36. Find the Fourier sine transform of $\dfrac{e^{-ax}}{x}(a > 0)$, using the definition directly.

37. Find the inverse Fourier cosine transform of $\dfrac{1}{1+s^2}$.

38. Find the inverse Fourier sine transform of $\dfrac{e^{-as}}{s}(a > 0)$, using the definition directly.

39. Find $f(x)$, if $\displaystyle\int_0^\infty f(x)\cos sx \; dx = \frac{\sin s}{s}$.

40. Find $f(x)$, if $\displaystyle\int_0^\infty f(x)\sin sx \; dx = e^{-as}$.

4.6 PROPERTIES OF FOURIER TRANSFORMS

1. Linearity property

F is a linear operator,i.e. $F[(c_1 f_1(x)+c_2 f_2(x)] = c_1 F\{f_1(x)\}+c_2 F\{f_2(x)\}$, where c_1 and c_2 are constants.

Proof

$$F\,[c_1 f_1(x) + c_2 f_2(x)] = \int_{-\infty}^{\infty} [c_1 f_1(x) + c_2 f_2(x)]\, e^{-isx}\, dx$$

$$= c_1 \int_{-\infty}^{\infty} f_1(x) e^{-isx}\, dx + c_2 \int_{-\infty}^{\infty} f_2(x) e^{-isx}\, dx$$

$$= c_1 F\{f_1(x)\} + c_2 F\{f_2(x)\}.$$

2. Change of scale property

If $\quad F\{f(x)\} = \bar{f}(s), \quad$ then

$$F\{f(ax)\} = \frac{1}{|a|}\bar{f}\left\{\frac{s}{a}\right\}$$

Proof

$$F\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{-isx}\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} f(t)e^{-ist/a}\cdot\frac{dt}{a}, \text{ on putting } ax = t \text{ and assuming that } a > 0.$$

$$= \frac{1}{a}\bar{f}\left\{\frac{s}{a}\right\}$$

But

$$F\{f(ax)\} = \int_{\infty}^{-\infty} f(t)e^{-ist/a}\cdot\frac{\mathrm{d}t}{a}, \text{ if } a < 0$$

$$= -\frac{1}{a}\bar{f}\left\{\frac{s}{a}\right\}$$

$\therefore$

$$F\{f(ax)\} = \frac{1}{|a|}\bar{f}\left\{\frac{s}{a}\right\}$$

Similarly, $\quad F_C\{f(ax)\} = \frac{1}{a}\cdot\bar{f}_C\left\{\frac{s}{a}\right\}$ and $F_S\{f(ax)\} = \frac{1}{a}\cdot\bar{f}_S\left\{\frac{s}{a}\right\}$.

3. Shifting property (Shifting in *x*)

If $\quad F\{f(x)\} = \bar{f}(s), \quad$ then $\quad F\{f(x-a)\} = e^{-ias}\bar{f}(s).$

Proof

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{-isx}\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} f(t)e^{-is(t+a)}\,\mathrm{d}t, \text{ on putting } t = x - a$$

$$= e^{-ias}\bar{f}(s).$$

4. Shifting in respect of *s*

If $\quad F\{f(x)\} = \bar{f}(s), \quad$ then $\quad F\{e^{-iax}f(x)\} = \bar{f}(s+a)$

Proof

$$F\{e^{-iax} f(x)\} = \int_{-\infty}^{\infty} e^{-iax} f(x) e^{-isx}\,dx$$
$$= \int_{-\infty}^{\infty} f(x) e^{-i(s+a)x}\,dx$$
$$= \bar{f}(s+a)$$

Similarly $\qquad F\{e^{iax} f(x)\} = \bar{f}(s-a).$

5. Modulation theorem

If $\quad F\{f(x)\} = \bar{f}(s), \quad$ then $\quad F\{f(x)\cos ax\} = \frac{1}{2}[\bar{f}(s+a) + \bar{f}(s-a)].$

Proof

$$F\{f(x)\cos ax\} = \frac{1}{2} F[f(x)(e^{iax} + e^{-iax})]$$
$$= \frac{1}{2}[F\{f(x)e^{iax}\} + F\{f(x)e^{-iax}\}]$$
$$= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)].$$

Corollaries

(i) $F_C\{f(x)\cos ax\} = \frac{1}{2}\{\bar{f}_C(s+a) + \bar{f}_C(s-a)\}$

(ii) $F_C\{f(x)\sin ax\} = \frac{1}{2}\{\bar{f}_S(a+s) + \bar{f}_S(a-s)\}$

(iii) $F_S\{f(x)\cos ax\} = \frac{1}{2}\{\bar{f}_S(s+a) + \bar{f}_S(s-a)\}$

(iv) $F_S\{f(x)\sin ax\} = \frac{1}{2}\{\bar{f}_C(s-a) - \bar{f}_C(s+a)\}$

6. Conjugate symmetry property

If $F\{f(x) = \bar{f}(s)$, then $F\{f^*(-x)\} = [\bar{f}(s)]^*$, where * denotes complex conjugate.

Proof

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-isx} dx$$

$$\therefore \qquad [\bar{f}(s)]^* = \int_{-\infty}^{\infty} f^*(x)e^{isx}dx$$

$$= \int_{-\infty}^{\infty} f^*(-t)e^{-ist}dt, \text{ on putting } x = -t$$

$$= F\{f^*(-x)\}$$

Note ✍

1. $F\{f^*(x)\} = [\bar{f}(-s)]^*$.
2. If $f(x)$ is a real valued function, then $F\{f(-x)\} = [\bar{f}(s)]^*$.

5. Transform of derivatives

If $f(x)$ is continuous, $f'(x)$ is piecewise continuously differentiable, $f(x)$ and $f'(x)$ are absolutely integrable in $(-\infty, \infty)$ and $\lim_{x\to\pm\infty} [f(x)] = 0$, then

$$F\{f'(x) = is\bar{f}(s), \text{ where } \bar{f}(s) = F\{f(x)\}$$

Proof

By the first three conditions given, $F\{f(x)$ and $F\{f'(x)\}$ exist.

$$F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-isx}\,\mathrm{d}x$$

$$= [e^{-isx}f(x)]_{-\infty}^{\infty} + is\int_{-\infty}^{\infty} e^{-isx}f(x)\,\mathrm{d}x, \text{ on integrating by parts.}$$

$$= 0 + \mathrm{is}F\{f(x)\}, \text{ by the given condition.}$$

$$= \mathrm{is}\ \bar{f}(s).$$

The theorem can be extended as follows.

If $f, f', f'', \ldots, f^{(n-1)}$ are continuous, $f^{(n)}$ is piecewise continuous, $f, f' f'' \ldots, f^{(n)}$ are absolutely integrable in $(-\infty, \infty)$ and $f, f' f'', \ldots, f^{(n-1)} \to 0$ as $x \to \pm\infty$, then

$$F\{f^{(n)}(x)\} = (is)^n \bar{f}(s)$$

where $\bar{f}(s) = F\{f(x)\}$.

Corollaries

(i) $F_C\{f'(x) = sF_S\{f(x)\} - f(0)$.
(ii) $F_S\{f'(x)\} = -sF_C\{f(x)\}$.
(iii) $F_C\{f''(x)\} = -s^2F_C\{f(x)\} - f'(0)$.
(iv) $F_S\{f''(x)\} = -s^2F_S\{f(x)\} + sf(0)$.

Note ✍

1. *When we solve boundary value problems (partial differential equations) by using Fourier cosine and sine transforms, the modified forms of the above formulas given below will be used. It is assumed that transforms are taken with respect to the variable x. We use the following notations.*

$$F_C\{f(x, y)\} = \bar{f}_C(s, y) \text{ and } F_S\{f(x, y)\} = \bar{f}_S(s, y)$$

1. $F_C\left\{\dfrac{\partial f}{\partial x}\right\} = s\bar{f}_s(s, y) - f(0, y).$
2. $F_S\left\{\dfrac{\partial f}{\partial x}\right\} = -s\bar{f}_C(s, y).$
3. $F_C\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = -s^2\bar{f}_C(s, y) - \dfrac{\partial f}{\partial x}(0, y).$
4. $F_S\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = -s^2\bar{f}_S(s, y) + sf(0, y).$

2. *If $f(0, y)$ is given but $\dfrac{\partial f}{\partial x}(0, y)$ is not known in a boundary value problem, Fourier sine transform is used. On the other hand, if $\dfrac{\partial f}{\partial x}(0, y)$ is given but $f(0, y)$ is not known, Fourier cosine transform is used.*
3. *When the transforms are taken with respect to x,*

$$F_C\left\{\frac{\partial f}{\partial y}\right\} = \int_0^\infty \frac{\partial f}{\partial y}\cos sx\,dx = \frac{\partial}{\partial y}\int_0^\infty f(x, y)\cos sx\,dx$$

$$= \frac{\partial}{\partial y}\bar{f}_C(s, y)$$

Extending, we get $$F_C\left\{\frac{\partial^r f}{\partial y^r}\right\} = \frac{\partial^r}{\partial y^r}\bar{f}_C(s, y).$$

Similarly, $$F_S\left\{\frac{\partial^r f}{\partial y^r}\right\} = \frac{\partial^r}{\partial y^r}\bar{f}_S(s, y).$$

8. Derivatives of the transform

If $\quad F\{f(x)\} = \bar{f}(s),\quad$ then $\quad -iF\{xf(x)\} = \dfrac{d}{ds}\bar{f}(s)$

Proof

$$\bar{f}(s) = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

$$\therefore \qquad \frac{d}{ds}\bar{f}(s) = \int_{-\infty}^{\infty} \frac{d}{ds}[e^{-isx} f(x)]\, dx$$

$$= (-i) \int_{-\infty}^{\infty} e^{-isx}[x f(x)]\, dx = -i \cdot F\{x f(x)\}$$

Extending, we get, $\dfrac{d^r}{ds^r}\bar{f}(s) = (-i)^r F\{x^r f(x)\}$

Definition

$\int_{-\infty}^{\infty} f(x-u)g(u)\, du$ is called *the convolution product* or simply *the convolution* of the functions $f(x)$ and $g(x)$ and is denoted by $f(x)^* g(x)$.

9. Convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms.

i.e if $F\{f(x)\} = \bar{f}(s)$ and $F\{g(x)\} = \bar{g}(s)$, then

$$F\{f(x)^* g(x)\} = \bar{f}(s) \cdot \bar{g}(s).$$

Proof

$$F\{f(x)^* g(x)\} = \int_{-\infty}^{\infty} f(x)^* g(x) e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-u)g(u)\, du\right] e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} g(u) \left[\int_{-\infty}^{\infty} f(x-u)e^{-isx}\, dx\right] du,$$

on changing the order of integration.

$$= \int_{-\infty}^{\infty} g(u)\left[e^{-ius}\bar{f}(s)\right] du, \text{ by the shifting property.}$$

$$= \bar{f}(s) \cdot \int_{-\infty}^{\infty} g(u)e^{-isu}\, du$$

$$= \bar{f}(s) \cdot \bar{g}(s)$$

Inverting, we get $\quad F^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(x)^{*}g(x)$

$$= F^{-1}\{\bar{f}(s)\}^{*}F^{-1}(\bar{g}(s)\}$$

10. Parseval's identity (or) energy theorem

If $\quad F\{f(x)\} = \bar{f}(s)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2 \, ds$$

By convolution theorem,

$$f(x)^{*}g(x) = F^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

i.e. $$\int_{-\infty}^{\infty} f(u)\cdot g(x-u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\bar{g}(s)e^{ixs}\, ds \qquad (1)$$

Putting $x = 0$ in (1), we get,

$$\int_{-\infty}^{\infty} f(u)g(-u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\bar{g}(s)\, ds \qquad (2)$$

(2) is true for any $g(u)$; take $g(u) = [f(-u)]^{*}$ and hence $g(-u) = [f(u)]^{*}$, where $[f(u)]^{*}$ is the complex conjugate of $f(u)$.

Also $\quad \bar{g}(s) = F\{g(x)\} = F\{f(-x)\}^{*} = [F\{f(x)\}]^{*} = [\bar{f}(s)]^{*}$

(by the conjugate symmetry property)

Using these in (2), we get

$$\int_{-\infty}^{\infty} f(u)[f(u)]^{*}\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)[\bar{f}(s)]^{*}\, ds$$

i.e. $$\int_{-\infty}^{\infty} |f(u)|^2\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds.$$

or $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds,$$ on changing the dummy variable.

Note ✍

Had we assumed that $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx}\,\mathrm{d}u$, *the Parseval's identity would have assumed the form* $\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |\bar{f}(s)|^2\,\mathrm{d}s$.

Property 11

If $\bar{f}_C(s), \bar{g}_C(s)$ are the Fourier cosine transforms and $\bar{f}_S(s), \bar{g}_S(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

(i) $\int_0^{\infty} f(x)g(x)\,\mathrm{d}x = \int_0^{\infty} \bar{f}_C(s)\bar{g}_C(s)\,\mathrm{d}s = \int_0^{\infty} \bar{f}_S(s)\bar{g}_S(s)\,\mathrm{d}s$

(ii) $\int_0^{\infty} |f(x)|^2\,\mathrm{d}x = \int_0^{\infty} |\bar{f}_C(s)|^2\,\mathrm{d}s = \int_0^{\infty} |\bar{f}_S(s)|^2\,\mathrm{d}s$,

which is Parseval's identity for Fourier cosine and sine transforms.

Proof

(i)
$$\int_0^{\infty} \bar{f}_C(s)\bar{g}_C(s)\,\mathrm{d}s = \int_0^{\infty} \bar{f}_C(s)\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} g(x)\cos sx\,\mathrm{d}x\right]\mathrm{d}s$$
$$= \int_0^{\infty} g(x)\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} \bar{f}_C(s)\cos xs\,\mathrm{d}s\right]\mathrm{d}x,$$
changing the order of integration
$$= \int_0^{\infty} f(x)g(x)\,\mathrm{d}x.$$

Note ✍

Had we used the definition $\bar{f}_C(s) = \int_0^{\infty} f(x)\cos sx\,\mathrm{d}x$ *and* $f(x) = \frac{2}{\pi}\int_0^{\infty} \bar{f}_C(s)\cos xs\,\mathrm{d}s$, *this result would have been* $\int_0^{\infty} f(x)g(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty} \bar{f}_C(s)\bar{g}_C(s)\,\mathrm{d}s$.

Similarly we can prove the other part of the result.

(ii) Replacing $g(x) = f^*(x)$ in (i) and noting that $F_C\{f^*(x)\} = \{\bar{f}_C(s)\}^*$ and

$F_S\{f^*(x)\} = \{\bar{f}_S(s)\}^*$, we get $\int_0^\infty f(x) f^*(x)\, dx = \int_0^\infty \bar{f}_C(s)\{\bar{f}_C(s)\}^*\, ds$

$= \int_0^\infty \bar{f}_S(s)\{\bar{f}_S(s)\}^*\, ds$ i.e. $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |\bar{f}_C(s)|^2\, ds = \int_0^\infty |\bar{f}_S(s)|^2\, ds$

Note ✍

Had we adopted the other definition, the result would have been

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty} |\bar{f}_C(s)|^2 ds = \frac{2}{\pi}\int_{-\infty}^{\infty} |\bar{f}_S(s)|^2 ds.]$$

Property 12

If $F_C\{f(x)\} = \bar{f}_C(s)$ and $F_S\{f(x)\} = \bar{f}_S(s)$, then

(i) $\dfrac{d}{ds}\{\bar{f}_C(s)\} = -F_S\{xf(x)\}$; and

(ii) $\dfrac{d}{ds}\{\bar{f}_S(s)\} = F_C\{xf(x)\}$.

Proof

$$\bar{f}_C(s) = \int_0^\infty f(x)\cos sx\, dx$$

$$\therefore \qquad \frac{d}{ds}\{\bar{f}_C(s)\} = \int_0^\infty f(x)\{-x\sin sx\}\, dx$$

$$= -\int_0^\infty \{xf(x)\}\sin sx\, dx$$

$$= -F_S\{xf(x)\}$$

Similarly the result (ii) follows.

Worked Examples 4(b)

Example 1

Find the function $f(x)$ for which the Fourier transform is

$$2\sin[3(s-2\pi)]/(s-2\pi)]$$

Let us first find $F^{-1}\left\{\dfrac{2\sin 3s}{s}\right\}$

$$F^{-1}\left\{\frac{2\sin 3s}{s}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}e^{ixs}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}(\cos xs + i\sin xs)\,ds$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{2\sin 3s\cos xs}{s}\,ds$$

{by the property of odd and even functions}

$$= \frac{1}{\pi}\int_{0}^{\infty}\left\{\frac{\sin(3+x)s}{s} + \frac{\sin(3-x)s}{s}\right\}ds$$

$$= \begin{cases} \dfrac{1}{\pi}\left(\dfrac{\pi}{2}+\dfrac{\pi}{2}\right), & \text{if } 3+x>0 \text{ and } 3-x>0 \\ 0, & \text{if } 3+x>0 \text{ and } 3-x<0 \text{ or} \\ & 3+x<0 \text{ and } 3-x>0 \end{cases}$$

$$\left[\because \int_{0}^{\infty}\frac{\sin mx}{x}\,dx = \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \text{ according as } m>0 \text{ or } <0\right]$$

$$= \begin{cases} 1, & \text{if } -3<x<3 \\ 0, & \text{if } x<-3 \text{ or } x>3 \end{cases}$$

$$= \begin{cases} 1, & \text{if } |x|<3 \\ 0, & \text{if } |x|>3 \end{cases} \tag{1}$$

By the shifting property, $F\{e^{iax}f(x)\} = \bar{f}(s-a)$

$$\therefore \qquad F^{-1}\{\bar{f}(s-a)\} = e^{iax}\cdot F^{-1}\{\bar{f}(s)\}$$

Thus,

$$F^{-1}\left[\frac{2\sin\{3(s-2\pi)\}}{s-2\pi}\right] = e^{-i2\pi x}\cdot F^{-1}\left[\frac{2\sin 3s}{s}\right]$$

$$= e^{i2\pi x}\times\begin{cases} 1, & \text{if } |x|<3 \\ 0, & \text{if } |x|>3 \end{cases} \qquad \text{from (1)}$$

$$= \begin{cases} e^{i2\pi x}, & \text{if } |x|<3 \\ 0, & \text{if } |x|>3 \end{cases}$$

Example 2

Find the Fourier transform of $f(x)$, defined as

$$f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

Hence find $\quad F\left[f(x)\left\{1+\cos\dfrac{\pi x}{a}\right\}\right]$

By Example (5) of Section 2(a),

$$F\{f(x)\} = \frac{2}{s}\sin as = \bar{f}(s), \text{ say} \tag{1}$$

By Modulation theorem,

$$\begin{aligned} F\left\{f(x)\cos\frac{\pi x}{a}\right\} &= \frac{1}{2}\left[\bar{f}\left(s+\frac{\pi}{a}\right) + \bar{f}\left(s-\frac{\pi}{a}\right)\right] \\ &= \frac{1}{s+\frac{\pi}{a}}\sin a\left(s+\frac{\pi}{a}\right) + \frac{1}{s-\frac{\pi}{a}}\sin a\left(s-\frac{\pi}{a}\right) \\ &= \frac{a}{as+\pi}\{-\sin as\} + \frac{a}{as-\pi}\{-\sin as\} \\ &= \frac{2a^2 s\sin as}{\pi^2 - a^2s^2} \end{aligned} \tag{2}$$

Adding (1) and (2), we get

$$\begin{aligned} F\left[f(x)\left\{1+\cos\frac{\pi x}{a}\right\}\right] &= 2\sin as\left[\frac{1}{s} + \frac{a^2 s}{\pi^2 - a^2 s^2}\right] \\ &= \frac{2\pi^2 \sin as}{s(\pi^2 - a^2s^2)} \end{aligned}$$

Example 3

Find the Fourier transform of $f(x) = x^n e^{-ax}U(x)$, where $U(x)$ is the unit step function.

$$\begin{aligned} F\{e^{-ax}U(x)\} &= \int_{-\infty}^{0} e^{-ax}\cdot 0 e^{-isx}\,\mathrm{d}x + \int_{0}^{\infty} e^{-ax}\cdot 1\cdot e^{-isx}\,\mathrm{d}x \\ &= \left[\frac{e^{-(a+is)x}}{-(a+is)}\right]_0^{\infty} = \frac{1}{a+is} \end{aligned}$$

By property (8),

$$\begin{aligned} F\{x^n e^{-ax}U(x)\} &= \frac{1}{(-i)^n}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left\{\frac{1}{a+is}\right\} \\ &= \frac{1}{(-i)^n}\frac{(-1)^n i^n n!}{(a+is)^{n+1}} = \frac{n!}{(a+is)^{n+1}} \end{aligned}$$

Example 4

Find the Fourier transform of $\left\{\dfrac{\sin ax}{x}\right\}$ and hence prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} = a\pi$.

$$F\left\{\frac{\sin ax}{x}\right\} = \int_{-\infty}^{\infty} \left\{\frac{\sin ax}{x}\right\} e^{-isx}\, \mathrm{d}x$$

$$= 2\int_{0}^{\infty} \frac{\sin ax \cos sx}{x}\, \mathrm{d}x$$

$$= \int_{0}^{\infty} \left\{\frac{\sin (a+s)x}{x} + \frac{\sin (a-s)x}{x}\right\} \mathrm{d}x$$

$$= \begin{cases} \pi, & \text{if } |s| < a \\ 0, & \text{if } |s| > a \end{cases} \quad \text{[proceeding as in Example (1)]}$$

Now $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, \mathrm{d}s$, by Parseval's identity.

$$\therefore \quad \int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = \frac{1}{2\pi}\int_{-a}^{a} \pi^2 ds = \frac{1}{2\pi}\cdot \pi^2 \cdot 2a = a\pi.$$

Example 5

Find the Fourier transform of $f(x)$, if

$$f(x) = \begin{cases} 1 - |x|, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence prove that $\displaystyle\int_{0}^{\infty} \frac{\sin^4 x}{x^4}\, \mathrm{d}x = \frac{\pi}{3}$

$$F\{f(x)\} = \int_{-1}^{1} \{1 - |x|\}\, e^{-isx}\, \mathrm{d}x$$

$$= 2\int_{0}^{1} (1-x)\cos sx\ \mathrm{d}x \quad \text{by property of even and odd functions.}$$

$$= 2\left[(1-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$= 2\frac{(1-\cos s)}{s^2} = \frac{4\sin^2\frac{s}{2}}{s^2}$$

By Parseval's identity,

$$\int_{-1}^{1}\{1-|x|\}^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{16\sin^4\left(\frac{s}{2}\right)}{s^4}\,\mathrm{d}s$$

i.e. $$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{16\sin^4\left(\frac{s}{2}\right)}{s^4}\,\mathrm{d}s = 2\int_0^1(1-x)^2\,\mathrm{d}x = \frac{2}{3}$$

Putting $\frac{s}{2} = t$, we get,

$$\frac{16}{\pi}\int_0^{\infty}\frac{\sin^4 t}{16t^4}\cdot 2\,\mathrm{d}t = \frac{2}{3}$$

$$\therefore \qquad \int_0^{\infty}\frac{\sin^4 t}{t^4}dt = \frac{\pi}{3} \quad \text{or} \quad \int_0^{\infty}\frac{\sin^4 x}{x^4}\,\mathrm{d}x = \frac{\pi}{3}.$$

Example 6

Using Parseval's identity for Fourier cosine and sine transforms of e^{-ax}, evaluate

(i) $$\int_0^{\infty}\frac{\mathrm{d}x}{(a^2+x^2)^2} \quad \text{and} \quad \text{(ii)} \int_0^{\infty}\frac{x^2}{(a^2+x^2)^2}\,\mathrm{d}x.$$

(i) $$F_C(e^{-ax}) = \int_0^{\infty}e^{-ax}\cos sx\,\,\mathrm{d}x = \frac{a}{s^2+a^2}$$

By Parseval's identity,

$$\int_0^{\infty}|f(x)|^2\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\infty}|\bar{f}_c(s)|^2\,\mathrm{d}s$$

$$\therefore \qquad \int_0^\infty e^{-2ax}\, dx = \frac{2}{\pi} a^2 \int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

i.e.
$$\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{2a^2} \cdot \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$
$$= \frac{\pi}{4a^3}, \quad \text{if} \quad a > 0$$

Changing the dummy variable s into x, we get the first result.

(ii) Now $\quad F_S(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx \; dx = \dfrac{s}{s^2+a^2}.$

By Parseval's identity,

$$\int_0^\infty |f(x)|^2\, dx = \frac{2}{\pi} \int_0^\infty |\bar{f}_s(s)|^2\, ds$$

i.e.
$$\frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds = \int_0^\infty e^{-2ax}\, dx$$

$$\therefore \qquad \int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^2} = \frac{\pi}{4a}, \quad \text{if } a > 0, \text{ on changing the dummy variables.}$$

Example 7

Use transfrom methods to evaluate

$$\text{(i)} \quad \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} \quad \text{and} \quad \text{(ii)} \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx.$$

(i) Let $\quad f(x) = e^{-x}$ and $g(x) = e^{-2x}$

Then $\quad \bar{f}_C(s) = \dfrac{1}{s^2+1}$ and $\bar{g}_C(s) = \dfrac{2}{s^2+4}$

By property (11),

$$\int_0^\infty f(x)g(x)\, dx = \frac{2}{\pi} \int_0^\infty \bar{f}_C(s)\bar{g}_C(s)\, ds$$

$$\therefore \qquad \int_0^\infty e^{-3x}\,\mathrm{d}x = \frac{2}{\pi}\int_0^\infty \frac{2}{(s^2+1)(s^2+4)}\,\mathrm{d}s$$

$$\text{i.e.} \qquad \int_0^\infty \frac{\mathrm{d}s}{(s^2+1)(s^2+4)} = \frac{\pi}{4}\left(\frac{e^{-3x}}{-3}\right)_0^\infty = \frac{\pi}{12}$$

Changing 's' into x, we get

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)(x^2+4)} = \frac{\pi}{12}$$

(ii) Let $\quad f(x) = e^{-ax}$ and $g(x) = e^{-bx}; \quad a, b > 0$

Then $\quad \bar{f}_S(s) = \dfrac{s}{s^2+a^2} \quad$ and $\quad \bar{g}_S(s) = \dfrac{s}{s^2+b^2}$

By property (11)

$$\int_0^\infty f(x)g(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\cdot \bar{g}_S(s)\,\mathrm{d}s$$

$$\therefore \qquad \int_0^\infty e^{-(a+b)x}\,\mathrm{d}x = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,\mathrm{d}s$$

$$\text{i.e.} \qquad \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,\mathrm{d}s = \frac{\pi}{2}\cdot\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2(a+b)}$$

Changing 's' into x, we get

$$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{2(a+b)}$$

Example 8

(i) Find $F_S(e^{-ax})$ and hence find $F_C(xe^{-ax})$

(ii) Find $F_C\left(e^{-a^2x^2}\right)$ and hence find $F_S\left(xe^{-a^2x^2}\right)$

(i) $F_S(e^{-ax}) = \dfrac{s}{s^2+a^2}, \quad$ if $a > 0$.

By property (12), $\quad \dfrac{\mathrm{d}}{\mathrm{d}s}\{F_S(e^{-ax})\} = F_C(xe^{-ax})$

$$\therefore \qquad F_C(xe^{-ax}) = \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{s}{s^2+a^2}\right\} = \frac{a^2-s^2}{(s^2+a^2)^2}.$$

(ii) In Worked Example 8 of Section 4(a), we have already found out $F_C\left(e^{-a^2x^2}\right)$ through $F\left(e^{-a^2x^2}\right)$. However, we shall find $F_C\left(e^{-a^2x^2}\right)$ by an alternative direct method.

Let $$I = F_C\left(e^{-a^2x^2}\right) = \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$$\therefore \quad \frac{dI}{ds} = -\int_0^\infty x e^{-a^2x^2} \sin sx \, dx$$

$$= \int_0^\infty \sin sx \, d\left[\frac{e^{-a^2x^2}}{2a^2}\right]$$

$$= \frac{1}{2a^2}\left[\left(e^{-a^2x^2} \sin sx\right)_0^\infty - s\int_0^\infty e^{-a^2x^2} \cos sx \, dx\right]$$

i.e. $$\frac{dI}{ds} = -\frac{s}{2a^2} I$$

Solving $$\log I = -\frac{s^2}{4a^2} + \log C$$

i.e., $$I = Ce^{-s^2/4a^2} \tag{1}$$

When $$s = 0, I = \int_0^\infty e^{-a^2x^2} \, dx = \frac{1}{a}\int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2a}$$

Using in (1), $$C = \frac{\sqrt{\pi}}{2a}$$

$$\therefore \quad F_C\left(e^{-a^2x^2}\right) = \frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}$$

By property (12), $$\frac{d}{ds}\left\{F_C\left(e^{-a^2x^2}\right)\right\} = -F_S\left(xe^{-a^2x^2}\right)$$

$$\therefore \quad F_S\left(xe^{-a^2x^2}\right) = -\frac{d}{ds}\left[\frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}\right]$$

$$= -\frac{\sqrt{\pi}}{2a}\left(-\frac{s}{2a^2}\right) e^{-s^2/4a^2}$$

$$= \frac{\sqrt{\pi}}{4a^3} s e^{-s^2/4a^2}$$

Example 9

Solve the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}, x > 0$, using Fourier transforms, given that $y(0) = 0$ and $y'(0) = 0$.

Taking Fourier complex transforms on both sides of the given differential equation, we have

$$(is)^2\bar{y}(s) + 3(is)\bar{y}(s) + 2\bar{y}(s) = F(e^{-x}), x > 0$$

$$= F\{U(x)\cdot e^{-x}\}, \quad \text{where} \quad U(x) \text{ is the unit step function}$$

i.e. $$[(is)^2 + 3(is) + 2]\bar{y}(s) = \int_0^\infty e^{-(1+is)x}\,dx = \frac{1}{1+is}$$

$\therefore$ $$\bar{y}(s) = \frac{1}{(is+1)^2(is+2)}$$

$$= -\frac{1}{is+1} + \frac{1}{(is+1)^2} + \frac{1}{is+2}, \text{ by partial fractions.}$$

$\therefore$ $$y = -F^{-1}\left\{\frac{1}{is+1}\right\} + F^{-1}\left\{\frac{1}{(is+1)^2}\right\} + F^{-1}\left\{\frac{1}{is+2}\right\}$$

$$= -U(x)e^{-x} + U(x)\cdot xe^{-x} + U(x)\cdot e^{-2x},$$

since $$F\{U(x)xe^{-x}\} = \int_0^\infty xe^{-(1+is)x}\,dx$$

$$= \left[x\left\{\frac{e^{-(1+is)x}}{-(1+is)}\right\} - \left\{\frac{e^{-(1+is)x}}{(1+is)^2}\right\}\right]_0^\infty$$

$$= \frac{1}{(1+is)^2}$$

i.e. $$y = -e^{-x} + xe^{-x} + e^{-2x}, x > 0$$

Note ✍

We have got the solution valid in $x > 0$, i.e. y and its derivatives are assumed to be zero in $x \le 0$. Though we have not explicitly used the conditions $y(0) = 0$ and $y'(0) = 0$, they have been taken care of, as the solution obtained satisfies both the conditions. However the Fourier complex transform method will fail if $y(0)$ and $y'(0)$ are prescribed non-zero values. In such situations, the Fourier sine and cosine transforms are used as explained in the following problem.

Example 10

Solve the equation $y'' + 3y' + 2y = e^{-x}$, $x > 0$, using transform method, given that $y(0) = 1$, $y'(0) = 2$.

The given conditions must be interpreted as $y(0+) = 1$ and $y'(0+) = 2$.

Taking Fourier cosine transforms on both sides of the given equation, we get

$$[-s^2\bar{y}_C(s) - y'(0)] + 3[s\bar{y}_S(s) - y(0)] + 2\bar{y}_C(s) = \frac{1}{s^2+1}$$

[Refer to Corollaries under property 7]

i.e. $$(2-s^2)\bar{y}_C(s)+3s\bar{y}_S(s)=\frac{1}{s^2+1}+5 \tag{1}$$

Taking Fourier sine transforms of the given equation, we get

$$[-s^2\bar{y}_S(s)+sy(0)]+3[-s\bar{y}_C(s)]+2\bar{y}_S(s)=\frac{s}{s^2+1}$$

i.e. $$-3s\bar{y}_C(s)+(2-s^2)\bar{y}_S(s)=\frac{s}{s^2+1}-s \tag{2}$$

Solving (1) and (2), we get

$$\begin{aligned}\bar{y}_S(s)&=\frac{5s-s^3}{\left(s^2+1\right)^2\left(s^2+4\right)}+\frac{s^3+13s}{\left(s^2+1\right)\left(s^2+4\right)}\\&=\frac{3s}{s^2+1}+\frac{2s}{\left(s^2+1\right)^2}-\frac{2s}{s^2+4},\text{ by partial fractions.}\end{aligned}$$

Taking inverse sine transforms, we get

$$y(x)=3e^{-x}-2e^{-2x}+xe^{-x}, x>0$$

Example 11

Solve the equation $(D^2-4D+4)y=xe^{-x}, x>0$, given that $y(0)=0$ and $y'(0)=0$.

Taking Fourier complex transforms of the given equation, we get,

$$\begin{aligned}(is-2)^2\bar{y}(s)&=F\{U(x)\cdot xe^{-x}\}\\&=\int_0^\infty xe^{-(1+is)x}\,dx\\&=\left[x\left\{\frac{e^{-(1+is)x}}{-(1+is)}\right\}-\frac{e^{-(1+is)x}}{(1+is)^2}\right]_0^\infty\\&=\frac{1}{(1+is)^2}\end{aligned}$$

$$\begin{aligned}\therefore\qquad \bar{y}(s)&=\frac{1}{(is-2)^2(is+1)^2}\\&=\frac{-2/27}{is-2}+\frac{1/9}{(is-2)^2}+\frac{2/27}{is+1}+\frac{1/9}{(is+1)^2}\end{aligned}$$

Inverting, we get

$$y=\frac{-2}{27}e^{2x}+\frac{1}{9}xe^{2x}+\frac{2}{27}e^{-x}+\frac{1}{9}xe^{-x};\quad x>0$$

Example 12

Solve the equation $y'' - 4y' + 5y = 1, \quad x > 0$,
given that $y(0) = 0$ and $y'(0) = 0$.

Taking complex Fourier transforms of the given equation, we get

$$[(is)^2 - 4is + 5]\bar{y}(s) = F\{1 \cdot U(x)\}$$

$$= \int_0^\infty e^{-isx}\,dx$$

$$= \left[\frac{e^{-isx}}{-is}\right]_0^\infty$$

$$= \frac{1}{is}$$

$$\therefore \quad \bar{y}(s) = \frac{1}{(is)(is-2-i)(is-2+i)}$$

$$= \frac{1/5}{is} + \frac{1/(-2+4i)}{is-2-i} + \frac{1/(-2-4i)}{is-2+i}, \quad \text{by partial fractions}$$

$$= \frac{1}{5}\cdot\frac{1}{is} - \frac{1}{10}(1+2i)\cdot\frac{1}{is-2-i} - \frac{1}{10}(1-2i)\cdot\frac{1}{is-2+i}$$

Taking inverse transforms, we get

$$y = \frac{1}{5} - \frac{1}{10}(1+2i)e^{(2+i)x} - \frac{1}{10}(1-2i)e^{(2-i)x}$$

$$= \frac{1}{5} - \frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x;\ x > 0$$

Example 13

Solve the equation $(D^2 - 4D + 3)y = \cos 3x, x > 0$, given that $y(0) = 0$ and $y'(0) = 0$.

Taking complex Fourier transforms of the given equation, we get,

$$[(is)^2 - 4is + 3]\bar{y}(s) = F[U(x)\cos 3x]$$

$$= \int_0^\infty e^{-isx}\cos 3x\,dx = \frac{is}{9 + (is)^2}$$

$$\therefore \quad \bar{y}(s) = \frac{is}{(is-1)(is-3)[(is)^2+9]}$$

$$= \frac{-1/20}{is-1} + \frac{1/12}{is-3} - \frac{(1/30)is}{(is)^2+9} - \frac{1/5}{(is)^2+9}$$

by partial fractions.

Inverting, we get

$$y = \frac{-1}{20}e^{x} + \frac{1}{12}e^{3x} - \frac{1}{30}\cos 3x - \frac{1}{15}\sin 3x; \qquad x > 0,$$

since $\qquad F[U(x)\sin 3x] = \dfrac{3}{(is)^2 + 9}$

Example 14

Solve the equation $y'' + \omega_0^2 y = \sin \omega x$, $x > 0$, given that $y(0) = 0$ and $y'(0) = 0$, when

(i) $\omega \neq \omega_0$ and $\qquad$ (ii) $\omega = \omega_0$

Taking complex Fourier transforms of the given equation, we get,

$$\begin{aligned}[(is)^2 + \omega_0^2]\bar{y}(s) &= F\{U(x)\cdot \sin \omega x\}\\ &= \int_0^\infty e^{-isx}\sin \omega x \; dx\\ &= \frac{\omega}{\omega^2 - s^2}\end{aligned}$$

$$\therefore \qquad \bar{y}(s) = \frac{\omega}{\left(\omega_0^2 - s^2\right)\left(\omega^2 - s^2\right)}, \quad \text{when } \omega \neq \omega_0$$

$$= \frac{\left[\dfrac{\omega}{\omega^2 - \omega_0^2}\right]}{\omega_0^2 - s^2} - \frac{\left[\dfrac{\omega}{\omega^2 - \omega_0^2}\right]}{\omega^2 - s^2}, \text{ by partial fractions.}$$

Inverting, we get

$$y = \frac{\omega}{\omega_0(\omega^2 - \omega_0^2)}\sin \omega_0 x - \frac{1}{\omega^2 - \omega_0^2}\sin \omega x.$$

When $\qquad \omega = \omega_0, \quad \bar{y}(s) = \dfrac{\omega_0}{(\omega_0^2 - s^2)^2}$

Inverting, $\qquad y = F^{-1}\left\{\dfrac{\omega_0}{(\omega_0^2 - s^2)^2}\right\} \qquad (1)$

Consider $\qquad \displaystyle\int_0^\infty (\sin ax)e^{-isx}\,dx = \frac{a}{a^2 - s^2}$

Differentiating both sides with respect to ‘a’,

$$\int_0^\infty (x\cos ax)e^{-isx}\,dx = \frac{1}{a^2 - s^2} - \frac{2a^2}{(a^2 - s^2)^2}$$

$$\therefore \quad \frac{a}{(a^2-s^2)^2} = \frac{1}{2a}\cdot\frac{1}{a^2-s^2} - \frac{1}{2a}\int_0^\infty (x\cos ax)e^{-isx}\,dx$$

$$\therefore \quad F^{-1}\left\{\frac{a}{(a^2-s^2)^2}\right\} = \frac{1}{2a^2}U(x)\sin ax - \frac{1}{2a}U(x)x\cos ax \tag{2}$$

Using (2) in (1), we get,

$$y = \frac{1}{2\omega_0^2}\sin\omega_0 x - \frac{1}{2\omega_0}x\cos\omega_0 x,\ x>0$$

Example 15

Solve the wave equation $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$, subject to the initial conditions $y(x,0) = f(x)$, $-\infty < x < \infty$, $\frac{\partial y}{\partial t}(x,0) = g(x)$ and the boundary conditions $y(x,t)\longrightarrow 0$, as $x\longrightarrow \pm\infty$.

Taking Fourier transforms of the equation with respect to x, we get,

$$\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial t^2}(x,t)\cdot e^{-isx}\,dx = a^2(is)^2\bar{y}(s,t)$$

i.e. $$\frac{\partial^2}{\partial t^2}\bar{y}(s,t) + a^2s^2\bar{y}(s,t) = 0 \tag{1}$$

Taking Fourier transforms of the initial conditions, we have

$$\bar{y}(s,0) = \bar{f}(s) \tag{2}$$

and $$\frac{\partial\bar{y}}{\partial t}(s,0) = \bar{g}(s) \tag{3}$$

(1) is an ordinary differential equation in $\bar{y}(s,t)$. Solving (1), we get

$$\bar{y}(s,t) = A\cos ast + B\sin ast \tag{4}$$

Using (2) in (4), we get $\quad A = \bar{f}(s)$

Using (3) in (4), we get $\quad B = \frac{1}{as}\bar{g}(s)$

Inserting these values in (4), we get

$$\begin{aligned}\bar{y}(s,t) &= \bar{f}(s)\cos ast + \frac{1}{as}\bar{g}(s)\sin ast\\ &= \frac{1}{2}\{\bar{f}(s)e^{iast} + \bar{f}(s)e^{-iast}\} + \frac{1}{2a}\left\{\frac{\bar{g}(s)}{is}e^{iast} - \frac{\bar{g}(s)}{is}e^{-iast}\right\}\end{aligned}$$

Taking inverse Fourier transforms, we get

$$y(x,t) = \frac{1}{4\pi}\left[\int_{-\infty}^{\infty} \bar{f}(s)e^{i(x+at)s}\,\mathrm{d}s + \int_{-\infty}^{\infty} \bar{f}(s)e^{i(x-at)s}\,\mathrm{d}s\right] +$$

$$\frac{1}{4\pi a}\left[\int_{-\infty}^{\infty} \frac{\bar{g}(s)}{is}e^{i(x+at)s}\,\mathrm{d}s - \int_{-\infty}^{\infty} \frac{\bar{g}(s)}{is}e^{i(x-at)s}\,\mathrm{d}s\right] \tag{5}$$

Now $\qquad F\{\phi'(x)\} = is\phi(s)$, by property (7)

Putting $\qquad \phi'(x) = g(x)$, $\quad$ we get $\phi(x) = \int_c^x g(u)\,\mathrm{d}u$

and $\qquad \dfrac{\bar{g}(s)}{is} = \bar{\phi}(s)$

$$\therefore \qquad F^{-1}\left\{\frac{\bar{g}(s)}{is}\right\} = \phi(x) = \int_c^x g(u)\,\mathrm{d}u$$

i.e. $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{g}(s)}{is}e^{ixs}\mathrm{d}s = \int_c^x g(u)\,\mathrm{d}u$$

Using (6) in (5), we get

$$y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\left[\int_c^{x+at} g(u)\,\mathrm{d}u - \int_c^{x-at} g(u)\,\mathrm{d}u\right]$$

i.e. $$y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(u)\,\mathrm{d}u$$

Example 16

Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $y \geq 0$ subject to the boundary conditions $u(x,0) = f(x)$, $-\infty < x < \infty$ and $u(x,y) \to 0$ as $y \to \infty$.

Taking Fourier transforms of the Laplace equation with respect to x, we get

$$(is)^2\bar{u}(s,y) + \frac{\partial^2}{\partial y^2}\bar{u}(s,y) = 0$$

i.e. $$\frac{\partial^2}{\partial y^2}\bar{u}(s,y) - s^2\bar{u}(s,y) = 0 \tag{1}$$

The transforms of the boundary conditions are

$$\bar{u}(s,0) = \bar{f}(s) \tag{2}$$

and $$\bar{u}(s, y) \to 0 \quad \text{as} \quad y \to \infty \tag{3}$$

Solving the ordinary differential equation (1), we get

$$\bar{u}(s, y) = Ae^{sy} + Be^{-sy} \tag{4}$$

For the solution (4) to be consistent with (3), $A = 0$

$\therefore$ $\bar{u}(s, y) = Be^{-|s|y}$, as the coefficient of y cannot be positive

Using (2), we get $B = \bar{f}(s)$.

$$\therefore \quad \bar{u}(s, y) = \bar{f}(s)e^{-|s|y} \tag{5}$$

Consider $$F^{-1}\{e^{|s|y}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|s|y}e^{isx}\, ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(y+ix)s}\, ds + \int_{0}^{\infty} e^{-(y-ix)s}\, ds\right]$$

[since $|s| = -s$, when $s < 0$ and $|s| = s$, when $s > 0$]

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{(y+ix)s}}{y+ix}\right\}_{-\infty}^{0} + \left\{\frac{e^{-(y-ix)s}}{-(y-ix)}\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{y+ix} + \frac{1}{y-ix}\right] = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2} \tag{6}$$

Taking inverse transforms on both sides of (5), we have

$$u(x, y) = F^{-1}\{\bar{f}(s)\bar{g}(s)\}, \quad \text{where } \bar{g}(s) = e^{-|s|y}$$

$$= \int_{-\infty}^{\infty} f(t)\cdot g(x-t)\, dt, \text{ by convolution theorem}$$

$$= \int_{-\infty}^{\infty} f(t)\cdot\frac{1}{\pi}\cdot\frac{y}{(x-t)^2+y^2}\, dt, \text{ by 6} \tag{7}$$

i.e. $$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(t)dt}{(x-t)^2+y^2}$$

Example 17

Solve the one-dimensional heat flow equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for a rod with insulated sides extending from $-\infty$ to ∞ and with initial temperature distribution given by $u(x, 0) = f(x)$.

Assume that $u(x, t) \to 0$ as $x \to \pm\infty$.

Taking Fourier transforms of the given equation with respect to x, we get

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x, t) e^{-isx}\,\mathrm{d}x = \alpha^2 (is)^2 \bar{u}(s, t)$$

i.e.
$$\frac{\partial}{\partial t}\bar{u}(s, t) = -\alpha^2 s^2 \bar{u}(s, t) \tag{1}$$

The Fourier transform of the initial condition is

$$\bar{u}(s, 0) = \bar{f}(s) \tag{2}$$

Solving the ordinary differential equation (1), we get

$$\bar{u}(s, t) = A e^{-\alpha^2 s^2 t} \tag{3}$$

Using (2) in (3),
$$A = \bar{f}(s).$$

$\therefore$
$$\bar{u}(s, t) = \bar{f}(s) \cdot e^{-\alpha^2 s^2 t} \tag{4}$$

From Example (8) in Section 2(a), we have

$$F(e^{-a^2 x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} \quad \text{or}$$

$$F^{-1}(e^{-s^2/4a^2}) = \frac{a}{\sqrt{\pi}} e^{-a^2 x^2}$$

Putting $\frac{1}{4a^2} = \alpha^2 t$, we get $a^2 = \frac{1}{4\alpha^2 t}$ or $a = \frac{1}{2\alpha\sqrt{t}}$ and hence

$$F^{-1}(e^{-\alpha^2 s^2 t}) = \frac{1}{2\alpha\sqrt{\pi t}} e^{-x^2/4\alpha^2 t} \tag{5}$$

Taking Fourier inverse transforms of (4) by using convolution theorem and using (5), we get

$$u(x, t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(z) e^{-(x-z)^2/4\alpha^2 t}\,\mathrm{d}z$$

$$= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(z) e^{-(z-x)^2/4\alpha^2 t}\,\mathrm{d}z \tag{6}$$

Putting $\dfrac{z-x}{2\alpha\sqrt{t}} = \omega$ in (6), the solution takes the following form:

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\sqrt{t}\alpha\omega)e^{-\omega^2}\,d\omega.$$

Example 18

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, satisfying the boundary conditions $u(0,t) = k, t \geq 0$ and $u(x,t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition $u(x,0) = 0$.

Refer to Note (2) under Property (7). Since $x > 0$ and $u(0,t)$ is given, we take Fourier sine transforms of the equation with respect to x.

Thus $$\frac{\partial}{\partial t}\bar{u}_S(s,t) = \alpha^2[-s^2\bar{u}_S(s,t) + su(0,t)]$$

i.e. $$\frac{\partial}{\partial t}\bar{u}\,(s,t) + \alpha^2 s^2 \bar{u}_S(s,t) = k\alpha^2 s \tag{1}$$

Transform of the initial condition is

$$\bar{u}_S(s,0) = 0 \tag{2}$$

Solving the ordinary differential equation (1), we get

$$\bar{u}_S(s,t) = Ae^{-\alpha^2 s^2 t} + \frac{1}{D+\alpha^2 s^2}\left(k\alpha^2 s\right), \left[D \equiv \frac{d}{dt}\right]$$

$$= Ae^{-\alpha^2 s^2 t} + \frac{k}{s} \tag{3}$$

Using (2) in (3), we get

$$A = -\frac{k}{s}$$

$$\bar{u}_S(s,t) = \frac{k}{s}(1 - e^{-\alpha^2 s^2 t})$$

Taking the inverse sine transforms, we get,

$$u(x,t) = \frac{2k}{\pi}\int_0^{\infty} \frac{1}{s}\left(1 - e^{-\alpha^2 s^2 t}\right)\sin xs\,ds$$

Example 19

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, satisfying the boundary conditions $\dfrac{\partial u}{\partial x}(0,t) = k, t \geq 0$ and $u(x,t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition $u(x,0) = 0$.

Refer to Note (2) under Property (7). Since $x > 0$ and $\frac{\partial u}{\partial x}(0, t)$ is given, we take Fourier cosine transforms of the equation with respect to x.

Thus $$\frac{\partial}{\partial t}\bar{u}_C(s, t) = \alpha^2\left[-s^2\bar{u}_C(s, t) - \frac{\partial u}{\partial x}(0, t)\right]$$

i.e. $$\frac{\partial}{\partial t}\bar{u}_C(s, t) + \alpha^2 s^2 \bar{u}_C(s, t) = -k\alpha^2 \tag{1}$$

Transform of the initial condition is

$$\bar{u}_C(s, 0) = 0 \tag{2}$$

Solving (1) and using (2), we get, as in the previous example

$$\bar{u}_C(s, t) = Ae^{-\alpha^2 s^2 t} - \frac{k}{s^2} \tag{3}$$

Using (2) in (3), we get $\quad A = \frac{k}{s^2}$

$\therefore \quad \bar{u}_C(s, t) = \frac{k}{s^2}\left(e^{-\alpha^2 s^2 t} - 1\right)$ Taking the inverse cosine transforms, we get

$$u(x, t) = \frac{2k}{\pi}\int_0^\infty \frac{1}{s^2}\left(e^{-\alpha^2 s^2 t} - 1\right)\cos xs \; ds$$

Example 20

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, satisfying the boundary conditions $u(0, t) = 0$ and $u(x, t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition $u(x, 0) = f(x), x > 0$.

Refer to Note (2) under Property (7). Since $x > 0$ and $u(0, t)$ is given, we take Fourier sine transforms of the equation with respect to x.

Thus $\quad \frac{\partial}{\partial t}\bar{u}_S(s, t) = \alpha^2[-s^2\bar{u}_S(s, t) + su(0, t)]$

i.e. $$\frac{\partial}{\partial t}\bar{u}_S(s, t) + \alpha^2 s^2 \bar{u}_S(s, t) = 0 \tag{1}$$

Sine transforms of the initial condition is

$$\bar{u}_S(s, 0) = \bar{f}_S(s) \tag{2}$$

Solving (1), we get $$\bar{u}_S(s, t) = Ae^{-\alpha^2 s^2 t} \tag{3}$$

Using (2) in (3), $$A = \bar{f}_S(s)$$

$\therefore$ $$\bar{u}_S(s, t) = \bar{f}_S(s)e^{-\alpha^2 s^2 t} \tag{4}$$

From Example (8), we have

$$F_C\left(e^{-a^2x^2}\right) = \frac{\sqrt{\pi}}{2a}e^{-s^2/4a^2}$$

Taking $$\frac{1}{4a^2} = \alpha^2 t \quad \text{or} \quad a = \frac{1}{2\alpha\sqrt{t}}$$

$$F_C\left\{e^{-x^2/4\alpha^2 t}\right\} = \alpha\sqrt{\pi t}\,e^{-\alpha^2 s^2 t}$$

or $$F_C^{-1}\left\{e^{-\alpha^2 s^2 t}\right\} = \frac{1}{\alpha\sqrt{\pi t}}e^{-x^2/4\alpha^2 t} = g(x,t), \text{ say.} \tag{5}$$

Then $$\bar{g}_C(s,t) = e^{-\alpha^2 s^2 t} \tag{6}$$

Using (6) in (4), we have

$$\bar{u}_S(s,t) = \bar{f}_S(s)\bar{g}_C(s,t)$$

Taking inverse sine transforms, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\cdot\bar{g}_C(s,t)\sin xs\ \mathrm{d}s$$

$$= \frac{2}{\pi}\int_0^\infty \bar{g}_C(s,t)\sin xs\ \mathrm{d}s\int_0^\infty f(z)\sin sz\ \mathrm{d}z,$$

on using the definition for $\bar{f}_S(s)$

$$= \int_0^\infty f(z)\,\mathrm{d}z\cdot\frac{2}{\pi}\int_0^\infty \bar{g}_C(s,t)\sin xs\sin zs\ \mathrm{d}s,$$

on changing the order of integration.

$$= \frac{1}{2}\int_0^\infty f(z)\,\mathrm{d}z\cdot\frac{2}{\pi}\int_0^\infty \bar{g}_C(s,t)\{\cos(x-z)s - \cos(x+z)s\}\,\mathrm{d}s$$

$$= \frac{1}{2}\int_0^\infty f(z)\ \mathrm{d}z[g(x-z,t) - g(x+z,t)]$$

$$= \frac{1}{2\alpha\sqrt{\pi t}}\int_0^\infty f(z)\left[e^{-(x-z)^2/4\alpha^2 t} - e^{-(x+z)^2/4\alpha^2 t}\right]\mathrm{d}z, \text{ from(5).}$$

Exercise 4(b)

Part A (Short-Answer Questions)

1. State the relation between $F\{f(x)\}$ and $F\{f(ax)\}$.
2. Derive the change of scale property for Fourier sine and cosine transforms.
3. Obtain $F\{f(x-a)\}$ in terms of $F\{f(x)\}$.
4. Find $F\{e^{iax} f(x)\}$ in terms of $F\{f(x)\}$.
5. State and prove the modulation theorem.
6. State and derive the conjugate symmetry property.
7. Express $F\{f'(x)\}$ in terms of $F\{f(x)\}$. State the conditions under which this relation holds good.
8. Express $F_S\{f''(x)\}$ and $F_C\{f''(x)\}$ in terms of $F_S\{f(x)\}$ and $F_C\{f(x)\}$ respectively.
9. Prove that $F\{xf(x)\} = i\dfrac{\mathrm{d}}{\mathrm{d}s}\bar{f}(s)$.
10. State convolution theorem in Fourier transforms.
11. State Parseval's identity in Fourier transforms. Also state the corresponding result in Fourier cosine and sine transforms.
12. Express $\dfrac{\mathrm{d}}{\mathrm{d}s}\{\bar{f}_C(s)\}$ and $\dfrac{\mathrm{d}}{\mathrm{d}s}\{\bar{f}_S(s)\}$ in terms of Fourier sine and cosine transforms respectively.
13. Prove that $F\{e^{-ax}U(x)\} = \dfrac{1}{a+is}$, where $U(x)$ is the unit step function.
14. Prove that $F\{xU(x)\} = \dfrac{1}{(is)^2}$.
15. Prove that $F\{xe^{-ax}U(x)\} = \dfrac{1}{(a+is)^2}$.
16. Prove that $F\{\sin ax \cdot U(x)\} = \dfrac{a}{a^2-s^2}$.
17. Prove that $F\{\cos ax \cdot U(x)\} = \dfrac{is}{a^2-s^2}$.

Part B

18. Prove that $F(1) = 2\pi\delta(s)$, where $\delta(s)$ is the unit impulse function. Hence find $F(\cos ax)$ and $F(\sin ax)$, using modulation theorem.
    [**Hint:** Use the definition of $\delta(s)$, proceed as in example 4(ii) of Section 4(a) and prove that $F^{-1}\{\delta(s)\} = \dfrac{1}{2\pi}\cdot 1$].
19. Find $F\{e^{-ax}\sin bxU(x)\}$ and $F\{e^{-ax}\cos bxU(x)\}$.
20. Find $F\{e^{-3|x|}\sin 2x\}$ and $F\{xe^{-2x}\sin 3xU(x)\}$.
21. Find $F(e^{-x^2})$ and hence find $F(e^{-x^2}\cos x)$, using modulation theorem.

22. Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

Hence prove that $$\int_0^\infty \frac{\sin^2 dx}{x^2}\, dx = \frac{\pi}{2}.$$

23. Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} a^2 - x^2 & \text{for } |x| < a \\ 0, & \text{for } |x| > a > 0. \end{cases}$$

Hence prove that

(*i*) $\displaystyle\int_0^\infty \frac{\sin x - x\cos x}{x^3}\, dx = \frac{\pi}{4}$ and

(ii) $\displaystyle\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.$

24. Find the Fourier cosine transform of (xe^{-ax}) and hence find the value of

$$\int_0^\infty \frac{(x^2 - a^2)^2}{(x^2 + a^2)^4}.$$

[**Hint:** See Problem (26) in Exercise 4(a).]

25. Find the Fourier sine transform of (xe^{-ax}) and hence evaluate

$$\int_0^\infty \frac{x^2\, dx}{(x^2 + a^2)^4}.$$

[**Hint:** See Example (13) in Section 4(a).]

26. Use transform methods to evaluate

(i) $\displaystyle\int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$; $a, b > 0$ and (ii) $\displaystyle\int_0^\infty \frac{x^2\, dx}{(x^2 + 4)(x^2 + 9)}.$

27. Use $F_C(e^{-ax})$ to find $F_S(xe^{-ax})$ and use the latter to find $F_C(x^2e^{-ax})$.

Solve the following initial value problems using Fourier transforms.

28. $(D^2 - D - 6)y = e^{3x}$; $y(0) = 0$, $y'(0) = 0$; $x > 0$.

29. $(D^2 + 6D + 9)y = e^{-3x}$; $y(0) = 0$, $y'(0) = 0$; $x > 0$.

30. $(D^2 - 2D + 5)y = xe^{2x}$; $y(0) = 0$, $y'(0) = 0$; $x > 0$.

31. $(D^2 - 3D + 2)y = 1 + x;\quad y(0) = 0,\quad y'(0) = 0;\ x > 0.$
32. $(D^2 + 2D + 2)y = \sin 2x;\quad y(0) = 0,\quad y'(0) = 0;\ x > 0.$
33. $(D^2 + 1)y = \cos x + \sin x;\quad y(0) = 0,\quad y'(0) = 0;\ x > 0.$
34. $(D - 1)^2 y = \cos x;\quad y(0) = 0,\quad y'(0) = 0;\ x > 0.$
35. Solve the equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, satisfying the initial conditions $y(x, 0) = f(x)$, $\dfrac{\partial y}{\partial t}(x, 0) = 0$ and the boundary conditions $y(0, t) = 0$, $y(x, t) \longrightarrow 0$ as $x \longrightarrow \infty$. Use transform method.
36. Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0;\ x \geq 0$, using transform method, given that $u(0, y) = f(y)$, $-\infty < y < \infty$ and $u(x, y) \longrightarrow 0$ as $x \longrightarrow \infty$.
37. Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, using transform method, given that $u(x, 0) = \begin{cases} k, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases}$ and $u(x, t) \longrightarrow 0$ as $x \longrightarrow \pm\infty$.
38. Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, $x \geq 0$, using transform method, given that $u(0, t) = f(t)$, $t \geq 0$, $u(x, t) \to 0$ as $x \to 0$ and $u(x, 0) = 0$.
39. Using transform method, solve the equation $\dfrac{\partial u}{\partial x} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, $x \geq 0$, subject to the boundary conditions $\dfrac{\partial u}{\partial x}(0, t) = f(t)$, $t \geq 0$ and $u(x, t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition $u(x, 0) = 0$.
40. Using transform method, solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, $x \geq 0$, subject to the boundary conditions $u(0, t) = 0$ and $u(x, t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition

$$u(x, 0) = \begin{cases} 1, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{for } x > 1 \end{cases}$$

4.7 FINITE FOURIER TRANSFORMS

Definitions

1. If the function $f(x)$ is piecewise continuous in the interval $(0, l)$, then $\displaystyle\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$, where n is an integer, is called the *Finite Fourier Sine Transform of* $f(x)$ in $(0, l)$ and denoted by $F_S\{f(x)\}$ or $\bar{f}_S(n)$.

i.e. $$F_S\{f(x)\} = \bar{f}_S(n) = \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

2. If the function $f(x)$ is piecewise continuous in the interval $(0, l)$, then $\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$, where n is an integer, is called *the Finite Fourier Cosine Transform of* $f(x)$ in $(0, l)$ and denoted by $F_C\{f(x)\}$ or $\bar{f}_C(n)$.

i.e. $$F_C\{f(x)\} = \bar{f}_C(n) = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

Inversion formulas

1. If $\bar{f}_S(n)$ is the finite Fourier sine transform of $f(x)$ in $(0, l)$, then $f(x) = \frac{2}{l}\sum_{n=1}^{\infty}\bar{f}_S(n)\sin\frac{n\pi x}{l}$ is called *the inverse finite Fourier sine transform* of $\bar{f}_S(n)$ and denoted as $F_S^{-1}\{\bar{f}_S(n)\}$. Once we assume the definition of $F_S\{f(x)\}$, the inversion formula is derived as follows:

Since $f(x)$ is piecewise continuous in $(0, l)$, it can be expanded as an infinite trigonometric series of the form $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ (i.e., Fourier half-range sine series)

i.e. $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (1)$$

Multiplying both sides of (1) by $\sin\frac{n\pi x}{l}$ and integrating with respect to x between the limits 0 and l, we get

$$\int_0^l f(x)\sin\frac{n\pi x}{l}dx = b_1\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx + b_2\int_0^l \sin\frac{2\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$+ \dots + b_n\int_0^l \sin^2\frac{n\pi x}{l}\,dx + \dots$$

$$= b_n\int_0^l \sin^2\frac{n\pi x}{l}\,dx \qquad [\because \text{all other integrals vanish}]$$

$$= \frac{b_n}{2}\int_0^l \left[1 - \cos\frac{2n\pi x}{l}\right]dx$$

$$= \frac{b_n}{2}\left\{x - \frac{\sin\left[\frac{2n\pi x}{l}\right]}{\left[\frac{2n\pi}{l}\right]}\right\}_0^l$$

$$= \frac{b_n}{2}\cdot l$$

$$\therefore \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\bar{f}_S(n), \text{ by definition} \tag{2}$$

Inserting (2) in (1), we get the following inversion formula.

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty}\bar{f}_S(n)\sin\frac{n\pi x}{l}$$

2. If $\bar{f}_C(n) = F_C\{f(x)\}$ in $(0, l)$, then $f(x) = \frac{1}{l}\bar{f}_C(0) + \frac{2}{l}\sum_{n=1}^{\infty}\bar{f}_C(n)\cos\frac{n\pi x}{l}$ is called the *inverse finite Fourier cosine transform of* $\bar{f}_C(n)$ and denoted as $F_C^{-1}\{\bar{f}_C(n)\}$.

As before, once the definition of $F_C\{f(x)\}$ is assumed, the inversion formula is derived as follows:

Since $f(x)$ is piecewise continuous in $(0, l)$, it can be expanded as an infinite trigonometric series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ (i.e., Fourier half-range cosine series).

i.e.
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} \tag{3}$$

Integrating both sides of (3) with respect to x between 0 and l, we get

$$\int_0^l f(x)\,dx = \frac{a_0}{2}\int_0^l dx = \frac{a_0}{2}\cdot l$$

$$\therefore \qquad \frac{a_0}{2} = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\bar{f}_C(0) \text{ (by definition)} \tag{4}$$

Multiplying both sides of (3) by $\cos\frac{n\pi x}{l}$ and integrating with respect to x between the limits 0 and l, we get

$$\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{a_0}{2}\int_0^l \cos\frac{n\pi x}{l}\,dx + a_1\int_0^l \cos\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$+a_2\int_0^l \cos\frac{2\pi x}{l}\cos\frac{n\pi x}{l}\,dx + \ldots + a_n\int_0^l \cos^2\frac{n\pi x}{l}\,dx + \ldots$$

$$= a_n \int_0^l \cos^2 \frac{n\pi x}{l}\, dx \qquad [\because \text{ all other integrals vanish}]$$

$$= \frac{a_n}{2} \int_0^l \left[1 + \cos \frac{2n\pi x}{l} \right] dx$$

$$= \frac{a_n}{2} \left\{ x + \frac{\sin\left(\frac{2n\pi x}{l}\right)}{\left(\frac{2n\pi}{l}\right)} \right\}_0^l = \frac{a_n}{2} \cdot l$$

$$\therefore \qquad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \bar{f}_C(n) \quad \text{(by definition)} \tag{5}$$

Inserting (4) and (5) in (3), we get the inversion formula

$$f(x) = \frac{1}{l} \bar{f}_C(0) + \frac{2}{l} \sum_{n=1}^{\infty} \bar{f}_C(n) \cos \frac{n\pi x}{l}$$

Finite Fourier transforms of derivatives

(i) $F_S\{f'(x)\} = -\frac{n\pi}{l} \bar{f}_C(n)$

(ii) $F_C\{f'(x)\} = (-1)^n f(l) - f(0) + \frac{n\pi}{l} \bar{f}_S(n)$

(iii) $F_S\{f''(x)\} = -\frac{n^2\pi^2}{l^2} \bar{f}_S(n) + \frac{n\pi}{l} \{f(0) - (-1)^n f(l)\}$

(iv) $F_C\{f''(x)\} = -\frac{n^2\pi^2}{l^2} \bar{f}_C(n) + (-1)^n f'(l) - f'(0).$

Proof

(i) $$F_S\{f'(x)\} = \int_0^l f'(x) \sin \frac{n\pi x}{l} dx$$

$$= \int_0^l \sin \frac{n\pi x}{l} d\{f(x)\}$$

$$= \left\{ f(x) \sin \frac{n\pi x}{l} \right\}_0^l - \frac{n\pi}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= -\frac{n\pi}{l} \bar{f}_C(n).$$

(iii) $$F_S\{f''(x)\} = \int_0^l \sin\frac{n\pi x}{l}\, d\{f'(x)\}$$

$$= \left\{f'(x)\sin\frac{n\pi x}{l}\right\}_0^l - \frac{n\pi}{l}\int_0^l f'(x)\cos\frac{n\pi x}{l}\, dx$$

$$= -\frac{n\pi}{l}\int_0^l \cos\frac{n\pi x}{l}\, d\{f(x)\}$$

$$= -\frac{n\pi}{l}\left[\left\{f(x)\cos\frac{n\pi x}{l}\right\}_0^l + \frac{n\pi}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx\right]$$

$$= -\frac{n^2\pi^2}{l^2}\bar{f}_S(n) + \frac{n\pi}{l}\{f(0) - (-1)^n f(l)\}$$

Similarly the results (ii) and (iv) may be proved.

Note ✍

Similar formulas hold good for finite Fourier transforms of partial derivatives, with minor changes.

Worked Examples 4(c)

Example 1

Find the finite Fourier sine and cosine transforms of $\left(\frac{x}{\pi}\right)$ in $(0, \pi)$.

$$F_S\left(\frac{x}{\pi}\right) = \int_0^\pi \frac{x}{\pi}\sin nx\, dx$$

$$= \frac{1}{\pi}\left\{x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right\}_0^\pi$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{n}(-1)^n\right\} = \frac{(-1)^{n+1}}{n}$$

$$F_C\left(\frac{x}{\pi}\right) = \frac{1}{\pi}\left\{x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right\}_0^\pi = \frac{1}{\pi n^2}\{(-1)^n - 1\},\ n\neq 0$$

Example 2

Find the finite Fourier cosine and sine transforms of $f(x)$, if

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < \pi/2 \\ -1, & \text{for } \pi/2 < x < \pi \end{cases}$$

$$F_C\{f(x)\} = \int_0^{\pi/2} 1 \cdot \cos nx \, dx - \int_{\pi/2}^{\pi} 1 \cdot \cos nx \, dx$$

$$= \left(\frac{\sin nx}{n}\right)_0^{\pi/2} - \left(\frac{\sin nx}{n}\right)_{\pi/2}^{\pi}$$

$$= \frac{1}{n}\left\{\sin\frac{n\pi}{2} - 0 - \sin n\pi + \sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{n}\sin\frac{n\pi}{2}, \quad n \neq 0.$$

$$F_S\{f(x)\} = \int_0^{\pi/2} \sin nx \, dx - \int_{\pi/2}^{\pi} \sin nx \, dx$$

$$= -\frac{1}{n}(\cos nx)_0^{\pi/2} + \frac{1}{n}(\cos nx)_{\pi/2}^{\pi}$$

$$= \frac{1}{n}\left\{1 - 2\cos\frac{n\pi}{2} + (-1)^n\right\}$$

Example 3

Find the finite Fourier sine and cosine transforms of $\left(1 - \frac{x}{\pi}\right)^2$ in $(0, \pi)$

$$F_S\left\{\left(1 - \frac{x}{\pi}\right)^2\right\} = \int_0^{\pi} \left(1 - \frac{x}{\pi}\right)^2 \sin nx \, dx$$

$$= \left[\left(1 - \frac{x}{\pi}\right)^2 \left(\frac{-\cos nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{1-x}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right) + \frac{2}{\pi^2}\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{1}{n} + \frac{2}{\pi^2 n^3}\{(-1)^n - 1\}$$

$$F_C\left\{\left(1 - \frac{x}{\pi}\right)^2\right\} = \int_0^{\pi} \left(1 - \frac{x}{\pi}\right)^2 \cos nx \, dx$$

$$= \left[\left(1 - \frac{x}{\pi}\right)^2 \frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(1 - \frac{x}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right) + \frac{2}{\pi^2}\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}, \quad n \neq 0.$$

Example 4

Find the finite Fourier sine and cosine transforms of e^{ax} in $(0, l)$.

$$F_S(e^{ax}) = \int_0^l e^{ax} \sin \frac{n\pi x}{l}\, dx$$

$$= \left[\frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left\{ a \sin \frac{n\pi x}{l} - \frac{n\pi}{l} \cos \frac{n\pi x}{l} \right\} \right]_0^l$$

$$= \frac{e^{al}}{a^2 + \frac{n^2\pi^2}{l^2}} \cdot \frac{n\pi}{l}(-1)^{n+1} + \frac{\frac{n\pi}{l}}{a^2 + \frac{n^2\pi^2}{l^2}}$$

$$= \frac{n\pi l}{n^2\pi^2 + l^2a^2} \left\{ (-1)^{n+1} e^{al} + 1 \right\}$$

$$F_C(e^{ax}) = \int_0^l e^{ax} \cos \frac{n\pi x}{l}\, dx$$

$$= \left[\frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left\{ a \cos \frac{n\pi x}{l} + \frac{n\pi}{l} \sin \frac{n\pi x}{l} \right\} \right]_0^l$$

$$= \frac{al^2}{n^2\pi^2 + a^2l^2} \left\{ (-1)^n e^{al} - 1 \right\}$$

Example 5

Find the finite Fourier sine transform of $\cos ax$ and finite Fourier cosine transform of $\sin ax$ in $(0, \pi)$.

$$F_S\,(\cos ax) = \int_0^\pi \cos ax \sin nx\, dx$$

$$= \frac{1}{2} \int_0^\pi [\sin (n + a)x + \sin (n - a)x]\, dx$$

$$= -\frac{1}{2} \left[\frac{\cos (n + a)x}{n + a} + \frac{\cos (n - a)x}{n - a} \right]_0^\pi$$

$$= \frac{1}{2} \left[\frac{1}{n + a}\{1 - \cos (n + a)\pi\} + \frac{1}{n - a}\{1 - \cos (n - a)\pi\} \right]$$

$$= \frac{1}{2} \left[\left\{ \frac{1}{n + a} + \frac{1}{n - a} \right\} - \left\{ \frac{1}{n + a} + \frac{1}{n - a} \right\} \cos n\pi \cos a\pi \right]$$

$$= \frac{n}{n^2 - a^2}[1 - (-1)^n \cos a\pi].$$

$$F_C(\sin ax) = \int_0^{\pi} \sin ax \cos nx \ \mathrm{d}x$$

$$= \frac{a}{a^2 - n^2}[1 - (-1)^n \cos a\pi], \text{ on interchanging } n \text{ and } a \text{ in (1).}$$

Example 6

Find $f(x)$, if its finite sine transform is given by

$$\bar{f}_S(n) = \frac{1 - \cos n\pi}{n^2\pi^2} \text{in } 0 < x < \pi$$

The inverse finite Fourier sine transform is given by

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \bar{f}_S(n) \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \left\{\frac{1 - \cos n\pi}{n^2\pi^2}\right\} \sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty} \left\{\frac{1 - (-1)^n}{n^2}\right\} \sin nx$$

$$= \frac{4}{\pi^3}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \sin nx$$

$$= \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2} \sin (2n - 1)x.$$

Example 7

Find $f(x)$, if its finite cosine transform is given by $\bar{f}_C(n) = \frac{1}{(2n+1)^2} \cos \frac{2n\pi}{3}$ in $0 < x < 1$.

The inverse finite Fourier cosine transform in $(0, l)$ is given by

$$f(x) = \frac{1}{l}\bar{f}_C(0) + \frac{2}{l}\sum_{n=1}^{\infty} \bar{f}_C(n) \cos \frac{n\pi x}{l}$$

Here $\qquad l = 1 \quad \text{and} \quad \bar{f}_C(0) = 1$

$\therefore \qquad f(x) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \cos \frac{2n\pi}{3} \cos n\pi x$

Example 8

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi, t > 0$, using finite Fourier transforms, given that $u(0, t) = 0, u(\pi, t) = 0$ for $t > 0$ and $u(x, 0) = 4 \sin^3 x$.

Since $u(0, t)$ and $u(\pi, t)$ are given, we take finite Fourier sine transforms on both sides of the given equation, with respect to x in $(0, \pi)$.

Then $$\frac{\partial}{\partial t}\bar{u}_S(n,t) = -n^2\bar{u}_S(n,t) + n\{u(0,t) - (-1)^n u(\pi,t)\}$$

i.e. $$\frac{\partial}{\partial t}\bar{u}_S = -n^2\bar{u}_S \tag{1}$$

on using the given boundary conditions.
Solving (1), we get

$$\bar{u}_S(n,t) = Ae^{-n^2 t} \tag{2}$$

Taking the finite sine transform of the initial condition $u(x,0) = 4\sin^3 x$, we get

$$\bar{u}_s(n,0) = \int_0^\pi 4\sin^3 x \sin nx \ dx$$

$$= \int_0^\pi (3\sin x - \sin 3x)\sin nx \ dx$$

$$= 0, \quad \text{when } n \neq 1 \text{ and } n \neq 3.$$

When $$n = 1, \bar{u}_S(1,0) = \int_0^\pi 3\sin^2 x \ dx$$

$$= \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{3\pi}{2} \tag{3}$$

When $$n = 3, \bar{u}_S(3,0) = \int_0^\pi (-\sin^2 3x)\ dx$$

$$= \frac{1}{2}\left[\frac{\sin 6x}{6} - x\right]_0^\pi = -\frac{\pi}{2} \tag{4}$$

Using (3) and (4) in (2), we get, $A = \frac{3\pi}{2}$, when $n = 1$, $A = -\frac{\pi}{2}$, when $n = 3$ and $A = 0$, for all other values of n.

By inversion formula,

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\bar{u}_S(n,t)\sin nx$$

$$= \frac{2}{\pi}\left\{\frac{3\pi}{2}\sin x \cdot e^{-t} - \frac{\pi}{2}\sin 3x \cdot e^{-9t}\right\}$$

$$= 3\sin x e^{-t} - \sin 3x \cdot e^{-9t}.$$

Example 9

Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 10$, using finite Fourier transforms, given that $u(0,t) = 0$, $u(10,t) = 0$, for $t > 0$ and $u(x,0) = 10x - x^2$ for $0 < x < 10$.
Since $u(0,t)$ and $u(10,t)$ are given, we take finite Fourier sine transforms on both sides of the given equation with respect to x in (0, 10).

Then $$\frac{\partial}{\partial t}\bar{u}_S(n,t) = -\frac{n^2\pi^2}{10^2}\bar{u}_S(n,t) + \frac{n\pi}{10}\{u(0,t) - (-1)^n u(10,t)\}$$

i.e. $$\frac{d}{dt}\bar{u}_S(n,t) = -\frac{n^2\pi^2}{100}\bar{u}_S(n,t) \tag{1}$$

on using the given boundary conditions.

Solving (1), we get

$$\bar{u}_S(n,t) = Ae^{-n^2\pi^2 t/100} \tag{2}$$

Taking the finite sine transform of the initial condition $u(x,0) = 10x - x^2$ in (0,10), we get

$$\bar{u}_S(n,0) = \int_0^{10} (10x - x^2)\sin\frac{n\pi x}{10}\,dx$$

$$= \left[(10x - x^2)\left\{\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right\} - (10-2x)\left\{\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right\} + (-2)\cdot\left\{\frac{\cos\frac{n\pi x}{10}}{\frac{n^3\pi^3}{10^3}}\right\}\right]_0^{10}$$

by Bernoulli's integration formula.

$$= \frac{2000}{n^3\pi^3}\{1 - (-1)^n\} = \begin{cases} \frac{4000}{n^3\pi^3}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases} \tag{3}$$

Using (3) in (2), $A = \begin{cases} \frac{4000}{n^3\pi^3}, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$

By inversion formula,

$$u(x,t) = \frac{2}{10}\sum_{n=1}^{\infty}\bar{u}_S(n,t)\sin\frac{n\pi x}{10}$$

$$= \frac{800}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{10}e^{-n^2\pi^2 t/100}$$

Example 10

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, $0 < x < l$, using finite Fourier transforms, given that $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(l,t) = 0$ for $t > 0$ and $u(x,0) = kx$, for $0 < x < l$.

Since $\frac{\partial u}{\partial x}(0, t)$ and $\frac{\partial u}{\partial x}(l, t)$ are given, we take finite Fourier cosine transforms on both sides of the given equation with respect to x in $(0, l)$.

Then $$\frac{\partial}{\partial t}\bar{u}_C(n, t) = \alpha^2\left[-\frac{n^2\pi^2}{l^2}\bar{u}_C(n, t) + (-1)^n\frac{\partial u}{\partial x}(l, t) - \frac{\partial u}{\partial x}(0, t)\right]$$

i.e. $$\frac{d}{dt}\bar{u}_C(n, t) = \frac{-n^2\pi^2\alpha^2}{l^2}\bar{u}_C(n, t) \tag{1}$$

on using the given boundary conditions.

Solving (1), we get

$$\bar{u}_C(n, t) = Ae^{-n^2\pi^2\alpha^2 t/l^2} \tag{2}$$

Taking the finite cosine transform of the initial conditon $u(x, 0) = \kappa x$, in $(0, l)$, we get

$$\bar{u}_C(n, 0) = \int_0^l kx \cos\frac{n\pi x}{l}\, dx$$

$$= k\left[x\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} + \frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l, \text{ by}$$

Bernoulli's integration formula,

$$= \frac{kl^2}{n^2\pi^2}\{(-1)^n - 1\}, \text{ when } n \neq 0 \tag{3}$$

When $n = 0$,

$$\bar{u}_C(0, 0) = \int_0^l kx\; dx = \frac{kl^2}{2} \tag{4}$$

Using (4) in (2), $\quad A = \frac{kl^2}{2}, \quad$ when $n = 0$

Using (3) in (2), $\quad A = \frac{kl^2}{n^2\pi^2}, \{(-1)^n - 1\}, \quad$ when $n \neq 0$.

By inversion formula,

$$u(x, t) = \frac{1}{l}\bar{u}_C(0, t) + \frac{2}{l}\sum_{n=1}^{\infty}\bar{u}_C(n, t)\cos\frac{n\pi x}{l}$$

$$= \frac{kl}{2} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{l}\cdot e^{-n^2\pi^2\alpha^2 t/l^2}$$

$$= \frac{kl}{2} - \frac{4kl}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}e^{-n^2\pi^2\alpha^2 t/l^2}$$

Exercise 4 (c)

Part A (Short Answer Questions)

1. Define finite Fourier sine and cosine transforms of $f(x)$.
2. Define the inverse finite fourier sine and cosine transforms.
3. Find the finite Fourier sine transform of $f'(x)$.
4. Find the finite Fourier cosine transform of $f'(x)$.
5. Write down the formulas for $F_S\{f''(x)\}$ and $F_C\{f''(x)\}$.
6. Write down the formulas for $F_S\left\{\frac{\partial u}{\partial x}(x,t)\right\}$ and $F_C\left\{\frac{\partial u}{\partial x}(x,t)\right\}$, where the transforms are taken with respect to x.
7. Write down the formulas for $F_S\left\{\frac{\partial^2 u}{\partial x^2}(x,t)\right\}$ and $F_C\left\{\frac{\partial^2 u}{\partial x^2}(x,t)\right\}$, where the transforms are taken with respect to x.

Part B

8. Find the finite Fourier sine and cosine transforms of $f(x) = 1$ in $(0, l)$.
9. Find the finite Fourier sine and cosine transforms of $f(x) = x$ in $(0, \pi)$.
10. Find the finite Fourier sine and cosine transforms of $f(x) = x^2$ in $(0, 1)$.
11. Find the finite Fourier sine and cosine transforms of $f(x) = x^3$ in $(0, 2)$.
12. Find the finite Fourier sine and cosine transforms of $f(x) = x(\pi - x)$ in $(0, \pi)$.
13. Find the finite Fourier sine and cosine transforms of
$$f(x) = \begin{cases} x, & \text{in } (0, l/2) \\ l - x & \text{in } (l/2, l) \end{cases} \quad \text{in the interval } (0, 1).$$
14. Find the finite Fourier cosine transform of $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$ in $(0, \pi)$.
15. Find $f(x)$, if (i) $\bar{f}_S(n) = \frac{16(-1)^{n-1}}{n^3}$, if $0 < x < 8$; and (ii) $\bar{f}_S(n) = \frac{2\pi(-1)^{n-1}}{n^3}$, if $0 < x < \pi$, where $n = 1, 2, 3, \ldots$
16. Find $f(x)$, if
$$\bar{F}_C(n) = \frac{\sin\left(\frac{n\pi}{2}\right)}{2n}, \quad n = 1, 2, 3, \ldots$$
$$= \pi/4, \quad n = 0;$$
where $0 < x < 2\pi$.
17. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0 < x < 1, t > 0$, using finite Fourier transforms, given that $u(0,t) = 0$, $u(1,t) = 0$ for $t > 0$ and $u(x, 0) = 2\sin 2\pi x \cos \pi x$.

18. Solve $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2}, 0 < x < 2, t > 0$, using finite Fourier transforms, given that

$$u(0, t) = 0, \quad u(2, t) = 0, \text{ for } t > 0 \text{ and}$$

$$u(x, 0) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2. \end{cases}$$

19. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, 0 < x < \pi, t > 0$, using finite Fourier transforms, given that $\dfrac{\partial u}{\partial x}(0, t) = 0, \dfrac{\partial u}{\partial x}(\pi, t) = 0$, for $t > 0$ and $u(x, 0) = 2\cos^2 x$, for $0 < x < \pi$.

20. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, 0 < x < l, t > 0$ using finite Fourier transforms, given that $\dfrac{\partial u}{\partial x}(0, t) = 0, \dfrac{\partial u}{\partial x}(l, t) = 0$ for $t > 0$ and $u(x, 0) = lx - x^2$ for $0 < x < l$.

Answers

Exercise 4 (a)

11. $\dfrac{a}{s^2 + a^2}$

12. $\dfrac{s}{s^2 + a^2}$

13. $\dfrac{2a}{s^2 + a^2}$

14. $10\left\{\dfrac{1}{s^2 + 4} + \dfrac{1}{s^2 + 25}\right\}$

15. $\dfrac{5s(s^2 + 5)}{(s^2 + 4)(s^2 + 9)}$

16. $\dfrac{2k \sin ls}{s}$

17. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{\sin s \cos xs}{s}\, ds$; $\quad I = \begin{cases} \pi/2, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| > 1. \end{cases}$

19. $\dfrac{\pi}{2}e^{-x} = \displaystyle\int_0^\infty \dfrac{s \sin xs}{s^2 + 1}\, ds$

22. $\dfrac{2i}{s^2}(as \cos as - \sin as)$

23. $\dfrac{2}{s^2}(1-\cos as)$

24. $\dfrac{1}{\pi x^3}(x^2 \sin x + x\cos x - \sin x)$

25. $\dfrac{1}{\pi x}\sin s_0 x;\ \dfrac{\pi}{2}$

26. $\dfrac{a}{s^2+a^2};\ \dfrac{a^2-s^2}{(a^2+s^2)^2};\ \dfrac{2(a^2-s^2)}{(a^2+s^2)^2}$

27. $\dfrac{s}{s^2+a^2};\ \dfrac{2as}{(s^2+a^2)^2};\ -\dfrac{4ias}{(s^2+a^2)^2}$

28. $\sqrt{\dfrac{\pi}{4a}}\left\{e^{-(s+b)^2/4a}+e^{-(s-b)^2/4a}\right\}$

29. $\dfrac{1}{2\sqrt{\pi}}e^{-x^2/4}$

30. $\dfrac{1}{4}(1+x)e^{-x}$

31. $\dfrac{1-\cos s}{s}$

32. $\dfrac{1}{s}(a\cos as - b\cos bs)+\dfrac{1}{b^2}(\sin bs - \sin as)$

33. $\dfrac{1}{2}\left\{\dfrac{\sin(s+1)a}{s+1}+\dfrac{\sin(s-1)a}{s-1}\right\}$

34. $\dfrac{\pi}{2}e^{-s}$

35. $\dfrac{\sqrt{\pi}}{2}e^{-s^2/4}$

36. $\tan^{-1}\left(\dfrac{s}{a}\right)$

37. e^{-x}

38. $\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{x}{a}\right)$

39. $f(x)=\begin{cases}1, & \text{for } 0<x<1\\ 0, & \text{for } x\geq 1.\end{cases}$

40. $\dfrac{2}{\pi}\cdot\dfrac{x}{a^2+x^2}$

Exercise 4(b)

18. $\pi\{\delta(s+a)+\delta(s-a)\}$; $\pi\left\{\delta(s+a)e^{i\pi/2}+\delta(s-a)e^{-\frac{i\pi}{2}}\right\}$

19. $\dfrac{b}{(a+is)^2+b^2}$; $\dfrac{a+is}{(a+is)^2+b^2}$

20. $\dfrac{-24is}{s^4+10s^2+169}$; $\dfrac{6(2+is)}{(2+is)^2+9}$

21. $\dfrac{\sqrt{\pi}}{2}\left\{e^{-(s-1)^2/4}+e^{-(s+1)^2/4}\right\}$

23. $\dfrac{4}{s^3}(\sin as - as\cos as)$

24. $\dfrac{\pi}{8a^3}$

25. $\dfrac{\pi}{32a^5}$

26. (i) $\pi/2ab(a+b)$; (ii) $\dfrac{\pi}{10}$

27. $\dfrac{2as}{(s^2+a^2)^2}$; $\dfrac{2a(a^2-3s^2)}{(s^2+a^2)^3}$

28. $y=\dfrac{1}{25}e^{-2x}-\dfrac{1}{25}e^{3x}+\dfrac{1}{5}xe^{3x}$

29. $y=\dfrac{1}{2}x^2e^{-3x}$

30. $y=\dfrac{2}{25}e^{x}\cos 2x-\dfrac{3}{50}e^{x}\sin 2x-\dfrac{2}{25}e^{2x}+\dfrac{1}{5}xe^{2x}$

31. $y=\dfrac{5}{4}+\dfrac{1}{2}x-2e^{x}+\dfrac{3}{4}e^{2x}$

32. $y=e^{-x}\left[\dfrac{1}{5}\cos x+\dfrac{2}{5}\sin x\right]-\dfrac{1}{10}(\sin 2x+2\cos 2x)$

33. $y=\dfrac{1}{2}\sin x+\dfrac{1}{2}x\sin x-\dfrac{1}{2}x\cos x$

34. $y=\dfrac{1}{2}xe^{x}-\dfrac{1}{2}\sin x$

35. $y(x,t)=\dfrac{1}{2}[f(x+ct)+f(x-ct)]$

36. $u(x, y) = \dfrac{x}{\pi} \displaystyle\int_{-\infty}^{\infty} f(z) \cdot \dfrac{1}{x^2 + (y - z)^2}\, dz (x > 0).$

37. $u(x, t) = \dfrac{k}{2} + \dfrac{k}{\sqrt{\pi}} \displaystyle\int_0^{x/2c\sqrt{t}} e^{-\omega^2}\, d\omega$

38. $u(x, t) = \dfrac{x}{2\alpha\sqrt{\pi}} \displaystyle\int_0^t \dfrac{f(z)}{(t - z)^{3/2}} e^{-x^2/4\alpha^2(t-z)}\, dz.$

39. $u(x, t) = -\dfrac{\alpha}{\sqrt{\pi}} \displaystyle\int_0^t \dfrac{f(z)}{\sqrt{t - z}} e^{-x^2/4\alpha^2(t-z)}\, dz.$

40. $u(x, t) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \left[\dfrac{1 - \cos s}{s}\right] e^{-s^2 t} \sin xs\, ds.$

Exercise 4(c)

8. $\dfrac{l}{n\pi}\{1 - (-1)^n\};\quad 0$, if $n \neq 0$ and l, if $n = 0$.

9. $\dfrac{\pi}{n}(-1)^{n+1};\ \dfrac{1}{n^2}\{(-1)^n - 1\}$, if $n \neq 0$ and $\dfrac{\pi^2}{2}$, if $n = 0$.

10. $\dfrac{(-1)^{n+1}}{n\pi} + \dfrac{2}{n^3\pi^3}\{(-1)^n - 1\},\ \dfrac{2(-1)^n}{n^2\pi^2}$ if $n \neq 0$ and $\dfrac{1}{3}$, if $n = 0$.

11. $\dfrac{(-1)^n}{n\pi} + \dfrac{6(-1)^n}{n^3\pi^3};\ \dfrac{3(-1)^n}{n^2\pi^2} - \dfrac{6}{n^4\pi^4}$, if $n \neq 0$ and 4, if $n = 0$.

12. $\dfrac{2}{n^3}\{1 - (-1)^n\};\ -\dfrac{\pi}{n^2}\{1 + (-1)^n\}$, if $n \neq 0$ and $\dfrac{\pi^3}{6}$, if $n = 0$.

13. $\dfrac{2l^2}{n^2} \sin \dfrac{n\pi}{2};\ \dfrac{l^2}{n^2\pi^2}\left(\cos \dfrac{n\pi}{2} - 1\right)$, if $n \neq 0$ and $\dfrac{l^2}{4}$, if $n = 0$.

14. $\dfrac{1}{n^2}$, if $n \neq 0$ and 0, if $n = 0$

15. (i) $4 \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^3} \sin \dfrac{n\pi x}{8}$; (ii) $4 \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^3} \sin nx.$

16. $\dfrac{1}{4} + \dfrac{1}{2\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{n\pi}{2} \cos \dfrac{nx}{2}$

17. $u(x, t) = e^{-2\pi^2 t} \sin \pi x + e^{-18\pi^2 t} \sin 3\pi x.$

18. $u(x,t) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}e^{-Kn^2\pi^2 t/4}$

19. $u(x,t) = 1 + \cos 2x \cdot e^{-4t}$

20. $u(x,t) = \dfrac{l^2}{6} - \dfrac{l^2}{\pi^2}\displaystyle\sum\frac{1}{n^2}\cos\frac{2n\pi x}{l}e^{-4n^2\pi^2\alpha^2 t/l^2}$