NUMERICAL METHODS

MA6459

Regulation 2013

As per Anna University Syllabus

(Common to CIVIL & E.E.E. Branches)

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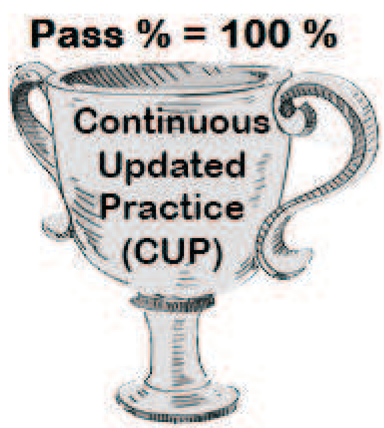

ALL THE BEST

MA6459 **NUMERICAL METHODS** **L T P C**

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OBJECTIVES:

- This course aims at providing the necessary basic concepts of a few numerical methods and give procedures for solving numerically different kinds of problems occurring in engineering and technology.

UNIT I Solution of Equations and Eigenvalue Problems **10+3**

Solution of algebraic and transcendental equations - Fixed point iteration method - Newton Raphson method- Solution of linear system of equations - Gauss elimination method - Pivoting - Gauss Jordan method - Iterative methods of Gauss Jacobi and Gauss Seidel - Matrix Inversion by Gauss Jordan method - Eigen values of a matrix by Power method.

UNIT II Interpolation and Approximation **8+3**

Interpolation with unequal intervals - Lagrange's interpolation - Newton's divided difference interpolation - Cubic Splines - Interpolation with equal intervals - Newton's forward and backward difference formulae.

UNIT III Numerical Differentiation and Integration **9+3**

Approximation of derivatives using interpolation polynomials - Numerical integration using Trapezoidal, Simpson's 1/3 rule - Romberg's method - Two point and three point Gaussian quadrature formulae - Evaluation of double integrals by Trapezoidal and Simpson's 1/3 rules.

UNIT IV Initial Value Problems for Ordinary Differential Equations **9+3**

Single Step methods - Taylor's series method - Euler's method - Modified Euler's method - Fourth order Runge-Kutta method for solving first order equations - Multi step methods - Milne's and Adams-Bash forth predictor corrector methods for solving first order equations.

UNIT V Boundary Value Problems in Ordinary and Partial Differential Equations **9+3**

Finite difference methods for solving two-point linear boundary value problems - Finite difference techniques for the solution of two dimensional Laplace's and Poisson's equations on rectangular domain - One dimensional heat flow equation by explicit and implicit (Crank Nicholson) methods - One dimensional wave equation by explicit method.

TOTAL (L:45+T:15): 60 PERIODS

OUTCOMES:

- The students will have a clear perception of the power of numerical techniques, ideas and would be able to demonstrate the applications of these techniques to problems drawn from industry, management and other engineering fields.

TEXT BOOKS:

1. Grewal. B.S., and Grewal. J.S.,"Numerical methods in Engineering and Science", Khanna Publishers, 9th Edition, New Delhi, 2007.

2. Gerald. C. F., and Wheatley. P. O., "Applied Numerical Analysis", Pearson Education, Asia, 6th Edition, New Delhi, 2006.

REFERENCES:

1. Chapra. S.C., and Canale.R.P., "Numerical Methods for Engineers, Tata McGraw Hill, 5th Edition, New Delhi, 2007.

2. Brian Bradie. "A friendly introduction to Numerical analysis", Pearson Education, Asia, New Delhi, 2007.

3. Sankara Rao. K., "Numerical methods for Scientists and Engineers", Prentice Hall of India Private, 3rd Edition, New Delhi, 2007.

Contents

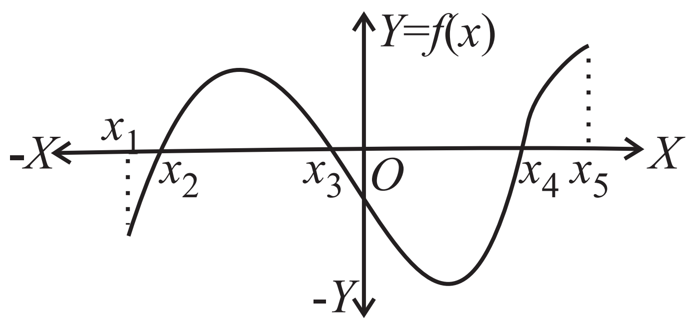
1 Solution of Equations and Eigenvalue Problems

Solution of algebraic and transcendental equations - Fixed point iteration method - Newton Raphson method- Solution of linear system of equations - Gauss elimination method - Pivoting - Gauss Jordan method - Iterative methods of Gauss Jacobi and Gauss Seidel - Matrix Inversion by Gauss Jordan method - Eigen values of a matrix by Power method.

1.1 Introduction

Solution of equation:

A value satisfies given equation is called solution of that equation.



Here the curve $f(x)$ with

$$f(x = x_2) = f(x = x_3) = f(x = x_4) = 0$$

$$\Rightarrow x_2, x_3 \text{ and } x_4 \text{ are solutions of } f(x).$$

$$\text{and } f(x) > 0 \text{ for } x_2 < x < x_3 \text{ and } x_4 < x < x_5$$

$$f(x) < 0 \text{ for } x_1 < x < x_2 \text{ and } x_3 < x < x_4.$$

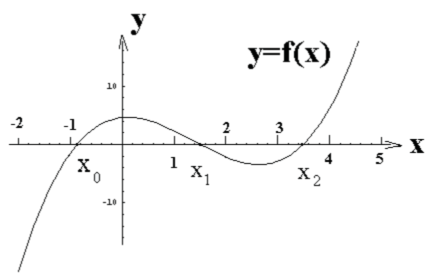
1.2 Solution of algebraic and transcendental equations

This chapter deals with finding solutions of algebraic and transcendental equations of either of the forms

$$f(x) = 0 \text{ or } f(x) = g(x) \tag{1}$$

where we want to solve for the unknown x. An algebraic equation is an equation constructed using the operations of $+, -, \times, \div$, and possibly root taking (radicals). Rational functions and polynomials are examples of algebraic functions. Transcendental equations in comparison are not algebraic. That is, they contain non-algebraic functions and possibly their inverses functions. Equations which contain either trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions are examples of non-algebraic functions which are called transcendental functions.

Example 1 : (Root of algebraic equation)
Estimate the solutions of the algebraic equation $f(x) = x^3 - \frac{132}{32}x^2 + \frac{28}{32}x + \frac{147}{32} = 0$.
Solution: We use a computer or calculator and plot a graph of the function $y = f(x)$ and obtain the figure as follows:



$$f(x) = x^3 - \frac{132}{32}x^2 + \frac{28}{32}x + \frac{147}{32} = 0$$

One can now estimate the solutions of the given equation by determining where the curve crosses the $x-$axis because these are the points where $y = 0$. Examining the graph in above figure we can place bounds on our estimates x_0, x_1, x_2 of the solutions. One such estimate is given by

$$-1.0 < x_0 < -0.8$$
$$1.4 < x_1 < 1.6$$
$$3.4 < x_2 < 3.6$$

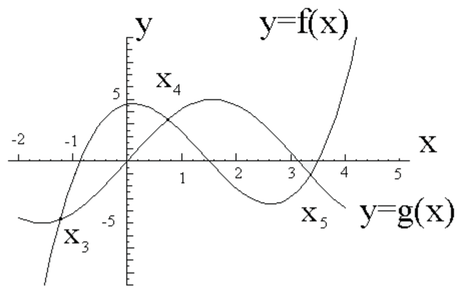
To achieve a better estimate for the roots one can plot three versions of the above graph which have some appropriate scaling in the neighborhood of the roots.

Finding values for x where $f(x) = g(x)$ can also be approached using graphics. One can plot graphs of the curves $y = f(x)$ and $y = g(x)$ on the same set of axes and then try to estimate where these curves intersect.

Example 2 : (Root of transcendental equation)
Estimate the solutions of the transcendental equation $x^3 - \frac{132}{32}x^2 + \frac{28}{32}x + \frac{147}{32} = 5\sin x$.
Solution: We again employ a computer or calculator and plot graphs of the functions

$$y = f(x) = x^3 - \frac{132}{32}x^2 + \frac{28}{32}x + \frac{147}{32} \text{ and } y = g(x) = 5\sin x$$

to obtain the following figure



Graph of $y = f(x)$ and $y = g(x)$

One can estimate the points where the curve $y = f(x)$ intersects the curve $y = g(x)$. If the curves are plotted to scale on the same set of axes, then one can place bounds on the estimates of the solution. One such set of bounds is given by

$$-1.5 < x_3 < -1.0$$

$$0.5 < x_4 < 1.0$$

$$3.0 < x_5 < 3.5$$

By plotting these graphs over a ner scale one can obtain better estimates for the solutions.

Soln. of 1 equation	**Soln. of n equations**	**Matrix Inversion**	**Eigen Val. & Vec.**
Fixed point (IM) Newton Raphson (IM)	Gauss elimination DM(RO) Gauss Jordan DM(RO) Gauss Jacobi IM(SVS) Gauss Seidel IM(LVS)	Gauss Jordan(RO)	Power method IM(MM)
Calculator Usage : Casio *fx*-991ms			
Polynomial: EQN→Deg?(2 or 3) **Trigonometry:** Deg.,Rad.,Grads **Exponential:** $\boxed{\text{Shift}}\boxed{\text{ln}}$ **Logarithmic:** log (def. base 10)	**Simultaneous Equations** EQN→Unknowns?(2 or 3)	**Matrix Entry *A*:** Press[MODE] [6:Matrix] 1:Mat*A* A^{-1}**:** $\boxed{\text{Shift}}\boxed{4}$ $\boxed{\text{MATRIX}}\boxed{3}$ $\boxed{\text{MAT}A}\boxed{x^{-1}}$	**Matrix Entry *A*&*B*:** Press[MODE] [6:Matrix] 1:Mat*A* & 2:Mat*B* $A * B$**:** $\boxed{\text{Shift}}\boxed{4}$ $\boxed{\text{MATRIX}}\boxed{3}$ $\boxed{\text{MAT}A} * \boxed{\text{MAT}B}$

where **IM:**Iterative method, **DM:** Direct method, **RO:** Row operations, **MM:** Matrix multiplication.

SVS: Step value substitution, **LVS:**Latest value substitution

Deg.: Degree Mode, **Rad.:**Radian Mode, **def.:**default $\left(90^\circ\text{degrees} = \frac{\pi}{2}\text{radians} = 100\text{Grads}\right)$

For example:

$\sin 90^\circ$(in degree mode) = $\sin(\pi \div 2)$(in radian mode) = $\sin(100)$(in Grads mode) = 1

*** $\sin(\pi \div 2) \neq \sin\pi \div 2$ $[\because \sin(\pi \div 2) = 1 \ \& \ \sin\pi \div 2 = 0 \div 2 = 0]$

Definition of natural logarithm : When $e^y = x$. Then base e logarithm of x is : $ln(x) = log_e(x) = y$

Default base of log is 10 and ln is e. [The e constant or Euler's number is: $e \approx 2.71828183$]

1.3 Fixed point iteration method

Write the given equation as $f(x) = 0$.
Find x_1, x_2 with opposite signs of $f(x)$. Find an relation as $x = g(x)$ such that

$$\left|g'(x_1)\right| < 1$$
$$\left|g'(x_2)\right| < 1$$

$$X_0 = \begin{cases} x_1, |f(x_1)| < |f(x_2)| \Rightarrow f(x_1) \text{ is nearer to zero.} \\ x_2, |f(x_1)| > |f(x_2)| \Rightarrow f(x_2) \text{ is nearer to zero.} \end{cases}$$

$$\begin{aligned} X_1 &= g(X_0) \\ X_2 &= g(X_1) \\ &\vdots \\ X_n &= g(X_{n-1}) \\ X_{n+1} &= g(X_n) \end{aligned}$$

Note: Stop method if the consecutive values of $X_n \& X_{n+1}$ are equal upto required place of decimal. i.e., $X_n = X_n + 1$.

1.3.1 Part A

1. If $g(x)$ is continuous in $[a, b]$ then under what condition the iterative method $x = g(x)$ has unique solution in $[a, b]$. (MJ2010)

   Solution: $|g'(x)| < 1$ in $[a, b]$ with order of convergence is one.

2. What are the advantages of iterative methods over direct methods for solving a system of linear equations. (ND2012)

   Solution:

   1. Iterative methods are suitable for solving linear equations when the number of equations in a system is very large(more than 100).
   2. Iterative methods are very effective concerning computer storage and time requirements.
   3. One of the advantages of using iterative methods is that they require fewer multiplications for large systems.
   4. Iterative methods automatically adjust to errors during study. They can be implemented in smaller programmes than direct methods.
   5. They are fast and simple to use when coefficient matrix is sparse(more zeros).
   6. Advantageously they have fewer rounds off errors as compared to other direct methods.
   7. The direct methods, aim to calculate an exact solution in a finite number of operations. Whereas iterative methods begins with an initial approximation and reproduce usually improved approximations in an infinite sequence whose limit is the exact solution.

8. Error is controlled by the number of itertaions and there is no round of error.

9. Iterative methods are self-correcting.

*. Direct methods : Gaussian Elimination, Gauss Jordan Method

   Iteration Methods : Fixed-point Iteration, Gauss-Seidel Iteration Method

3. What do you mean by the order of convergence of an iterative method for finding the root of the equation $f(x) = 0$? (ND2013)

Solution: For Fixed point iteration method, $|g'(x)| < 1$ in $[a, b]$, with order of convergence is one.

$$\text{where } f(x) = 0 \Rightarrow x = g(x)$$

For Newton-Raphson iteration method, $|f(x)f''(x)| < |f'(x)|^2$, with order of convergence is two.

Worked Examples

1.3.2 Part B

Example 1.1. Solve the equation $x^3 + x^2 - 100 = 0$ by iteration method.

Solution: Let $f(x) = x^3 + x^2 - 100 = 0$

$$f(x=0) = (0)^3 + (0)^2 - 100 = -100 < 0 \ ('-'\, ve)$$
$$f(x=1) = (1)^3 + (1)^2 - 100 = -98 < 0 \ ('-'\, ve)$$
$$f(x=2) = (2)^3 + (2)^2 - 100 = -88 < 0 \ ('-'\, ve)$$
$$f(x=3) = (3)^3 + (3)^2 - 100 = -64 < 0 \ ('-'\, ve)$$
$$\boxed{\boldsymbol{f(x=4) = (4)^3 + (4)^2 - 100 = -20 < 0 \ ('-'\, ve)}}$$
$$\boxed{\boldsymbol{f(x=5) = (5)^3 + (5)^2 - 100 = 50 > 0 \ ('+'\, ve)}}$$

$\therefore$ $f(x)$ has a root between 4 and 5.
i.e, $f(x)$ has a positive root.
This equation $f(x) = x^3 + x^2 - 100 = 0$ can be written as

$$x^3 + x^2 - 100 = 0$$
$$i.e., x^2 (x+1) = 100$$
$$\text{i.e., } x = \frac{10}{\sqrt{x+1}} \ [= g(x), \text{ say}]$$
$$\boxed{\therefore g(x) = \frac{10}{\sqrt{x+1}} = 10(x+1)^{\frac{-1}{2}}}$$

Therefore, $g'(x) = \frac{d}{dx}[g(x)]$

$$= \frac{d}{dx}\left[10(x+1)^{\frac{-1}{2}}\right] = 10\left(\frac{-1}{2}\right)(x+1)^{\frac{-1}{2}-1} = -5(x+1)^{\frac{-3}{2}} = \frac{-5}{(x+1)^{\frac{3}{2}}}$$

$$\therefore |g'(x)| = \left|\frac{-5}{(x+1)^{\frac{3}{2}}}\right| = \frac{5}{(x+1)^{\frac{3}{2}}}$$

At $x = 4, |g'(x=4)| = \frac{5}{(4+1)^{3/2}} = \frac{5}{(5)^{3/2}} = \frac{5}{5\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{1}{2.23606} = 0.57735 < 1$

At $x = 5, |g'(x=5)| = \frac{5}{(5+1)^{3/2}} = \frac{5}{(6)^{3/2}} = 0.34020 < 1$

$$\boxed{|g'(x)| < 1 \text{ in } (4,5)}$$

So the fixed point iteration method can be applied. Let $x_0 = 4$ $[|f(x=4)| < |f(x=5)|]$

Use $x_{n+1} = g(x_n)$

$$x_1 = g(x_0) = \frac{10}{\sqrt{x_0+1}} = \frac{10}{\sqrt{4+1}} = \frac{10}{\sqrt{5}} = 4.47214$$ (round off at 5^th decimal)

$$x_2 = g(x_1) = \frac{10}{\sqrt{4.47214+1}} = 4.27486$$

$$x_3 = g(x_2) = \frac{10}{\sqrt{4.27486+1}} = 4.35406$$

$$x_4 = g(x_3) = \frac{10}{\sqrt{4.35406+1}} = 4.32174$$

$$x_5 = g(x_4) = \frac{10}{\sqrt{4.32174+1}} = 4.33484$$

$$x_6 = g(x_5) = \frac{10}{\sqrt{4.33484+1}} = 4.32952$$

$$x_7 = g(x_6) = \frac{10}{\sqrt{4.32952+1}} = 4.33168$$

$$x_8 = g(x_7) = \frac{10}{\sqrt{4.33168+1}} = 4.33080$$

$$x_9 = g(x_8) = \frac{10}{\sqrt{4.33080+1}} = 4.33116$$

$$x_{10} = g(x_9) = \frac{10}{\sqrt{4.33116+1}} = 4.33101$$

$$x_{11} = g(x_{10}) = \frac{10}{\sqrt{4.33101+1}} = 4.33107$$

$$\boxed{\mathbf{x_{12} = g(x_{11}) = \frac{10}{\sqrt{4.33107+1}} = 4.33105}}$$

$$\boxed{\mathbf{x_{13} = g(x_{12}) = \frac{10}{\sqrt{4.33105+1}} = 4.33105}}$$

These values are same at fifth decimal, i.e., $x_{12} = x_{13}$. So stop the method.

$$\boxed{\therefore \text{The root of the given equation } x^3 + x^2 - 100 = 0 \text{ is } 4.33105}$$

Note: In above problem, the root is 4.33105(at 5th decimal) or 4.3311(at 4th decimal) or 4.331(at 3rd decimal).

Example 1.2. Solve the equation $x^3 + 2x^2 + 10x = 20$ by fixed point iteration method.

Solution: $[x_{13} = x_{14} = 1.36880]$

Example 1.3. Solve the equation $x^3 + x + 1 = 0$ by fixed point iteration method.

Solution: $\left[g(x) = (-1-x)^{1/3}; x_{23} = x_{24} = -0.682\right]$

Example 1.4. Solve the equation $3x - \cos x - 2 = 0$ by fixed point iteration method correct to 3 decimal places.

Solution: Let $f(x) = 3x - \cos x - 2 = 0$

$$\boxed{f(x=0) = 3(0) - \cos(0) - 2 = 0 - 1 - 2 = -3 < 0\,('-'\,ve)}$$

$$\boxed{f(x=1) = 3(1) - \cos(1) - 2 = 0.459697 > 0\,('+'\,ve)}$$

$\therefore f(x)$ has a root between 0 and 1.

i.e, $f(x)$ has a positive root.

This equation $f(x) = 3x - \cos x - 2 = 0$ can be written as

$$3x - \cos x - 2 = 0$$

$$\text{i.e., } 3x = 2 + \cos x$$

$$\text{i.e., } x = \frac{2+\cos x}{3} \quad [= g(x), \text{ say}]$$

$$\boxed{\therefore g(x) = \frac{2+\cos x}{3}}$$

$$\text{Therefore, } g'(x) = \frac{d}{dx}[g(x)]$$

$$= \frac{d}{dx}\left[\frac{2+\cos x}{3}\right] = \left(\frac{1}{3}\right)\frac{d}{dx}[2+\cos x] = \left(\frac{1}{3}\right)[0 - \sin x] = \frac{-\sin x}{3}$$

$$\therefore |g'(x)| = \left|\frac{-\sin x}{3}\right| = \frac{\sin x}{3}$$

$$\text{At } x = 0, |g'(x=0)| = \frac{\sin(0)}{3} = 0 < 1$$

$$\text{At } x = 1, |g'(x=1)| = \frac{\sin(1)}{3} = 0.2804 < 1$$

$$\boxed{|g'(x)| < 1 \text{ in } (0,1)}$$

So the fixed point iteration method can be applied. Let $x_0 = 0$ $\quad [|f(x=1)| < |f(x=0)|]$

Use $x_{n+1} = g(x_n)$ Now, put $n = 0$ in $x_{n+1} = g(x_n)$, we get

$$x_1 = g(x_0) = \frac{2+\cos(x_0)}{3} = \frac{2+\cos(0)}{3} = \frac{2+1}{3} = 1$$
$$x_2 = g(x_1) = \frac{2+\cos(x_1)}{3} = \frac{2+\cos(1)}{3} = 0.84677$$
$$x_3 = g(x_2) = \frac{2+\cos(x_2)}{3} = \frac{2+\cos(0.84677)}{3} = 0.88747$$
$$x_4 = g(x_3) = \frac{2+\cos(x_3)}{3} = \frac{2+\cos(0.84747)}{3} = 0.87713$$
$$\boxed{x_5 = g(x_4) = \frac{2+\cos(x_4)}{3} = \frac{2+\cos(0.87713)}{3} = 0.87979}$$
$$\boxed{x_6 = g(x_5) = \frac{2+\cos(x_5)}{3} = \frac{2+\cos(0.87979)}{3} = 0.87910}$$

These values are same at third decimal, i.e., $x_5 = x_6$. So stop the method.

$$\boxed{\therefore \text{The root of the given equation } 3x - \cos x - 2 = 0 \text{ is } 0.879}$$

Example 1.5. Solve the equation $\cos x = 3x - 1$ by iteration method upto 5 decimal.
(Or) Find a positive root of the equation $cosx - 3x + 1 = 0$ by using iteration method. (AM13)
Solution: $[x = 0.60710]$

Example 1.6. Solve the equation $2\sin x = x$ by fixed point iteration method.
Solution: $[x = 1.89549]$

{Hint: $|g'(x=1)| > 1$, so use $|g'(x=1.5)| < 1 \& |g'(x=2)| < 1$}

Example 1.7. Solve the equation $e^x - 3x = 0$ by iteration method correct to 3 decimal.(MJ12)
Solution: Let $f(x) = e^x - 3x = 0$

$$\boxed{\begin{aligned} &\boldsymbol{f(x=0) = e^0 - 3(0) = 1 > 0 \; ('+' \, ve)} \\ &\boldsymbol{f(x=1) = e^1 - 3(1) = -0.2817 < 0 \; ('-' \, ve)} \end{aligned}}$$

$\therefore f(x)$ has a root between 0 and 1.
i.e, $f(x)$ has a positive root.
This equation $f(x) = e^x - 3x = 0$ can be written as

$$3x = e^x$$
$$i.e., x = \frac{e^x}{3} \; [= g(x), \text{ say}]$$
$$\boxed{\therefore g(x) = \frac{e^x}{3}}$$

$$\text{Therefore, } g'(x) = \frac{d}{dx}[g(x)] = \frac{d}{dx}\left[\frac{e^x}{3}\right] = \frac{e^x}{3}$$
$$\therefore |g'(x)| = \left|\frac{e^x}{3}\right| = \frac{e^x}{3}$$

$$\text{At } x = 0, |g'(x=0)| = \frac{e^0}{3} = \frac{1}{3} = 0.333 < 1$$
$$\text{At } x = 1, |g'(x=1)| = \frac{e^1}{3} = 0.906 < 1$$

$$\boxed{|g'(x)| < 1 \text{ in } (0,1)}$$

So the fixed point iteration method can be applied. Let $x_0 = 0$ use $x_{n+1} = g(x_n)$

Now, put $n = 0$ in $x_{n+1} = g(x_n)$, we get

$x_1 = g(x_0) = \frac{e^{x_0}}{3} = \frac{e^0}{3} = 0.33333$	$x_8 = \frac{e^{x_7}}{3} = \frac{e^{0.60727}}{3} = 0.61180$
$x_2 = \frac{e^{x_1}}{3} = \frac{e^{0.33333}}{3} = 0.46520$	$x_9 = \frac{e^{x_8}}{3} = \frac{e^{0.61180}}{3} = 0.61458$
$x_3 = \frac{e^{x_2}}{3} = \frac{e^{0.46520}}{3} = 0.53078$	$x_{10} = \frac{e^{x_9}}{3} = \frac{e^{0.61458}}{3} = 0.61629$
$x_4 = \frac{e^{x_3}}{3} = \frac{e^{0.53078}}{3} = 0.56675$	$x_{11} = \frac{e^{x_{10}}}{3} = \frac{e^{0.61629}}{3} = 0.61735$
$x_5 = \frac{e^{x_4}}{3} = \frac{e^{0.56675}}{3} = 0.58751$	$\boxed{x_{12} = g(x_{11}) = \frac{e^{x_{11}}}{3} = \frac{e^{0.61735}}{3} = 0.61800}$
$x_6 = \frac{e^{x_5}}{3} = \frac{e^{0.58751}}{3} = 0.59983$	$\boxed{x_{13} = g(x_{12}) = \frac{e^{x_{12}}}{3} = \frac{e^{0.61800}}{3} = 0.61840}$
$x_7 = \frac{e^{x_6}}{3} = \frac{e^{0.59983}}{3} = 0.60727$	

These values are same at third decimal, i.e., $x_{12} = x_{13}$. So stop the method.

$$\boxed{\therefore \text{ The root of the given equation } e^x - 3x = 0 \text{ is } 0.618}$$

Example 1.8. Solve the equation $2x - \log x = 7$ by iteration method correct to 5 decimal.

Solution: Let $f(x) = 2x - \log x = 7 = 0$

$$f(x=0) = 2(0) - \log_{10}(0) - 7 = \text{(Undefined)}$$
$$f(x=1) = 2(1) - \log_{10}(1) - 7 = -5 < 0 \ ('-' ve)$$
$$f(x=2) = 2(2) - \log_{10}(2) - 7 = -3 < 0 \ ('-' ve)$$
$$f(x=3) = 2(3) - \log_{10}(3) - 7 = -1 < 0 \ ('-' ve)$$
$$f(x=4) = 2(4) - \log_{10}(4) - 7 = 0.39 > 0 \ ('+' ve)$$

$\therefore$ $f(x)$ has a root between 3 and 4.

i.e, $f(x)$ has a positive root.

This equation $f(x) = 2x - \log x - 7 = 0$ can be written as

$$2x = 7 + \log x$$
$$\text{i.e.,} x = \frac{7 + \log x}{2} \ [= g(x), \text{ say}]$$
$$\boxed{\therefore g(x) = \frac{7 + \log x}{2}}$$

$$\text{Therefore, } g'(x) = \frac{d}{dx}[g(x)] = \frac{d}{dx}\left[\frac{7 + \log x}{2}\right] = \frac{\frac{1}{x}}{2} = \frac{1}{2x}$$
$$\therefore |g'(x)| = \left|\frac{1}{2x}\right| = \frac{1}{2x}$$

$$\text{At } x = 3, |g'(x=3)| = \frac{1}{2(3)} = \frac{1}{6} = 0.166 < 1$$
$$\text{At } x = 4, |g'(x=4)| = \frac{1}{2(4)} = \frac{1}{8} = 0.125 < 1$$

$$\boxed{|g'(x)| < 1 \text{ in } (3,4)}$$

So the fixed point iteration method can be applied. Let $x_0 = 4$, use $x_{n+1} = g(x_n)$

Now, put $n = 0$ in $x_{n+1} = g(x_n)$, we get

$$\begin{aligned}
x_1 &= g(x_0) = \frac{7+\log(x_0)}{2} = \frac{7+\log(4)}{2} = 3.80103 \\
x_2 &= g(x_1) = \frac{7+\log(x_1)}{2} = \frac{7+\log(3.80103)}{2} = 3.78995 \\
x_3 &= g(x_2) = \frac{7+\log(x_2)}{2} = \frac{7+\log(3.78995)}{2} = 3.78932 \\
x_4 &= g(x_3) = \frac{7+\log(x_3)}{2} = \frac{7+\log(3.78932)}{2} = 3.78928 \\
x_5 &= g(x_4) = \frac{7+\log(x_4)}{2} = \frac{7+\log(3.78928)}{2} = 3.78928
\end{aligned}$$

These values are same at fifth decimal, i.e., $x_4 = x_5$. So stop the method.

$$\boxed{\therefore \text{ The root of the given equation } 2x - \log x = 7 \text{ is } 3.78928}$$

Example 1.9. Solve the equation $3x - \log_{10} x = 6$ by fixed point iteration method.

Solution: [$x = 2.1080$]

1.3.3 Anna University Questions

1. Solve $e^x - 3x = 0$ by the method of fixed point iteration. (AM12)

   Solution : [$x = 0.618$]

2. Find a positive root of the equation $cosx - 3x + 1 = 0$ by using iteration method. (AM13)

   Solution : [$x = 0.60710$]

1.4 Newton Raphson method

(Newton's method or method of tangents)

Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = \phi(x_n)$, where $n = 0, 1, 2, \cdots$

1.4.1 Part A

1. State the order of convergence and convergence condition for Newton-Raphson method.

   (Or) What is the criterion for the convergence in Newton's method? (MJ2011)

   (Or) State the order of convergence and the criterion for the convergence in Newton's method. (AM2012)

   (Or) Write down the condition for convergence of Newton-Raphson method for $f(x) = 0$. (ND14)

   Solution: Order of convergence is 2. Convergence condition is

$$|f(x)\, f''(x)| < |f'(x)|^2$$

2. Find an iterative formula to find $\sqrt{N}$, where N is positive number.

Solution: Let $x = \sqrt{N}$

$$\Rightarrow x^2 = N \Rightarrow x^2 - N = 0$$

Let $f(x) = x^2 - N \Rightarrow f'(x) = 2x$

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

3. Derive Newton's algorithm for finding the p^{th} root of a number N.

Solution: Let $x = N^{1/p} = \sqrt[p]{N} \Rightarrow x^p - N = 0$

Let $f(x) = x^p - N \Rightarrow f'(x) = px^{p-1}$

By Newton's formula

$$x_{n+1} = x_n - \frac{x_n^p - N}{px^{p-1}} = \frac{px_n^p - x_n^p + N}{px^{p-1}} = \frac{(p-1)\,x_n^p + N}{px^{p-1}}$$

4. Establish an iteration formula to find the reciprocal of a positive number N by Newton-Raphson method. (Or)

Find an iterative formula to find the reciprocal of a given number $N(N \neq 0)$. (AM2013)

Solution : Let $x = \dfrac{1}{N}$

i.e., $N = \dfrac{1}{x}$

Let $f(x) = \dfrac{1}{x} - N$

$f'(x) = -\dfrac{1}{x^2}$

We know that by Newton's iterative formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \left[\dfrac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}\right]$

$$= x_n + x_n^2\left[\frac{1}{x_n} - N\right] = x_n + x_n - Nx_n^2 = 2x_n - Nx_n^2$$

$$x_{n+1} = x_n\,[2 - Nx_n]$$

which is the required iterative formula for the reciprocal of a given number N.

5. Locate the negative root of $x^3 - 2x + 5 = 0$, approximately.

Solution: Let $f(x) = x^3 - 2x + 5 = 0$

$$f(-1) = -1 + 2 + 5 = 6\,(+ve)$$

$$f(-2) = -8 + 4 + 5 = 1\,(+ve)$$

$$f(-3) = -27 + 6 + 5 = -16\,(-ve)$$

$\therefore$ Root lies between -2 and -3.

and root is closer to -2 since $|f(-2)| < |f(-3)|$

6. Evaluate $\sqrt{12}$ applying Newton formula.

Solution: Let $x = \sqrt{12}$

$$\Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$$

Let $f(x) = x^2 - 12 \Rightarrow f'(x) = 2x$

$$f(3) = 9 - 12 = -3\,(-ve)$$

$$f(4) = 16 - 12 = 4\,(+ve)$$

$\therefore$ Root lies between 3 and 4.
and root is closer to 3 since $|f(3)| < |f(4)|$
Take $x_0 = 3$
Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

when $n = 0, x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3 - \dfrac{f(3)}{f'(3)} = 3 - \dfrac{(3)^2 - 12}{2(3)} = 3.5$

when $n = 1, x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3.5 - \dfrac{f(3.5)}{f'(3.5)} = 3.5 - \dfrac{(3.5)^2 - 12}{2(3.5)} = 3.46429$

when $n = 2, x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 3.46429 - \dfrac{f(3.46429)}{f'(3.46429)} = 3.46429 - \dfrac{(3.46429)^2 - 12}{2(3.46429)} = 3.46410$

when $n = 3, x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} = 3.46410 - \dfrac{f(3.46410)}{f'(3.46410)} = 3.46410 - \dfrac{(3.46410)^2 - 12}{2(3.46410)} = 3.46410$

Here $x_3 = x_4 = 3.46410$
$\therefore$ The root is 3.46410

7. What is Newton's algorithm to solve the equation $x^2 = 12$? (ND2010)
Solution : Let $f(x) = x^2 - 12 \Rightarrow f'(x) = 2x$
We know that by

Newton Raphson iteration formula
$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - 12}{2x_n} \\ &= \frac{2x_n^2 - x_n^2 + 12}{2x_n} \\ x_{n+1} &= \frac{x_n^2 + 12}{2x_n} \end{aligned}$$

8. Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$. (ND2011)
Solution: Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $n = 0, 1, 2, \cdots$ (1)

Let $f(x) = x^3 - 6x + 4 \Rightarrow f'(x) = 3x^2 - 6$

$f(x = 0) = 0^3 - 6(0) + 4 = 4 > 0$ ('+' ve)

$f(x = 1) = 1^3 - 6(1) + 4 = -1 < 0$ ('−' ve)

$\therefore$ A root lies between 0 and 1.

But $|f(0)| > |f(1)|$

$\therefore$ The root is nearer to 1.

Take $x_0 = 1$

Now, put $n = 0$ in (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0)^3 - 6(x_0) + 4}{\left[3(x_0)^2 - 6\right]} = 1 - \frac{\left[1^3 - 6(1) + 4\right]}{\left[3(1)^2 - 6\right]}$$

$$= 1 - \frac{(-1)}{-3} = 1 - \frac{1}{3} = 0.666 = 0.67 \qquad \text{[Corrected to 2 dec. places]}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1)^3 - 6(x_1) + 4}{\left[3(x_1)^2 - 6\right]} = 0.67 - \frac{\left[(0.67)^3 - 6(0.67) + 4\right]}{\left[3(0.67)^2 - 6\right]}$$

$$= 0.67 - \frac{0.28}{-4.65} = 0.67 + \frac{0.28}{4.68} = 0.73$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2)^3 - 6(x_2) + 4}{\left[3(x_2)^2 - 6\right]} = 0.73 - \frac{\left[(0.73)^3 - 6(0.73) + 4\right]}{\left[3(0.73)^2 - 6\right]}$$

$$= 0.73 - \frac{0.009}{-4.4013} = 0.73 + \frac{0.009}{4.4013} = 0.7320 = 0.73 \qquad \text{[Corrected to 2 dec. places]}$$

Here $x_2 = x_3 = 0.73$

$\therefore$ A root of given equation is 0.73 corrected to two decimal places.

9. Evaluate $\sqrt{15}$ using Newton-Raphson's formula. (AM2014)

Solution : W.K.T. The Newton Raphson iterative formula for $\sqrt{N}$ is

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

To find $\sqrt{15}$

W.K.T. $\sqrt{15}$ is nearer to 4

$\therefore$ Let $x_0 = 4$

$$\begin{aligned} x_1 &= \frac{1}{2}\left[x_0 + \frac{N}{x_0}\right] = \frac{1}{2}\left[4 + \frac{15}{4}\right] = 3.87500 \\ x_2 &= \frac{1}{2}\left[x_1 + \frac{N}{x_1}\right] = \frac{1}{2}\left[3.875 + \frac{15}{3.875}\right] = 3.87298 \\ x_3 &= \frac{1}{2}\left[x_2 + \frac{N}{x_2}\right] = \frac{1}{2}\left[3.87298 + \frac{15}{3.87298}\right] = 3.87298 \end{aligned}$$

Here $x_2 = x_3 = 3.87298$

$\therefore$ The root is 3.87298

1.4.2 Part B

Example 1.10. Using Newton-Raphson method, find a ' + 've root correct to 3 decimal places for the equation $x^3 - x - 2 = 0$.

Solution: Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $n = 0, 1, 2, \cdots$

Let $f(x) = x^3 - x - 2$

$\Rightarrow f'(x) = 3x^2 - 1$

$f(x = 0) = 0^3 - 0 - 2 = -2 < 0$ ('$-$'ve)

$f(x = 1) = 1^3 - 1 - 2 = -2 < 0$ ('$-$'ve)

$f(x = 2) = 2^3 - 2 - 2 = 4 > 0$ ('$+$' ve)

$\therefore$ A root lies between 1 and 2.

Take $x_0 = 1$

Now, put $n = 0$ in $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0)^3 - x_0 - 2}{\left[3(x_0)^2 - 1\right]} = 1 - \frac{\left[1^3 - 1 - 2\right]}{\left[3(1)^2 - 1\right]} = 2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1)^3 - x_1 - 2}{\left[3(x_1)^2 - 1\right]} = 2 - \frac{\left[2^3 - 2 - 2\right]}{\left[3(2)^2 - 1\right]} = 1.63636$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2)^3 - x_2 - 2}{\left[3(x_2)^2 - 1\right]} = 1.63636 - \frac{1.63636^3 - 1.63636 - 2}{\left[3(1.63636)^2 - 1\right]} = 1.53039$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{(x_3)^3 - x_3 - 2}{\left[3(x_3)^2 - 1\right]} = 1.53039 - \frac{1.53039^3 - 1.53039 - 2}{\left[3(1.53039)^2 - 1\right]} = 1.52144$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = x_4 - \frac{(x_4)^3 - x_4 - 2}{\left[3(x_4)^2 - 1\right]} = 1.52144 - \frac{1.52144^3 - 1.52144 - 2}{\left[3(1.52144)^2 - 1\right]} = 1.52138$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = x_5 - \frac{(x_5)^3 - x_5 - 2}{\left[3(x_5)^2 - 1\right]} = 1.52138 - \frac{1.52138^3 - 1.52138 - 2}{\left[3(1.52138)^2 - 1\right]} = 1.52138$$

$x_5 = x_6 = 1.52138$ upto 5 decimal places.

$\therefore$ A root of given equation is 1.521

Example 1.11. Find the real root of $x^3 - 3x + 1 = 0$ lying between 1 and 2 by Newton Raphson method.

Solution: Let $f(x) = x^3 - 3x + 1$

$f(1) = 1 - 3 + 1 = -1 < 0$

$f(2) = 8 - 6 + 1 = 3 > 0$

$\therefore$ A root lies between 1 and 2.

Since $|f(1)| < |f(2)|$, the root is nearer to 1.

Let $x_0 = 1$.

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3$$

Since $f'(1) = 3 - 30$, Newton's approximation formula cannot be applied for the initial approximation at $x = 1$.

Let us take $x_0 = 1.5$ Now,

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ $= 1.5 - \dfrac{f(1.5)}{f'(1.5)}$ $= 1.5 - \left[\dfrac{(1.5)^3 - 3(1.5) + 1}{3(1.5)^2 - 3}\right] = 1.5 - \left[\dfrac{-0.125}{3.75}\right]$ $= 1.5 + 0.0333$ $x_1 = 1.5333$	$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ $= 1.5333 - \dfrac{f(1.5333)}{f'(1.5333)}$ $= 1.5333 - \left[\dfrac{(1.5333)^3 - 3(1.5333) + 1}{3(1.5333)^2 - 3}\right]$ $= 1.5333 - \left[\dfrac{0.0049}{4.0530}\right] = 1.5333 - 0.0012$ $x_2 = 1.5321$
$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$ $= 1.5321 - \dfrac{f(1.5321)}{f'(1.5321)}$ $= 1.5321 - \left[\dfrac{(1.5321)^3 - 3(1.5321) + 1}{3(1.5321)^2 - 3}\right]$ $= 1.5321 - \left[\dfrac{0.00004}{4.042}\right] = 1.5321 - 0.00001$ $x_3 = 1.5321$	Here $x_2 = 1.5321$ and $x_3 = 1.5321.$ $\therefore$**The required root, correct to four places of decimal is 1.5321.**

Example 1.12. Solve the equation $x^3 - 5x + 3 = 0$ by Newton-Raphson method.

Solution: [$x_3 = x_4 = 0.65662$]

Example 1.13. Solve the equation $x^4 - x - 9 = 0$ by Newton-Raphson method.

Solution: [$x_6 = x_7 = 1.813$] upto 3rd decimal.

Example 1.14. Find a negative root of $x^3 - 2x + 5 = 0$ by Newton Raphson method.

Solution: (Home work)

[Hint: $f(-2) > 0$, $f(-3) < 0$

$\therefore$ a root lies between -2 and -3

Ans : -2.095]

[Or]

[Given: $x^3 - 2x + 5 = 0$

Replace x by $-x^3 + 2x + 5 = 0$

i.e., $x^3 - 2x - 5 = 0$

Let $f(x) = x^3 - 2x - 5 = 0$

Now, we find the positive root of $f(x) = x^3 - 2x - 5$

$f(2) = -1$ and $f(3) = 16$

$\therefore$ a root lies between 2 and 3

The root of $f(x) = x^3 - 2x - 5$ is 2.095

$\therefore$ The root of $x^3 - 2x + 5$ is -2.095]

Example 1.15. Using Newton-Raphson method, find a '+'ve root correct to 5 decimal places for the equation $x \tan x = 1.28$

Solution: Newton-Raphson method formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } n = 0, 1, 2, \cdots$$

Let $f(x) = x\tan x - 1.28 \Rightarrow f'(x) = x\sec^2 x + \tan x$

$$f(x=0) = (0)\tan(0) - 1.28 = -1.28 < 0 \ ('-'\text{ve})$$
$$f(x=0.5) = (0.5)\tan(0.5) - 1.28 = -1.006 < 0 \ ('-'\text{ve})$$
$$f(x=1) = (1)\tan(1) - 1.28 = 0.2774 > 0 \ ('+'\text{ ve})$$

$\therefore$ A root lies between 0.5 and 1.

Take $x_0 = 0.5$

Now, put $n = 0$ in $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0\tan x_0 - 1.28}{x_0\sec^2 x_0 + \tan x_0} = 0.5 - \frac{(0.5)\tan(0.5) - 1.28}{(0.5)\sec^2(0.5) + \tan(0.5)} = 1.34218$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.34218 - \frac{(1.34218)\tan(1.34218) - 1.28}{(1.34218)\sec^2(1.34218) + \tan(1.34218)} = 1.19469$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.19469 - \frac{(1.19469)\tan(1.19469) - 1.28}{(1.19469)\sec^2(1.19469) + \tan(1.19469)} = 1.04143$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.04143 - \frac{(1.04143)\tan(1.04143) - 1.28}{(1.04143)\sec^2(1.04143) + \tan(1.04143)} = 0.95512$$
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.95512 - \frac{(0.95512)\tan(0.95512) - 1.28}{(0.95512)\sec^2(0.95512) + \tan(0.95512)} = 0.93871$$
$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 0.93871 - \frac{(0.93871)\tan(0.93871) - 1.28}{(0.93871)\sec^2(0.93871) + \tan(0.93871)} = 0.93826$$
$$x_7 = x_6 - \frac{f(x_6)}{f'(x_6)} = 0.93826 - \frac{(0.93826)\tan(0.93826) - 1.28}{(0.93826)\sec^2(0.93826) + \tan(0.93826)} = 0.93826$$

$x_6 = x_7 = 0.93826$ upto 5 decimal places.

$\therefore$ A root of given equation is 0.93826

Example 1.16. Solve the equation $x\sin x + \cos x = 0$ by Newton-Raphson method.

Solution: $[x_4 = x_5 = 2.79839]$

Example 1.17. Using Newton-Raphson method, find a root correct to 5 decimal places for the equation $e^x - 4x = 0$

Solution: Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Let $f(x) = e^x - 4x \Rightarrow f'(x) = e^x - 4$

$$f(x=0) = e^0 - 4(0) = 1 > 0 \ ('+'\ ve)$$
$$f(x=1) = e^1 - 4(1) = -1.28272 < 0 \ ('-'ve)$$

$\therefore$ A root lies between 0 and 1.

Here $|f(1)| = 1.28172 > |f(0)| = 1$

$\therefore$ Root is nearer to 0.

Take $x_0 = 0$

Now, put $n = 0$ in $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{e^{x_0} - 4x_0}{e^{x_0} - 4} = 0 - \frac{e^0 - 4(0)}{e^0 - 4} = 0.33333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{e^{x_1} - 4x_1}{e^{x_1} - 4} = 0.33333 - \frac{e^{0.33333} - 4(0.33333)}{e^{0.33333} - 4} = 0.35725$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{e^{x_2} - 4x_2}{e^{x_2} - 4} = 0.35725 - \frac{e^{0.35725} - 4(0.35725)}{e^{0.35725} - 4} = 0.35740$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{e^{x_3} - 4x_3}{e^{x_3} - 4} = 0.35740 - \frac{e^{0.35740} - 4(0.35740)}{e^{0.35740} - 4} = 0.35740$$

$\therefore$ A root of given equation is 0.35740

Example 1.18. Find the real root of $xe^x - 2 = 0$ correct to three places of decimals using Newton Raphson method.

Solution: Let $f(x) = xe^x - 2$

$f(0) = -2 < 0$

$f(1) = e - 2 = 0.7183 > 0$

Since $f(0) < 0$ and $f(1) > 0$.

$\therefore$ A root lies between 0 and 1.

Since $|f(1)| < |f(0)|$, the root is nearer to 1.

Let $x_0 = 1$.

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = xe^x - 2 \Rightarrow f'(x) = xe^x - e^x$$

Now,

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ $= 1 - \dfrac{f(1)}{f'(1)}$ $= 1 - \left[\dfrac{e-2}{e+e}\right]$ $= 1 - \left[\dfrac{0.7183}{5.4366}\right]$ $= 1 - 01321$ $x_1 = 0.8679$	$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ $= 0.8679 - \dfrac{(0.8679)e^{0.8679} - 2}{(0.8679)e^{0.8679} + e^{0.8679}}$ $= 0.8679 - \left[\dfrac{0.0673}{4.4492}\right]$ $= 0.8679 - 0.0151$ $x_2 = 0.8528$	$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$ $= 0.8528 - \dfrac{(0.8528)e^{0.8528} - 2}{(0.8528)e^{0.8528} + e^{0.8528}}$ $= 0.8528 - \left[\dfrac{0.0008}{4.3471}\right]$ $= 0.8528 - 0.0002$ $x_3 = 0.8526$

Here $x_2 = 0.853$ and $x_3 = 0.853$.

$\therefore$ The required root, correct to three places of decimal is 0.853.

Example 1.19. Solve the equation $\cos x = xe^x$, by $x_0 = 0.5$ by Newton-Raphson method.

Solution: [$x_2 = x_3 = 0.51776$]

Example 1.20. Solve the equation $xe^{-2x} = 0.5 \sin x$ by Newton-Raphson method.

Solution: [$x_2 = x_3 = 3.12962$ upto 5th decimal]

Example 1.21. Solve the equation $2x - \log_{10} x = 7$ by Newton-Raphson method.

Solution: $[x_5 = x_6 = 3.78928$upto 5th decimal$]$

Example 1.22. Find the Newton-Raphson formula to find the value of $\frac{1}{N}$ where N is a real number, hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Solution: Let $x = \frac{1}{N}$

$\Rightarrow N = \frac{1}{x} \Rightarrow \frac{1}{x} - N = 0$

Let $f(x) = \frac{1}{x} - N \Rightarrow f'(x) = -\frac{1}{x^2}$

By Newton's formula ,

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}} = x_n + \left(\frac{1}{x_n} - N\right) x_n^2 = x_n (2 - N x_n)$$

This is the Newton's-Raphson formula to find the value of $\frac{1}{N}$.

To find $\frac{1}{26}$

Put $N = 26$ and $n = 0$ in $x_{n+1} = x_n (2 - N x_n)$, we get

$$x_1 = x_0 (2 - 26x_0) \tag{1}$$

Take $\frac{1}{25} = 0.04$ as x_0 $\left(\because \frac{1}{25} \text{ is nearer to } \frac{1}{26}\right)$

$$\begin{aligned}
\therefore (1) \Rightarrow x_1 &= x_0 (2 - 26x_0) = 0.04 [2 - 26 (0.04)] = 0.0384 \\
x_2 &= x_1 (2 - 26x_1) = 0.0384 [2 - 26 (0.0384)] = 0.03846 \\
x_3 &= x_2 (2 - 26x_2) = 0.03846 [2 - 26 (0.03846)] = 0.03846
\end{aligned}$$

$\therefore$ A root of given equation is 0.03846 or 0.0385

Example 1.23. Find the Newton-Raphson formula to find the value of $\sqrt{N}$ where N is a real number, hence evaluate $\sqrt{142}$ correct to 5 decimal places.

Solution: $[x_2 = x_3 = 11.91638]$

Example 1.24. Find the Newton-Raphson formula to find the value of $N^{\frac{1}{p}}$ where N is a real number, hence evaluate $\sqrt[3]{17}$ correct to 5 decimal places.

Solution: $[x_4 = x_5 = 2.57128]$

1.4.3 Anna University Questions

1. Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton - Raphson method.(MJ2010)

   Solution : $[x_2 = x_3 = 1.856]$

2. Find the approximate root of $xe^x = 3$ by Newton's method correcot to 3 decimal places. (MJ2011)

   Solution : $[x_2 = x_3 = 1.050]$

3. Find the Newton's iterative formula to calculate the reciprocal of N and hence find the value of $\frac{1}{23}$. (ND2012)
   Solution : $[x_0 = x_1 = 0.0435]$

4. Using Newton's method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.(ND2013)
   Solution : $[x_3 = x_4 = 2.740646096 \equiv 2.74065]$

5. Find by Newton-Raphson method a positive root of the equation $3x - \cos x - 1 = 0$. (ND14)
   Solution : $[x = 0.607102]$

6. Find Newton's iterative formula for the reciprocal of a number N and hence find the value of $\frac{1}{23}$, correct to five decimal places.
   Solution : $[x = 0.04347]$

7. Prove the quadratic convergence of Newton's-Raphson method. Find a positive root of $f(x) = x^3 - 5x + 3 = 0$, using this method.
   Solution : $[x = 0.6566]$

8. Use Newton's method to find the real root of $3x - \cos x - 1 = 0$.
   Solution : $[x = 0.607102]$

9. Obtain the positive root of $2x^3 - 3x - 6 = 0$ that lies between 1 and 2 by using Newton's-Raphson method.
   Solution : [1.7838]

Solution of linear system of equations

Let us consider a system of n linear algebraic equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

where the coefficients a_{ij} and the constants b_i are real and known. This system of equations in matrix form may be written as

$$AX = b$$

where $A = \left(a_{ij}\right)_{n\times n}$

$X = (x_1, x_2, \cdots, x_n)^T$

and $b = (b_1, b_2, \cdots, b_n)^T$.

A is called the coefficient matrix.

We are interested in finding the values $x_i, i = 1, 2, \cdots, n$ if they exist, satisfying Equation.

Methods to solve linear system of equations:

1. Gaussian elimination method

2. Pivoting-Gauss Jordan method

1.5 Gauss elimination method

Step 1: Write the augmented matrix for the given system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & b_{33} & b_3 \end{pmatrix}$$

Step 2: Using elementary row operations reduce the given matrix into an upper-triangular matrix say

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

Step 3: By back substitution we get the values for unknowns.

1.5.1 Part A

1. By Gauss elimination method solve $x + y = 2, 2x + 3y = 5.$ (MJ2011)

Solution : Given $x + y = 2$

$$2x + 3y = 5$$

The given system can be written as

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$AX = B$$

$\therefore$ The augmented matrix is $[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right]$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] R_2 \to R_2 - 2R_1 \qquad (1)$$

Now From (1), by back substitution we get

$$0x + y = 1 \Rightarrow \boxed{y = 1}$$

$$x + y = 2 \Rightarrow x + 1 = 2 \Rightarrow \boxed{x = 1}$$

$\therefore x = 1, y = 1.$

2. Solve the equations $x + 2y = 1$ and $3x - 2y = 7$ by Gauss-Elimination method. (ND2013)

   Solution : $\left[x = 2, y = -\frac{1}{2}\right]$.

3. Using Gauss elimination method solve : $5x + 4y = 15, 3x + 7y = 12$. (AM2014)

   Solution : Given $5x + 4y = 15$

   $3x + 7y = 12$

   The given system can be written as

$$\begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$$

$$AX = B$$

$\therefore$ The augmented matrix is $[A|B] = \left[\begin{array}{cc|c} 5 & 4 & 15 \\ 3 & 7 & 12 \end{array}\right]$

$$\sim \left[\begin{array}{cc|c} 5 & 4 & 15 \\ 3 - \frac{3}{5}(5) & 7 - \frac{3}{5}(4) & 12 - \frac{3}{5}(15) \end{array}\right] R_2 \to R_2 - \frac{3}{5}(R_1)$$

$$\sim \left[\begin{array}{cc|c} 5 & 4 & 15 \\ 0 & \frac{23}{5} & \frac{15}{5} \end{array}\right] \tag{1}$$

Now From (1), by back substitution we get

$$0x + \frac{23}{5}y = \frac{15}{5} \Rightarrow \boxed{y = \frac{15}{23}}$$

$$5x + 4y = 15 \Rightarrow 5x = \frac{285}{23} \Rightarrow \boxed{x = \frac{57}{23}}$$

$\therefore x = \frac{57}{23}, y = \frac{15}{23}$.

1.5.2 Part B

Example 1.25. Solve the system of equations by Gauss elimination method

$$\mathbf{-3x_1 + 2x_2 - 3x_3 = -6}$$

$$\mathbf{x_1 - x_2 + x_3 = 1}$$

$$\mathbf{2x_1 - 5x_2 + 4x_3 = 5.}$$

Solution: The given system of equations can be written as

$$x_1 - x_2 + x_3 = 1$$

$$-3x_1 + 2x_2 - 3x_3 = -6$$

$$2x_1 - 5x_2 + 4x_3 = 5$$

Given system of the form $AX = B$

Gauss elimination method:

The augmented matrix form is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3+3 & 2-3 & -3+3 & -6+3 \\ 2-2 & -5+2 & 4-2 & 5-2 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3+3 & 2-0 & 3+9 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{array}\right]$$

$\therefore$ By back substitution, $(-2, 3, 6)$ is the solution for the system of equations.

Example 1.26. Solve the system of equations by Gauss elimination method $10x - 2y + 3z = 23$, $2x + 10y - 5z = -33$, $3x - 4y + 10z = 41$.

Solution: The matrix form is $\begin{pmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ -33 \\ 41 \end{pmatrix}$

i.e., $AX = B$

The augmented matrix

$$[A, B] = \begin{pmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix}$$

$$\sim \begin{pmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{pmatrix} \begin{array}{l} \\ R_2 \to 5R_2 - R_1 \\ R_3 \to 10R_3 - 3R_1 \end{array}$$

$$\sim \begin{pmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{pmatrix} \begin{array}{l} \\ \\ R_3 \to 52R_3 + 34R_2 \end{array}$$

The reduced system of equations is given by

$$10x - 2y + 3z = 23 \quad (1)$$
$$52y - 28z = -188 \quad (2)$$
$$3780z = 11340 \quad (3)$$

From (3) $\Rightarrow z = \dfrac{11340}{3780} = 3$ ($\because$ by back substitution)

From (2) $\Rightarrow 52y - 28(3) = -188$
$$52y = -104$$
$$y = -2$$

From (1) $\Rightarrow 10x + 4 + 9 = 23$
$$10x = 10$$
$$x = 1$$

Hence the solution is $x = 1, y = -2, z = 3$.

Example 1.27. Using Gauss elimination method, solve the system $3.15x - 1.96y + 3.85z = 12.95$, $2.13x + 5.12y - 2.89z = -8.61$, $5.92x + 3.05y + 2.15z = 6.88$.

Solution: The matrix form is $\begin{pmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12.95 \\ -8.61 \\ 6.88 \end{pmatrix}$

i.e., $AX = B$

The augmented matrix

$$[A, B] = \begin{pmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 21.2107 & -16.0195 & -54.992 \end{pmatrix} \begin{matrix} \\ R_2 \to 3.15R_2 - 2.13R_1 \\ R_3 \to 3.5R_3 - 5.92R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 0 & 41.7892 & 43.8398 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to 20.3028R_3 - 21.2107R_2 \end{matrix}$$

The reduced system of equations is given by

$$3.15x - 1.96y + 3.85z = 12.95 \quad (1)$$
$$20.3028y - 17.304z = -54.705 \quad (2)$$
$$41.7892z = 43.8398 \quad (3)$$

From (3) $\Rightarrow z = \frac{43.8398}{41.7892} = 1.049$(correct to 3 decimals)

From (2) $\Rightarrow 20.3028y - 17.304(1.049) = -54.705$

$$\Rightarrow y = -\frac{36.5531}{20.3028} = -1.800$$

From (1) $\Rightarrow 3.15x - 1.96(-1.8) + 3.85(1.049) = 12.95$

$$3.15x + 7.5667 = 12.95$$

$$x = \frac{5.3834}{3.15} = 1.709$$

Hence the solution is $x = 1.709, y = -1.8, z = 1.049$.

1.5.3 Anna University Questions

1. Solve the given system of equations by Gauss elimination method: $-x_1 + x_2 + 10x_3 = 35.61, 10x_1 + x_2 - x_3 = 11.19, x_1 + 10x_2 + x_3 = 20.08$.
   Solution: $\left[x = \frac{1453}{1100}, y = \frac{837}{550}, z = \frac{779}{220}\right]$

1.6 Pivoting - Gauss Jordan method

Step 1: Write the augmented matrix for the given system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & b_{33} & b_3 \end{pmatrix}$$

Step 2: Using elementary row operations reduce the given matrix into a diagonal matrix say

$$\begin{pmatrix} c_{11} & 0 & 0 & d_1 \\ 0 & c_{22} & 0 & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

$$\left[\text{Here} \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \text{is a} \begin{pmatrix} \text{diagonal matrix} \\ \text{or} \\ \text{unit matrix} \end{pmatrix}\right]$$

Step 2: By direct substitution we get the values for unknowns.

1.6.1 Part A

1. **For solving a linear system $AX = B$, compare Gauss elimination method and Gauss Jordan method.**
   Solution:

Gauss elimination method	**Gauss Jordan method**
Coefficient matrix is transmitted into upper triangular matrix	Coefficient matrix is transmitted into diagonal matrix or unit matrix
Direct method	Direct method
Obtain the solution by back substitution method	No need of back substitution method

2. **State the two difference between direct and iterative methods for solving system of equations.**

   Solution:

Direct Method	**Iterative Method**
It gives exact value	It gives only approximate value
Simple, takes less time	Time consuming and labourious
This method determine all the roots at the same time	This method determine only one root at a time

1.6.2 Part B

Example 1.28. Solve the system of equations by Gauss Jordan method

$$\mathbf{-3x_1 + 2x_2 - 3x_3 = -6}$$

$$\mathbf{x_1 - x_2 + x_3 = 1}$$

$$\mathbf{2x_1 - 5x_2 + 4x_3 = 5.}$$

Solution: The given system of equations can be written as

$$x_1 - x_2 + x_3 = 1$$

$$-3x_1 + 2x_2 - 3x_3 = -6$$

$$2x_1 - 5x_2 + 4x_3 = 5$$

Given system of the form $AX = B$

Gauss-Jordan method:

The augmented matrix form is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3+3 & 2-3 & -3+3 & -6+3 \\ 2-2 & -5+2 & 4-2 & 5-2 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -3 & 2 & 3 \end{array}\right] R_2 \to R_2 \times (-1)$$

$$\sim \left[\begin{array}{ccc|c} 1+0 & -1+1 & 1+0 & 1+3 \\ 0 & 1 & 0 & 3 \\ 0 & -3+3(1) & 2+3(0) & 3+3(3) \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right] R_3 \to \frac{R_3}{2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right] R_1 \to R_1 - R_3$$

$\therefore x_1 = -2, x_2 = 3, x_3 = 6$

$\Rightarrow (x_1, x_2, x_3) = (-2, 3, 6).$

Example 1.29. Solve the system of equation by (i) Gauss elimination method (ii) Gauss Jordan method

$$\begin{aligned} 28x + 4y - z &= 32 \\ x + 3y + 10z &= 24 \\ 2x + 17y + 4z &= 35 \end{aligned}$$

Solution: $[(x, y, z) = (0.99359, 1.50697, 1.84855)]$

Example 1.30. Solve the system of equation by (i) Gauss elimination method (ii) Gauss Jordan method

$$\begin{aligned} p + q + r + s &= 2 \\ p + q + 3r - 2s &= -6 \\ 2p + 3q - r + 2s &= 7 \\ p + 2q + r - s &= -2 \end{aligned}$$

Solution: $[(p, q, r, s) = (1, 0, -1, 2)]$

1.6.3 Anna University Questions

1. Apply Gauss-Jordan method to solve the following system of equations $x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40$. (AM11)

   Solution: $[(x, y, z) = (1, 3, 5)]$

2. Apply Gauss-Jordan method to find the solution of the following system: $10x+y+z = 12$, $2x+10y+z = 13$, $x + y + 5z = 7$ (ND11)

   Solution: $[(x, y, z) = (1, 1, 1)]$

3. Solve the system of equations by Gauss-Jordan method: $5x_1-x_2 = 9; -x_1+5x_2-x_3 = 4; -x_2+5x_3 = -6$ (AM14)

   Solution: $[(x, y, z) = (2, 1, -1)]$

4. Using Gauss-Jordan method to solve $2x - y + 3z = 8; -x + 2y + z = 4, 3x + y - 4z = 0.$ (ND14)

   Solution: $[x = 2, y = 2, z = 2]$

5. Solve by Gauss Jordan method, the following system $10x+y-z = 11.19; x+10y+z = 20.08; -x+y+10z = 35.61.$

   Solution: $\left[x = \frac{1453}{1100}, y = \frac{837}{550}, z = \frac{779}{220}\right]$

6. Apply Gauss Jordan method to solve the equations $x + y + z = 9; 2x - 3y + 4z = 13; 3x + 4y + 5z = 40.$

   Solution: $\left[x = \frac{11}{12}, y = \frac{19}{6}, z = \frac{59}{12}\right]$

Diagonal system

In the system of linear equations in n unknowns $AX = B$. If the coefficient matrix A is diagonally dominant then the system is said to be a diagonal system.

Thus the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

i.e., $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

i.e., $AX = B$

is a diagonal system if

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$
$$|c_3| \geq |a_3| + |b_3|$$

Note : For the Gauss Seidal method to converge quickly, the coefficient matrix A must be diagonally dominant.

If the coefficient matrix A is not diagonally dominant we must rearrange the equations in such a way that the resulting matrix becomes dominant, and then only we can apply Gauss Seidel method.

Iterative Methods:

Let us consider a system of n linear algebraic equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where the coefficients a_ij and the constants b_i are real and known. This system of equations in matrix form may be written as

$$AX = b$$

where $A = \left(a_{ij}\right)_{n\times n}$

$X = (x_1, x_2, \cdots, x_n)^T$

and $b = (b_1, b_2, \cdots, b_n)^T$.

A is called the coefficient matrix.

We are interested in finding the values $x_i, i = 1, 2n$ if they exist, satisfying Equation.

A square matrix A is called **diagonally dominant** if $|A_{ii}| \geq \sum_{j\neq i} |A_{ij}|$ for all i.

i.e.,

$$|A_{11}| \geq \sum_{j\neq 1} |A_{1j}|$$
$$|A_{22}| \geq \sum_{j\neq 2} |A_{2j}|$$
$$|A_{33}| \geq \sum_{j\neq 3} |A_{3j}|$$

A square matrix A is called **strictly diagonally dominant** if $|A_{ii}| > \sum_{j\neq i} |A_{ij}|$ for all i.

i.e.,

$$|A_{11}| > \sum_{j\neq 1} |A_{1j}|$$
$$|A_{22}| > \sum_{j\neq 2} |A_{2j}|$$
$$|A_{33}| > \sum_{j\neq 3} |A_{3j}|$$

The following methods to solve the system of equations by iterative methods:

(a) Gauss Jacobi Method

(a) Gauss-Seidel Method

1.7 Iterative methods of Gauss-Jacobi

Step 1: Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

with diagonally dominant.

Step 2: The above system should write into the form

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \tag{1}$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \tag{2}$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \tag{3}$$

Step 3: Start with the initial values $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ for x, y, z and get $x^{(1)}, y^{(1)}, z^{(1)}$

$\therefore$ (1),(2),(3) become

$$x^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)}\right)$$
$$y^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(0)} - a_{23}z^{(0)}\right)$$
$$z^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(0)} - a_{32}y^{(0)}\right)$$

Step 4: Using this $x^{(1)}$ for x, $y^{(1)}$ for y, $z^{(1)}$ for z in (1),(2),(3) respectively, we get

$$x^{(2)} = \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(1)} - a_{13}z^{(1)}\right)$$
$$y^{(2)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(1)} - a_{23}z^{(1)}\right)$$
$$z^{(2)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)}\right)$$

Continuing in the same procedure until the convergence is confirmed.

The general iterative formula of Gauss-Jacobi is

$$x_1^{(j+1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$
$$x_2^{(j+1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(j)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$
$$x_3^{(j+1)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(j)} + a_{32}x_2^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$
$$\vdots$$
$$x_n^{(j+1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(j)} + a_{n2}x_2^{(j)} + \cdots + a_{n,n-1}x_{n-1}^{(j)}\right)\right]$$

Note : Suppose n equations with n unknown variables $x_1, x_2, \cdots, x_n$, then

In Gauss Jacobi iteration method, for $x_n^{(j+1)}$, use $x_n^{(j)}$ values only. [Start with $x_1 = x_2 = \cdots = 0$]

In Gauss Seidel iteration method, for $x_n^{(j+1)}$, use latest values of $x_n^{(j)}$ or $x_n^{(j+1)}$. [Start with $x_2 = x_3 = \cdots = 0$]

1.7.1 Part A

1. Why use Jacobi ?

   Solution : Because you can separate the n−equations into n independent tasks; it is very well suited to computers with parallel processors.

1.7.2 Part B

Example 1.31. Solve by Gauss-Jacobi iterative method

$$\begin{aligned} x + y + 54z &= 110 \\ 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \end{aligned}$$

Solution: This can be written in diagonally dominant matrix as

$$27x + 6y - z = 85 \tag{1}$$

$$6x + 15y + 2z = 72 \tag{2}$$

$$x + y + 54z = 110 \tag{3}$$

$$(1) \Rightarrow x = \frac{1}{27}(85 - 6y + z) \tag{4}$$

$$(2) \Rightarrow y = \frac{1}{15}(72 - 6x - 2z) \tag{5}$$

$$(3) \Rightarrow z = \frac{1}{54}(110 - x - y) \tag{6}$$

Iteration 1: Put the initial values $x^{(0)} = y^{(0)} = z^{(0)} = 0$ in RHS of (4),(5),(6)

$$(4) \Rightarrow x^{(1)} = \frac{1}{27}\left[85 - 6y^{(0)} + z^{(0)}\right] = \frac{1}{27}[85 - 6(0) + 0] = 3.14815$$

$$(5) \Rightarrow y^{(1)} = \frac{1}{15}\left[72 - 6x^{(0)} - 2z^{(0)}\right] = \frac{1}{15}[72 - 6(0) - 2(0)] = 4.80000$$

$$(6) \Rightarrow z^{(1)} = \frac{1}{54}\left(110 - x^{(0)} - y^{(0)}\right] = \frac{1}{54}(110 - 0 - 0) = 2.03704$$

Iteration 2: Put the intial values $x = 3.14815, y = 4.80000, z = 2.03704$ in RHS of (4),(5),(6)

$$(4) \Rightarrow x^{(2)} = \frac{1}{27}\left[85 - 6y^{(1)} + z^{(1)}\right] = \frac{1}{27}[85 - 6(4.80000) + 2.03704] = 2.15693$$

$$(5) \Rightarrow y^{(2)} = \frac{1}{15}\left[72 - 6x^{(1)} - 2z^{(1)}\right] = \frac{1}{15}[72 - 6(3.14815) - 2(2.03704)] = 3.26914$$

$$(6) \Rightarrow z^{(2)} = \frac{1}{54}\left[110 - x^{(1)} - y^{(1)}\right] = \frac{1}{54}(110 - 3.14815 - 4.80000) = 1.88985$$

Iteration 3:

$$(4) \Rightarrow x^{(3)} = \frac{1}{27}\left[85 - 6y^{(2)} + z^{(2)}\right] = \frac{1}{27}[85 - 6(3.26914) + 1.88985] = 2.49167$$

$$(5) \Rightarrow y^{(3)} = \frac{1}{15}\left[72 - 6x^{(2)} - 2z^{(2)}\right] = \frac{1}{15}[72 - 6(2.15693) - 2(1.88985)] = 3.68525$$

$$(6) \Rightarrow z^{(3)} = \frac{1}{54}\left[110 - x^{(2)} - y^{(2)}\right] = \frac{1}{54}(110 - 2.15693 - 3.26914) = 1.93655$$

Iteration 4:

$$(4) \Rightarrow x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right] = \frac{1}{27}[85 - 6(3.68525) + 1.93655] = 2.40093$$
$$(5) \Rightarrow y^{(4)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(3)}\right] = \frac{1}{15}[72 - 6(2.49167) - 2(1.93655)] = 3.54513$$
$$(6) \Rightarrow z^{(4)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}(110 - 2.49167 - 3.68525) = 1.92265$$

Iteration 5:

$$(4) \Rightarrow x^{(5)} = \frac{1}{27}\left[85 - 6y^{(4)} + z^{(4)}\right] = \frac{1}{27}[85 - 6(3.54513) + 1.92265] = 2.43155$$
$$(5) \Rightarrow y^{(5)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(4)}\right] = \frac{1}{15}[72 - 6(2.40093) - 2(1.92265)] = 3.58328$$
$$(6) \Rightarrow z^{(5)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right] = \frac{1}{54}(110 - 2.40093 - 3.54513) = 1.92692$$

Iteration 6:

$$(4) \Rightarrow x^{(6)} = \frac{1}{27}\left[85 - 6y^{(5)} + z^{(5)}\right] = \frac{1}{27}[85 - 6(3.58328) + 1.92692] = 2.42323$$
$$(5) \Rightarrow y^{(6)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(5)}\right] = \frac{1}{15}[72 - 6(2.43155) - 2(1.92692)] = 3.57046$$
$$(6) \Rightarrow z^{(6)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right] = \frac{1}{54}(110 - 2.43155 - 3.58328) = 1.92565$$

Iteration 7:

$$(4) \Rightarrow x^{(7)} = \frac{1}{27}\left[85 - 6y^{(6)} + z^{(6)}\right] = \frac{1}{27}[85 - 6(3.57046) + 1.92565] = 2.42603$$
$$(5) \Rightarrow y^{(7)} = \frac{1}{15}\left[72 - 6x^{(6)} - 2z^{(6)}\right] = \frac{1}{15}[72 - 6(2.42323) - 2(1.92565)] = 3.57395$$
$$(6) \Rightarrow z^{(7)} = \frac{1}{54}\left[110 - x^{(6)} - y^{(6)}\right] = \frac{1}{54}(110 - 2.42323 - 3.57046) = 1.92604$$

Iteration 8:

$$(4) \Rightarrow x^{(8)} = \frac{1}{27}\left[85 - 6y^{(7)} + z^{(7)}\right] = \frac{1}{27}[85 - 6(3.57395) + 1.92604] = 2.42527$$
$$(5) \Rightarrow y^{(8)} = \frac{1}{15}\left[72 - 6x^{(7)} - 2z^{(7)}\right] = \frac{1}{15}[72 - 6(2.42603) - 2(1.92604)] = 3.57278$$
$$(6) \Rightarrow z^{(8)} = \frac{1}{54}\left[110 - x^{(7)} - y^{(7)}\right] = \frac{1}{54}(110 - 2.42603 - 3.57395) = 1.92593$$

Iteration 9:

$$(4) \Rightarrow x^{(9)} = \frac{1}{27}\left[85 - 6y^{(8)} + z^{(8)}\right] = \frac{1}{27}[85 - 6(3.57278) + 1.92593] = 2.42553$$
$$(5) \Rightarrow y^{(9)} = \frac{1}{15}\left[72 - 6x^{(8)} - 2z^{(8)}\right] = \frac{1}{15}[72 - 6(2.42527) - 2(1.92593)] = 3.57310$$
$$(6) \Rightarrow z^{(9)} = \frac{1}{54}\left[110 - x^{(8)} - y^{(8)}\right] = \frac{1}{54}(110 - 2.42527 - 3.57278) = 1.92596$$

Iteration 10:

$$(4) \Rightarrow x^{(10)} = \frac{1}{27}\left[85 - 6y^{(9)} + z^{(9)}\right] = \frac{1}{27}[85 - 6(3.57310) + 1.92596] = 2.42546$$
$$(5) \Rightarrow y^{(10)} = \frac{1}{15}\left[72 - 6x^{(9)} - 2z^{(9)}\right] = \frac{1}{15}[72 - 6(2.42553) - 2(1.92596)] = 3.57299$$
$$(6) \Rightarrow z^{(10)} = \frac{1}{54}\left[110 - x^{(9)} - y^{(9)}\right] = \frac{1}{54}(110 - 2.42553 - 3.57310) = 1.92595$$

Iteration 11:

$$(4) \Rightarrow x^{(11)} = \frac{1}{27}\left[85 - 6y^{(10)} + z^{(10)}\right] = \frac{1}{27}[85 - 6(3.57299) + 1.92595] = 2.42548$$
$$(5) \Rightarrow y^{(11)} = \frac{1}{15}\left[72 - 6x^{(10)} - 2z^{(10)}\right] = \frac{1}{15}[72 - 6(2.42546) - 2(1.92595)] = 3.57302$$
$$(6) \Rightarrow z^{(11)} = \frac{1}{54}\left[110 - x^{(10)} - y^{(10)}\right] = \frac{1}{54}(110 - 2.42546 - 3.57299) = 1.92595$$

Iteration 12:

$$(4) \Rightarrow x^{(12)} = \frac{1}{27}\left[85 - 6y^{(11)} + z^{(11)}\right] = \frac{1}{27}[85 - 6(3.57302) + 1.92595] = 2.42547$$
$$(5) \Rightarrow y^{(12)} = \frac{1}{15}\left[72 - 6x^{(11)} - 2z^{(11)}\right] = \frac{1}{15}[72 - 6(2.42548) - 2(1.92595)] = 3.57301$$
$$(6) \Rightarrow z^{(12)} = \frac{1}{54}\left[110 - x^{(11)} - y^{(11)}\right] = \frac{1}{54}(110 - 2.42548 - 3.57302) = 1.92595$$

Iteration 13:

$$(4) \Rightarrow x^{(13)} = \frac{1}{27}\left[85 - 6y^{(12)} + z^{(12)}\right] = \frac{1}{27}[85 - 6(3.57301) + 1.92595] = 2.42548$$
$$(5) \Rightarrow y^{(13)} = \frac{1}{15}\left[72 - 6x^{(12)} - 2z^{(12)}\right] = \frac{1}{15}[72 - 6(2.42547) - 2(1.92595)] = 3.57302$$
$$(6) \Rightarrow z^{(13)} = \frac{1}{54}\left[110 - x^{(12)} - y^{(12)}\right] = \frac{1}{54}(110 - 2.42547 - 3.57301) = 1.92595$$

Iteration 14:

$$(4) \Rightarrow x^{(14)} = \frac{1}{27}\left[85 - 6y^{(13)} + z^{(13)}\right] = \frac{1}{27}[85 - 6(3.57302) + 1.92595] = 2.42548$$
$$(5) \Rightarrow y^{(14)} = \frac{1}{15}\left[72 - 6x^{(13)} - 2z^{(13)}\right] = \frac{1}{15}[72 - 6(2.42548) - 2(1.92595)] = 3.57302$$
$$(6) \Rightarrow z^{(14)} = \frac{1}{54}\left[110 - x^{(13)} - y^{(13)}\right] = \frac{1}{54}(110 - 2.42548 - 3.57302) = 1.92595$$

$\therefore$ Solution is $(x, y, z) = (2.42548, 3.57302, 1.92595)$

1.8 Iterative methods of Gauss-Seidel

Step 1: Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

with diagonally dominant.

Step 2: The above system should write into the form

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \tag{1}$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \tag{2}$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \tag{3}$$

Step 3: Start with the initial values $y^{(0)} = 0, z^{(0)} = 0$ for y, z and get $x^{(1)}$ from the first equation.

$\therefore$ (1) becomes $x^{(1)} = \dfrac{1}{a_{11}}\left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)}\right)$

Step 4: Using this $x^{(1)}$ in (2), we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of $x^{(0)}$, we get

$\therefore$ (2) becomes $y^{(1)} = \dfrac{1}{a_{22}}\left(b_2 - a_{21}x^{(1)} - a_{23}z^{(0)}\right)$

Step 5: Substitute $x^{(1)}, y^{(1)}$ for x, y in the third equation.

$\therefore$ (3) becomes $z^{(1)} = \dfrac{1}{a_{33}}\left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)}\right)$

Step 6: To find the values of unknowns, use the latest available values on the right side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterate values, then the next iteration will be

$$\begin{aligned}
x^{(r+1)} &= \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(r)} - a_{13}z^{(r)}\right)\\
y^{(r+1)} &= \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(r+1)} - a_{23}z^{(r)}\right)\\
z^{(r+1)} &= \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(r+1)} - a_{32}y^{(r+1)}\right)
\end{aligned}$$

Step 7: This process of continued until the convergence is confirmed.

The general iterative formula of Gauss-Seidel is

$$\begin{aligned}
x_1^{(j+1)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
x_2^{(j+1)} &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(j+1)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
x_3^{(j+1)} &= \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(j+1)} + a_{32}x_2^{(j+1)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
&\vdots\\
x_n^{(j+1)} &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(j+1)} + a_{n2}x_2^{(j+1)} + \cdots + a_{n,n-1}x_{n-1}^{(j+1)}\right)\right]
\end{aligned}$$

Note : If either method converges, Gauss-Seidel converges faster than Jacobi.

1.8.1 Part A

1. Write the first iteration values of x, y, z when the equations $27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110$ are solved by Gauss-Seidel method.

   Solution: Since the given equations are in the diagonally dominant and find the x, y, z as follows

$$\begin{aligned}
x &= \frac{1}{27}[85 - 6y + z] = \frac{1}{27}[85 - 6(0) + (0)] = \frac{85}{27} = 3.14815 && \text{(by putting } y = z = 0)\\
y &= \frac{1}{15}[72 - 6x - 2z] = \frac{1}{15}[72 - 6(3.14815) - 2(0)] = 3.54074 && \text{(by putting } x = 3.14815, z = 0)\\
z &= \frac{1}{54}[110 - x - y] = \frac{1}{54}[110 - 3.14815 - 3.54074] = 1.91317 && \text{(by putting } x = 3.14815, y = 3.54074)
\end{aligned}$$

   $\therefore$ First iteration values of x, y, z is $(x = 3.14815, y = 3.54074, z = 1.91317)$

1.8.2 Part B

Example 1.32. Solve by Gauss-Seidel iterative method **[AM 2011]**

$$\boldsymbol{x + y + 54z = 110}$$
$$\boldsymbol{27x + 6y - z = 85}$$
$$\boldsymbol{6x + 15y + 2z = 72}$$

Solution: This can be written in diagonally dominant matrix as

$$27x + 6y - z = 85 \quad (1)$$
$$6x + 15y + 2z = 72 \quad (2)$$
$$x + y + 54z = 110 \quad (3)$$
$$(1) \Rightarrow x = \frac{1}{27}[85 - 6y + z] \quad (4)$$
$$(2) \Rightarrow y = \frac{1}{15}[72 - 6x - 2z] \quad (5)$$
$$(3) \Rightarrow z = \frac{1}{54}[110 - x - y] \quad (5)$$

Iteration 1: Put Initial values $y^{(0)} = z^{(0)} = 0$ in (4)

$$(4) \Rightarrow x^{(1)} = \frac{1}{27}\left[85 - 6y^{(0)} + z^{(0)}\right] = \frac{1}{27}[85 - 6(0) + 0] = 3.14815$$
$$(5) \Rightarrow y^{(1)} = \frac{1}{15}\left[72 - 6x^{(1)} - 2z^{(0)}\right] = \frac{1}{15}[72 - 6(3.14815) - 2(0)] = 3.54074$$
$$(6) \Rightarrow z^{(1)} = \frac{1}{54}\left[110 - x^{(1)} - y^{(1)}\right] = \frac{1}{54}[110 - 3.14815 - 3.54074] = 1.91317$$

Iteration 2: [Substitute the latest values of x, y, z in (4), (5), (6)]

$$x^{(2)} = \frac{1}{27}\left[85 - 6y^{(1)} + z^{(1)}\right] = \frac{1}{27}[85 - 6(3.54074) + 1.91317] = 2.43218$$
$$y^{(2)} = \frac{1}{15}\left[72 - 6x^{(2)} - 2z^{(1)}\right] = \frac{1}{15}[72 - 6(2.43218) - 2(1.91317)] = 3.57204$$
$$z^{(2)} = \frac{1}{54}\left[110 - x^{(2)} - y^{(2)}\right] = \frac{1}{54}[110 - 2.43218 - 3.57204] = 1.92585$$

Iteration 3:

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.57204) + 1.92585] = 2.42569$$
$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(2)}\right] = \frac{1}{15}[72 - 6(2.42569) - 2(1.92585)] = 3.57294$$
$$z^{(3)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}[110 - 2.42569 - 3.57294] = 1.92595$$

Iteration 4:

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right] = \frac{1}{27}[85 - 6(3.57294) + 1.92595] = 2.42549$$
$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(3)}\right] = \frac{1}{15}[72 - 6(2.42549) - 2(1.92595)] = 3.57301$$
$$z^{(4)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}[110 - 2.42549 - 3.57301] = 1.92595$$

Iteration 5:

$$x^{(5)} = \frac{1}{27}\left[85 - 6y^{(4)} + z^{(4)}\right] = \frac{1}{27}[85 - 6(3.57301) + 1.92595] = 2.42548$$
$$y^{(5)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(4)}\right] = \frac{1}{15}[72 - 6(2.42548) - 2(1.92595)] = 3.57301$$
$$z^{(5)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right] = \frac{1}{54}[110 - 2.42548 - 3.57301] = 1.92595$$

$\therefore$ Solution is $(x, y, z) = (2.42548, 3.57301, 1.92595)$

Note : Iterations of the above given problem are given as follows:

Iteration	x	y	z
1	3.14815	3.54074	1.91317
2	1.91317	3.57204	1.92585
3	2.42569	3.57294	1.92595
4	2.42549	3.57301	1.92595
5	2.42548	3.57301	1.92595

Example 1.33. Solve by Gauss-Seidel iterative method

$$\mathbf{8x - 3y + 2z = 20}$$
$$\mathbf{6x + 3y + 12z = 35}$$
$$\mathbf{4x + 11y - z = 33}$$

Solution: $[(x, y, z) = (3.016, 1.985, 0.911)]$

Example 1.34. Apply Gauss-Seidel method to solve the system of equations

$20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$

Solution: $[(x, y, z) = (0.999969 \approx 1, -1064 \approx -1, 102 \approx 1)]$

Example 1.35. Solve the following system by Gauss-Seidal method :

$28x + 4y - z = 32; x + 3y + 10z = 24; 2x + 17y + 4z = 35$

Solution: $[x = 0.9936, y = 1.507, z = 1.8486]$

Example 1.36. By using Gauss-Seidel method, solve the system of equations

$6x + 3y + 12z = 35; 8x - 3y + 2z = 20; 4x + 11y - z = 33.$

Solution: $[(x, y, z) = (3.016, 1.985, 0.911)]$

Example 1.37. By using Gauss-Seidel iteration method, solve the following system of equations upto four decimals.

$10x - 2y - z - w = 3; -2x + 10y - z - w = 15; -x - y + 10z - 2w = 27; -x - y - 2z + 10w = -9.$

Solution: $[x = 3.017, y = 1.986, z = 0.91]$

Example 1.38. Using Gauss Seidel iteration method solve the following system start with $x = 1, y = -2, z = 3$: $x + 3y + 52z = 173.61, x - 27y + 2z = 71.31, 41x - 2y + 3z = 65.46.$

Solution: Given

$$x + 3y + 52z = 173.61$$
$$x - 27y + 2z = 71.31$$
$$41x - 2y + 3z = 65.46$$

Since the coefficient matrix A of the given system of equations is not diagonally dominant, we rewrite the equations

$$41x - 2y + 3z = 65.46$$
$$x - 27y + 2z = 71.31$$
$$x + 3y + 52z = 173.61$$

From the above system of equations

$$x = \frac{1}{41}[65.46 + 2y - 3z]$$
$$y = -\frac{1}{27}[71.31 - x - 2z]$$
$$z = \frac{1}{52}[173.61 - x - 3y]$$

Let the initial values be $x = 1, y = -2, z = 3$

Iteration 1: $x^{(1)} = \frac{1}{41}[65.46 + 2(-2) - 3(3)] = 1.2795$; $y^{(1)} = -\frac{1}{27}[71.31 - 1.2795 - 2(3)] = -2.3715$; $z^{(1)} = \frac{1}{52}[173.61 - 1.2795 - 3(-2.3715)] = 3.4509$	**Iteration 2:** $x^{(2)} = \frac{1}{41}[65.46 + 2(-2.3715) - 3(3.4509)] = 1.2284$; $y^{(2)} = -\frac{1}{27}[71.31 - 1.2284 - 2(3.4509)] = -2.34$; $z^{(2)} = \frac{1}{52}[173.61 - 1.2284 - 3(-2.34)] = 3.4477$
Iteration 3: $x^{(3)} = \frac{1}{41}[65.46 + 2(-2.34) - 3(3.4477)] = 1.2302$; $y^{(3)} = -\frac{1}{27}[71.31 - 1.2302 - 2(3.4477)] = -2.3402$; $z^{(3)} = \frac{1}{52}[173.61 - 1.2302 - 3(-2.3402)] = 3.45$	**Iteration 4:** $x^{(4)} = \frac{1}{41}[65.46 + 2(-2.3402) - 3(3.45)] = 1.23$; $y^{(4)} = -\frac{1}{27}[71.31 - 1.23 - 2(3.45)] = -2.34$; $z^{(4)} = \frac{1}{52}[173.61 - 1.23 - 3(-2.34)] = 3.45$

From third and fourth iterations

$$x = 1.23, y = -2.34, z = 3.45$$

1.8.3 Anna University Questions

1. Use Gauss-Seidel iterative method to obtain the solution of the equations $9x-y+2z = 9$, $x+10y-2z = 15$, $2x - 2y - 13z = -17$. (MJ10)
   Solution : $[(x, y, z)^{(5)} = (x, y, z)^{(6)} = (0.917, 1.647, 1.195)]$

2. Apply Gauss-Seidel method to solve the system of equations $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$. (ND10)
   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -1, 1)]$

3. Solve by Gauss-Seidel iterative method $x + y + 54z = 110$, $27x + 6y - z = 85$, $6x + 15y + 2z = 72$.(AM11)
   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (2.426, 3.573, 1.926)]$

4. Solve the following system by Gauss-Seidal method :
   $28x + 4y - z = 32; x + 3y + 10z = 24; 2x + 17y + 4z = 35$ (ND11)
   Solution : $[(x, y, z)^{(5)} = (x, y, z)^{(6)} = (0.9936, 1.507, 1.8486)]$

5. Apply Gauss-Seidel method to solve the system of equations
$20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$ (MJ12,ND10)
Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -1, 1)]$

6. Solve the following system of equations using Gauss-Seidel method: $10x+2y+z = 9$, $x+10y-z = -22$, $-2x + 3y + 10z = 22$ (ND12)
Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -2, 3)]$

7. Solve by Gauss-Seidel method, the equations $27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110$. (AM13)
Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (2.426, 3.573, 1.926)]$

8. Solve the equations by Gauss-Seidel method of iteration. $10x + 2y + z = 9, x + 10y - z = -22, -2x + 3y + 10z = 22$. (ND13)
Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -2, 3)]$

9. Using Gauss-Seidel method, solve the following system of linear equation s $4x+2y+z = 14; x+5y-z = 10; x + y + 8z = 20$. (AM14)
Solution : $[(x, y, z) = (2, 2, 2)]$

10. Apply Gauss-Seidal method to solve the system of equations $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$. (ND14)
Solution : $[(x, y, z) = (1, -1, 1)]$

1.9 Matrix Inversion by Gauss Jordan method

Given Matrix $A_{n\times n}$,
Form

$$[A_{n\times n} \mid I_{n\times n}] \quad \underset{\text{Operations}}{\overset{\text{Row}}{\sim}} \quad \left[I_{n\times n} \mid A^{-1}_{n\times n}\right]$$

1.9.1 Part A

1. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ by Gauss-Jordan method. (ND14)

Solution : Consider $[A|I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] R_2 \to R_2 - 2R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right] R_1 \to R_1 - 3R_2$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

1.9.2 Part B

Example 1.39. Using Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{0} & \mathbf{-6} & \mathbf{-7} \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

Consider $[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3-3(1) & 4-3(0) & 5-3(-1) & 0-3(1) & 1-3(0) & 0-3(0) \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 8 & -3 & 1 & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] R_2 \to \frac{R_2}{4}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3/4 & 1/4 & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] R_3 \to R_3 + 6(R_2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3/4 & 1/4 & 0 \\ 0 & -6+6(1) & -7+6(2) & 0+6(-3/4) & 0+6(1/4) & 1+6(0) \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3/4 & 1/4 & 0 \\ 0 & 0 & 5 & -9/2 & 3/2 & 1 \end{array}\right] R_3 \to \frac{R_3}{5}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3/4 & 1/4 & 0 \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2(R_3) \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1+0 & 0+0 & -1+1 & 1+(-9/10) & 0+(3/10) & 0+(1/5) \\ 0-2(0) & 1-2(0) & 2-2(1) & (-3/4)-2(-9/10) & (1/4)-2(3/10) & 0-2(1/5) \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/10 & 3/10 & 1/5 \\ 0 & 1 & 0 & 21/20 & -7/20 & -2/5 \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{array}\right]$$

$$= \left[I|A^{-1}\right]$$

$$\therefore A^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 3 & 2 \\ 21/2 & -7/2 & -4 \\ -9 & 3 & 2 \end{bmatrix}$$

Example 1.40. Find the inverse of a matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by Gauss-Jordan method.

Solution: $\left[A^{-1} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}\right]$

Example 1.41. Using Gauss-Jordan method, find the inverse of the matrix $\begin{bmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & -8 \end{bmatrix}$

Solution: $\left[A^{-1} = \frac{1}{56}\begin{bmatrix} 12 & 4 & 6 \\ 1 & 5 & -3 \\ 5 & -3 & -1 \end{bmatrix}\right]$

Example 1.42. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ by using Gauss-Jordan method.

Solution: $\left[A^{-1} = \begin{bmatrix} -1/5 & 1/3 & -1/15 \\ 4/5 & -1/3 & 4/15 \\ 2/5 & -1/3 & 7/15 \end{bmatrix}\right]$

Example 1.43. Find the inverse of the matrix by Gauss-Jordan method: $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$.

Solution: $\left[A^{-1} = \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -2 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix}\right]$

1.9.3 Anna University Questions

1. Find the inverse of the matrix by Gauss - Jordan method: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$. (AM10)

   Solution: $\left\{ A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} \right\}$

2. Find the inverse of $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ by using Gauss-Jordan method. (ND10)

   Solution: $\left\{ A^{-1} = \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -3 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix} \right\}$

3. Using Gauss Jordan method, find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{pmatrix}$. (MJ12)

   Solution: $\left\{ A^{-1} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix} \right\}$

4. Using Gauss-Jordan method, find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$. (ND12)

   Solution: $\left\{ A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \right\}$

5. Find, by Gauss-Jordan method, the inverse of the matrix $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$. (MJ13)

   Solution: $\left\{ A^{-1} = \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -3 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix} \right\}$

6. Find the inverse of the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ using Gauss-JOrdan method. (ND13)

   Solution: $\left\{ A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \right\}$

7. Using Gauss-Jordan method, find the inverse of $\begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & 8 \end{pmatrix}$. (AM14)

Solution: $\left\{ A^{-1} = \begin{bmatrix} 0 & 1/5 & 3/20 \\ 1/8 & 1/40 & -3/40 \\ 1/8 & -3/40 & -1/40 \end{bmatrix} \right\}$

1.10 Eigen values of a matrix by Power method

Suppose a given square matrix is A.

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial eigen vector of the matrix A.

$$AX_1 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \begin{pmatrix} \text{numerically largest value} \\ \text{of matrix } A \text{ say} \\ \text{value 1} \end{pmatrix} \begin{bmatrix} 1 \\ \text{value 2/value 1} \\ \text{value 3/value 1} \end{bmatrix} = \lambda_1 [X_2]$$

$$AX_2 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \begin{pmatrix} \text{numerically largest value} \\ \text{of matrix } A \text{ say} \\ \text{value 2} \end{pmatrix} \begin{bmatrix} \text{value 1/value 2} \\ 1 \\ \text{value 3/value 2} \end{bmatrix} = \lambda_2 [X_3]$$

$$\vdots$$

$$\left. \begin{aligned} AX_n &= \lambda_n [X_{n+1}] \\ AX_{n+1} &= \lambda_{n+1} [X_{n+2}] \end{aligned} \right\} \begin{array}{c} \text{If } \lambda_n = \lambda_{n+1} \text{ and } X_{n+1} = X_{n+2} \text{ are same} \\ \text{upto required decimals,then stop iteration.} \end{array}$$

$$\therefore \text{ Dominant eigen value } = \lambda_n \,(or)\, \lambda_{n+1} \text{and the corresponding}$$
$$\text{eigen vector } = X_{n+1} \,(or)\, X_{n+2}$$

To find the smallest eigen value of A

Form $B = A - \lambda I$ & $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary eigen vector of the matrix B.

$$BY_1 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \begin{pmatrix} \text{numerically largest value} \\ \text{of matrix } B \text{ say} \\ \text{value 1} \end{pmatrix} \begin{bmatrix} 1 \\ \text{value 2/value 1} \\ \text{value 3/value 1} \end{bmatrix} = \lambda_1 [Y_2]$$

$$BY_2 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \begin{pmatrix} \text{numerically largest value} \\ \text{of matrix } B \text{ say} \\ \text{value 2} \end{pmatrix} \begin{bmatrix} \text{value 1/value 2} \\ 1 \\ \text{value 3/value 2} \end{bmatrix} = \lambda_2 [Y_3]$$

$$\vdots$$

$$\left. \begin{aligned} BY_n &= \lambda_n [Y_{n+1}] \\ BY_{n+1} &= \lambda_{n+1} [Y_{n+2}] \end{aligned} \right\} \begin{array}{c} \text{If } \lambda_n = \lambda_{n+1} \text{ and } Y_{n+1} = Y_{n+2} \text{ are same} \\ \text{upto required decimals,then stop iteration.} \end{array}$$

$\therefore$ The smallest eigen value of $A = \begin{cases} \text{Dominant eigen value of } B \\ + \\ \text{Dominant eigen value of } A \end{cases}$

To find third eigen value of A:

$\lambda_1 + \lambda_2 + \lambda_3 =$ Sum of the main diagonal elements.

1.10.1 Part A

1. What type of eigen value can by obtained by using power method.

   Solution: We can obtain dominant eigen value of the given matrix.

2. Write down all possible initial vectors of a matrix 2×2 order.

   Solution: The initial vectors are $(0, 1)^T, (1, 0)^T, (1, 1)^T$.

3. Find the dominant eigen value of $A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ by power method upto two decimals and choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the initial eigen vector.

   Solution: Let $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$AX_1 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = 5X_2$$

$$AX_2 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 4.8 \\ 3.4 \end{bmatrix} = 4.8\begin{bmatrix} 1 \\ 0.71 \end{bmatrix} = 4.8X_3$$

$$AX_3 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.71 \end{bmatrix} = \begin{bmatrix} 4.71 \\ 3.13 \end{bmatrix} = 4.71\begin{bmatrix} 1 \\ 0.67 \end{bmatrix} = 4.71X_4$$

$$AX_4 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 4.67 \\ 3.01 \end{bmatrix} = 4.67\begin{bmatrix} 1 \\ 0.65 \end{bmatrix} = 4.67X_5$$

$$AX_5 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.65 \end{bmatrix} = \begin{bmatrix} 4.65 \\ 2.95 \end{bmatrix} = 4.65\begin{bmatrix} 1 \\ 0.63 \end{bmatrix} = 4.65X_6$$

$$AX_6 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.63 \end{bmatrix} = \begin{bmatrix} 4.63 \\ 2.89 \end{bmatrix} = 4.63\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = 4.63X_7$$

$$AX_8 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \begin{bmatrix} 4.62 \\ 2.86 \end{bmatrix} = 4.62\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = 4.62X_8$$

$\therefore$ The eigen value $= \lambda = 4.62$ and the corresponding eigen vector $= X = \begin{bmatrix} 1 \\ 0.62 \end{bmatrix}$.

1.10.2 Part B

Example 1.44. Find the dominant eigen value and eigen vector of the matrix $\begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}$ by power method.

Solution: Let $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be an arbitrary initial eigen vector.

$$AX_1 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (-4\times 1) & + & (-5\times 0) \\ (1\times 1) & + & (2\times 0) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -4\begin{bmatrix} 1 \\ -0.25 \end{bmatrix} = -4X_2, \text{ say}$$

$$AX_2 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.25 \end{bmatrix} = \begin{bmatrix} (-4\times 1) & + & (-5\times(-0.25)) \\ (1\times 1) & + & (2\times(-0.25)) \end{bmatrix} = \begin{bmatrix} -2.75 \\ 0.5 \end{bmatrix} = -2.75\begin{bmatrix} 1 \\ -0.18182 \end{bmatrix} = -2.75X_3$$

$$AX_3 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.18182 \end{bmatrix} = \begin{bmatrix} -3.0909 \\ 0.63636 \end{bmatrix} = -3.0909\begin{bmatrix} 1 \\ -0.20588 \end{bmatrix} = -3.0909X_4$$

$$AX_4 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.20588 \end{bmatrix} = \begin{bmatrix} -2.9706 \\ 0.58824 \end{bmatrix} = -2.9706\begin{bmatrix} 1 \\ -0.06666 \end{bmatrix} = -2.9706X_5$$

$$AX_5 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.06666 \end{bmatrix} = \begin{bmatrix} -3.6667 \\ 0.86668 \end{bmatrix} = -3.6667\begin{bmatrix} 1 \\ -0.23637 \end{bmatrix} = -3.6667X_6$$

$$AX_6 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.23637 \end{bmatrix} = \begin{bmatrix} -2.81815 \\ 0.52726 \end{bmatrix} = -2.81815\begin{bmatrix} 1 \\ -0.18709 \end{bmatrix} = -2.81815X_7$$

$$AX_7 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.18709 \end{bmatrix} = \begin{bmatrix} -3.06455 \\ 0.62582 \end{bmatrix} = -3.06455\begin{bmatrix} 1 \\ -0.19579 \end{bmatrix} = -3.06455X_8$$

$$AX_8 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.19579 \end{bmatrix} = \begin{bmatrix} -3.02105 \\ 0.60842 \end{bmatrix} = -3.022105\begin{bmatrix} 1 \\ -0.20133 \end{bmatrix} = -3.022105X_9$$

Dominant Eigen value $= \lambda = -3$ and the corresponding

$$\text{Eigen vector } = X = \begin{bmatrix} 1 \\ -0.19529 \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.2 \end{bmatrix}$$

Example 1.45. Find by power method, the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-1} \\ \mathbf{3} & \mathbf{2} & \mathbf{4} \\ \mathbf{-1} & \mathbf{4} & \mathbf{10} \end{bmatrix}$**.**

Solution: Dom. Eigen value $= \lambda = 11.72$ and Eigen vector $= X = \begin{bmatrix} 0.02404 \\ 0.42582 \\ 1 \end{bmatrix}$

Example 1.46. Find all the eigen values of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$ **by power method.**

Solution: Dominant Eigen value of $A = \lambda_1 = 4,$ Dominant Eigen value of $B = -5$

Smallest Eigen value of $A = \lambda_2 =$ Dom. Eig. val. of $A +$ Dom. Eig. val. of $B = 4 + (-5) = -1$

Other Eigen value of $A = \lambda_3 =$ Sum of the main diagonal elements of $A - (\lambda_1 + \lambda_2) = 6 - 3 = 3$

The required eigen values of the given matrix are $4, -1, 3$

1.10.3 Anna University Questions

1. Find the dominant eigen value and the corresponding eigen vector of the matrix $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. (AM10)

   Solution: $\left\{ \text{In } 8^{th} \text{ iteration, Dominant E. value} = 4, \text{corresponding E. vector } = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} \right\}$

2. Find, by power method, the largest eigenvalue and the corresponding eigenvector of a matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ with initial vector $(1 \quad 1 \quad 1)^T$. (ND10)

   Solution: $\left\{ \text{In } 8^{th} \text{ iteration, Dom. E. value} = 11.663, \text{corres. E. vector } = (0.025 \quad 0.422 \quad 1)^T \right\}$

3. Using Jacobi's method, find all the eigenvalues and eigenvectors for the given matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$. (AM11)

   Solution: $\left\{ \text{E. values} = \{-2, 4, 6\}, \text{ E. vectors } = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$

4. Find the largest eigenvalue of $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ by using Power method. (ND11)

   Solution: $\left\{ \text{Eigen values} = \{-1, \mathbf{4}, 3\}, \text{ Eigen vectors } = \begin{bmatrix} -0.9487 \\ 0.3162 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{-0.8944} \\ \mathbf{-0.4472} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} -0.2357 \\ -0.2357 \\ 0.9428 \end{bmatrix} \right\}$

5. Determine the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

   (MJ12)

   Solution: $\left\{ \text{Eigen value} = \{0.5858, 2, \mathbf{3.4142}\}, \text{ Eigen vectors } = \begin{bmatrix} 0.5 \\ \frac{1}{\sqrt{2}} \\ 0.5 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \mathbf{-0.5} \\ \mathbf{\frac{1}{\sqrt{2}}} \\ \mathbf{-0.5} \end{bmatrix} \right\}$

6. Find all the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$ using Jacobi method. (ND12)

Solution: $\left\{\text{Eigen values} = \{-1, 1, 5\},\ \text{Eigen vectors} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0.5 \\ -\frac{1}{\sqrt{2}} \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ \frac{1}{\sqrt{2}} \\ 0.5 \end{bmatrix}\right\}$

7. Using Jacobi method find the all eigen values and their corresponding eigen vectors of the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. (MJ13)

Solution: $\left\{\text{Eigen values} = \{-1, 5\},\ \text{Eigen vectors} = \left(-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^T, \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^T\right\}$

8. Determine the largest eigenvalue and the corresponding eigenvector of a matrix $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ with $(1 \quad 1 \quad 1)^T$ as initial vector by power method. (ND13)

Solution: $\left\{\text{Dominant E. value} = 11.662, \text{corresponding E. vector} = (0.0229 \quad 0.3885 \quad 0.9212)^T\right\}$

9. Find the numerically largest eigen value of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ and its corresponding eigen vector by power method, taking the initial eigen vector as $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ (upto three decimal places). (AM14)

Solution: $\left\{\text{Dom. E. value} = 25.1822, \text{corres. E. vector} = (-0.9967 \quad -0.0449 \quad -0.0683)^T\right\}$

10. Find the numerically largest eigenvalue if $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and the corresponding eigen vector. (ND14)

Solution: $\left\{\text{Dom. E. value} = 25.1822, \text{corres. E. vector} = (-0.9967 \quad -0.0449 \quad -0.0683)^T\right\}$

11. Obtain by power method the numerically largest eigen value of the matrix $\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -21 \end{bmatrix}$.

Solution: $\left\{\text{E. values} = -22.2315, 20, 8.2315, \text{E. vectors} = \begin{bmatrix} 0.1000 \\ 0.2001 \\ 0.9747 \end{bmatrix}, \begin{bmatrix} 0.7068 \\ -0.5823 \\ -0.4016 \end{bmatrix}, \begin{bmatrix} -0.4399 \\ -0.8797 \\ 0.1806 \end{bmatrix}\right\}$

12. Solve by power method, to find the dominant Eigen value for the following matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: $\left\{\text{Eigen values} = \{-2, 3, 6\},\ \text{Eigen vectors} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}, \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix}\right\}$

[Assignment Problems, P.T.O.]

1.10.4 Assignment Problems

1. Solve $e^x - 3x = 0$ by the method of fixed point iteration. (AM12)

2. Find a positive root of the equation $cosx - 3x + 1 = 0$ by using iteration method. (AM13)

3. Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton - Raphson method.(MJ2010)

4. Using Newton's method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.(ND2013)

5. Solve the given system of equations by Gauss elimination method: $-x_1 + x_2 + 10x_3 = 35.61, 10x_1 + x_2 - x_3 = 11.19, x_1 + 10x_2 + x_3 = 20.08$.

6. Solve the system of equations by Gauss-Jordan method: $5x_1 - x_2 = 9; -x_1 + 5x_2 - x_3 = 4; -x_2 + 5x_3 = -6$ (AM14)

7. Using Gauss-Jordan method to solve $2x - y + 3z = 8; -x + 2y + z = 4, 3x + y - 4z = 0$. (ND14)

8. Using Gauss-Seidel method, solve the following system of linear equation s $4x+2y+z = 14; x+5y-z = 10; x + y + 8z = 20$. (AM14)

9. Apply Gauss-Seidal method to solve the system of equations $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$. (ND14)

10. Find the inverse of the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ using Gauss-Jordan method. (ND13)

11. Using Gauss-Jordan method, find the inverse of $\begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & 8 \end{pmatrix}$. (AM14)

12. Find the largest eigenvalue of $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ by using Power method. (ND11)

13. Find the numerically largest eigen value of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ and its corresponding eigen vector by power method, taking the initial eigen vector as $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ (upto three decimal places). (AM14)

2 Interpolation and Approximation

Interpolation with unequal intervals - Lagrange's interpolation - Newton's divided difference interpolation - Cubic Splines - Interpolation with equal intervals - Newton's forward and backward difference formulae.

2.1 Introduction

The process of finding the value of a function inside the given range of discrete points are called interpolation. We have

1. Interpolation with unequal intervals
2. Interpolation with equal intervals

Methods of Equal or Unequal intervals	**Methods of only Equal intervals**
Lagrange's interpolation (Lagrange & Inverse Lagrange)	Newton's forward difference method
Newton's divided difference interpolation	Newton's backward difference method
Cubic Splines	

2.2 Lagrange's interpolation

Lagrangian Polynomials(Equal and unequal intervals):

Let $y = f(x)$ be a function which takes the values $y = y_0, y_1, \cdots, y_n$ corresponding to $x = x_0, x_1, \cdots, x_n$.

Lagrange's interpolation formula(x given, finding y in terms of x)

$$\begin{aligned} y = y(x) = f(x) = & \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)} y_0 \\ & + \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)} y_1 + \dots \\ & + \frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_2 - x_0)(x_2 - x_1)\dots(x - x_{n-1})} y_n \end{aligned}$$

Inverse Lagrange's interpolation formula(y given, finding x in terms of y)

$$\begin{aligned} x = x(y) = f(y) = & \frac{(y - y_1)(y - y_2)\dots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\dots(y_0 - y_n)} x_0 \\ & + \frac{(y - y_0)(y - y_2)\dots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\dots(y_1 - y_n)} x_1 + \dots \\ & + \frac{(y - y_0)(y - y_1)\dots(y - y_{n-1})}{(y_2 - y_0)(y_2 - y_1)\dots(y - y_{n-1})} x_n \end{aligned}$$

Note: Lagrange's interpolation formula can be used for equal and unequal intervals.

2.2.1 Part A

1. State Lagrange's interpolation formula for unequal intervals. (ND11)

   Solution: Lagrange's interpolation formula for unequal intervals is

$$\begin{aligned} y = y(x) = f(x) = & \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)} y_0 \\ & + \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)} y_1 + \dots \\ & + \frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_2 - x_0)(x_2 - x_1)\dots(x - x_{n-1})} y_n \end{aligned}$$

2. Using Lagrange's formula, find the polynomial to the given data. (MJ13)

$X:$	0	1	3
$Y:$	5	6	50

   Solution: Lagrange's formula to find 'y' for three sets of given values $(x_0 = 0, y_0 = 5), (x_1 = 1, y_1 = 6)$ and $(x_2 = 3, y_2 = 50)$

$$\begin{aligned} y = f(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \\ &= \frac{(x-1)(x-3)}{(0-1)(0-3)}(5) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(6) + \frac{(x-0)(x-1)}{(3-0)(3-1)}(50) \\ &= \frac{5}{3}(x-1)(x-3) - 3x(x-3) + \frac{25}{3}x(x-1) \\ &= \frac{5}{3}(x-1)[x-3+5x] - 3\left(x^2 - 3x\right) \\ &= \frac{5}{3}(x-1)[6x-3] - 3\left(x^2 - 3x\right) \\ &= 5(x-1)[2x-1] - \left(3x^2 - 9x\right) \\ &= 5\left[2x^2 - 3x + 1\right] - 3x^2 + 9x \\ &= 7x^2 - 6x + 5 \end{aligned}$$

3. Find the second degree polynomial through the points $(0, 2)$, $(2, 1)$,$(1, 0)$ using Lagrange's formula. (ND14)

4. What is the assumptions we make when Lagrange's formula is used?

   Solution: Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not whether the difference of y become smaller or not.

5. What is the disadvantage in practice in applying Lagrange's interpolation formula?

   Solution: Though Lagrange's formula is simple and easy to remember, its application is not speedy. It requires close attention to sign and there is always a chance of committing some error due to a number of positive and negative signs in the numerator and the denominator.

6. What is 'inverse interpolation'?

   Solution: Suppose we are given a table of values of x and y. Direct interpolation is the process if finding the values of y corresponding to a value of x, not present in the table. Inverse interpolation is the process of finding the values of x corresponding to a value of y, not present in the table.

7. Construct a linear interpolating polynomial given the points (x_0, y_0) and (x_1, y_1).

Solution: $y = y(x) = f(x) = \dfrac{(x - x_1)}{(x_0 - x_1)}y_0 + \dfrac{(x - x_0)}{(x_1 - x_0)}y_1$

8. What is the Lagrange's formula to find 'y' if three sets of values $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) are given.

Solution: $y = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \dfrac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$

9. Find the second degree polynomial fitting the following data:

x	1	2	4
y	4	5	13

Solution: Here $x_0 = 1, x_1 = 2, x_2 = 4$

$y_0 = 4, y_2 = 5, y_2 = 13$

By Lagrange's formula for three points is

$$
\begin{aligned}
y &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \\
y &= \frac{(x^2 - 6x + 8)}{(-1)(-3)}(4) + \frac{(x^2 - 5x + 4)}{(1)(-2)}(5) + \frac{(x^2 - 3x + 2)}{(3)(2)}(13) \\
&= \frac{(x^2 - 6x + 8)}{3}(4) + \frac{(x^2 - 5x + 4)}{-2}(5) + \frac{(x^2 - 3x + 2)}{6}(13) \\
&= \frac{1}{6}\left[8x^2 - 48x + 64 - 15x^2 + 75x - 60 + 13x^2 - 39x + 26\right] \\
&= \frac{1}{6}\left[6x^2 - 12x + 30\right] \\
y = f(x) &= x^2 - 2x + 5
\end{aligned}
$$

2.2.2 Part B

Example 2.1. Using Lagrange interpolation formula, find $f(4)$ given that $f(0) = 2, f(1) = 3, f(2) = 12, f(15) = 3587$.

Solution: Given

x	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 15$
$y = f(x)$	$y_0 = 2$	$y_1 = 3$	$y_2 = 12$	$y_3 = 3587$

Lagrange interpolation formula is

$$
\begin{aligned}
y &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 \\
&\quad + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3 \\
f(4) &= \frac{(4 - 1)(4 - 2)(4 - 15)}{(0 - 1)(0 - 2)(0 - 15)}(2) + \frac{(4 - 0)(4 - 2)(4 - 15)}{(1 - 0)(1 - 2)(1 - 15)}(3) \\
&\quad + \frac{(4 - 0)(4 - 1)(4 - 15)}{(2 - 0)(2 - 1)(2 - 15)}(12) + \frac{(4 - 0)(4 - 1)(4 - 2)}{(15 - 0)(15 - 1)(15 - 2)}(3587) \\
&= \frac{(3)(2)(-11)}{(-1)(-2)(-15)}(2) + \frac{(4)(2)(-11)}{(1)(-1)(-14)}(3) \\
&\quad + \frac{(4)(3)(-11)}{(2)(1)(-13)}(12) + \frac{(4)(3)(2)}{(15)(14)(13)}(3587) \\
&= 77.99 = 78
\end{aligned}
$$

Example 2.2. Find polynomial $f(x)$ by using Lagrange formula from the given data and find $f(8)$.

x	3	7	9	10
$f(x)$	168	120	72	63

Solution: Lagrange polynomial $f(x) = x^3 - 21x^2 + 119x - 27$.

[$y_{(x=8)}$ or $y(x = 8)$ or $f(x = 8) = 93$]

Example 2.3. Use Lagrange's formula to fit a polynomial to the data

x	−1	0	2	3
$f(x)$	−8	3	1	12

and hence find $y(1)$.

Solution: Lagrange polynomial $f(x) = 2x^3 - 6x^2 + 3x + 3$.

[$y_{(x=1)} = y(x = 1) = f(x = 1) = 2.$]

Example 2.4. Using Lagrange's formula, prove that

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5}).$$

Solution: From the equation, the values of x are

$x_0 = -5$	$x_1 = -3$	$x_2 = 3$	$x_3 = 5$
$y_0 = y_{-5}$	$y_1 = y_{-3}$	$y_2 = y_3$	$y_3 = y_5$

The x values are not equally space, so use Lagrange's formula to find $y = f(x)$.

Lagrange's formula for a set of 4 pair of values is

$$\begin{aligned} y = y_x = f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \\ &= \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)}y_{-5} + \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)}y_{-3} \\ &+ \frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)}y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)}y_5 \end{aligned}$$

Put $x = 1$, we get

$$\begin{aligned} y_1 &= \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)}y_{-5} + \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)}y_{-3} \\ &+ \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)}y_3 + \frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)}y_5 \\ &= -0.2y_{-5} + 0.5y_{-3} + y_3 - 0.3y_5 \\ &= -0.2y_{-5} + 0.2y_{-3} + 0.3y_{-3} + y_3 - 0.3y_5 \\ y_1 &= y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5}) \end{aligned}$$

Example 2.5. Find the age corresponding to the annuity value 13.6 from the given table

Age (x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

Solution: $[x_{13.6} \text{ or } (y = 13.6) = 43]$

Example 2.6. Find x for which $y = 7$, given

x	**1**	**3**	**4**
y	**4**	**12**	**19**

Solution: $[x(y = 7) = 1.8565]$

2.2.3 Anna University Questions

1. Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12$, $f(1) = 0$, $f(3) = 6$ and $f(4) = 12$. Hence find $f(2)$. (AM10)

   Solution: $\left[f(x) = x^3 - 7x^2 + 18x - 12 \Rightarrow f(2) = 4\right]$

2. Using Lagrange's interpolation formula to fit a polynomial to the given data $f(-1) = -8$, $f(0) = 3$, $f(2) = 1$ and $f(3) = 12$. Hence find the valur of $f(1)$. (ND10)

   Solution: $\left[f(x) = 2x^3 - 6x^2 + 3x + 3 \Rightarrow f(1) = 2\right]$

3. Find the expression of $f(x)$ using Lagrange's formula for the followin data. (AM11)

$x:$	0	1	4	5
$f(x):$	4	3	24	39

   Solution: $\left[f(x) = 2x^2 - 3x + 4\right]$

4. Find the value of x when $y = 20$ using Lagrange's formula from the following table. (AM11)

$x:$	1	2	3	4
$y = f(x):$	1	8	27	64

   Solution: $[f(y = 20) = x(y = 20) = 2.84675]$

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data: (MJ12)

Year:	1997	1999	2001	2002
Profit in Lakhs Rs. :	43	65	159	248

   Solution: $[f(x = 2000) = 100]$

6. Use Lagrange's method to find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, and $\log_{10} 661 = 2.8202$. (ND12)

   Solution: $[f(656) = 2.8168]$

7. Apply Lagrange's formula, to find $y(27)$ to the data given below. (MJ13)

$x:$	14	17	31	35
$y:$	68.8	64	44	39.1

   Solution: $[f(27) = 49.3]$

8. Use Lagrange's formula to find the value of y at $x = 6$ from the following data: (ND13)

x :	3	7	9	10
y :	168	120	72	63

Solution: $[f(6) = 147]$

9. Using Lagrange's interpolation formula, find $y(2)$ from the following data: $y(0) = 0; y(1) = 1; y(3) = 81; y(4) = 256; y(5) = 625$ (AM14)

Solution: $[f(2) = \qquad]$

10. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for the following values of x and y: (ND14)

x	0	1	2	5
y	2	3	12	147

Solution: $[f(x) = \qquad , f(2) = \qquad]$

11. Find the Lagrange's polynomial of degree 3 to fit the data : $y(0) = -12$, $y(1) = 0$, $y(3) = 6$ and $y(4) = 12$. Hence find $y(2)$. $\left[\text{Ans: } f(x) = x^3 - 7x^2 + 18x - 12; y(2) = 4\right]$

12. Find the missing term in the following table using Lagrange's interpolation.

x	0	1	2	3	4
y	1	3	9	–	81

[Ans : 31]

13. Find the value of x corresponding to $y = 100$ from the table.

x	3	5	7	9	11
y	6	24	58	108	174

[Ans : 8.656]

2.3 Newton's divided difference interpolation

First divided difference for arguments x_0, x_1 :

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1) = \underset{x_1}{\Delta} f(x_0) = [x_0, x_1](or)[x_1, x_0] = \underset{x_0}{\Delta} f(x_1)$$

First divided difference for arguments x_1, x_2 :

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = [x_1, x_2](or)[x_2, x_1] = \underset{x_1}{\Delta} f(x_2)$$

Second divided difference for arguments x_1, x_2, x_3 :

$$\frac{\underset{x_3}{\Delta} f(x_2) - \underset{x_2}{\Delta} f(x_1)}{x_3 - x_1} = f(x_1, x_2, x_3) = \underset{x_3 x_2}{\Delta^2} f(x_1) = [x_1, x_2, x_3]$$

Third divided difference for arguments x_0, x_1, x_2, x_3 :

$$\frac{\underset{x_3 x_2}{\Delta^2} f(x_1) - \underset{x_2 x_1}{\Delta^2} f(x_0)}{x_3 - x_0} = f(x_0, x_1, x_2, x_3) = \underset{x_3 x_2 x_1}{\Delta^3} f(x_0) = [x_0, x_1, x_2, x_3]$$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

2.3.1 Properties of divided differences

1. The divided difference are symmetric functions of their arguments.

   For example, (1) $\underset{y}{\Delta} f(x) = \underset{x}{\Delta} f(y)$

   (2) $\underset{yz}{\Delta^2} f(x) = \underset{xz}{\Delta^2} f(y) = \underset{xy}{\Delta^2} f(z)$

2. The n^{th} divided differences of a polynomial of degree n are constants.

3. The divided difference operator(Δ) is a linear operator.

   $\Delta[f(x) + g(x)] = \Delta[f(x)] + \Delta[g(x)]$ and

   $\Delta[cf(x)] = c\Delta[f(x)]$, where c is constant

Note : For n set of (x, y) values, we get upto $(n-1)^{\text{th}}$ divided difference values.

2.3.2 Part A

1. Form the divided difference table for the data (0,1) , (1,4) , (3,40) and (4,85). (AM10)

   Solution: Newton's divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		$\frac{4-1}{1-0} = 3$		
1	4		$\frac{18-3}{3-0} = 5$	
		$\frac{40-4}{3-1} = 18$		$\frac{6.75-5}{4-0} = 0.44$
3	40		$\frac{45-18}{4-0} = 6.75$	
		$\frac{85-40}{4-3} = 45$		
4	85			

2. Find the first and second divided differences with arguments a, b, c of the function $f(x) = \frac{1}{x}$.(ND10)

   Solution : If $f(x) = \frac{1}{x} \Rightarrow f(a) = \frac{1}{a}$

$$f(a,b) = \underset{b}{\Delta} \frac{1}{a} = \frac{f(b) - f(a)}{b-a} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$$f(a,b,c) = \underset{bc}{\Delta^2} \frac{1}{a} = \frac{f(b,c) - f(a,b)}{c-a} = \frac{-\frac{1}{bc} + \frac{1}{ab}}{c-a} = \frac{-a+c}{abc(c-a)} = \frac{1}{abc}$$

$$\therefore \underset{bc}{\Delta^2} \frac{1}{a} = \frac{1}{abc}$$

3. Find the divided differences of $f(x) = x^3 - x + 2$ for the arguments $1, 3, 6, 11$. (AM11)

   Solution: Newton's divided difference table is

x	$f(x) = x^3 - x + 2$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	2			
		$\frac{26-2}{3-1} = 12$		
3	26		$\frac{62-12}{6-1} = 10$	
		$\frac{212-26}{6-3} = 62$		$\frac{20-10}{11-1} = 1$
6	212		$\frac{222-62}{11-3} = 20$	
		$\frac{1322-212}{11-6} = 222$		
11	1322			

4. Construct the divided difference table for the following data: (MJ12)

$$\begin{array}{lcccc} x: & 0 & 1 & 2 & 5 \\ f(x): & 2 & 3 & 12 & 147 \end{array}$$

Solution: [Ref : Part A : Example 1 (AM10)]

5. Find the divided differences of $f(x) = x^3 - x^2 + 3x + 8$ for arguments $0, 1, 4, 5$. (ND13)

Solution: [Ref : Part A : Example 3 (AM11)]

6. Find the second divided difference with arguments a, b, c, if $f(x) = \frac{1}{x}$. (AM14, ND10)

Solution: [Ref : Part A : Example 2 (ND10)]

7. Prove that $\underset{yz}{\Delta^2} x^3 = x + y + z$.

Solution : Given the function $f(x) = x^3$ and the arguments are x, y, z.

$$\underset{y}{\Delta} f(x) = \frac{f(y) - f(x)}{y - x} = \frac{y^3 - x^3}{y - x} = \frac{(y-x)(x^2 + xy + y^2)}{y - x} = x^2 + xy + y^2$$

Similarly, $\underset{z}{\Delta} f(y) = y^2 + yz + z^2$

$$\text{Now, } \underset{yz}{\Delta^2} f(x) = \frac{\underset{z}{\Delta} f(y) - \underset{y}{\Delta} f(x)}{z - x} = \frac{y^2 + yz + z^2 - (x^2 + xy + y^2)}{z - x}$$

$$= \frac{z^2 - x^2 + yz - xy}{z - x} = \frac{(z+x)(z-x) + y(z-x)}{z - x} = x + y + z$$

8. Show that $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

Solution : If $f(x) = \frac{1}{x}$, $f(a) = \frac{1}{a}$

$$f(a,b) = \underset{b}{\Delta}\frac{1}{a} = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab}$$

Similarly, $f(b,c) = \underset{c}{\Delta}\frac{1}{b} = -\frac{1}{bc}$, $f(c,d) = \underset{d}{\Delta}\frac{1}{c} = -\frac{1}{cd}$

$$f(a,b,c) = \underset{bc}{\Delta^2}\frac{1}{a} = \frac{\underset{c}{\Delta}\frac{1}{b} - \underset{b}{\Delta}\frac{1}{b}}{c - a} = \frac{-\frac{1}{bc} + \frac{1}{ab}}{c - a} = \frac{-a + c}{abc(c - a)} = \frac{1}{abc}$$

$$\text{Similarly, } f(b,c,d) = \underset{cd}{\Delta^2}\frac{1}{b} = \frac{1}{bcd}$$

$$\therefore\ f(a,b,c,d) = \underset{bcd}{\Delta^3}\frac{1}{a} = \frac{\underset{cd}{\Delta^2}\frac{1}{b} - \underset{bc}{\Delta^2}\frac{1}{a}}{d-a} = \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a}$$

$$= \frac{a-d}{abc(d-a)} = -\frac{1}{abcd}$$

2.3.3 Part B

Example 2.7. Construct the divided difference table for the following data and find the value $f(2)$.

x	4	5	7	10	11	12
$y = f(x)$	50	102	296	800	1010	1224

Solution: Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3) + \cdots$$

Newton's divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	50					
		$\frac{102-50}{5-4} = 52$				
5	102		$\frac{97-52}{7-4} = 15$			
		$\frac{296-102}{7-5} = 97$		$\frac{14.2-15}{10-4} = -0.133$		
7	296		$\frac{168-97}{10-5} = 14.2$		$\frac{-.617+.133}{11-4} = -.069$	
		$\frac{800-296}{10-7} = 168$		$\frac{10.5-14.2}{11-5} = -.617$		$\frac{-.158+.069}{12-4} = -.011$
10	800		$\frac{210-168}{11-7} = 10.5$		$\frac{-1.7+.617}{12-5} = -.158$	
		$\frac{1010-800}{11-10} = 210$		$\frac{2-10.5}{12-7} = -1.7$		
11	1010		$\frac{214-210}{12-10} = 2$			
		$\frac{1224-1010}{12-11} = 214$				
12	1224					

$$f(x) = 50 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(-0.133)$$
$$+ (x-4)(x-5)(x-7)(x-10)(-0.069)$$
$$+ (x-4)(x-5)(x-7)(x-10)(x-11)(-0.011)$$
$$f(x=2) = 50 + (2-4)(52) + (2-4)(2-5)(15) + (2-4)(2-5)(2-7)(-0.133)$$
$$+ (2-4)(2-5)(2-7)(2-10)(-0.069)$$
$$+ (2-4)(2-5)(2-7)(2-10)(2-11)(-0.011)$$
$$= 47.19$$

Example 2.8. If $f(0)=0, f(1)=0, f(2)=-12, f(4)=0, f(5)=600, f(7)=7308$, find a polynomial that satisfies this data using Newton's devided difference formula. Hence find $f(6), f(-1)$.

Solution: Hint : $f(x)=x(x-1)[x^3-2x^2-13x+32]$

$$f(6)=2940$$

$$f(-1)=84$$

Example 2.9. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x)=x^3-2x$.

Solution: [Ref : Part A : Example 3, 5]

2.3.4 Anna University Questions

1. Find the function $f(x)$ from the following table using Newton's divided difference formula: (AM10)

$x:$	0	1	2	4	5	7
$f(x):$	0	0	−12	0	600	7308

   Solution: $\left[f(x)=x^5-3x^4-11x^3+33x^2-20x \Rightarrow f(6)=2580\right]$

2. Given tha table

$x:$	5	7	11	13	17
$f(x):$	150	392	1452	2366	5202

   Evaluate $f(9)$ using Newton's divided difference formula. (AM11)

   Solution: $\left[f(x)=x^3-x^2+24x-70 \Rightarrow f(9)=794\right]$

3. Determine $f(x)$ as a polynomial in x for the following data, using Newton's divided difference formulae. Also find $f(2)$. (ND11)

x:	−4	−1	0	2	5
$f(x)$:	1245	33	5	9	1335

   Solution: $\left[f(x)=3x^4-5x^3+6x^2-14x+5 \Rightarrow f(2)=9\right]$

4. Use Newton's divided difference formula to find $f(x)$ from the following data. (MJ13)

$x:$	1	2	7	8
$y:$	1	5	5	4

   Solution: $\left[f(x)=\frac{1}{42}\left(3x^3-58x^2+321x-224\right)\right]$

5. Find $f(3)$ by Newton's divided difference formula for the following data: (AM14, ND2004)

$x:$	−4	−1	1	2	5
$y:$	1245	33	5	9	1335

   Solution: $\left[f(x)=3x^4-5x^3+6x^2-14x+5 \Rightarrow f(3)=125\right]$

6. By using Newton's divided difference formula find $f(8)$, gives (ND14)

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

7. If $f(0) = f(1) = 0$, $f(2) = -12$, $f(4) = 0$, $f(5) = 600$ and $f(7) = 7308$, find a polynomial that satisfies this data using Newton's divided difference interpolation formula. Hence, find $f(6)$. (MJ2007)

[Ans : $f(6) = 2580$]

8. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's divided difference formula. (N/D2007)

$\left[\text{Ans : } f(x) = x^3 - x^2 + 24x - 70;\ f(9) = 794\right]$

9. Using Newton's divided difference interpolation, find the polynomial of the given data

x	-1	0	1	3
$f(x)$	2	1	0	-1

(AU Nov/Dec 2007)

$\left[\text{Ans : } f(x) = \frac{1}{24}\left(x^3 - 25x + 24\right)\right]$

2.4 Cubic Splines

Interpolating with a cubic spline

The cubic spline interpolation formula is

$$S(x) = y(x) = y = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}(x_i - x)[y_{i-1} - \frac{h^2}{6}M_{i-1}] + \frac{1}{h}(x - x_{i-1})[y_i - \frac{h^2}{6}M_i]$$

where

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

n = number of data

i = number of intervals [i.e., $i = 1, 2, 3, , (n-1)$]

h = length of interval = interval length.

Note : If M_i and y_i'' values are not given, then assume $M_0 = M_n = 0$ $\left[\text{or } y_0'' = y_n'' = 0\right]$, and find $M_1, M_2, \cdots, M_{n-1}$ in 1st interval, 2nd interval, $\cdots$, $(n-1)$th interval value.

Note : Order of convergence of the cubic spline is 4.

2.4.1 Part A

1. Define a cubic spline $S(x)$ which is commonly used for interpolation. (AM10)

Solution: Definition(Cubic Spline Interpolation): Given a function f defined on $[a, b]$ and

$a = x_0 < x_1 < \cdots < x_n = b$, a cubic spline interpolant S for f is a function that satisfies the following conditions:

1. For each $j = 1, \cdots, n, S(x)$ is a cubic polynomial, denoted by $S_j(x)$, on the subinterval $[x_{j-1}, x_j]$.
2. $S(x_j) = f(x_j)$ for each $j = 0, 1, \cdots, n$.
3. $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$.
4. $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$.
5. $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$.
6. One of the following sets of boundary conditions is satisfied:
   i. $S''(x_0) = S''(x_n) = 0$ (natural or free boundary);
   ii. $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (clamped boundary).

2. Define cubic spline function. (ND11)

Solution : A function $s \in C^2[a, b]$ is called a cubic spline on $[a, b]$, if s is a cubic polynomial s_i in each interval $[x_i, x_{i+1}]$. It is called a cubic interpolating spline if $s(x_i) = y_i$ for given values y_i.

3. For cubic splines, what are the 4n conditions required to evaluate the unknowns. (MJ12)

Solution: We need $4n$ conditions to fix the coefficients:

1. $S(x_j) = f(x_j)$ for each $j = 1, \cdots, n \Rightarrow$ (n conditions)
2. $S_{n1} = y_n$, 1 condition $\Rightarrow$ (1 condition)
3. $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$ for each $j = 1, \cdots, n-1. \Rightarrow$ ($n-1$ conditions)
4. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$ for each $j = 1, \cdots, n-1. \Rightarrow$ ($n-1$ conditions)
5. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$ for each $j = 1, \cdots, n-1. \Rightarrow$ ($n-1$ conditions)

$\Rightarrow$ Totally ($4n-2$ conditions)

These are $4n - 2$ conditions. We need two extra.

We can define two extra boundary conditions.

Natural Spline: $M_0 = S''_0(x_0) = 0$ and $M_n = S''_n(x_n) = 0$. (2 conditions)

$\therefore 4n - 2 + 2 = 4n$ conditions required to evaluate the unknowns for cubic spline.

4. Define cubic spline. (ND12, ND11)

5. What is a cubic spline?

Solution : A cubic spline which has continuous slope and curvature is called a cubic spline.

6. What is a natural cubic spline?

Solution : A cubic spline fitted to the given data such that the end cubics approach linearity at their extremities is called a natural cubic spline.

7. State the conditions required for a natural cubic spline.

Solution : A cubic spline $g(x)$ fits to each of the points is continuous and is continuous in slope and curvature such that $M_0 = S_0 = g''_0(x_0) = 0$ and $M_n = S_n = g''_{n-1}(x_n) = 0$ is called a natural cubic spline. Let us assume that $(x_i, y_i), i = 0, 1, 2, \cdots, n$ are data points.

8. What are the advantages of cubic spline fitting?

   Solution : Cubic spline provide better approximation to the behavior of functions that have abrupt local changes. Further, spline perform better than higher order polynomial approximation.

9. Write the end conditions on $M_i(x)$ in natural cubic spline.

   Solution : $M_0(x) = 0, M_n(x) = 0.$

10. Write the relation between the second derivatives $M_i(x)$ in cubic splines with equal mesh spacing.

    Solution :

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}], i = 1, 2, \cdots, n-1.$$

Or

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[f_{i-1} - 2f_i + f_{i+1}], i = 1, 2, \cdots, n-1$$

2.4.2 Part B

Example 2.10. Find the cubic spline approximation for the function f(x) given by the data:

x	**0**	**1**	**2**	**3**
$y = f(x)$	**1**	**2**	**33**	**244**

with $M_0 = 0 = M_3$. Hence estimate the value $f(0.5), f(1.5), f(2.5)$. **{AU2010}**

Solution: We know that cubic spline interpolation formula for $x_{i-1} \le x < x_i, i = 1, 2, 3$ is

$$\begin{aligned} S_i(x) = y(x) = y = {} & \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] \\ & + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right] \\ & + \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \end{aligned} \quad (1)$$

$$\text{where } M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \quad (2)$$

n = number of data = 4

i = number of intervals = 3 i.e., $i = 1, 2, 3$.

h = length of inteval = 1

Given $M_0 = M_2 = 0$, so find M_1, M_2.

Suppose M_i or y_i'' values are not given, then assume $M_0 = M_3 = 0$ and find M_1, M_2.

<u>To find M_1, M_2</u>

$$\text{When } i = 1, (2) \Rightarrow M_0 + 4M_1 + M_2 = \frac{6}{1^2}[y_0 - 2y_1 + y_2]$$

$$\Rightarrow 0 + 4M_1 + M_2 = 6[1 - 2(2) + 33]$$

$$\Rightarrow 4M_1 + M_2 = 180 \quad (3)$$

When $i = 2, (2) \Rightarrow M_1 + 4M_2 + M_3 = \dfrac{6}{1^2}\left[y_1 - 2y_2 + y_3\right]$

$$\Rightarrow M_1 + 4M_2 + 0 = 6\left[2 - 2(2) + 244\right]$$

$$\Rightarrow 4M_1 + M_2 = 180 \qquad (4)$$

Solving (3)&(4), $\quad (3) \Rightarrow 4M_1 + M_2 = 180$

$$4 \times (4) \Rightarrow 4M_1 + 16M_2 = 4320$$

$$\text{i.e., } (3) + 4 \times (4) \Rightarrow -15M_2 = 4140$$

$$\Rightarrow M_2 = 276$$

$$(3) \Rightarrow 4M_1 = 180 - 276$$

$$\Rightarrow M_1 = -24$$

<u>To find Cubic spline</u>

When $i = 1$,

Cubic spline in $x_{i-1} \le x \le x_i$

i.e., $x_0 \le x \le x_1$

i.e., $0 \le x \le 1$

i.e., Cubic spline in $0 \le x \le 1$ is

$$\begin{aligned}
y_1(x) = S_1(x) &= \frac{1}{6(1)}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right] \\
&\quad + \frac{1}{1}(x_1 - x)\left[y_0 - \frac{1}{6}M_0\right] + \frac{1}{1}(x - x_0)\left[y_1 - \frac{1}{6}M_1\right] \\
&= \frac{1}{6}\left[(1 - x)^3 (0) + (x - 0)^3 (-24)\right] \\
&\quad + (1 - x)[1 - 0] + (x - 0)[2 - (-24)] \\
&= -4x^3 + (1 - x) + 6x \\
&= -4x^3 + 5x + 1
\end{aligned}$$

When $i = 2$,

Cubic spline in $x_{i-1} \le x \le x_i$

i.e., $x_1 \le x \le x_2$

i.e., $1 \le x \le 2$

i.e., Cubic spline in $1 \le x \le 2$ is

$$\begin{aligned}
y_2(x) = S_2(x) &= \frac{1}{6(1)}\left[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2\right] \\
&\quad + \frac{1}{1}(x_2 - x)\left[y_1 - \frac{1}{6}M_1\right] + \frac{1}{1}(x - x_1)\left[y_2 - \frac{1}{6}M_2\right] \\
&= \frac{1}{6}\left[(2 - x)^3 (-24) + (x - 1)^3 (276)\right] \\
&\quad + (2 - x)\left[2 - \frac{1}{6}(-24)\right] + (x - 1)\left[33 - \frac{1}{6}(276)\right]
\end{aligned}$$

$$= -4\,(2-x)^3 + 46\,(x-1)^3 + 6\,(2-x) - 13\,(x-1)$$
$$= 50x^3 - 162x^2 + 162x - 53$$

When $i = 3$,

Cubic spline in $x_{i-1} \le x \le x_i$

i.e., $x_2 \le x \le x_3$

i.e., $2 \le x \le 3$

i.e., Cubic spline in $2 \le x \le 3$ is

$$y_3\,(x) = S_3\,(x) = \frac{1}{6\,(1)}\left[(x_3 - x)^3\, M_2 + (x - x_2)^3\, M_3\right]$$
$$+ \frac{1}{1}\,(x_3 - x)\left[y_2 - \frac{1}{6}M_2\right] + \frac{1}{1}\,(x - x_2)\left[y_3 - \frac{1}{6}M_3\right]$$
$$= \frac{1}{6}\left[(3-x)^3\,(276) + 0\right]$$
$$+ (3-x)\left[33 - \frac{1}{6}\,(276)\right] + (x-2)\,[244 - 0]$$
$$= 46\left(27 - x^3 + 9x^2 - 27x\right) - 13\,(3-x) + 244x - 488$$
$$= -46x^3 + 414x^2 - 985x + 715$$

$\therefore$ Cubic spline is

$$S\,(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & 0 \le x \le 1 \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 167x - 53, & 1 \le x \le 2 \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & 2 \le x \le 3 \end{cases}$$

When $x = 0.5, y_1(x = 0.5) = S_1(x = 0.5) = -4\,(0.5)^3 + 5\,(0.5)^2 + 1 = 3$

When $x = 1.5, y_2(x = 1.5) = S_2(x = 1.5) = 50\,(1.5)^3 - 162\,(1.5)^2 + 167\,(1.5) - 53 = 1.75$

When $x = 2.5, y_3(x = 2.5) = S_3(x = 2.5) = -46\,(2.5)^3 + 414\,(2.5)^2 - 985\,(2.5) + 715 = 121.25$

Example 2.11. From the following table

x	1	2	3
$y = f(x)$	−8	−1	18

. Find cubic spline and compute $y(1.5), y'(1), y(2.5)$ and $y'(3)$.

Solution:

$$S\,(x) = \begin{cases} S_1(x) = y_1(x) = 3(x-1)^3 + 4x - 12, & 1 \le x \le 2 \\ S_2(x) = y_2(x) = 3(3-x)^3 + 22x - 48, & 2 \le x \le 3 \end{cases}$$

&

$$y(x = 1.5) = S_1(x = 1.5) = -\frac{45}{8}, y'(x = 1) = S_1'(x = 1) = 4$$
$$y(x = 2.5) = S_2(x = 2.5) = 7.375, y'(x = 3) = S_2'(x = 3) = 22.$$

Example 2.12. Fit a natural cubic spline for the following data:

x	0	1	2	3
$y = f(x)$	1	4	0	−2

{AU 2008}

Solution: Assume $M_0 = 0 = M_3$.

$$S(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 + 5x + 1, & [0,1] \\ S_2(x) = y_2(x) = 3x^3 - 15x^2 + 20x - 4, & [1,2] \\ S_3(x) = y_3(x) = -x^3 + 9x^2 - 28x + 28, & [2,3] \end{cases}$$

2.4.3 Anna University Questions

1. If $f(0) = 1$, $f(1) = 2$, $f(2) = 33$ and $f(3) = 244$, find a cubic spline approximation, assuming $M(0) = M(3) = 0$. Also, find $f(2.5)$. (AM10)

   Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & x \in [0,1] \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 1670x - 53, & x \in [1,2] \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & x \in [2,3] \end{cases}$$

$$f(x) = -46x^3 + 414x^2 - 985x + 715 \quad x \in [2,3]$$

$$f'(x) = -138x^2 + 828x - 985$$

$$f'(x = 2.5) = -138(2.5)^2 + 828(2.5) - 985$$

$$= 222.5$$

2. Find the natural cubic spline approximation for the function $f(x)$ defined by the following data:

$x:$	0	1	2	3
$f(x):$	1	2	33	244

(ND10)

   Solution: Hint : Ref : Previous AU 1 (AM10)

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & x \in [0,1] \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 1670x - 53, & x \in [1,2] \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & x \in [2,3] \end{cases}$$

$$\left[f(x) = -46x^3 + 414x^2 - 985x + 715 \Rightarrow f(x = 2.5) = y(x = 2.5) = 121.25, \because x = 2.5 \in [2,3]\right]$$

3. Find the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y'_0 = y'_3 = 0$. (AM11)

$x:$	−1	0	1	2
$y:$	−1	1	3	35

   Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 - 6x^2 - 2x + 1, & x \in [-1,0] \\ S_2(x) = y_2(x) = 10x^3 - 6x^2 - 2x + 1, & x \in [0,1] \\ S_3(x) = y_3(x) = -8x^3 + 48x^2 - 56x + 19, & x \in [1,2] \end{cases}$$

4. The following values of x and y are given:

x :	1	2	3	4
y :	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$ (MJ12)

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & x \in [1,2] \\ S_2(x) = y_2(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & x \in [2,3] \\ S_3(x) = y_3(x) = \frac{1}{3}\left(-2x^3 + 24x^2 - 76x + 81\right), & x \in [3,4] \end{cases}$$

$$y(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right) \Rightarrow y(1.5) = 1.375, \quad x \in [1,2]$$

$$y'(x) = \frac{1}{3}\left(3x^2 - 6x + 5\right) \Rightarrow y'(3) = 4.666666667, \quad x \in [2,3]$$

(or)

$$y'(x) = \frac{1}{3}\left(-6x^2 + 48x - 76\right) \Rightarrow y'(3) = 4.666666667, \quad x \in [3,4]$$

5. Obtain the cubic spline for the following data to find $y(0.5)$. (ND12)

x :	− 1	0	1	2
y :	−1	1	3	35

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 - 6x^2 - 2x + 1, & x \in [-1,0] \\ S_2(x) = y_2(x) = 10x^3 - 6x^2 - 2x + 1, & x \in [0,1] \\ S_3(x) = y_3(x) = -8x^3 + 48x^2 - 56x + 19, & x \in [1,2] \end{cases}$$

6. Using cubic spline, compute $y(1.5)$ from the given data. (MJ13)

x :	1	2	3
y :	−8	−1	18

Solution: Hint :

$$S(x) = y(x) = 3x^3 - 9x^2 + 13x - 15$$

$$y(1.5) = 3(1.5)^3 - 9(1.5)^2 + 13(1.5) - 15 = -\frac{45}{8} = -5.625, \quad x \in [1,2]$$

7. Find the natural cubic spline to fit the data:

x :	0	1	2
$f(x)$:	−1	3	29

Hence find $f(0.5)$ and $f(1.5)$. (ND13)

Solution: Hint :

$$S(x) = y(x) = \frac{11}{2}x^3 - \frac{3}{2}x - 1, \quad x \in [0,1]$$

$$y(0.5) = -1.0625, \quad x \in [0,1]$$

$$S(x) = y(x) = \frac{11}{2}(2-x)^3 - \frac{5}{2}(2-x) + 29x - 29, \quad x \in [1,2]$$

$$y(1.5) = 13.9375, \quad x \in [1,2]$$

8. Fit the cubic splines for the following data. (16)

x :	1	2	3	4	5
y :	1	0	1	0	1

(AM14)

Solution : Assume $M_0 = 0 = M_4$.

$$f(x) = S(x) = \begin{cases} S_1(x) = y_1(x) = 2 - x, & 1 \le x \le 2 \\ S_2(x) = y_2(x) = \frac{1}{7}[-5x^3 + 45x^2 - 123x + 106], & 2 \le x \le 3 \\ S_3(x) = y_3(x) = \frac{1}{7}[6x^3 - 72x^2 + 275x - 332], & 3 \le x \le 4 \\ S_4(x) = y_4(x) = \frac{1}{7}[-5x^3 + 75x^2 - 363x + 772], & 4 \le x \le 5 \end{cases}$$

9. Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$. (ND14)

x	-1	0	1	2
y	-1	1	3	35

10. Find the cubic Spline interpolation. (AU N/D, 2007,AM2014)

x	1	2	3	4	5
$f(x)$	1	0	1	0	1

11. Given the following table, find $f(2.5)$ using cubic spline functions : (AU May/June 2007)

x	1	2	3	4
$f(x)$	0.5	0.3333	0.25	0.2

Solution : [Ans: $S_2(2.5) = 0.2829$]

12. Fit the st.line for the data.

x	0	1	2	3
$f(x)$	1	2	9	28

Solution :

$$f(x) = \begin{cases} y_1(x) = \frac{4}{5}x^3 - \frac{4}{5}x + 1, & 0 \le x \le 1 \\ y_2(x) = \frac{1}{5}[10x^3 - 18x^2 + 19x - 1, & 1 \le x \le 2 \\ y_3(x) = -2x^3 + \frac{102}{5}x^2 - \frac{333}{5}x + \frac{159}{5}, & 2 \le x \le 3 \end{cases}$$

2.5 Newton's forward and backward difference formulae

Newton's forward and backward difference formulae for Uniform (or) equal intervals only.

Newton's forward interpolation difference formula:

[If y(required x near to x_0) =? and use Δ]

$$\begin{aligned} y(x) &= f(x) = f(x_0 + uh) \\ &= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \end{aligned}$$

where $u = \frac{x - x_0}{h}$, h = length of interval.

2.6 Newton's backward interpolation difference formula:

[If y (required x near to x_n) =? and use ∇]

$$y(x) = f(x) = f(x_n + vh)$$
$$= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

where $v = \frac{x - x_n}{h}$, h = length of interval.

2.6.1 Part A

1. When to use Newton's forward interpolation and when to use Newton's backward interpolation? (ND10)

   Solution :

   Use of Newton's forward interpolation : The formula is used to interpolate the values of y near the begining of the table value and also for extrapolation the values of y short distance ahead (to the left) of y_0.

   Use of Newton's backward interpolation : The formula is used to interpolate the values of y near the end of the table value and also for extrapolation the values of y short distance ahead (to the right) of y_0.

2. State Newton's backward difference formula. (ND12)

   Solution : Newton's backward interpolation difference formula:

   [If y (required x near to x_n) =? and use ∇]

$$y(x) = f(x) = f(x_n + vh)$$
$$= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

   where $v = \frac{x - x_n}{h}$, h = length of interval.

3. State Newton's forward interpolation formula. (MJ13)

   Solution : Newton's forward interpolation difference formula:

   [If y (required x near to x_0) =? and use Δ]

$$y(x) = f(x) = f(x_0 + uh)$$
$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

   where $u = \frac{x - x_0}{h}$, h = length of interval

4. State Newton's forward difference formula for equal intervals. (ND13)

   Solution : [Ref : Part A : 3 (MJ13)]

5. State Newton's backward formula for interpolation. (ND14,ND12)

6. What advantage has Lagrange's formula over Newton?

   Solution : The forward and backward interpolation formulae of Newton can be used only when the values of the independent variable x are equally spaced can also be used when the differences of the dependent variable y become smaller ultimately. But Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not and whether the difference of y become smaller or not.

7. Derive Newton's forward difference formula by using operator method. (or) Derive Gregory - Newton forward difference interpolation formula.

Solution : $P_n(x) = P_n(x_0 + uh) = E^u P_n(x_0) = E^u y_0$

$$= (1+\Delta)^u y_0$$
$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$
where $u = \dfrac{x-x_0}{h}$.

8. Derive Newton's backward difference formula by using operator method.

Solution :

$$P_n(x) = P_n(x_0 + vh) = E^v P_n(x_n) = E^v y_n$$
$$= (1-\nabla)^{-v} y_n \text{ where } E = (1-\nabla)^{-1}$$
$$= \left(1 + v\nabla + \frac{v(v+1)}{2!}\nabla^2 + \frac{v(v+1)(v+2)}{3!}\nabla^3 + \cdots\right) y_n$$
$$= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$
where $v = \dfrac{x-x_n}{h}$

2.6.2 Part B

Example 2.13. Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values

x	**0**	**1**	**2**	**3**
$y = f(x)$	**1**	**2**	**1**	**10**

.

Evaluate $f(4)$. **(AU 2000, 2009)**

Solution: WKT, Newton's forward formula to find the polynomial in x.

There are only 4 data given. Hence the polynomial will be degree 3. Newton's forward formula is

$$y(x) = f(x) = f(x_0 + uh)$$
$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$
where $u = \dfrac{x-x_0}{h}$

h = length of interval.

Newton's divided difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 1$			
		$2 - 1 = 1$		
$x_1 = 1$	$y_1 = 2$		$-1 - 1 = -2$	
		$1 - 2 = -1$		$10 - (-2) = 12$
$x_2 = 2$	$y_2 = 1$		$9 - (-1) = 10$	
		$10 - 1 = 9$		
$x_3 = 3$	$y_3 = 10$			

$$\begin{aligned} f(x) &= 1 + \frac{x}{1!}(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12) \\ &= 2x^3 - 7x^2 + 6x + 1 \end{aligned}$$

When $x = 4$,

$$f(4) = 2 \times 4^3 - 7 \times 4^2 + 6 \times 4 + 1 = 128 - 92 + 24 + 1 = 41$$

Example 2.14. The population of a city in a census takes once in 10 years is given below. Estimate the population in the year 1955.

Year	**1951**	**1961**	**1971**	**1981**
Population in lakhs	**35**	**42**	**58**	**84**

Solution: $y(x = 1955) = f(x = 1955) = 36.784$

Example 2.15. From the table given below find $\sin 52°$ by using Newton's forward interpolation formula.

x	**45°**	**50°**	**55°**	**60°**
$y = \sin x$	**0.7071**	**0.7660**	**0.8192**	**0.8660**

Solution: $y(x = 52) = \sin 52° = 0.788$ approximately.

Example 2.16. From the data given below find the number of students whose weight is between 60 and 70.

Weight in lbs	**0-40**	**40-60**	**60-80**	**80-100**	**100-120**
Number of students	**250**	**120**	**100**	**70**	**50**

Solution: Let weight be denoted by x and

Number of students be denoted by y, i.e., $y = f(x)$.

Use Newton's forward formula to find y where x lies between 60 – 70.

Newton's forward formula is

$$\begin{aligned} y(x) = f(x) &= f(x_0 + uh) \\ &= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \end{aligned}$$

We rewrite the table as cumulative table showing the number of students less than x lbs.

x	Below 40	Below 60	Below 80	Below 100	Below 120
y	250	370	470	540	590

Newton's forward difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		–20		
		120		–10	
Below 80	470		–30		20
		70		10	
Below 100	540		–20		
		50			
Below 120	590				

Here $u = \dfrac{x - x_0}{h} = \dfrac{x - 40}{20}$.

When $x = 70, u = \dfrac{70 - 40}{20} = 1.5$

$$y(x = 70) = 250 + \frac{1.5}{1!}(120) + \frac{1.5(1.5-1)}{2!}(-20) + \frac{1.5(1.5-1)(1.5-2)}{3!}(-10) + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!}(20)$$
$$= 423.59 = 424$$

$\therefore$ Number of students whose weight is 70 = 424

$\therefore$ Number of students whose weight is between 60 – 70 = 424 – 370 = 54.

Example 2.17. Use Newton's backward difference formula to construct as interpolating polynomial of degree 3 for the data.
$f(0.75) = 0.07181250$, $f(0.5) = 0.024750$, $f(0.25) = 0.33493750$, $f(0) = 1.10100$. Find $f\left(-\frac{1}{3}\right)$.

Solution: $y(x) = x^3 + 4.001x^2 + 4.002x + 1.101$, $f\left(-\frac{1}{3}\right) = 0.174518518$.

Example 2.18. From the following data, find θ at $x = 43$ and $x = 84$.

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

. Also express θ in terms of x.

Solution: $\theta(x = 43) = 189.79$ (by Newton's forward formula)

$\theta(x = 84) = 286.96$ (by Newton's backward formula)

$\theta(x) = 0.01x^2 + 1.1x + 124$ (by Newton's forward formula)

Example 2.19. Find a polynomial of degree two for the data by Newton's forward difference method :

x	0	1	2	3	4	5	6	7
$f(x)$	1	2	4	7	11	16	22	29

(AU May/June, 2007)

Solution: $\left[\text{Ans : } y(x) = \frac{1}{2}\left(x^2 + x + 2\right)\right]$

2.6.3 Anna University Questions

1. Find the value of $\tan 45°15'$ by using Newton's forward difference interpolation formula for

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

(ND10)

Solution : [Ans: $\tan 45°15' = 1.00876$, by Newton's forward difference formula]

2. Derive Newton's backward difference formula by using operator method. (MJ12)

3. Find the value of y when $x = 5$ using Newton's interpolation formular from the following table:

x :	4	6	8	10
y :	1	3	8	16

(ND12)

Solution : [$y(x = 5) = 1.625$, by Newton's forward difference formula]

4. Fit a polynomial, by using Newton's forward interpolation formula, to the data given below. (8)

x :	0	1	2	3
y :	1	2	1	10

(MJ13)

Solution :

$$y(x) = 2x^3 - 7x^2 + 6x + 1, \quad \text{by Newton's forward \& backward formula}$$

$$y(x = 4) = 41$$

5. Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values: (ND14)

x	0	1	2	3
$f(x)$	1	2	1	10

2.6.4 Assignment problems

1. Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12$, $f(1) = 0$, $f(3) = 6$ and $f(4) = 12$. Hence find $f(2)$. (AM10)

2. Find the value of x when $y = 20$ using Lagrange's formula from the following table. (AM11)

x :	1	2	3	4
$y = f(x)$:	1	8	27	64

3. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data: (MJ12)

Year:	1997	1999	2001	2002
Profit in Lakhs Rs. :	43	65	159	248

4. Use Lagrange's method to find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, and $\log_{10} 661 = 2.8202$. (ND12)

5. Find the function $f(x)$ from the following table using Newton's divided difference formula: (AM10)

x :	0	1	2	4	5	7
$f(x)$:	0	0	−12	0	600	7308

6. Given tha table

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

   Evaluate $f(9)$ using Newton's divided difference formula. (AM11)

7. Find $f(3)$ by Newton's divided difference formula for the following data: (AM14, ND2004)

x :	−4	−1	1	2	5
y :	1245	33	5	9	1335

8. By using Newton's divided difference formula find $f(8)$, gives (ND14)

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

9. If $f(0) = 1$, $f(1) = 2$, $f(2) = 33$ and $f(3) = 244$, find a cubic spline approximation, assuming $M(0) = M(3) = 0$. Also, find $f(2.5)$. (AM10)

10. Find the natural cubic spline to fit the data:

x :	0	1	2
$f(x)$:	−1	3	29

   Hence find $f(0.5)$ and $f(1.5)$. (ND13)

11. Fit the cubic splines for the following data. (AM14)

x:	1	2	3	4	5
y:	1	0	1	0	1

12. Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$. (ND14)

x	-1	0	1	2
y	-1	1	3	35

13. Find the value of $\tan 45°15'$ by using Newton's forward difference interpolation formula for (ND10)

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

14. Find the value of y when $x = 5$ using Newton's interpolation formular from the following table: (ND12)

x:	4	6	8	10
y:	1	3	8	16

15. Fit a polynomial, by using Newton's forward interpolation formula, to the data given below. (MJ13)

x:	0	1	2	3
y:	1	2	1	10

16. Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values: (ND14)

x	0	1	2	3
$f(x)$	1	2	1	10

3 Numerical Differentiation and Integration

Approximation of derivatives using interpolation polynomials - Numerical integration using Trapezoidal, Simpson's 1/3 rule - Romberg's method - Two point and three point Gaussian quadrature formulae - Evaluation of double integrals by Trapezoidal and Simpson's 1/3 rules.

3.1 Introduction

Numerical Differentiation	**Numerical Integration**
Newton's forward difference formula to compute derivatives (Equal interval)	Trapezoidal rule [$n = 1$ in Quadrature formula]
Newton's backward difference formula to compute derivatives(Equal interval)	Simpson's one third rule [$n = 2$ in Quadrature formula]
Lagrange's Interpolation formula (Equal or unequal intervals)	Simpson's three eighth rule [$n = 3$ in Quadrature formula]
Newton's divided difference Interpolation formula (Equal or unequal intervals)	Romberg's method
Maxima and minima of a tabulated function	Two point Gaussian's quadrature formula
	Three point Gaussian's quadrature formula
	Double integrals by Trapezoidal rule
	Double integrals by Simpson's 1/3 rule

3.2 Approximation of derivatives using interpolation polynomials

Numerical Differentiation

Given $y = y(x) = f(x)$ [in a table]

$\frac{dy}{dx} = y'(x) = f'(x)$ is the first numerical derivative

$\frac{d^2y}{dx^2} = y''(x) = f''(x)$ is the second numerical derivative

$\frac{d^ny}{dx^n} = y^{1\cdots n}(x) = f^{1\cdots n}(x)$ is the nth numerical derivative

3.2.1 Newton's forward difference formula to compute derivative

WKT, Newton's forward difference interpolation formula is

$$y(x) = f(x_0+uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$
$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \frac{u^4-6u^3+11u^2-6u}{4!}\Delta^4 y_0 + \cdots$$

where $u = \dfrac{x-x_0}{h}$

First derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{du}$$

i.e., $$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\begin{bmatrix}\text{at } x = x_0\\ \Rightarrow u=0\end{bmatrix}} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{du}\frac{du}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{du}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h}\frac{d}{du}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h^2}\frac{d^2y}{du^2}$$
$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{12u^2-36u+22}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{\begin{bmatrix}\text{at } x = x_0\\ \Rightarrow u=0\end{bmatrix}} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{22}{24}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

Third derivative

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \left(\frac{24u-36}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{\begin{bmatrix}\text{at } x = x_0\\ \Rightarrow u=0\end{bmatrix}} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{36}{24}\Delta^4 y_0 + \cdots\right]$$

3.2.2 Newton's backward difference formula to compute derivatives

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n+vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$
$$= y_n + \frac{v}{1!}\nabla y_n + \left(\frac{v^2+v}{2!}\right)\nabla^2 y_n + \left(\frac{v^3+3v^2+2v}{3!}\right)\nabla^3 y_n + \left(\frac{v^4+6v^3+11v^2+6v}{4!}\right)\nabla^4 y_n + \cdots$$

where $v = \dfrac{x-x_n}{h}$

First derivative

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{dv}$$

$$= \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{3!}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\left[\begin{array}{c}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}\right]} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{dv}\frac{dv}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{d}{dv}\frac{dv}{dx}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{d}{dv}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{1}{h^2}\left[\frac{d^2y}{dv^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \left(\frac{6v+6}{3!}\right)\nabla^3 y_n + \left(\frac{12v^2+36v+22}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{\left[\begin{array}{c}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}\right]} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6}{3!}\nabla^3 y_n + \frac{22}{4!}\nabla^4 y_n + \cdots\right]$$

Third derivative

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \left(\frac{24v+36}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{\left[\begin{array}{c}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}\right]} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \frac{36}{4!}\nabla^4 y_n + \cdots\right]$$

3.2.3 Maxima and Minima of a tabulated function

(If the intervals are same)

WKT, Newton's forward difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

First derivative

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots\right] \tag{1}$$

substitute $h, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \cdots$ gives the equation

$$\frac{dy}{dx} = \text{ an equation in } u \tag{2}$$

For maxima or minima is obtained by the equation by

$$\frac{dy}{dx} = 0 \text{ (or) } y'(x) = 0$$

Put RHS of (1) = 0, find $u = u_1, u_2, \cdots$

Find $\dfrac{d^2y}{dx^2}$ from (2):

Suppose $\left(\dfrac{d^2y}{dx^2}\right)_{\text{at } u_1} = -\text{ve} \Rightarrow u_1$ is maximum point

$$\Rightarrow y(u_1) = \text{ maximum value of } y$$

Suppose $\left(\frac{d^2y}{dx^2}\right)_{\text{at } u_2} = +\text{ve} \Rightarrow u_2$ is minimum point

$\Rightarrow y(u_2) =$ minimum value of y

3.2.4 Part A

1. Write down the formulae for finding the first derivative using Newton's forward difference at $x = x_0$ Newton's backward difference at $x = x_n$. (ND2010)

   Solution : First derivative by Newton's forward difference interpolation formula is

$$\left[\frac{dy}{dx}\right]_{\begin{bmatrix} \text{at } x = x_0 \\ \Rightarrow u = 0 \end{bmatrix}} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

   First derivative by Newton's backward difference interpolation formula is

$$\left[\frac{dy}{dx}\right]_{\begin{bmatrix} \text{at } x = x_n \\ \Rightarrow v = 0 \end{bmatrix}} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

2. Write down the expression for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_n$ by Newton's backward difference formula. (AM2014)

   First derivative by Newton's backward difference interpolation formula is

$$\left[\frac{dy}{dx}\right]_{\begin{bmatrix} \text{at } x = x_n \\ \Rightarrow v = 0 \end{bmatrix}} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

   Second derivative by Newton's backward difference interpolation formula is

$$\left[\frac{d^2y}{dx^2}\right]_{\begin{bmatrix} \text{at } x = x_n \\ \Rightarrow v = 0 \end{bmatrix}} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6}{3!}\nabla^3 y_n + \frac{22}{4!}\nabla^4 y_n + \cdots\right]$$

3.2.5 Part B

Example 3.1. The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds.

Time(sec)	**0**	**5**	**10**	**15**	**20**
Velocity (m/sec)	**0**	**3**	**14**	**69**	**228**

Find (a) Initial acceleration using the entire data (b) Final acceleration.

Solution: The difference table is

Time $t = x$	Velocity $v = y(x)$	$\Delta y(x)$	$\Delta^2 y(x)$	$\Delta^3 y(x)$	$\Delta^4 y(x)$
$0 (= x_0)$	$0 (= y_0)$				
		$3 (= \Delta y_0)$			
5	3		$8 \left(= \Delta^2 y_0\right)$		
		11		$36 \left(= \Delta^3 y_0\right)$	
10	14		44		$24 \left(= \Delta^4 y_0 \text{ or } \nabla^4 y_n\right)$
		55		$60 \left(= \nabla^3 y_n\right)$	
15	69		$104 \left(= \nabla^2 y_n\right)$		
		$159 (= \nabla y_n)$			
20	$228 (= y_n)$				

Here $x_0 = 0, h =$ interval length $= 5$

(a) WKT, acceleration $= \dfrac{dv}{dt}$ = rate of change of velocity

To find initial acceleration, put $\left(\dfrac{dv}{dt}\right)_{t=t_0} = \left(\dfrac{dv}{dt}\right)_{t=0}$.

i.e., initial acceleration exists at $t = 0 = x_0$ [which is nearer to beginning of the table], so we use Newton's forward difference formula for first derivative.

$\therefore$ Newton's forward difference interpolation formula is

$$y(x) = f(x_0 + uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Now, Newton's forward difference formula for first derivative at $x = x_0 = 0\,[\Rightarrow u = 0]$

$$\text{i.e., } y'(x_0) = \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \qquad [\text{Here } h = 5]$$

$$= \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right] = 1$$

$\Rightarrow y'(0) = 1$

$\therefore$ Initial acceleration (acceleration when $t = 0$) is 1 m/sec^2.

(b) Final acceleration exists at $t = 20 = x_4$ [Nearer to ending of table], so use Newton's backward difference interpolation formula for first derivative, put $\left(\dfrac{dv}{dt}\right)_{t=t_n} = \left(\dfrac{dv}{dt}\right)_{t=20}$.

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{5}\left[159 + \frac{1}{2}(104) + \frac{2}{6}(60) + \frac{6}{24}(24)\right]$$

$$= \frac{1}{5}(237) = 47.2 \text{ m / sec}^2$$

Example 3.2. Consider the following table of data :

x	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
$f(x)$	**0.9798652**	**0.9177710**	**0.8080348**	**0.636093**	**0.3843735**

Find (1) $f'(0.25)$ using Newton's forward difference approximation

(2) $f'(0.95)$ using Newton's backward difference approximation

Solution: Here $h = 0.2$

$$(1)\ u = \frac{x - x_0}{h} = \frac{x - 0.2}{0.2} = \frac{0.25 - 0.2}{0.2} = 0.25$$

$$\therefore f'(0.25) = -0.25828$$

$$(2)\ u = \frac{x - x_n}{h} = \frac{0.95 - 1}{0.2} = -0.25$$

$$f'(0.95) = -1.367948$$

Example 3.3. Find the value of $f'(8)$, $f''(9)$, maximum and minimum value from the following data, using an approximate interpolation formula.

x	4	5	7	10	11
$f(x)$	48	100	294	900	1210

Solution: The values of x are unequally spaced.

To find $f(x)$, we use Newton's divided difference formula (or) Lagrange formula.

WKT, Newton's Divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\, f(x_0, x_1, x_2, x_3, x_4) + \cdots \quad (1)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		
		310			
11	1210				

$$\therefore (1) \Rightarrow f(x) = 48 + (x - 4)(52) + (x - 4)(x - 5)(15) + (x - 4)(x - 5)(x - 7)(1)$$
$$= 48 + 52x - 208 + 15\left[x^2 - 9x + 20\right] + x^3 - x^2(16) + x(83) - 140$$
$$= x^3 + x^2(-16 + 15) + x(83 - 135 + 52) - 208 + 300 - 140 + 48$$
$$= x^3 - x^2 + x(0) + 0$$
$$\therefore f(x) = x^3 - x^2$$
$$f'(x) = 3x^2 - 2x, \quad f'(8) = 176$$
$$f''(x) = 6x - 2, \quad f''(9) = 52$$

<u>To find maximum and minimum</u>

Put $f'(x) = 0 \Rightarrow 3x^2 - 2x = 0 \Rightarrow x = 0, x = \frac{2}{3}$

At $x = 0,\ f''(x=0) = 6(0) - 2 = -2 < 0$

$\Rightarrow x = 0$ is a maximum point & maximum value is $f(x=0) = 0$.

At $x = \frac{2}{3},\ f''\left(x = \frac{2}{3}\right) = 6\left(\frac{2}{3}\right) - 2 = 2 > 0.$

$\Rightarrow x = \frac{2}{3}$ is a minimum point & minimum value is $f\left(x = \frac{2}{3}\right) = -\frac{4}{27}$

Example 3.4. Evaluate y' and y'' at $x = 2$ given

x	**0**	**1**	**3**	**6**
y	**18**	**10**	**−18**	**40**

Solution:

$$\left[\text{Ans}: y(x) = x^3 - \frac{70}{9}x^2 - \frac{15}{9}x + 18, y'(x=2) = -\frac{187}{9}, y''(x=2) = -\frac{22}{9}\right]$$

Example 3.5. Find the value of $\cos(1.747)$ using the values given in the table below :

x	**1.70**	**1.74**	**1.78**	**1.82**	**1.86**
$\sin x$	**0.9916**	**0.9857**	**0.9781**	**0.9691**	**0.9584**

Solution: [Ans : − 0.175]

Example 3.6. Find $\sec 31°$ from the following data :

θ	**31**	**32**	**33**	**34**
$\tan\theta$	**0.6008**	**0.6249**	**0.6494**	**0.6745**

Solution:

$$\left[\text{Ans}: \sec^2 31 = 1.3835 \Rightarrow \sec 31 = 1.174,\ \text{Hint}: 1° = \frac{\pi}{180} = 0.017453292\right]$$

3.2.6 Anna University Questions

1. Given the following data, find $y'(6)$ and the maximum value of y (if it exists). (AM10)

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922

2. Find $f'(x)$ at $x = 1.5$ and $x = 4.0$ from the following data using Newton's formulae for differentiation.

x :	1.5	2.0	2.5	3.0	3.5	4.0
$y = f(x)$:	3.375	7.0	13.625	24.0	38.875	59.0

(MJ12)

3. Find the first three derivatives of $f(x)$ at $x = 1.5$ by using Newton's forward interpolation formula to the data given below.

x :	1.5	2	2.5	3	3.5	4
y :	3.375	7	13.625	24	38.875	59

(MJ13)

Numerical integration

Newton-Cote's quadrature formula is

$$\int_a^{a+nh} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

where n = number of intervals

h = length of interval

3.3 Numerical integration using Trapezoidal

Trapezoidal rule($n = 1$ in quadrature formula)

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \\ \text{(or)} \\ \frac{h}{2}\left[(\text{first term} + \text{last term}) + 2(\text{remaining terms})\right] \end{cases}$$

3.3.1 Part A

1. Write down the Newton-Cote's formula for the equidistant ordinates. (MJ2011)

   Solution : The general Newton-Cote's quadrature formula is

$$\int_a^{a+nh} f(x)\,dx = nh\left[y_0 + \frac{n\Delta}{2}y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

   This is known as the general Gauss-Lagendre integration formula. Putting $n = 1$ and omitting second and hyper differences in the above, we get

$$\int_a^{a+h} f(x)\,dx = \frac{h}{2}\left[f(a) + f(a+h)\right] = \frac{h}{2}(y_1 + y_2)$$

   which is the trapezoidal rule.

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule. (ND2012)

   Solution : Let $y = y(x) = f(x) = \frac{1}{1+x^2}$, $h = 0.2 = \frac{1}{5}$

x	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$	1	0.96154	0.86207	0.73529	0.60976	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \frac{h}{2}[(\text{first term} + \text{last term}) + 2(\text{remaining terms})] \\ \text{(or)} \\ \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})] \end{cases}$$

$$= \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}[(1 + 0.5) + 2(0.96154 + 0.86207 + 0.73529 + 0.60976)]$$

$$\equiv 0.78373$$

3. Evaluate $\int_0^{\pi} \sin x dx$ by Trapezoidal rule by dividing ten equal parts. (AM2013)
   Solution : [1.9843]

4. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Trapezoidal rule. (ND2013)
   Solution : [1.41079950]

5. Taking $h = 0.5$, evaluate $\int_1^2 \frac{dx}{1+x^2}$ using Trapezoidal rule. (AM2014)
   Solution : [0.3289]

3.3.2 Part B

Example 3.7. Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{1}{(1+x^2)} dx$ by taking eight equal intervals. (MJ13)

Solution: Let $y = y(x) = f(x) = \frac{1}{1+x^2}$

Lenth of the given interval $[a, b] = b - a = 1 - (-1) = 2$

Lenth of the 8 equal intervals $= h = \frac{2}{8} = 0.25$

Form the table for the ordinates of the function $y = y(x) = f(x) = \frac{1}{1+x^2}$

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
$y = \frac{1}{1+x^2}$	0.5	0.64	0.8	0.94118	1	0.94118	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \frac{h}{2}[(\text{first term} + \text{last term}) + 2(\text{remaining terms})] \\ \text{(or)} \\ \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})] \end{cases}$$

$$= \frac{h}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.2}{2}[(0.5 + 0.5) + 2(0.64 + 0.8 + 0.94118 + 1 + 0.94118 + 0.8 + 0.64)]$$

$$\equiv 1.56559$$

3.3.3 Anna University Questions

1. Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{1}{(1+x^2)} dx$ by taking eight equal intervals. (MJ13)

   Solution: [1.5656]

3.4 Numerical integration using Simpson's 1/3 rule

Simpson's one third rule (Simpson's 1/3 rule) ($n = 2$ in quadrature form)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\right] \\ \text{(or)} \\ \frac{h}{3}\begin{bmatrix} \text{(first term + last term)} \\ +4\,\text{(odd suffices)} \\ +2\,\text{(even suffices)} \end{bmatrix} \end{cases}$$

3.4.1 Part A

1. When do you apply Simpson's $\frac{1}{3}$ rule, and what is the order of the error in Simpson's $\frac{1}{3}$ rule. (MJ2011)

   Solution : Let n = number of intervals
   Simpson's $\frac{1}{3}$ rule : The number of ordinates is odd (or) the intervals number is even.
   The error in the Simpson's 1/3 rule is of the order h^4.

2. State Simpson's one-third rule. (ND2011,AM2013)

   Solution : Let DC be the curve $y = f(x)$ and DA, CB be the terminal ordinates. Let $OA = a$ and $OB = b$. Divide AB into even number(say $2n$) of equal parts, equal to h. Let $x_1, x_2, \cdots, x_{2n+1}$ be the abscissae of the points, A, A_1, B and $y_1, y_2, \cdots, y_{2n+1}$ be the corresponding ordinates. Then

   $$\int_a^b y dx \text{ is approximately } = \frac{h}{3}(A + 4B + 2C) \tag{1}$$

   where $A = y_1 + y_{2n+1}$ = sum of the first and last ordinates.

   $B = y_2 + y_1 + \cdots + y_{2n}$ = sum of the even ordinates and

   $C = y_3 + y_5 + \cdots + y_{2n-1}$ = sum of the remaining ordinates (1)

   known as Simpson's rule or Simpson's one - third rule.

3. State the local error term in Simpson's $\frac{1}{3}$ rule. (ND14)

3.4.2 Part B

Example 3.8. A curve passes through the points $(1,2)$, $(1.5,2.4)$, $(2,2.7)$, $(2.5,2.8)$, $(3,3)$, $(3.5,2.6)$ & $(4,2.1)$. Obtain Area bounded by the curve, x axis between $x = 1$ and $x = 4$. Also find the

volume of solids of revolution by revolving this area about $x-$ axis.

Solution:

$$\text{WKT, Area } = \int_a^b y dx = \int_1^4 y dx$$

$$\text{Simpson's 1/3 rule, Area } = \int_1^4 y dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= 7.783$$

$$\text{Volume } = \pi \int_a^b y^2 dx = \pi \int_1^4 y^2 dx$$

To find y^2 :

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1
y^2	4	5.76	7.29	7.84	9	6.76	4.41

$$\therefore \text{ Volume } = \pi \int_1^4 y^2 dx = \pi \left\{ \frac{h}{3}\left[\left(y_0^2 + y_6^2\right) + 4\left(y_1^2 + y_3^2 + y_5^2\right) + 2\left(y_2^2 + y_4^2\right)\right]\right\} \quad \text{[by Simpson's 1/3 rule]}$$

$$= 64.07 \text{ cubic units}$$

3.4.3 Anna University Questions

1. The velocity v of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
v (m / sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule. (AM10)

Solution: [By S(1/3) : $I = 1.06338$, By S(3/8) : $I = 1.06425$]

3.5 Numerical integration using Simpson's 3/8 rule

Simpson's three eighth rule (Simpson's 3/8 rule) ($n = 3$ in quadratic form)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \dfrac{3h}{8}\begin{bmatrix} (y_0 + y_n) \\ +2(y_3 + y_6 + \cdots) \\ +3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) \end{bmatrix} \\ \text{(or)} \\ \dfrac{3h}{8}\begin{bmatrix} \text{(first term + last term)} \\ +2\,\text{(suffices with a multiple of 3)} \\ +3\,\text{(remaining terms)} \end{bmatrix} \end{cases}$$

3.5.1 Part A

1. Under what condition, Simpson's 3/8 rule can be applied and state the formula. (AM2012)

   Solution: Condition : The number of intervals should be multiple of three.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \dfrac{3h}{8}\begin{bmatrix} (y_0+y_n) \\ +2\,(y_3+y_6+\cdots) \\ +3\,(y_1+y_2+y_4+y_5+y_7+\cdots) \end{bmatrix} \\ \text{(or)} \\ \dfrac{3h}{8}\begin{bmatrix} \text{(first term + last term)} \\ +2\,\text{(suffices with a multiple of 3)} \\ +3\,\text{(remaining terms)} \end{bmatrix} \end{cases}$$

3.5.2 Part B

Example 3.9. Compute the value of $\int_1^2 \frac{dx}{x}$ using Simpson's 3/8 rule

Solution: Let $y = f(x) = \dfrac{1}{x}$,

$h = 1/3$

The tabulated values of $y = f(x)$ are

	x_0	x_1	x_2	x_3
x	1	$4/3$	$5/3$	$6/3 = 2$
$f(x)$	$\frac{1}{1} = 1$	$\frac{1}{4/3} = 3/4 = .75$	$\frac{1}{5/3} = 0.6$	$\frac{1}{2} = 0.5$
	y_0	y_1	y_2	y_3

WKT, Simpson's 3/8 rule is

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_4)+3\,(y_1+y_2)+2\,(y_3)\right]$$

$$\text{i.e., } \int_1^2 \left(\frac{1}{x}\right) dx = \frac{3\,(1/3)}{8}\left[(1+0.5)+3\,(0.75+0.6)+2\,(0)\right]$$

$$= 0.69375$$

By actual integration,

$$\int_1^2 \frac{1}{x}\,dx = (\log_e x)_1^2 = (\ln 2 - \ln 1) = 0.69315$$

3.5.3 Single integrals by Trapezoidal, Simpson 1/3 & 3/8

Example 3.10. Compute the value of $\int_1^2 \frac{dx}{x}$ using (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule

Solution: Here $h = 0.25$,

$$y = f(x) = \frac{1}{x}$$

The tabulated values of $y = f(x)$ are

	x_0	x_1	x_2	x_3	x_4
x	1	1.25	1.50	1.75	2
$f(x)$	$\frac{1}{1} = 1$	$\frac{1}{1.25} = 0.8$	$\frac{1}{1.5} = 0.66667$	$\frac{1}{1.75} = 0.57143$	$\frac{1}{1.2} = 0.5$
	y_0	y_1	y_2	y_3	y_4

(a) WKT, Trapezoidal rule is

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$\Rightarrow \int_1^2 \left(\frac{1}{x}\right) dx = \frac{0.25}{2}\left[(1 + 0.5) + 2(0.8 + 0.66667 + 0.57143)\right]$$

$$= \frac{0.25}{2}\left[1.5 + 4.0762\right] = \frac{0.25}{2}\left[5.5762\right]$$

$$= 0.697025$$

(b) WKT, Simpson's 1/3 rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3) + 2(y_2)\right]$$

$$\Rightarrow \int_1^2 \left(\frac{1}{x}\right) dx = \frac{0.25}{3}\left[(1 + 0.5) + 4(0.8 + 0.57143) + 2(0.66667)\right]$$

$$= 0.693255$$

(c) WKT, Simpson's 3/8 rule is

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3)\right]$$

$$\text{i.e., } \int_1^2 \left(\frac{1}{x}\right) dx = \frac{3(0.25)}{8}\left[(1 + 0.5) + 3(0.8 + 0.66667) + 2(0.57143)\right]$$

$$= 0.66024$$

By actual integration,

$$\int_1^2 \frac{1}{x}\,dx = (\log_e x)_1^2 = (\ln 2 - \ln 1) = 0.69315$$

Example 3.11. Evaluate $\int_0^{10} \frac{dx}{1+x}$ by dividing the range into 8 equal parts by (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule

Solution: Here $h = 1.25,\ y = f(x) = \frac{1}{1+x}$

(a) I = 2.51368 , (b) I = 2.42200 , (c) I = 2.41838 , Actual integration, $\int_0^{10} \frac{dx}{1+x} = 2.39790.$

Example 3.12. Evaluate $\int_0^1 x\, e^x dx$ taking 4 equal intervals by (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule

Solution: (a) 1.02307 , (b) 1.00017 , (c) 0.87468 , AI = 1

Example 3.13. Calculate $\int_0^{\pi} \sin^3 x dx$ taking 7 ordinates (6 intervals) using a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule

Solution: (a)1.33467 , (b)1.32612 , (c)1.30516 , AI = 1.3333.

3.5.4 Anna University Questions

1. The velocity υ of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
υ (m / sec)	47	58	64	65	61	52	38

   Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule. (AM10)

   Solution: [S(1/3) : $I = 1.06338$, S(3/8) : $I = 1.06425$]

2. Evaluate $I = \int_0^6 \frac{1}{1+x} dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) Simpson's one-third rule (iv) Simpson's three-eighth rule. (ND11)

   Solution: [By Dir. Int. : $I = 1.9459$, Trap. : $I = 2.022$, S(1/3) : $I = 1.9587$, S(3/8) : $I = 1.966$]

3. Compute $\int_0^{\pi/2} \sin x \mathrm{dx}$ using Simpson's 3/8 rule. (ND12)

   Solution: [S(3/8) : $I = .9999988 \approx 1$]

4. The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time(min):	0	2	4	6	8	10	12
Velocity(km/hr):	0	22	30	27	18	7	0

   Using Simpson's $\frac{1}{3}$–rd rule find the distance covered by the car. (ND13)

   Solution: [By S(1/3) rule, the distance covered by the car : $I = 3.55$ km,]

5. Taking $h = 0.05$ evaluate $\int_1^{1.3} \sqrt{x} dx$ using Trapezoidal rule and Simpson's three-eighth rule. (AM14)

   Solution: [By Trap. : $I = 0.32147$, S(3/8) : $I = 0.321485354$]

6. The velocity υ of a particle at a distance α from a point on its path is given by the table: (ND14)

$s(ft)$	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet by using Simpson's $\frac{3}{8}$ rule.

3.6 Romberg's method

Romberg's method for a given interval $\left(I = \int_a^b f(x)dx\right)$

when $h = \dfrac{b-a}{2}$, by trapezoidal rule, we get I_1

when $h = \dfrac{b-a}{4}$, by trapezoidal rule, we get I_2

when $h = \dfrac{b-a}{8}$, by trapezoidal rule, we get I_3

$$\text{Romberg's formula for } I_1\&I_2 = I_{RM_{1,2}} = I_2 + \frac{(I_2 - I_1)}{3}$$

$$\text{Romberg's formula for } I_2\&I_3 = I_{RM_{2,3}} = I_3 + \frac{(I_3 - I_2)}{3}$$

$$\text{If } I_{RM_{1,2}} = I_{RM_{2,3}}, \text{ then we can equal } I = I_{RM_{1,2}} = I_{RM_{2,3}}$$

Note : Check, use actual integration, we get $I_{AI} = I = I_{RM_{1,2}} = I_{RM_{2,3}}$

3.6.1 Part A

1. State the Romberg's integration formula with h_1 and h_2. Further, obtain the formula when $h_1 = h$ and $h_2 = \frac{h}{2}$. (MJ2010)

   Solution : $I = \dfrac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$, where I_1 the value of the integral with h_1

   I_2 the value of the integral with h_2

   If $h_1 = h$ & $h_2 = \frac{h}{2}$ we get

$$I = \frac{4I_2 - I_1}{3} = I_2 + \frac{1}{3}(I_2 - I_1)$$

2. State Romberg's integration formula to find the value of $I = \int_a^b f(x)dx$ for first two intervals.(ND14)

3.6.2 Part B

Example 3.14. Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence obtain an approximate value for π.

Solution: To find I_1

When $h = \dfrac{2-0}{2} = 1, y = f(x) = \dfrac{1}{x^2+4}$

$$\text{Let } I = \int_0^2 \frac{dx}{x^2+4}$$

The tabulated values of y are

x	0	1	2
$f(x) = \dfrac{1}{x^2+4}$	$\dfrac{1}{0^2+4} = 0.25$	0.2	0.125
	y_0	y_1	y_2

Using Trapezoidal rule,

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}[(y_0+y_2)+2(y_1)]$$
$$= \frac{1}{2}[(0.25+0.125)+2(0.2)]$$
$$= 0.3875$$

<u>To find I_2</u>

$$h = \frac{2-0}{4} = \frac{1}{2} = 0.5$$

The tabulated values of y are

x	0	0.5	1	1.5	2
$f(x) = \dfrac{1}{x^2+4}$	0.25	0.23529	0.2	0.160	0.125
	y_0	y_1	y_2	y_3	y_4

Using Trapezoidal rule,

$$I_2 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]$$
$$= 0.25[(0.25+0.125)+2(0.23529+0.2+0.160)]$$
$$= 0.39136$$

<u>To find I_3</u>

$$h = \frac{2-0}{8} = \frac{1}{4} = 0.25$$

The tabulated values of y are

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x) = \dfrac{1}{x^2+4}$	0.25	0.24615	0.23529	0.21918	0.2	0.17918	0.160	0.14159	0.125
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Trapezoidal rule,

$$I_3 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)]$$
$$= 0.125\Big[(0.25+0.125)+2(0.24615+0.23529+0.21918+0.2+0.17918+0.16+0.14159)\Big]$$
$$= 0.39237$$

Romberg's formula for I_1&I_2 is

$$I_{\mathrm{RM}_{1,2}} = I_2 + \frac{(I_2-I_1)}{3} = 0.39136 + \frac{(0.39136-0.39136)}{3}$$
$$= 0.39265$$

Romberg's formula for $I_2 \& I_3$ is

$$I_{\mathrm{RM}_{2,3}} = I_3 + \frac{(I_3 - I_2)}{3} = 0.39237 + \frac{(0.39237 - 0.39237)}{3}$$
$$= 0.39271$$

$$\therefore I_{\mathrm{RM}_{1,2}} \cong 0.3927$$
$$I_{\mathrm{RM}_{2,3}} \cong 0.3927$$

Here $I_{\mathrm{RM}_{1,2}} \& I_{\mathrm{RM}_{2,3}}$ are almost equal and $I = 0.3927$ (1)

By actual integration,

$$\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{x=0}^{x=2} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8} \tag{2}$$

From (1) & (2),

$$\frac{\pi}{8} = 0.3927$$
$$\Rightarrow \pi = 8(0.3927) = 3.1416$$

Example 3.15. Using Romberg's method, evaluate $\int_0^1 \frac{1}{1+x} dx$ correct to 3 places of decimals.

Solution: $I_1 = 0.7083$

$I_2 = 0.6970$

$I_3 = 0.6941$

$I_{\mathrm{RM}_{1,2}} = 0.6932$

$I_{\mathrm{RM}_{2,3}} = 0.6931$

$\therefore I = 0.693$

3.6.3 Anna University Questions

1. Using Romberg's integration to evaluate $\int_0^1 \frac{dx}{1+x^2}$. (AM10)

   Solution: [0.7854]

2. Using Romberg's rule evaluate $\int_0^1 \frac{1}{1+x} dx$ correct to three decimal places by taking $h = 0.5, 0.25,$ and 0.125. (ND10)

   Solution: [0.6931]

3. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ using Romberg's method. (AM11,AM10)

   Solution: [0.7854]

4. Use Romberg's method to compute $\int_0^1 \frac{1}{1+x^2}dx$ correct to 4 decimal places. Also evaluate the same integral using tree-point Gaussian quadrature formula. Comment on the obtained values by comparing with the exact value of the integral which is equal to $\frac{\pi}{4}$. (MJ12)
Solution: [By Rom. : $I = 0.7854$, Dir. Int. : $I = 0.7853982$]

5. Evaluate $\int_0^{\frac{1}{2}} \frac{x}{\sin x}dx$ correct to three decimal places using Romberg's method. (AM14)
Solution: [0.5070676]

6. Evaluate $\int_0^1 \frac{dx}{1+x}$ and correct to 3 decimal places using Romberg's method and hence find the value of $\log_e 2$. (ND14)

3.7 Two point Gaussian quadrature formula

Two point Gaussian quadrature formula : (Guass two point formula)
Given $I = \int_a^b f(x)dx$
Case (i) If $a = -1, b = +1$, then

$$I = \int_{-1}^{1} f(x)dx = f\left[-\frac{1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right]$$

Case (ii) If $a = 0, b = 1$, then

$$I = \int_0^1 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f(x)\,dx, \text{ if } f(x) \text{ is an even function}$$

Case (iii) If $(a \neq -1 \& b \neq 1)$, then $x = \frac{b-a}{2}z + \frac{b+a}{2} = mz + c \Rightarrow dx = mdz$

$$I = \int_a^b f(x)dx = \int_{-1}^{1} f(z)\,mdz = m\int_{-1}^{1} f(z)\,dz$$
$$= m\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

3.7.1 Part A

1. Use two-point Gaussian quadrature formula to solve $\int_{-1}^{1} \frac{dx}{1+x^2}$ & also find error. (MJ2010,AM2012)
Solution : Given interval is -1 to 1 so we apply formula

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Here $f(x) = \dfrac{1}{1+x^2}$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

$$\therefore \int_{-1}^{1} \frac{1}{1+x^2}dx = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} = 1.5$$

But actual integration

$$\int_{-1}^{1} \frac{1}{1+x^2}dx = \left[\tan^{-1} x\right]_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \tan^{-1}(1) + \tan^{-1}(1)$$

$$= 2\tan^{-1}(1)$$

$$= 2\frac{\pi}{4} = \frac{\pi}{2}$$

$$= 1.5708$$

Here the error due to two-point formula is 0.0708.

2. Evaluate $\int_0^2 e^{-x^2} dx$ by two point Gaussian quadrature formula. (ND2010)

Solution : Here $f(x) = e^{-x^2}$ [Given range is not in exact form]

Let $x = \dfrac{b-a}{2}z + \dfrac{b+a}{2}$ [Here $a = 0, b = 2$]

$= \dfrac{2-0}{2}z + \dfrac{2+0}{2}$

$x = z + 1$ when $x = 0 \Rightarrow z = -1$

$dx = dz$ when $x = 2 \Rightarrow z = 1$

$$\int_0^2 e^{-x^2} dx = \int_{-1}^{1} e^{-(z+1)^2} dz$$

$$= f\left[\frac{-1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right] \quad (1)$$

Here $f(z) = e^{-(z+1)^2}$

$$f\left[\frac{-1}{\sqrt{3}}\right] = e^{-\left[\frac{-1}{\sqrt{3}}+1\right]^2} = e^{-0.1786} = 0.8364$$

$$f\left[\frac{1}{\sqrt{3}}\right] = e^{-\left[\frac{1}{\sqrt{3}}+1\right]^2} = e^{-2.488} = 0.0831$$

$$\therefore (1) \Rightarrow \int_0^2 e^{-x^2} dx = 0.8364 + 0.0831$$
$$= 0.9195$$

3. Write down two point Gaussian quadrature formula. (ND2011)

   Solution: $\left[\int_{-1}^{1} f(x)dx = f\left[-\frac{1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right]\right]$

4. Evaluate $\int_{-2}^{2} e^{\frac{-x}{2}}$ by Gauss two point formula. (ND2013)

   Solution: [4.6854]

3.7.2 Part B

Example 3.16. Evaluate $\int_1^2 \frac{dx}{x}$ by using Gaussian two point formula.

Solution: Here $a \neq -1, b = 2 \neq 1,$

so use $x = \dfrac{(b-a)z + (b+a)}{2}$

$x = (z+3)/2 \Rightarrow \ dx = dz/2$

$\therefore$ The above integral becomes

$$I = \int_1^2 \frac{dx}{x} = \int_{-1}^{1} \frac{dz/2}{(z+3)/2} = \int_{-1}^{1} \frac{1}{z+3} dz = \int_{-1}^{1} f(z)\, dz, \text{ where } f(z) = \frac{1}{z+3}$$

$\therefore$ By Gaussian two point formula

$$\int_{-1}^{1} f(z)\, dz = f\left(z = \frac{1}{\sqrt{3}}\right) + f\left(z = -\frac{1}{\sqrt{3}}\right)$$
$$= \left(\frac{1}{z+3}\right)_{z=\frac{1}{\sqrt{3}}} + \left(\frac{1}{z+3}\right)_{z=-\frac{1}{\sqrt{3}}}$$
$$= 0.693$$

Example 3.17. Using Gaussian two point formula evaluate $\int_0^{\pi/2} \log(1+x)dx$

Solution: $I = \displaystyle\int_{-1}^{1} \log\left[1 + \frac{\pi}{4}(1+z)\right] \frac{\pi}{4} dz$

$$= \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$
$$= 0.858$$

Example 3.18. Evaluate $\int_1^2 \frac{dx}{1+x^3}$ by using Gaussian two point formula.

Solution: $I = 0.2544.$

3.8 Three point Gaussian quadrature formula

Case (i) $\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$

Case (ii) $\int_{0}^{1} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f(x)dx$ [for even function $f(x)$]

$$= -\frac{1}{2}\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right\}$$

Case (iii) $(a \neq -1 \& b \neq 1)$, then $x = \frac{b-a}{2}z + \frac{b+a}{2} = mz + c \Rightarrow dx = m\,dz$

$$I = \int_{a}^{b} f(x)dx = \int_{-1}^{1} f(z)\,m\,dz = m\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right\}$$

3.8.1 Part A

1. Write down the three point Gaussian quadrature formula to evaluate $\int_{-1}^{1} f(x)dx$. (ND2012)

Solution: $\left[\text{Ans}: \int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right]$

3.8.2 Part B

Example 3.19. Evaluate $\int_{1}^{2} \frac{1}{1+x^3}dx$ by using Gaussian three point formula.

Solution: Using the substitution $x = \frac{(b-a)z + (b+a)}{2}, a = 1, b = 2$

$$x = \frac{z+3}{2}$$

$$I = \int_{-1}^{1} \frac{1}{1+\left(\frac{z+3}{2}\right)^3}\left(\frac{dz}{2}\right)$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1}{1+\left(\frac{z+3}{2}\right)^3}dz$$

$$= \frac{8}{2}\int_{-1}^{1} \frac{1}{8+(z+3)^3}dz$$

$$= \int_{-1}^{1} \frac{4}{8+(z+3)^3}dz, \qquad \text{where } f(z) = \frac{4}{8+(z+3)^3}$$

$$= \left\{\frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + \frac{8}{9}f(0)\right\}$$

$$= \frac{5}{9}[0.27505] + \frac{8}{9}(0.11429)$$

$$= 0.25439 \qquad \left[\because \frac{5}{9} = 0.5555, \frac{8}{9} = 0.8888\right]$$

$$I = 0.02544$$

Example 3.20. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by two and three point Gaussian quadrature formula & hence find the value of π.

Solution:

$$\text{Now}, I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{1}{2}\frac{1}{1+x^2}$$

$$= \frac{1}{2}\int_{-1}^{1} f(x)dx, \qquad \text{where } f(x) = \frac{1}{2}\frac{1}{1+x^2}$$

By Gaussian two point formula

$$I = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right] = 0.75 \tag{1}$$

By Guassain three point formula

$$I = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[\frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + \frac{8}{9}f(0)\right] = 0.79166 \tag{2}$$

By actual integration

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}(x)\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} \tag{3}$$

From (2) & (3),

$$0.79166 = \frac{\pi}{4}$$

$$\Rightarrow \pi = 3.16664$$

Example 3.21. Find $\int_0^{\frac{\pi}{2}} \sin x dx$ by two & three point Gaussian quadrature formula.

Solution: $x = \dfrac{\pi(z+1)}{4}$

$$I = 0.9985 \qquad \text{(by two point formula)}$$

$$I = 1.0000 \qquad \text{(by three point formula)}$$

Example 3.22. Find $\int_0^1 \frac{1}{t}dt$ by using Gaussian three point formula.

Solution: [$I = 1.6027$].

3.8.3 Anna University Questions

1. Evaluate $\int_1^2 \frac{1}{1+x^3}dx$ using Using Gauss three point formula. (AM11)
   Solution: [0.2544]

2. Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^2}dx$ by Gaussian three point formula. (MJ13)
   Solution: [1.5363]

3. Apply three point Gaussian quadrature formula to evaluate $\int_0^1 \frac{\sin x}{x}dx$. (ND13)
   Solution: [0.94616]

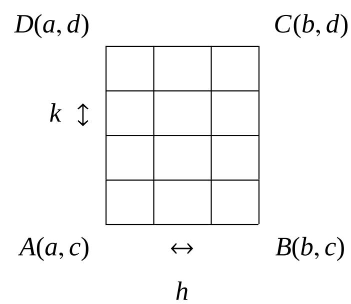
4. Evaluate $\int_1^2 \frac{dx}{1+x^2}$ using 3 point Gaussian formula. (ND14)

3.9 Double integrals by Trapezoidal

Double integration by trapezoidal rule is

Given $\int_c^d \int_a^b f(x)\, dx\, dy [= I(Say)]$

$$\text{i.e., } I = \frac{hk}{4}\left\{\begin{array}{l}[\text{sum of values of } f \text{ at the four corners}] \\ +2\left[\begin{array}{l}\text{sum of values of } f \text{ at the nodes} \\ \text{on the boundary except the corners}\end{array}\right] \\ +4\,[\text{sum of the values at the interior nodes}]\end{array}\right\}$$



where $h = \frac{b-a}{n}$, $k = \frac{d-c}{m}$

where $n =$ is number of equal intervals in (a,b).

where $m =$ is number of equal intervals in (c,d).

3.9.1 Part B

Example 3.23. Evaluate the integral $\int_1^{1.4}\int_2^{2.4} \frac{dxdy}{xy}$ using Trapezoidal rule. Verify your results by actual integration.

Solution: $f(x,y) = \frac{1}{xy}$, x varies from $(2, 2.4)$

y varies from $(1, 1.4)$

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4-2}{4} = 0.1,\ k = \frac{1.4-1}{4} = 0.1$$

The values of $f(x,y)$ at the nodal points are given in the table :

$y \backslash x$	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

By Trapezoidal rule for double integration

$$I = \frac{hk}{4}\left[\begin{array}{l}\text{Sum of values of } f \text{ at the four corners}\\ +2\left(\begin{array}{l}\text{Sum of values of } f \text{ at the nodes}\\ \text{on the boundary except the corners}\end{array}\right)\\ +4\,(\text{Sum of the values at the interior nodes})\end{array}\right]$$

$$= \frac{(0.1)(0.1)}{4}\left[\begin{array}{l}(0.5+0.4167+0.2976+0.3571)\\ +2\left(\begin{array}{l}0.4762+0.4545+0.4348+0.3788+0.3472+0.3205\\ +0.3106+0.3247+0.3401+0.3846+0.4167+0.4545\end{array}\right)\\ +4\left(\begin{array}{l}0.4329+0.4132+0.3953+0.3623+0.3344\\ +0.3497+0.3663+0.3698+0.3788\end{array}\right)\end{array}\right]$$

$$= 0.0614$$

By actual integration

$$\int_1^{1.4}\int_2^{2.4}\frac{1}{xy}dxdy = \int_1^{1.4}\left(\int_2^{2.4}\frac{1}{x}dx\right)\frac{1}{y}dy = \int_1^{1.4}(\log x)_2^{2.4}\frac{1}{y}dy$$

$$= (\log 2.4 - \log 2)(\log y)_1^{1.4}$$

$$= 0.0613$$

3.9.2 Anna University Questions

1. Evaluate $\int_1^{1.2}\int_1^{1.4}\frac{dxdy}{x+y}$ by trapezoidal formula by taking $h = k = 0.1$. **(AM10)**

   Solution: [0.0349]

2. Evaluate $\int_1^5\left[\int_1^4\frac{1}{x+y}dx\right]dy$ by Trapezoidal rule in $x-$direction with $h = 1$ and Simpson's one-third rule in $y-$direction with $k = 1$. **(ND10)**

   Solution: [By Trap. : $I = 2.4053$, Simp. : $I = 2.122$]

3. Evaluate $\int_0^1\int_0^1\frac{1}{x+y+1}dx\,dy$ by using Trapezoidal rule taking $h = 0.5$ and $k = 0.25$. **(AM11)**

   Solution: [$0.5319 \approx 0.532$]

4. Using Trapezoidal rule, evaluate $\int_1^2\int_1^2\frac{dxdy}{x^2+y^2}$ numerically with $h = 0.2$ along $x-$direction and $k = 0.25$ along $y-$direction. **(MJ12)**

   Solution: [0.2643]

5. Evaluate $\int_{2}^{2.4}\int_{4}^{4.4} xy\,dx\,dy$ by Trapezoidal rule taking $h = k = 0.1$. (ND13)

Solution: [1.4784]

6. Evaluate $\int_{0}^{1}\int_{0}^{1} \frac{1}{1+x+y} dxdy$. (ND14)

3.10 Double integrals by Simpson's 1/3 rules

Double integration by Simpson's 1/3 rule is Given $\int_{c}^{d}\int_{a}^{b} f(x)\,dx\,dy[= I(Say)]$

$$\text{i.e., } I = \frac{hk}{9}\left\{\begin{array}{l} [\text{sum of values of } f \text{ at the four corners}] \\ +2\left[\begin{array}{l}\text{sum of values of } f \text{ at the odd positions}\\ \text{on the boundary except the corners}\end{array}\right] \\ +4\left[\begin{array}{l}\text{sum of the values of } f \text{ at the even positions}\\ \text{on the boudary except the corners}\end{array}\right] \\ +4\left[\begin{array}{l}\text{sum of the values of } f \text{ at odd positions}\\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right] \\ +8\left[\begin{array}{l}\text{sum of the values of } f \text{ at even positions}\\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right] \\ +8\left[\begin{array}{l}\text{sum of the values of } f \text{ at odd positions}\\ \text{on the even rows of the matrix except boundary rows}\end{array}\right] \\ +16\left[\begin{array}{l}\text{sum of the values of } f \text{ at even positions}\\ \text{on the even rows of the matrix except boundary rows}\end{array}\right] \end{array}\right\}$$

3.10.1 Part B

Example 3.24. Evaluate the integral $\int_{1}^{1.4}\int_{2}^{2.4} \frac{dxdy}{xy}$ using Simpson's rule. Verify your results by actual integration.

Solution: $f(x,y) = \frac{1}{xy}$, x varies from $(1, 1.4)$

y varies from $(2, 2.4)$

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4-2}{4} = 0.1,\ k = \frac{1.4-1}{4} = 0.1$$

The values of $f(x,y)$ at the nodal points are given in the table :

y \ x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

By Extended Simpson's rule

$$I = \frac{hk}{9}\left[\begin{array}{l} \text{(Sum of the values of } f \text{ at the four corners)} \\ +\,2\,\text{(Sum of the values of } f \text{ at the odd positions on the boundary except the corners)} \\ +\,4\,\text{(Sum of the values of } f \text{ at the even positions on the boundary except the corners)} \\ +\,4\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the odd positions} \\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right) \\ +\,8\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the even positions} \\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right) \\ +8\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the odd positions} \\ \text{on the even rows of the matrix except boundary rows}\end{array}\right) \\ +16\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the even positions} \\ \text{on the even rows of the matrix except boundary rows}\end{array}\right) \end{array}\right]$$

$$= \frac{(0.1)(0.1)}{9}\left[\begin{array}{l} (0.5 + 0.4167 + 0.2976 + 0.3571) \\ +2\,(0.4545 + 0.3472 + 0.3247 + 0.4167) \\ +4\left(\begin{array}{c} 0.4762 + 0.4348 + 0.3788 + 0.3205 + 0.3106 \\ +0.3401 + 0.3846 + 0.4545 \end{array}\right) \\ +4\,(0.3788) \\ +8\,(0.3968 + 0.3623) \\ +8\,(0.3497 + 0.4132) \\ +16\,(0.3663 + 0.3344 + 0.4329 + 0.3953) \end{array}\right]$$

$$= \frac{0.01}{9}(55.2116) = 0.0613$$

By actual integration

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}dxdy = \int_1^{1.4}\left(\int_2^{2.4} \frac{1}{x}dx\right)\frac{1}{y}dy = \int_1^{1.4} (\log x)_2^{2.4}\,\frac{1}{y}dy$$

$$= (\log 2.4 - \log 2)\,(\log y)_1^{1.4}$$

$$= 0.0613$$

There is no error(or deviation) netween actual integration and Simpson's rule.

Example 3.25. Evaluate $I = \int_0^{0.5}\int_0^{0.5} \frac{\sin xy}{1+xy}dxdy$ by using Simpson's rule with step size 0.25.

Solution: [Ans : $I = 0.000216$(Simpson's rule)].

3.10.2 Double integrals by Trapezoidal, Simpson rule

Example 3.26. Evaluate the integral $\int_1^{1.4}\int_2^{2.4} \frac{dxdy}{xy}$ using Trapezoidal rule and Simpson's rule. Verify your results by actual integration.

Solution: [$I = 0.0614$(Trapezoidal rule), $I = 0.0613$(Simpson's rule)]

Example 3.27. Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x+y)dxdy$ by using trapezoidal rule, Simpson's rule and also by actual integration.

Solution: Divide into 2 intervals (3 ordinates) on $x\&y$

By Trapezoidal rule, $I = 1.7975$

Simpson's rule, $I = 2.0080$

Actual integration, $I = 2$

3.10.3 Anna University Questions

1. Evaluate $\int_1^5 \left[\int_1^4 \frac{1}{x+y} dx\right] dy$ by Trapezoidal rule in x–direction with $h = 1$ and Simpson's one-third rule in y–direction with $k = 1$. (ND10)

   Solution: [By Trap. : $I = 2.4053$, Simp. : $I = 2.122$]

2. Evaluate $\int_0^2 \int_0^1 4xy\,dx\,dy$ using Simpson's rule by taking $h = \frac{1}{4}$ and $k = \frac{1}{2}$. (ND12)

   Solution: [3.1111]

3. Evaluate $\int_1^{1.4} \int_2^{2.4} \frac{1}{xy} dx\,dy$ using Simpson's one-third rule. (MJ13)

   Solution: [0.0613]

4. Taking $h = k = \frac{1}{4}$, evaluate $\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\sin(xy)}{1+xy} dx\,dy$ using Simpson's rule . (AM14)

   Solution: [0.0141]

3.10.4 Assignment problems

1. Given the following data, find $y'(6)$ and the maximum value of y (if it exists). (AM10)

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922

2. Find $f'(x)$ at $x = 1.5$ and $x = 4.0$ from the following data using Newton's formulae for differentiation. (MJ12)

x :	1.5	2.0	2.5	3.0	3.5	4.0
$y = f(x)$:	3.375	7.0	13.625	24.0	38.875	59.0

3. Find the first three derivatives of $f(x)$ at $x = 1.5$ by using Newton's forward interpolation formula to the data given below. (MJ13)

x :	1.5	2	2.5	3	3.5	4
y :	3.375	7	13.625	24	38.875	59

4. Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{1}{(1+x^2)} dx$ by taking eight equal intervals. (MJ13)

5. The velocity v of a particle at a distance S from a point on its path is given by the table below: (AM10)

S (meter)	0	10	20	30	40	50	60
v (m / sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule.

6. Evaluate $I = \int_0^6 \frac{1}{1+x}dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) Simpson's one-third rule (iv) Simpson's three-eighth rule. (ND11)

7. Taking $h = 0.05$ evaluate $\int_1^{1.3} \sqrt{x}dx$ using Trapezoidal rule and Simpson's three-eighth rule. (AM14)

8. Use Romberg's method to compute $\int_0^1 \frac{1}{1+x^2}dx$ correct to 4 decimal places. Also evaluate the same integral using tree-point Gaussian quadrature formula. Comment on the obtained values by comparing with the exact value of the integral which is equal to $\frac{\pi}{4}$. (MJ12)

9. Evaluate $\int_0^{\frac{1}{2}} \frac{x}{\sin x}dx$ correct to three decimal places using Romberg's method. (AM14)

10. Evaluate $\int_0^1 \frac{dx}{1+x}$ and correct to 3 decimal places using Romberg's method and hence find the value of $\log_e 2$. (ND14)

11. Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^2}dx$ by Gaussian three point formula. (MJ13)

12. Apply three point Gaussian quadrature formula to evaluate $\int_0^1 \frac{\sin x}{x}dx$. (ND13)

13. Evaluate $\int_0^1\int_0^1 \frac{1}{x+y+1}dx\,dy$ by using Trapezoidal rule taking $h = 0.5$ and $k = 0.25$. (AM11)

14. Using Trapezoidal rule, evaluate $\int_1^2\int_1^2 \frac{dxdy}{x^2+y^2}$ numerically with $h = 0.2$ along x–direction and $k = 0.25$ along y–direction. (MJ12)

15. Evaluate $\int_2^{2.4}\int_4^{4.4} xy\,dx\,dy$ by Trapezoidal rule taking $h = k = 0.1$. (ND13)

16. Taking $h = k = \frac{1}{4}$, evaluate $\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \frac{\sin(xy)}{1+xy}dxdy$ using Simpson's rule . (AM14)

4 Initial Value Problems for Ordinary Differential Equations

Single Step methods – Taylor's series method – Euler's method – Modified Euler's method
– Fourth order Runge – Kutta method for solving first order equations
Multi step methods – Milne's and Adams – Bash forth predictor corrector methods
for solving first order equations

4.1 Introduction

Single Step methods	**Multi step methods**
Taylor's series method	Milne's forth predictor corrector method
Taylor's series method for simultaneous first ODE	Adams-Bash forth predictor corrector method
Taylor's series method for II order ODE	
Euler's method	
Modified Euler's method	
Runge-Kutta method for first order ODE	
Runge - Kutta method for solving II order ODE	
R. K. Method for simultaneous first order ODE	

4.2 Taylor's series method

$$\text{Given } \frac{dy}{dx} = y' = f(x,y) \text{ with } y(x_0) = y_0$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$
$$\text{i.e., } y'' = f_x + f_y(y')$$

lll^{ly} find $y''', y'''', \cdots$

Taylor's series expansion of $y(x)$ above $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$
$$= y_0 + \frac{(x-x_0)}{1!}y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \cdots$$

(or)

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots, \quad \text{where } x_1 = x_0 + h, h = x_1 - x_0$$
$$y(x_2) = y_2 = y_1 + \frac{h}{1!}y'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \cdots, \quad \text{where } x_2 = x_1 + h, h = x_2 - x_1$$

4.2.1 Part A

1. Using Taylor series method find $y(1.1)$ given that $y' = x + y, y(1) = 0$. (MJ2011)

Solution: Given $y' = x + y, y(x=1) = 0 \Rightarrow x_0 = 1,\ y_0 = 0, x_1 = 1.1, h = x_1 - x_0 = 1.1 - 1 = 0.1$

$\therefore$ Taylor's series formula is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \cdots \quad (1)$$

$y' = x + y$	$y'_0 = x_0 + y_{0(x_0=1,y_0=0)} = 1 + 0 = 1$		(2)
$y'' = 1 + y'$	$y''_0 = 1 + y'_0 = 1 + 1 = 2$	[by (2)]	(3)
$y''' = 0 + y'' = y''$	$y'''_0 = y''_0 = 2$	[by (3)]	(4)
$y^{iv} = y'''$	$y^{iv}_0 = y'''_0 = 2$	[by (4)]	

$$\therefore (1) \Rightarrow y_1 = 0 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$
$$\text{i.e., } y(1.1) = (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \cdots$$
$$= 0.1103083$$
$$\equiv 0.11031$$

2. What is the major drawback of Taylor series method? (AM2012)

Solution: The major drawbacks of Taylor series method are :

   1. In the differential equation, $\frac{dy}{dx} = f(x,y)$ may have a complicated algebraic structure.
   2. The evaluation of higher order derivatives may become tedious.

3. Find $y(0.1)$ if $\frac{dy}{dx} = 1 + y, y(0) = 1$ using Taylor series method. (ND2012)

Solution: Given $y' = 1 + y, y(x=0) = 1 \Rightarrow x_0 = 0,\ y_0 = 1, x_1 = 0.1, h = x_1 - x_0 = 0.1 - 0 = 0.1$

$\therefore$ Taylor's series formula is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \cdots \quad (1)$$

$y' = 1 + y$	$y'_0 = 1 + y_{0(x_0=0,y_0=1)} = 1 + 1 = 2$		(2)
$y'' = 0 + y' = y'$	$y''_0 = y'_0 = 2$	[by (2)]	(3)
$y''' = y''$	$y'''_0 = y''_0 = 2$	[by (3)]	(4)
$y^{iv} = y'''$	$y^{iv}_0 = y'''_0 = 2$	[by (4)]	

$$\therefore (1) \Rightarrow y_1 = 1 + \frac{0.1}{1!}(2) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$

$$\text{i.e., } y(1.1) = 1.210341667$$

$$\equiv 1.21034$$

4. Find $y(1.1)$ if $y' = x + y, y(1) = 0$ by Taylor series method. (AM2013)

   Solution: $[Ref : (MJ2011)]$

5. State the advantages and disadvantages of the Taylor's series method. (AM2014)

   Solution:

   The advantages of the Taylor's series method are :

   1. It is a powerful single step method if we are able to find the successiove derivatives easily.
   2. This method gives a straight forward adaptation of classic calculus to develop the solution as an infinite aseries.
   3. This method will be very useful for finding the starting values for powerful methods like Runge-Kutta method, Milne's method etc.

   The disadvantages of the Taylor's series method are :

   1. In the differential equation, $\frac{dy}{dx} = f(x,y)$ may have a complicated algebraic structure.
   2. The evaluation of higher order derivatives may become tedious.

4.2.2 Part B

Example 4.1. Using Taylor's series method, find y at $x = 0.1$, if $\frac{dy}{dx} = x^2y - 1, y(0) = 1$

Solution: Given $y' = x^2y - 1,\ x_0 = 0,\ y_0 = 1, h = 0.1$ (G)

$\therefore$ Taylor's series formula is

$$y_1 = y_0 + \frac{h}{1!}y' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots \quad (1)$$

$y' = x^2y - 1$	$y_0' = x_0^2y_0 - 1_{(x_0=0,y_0=1)} = 0 - 1 = -1$ [by (G)] (2)
$y'' = 2xy + x^2y'$	$y_0'' = 2x_0y_0 + x_0^2y_0' = 0$ [by (G) & (2)] (3)
$y''' = 2(xy' + y) + x^2y'' + y'2x$	$y_0''' = 2\left(x_0y_0' + y_0\right) + x_0^2y_0'' + y_0'2x_0 = 2$ [by (G) & (3)] (4)
$y^{iv} = 2y' + 2xy'' + 2y' + 2xy'' + x^2y'' + y''(2x) + 2y'$	$y_0^{iv} = 2y_0' + 2x_0y_0'' + 2y_0' + 2x_0y_0'' + x_0^2y_0'' + y_0''(2x_0) + 2y_0' = -6$ [by (G) & (4)]

$$\therefore (1) \Rightarrow y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(-6)$$

$$\text{i.e., } y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

$$= 0.900308$$

Example 4.2. Use Taylor series method to solve $\frac{dy}{dx} = x - y^2, y(0) = 1$ and hence compute $y(0.1)$ and $y(0.2)$ correct to 4 places of decimals.

Solution: Given $y' = x - y^2$ and $x_0 = 0, y_0 = 1$.

Take $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = y(0.1)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots \tag{1}$$

$y' = x - y^2$	$y_0' = x_0 - y_0^2$ $= 0 - 1 = -1$
$y'' = 1 - 2yy'$	$y_0'' = 1 - 2y_0y_0'$ $= 1 - 2(1)(-1) = 3$
$y''' = -2(yy'' + y'y')$ $= -2yy'' - 2y'^2$	$y_0''' = -2y_0y_0'' - 2y_0'^2$ $= -2(1)(3) - 2 = -8$
$y^{iv} = -2(yy''' + y''y') - 4y'y''$ $= -6y'y'' - 2yy'''$	$y_0^{iv} = -6y_0'y_0'' - 2y_0y_0'''$ $= -6(-1)(3) - 2(1)(-8) = 34$

From (1)

$$\therefore (1) \Rightarrow y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34)$$

$$= 1 - 0.1 + 0.015 - 0.001333 + 0.0001417$$

$$= 0.9138$$

$$\therefore y(0.1) = 0.9138$$

To find y_2

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

$$y_2 = y(x_2) = y(0.2)$$

The Taylor's series formula for y_2 is

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots \tag{2}$$

$y' = x - y^2$	$y_1' = x_1 - y_1^2$ $= 0.1 - (0.9138)^2 = -0.735$
$y'' = 1 - 2yy'$	$y_1'' = 1 - 2y_1y_1'$ $= 1 - 2(0.9138)(-0.735) = 2.3433$
$y''' = -2(yy'' + y'y')$ $= -2yy'' - 2y'^2$	$y_1''' = -2y_1y_1'' - 2y_1'^2$ $= -2(0.9138)(2.3433) - 2(-0.735)^2$ $= -5.363$

$y^{iv} = -2(yy''' + y''y') - 4y'y''$ $= -6y'y'' - 2yy'''$	$y_0^{iv} = -6y'_1y''_1 - 2y_1y'''_1$ $= -6(-0.735)(2.3433) - 2(0.9138)(-5.363)$ $= 20.1354$

From (2)

$$y_2 = 0.9138 + \frac{0.1}{1}(-0.735) + \frac{(0.1)^2}{2}(2.3433) + \frac{(0.1)^3}{6}(-5.363) + \frac{(0.1)^4}{24}(20.1354)$$

$$y_2 = 0.8512$$

$$\therefore y(0.2) = 0.8512$$

Example 4.3. Using Taylor series method, find y to five places of decimals when $x = 1.3$ given that $dy = \left(x^2y - 1\right)dx$ and $y = 2$ when $x = 1$.

Solution: Given $y' = x^2y - 1$ and $x_0 = 1, y_0 = 2$.

Given $h = 0.3$

$$x_1 = x_0 + h = 1 + 0.3 = 1.3$$

$$y_1 = y(x_1) = y(1.3)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \cdots \tag{1}$$

$y' = x^2y - 1$	$y'_0 = x_0^2y_0 - 1$ $= 1^2(2) - 1 = 1$
$y'' = x^2y' + 2xy$	$y''_0 = x_0^2y'_0 + 2x_0y_0$ $= 1^2(1) + 2(1)(2) = 5$
$y''' = x^2y'' + 2xy' + 2[xy' + y]$ $= x^2y'' + 4xy' + 2y$	$y'''_0 = x_0^2y''_0 + 4x_0y'_0 + 2y_0$ $= 1^2(5) + 4(1)(1) + 2(2) = 13$
$y^{iv}_0 = x^2y''' + 2xy'' + 4[xy'' + y'] + 2y'$ $= x^2y''' + 6xy'' + 6y'$	$y^{iv}_0 = x_0^2y'''_0 + 6x_0y''_0 + 6y'_0$ $= 1^2(13) + 6(1)(5) + 6(1) = 49$

From (1),

$$y_1 = 2 + \frac{0.3}{1!}(1) + \frac{(0.3)^2}{2!}(5) + \frac{(0.3)^3}{3!}(13) + \frac{(0.3)^4}{4!}(49)$$

$$= 2 + 0.3 + 0.225 + 0.0585 + 0.01654$$

$$= 2.60004$$

Example 4.4. Using Taylor series method find y at $x = 0.1$ correct to four decimal places from $\frac{dy}{dx} = x^2 - y, y(0) = 1$ with $h = 0.1$. Compute terms upto x^4.

Solution: Given $y' = x^2 - y$ and $x_0 = 0, y_0 = 1$.

Given $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = y(0.1)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \cdots \tag{1}$$

$y' = x^2 - y$	$y'_0 = x_0^2 - y_0$ $= 0^2 - 1 = -1$
$y'' = 2x - y'$	$y''_0 = 2x_0 - y'_0$ $= 2(0) - (-1) = 1$
$y''' = 2 - y''$	$y'''_0 = 2 - y''_0$ $= 2 - 1 = 1$
$y^{iv} = -y'''$	$y^{iv}_0 = -y'''_0$ $= -1$

From (1),

$$y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(-1)$$
$$= 1 - 0.1 + 0.005 + 0.000167 - 0.000004$$
$$= 0.905163$$

Example 4.5. Using Taylor series method, find $y(1.1)$ and $y(1.2)$ correct to four decimal places given $\frac{dy}{dx} = xy^{\frac{1}{3}}$ and $y(1) = 1$.

Solution: Given $y' = xy^{\frac{1}{3}}$ and $y(1) = 1$.

Given $h = 0.1$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

$$y_1 = y(x_1) = y(1.1)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots \tag{1}$$

$y' = xy^{\frac{1}{3}}$	$y'_0 = x_0 y_0^{\frac{1}{3}}$ $= 1(1)^{\frac{1}{3}} = 1$

$y'' = x\frac{1}{3}y^{-\frac{2}{3}}y' + y^{\frac{1}{3}}$ $= \frac{1}{3}x^2y^{-\frac{1}{3}} + y^{\frac{1}{3}}$	$y_0'' = \frac{1}{3}x_0^2y_0^{-\frac{1}{3}} + y_0^{\frac{1}{3}}$ $= \frac{1}{3}(1)^2(1)^{-\frac{1}{3}} + (1)^{\frac{1}{3}} = \frac{4}{3}$
$y''' = \frac{1}{3}\left[x^2\left(-\frac{1}{3}\right)y^{-\frac{4}{3}}y' + y^{-\frac{1}{3}}(2x)\right]$ $+\frac{1}{3}y^{-\frac{2}{3}}y'$ $= -\frac{1}{9}x^2y^{-\frac{4}{3}}y' + \frac{2}{3}xy^{-\frac{1}{3}} + \frac{1}{3}y^{-\frac{2}{3}}y'$	$y_0''' = -\frac{1}{9}x_0^2y_0^{-\frac{2}{3}}y_0' + \frac{2}{3}x_0y_0^{-\frac{1}{3}}$ $+\frac{1}{3}y_0^{-\frac{2}{3}}y_0'$ $= -\frac{1}{9}(1)^2(1)^{-\frac{2}{3}}(1)$ $+\frac{2}{3}(1)(1)^{-\frac{1}{3}} + \frac{1}{3}(1)^{-\frac{2}{3}}(1)$ $= \frac{8}{9}$

From (1),

$$y_1 = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}\left(\frac{4}{3}\right) + \frac{(0.1)^3}{3!}\left(\frac{8}{9}\right)$$

$$= 1 + 0.1 + 0.00666 + 0.000148$$

$$= 1.10681$$

$$\therefore y(1.1) = 1.10681$$

<u>To find y_2</u>

$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$y_1 = y(x_2) = y(1.2)$

The Taylor's series formula for y_2 is

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \tag{2}$$

$y' = xy^{\frac{1}{3}}$	$y_1' = x_1y_1^{\frac{1}{3}}$ $= (1.1)(1.10681)^{\frac{1}{3}} = 1.13785$
$y'' = \frac{1}{3}x^2y^{-\frac{1}{3}} + y^{\frac{1}{3}}$	$y_1'' = \frac{1}{3}x_1^2y_1^{-\frac{1}{3}} + y_1^{\frac{1}{3}}$ $= \frac{1}{3}(1.1)^2(1.10681)^{-\frac{1}{3}} + (1.10681)^{\frac{1}{3}}$ $= 1.42433$

$y''' = -\frac{1}{9}x^2y^{-\frac{4}{3}}y' + \frac{2}{3}xy^{-\frac{1}{3}} + \frac{1}{3}y^{-\frac{2}{3}}y'$	$y_1''' = -\frac{1}{9}x_1^2y_1^{-\frac{4}{3}}y_1' + \frac{2}{3}x_1y_1^{-\frac{1}{3}} + \frac{1}{3}y_1^{-\frac{2}{3}}y_1'$ $= -\frac{1}{9}(1.1)^2(1.10681)^{-\frac{4}{3}}(1.13785)$ $+\frac{2}{3}(1.1)(1.10681)^{-\frac{1}{3}}$ $+\frac{1}{3}(1.10681)^{-\frac{2}{3}}(1.13785)$ $= 0.92979$

From (2),

$$\begin{aligned} y_2 &= 1.10681 + \frac{0.1}{1!}(1.13785) + \frac{(0.1)^2}{2!}(1.42433) + \frac{(0.1)^3}{3!}(1.92979) \\ &= 1.2277 \\ \therefore y(1.2) &= 1.2277 \end{aligned}$$

Example 4.6. Find the Taylor series solution with three terms for the initial value problem $\frac{dy}{dx} = x^3 + y, y(1) = 1$.

Solution: Given $y' = x^3 + y$ and $x_0 = 1, y_0 = 1$.

The Taylor's series formula for y_1 is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \tag{1}$$

$y' = x^3 + y$	$y_0' = x_0^3 + y_0$ $= 1^3 + 1 = 2$
$y'' = 3x^2 + y'$	$y_0'' = 3x_0^2 + y_0'$ $= 3\left(1^2\right) + 2 = 5$
$y''' = 6x + y''$	$y_0''' = 6x_0 + y_0''$ $= 6(1) + 5 = 11$

Substituting in (1)

$$\begin{aligned} y_1 &= 1 + \frac{x-1}{1!}(2) + \frac{(x-1)^2}{2!}(5) + \frac{(x-1)^3}{3!}(11) \\ &= 1 + 2(x-1) + \frac{5}{4}(x-1)^2 + \frac{11}{6}(x-1)^3 \end{aligned}$$

Example 4.7. Using Taylor's method, compute $y(0.2)$ and $y(0.4)$ correct to 4 decimal places given $\frac{dy}{dx} (= y') = 1 - 2xy$ & $y(0) = 0$.

Solution: $y(0.2) = 0.194752003, y(0.4) = 0.359883723$

Example 4.8. Find the Taylor series solution of $y(0.1)$ given that $\frac{dy}{dx} + y^2 = e^x, y(0) = 1$. Compute using first five terms.

Solution: Given $y' = e^x - y^2$ and $x_0 = 0, y_0 = 1$.

Take $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = y(0.1)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \frac{h^5}{5!}y^{v}_0 + \cdots \tag{1}$$

$y' = e^x - y^2$	$y'_0 = e^{x_0} - y_0^2 = e^0 - 1^2 = 0$
$y'' = e^x - 2yy'$	$y''_0 = e^{x_0} - 2y_0y'_0 = e^0 - 2(1)(0) = 1$
$y''' = e^x - 2[yy'' + y'y'] = e^x - 2[yy'' + y'^2]$	$y'''_0 = e^{x_0} - 2[y_0y''_0 - y'^2_0] = e^0 - 2[1(1) + 0^2] = -1$
$y^{iv} = e^x - 2[yy''' + y''y' + 2y'y''] = e^x - 2[y'y''' + 3y'y'']$	$y^{iv}_0 = e^{x_0} - 2[y'_0y'''_0 + 3y'_0y''_0] = e^0 - 2[1(-1) + 3(0)(1)] = 3$
$y^{v} = e^x - 2[yy^{iv} + y'''y' + 3(y'y''' + y''y'')] = e^x - 2[y'y^{iv} + 4y'y''' + 3y''^2]$	$y^{v}_0 = e^{x_0} - 2[y'_0y^{iv}_0 + 4y'_0y'''_0 + 3y''^2_0] = 1 - 2[1(3) + 4(0)(-1) + 3(1)^2] = -11$

From (1)

$$\begin{aligned} y_1 &= 1 + \frac{0.1}{1}(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(-1) + \frac{(0.1)^4}{24}(3) + \frac{(0.1)^5}{120}(-11) \\ &= 1 + 0 + 0.005 - 0.000167 + 0.000125 - 0.000000917 \\ &= 1.00496 \end{aligned}$$

$$\therefore y(0.1) = 1.00496$$

Example 4.9. Using Taylor's series method, with the first five terms in the expansion; find $y(0.1)$ correct to 3 decimal places, given that $\frac{dy}{dx} = e^x - y^2, y(0) = 1$

Solution: $y(0.1) = 1.0049891 \cong 1.005$ (correct to 3 decimal places)

4.2.3 Anna University Questions

1. Evaluate the value of y at $x = 0.1, 0.2$ given $\frac{dy}{dx} = x^2y - 1, y(0) = 1$, by Taylor's series method upto four terms. (ND10)

   Solution: $[y(0.1) = 0.900308, y(0.2) = 0.802269]$

2. Using Taylor series method to find $y(0.1)$ if $y' = x^2 + y^2, y(0) = 1$. (MJ13)

   Solution: $[y(0.1) = 1.1115]$

3. Obtain y by Taylor series method, given that $y' = xy + 1, y(0) = 1$, for $x = 0.1$ and 0.2 correct to four decimal places. (ND13)

   Solution: $[y(0.1) = 1.1053, y(0.2) = 1.2224]$

4. Using Taylor's series method, find y at $x = 1.1$ by solving the equation $\frac{dy}{dx} = x^2 + y^2; y(1) = 2$. Carryout the computations upto fourth order derivative. (AM14)

   Solution: $[y(0.1) = 2.64333]$

5. Using Taylor's series method, find y at $x = 0$ if $\frac{dy}{dx} = x^2y - 1, y(0) = 1$. (ND14)

4.3 Taylor's series method for simultaneous first order differential equations

Given $\frac{dy}{dx} = f_1(x,y,z), \frac{dz}{dx} = f_2(x,y,z)$ with initial conditions $y(x_0) = y_0, z(x_0) = z_0$

4.3.1 Part B

Example 4.10. Solve the system of equations $\frac{dy}{dx} = z - x^2, \frac{dz}{dx} = y + x$ with $y(0) = 1, z(0) = 1$ by taking $h = 0.1$ to get $y(0.1)$ and $z(0.1)$.

Solution: Given $x_0 = 0, y_0 = 1, z_0 = 1$

WKT, Taylor's series for y_1 is

$$y_1 = y(0.1) = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \cdots \tag{1}$$

& Taylor's series for z_1 is

$$z_1 = z(0.1) = z_0 + \frac{h}{1!}z'_0 + \frac{h^2}{2!}z''_0 + \frac{h^3}{3!}z'''_0 + \frac{h^4}{4!}z^{iv}_0 + \cdots \tag{2}$$

$y' = \frac{dy}{dx} = z - x^2$	$y'_0 = z_0 - x_0^2 = 1 - 0 = 1$	$z' = \frac{dz}{dx} = x + y$	$z'_0 = x_0 + y_0 = 0 + 1 = 1$
$y'' = z' - 2x$	$y''_0 = 1$	$z'' = 1 + y'$	$z''_0 = 1 + y'_0 = 2$
$y''' = z'' - 2$	$y'''_0 = 0$	$z''' = y''$	$z'''_0 = 1$
$y^{iv} = z'''$	$y^{iv}_0 = 1$	$z^{iv} = y'''$	$z^{iv}_0 = 0$

$$\therefore (1) \Rightarrow y(0.1) = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(1)$$
$$= 1.1050 \quad \text{[correct to 4 decimal places]}$$

$$\text{Now } (2) \Rightarrow z(0.1) = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(0)$$
$$= 1.110167$$
$$\simeq 1.1102 \quad \text{[correct to 4 decimal places]}$$

4.4 Taylor's series for II order differential equations

4.4.1 Part B

Example 4.11. By Taylor's series, find $y(0.1)$ and $y(0.2)$ given that $y'' = y + xy', y(0) = 1, y'(0) = 0$.

Solution: Given $x_0 = 0, y_0 = 1, y'_0 = 0$ & $y'' = y + xy'$

WKT, Taylor's series formula is

$$y(x) = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \frac{x^4}{4!}y^{iv}_0 + \cdots \tag{1}$$

$y'' = y + xy'$	$y''_0 = y_0 + x_0 y'_0 = 1 + 0(0) = 1$
$y''' = y' + xy'' + y' = 2y' + xy''$	$y'''_0 = 2(0) + (0)(1) = 0$
$y^{iv} = 2y'' + xy''' + y'' = 3y'' + xy'''$	$y^{iv}_0 = 3(1) + (0)(0) = 3$

$$\begin{aligned}
\therefore (1) \Rightarrow y(x) &= 1 + 0 + \frac{x^2}{2!}(1) + 0 + \frac{x^4}{4!}(3) + \cdots \\
&= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \cdots \\
y(0.1) &= 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8} + \cdots \\
&= 1.0050125 \simeq 1.0050 \qquad \text{(correct to 4 decimal places)} \\
y(0.2) &= 1 + \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} + \cdots \\
&= 1.0202
\end{aligned}$$

4.5 Euler's method

Given $y' = f(x, y), x_0, y_0, h$

Euler algorithm is

$$\begin{aligned}
y_1 &= y_0 + hf(x_0, y_0) \\
y_2 &= y_1 + hf(x_1, y_1) \\
y_3 &= y_2 + hf(x_2, y_2) \\
&\vdots \\
\text{In general, } y_n &= y_{n-1} + hf(x_{n-1}, y_{n-1})
\end{aligned}$$

4.5.1 Part A

1. Use Euler's method to find $y(0.2)$ and $y(0.4)$ given $\frac{dy}{dx} = x + y, y(0) = 1$. (MJ2010)

   Solution : Given $f(x, y) = x + y, x_0 = 0, y_0 = 1, x_1 = 0.2, x_2 = 0.4$. (Here $h = x_1 - x_0 = x_2 - x_1 = 0.2$)

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.2)[x_0 + y_0] = 1 + (0.2)[0 + 1]$$

$$\text{i.e., } y_1 = y(0.2) = 1.2$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.2 + (0.2)[x_1 + y_1] = 1.2 + (0.2)[0.2 + 1.2]$$

$$= 1.2 + 0.28$$

$$\text{i.e., } y_2 = y(0.4) = 1.48$$

2. Find $y(0.1)$ by using Euler's method given that $\frac{dy}{dx} = x + y, y(0) = 1$. (ND10)

Solution : Given, $\frac{dy}{dx} = x + y = f(x,y), x_0 = 0, y_0 = 1, x_1 = 0.1, h = 0.1$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.1)[x_0 + y_0]$$

$$= 1 + (0.1)[0 + 1]$$

$$= 1 + 0.1$$

$$= 1.01$$

$$\text{i.e., } y(0.1) = 1.01$$

3. Find $y(0.2)$ for the equation $y' = y + e^x$, given that $y(0) = 0$ by using Euler's method. (AM11)

Solution : Given, $f(x,y) = y + e^x, x_0 = 0, y_0 = 0, h = 0.2$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 0 + 0.2 f(0, 0)$$

$$= 0.2\left[0 + e^0\right] = 0.2$$

$$\text{i.e., } y(0.2) = 0.2$$

4. State Euler's method to solve $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$. (ND11)

Solution: $[y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})]$

5. Using Euler's method, find the solution of the initial value problem $\frac{dy}{dx} = \log(x + y)$, $y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$. (MJ12)

Solution : Given, $\frac{dy}{dx} = \log(x + y) = f(x,y), x_0 = 0, y_0 = 2, h = 0.2$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + 0.2 f(0, 2)$$

$$= 2 + 0.2[\log(0 + 2)] = 2 + 0.2[\log 2] = 2 + 0.2(0.30103)$$

$$\text{i.e., } y(0.2) = 2.060206$$

6. State Euler's formula. (MJ13)

   Solution: $[y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})]$

7. Using Euler's method find the solution of the initial value problem $y' = y - x^2 + 1, y(0) = 0.5$ at $x = 0.2$ taking $h = 0.2$. (ND13)

   Solution : Given, $y' = y - x^2 + 1 = f(x, y), x_0 = 0, y_0 = 0.5, h = 0.2$

   By Euler algorithm,

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 0.5 + 0.2f(0, 0.5) \\ &= 0.5 + 0.2\left[y_0 - x_0^2 + 1\right] = 0.5 + 0.2\left[0.5 - 0^2 + 1\right] = 2 + 0.2(1.5) \\ \text{i.e., } y(0.2) &= 0.8 \end{aligned}$$

8. Given $y' = x + y, y(0) = 1$, find $y(0.1)$ by Euler's method. (ND14)

4.5.2 Part B

Example 4.12. Using Euler's method, find $y(0.2)$, $y(0.4)$ and $y(0.6)$ from $\frac{dy}{dx} = x + y, y(0) = 1$ with $h = 0.2$.

Solution: Given $y(0)\,[(\text{or})\,y_0] = 1 \Rightarrow x_0 = 0$

$$\begin{aligned} &\& x_1 = 0.2, x_2 = 0.4, x_3 = 0.4, h = 0.2 \\ &y' = x + y \end{aligned}$$

We have to find $y(0.2) = y_1$

Now, by Euler algorithm,

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ \text{i.e., } y(0.2) &= [1 + (0.2)(x + y)]_{\binom{x=x_0}{y=y_0}} = 1 + 0.2(x_0 + y_0) = 1 + (0.2)(0 + 1) \\ y(0.2) &= 1.2 = y_1 \\ y_2 &= y_1 + hf(x_1, y_1) = 1.2 + (0.2)[x_1 + y_1] \\ &= 1.2 + 0.2[0.2 + 1.2] = 1.48 \Rightarrow y(0.4) = 1.48 \\ y_3 &= y_2 + hf(x_2, y_2) = 1.48 + (0.2)[x_2 + y_2] \\ &= 1.856 \Rightarrow y(0.6) = 1.856 \end{aligned}$$

∴ The result of the problem:

x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.2$	$y_1 = y_0 + hf(x_0, y_0) = 1.2$
$x_2 = 0.4$	$y_2 = y_1 + hf(x_1, y_1) = 1.48$
$x_3 = 0.4$	$y_3 = y_2 + hf(x_2, y_2) = 1.856$

Example 4.13. Using Euler's method solve $y' = x + y + xy, y(0) = 1$ compute y at $x = 0.1$ by taking $h = 0.05$.

Solution: Given $y' = x + y + xy, x_0 = 0, y_0 = 1$
$h = 0.05$.

$$x_1 = x_0 + h = 0 + 0.05 = 0.05$$
$$y_1 = y(x_1) = y(0.05)$$

To find y_1

By Euler algorithm

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.05f(0, 1) \\ &= 1 + 0.05[0 + 1 + (0)(1)] \\ y_1 &= 1.05 \\ \therefore y(0.05) &= 1.05 \end{aligned}$$

$$x_2 = x_1 + h = 0.05 + 0.05 = 0.1$$
$$y_2 = y(x_2) = y(0.1)$$

To find y_2

By Euler algorithm

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) \\ &= 1.05 + 0.05f(0.05, 1.05) \\ &= 105 + 0.05[0.05 + 1.05 + (0.05)(1.05)] \\ y_2 &= 1.10762 \\ \therefore y(0.1) &= 1.10762 \end{aligned}$$

Example 4.14. Using Euler's method find $y(0.3)$ of $y(x)$ satisfies the initial value problem $\frac{dy}{dx} = \frac{1}{2}\left(x^2 + 1\right)y^2, y(0.2) = 1.1114$.

Solution: Given $f(x, y) = \frac{1}{2}\left(x^2 + 1\right)y^2$ and $x_0 = 0.2, y_0 = 1.1114$.
Take $h = 0.1$

$$x_1 = x_0 + h = 0.2 + 0.1 = 0.3$$
$$y_1 = y(x_1) = y(0.3)$$

To find y_1

By Euler's method

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1.1114 + 0.1f(0.2, 1.1114) \end{aligned}$$

$$= 1.1114 + 0.1\left[\frac{1}{2}\left((0.2)^2 + 1\right)(1.1114)^2\right]$$

$$y_1 = 1.1756$$

$$\therefore y(0.3) = 1.1756$$

Example 4.15. Using Euler's method with $h = 0.1$ to solve the equation $\frac{dy}{dx} = \frac{y}{1+x}$, $y(0) = 2$ in the range $0 \le x \le 1$.

Solution: Given $f(x,y) = \frac{y}{1+x}$ and $x_0 = 0, y_0 = 2, h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = y(0.1)$$

<u>To find y_1</u>

By Euler algorithm

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 2 + 0.1 f(0, 2) \\ &= 2 + 0.1\left[\frac{2}{1+0}\right] \\ y_1 &= 2.2 \\ \therefore y(0.1) &= 2.2 \end{aligned}$$

Similarly

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ $y_2 = y(x_2) = y(0.2)$ $\therefore y_2 = y_1 + hf(x_1, y_1)$ $= 2.2 + 0.1 f(0.1, 2.2)$ $= 2.2 + 0.1\left[\frac{2.2}{1+0.1}\right]$ $y_2 = 2.4$ $\therefore y(0.2) = 2.4$	$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$ $y_3 = y(x_3) = y(0.3)$ $\therefore y_3 = y_2 + hf(x_2, y_2)$ $= 2.4 + 0.1 f(0.2, 2.4)$ $= 2.4 + 0.1\left[\frac{2.4}{1+0.2}\right]$ $y_3 = 2.6$ $\therefore y(0.3) = 2.6$	$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$ $y_4 = y(x_4) = y(0.4)$ $\therefore y_4 = y_3 + hf(x_3, y_3)$ $= 2.6 + 0.1 f(0.3, 2.6)$ $= 2.6 + 0.1\left[\frac{2.4}{1+0.3}\right]$ $y_4 = 2.8$ $\therefore y(0.3) = 2.8$
$y_5 = y_4 + hf(x_4, y_4)$ $= 2.8 + 0.1 f(0.4, 2.8)$ $= 2.8 + 0.1\left[\frac{2.8}{1+0.4}\right]$ $y_5 = 3$ $\therefore y(0.5) = 3$	$y_6 = y_5 + hf(x_5, y_5)$ $= 3 + 0.1 f(0.5, 3)$ $= 3 + 0.1\left[\frac{3}{1+0.5}\right]$ $y_6 = 3.2$ $\therefore y(0.6) = 3.2$	$y_7 = y_6 + hf(x_6, y_6)$ $= 3.2 + 0.1 f(0.6, 3.2)$ $= 3.2 + 0.1\left[\frac{3.2}{1+0.6}\right]$ $y_7 = 3.4$ $\therefore y(0.7) = 3.4$

$y_8 = y_7 + hf(x_7, y_7)$	$y_9 = y_8 + hf(x_8, y_8)$	$y_{10} = y_9 + hf(x_9, y_9)$
$= 3.4 + 0.1 f(0.7, 3.4)$	$= 3.6 + 0.1 f(0.8, 3.6)$	$= 3.8 + 0.1 f(0.9, 3.8)$
$= 3.4 + 0.1\left[\dfrac{3.4}{1+0.7}\right]$	$= 3.6 + 0.1\left[\dfrac{3.6}{1+0.8}\right]$	$= 3.8 + 0.1\left[\dfrac{3.8}{1+0.9}\right]$
$y_8 = 3.6$	$y_9 = 3.8$	$y_{10} = 4$
$\therefore y(0.8) = 3.6$	$\therefore y(0.9) = 3.8$	$\therefore y(1) = 4$

Example 4.16. Solve $\dfrac{dy}{dx} = y + e^x, y(0) = 0$ for $x = 0.2, 0.4$ by using Euler's method.

Solution: $[y_1 = 0.2, y_2 = 0.484281]$

Example 4.17. Using Euler's method, find the solution of the initial value problem $\dfrac{dy}{dx} = \log(x+y), y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$.

Solution: $[y(0.2) = 2.0602]$

4.5.3 Anna University Questions

1. Solve $y' = \dfrac{y-x}{y+x}, y(0) = 1$ at $x = 0.1$ by taking $h = 0.02$ by using Euler's method. (MJ13)
   Solution: $[y(0.02) = 1.02, y(0.04) = 1.0392, y(0.06) = 1.0577, y(0.08) = 1.0756, \therefore y(0.1) = 1.0928]$

4.6 Modified Euler's method

Modified Euler's method formula is

$$\text{In general, } y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]$$

$$\text{When } n = 0, y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$\text{When } n = 1, y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x, y)\right]$$

$$\text{When } n = 2, y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f(x_2, y_2)\right]$$

4.6.1 Part B

Example 4.18. Solve $y' = 1 - y, y(0) = 0$ by modified Euler method.

Solution: Given $y' = f(x, y) = 1 - y, x_0 = 0, y_0 = 0$.

Let $h = 0.1, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$

We have to find y_1, y_2, y_3.

By Modified Euler's method:

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{1}{2}hf(x_n, y_n)\right]$$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right] \qquad [\text{Here } f(x_0, y_0) = f(x = x_0, y = y_0) = 1 - y_0 = 1 - 0 = 1]$$

$$\therefore y_1 = 0 + (0.1)\, f\left[0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}(1)\right]$$

$$= 0.1 f[0.05, 0.05]$$

$$= \{(0.1)[1 - y]\}_{\binom{x=0.05}{y=0.05}} = 1 - 0.05$$

$$y_1 = 0.095$$

$$\text{Now } y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1, y_1)\right] \qquad [\text{Here } f(x_1, y_1) = 1 - y_1 = 0.905]$$

$$\text{i.e., } y_2 = 0.905 + (0.1)\, f\left[0.1 + \frac{0.1}{2}, 0.095 + \frac{0.1}{2}f(0.1, 0.095)\right]$$

$$y_2 = 0.18098$$

$$\text{Now } y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2}f(x_2, y_2)\right] \qquad [\text{Here } f(x_2, y_2) = 1 - y_2 = 1 - 0.18098 = 0.81902]$$

$$\text{i.e., } y_3 = 0.18098 + (0.1)\, f\left[0.2 + \frac{0.1}{2}, 0.18098 + \frac{0.1}{2}f(0.2, 0.18098)\right]$$

$$y_3 = 0.258787$$

$$\boxed{\therefore y_0 = 0, y_1 = 0.095, y_2 = 0.18098, y_3 = 0.258787}$$

Example 4.19. By using modified Euler method, find $y(1.8)$ if $\frac{dy}{dx} = \frac{x - y}{x + y}, y(2) = 1$ [Hint $h = -0.2$]

Solution: [$y(1.8) = 0.9349$].

Example 4.20. Given $\frac{dy}{dx} + y - x^2 = 0, y(0) = 1, y(0.1) = 0.9052, y(0.2) = 0.8213$, find correct to four decimal places $y(0.3)$, using modified Euler's method.

Solution: $\left[\text{Hint } y' = x^2 - y\right]$ $\left[\text{Ans}: y(0.3) = y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2}f(x_2, y_2)\right] = 0.7493\right]$

4.6.2 Anna University Questions

1. Using Modified Euler's method, find $y(4.1)$ and $y(4.2)$ if $5x\frac{dy}{dx} + y^2 - 2 = 0; y(4) = 1.$ (ND12)

   Solution: [$y(4.1) = 1.005,\ y(4.2) = 1.0098$]

2. Apply modified Euler's method to find $y(0.2)$ and $y(0.4)$ given $y' = x^2 + y^2$, $y(0) = 1$ by taking $h = 0.2$. (ND14)

4.7 Fourth order Runge-Kutta method for solving first order equations

Fourth order Runge-Kutta method for solving I order Differential Equations

[Single step method]

$$\text{Given } \frac{dy}{dx} = y' = f(x,y)$$
$$\& \ y(x_0) = y_0$$

We have to find $y(x_1) =? = y_1, y(x_2) =? = y_2, y(x_3) =? = y_3, \cdots$

To find $f(x_{n+1}) = y(x_{n+1}) = y_{n+1}$:

$$y_{n+1} = y(x_n + h) = y(x_n) + \Delta y = y_n + \Delta y$$
$$\text{where } \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$\text{where } k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

To find $f(x_1) = y(x_1) = y_1$

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$\therefore y_1 = y(x_0 + h) = y(x_0) + \Delta y = y_0 + \Delta y$$

4.7.1 Part A

1. State the fourth order Runge - Kutta algorithm. (ND2012)

   Solution: The fourth order Runge - Kutta algorithm is

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$\therefore y_1 = y(x_0 + h) = y(x_0) + \Delta y = y_0 + \Delta y$$

4.7.2 Part B

Example 4.21. Given $\frac{dy}{dx} = x + y^2, y(0) = 1$, find $y(0.1)$ & $y(0.2)$ by Runge-Kutta method for IV order.

Solution: Given $y' = f(x,y) = x + y^2, h = 0.1$

$$\& x_0 = 0, y(x_0 = 0) = 1$$

To find $y(0.1) = y_1$

$$k_1 = hf[x_0, y_0] = (0.1)f(0.1) = (0.1)\left[0 + 1^2\right] = 0.1$$
$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = (0.1)f[0 + 0.05, (1 + 0.05)] = (0.1)\left[0.05 + (1.05)^2\right] = 0.11525$$
$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = (0.1)f\left[0.05, 1 + \frac{0.11525}{2}\right] = (0.1)\left[0.05 + (1.057625)^2\right] = 0.116857$$
$$k_4 = hf[x_0 + h, y_0 + k_3] = (0.1)f[0.05, 1 + 0.116857] = (0.1)\left[0.05 + (1.116857)^2\right] = 0.134737$$
$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \frac{1}{6}[(0.1) + 2(0.1) + 2(0.11525) + (0.116857)]$$
$$= 0.11649$$

$\therefore y(0.1) = y_1 = y_0 + \Delta y = 1 + 0.11649 = 1.11649 \Rightarrow y_1 = 1.11649$

To find $y(0.2) = y_2$

$$k_1 = hf[x_1, y_1] = (0.1)f[0.1, 1.11649] = (0.1)[0.1 + (1.116492)] = 0.1347$$
$$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = (0.1)f\left[0.1 + 0.05, 1.11649 + \frac{0.1347}{2}\right] = (0.1)f[0.15, 1.18385] = 0.1552$$
$$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right] = (0.1)f\left[0.1 + 0.05, 1.11649 + \frac{0.1552}{2}\right] = (0.1)f[0.15, 1.1941] = 0.1576$$
$$k_4 = hf[x_1 + h, y_1 + k_3] = (0.1)f[0.1 + 0.1, 1.11649 + 0.1576] = 0.18233$$
$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.1571$$
$$y_2 = y_1 + \Delta y = 1.11649 + 0.1571 = 1.27359$$

i.e.,

x	0	0.1	0.2
y	1	1.11649	1.27359

Example 4.22. Using R.K. method of fourth order solve for $x = 1.4$, from $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$ with $y = 0$ at $x = 1$.

Solution: $[y_1 = y(1.2) = 0.140, \ \therefore y_2 = y(1.4) = 0.27, \text{ with } h = 0.2]$

Example 4.23. Solve y for $x = 0.2, 0.4$ from the given O.D.E. $\frac{dy}{dx} = \sqrt{x^2 + y}$ and $y = 0.8$ at $x = 0$.

Solution: $[y_1 = y(0.2) = 0.99029, \ y_2 = y(0.4) = 0.120828]$

4.7.3 Anna University Questions

1. Using Runge-Kutta method of order four, find y when $x = 1.2$ in steps of 0.1 given that $y' = x^2 + y^2$ and $y(1) = 0.5$. (ND13)

   Solution: $[y_1 = y(1.1) = y_0 + \nabla y = 0.6428, \ \therefore y_2 = y(1.2) = y_1 + \nabla y = 0.8278$ with $h = 0.1]$

4.8 Fourth order Runge - Kutta method for solving II order differential equation

Given $f(y'', y', y) = g(x)$ with $y(x_0), y'(x_0)$. Find $y(x_1) = y_1, y'(x_1) = y_1'$ [or $z(x_1) = z_1$].

Let us set $y' = z = f_1(x, y, z)$

$y'' = f_2(x, y, z)$

To find y_1	**To find z_1**
$k_1 = hf_1(x_0, y_0, z_0)$	$\ell_1 = hf_2(x_0, y_0, z_0)$
$k_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{\ell_1}{2}\right]$	$\ell_2 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{\ell_1}{2}\right]$
$k_3 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right]$	$\ell_3 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right]$
$k_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + \ell_3]$	$\ell_4 = hf_2[x_0 + h, y_0 + k_3, z_0 + \ell_3]$
$\Delta y = \frac{1}{6}[k_1 + 2k_1 + 2k_2 + 2k_3]$	$\Delta z = \frac{1}{6}[\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4]$
$y_1 = y_0 + \Delta y$	$z_1 = z_0 + \Delta z$

Example 4.24. Consider the II order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using fourth order Runge-Kutta method, find $y(0.2), z(0.2)$.

Solution: Let $t = x$. $\therefore$ Given equations can be written as

$$y'' - 2y' + 2y = e^{2x}\sin x \tag{1}$$

$$y(x = 0) = -0.4$$

$$y'(x = 0) = -0.6$$

Here $h = 0.2$

Setting $y' = z, y'' = z'$

$$\therefore (1) \Rightarrow z' = 2z - 2y + e^{2x}\sin x$$

Let $f_1(x, y, z) = \frac{dy}{dx} = z$,

$\& f_2(x, y, z) = \frac{d^2y}{dx^2} = \frac{dz}{dx} = 2z - 2x + e^{2x}\sin x$

$k_1 = hf_1(x_0, y_0, z_0) = (0.2) f_1(0, -0.4, -0.6)$ $= (0.2)(z_0)$ $= (0.2)(-0.6) = -0.12$	$\ell_1 = hf_2(x_0, y_0, z_0) = (0.2) f_2(0, -0.4, -0.6)$ $= (0.2)\left[2z_0 - 2y_0 + e^{2x_0}\sin x_0\right]$ $= (0.2)\left[2(-0.6) - 2(-0.4) + e^{2(0)}\sin(0)\right]$ $= -0.08$
$k_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{\ell_1}{2}\right]$ $= (0.2) f_1\left[0 + \frac{0.2}{2}, (-0.4) - \frac{0.12}{2}, (-0.6) - \frac{0.8}{2}\right]$ $= (0.2) f_1[0.1, -0.46, -0.64]$ $= (0.2)(-0.64)$ $= -0.128$	$\ell_2 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{\ell_1}{2}\right]$ $= (0.2) f_2\left[0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 - \frac{0.08}{2}\right]$ $= (0.2) f_2[0.1, -0.46, -0.64]$ $= (0.2)\left[2(-0.64) - 2(-0.46) + e^{2(0.1)}\sin(0.1)\right]$ $= -0.0476$
$k_3 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right]$ $= (0.2) f_1\left[0+\frac{0.2}{2}, -0.4- \frac{0.128}{2}, -0.6- \frac{0.0476}{2}\right]$ $= (0.2) f_1[0.1, -0.464, -0.62381]$ $= (0.2)[-0.62381]$ $= -0.12476$	$\ell_3 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right]$ $= (0.2) f_2\left[0+\frac{0.2}{2}, -0.4- \frac{0.128}{2}, -0.6- \frac{0.0476}{2}\right]$ $= (0.2) f_2[0.1, -0.464, -0.62381]$ $= (0.2)\left[2(-0.62381) - 2(-0.464) + e^{2(0.1)}\sin(0.1)\right]$ $= -0.03954$
$k_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + \ell_3]$ $= (0.2) f_1[0+0.2, -0.4-0.12476, -0.6-0.03954]$ $= (0.2) f_1[0.2, -0.52476, -0.63954]$ $= (0.2)[-0.63954]$ $= -0.12791$	$\ell_4 = hf_2[x_0 + h, y_0 + k_3, z_0 + \ell_3]$ $= (0.2) f_2[0+0.2, -0.4-0.12476, -0.6-0.03954]$ $= (0.2) f_2[0.2, -0.52476, -0.63954]$ $= (0.2)\left[2(-0.63954) - 2(-0.52476) + e^{2(0.2)}\sin(0.2)\right]$ $= 0.013366$
$\Delta y = \frac{1}{6}[k_1 + 2k_1 + 2k_2 + 2k_3]$ $= -0.12557$ $y_1 = y(0.2) = y_0 + \Delta y$ $= -0.52557$	$\Delta z = \frac{1}{6}[\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4]$ $= -0.04015$ $z_1 = z(0.2) = z_0 + \Delta z$ $= -0.64015$

Example 4.25. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$, find the value of $y(0.1)$ by using R. K. method of fourth order.

Solution: [$y(0.1) = 0.9950$].

Example 4.26. Given $y'' - x(y')^2 + y^2 = 0, y(0) = 1, y'(0) = 0$, find the value of $y(0.2)$ by using R. K. method of fourth order.

Solution: [$y(0.2) = 0.9801$].

4.8.1 Anna University Questions

1. Find the value of $y(0.1)$ by Runge-Kutta method of fourth order given $y'' + xy' + y = 0, y(0) = 1$ and $y'(0) = 0$. (ND10)

   Solution: [$y_1 = y(0.1) = y_0 + \nabla y = 0.9950$]

2. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$. Find the value of $y(0.1)$ by using Runge-Kutta method of fourth order. (ND11)

   Solution: [$y_1 = y(0.1) = y_0 + \nabla y = 0.9950$]

3. Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using Fourth order Runge Kutta algorithm, find $y(0.2)$. (MJ12)

   Solution: $[y_1 = y(0.2) = y_0 + \nabla y = -0.5159]$

4. Using Runge-Kutta method find $y(0.2)$ if $y'' = xy'^2 - y^2, y(0) = 1, y'(0) = 0, h = 0.2$. (MJ13)

   Solution: $[y_1 = y(0.2) = y_0 + \nabla y = 0.9801]$

5. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ find the value of $y(0.1)$ by Runge-Kutta's method of fourth order. (ND14)

   Solution: $[y_1 = y(0.1) = y_0 + \nabla y = 0.9950]$

4.9 Fourth order R. K. Method for simultaneous first order differential equations

Solving the equation $\frac{dy}{dx} = f_1(x,y,z) \& \frac{dz}{dx} = f_2(x,y,z)$ with the initial conditions $y(x_0) = y_0, z(x_0) = z_0$.
Now starting from with increments, (x_0, y_0, z_0) with increments, Δy & Δz in y & z respectively.
Use formula

$k_1 = hf_1(x_0, y_0, z_0)$ $k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{\ell_1}{2}\right)$ $k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right)$ $k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + \ell_3)$	$\ell_1 = hf_2(x_0, y_0, z_0)$ $\ell_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{\ell_1}{2}\right)$ $\ell_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right)$ $\ell_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + \ell_3)$
$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$\Delta z = \frac{1}{6}(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$
$y_1 = y_0 + \Delta y$	$z_1 = z_0 + \Delta z$

4.9.1 Part B

Example 4.27. Solving the system of differential equation $\frac{dy}{dx} = xz + 1, \frac{dz}{dx} = -xy$ for $x = 0.3$ using fourth order R. K. method, the initial values are $x = 0, y = 0, z = 1$.

Solution: Given $x_0 = 0, y_0 = 0, z_0 = 1, h = 0.3$.

$f_1(x,y,z) = xz + 1$	$f_2(x,y,z) = -xy$
$k_1 = hf_1(x_0,y_0,z_0) = (0.3)(x_0z_0+1)$	$\ell_1 = hf_2(x_0,y_0,z_0) = (0.3)[-0(0)]$
$= (0.3)(0+1) = 0.3$	$= 0$
$k_2 = hf_1\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{\ell_1}{2}\right)$	$\ell_2 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{\ell_1}{2}\right)$
$= 0.345$	$= -0.007$
$k_3 = hf_1\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{\ell_2}{2}\right)$	$\ell_3 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{\ell_2}{2}\right)$
$= 0.3448$	$= -0.0078$
$k_4 = hf_1(x_0+h, y_0+k_3, z_0+\ell_3)$	$\ell_4 = hf_2(x_0+h, y_0+k_3, z_0+\ell_3)$
$= 0.3893$	$= -0.031032$
$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.34482$	$\Delta z = \frac{1}{6}(\ell_1+2\ell_2+2\ell_3+\ell_4) = -0.01011$
$y_1 = y_0 + \Delta y = 0 + 0.34482$	$z_1 = z_0 + \Delta z = 1 + (-0.01011)$
$\Rightarrow y(0.3) = 0.34482$	$\Rightarrow z(0.3) = 0.98989$

Example 4.28. Solve the simultaneous differential equation $\frac{dy}{dx} = 2y + z, \frac{dz}{dx} = y - 3z, y(0) = 0, z(0) = 0.5$ for $y(0.1)$ and $z(0.1)$ using R. K. method of fourth method.

Solution: Given $x_0 = 0, y_0 = 0, z_0 = 0.5, h = 0.1$

$[y(0.1) = 0.04814, z(0.1) = 0.37263]$.

Example 4.29. Using the R. K. method, tabulate the solution of the system $\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y, y = 0, z = 0$, when $x = 0$ at intervals of $h = 0.1$ from $x = 0.0$ to $x = 0.2$.

Solution: Given $x_0 = 0, y_0 = 0, z_0 = 1, h = 0.1$

[Hint: $y(0.1) = 0.1050, z(0.1) = 0.9998$

$y(0.2) = 0.2199, z(0.2) = 0.9986$]

4.9.2 Anna University Questions

1. Solve $y(0.1)$ and $z(0.1)$ from the simultaneous differential equations $\frac{dy}{dx} = 2y + z; \frac{dz}{dx} = y - 3z; y(0) = 0, z(0) = 0.5$ using Runge-Kutta method of the fourth order. (ND12)

   Solution: Given $x_0 = 0, y_0 = 0, z_0 = 0.5, h = 0.1$

   $[y(0.1) = 0.04814, z(0.1) = 0.37263]$.

Multi step methods

h(= Interval length)

Given $(x_0, y_0), x_1, x_2, x_3, x_4$: $y_{4,p}$ = known by Multistep predictor method

y_1, y_2, y_3 are given (or) known by Singular method $y_{4,c}$ = known by Multistep corrector method

i.e.,

Given x :	x_0	x_1	x_2	x_3	x_4	x_4
Given or find y :	y_0	y_1	y_2	y_3	$\boldsymbol{y_{4,p}}$	$\boldsymbol{y_{4,c}}$
					find by Multistep predictor method	find by Multistep corrector method

Multi step methods:

1. Milne's forth predictor corrector methods for solving first order equations
2. Adam's Bash-forth predictor corrector methods for solving first order equations

4.10 Milne's forth predictor corrector methods for solving first order equations

Milne's predictor and corrector methods (multistep method)

Milne's predictor formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_{n}\right]$$

Milne's corrector formula

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_{n} + y'_{n+1}\right]$$

4.10.1 Part A

1. What are multi-step methods? How are they better than single step methods? (ND2010)

   Solution : 1. Milne's predictor - correction method.

   2. Adams-Bashforth predictor - correction method.

   In the single step methods, it is not possible to get any information about truncation error.
   In the multi step methods, it is possible to get easily a good estimate of the truncation error.

2. State the Milne's predictor and corrector formulae. (AM2014)
3. State the Milne's predictor-corrector formulae. (ND14)

4.10.2 Part B

Example 4.30. Given $\frac{dy}{dx} = \frac{1}{2}[x + y]$, $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$ by Milne's method to find $y(2)$.

Solution: Given

$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
$y_0 = 2$	$y_1 = 2.636$	$y_2 = 3.595$	$y_3 = 4.968$	$y_4 = ?$

Here $h = 0.5$ Given $y' = \frac{1}{2}[x + y] = f(x, y)$ By Milne's predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_{n}\right]$$

$$\text{when } n = 3, y_{3+1,p} = y_0 + \frac{4h}{3}\left[2y'_{1} - y'_{2} + 2y'_{3}\right] \quad (1)$$

$$\text{Now, } y'_1 = \frac{1}{2}[x_1 + y_1] = \frac{1}{2}[0.5 + 2.636] = 1.568$$
$$y'_2 = \frac{1}{2}[x_2 + y_2] = \frac{1}{2}[1 + 3.595] = 2.2975$$
$$y'_3 = \frac{1}{2}[x_3 + y_3] = \frac{1}{2}[1.5 + 4.968] = 3.234$$
$$(1) \Rightarrow y_{4,p} = 2 + \frac{4(0.5)}{3}[2(1.568) - 2.2975 + 2(3.234)]$$
$$= 6.871\,(= y_4, \text{say})$$

By Milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$
$$\text{when } n = 3, y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right] \tag{2}$$

$$\text{Now, } y'_4 = \frac{1}{2}[x_4 + y_4] = \frac{1}{2}[2 + 6.871] = 4.4355$$
$$(2) \Rightarrow y_{4,c} = 3.595 + \frac{0.5}{3}[2.2975 + 4(3.234) + 4.4355]$$
$$= 3.595 + \frac{0.5}{3}(19.669) = 6.8732$$

$\therefore$ Corrected value of y at 2 is $y(2) = 6.8732$.

Example 4.31. Given $\frac{dy}{dx} = x^3 + y, y(0) = 2$, the values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, $y(0.6) = 3.023$ are got by R.K. Method. Find $y(0.8)$ by Milne's method.

Solution: $\left[y_{4,p} = 4.1659, y_{4,c} = 3.7953\right]$

Example 4.32. Solve $y' = x - y^2, 0 \le x \le 1, y(0) = 0, y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$ by Milne's method to find $y(0.8)$ and $y(1)$.

Solution: $[y(0.8) = 0.3046, y(1) = 0.4515]$.

4.10.3 Anna University Questions

1. Use Milne's predictor-corrector formula to find $y(0.4)$, given $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}, y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12$ and $y(0.3) = 1.21$. (AM10)

   Solution: $\left[y_{4,p} = 1.2771, y_{4,c} = 1.2797\right]$

2. Given that $\frac{dy}{dx} = \frac{1}{2}\left(1 + x^2\right)y^2; y(0) = 1; y(0.1) = 1.06; y(0.2) = 1.12$ and $y(0.3) = 1.21$, evaluate $y(0.4)$ and $y(0.5)$ by Milne's predictor corrector method. (ND11)

   Solution: $\left[y_{4,p} = 1.2771, y_{4,c} = 1.2797\right]$

3. Given that $\frac{dy}{dx} = 1 + y^2; y(0.6) = 0.6841, y(0.4) = 0.4228, y(0.2) = 0.2.27, y(0) = 0$, find $y(-0.2)$ using Milne's method. (ND12)

   Solution: $\left[y_{4,p} = -0.2003, y_{4,c} = -0.2027\right]$

4. Use Milne's method to find $y(0.8)$, given $y' = \frac{1}{x+y}, y(0) = 2, y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) = 2.2493$. (ND13)

   Solution: $\left[y_{4,p} = 2.3162, y_{4,c} = 2.3164\right]$

5. Given $5xy' + y^2 = 2, y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$. Compute $y(4.4)$ using Milne's method. (ND14)

   Solution: $\left[y_{4,p} = 1.01897, y_{4,c} = 1.01874\right]$

4.11 Adams-Bash forth predictor corrector methods for solving first order equations

Adams predictor and corrector methods(efficient multistep method) [Adam's-Bashforth method]

Adam's predictor formula :

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]$$

Adam's corrector formula :

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$

4.11.1 Part A

1. Write the Adam-Bashforth predictor and corrector formulae. (MJ2010)

   (Or) State Adam's predictor-corrector formulae. (ND2011)

   Solution : Adam's predictor and corrector formulas are

$$y_{k+1,p} = y_k + \frac{h}{24}\left[55y'_k - 59y'_{k-1} + 37y'_{k-2} - 9y'_{k-3}\right]$$
$$y_{k+1,c} = y_k + \frac{h}{24}\left[9y'_{k+1} + 19y'_k - 5y'_{k-1} + y'_{k-2}\right]$$

4.11.2 Part B

Example 4.33. Given $\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adam's-Basforth method.

Solution: Given

$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$	$x_4 = 1.4$
$y_0 = 1$	$y_1 = 1.233$	$y_2 = 1.548$	$y_3 = 1.979$	$y_4 = ?$

Here $h = 0.1$

By Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]$$
$$\text{when } n = 3, y_{4,p} = y_3 + \frac{h}{24}\left[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0\right] \quad (1)$$

Here $y'_0 = x_0^2(1+y_0) = 1^2[1+1] = 2$

$y'_1 = x_1^2(1+y_1) = (1.1)^2[1+1.233] = 2.70193$

$y'_2 = x_2^2(1+y_2) = (1.2)^2[1+1.548] = 3.66912$

$y'_3 = x_3^2(1+y_3) = (1.3)^2[1+1.979] = 5.0345$

$$(1) \Rightarrow y_{4,p} = 1.979 + \frac{0.1}{24}\left[55(5.0345) - 59(3.66912) + 37(2.70193) - 9(2)\right]$$
$$= 2.5723$$

By Adam's corrector method,

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$
$$(\text{when } n = 3), y_{4,c} = y_3 + \frac{h}{24}\left[9y'_4 + 19y'_3 - 5y'_2 + y'_1\right] \quad (2)$$
$$y'_4 = (x_4)^2(1 + y_4) = (1.4)^2[1 + 2.5871] = 7.0017$$

$$\therefore (2) \Rightarrow y_{4,c} = 1.979 + \frac{0.1}{24}\left[9(7.030716) + 19(5.0345) - 5(3.60912) + (2.70193)\right]$$
$$= 2.5749$$

Example 4.34. Using Adam's Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ & $y(4.3) = 1.0143$.

Solution: Here $y' = \dfrac{2-y^2}{2x}$, $\left[\therefore y_{4,p} = 1.0186 \& y_{4,c} = 1.0187\right]$.

Example 4.35. Evaluate $y(0.9)$, using Adam Bashforth's predictor-corrector method, given that $\dfrac{dy}{dx} = xy^{\frac{1}{3}}$, $y(1) = 1$, $y(1.1) = 1.106814$, $y(1.2) = 1.22787$ and $y(1.3) = 1.36412$.

Solution: Here $h = -0.1$, $\left[\therefore y_{4,p} = 0.906518 \& y_{4,c} = 0.906520\right]$.

4.11.3 Anna University Questions

1. Given $\dfrac{dy}{dx} = xy + y^2, y(0) = 1, y(0.1) = 1.1169, y(0.2) = 1.2774, y(0.3) = 1.5041$. Use Adam's method to estimate $y(0.4)$. (AM10)

   Solution: $\left[y_{4,p} = 1.8341, y_{4,c} = 1.8389\right]$

2. Using Adam's method to find $y(2)$ if $y' = (x+y)/2, y(0) = 2, y(0.5) = 2.636, y(1) = 3.595, y(1.5) = 4.968$. (MJ13)

   Solution: $\left[y_{4,p} = 6.8708, y_{4,c} = 6.8731\right]$

3. Using Adam's Bashforth method, find $y(4.4)$ given that $5xy' + y^2 = 2, y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097$ and $y(4.3) = 1.0143$. (AM14)

   Solution: $\left[y_{4,p} = 1.0186, y_{4,c} = 1.0187\right]$

4.11.4 Anna University Questions (Taylor's, RK, Adam, Milne)

1. Given $\dfrac{dy}{dx} = xy + y^2, y(0) = 1, y(0.1) = 1.1169$ and $y(0.2) = 1.2774$, find (i) $y(0.3)$ by Runge-Kutta method of fourth order and (ii) $y(0.4)$ by Milne's method. (ND10)

   Solution: $\left[(i) y(0.3) = 1.5041, (ii) y_{4,p} = 1.8344, y_{4,c} = 1.8387\right]$

2. Using Runge-Kutta method of fourth order, find y for $x = 0.1, 0.2, 0.3$ given that $y' = xy + y^2, y(0) = 1$. Continue the solution at $x = 0.4$ usign Milne's method. (AM11)

   Solution: $\left[y(0.1) = 1.11689, y(0.2) = 1.2774, y(0.3) = 1.5041; y_{4,p} = 1.8344, y_{4,c} = 1.8387\right]$

3. Solve $y' = x - y^2, y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Taylor's method using the value $h = 0.1$. (AM11)

   Solution: $\left[\text{TS:P}y(0.1) = 0.9138, y(0.2) = 0.8512, y(0.3) = 0.8076; \text{MM}y_{4,p} = 0.7799, y_{4,c} = 0.7797\right]$

4. Given that $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ obtain y for $x = 0.1, 0.2$ and 0.3 by Taylor's series method and find the solution for $y(0.4)$ by Milne's method. (MJ12)

   Solution: $\left[\text{TS: } y(0.1) = 0.995, y(0.2) = 0.9802, y(0.3) = 0.956; \text{MM: } y_{4,p} = 0.9232, y_{4,c} = 0.9232\right]$

5. Using Runge Kutta method of fourth order, find the value of y at $x = 0.2, 0.4, 0.6$ given $\frac{dy}{dx} = x^3 + y, y(0) = 2$. Also find the value of y at $x = 0.8$ using Milne's predictor and corrector method. (AM14)

   Solution: $\left[\text{RK: } y(0.2) = 2.073, y(0.4) = 2.452, y(0.6) = 3.023; \text{MM: } y_{4,p} = 4.1664, y_{4,c} = 3.79536\right]$

4.11.5 Assignment problems

1. Using Taylor series method to find $y(0.1)$ if $y' = x^2 + y^2, y(0) = 1$. (MJ13)
2. Using Taylor's series method, find y at $x = 1.1$ by solving the equation $\frac{dy}{dx} = x^2 + y^2; y(1) = 2$. Carryout the computations upto fourth order derivative. (AM14)
3. Using Taylor's series method, find y at $x = 0$ if $\frac{dy}{dx} = x^2y - 1, y(0) = 1$. (ND14)
4. Solve $y' = \frac{y - x}{y + x}, y(0) = 1$ at $x = 0.1$ by taking $h = 0.02$ by using Euler's method. (MJ13)
5. Using Modified Euler's method, find $y(4.1)$ and $y(4.2)$ if $5x\frac{dy}{dx} + y^2 - 2 = 0; y(4) = 1$. (ND12)
6. Apply modified Euler's method to find $y(0.2)$ and $y(0.4)$ given $y' = x^2 + y^2$, $y(0) = 1$ by taking $h = 0.2$. (ND14)
7. Using Runge-Kutta method of order four, find y when $x = 1.2$ in steps of 0.1 given that $y' = x^2 + y^2$ and $y(1) = 0.5$. (ND13)
8. Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using Fourth order Runge Kutta algorithm, find $y(0.2)$. (MJ12)
9. Using Runge-Kutta method find $y(0.2)$ if $y'' = xy'^2 - y^2, y(0) = 1, y'(0) = 0, h = 0.2$. (MJ13)
10. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ find the value of $y(0.1)$ by Runge-Kutta's method of fourth order. (ND14)
11. Solve $y(0.1)$ and $z(0.1)$ from the simultaneous differential equations $\frac{dy}{dx} = 2y + z; \frac{dz}{dx} = y - 3z; y(0) = 0, z(0) = 0.5$ using Runge-Kutta method of the fourth order. (ND12)
12. Use Milne's predictor-corrector formula to find $y(0.4)$, given $\frac{dy}{dx} = \frac{\left(1 + x^2\right)y^2}{2}, y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12$ and $y(0.3) = 1.21$. (AM10)
13. Use Milne's method to find $y(0.8)$, given $y' = \frac{1}{x + y}, y(0) = 1$, for $x = 0.1$ and 0.2 correct to four decimal places. (ND13)

14. Given $5xy' + y^2 = 2, y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0142$. Compute $y(4.4)$ using Milne's method. (ND14)

15. Using Adam's method to find $y(2)$ if $y' = (x+y)/2, y(0) = 2, y(0.5) = 2.636, y(1) = 3.595, y(1.5) = 4.968$. (MJ13)

16. Using Adam's Bashforth method, find $y(4.4)$ given that $5xy' + y^2 = 2, y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097$ and $y(4.3) = 1.0143$. (AM14)

17. Given $\frac{dy}{dx} = xy + y^2, y(0) = 1, y(0.1) = 1.1169$ and $y(0.2) = 1.2774$, find (i) $y(0.3)$ by Runge-Kutta method of fourth order and (ii) $y(0.4)$ by Milne's method. (ND10)

18. Using Runge Kutta method of fourth order, find the value of y at $x = 0.2, 0.4, 0.6$ given $\frac{dy}{dx} = x^3 + y, y(0) = 2$. Also find the value of y at $x = 0.8$ using Milne's predictor and corrector method. (AM14)

5 Boundary Value Problems in Ordinary and Partial Differential Equations

Finite difference methods for solving two-point linear boundary value problems - Finite difference techniques for the solution of two dimensional Laplace's and Poisson's equations on rectangular domain - One dimensional heat flow equation by explicit and implicit (Crank Nicholson) methods - One dimensional wave equation by explicit method.

5.1 Introduction

5.1.1 Part A

1. What is the central difference approximation for y''. (AM2012,ND2014)

   Solution: The second-order central difference for the second derivative is given by:

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}=y''(x_i)+o\left(h^2\right)$$

$$(or)$$

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x)+o\left(h^2\right)$$

5.2 Finite difference methods for solving two-point linear boundary value problems

Consider the problem

$$y''(x)+f(x)g'(x)+g(x)y(x)=r(x)$$

with the boundary conditions $y(x_0)=a$ and $y(x_n)=b$.
For solving the equation, replace y'' and y' by the formulae

$$y''=\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}$$
$$y'=\frac{y_{i+1}-y_{i-1}}{2h}$$

5.2.1 Part A

1. Obtain the finite difference scheme for the differential equation $2y'' + y = 5$. (AM2013)

2. Using finite difference solve $y'' - y = 0$ given $y(0) = 0, y(1) = 1, n = 2$. (ND2013)

3. Obtain the finite difference scheme for the differential equation $2y''(x) + y(x) = 5$. (AM2014)

5.2.2 Part B

Example 5.1. Solve the differential equation $\frac{d^2y}{dx^2} - y = x$ with $y(0) = 0, y(1) = 0$ with $h = \frac{1}{4}$ by finite difference method.

Solution: Given differential equation

$$\frac{d^2y}{dx^2} - y = x \tag{1}$$

$$\text{i.e., } \frac{d^2y(x)}{dx^2} - y(x) = x$$

Using the central difference approximation, we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \tag{2}$$

Substitute (2) in (1), we get

$$y_{i-1} - 2y_i + y_{i+1} - h^2 y_i = h^2 x_i$$

$$y_{i-1} - 2y_i + y_{i+1} - \frac{1}{16} y_i = \frac{1}{16} x_i \qquad \left(\because h = \frac{1}{4}\right)$$

$$16y_{i-1} - 33y_i + 16y_{i+1} = x_i \tag{3}$$

Put $k = 1, 2, 3$ in (3), we get

$$\left.\begin{aligned} 16y_0 - 33y_1 + 16y_2 &= \frac{1}{4} \\ 16y_1 - 33y_2 + 16y_3 &= \frac{1}{2} \\ 16y_2 - 33y_3 + 16y_4 &= \frac{3}{4} \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \text{Given } y(0) &= 0 \text{ i.e., } y(x_0 = 0) = 0 \text{ i.e., } y_0 = 0 \\ y(1) &= 0 \text{ i.e., } y(x_4 = 1) = 0 \text{ i.e., } y_4 = 0 \end{aligned}\right\} \tag{5}$$

Since $h = \frac{1}{4}$, we have $x_1 = x_0 + h = \frac{1}{4}, x_2 = x_1 + h = \frac{1}{2}, x_3 = x_2 + h = \frac{3}{4}$

Substitute (5) in (4), we get

$$0 - 33y_1 + 16y_2 = \frac{1}{4} \tag{6}$$

$$16y_1 - 33y_2 + 16y_3 = \frac{1}{2} \tag{7}$$

$$16y_2 - 33y_3 + 0 = \frac{3}{4} \tag{8}$$

Solving (6) & (8), we get

$$y_1 - y_3 = \frac{1}{66} \tag{9}$$

Solving (7) & (8), we get

$$256y_1 - 833y_3 = \frac{131}{4} \tag{10}$$

Solving (9) & (10), we get

$$y_3 = -0.05004$$

Substitute y_3 in (9), we get

$$y_1 = -0.03488$$

Substitute $y_1 \& y_3$ in (6), we get

$$y_2 = -0.05632$$

∴ Solution of the given differential equation is

$$\boxed{\begin{array}{l} y_1 = -0.03488, \\ y_2 = -0.05632, \\ y_3 = -0.05004 \end{array}}$$

Example 5.2. Solve by finite difference method, the boundary value problem $y''(x) - y(x) = 2$, when $y(0) = 0, y(1) = 1$, taking $h = \frac{1}{4}$.

Solution: $\left[\text{Ans}: y_1 = y\left(\frac{1}{4}\right) = 0.0451, y_2 = y\left(\frac{2}{4}\right) = 0.2183, y_3 = y\left(\frac{3}{4}\right) = 0.5301\right]$

Example 5.3. Solve by finite difference method, the boundary value problem $y''(x) - 3y'(x) + 2y(x) = 0$ when $y(0) = 2, y(1) = 10.1$.

Solution: $[\text{Ans}: y(0.5) = 5.32]$

Example 5.4. Using the finite difference method, find $y(0.25)$, $y(0.5)$ and $y(0.75)$ satisfying the differential equation $y''(x) + y(x) = x$ subject to the boundary conditions $y(0) = 0, y(1) = 2$.

Solution: $\left[\text{Ans}: y_1 = y\left(\frac{1}{4}\right) = 0.5443, y_2 = y\left(\frac{2}{4}\right) = 1.0701, y_3 = y\left(\frac{3}{4}\right) = 1.5604\right]$

5.2.3 Anna University Questions

1. Solve the boundary value problem $y'' = xy$ subject to the condition $y(0) + y'(0) = 1, y(1) = 1$, taking $h = 1/3$, by finite difference method. (ND10)

   Solution: $[y(0) = -0.9880, y(1/3) = -0.3253, y(2/3) = 0.3253]$

2. Using the finite difference method, compute $y(0.5)$, given $y'' - 64y + 10 = 0, x \in (0,1), y(0) = y(1) = 0$, subdividing the interval into (i) 4 equal parts (ii) 2 equal parts. (ND11)

   Solution: $[(i) y(0.25) = 0.129, y(0.5) = 0.147, y(0.75) = 0.129; (ii) y(0.5) = 0.1389]$

3. Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$. (MJ12)

   Solution: $[y(0.25) = -0.0349, y(0.5) = -0.0563, y(0.75) = -0.050]$

4. Solve $y'' - y = 0$ with the boundary condition $y(0) = 0$ and $y(1) = 1$. (ND12)

   Solution: $[y(0.25) = 0.2151, y(0.5) = 0.4437, y(0.75) = 0.7]$

5. Solve $y'' - y = x, x \in (0,1)$ given $y(0) = y(1) = 0$ using finite differences by dividing the interval into four equal parts. (AM14)

5.3 Classification of PDE of second order

The general second order p.d.e. is

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0$$

If $B^2 - 4AC < 0$ [Elliptic form]

$B^2 - 4AC = 0$ [Parabola form]

$B^2 - 4AC > 0$ [Hyperbola form]

5.3.1 Part A

1. Classify the partial differential equation $u_{xx} + 2u_{xy} + 4u_{yy} = 0, x, y > 0$.
   Solution : Given $u_{xx} + 2u_{xy} + 4u_{yy} = 0, x, y > 0$.
   Here $A = 1, B = 2, C = 4$
   $$B^2 - 4AC = -12 < 0$$
   $\therefore$ The given equation is elliptic.

2. Classify the PDE $y_{xx} - xu_{yy} = 0$.
   Solution : Given $y_{xx} - xu_{yy} = 0$.
   Here $A = 1, B = 0, C = -x$
   $$B^2 - 4AC = 4x$$
   $\therefore$ The given equation is elliptic if $x < 0$
   parabolic if $x = 0$
   hyperbolic if $x > 0$

3. Classify $f_{xx} - 2f_{xy} = 0, x > 0, y > 0$.
   Solution : Here $A = 1, B = -2, C = 0$.
   $$B^2 - 4AC = 4 > 0$$
   $\Rightarrow$ Given equation is Hyperbolic form.

4. Classify $xf_{xx} + yf_{xy} = 0, x > 0, y > 0$.
   Solution : Here $A = x, B = 0, C = y$.
   $$B^2 - 4AC = -4xy < 0 \text{ when } x > 0, y > 0$$
   $\Rightarrow$ Given equation is Elliptic form.

5. Classify $u_{xx} - 2u_{xy} + u_{yy} = 0$.

   Solution : Here $A = 1, B = -2, C = 1$.

$$B^2 - 4AC = 0$$

   $\Rightarrow$ Given equation is Parabolic form.

6. Classify the partial differential equation $u_{xx} - 2u_{xy} + 4u_{yy} = 0, x, y > 0$. (MJ2011)

7. Classify $x^2 f_{xx} + \left(1 - y^2\right) f_{yy} = 0, -1 < y < 1, -\infty < x < \infty$.

   Solution : Here $A = x^2, B = 0, C = \left(1 - y^2\right)$.

$$B^2 - 4AC = -4x^2\left(1 - y^2\right) = 4x^2\left(y^2 - 1\right).$$

   x^2 is always ' + ' in $-\infty < x < \infty$
   In $-1 < y < 1, y^2 - 1$ is ' − '.

$$B^2 - 4AC = -ve\,(x \neq 0)$$

   $\Rightarrow$ Given equation is Elliptic form.
   If $x = 0, B^2 - 4AC = 0 \Rightarrow$ Given equation is Parabolic form.
   If $y < -1, B^2 - 4AC > 0 \Rightarrow$ Given equation is Hyperbolic form.

8. Classify the PDE $y(x_0) = y_0$. (ND2011)

9.

<table>
<tr><th colspan="2">Parabolic Type</th><th>Hyperbolic Type</th><th>Elliptic Type</th></tr>
<tr><td colspan="2">$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$
(1D Heat eqn.)</td><td rowspan="2">$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$
(1D Wave eqn.)</td><td>$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
(Laplace eqn in 2D Heat eqn.)</td></tr>
<tr><td>$\swarrow$
Explicit form
(Bender - Schmidt's method)</td><td>$\searrow$
Implicit form
(Crank - Nicolson's method)</td><td>$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$
(Poisson eqn. in 2D Heat eqn.)</td></tr>
</table>

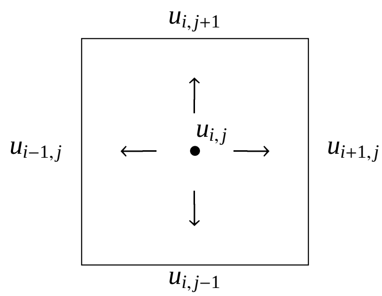
5.4 Finite difference technique for the soln. of 2D Laplace's equations on rectangular domain

Two dimensional Laplace equation is

$$\nabla^2 u = 0 \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

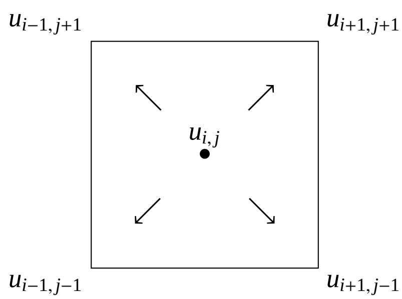
Standard five - point formula(SFPF):

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

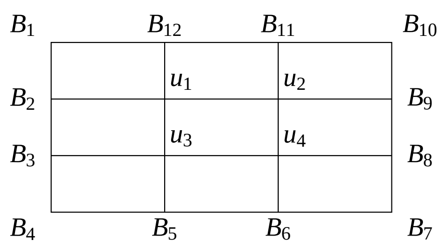


Diagonal five - point formula(DFPF):

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$



<u>For 3×3 matrix form:</u>



To find u_1, u_2, u_3, u_4 :

Assume $u_i = 0$, for any one i

Initially take $u_4 = 0$ or take $u_4 =$ some finite value

Find Rough Values [Initial values (or) Starting values] :

$$u_1 = \frac{1}{4}\left[B_1 + B_3 + u_4 + B_{11}\right] \quad \text{(by DFPF)}$$

$$u_2 = \frac{1}{4}\left[u_1 + u_4 + B_9 + B_{11}\right] \quad \text{(by SFPF)}$$

$$u_3 = \frac{1}{4}\left[u_1 + B_3 + B_5 + u_4\right] \quad \text{(by SFPF)}$$

$$u_4 = \frac{1}{4}\left[u_2 + u_3 + B_6 + B_8\right] \quad \text{(by SFPF)}$$

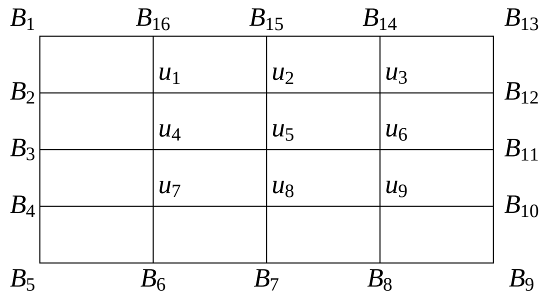
First Iteration and onwards (apply only SFPF)

$$\left.\begin{array}{l}\text{The Liebmann's iterative}\\ \text{formula is}\end{array}\right\} \quad u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

Stop procedure until two consecutive iteration values are same.

$$\text{i.e., stop if } u_i^{(n)} = u_i^{(n+1)}, \forall i \text{ with error} < 0.1$$

<u>For 4×4 matrix form:</u>



To find $u_i, i = 1, 2, \cdots 9$:

Find Rough Values [Initial values (or) Starting values] :

$$u_5 = \frac{1}{4}[B_3 + B_7 + B_{11} + B_{15}] \qquad \left(\text{by SFPF, extreme boundary values only}\right)$$

$$u_1 = \frac{1}{4}[B_1 + B_3 + u_5 + B_{15}] \qquad \text{(by DFPF)}$$

$$u_3 = \frac{1}{4}[B_{15} + u_5 + B_{11} + B_{13}] \qquad \text{(by DFPF)}$$

$$u_7 = \frac{1}{4}[B_3 + B_5 + B_7 + u_5] \qquad \text{(by DFPF)}$$

$$u_9 = \frac{1}{4}[u_5 + B_7 + B_9 + B_{11}] \qquad \text{(by DFPF)}$$

$$u_2 = \frac{1}{4}[u_1 + u_5 + u_3 + B_{15}] \qquad \text{(by SFPF)}$$

$$u_4 = \frac{1}{4}[u_1 + B_3 + u_7 + u_5] \qquad \text{(by SFPF)}$$

$$u_6 = \frac{1}{4}[u_5 + u_9 + B_{11} + u_3] \qquad \text{(by SFPF)}$$

$$u_8 = \frac{1}{4}[u_7 + B_7 + u_9 + u_5] \qquad \text{(by SFPF)}$$

First Iteration and onwards (apply only SFPF)

$$\left.\begin{array}{l}\text{The Liebmann's iterative}\\ \text{formula is}\end{array}\right\} \quad u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

Stop procedure until two consecutive iteration values are same.

$$\text{i.e., stop if } u_i^{(n)} = u_i^{(n+1)}, \forall i \text{ with error} < 0.1$$

5.4.1 Part A

1. Write down the standard five point formula to find the numerical solution of Laplace equation. (MJ2010)

   Solution : Standard five - point formula(SFPF) for $\nabla^2\phi = 0$ is

   $$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

2. State Standard Five Point formula with relevant diagram. (ND2011)

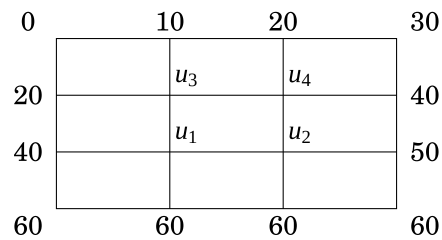
3. Write Liebmann's iteration process. (AM2013)

4. Write the diagonal five point five point formula for solving the two dimensional Laplace equation $\nabla^2 u = 0$. (ND2013)

5. Write down the standard five-point formula to find the numerical solution of Laplace equation. (ND14)

5.4.2 Part B

Example 5.5. Solve $\nabla^2 u = 0$, the boundary conditions are given below (give only 3 iterations)



Solution: Assume $u_4 = 0$

Find Rough Values [Initial values (or) Starting values] :

$$u_1 = \frac{1}{4}[20 + 0 + 60 + 60] = 35 \qquad \text{(by DFPF)}$$
$$u_2 = \frac{1}{4}[35 + 0 + 50 + 60] = 36.25 \qquad \text{(by SFPF)}$$
$$u_3 = \frac{1}{4}[10 + 20 + 35 + 0] = 16.25 \qquad \text{(by SFPF)}$$
$$u_4 = \frac{1}{4}[16.25 + 20 + 40 + 36.25] = 28.125 \qquad \text{(by SFPF)}$$

First Iteration and onwards (apply only SFPF)

$$\left.\begin{array}{l}\text{The Liebmann's iterative}\\ \text{formula is}\end{array}\right\} \quad u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

I iteration :

$$u_1^{(1)} = \frac{1}{4}[40 + 60 + 16.25 + 36.25] = 38.125 \qquad \text{(by SFPF)}$$
$$u_2^{(1)} = \frac{1}{4}[60 + 50 + 38.125 + 28.125] = 44.0625 \qquad \text{(by SFPF)}$$
$$u_3^{(1)} = \frac{1}{4}[20 + 10 + 38.125 + 28.125] = 24.0625 \qquad \text{(by SFPF)}$$
$$u_4^{(1)} = \frac{1}{4}[20 + 40 + 24.0625 + 44.0625] = 32.0313 \qquad \text{(by SFPF)}$$

II iteration :

$$u_1^{(2)} = \frac{1}{4}[60 + 40 + 24.0625 + 44.0625] = 42.0313 \qquad \text{(by SFPF)}$$
$$u_2^{(2)} = \frac{1}{4}[60 + 50 + 42.0313 + 32.0313] = 46.0157 \qquad \text{(by SFPF)}$$
$$u_3^{(2)} = \frac{1}{4}[20 + 10 + 42.0313 + 32.0313] = 26.0157 \qquad \text{(by SFPF)}$$
$$u_4^{(2)} = \frac{1}{4}[20 + 40 + 26.0157 + 46.0157] = 33.0079 \qquad \text{(by SFPF)}$$

III iteration :

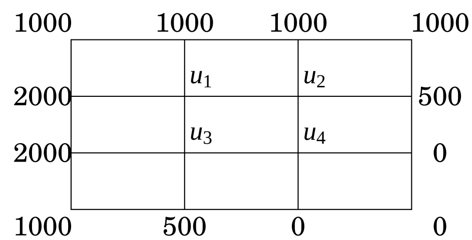
$$u_1^{(3)} = \frac{1}{4}[60 + 40 + 46.0157 + 26.0157] = 43.0079 \qquad \text{(by SFPF)}$$

$$u_2^{(3)} = \frac{1}{4}[60 + 50 + 43.0079 + 33.0079] = 46.5040 \qquad \text{(by SFPF)}$$

$$u_3^{(3)} = \frac{1}{4}[20 + 10 + 43.0079 + 33.0079] = 26.5040 \qquad \text{(by SFPF)}$$

$$u_4^{(3)} = \frac{1}{4}[20 + 40 + 46.5040 + 26.5040] = 33.252 \qquad \text{(by SFPF)}$$

Example 5.6. Obtain a finite difference scheme to solve the Laplace equation solve $\nabla^2 u = 0$ at the pivotal points in the square mesh. Use Liebmann's iterative procedure.



Solution: $u_1 = 1208.3, u_2 = 791.7, u_3 = 1041.7, u_4 = 458.4$

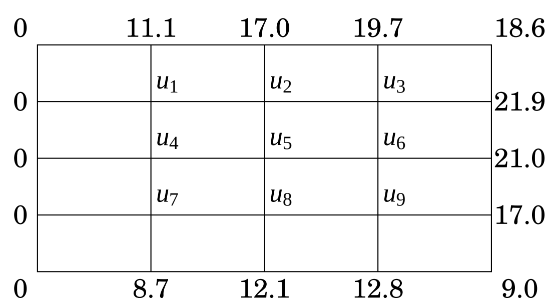
Example 5.7. By iteration method, solve the Laplace equation $\nabla^2 u = 0$ over the square region, satisfying the boundary conditions

$$u(0, y) = 0, 0 \le y \le 3, u(3, y) = 9 + y, 0 \le y \le 3,$$

$$u(x, 0) = 3x, 0 \le x \le 3, u(x, 3) = 4x, 0 \le x \le 3.$$

Solution: $u_1 = 3.67, u_2 = 7.34, u_3 = 3.34, u_4 = 6.67$

Example 5.8. Find by Liebmann's method the values at the interior lattice points of a square region of the harmonic function u whose boundary values are as shown.



Solution: Find Rough Values [Initial values (or) Starting values] :

$$u_5 = \frac{1}{4}[0 + 17.0 + 21.0 + 12.1] = 12.5 \qquad \text{(by SFPF, ext. bound. vals. only)}$$

$$u_1 = \frac{1}{4}[0 + 0 + 12.5 + 17.0] = 7.4 \qquad \text{(by DFPF)}$$

$$u_3 = \frac{1}{4}[12.5 + 18.6 + 17.0 + 1.0] = 17.3 \qquad \text{(by DFPF)}$$

$$u_7 = \frac{1}{4}[12.5 + 0 + 0 + 12.1] = 6.2 \qquad \text{(by DFPF)}$$

$$u_9 = \frac{1}{4}[12.5 + 9.0 + 12.1 + 21.0] = 13.7 \qquad \text{(by DFPF)}$$

$$u_2 = \frac{1}{4}[17.0 + 12.5 + 7.5 + 17.3] = 13.6 \qquad \text{(by SFPF)}$$

$$u_4 = \frac{1}{4}[7.4 + 6.2 + 0 + 12.5] = 6.5 \qquad \text{(by SFPF)}$$

$$u_6 = \frac{1}{4}[12.5 + 21.0 + 17.3 + 13.7] = 16.1 \qquad \text{(by SFPF)}$$

$$u_8 = \frac{1}{4}[12.5 + 12.1 + 6.2 + 13.7] = 11.1 \qquad \text{(by SFPF)}$$

First Iteration and onwards (apply only SFPF)

$$\left.\begin{array}{l}\text{The Liebmann's iterative}\\ \text{formula is}\end{array}\right\} \quad u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

I iteration :

$$\begin{array}{ll|ll|ll}
u_1^{(1)} = 7.8 & \text{(by SFPF)} & u_4^{(1)} = 6.6 & \text{(by SFPF)} & u_7^{(1)} = 6.6 & \text{(by SFPF)} \\
u_2^{(1)} = 13.7 & \text{(by SFPF)} & u_5^{(1)} = 11.9 & \text{(by SFPF)} & u_8^{(1)} = 11 & \text{(by SFPF)} \\
u_3^{(1)} = 17.9 & \text{(by SFPF)} & u_6^{(1)} = 16.1 & \text{(by SFPF)} & u_9^{(1)} = 14.3 & \text{(by SFPF)}
\end{array}$$

II iteration :

$$\begin{array}{ll|ll|ll}
u_1^{(2)} = 7.9 & \text{(by SFPF)} & u_4^{(2)} = 6.6 & \text{(by SFPF)} & u_7^{(2)} = 6.6 & \text{(by SFPF)} \\
u_2^{(2)} = 13.7 & \text{(by SFPF)} & u_5^{(2)} = 11.9 & \text{(by SFPF)} & u_8^{(2)} = 11.2 & \text{(by SFPF)} \\
u_3^{(2)} = 17.9 & \text{(by SFPF)} & u_6^{(2)} = 16.3 & \text{(by SFPF)} & u_9^{(2)} = 14.3 & \text{(by SFPF)}
\end{array}$$

III iteration :

$$\begin{array}{ll|ll|ll}
u_1^{(3)} = 7.9 & \text{(by SFPF)} & u_4^{(3)} = 6.6 & \text{(by SFPF)} & u_7^{(3)} = 6.6 & \text{(by SFPF)} \\
u_2^{(3)} = 13.7 & \text{(by SFPF)} & u_5^{(3)} = 11.9 & \text{(by SFPF)} & u_8^{(3)} = 11.2 & \text{(by SFPF)} \\
u_3^{(3)} = 17.9 & \text{(by SFPF)} & u_6^{(3)} = 16.3 & \text{(by SFPF)} & u_9^{(3)} = 14.3 & \text{(by SFPF)}
\end{array}$$

Since $u_i^{(2)} = u_i^{(3)}, \forall i$.
Therefore, the solution of the given Laplace equation is

$$u_1 = 7.9, u_2 = 13.7, u_3 = 17.9, u_4 = 6.6, u_5 = 11.9,$$
$$u_6 = 16.3, u_7 = 6.6, u_8 = 11.2, u_9 = 14.3$$

Example 5.9. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, |x| < 1, |y| < 1$ with $h = \frac{1}{2}$ and $u(x, \pm 1) = x^2, u(\pm 1, y) = y^2$.

Solution: $u_1 = 0.9, u_2 = 0.13, u_3 = 0.19, u_4 = 0.13, u_5 = 0.13,$

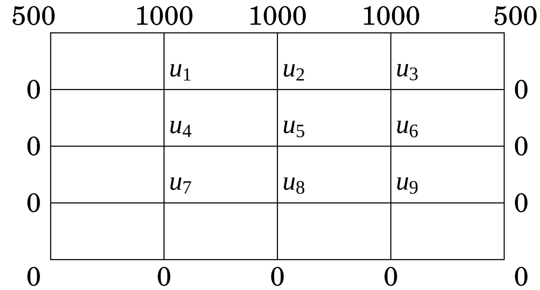
$u_6 = 0.13, u_7 = 0.19, u_8 = 0.13, u_9 = 0.19$

Example 5.10. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 \le x, y \le 1$ with $u(0, y) = 10 = u(1, y)$ and $u(x, 0) = 20 = u(x, 1)$. Take $h = 0.25$ and apply Liebmann's method to 3 decimal places.

Solution: $u_1 = 14.999, u_2 = 16.249, u_3 = 14.999, u_4 = 13.749, u_5 = 14.999,$

$u_6 = 13.749, u_7 = 14.999, u_8 = 16.249, u_9 = 14.999$

Example 5.11. Solve the Laplace equation at the interior poins of the square region given below.

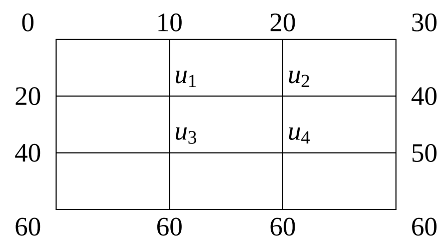


Solution: $u_1 = 71.5, u_2 = 98.3, u_3 = 71.5, u_4 = 187.6, u_5 = 250.1,$

$u_6 = 187.6, u_7 = 428.6, u_8 = 526.8, u_9 = 428.6$

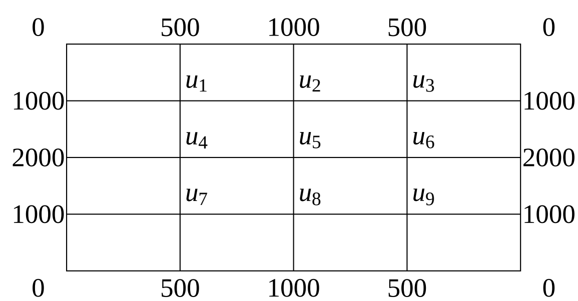
5.4.3 Anna University Questions

1. Deduce the standard five point formula for $\nabla^2 u = 0$. Hence, solve it over the square region given by the boundary conditions as in the figure below: (AM10)



Solution: $[u_1 = 43.1, u_2 = 46.6, u_3 = 26.6, u_4 = 33.3]$

2. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown: (MJ12)



Solution:
$[u_1 = 937.6, u_2 = 1000.1, u_3 = 937.6, u_4 = 1250.1, u_5 = 1125.1, u_6 = 1250.1, u_7 = 937.6, u_8 = 1001.1, u_9 = 937.6]$

3. By iteration method solve the elliptic equation $u_{xx} + u_{yy} = 0$ over the square region of side 4, satifying the boundary conditions.

(i) $u(0, y) = 0, 0 \leq y \leq 4$

(ii) $u(4, y) = 8 + 2y, 0 \leq y \leq 4$

(iii) $u(x, 0) = \frac{x^2}{2}, 0 \leq x \leq 4$

(iv) $u(x, 4) = x^2, 0 \leq x \leq 4$

Compute the values at the interior points correct to one decimal with $h = k = 1$. (ND13)

Solution: $[u_1 = 2, u_2 = 4.9, u_3 = 9, u_4 = 2.1, u_5 = 4.7, u_6 = 8.1, u_7 = 1.6, u_8 = 3.7, u_9 = 6.6]$

4. By iteratin method, solve the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ over a square region of side 4, satisfying the boundary conditions.

(i). $u(0, y) = 0, 0 \leq y \leq 4$

(ii). $u(4, y) = 12 + y, 0 \leq y \leq 4$

(iii). $u(x, 0) = 3x, 0 \leq x \leq 4$

(iv). $u(x, 4) = x^2, 0 \leq x \leq 4$

By dividing the square into 16 square meshes of side 1 and always correcting the computed values to two places of decimals, obtain the values of u at 9 interior pivotal points. (ND14)

5.5 Finite difference technique for the soln. of 2D Poisson's equations on rectangular domain

Two dimensional Poisson equation is

$$\nabla^2 u = f(x, y) \Rightarrow u_{xx} + u_{yy} = f(x, y) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

where $x = ih, y = jk = jh$ $(\because k = h)$

Standard five - point formula(SFPF) for Poisson equation :

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 \cdot f(ih, jh)$$

5.5.1 Part A

1. What is the error for solving Laplace and Poisson's equations by finite difference method? (ND2010)

Solution : For SFPF

$$\text{T.E.} = \frac{h^4}{12}\left[\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right]_{i,j} + \cdots$$

For DFPF

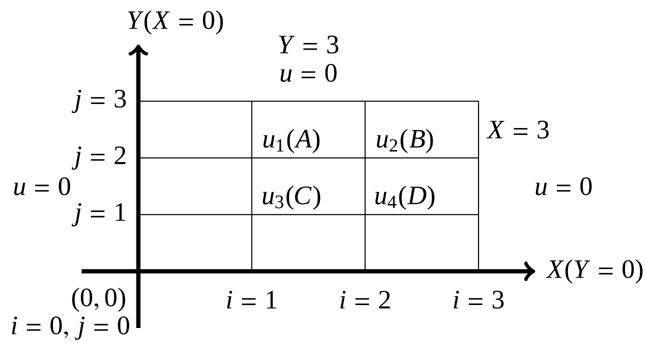
$$\text{T.E.} = \frac{h^4}{6}\left[\frac{\partial^4 u}{\partial x^4} + 6\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}\right]_{i,j} + \cdots$$

2. Write the difference scheme for solving the Poisson equation $\nabla^2 u = f(x, y)$. (AM2012)

5.5.2 Part B

Example 5.12. Solve the Poisson equation $\nabla^2 u = -10\left(x^2 + y^2 + 10\right)$, over the square with sides $x = 0, y = 0, x = 3, y = 3$ with $u = 0$ on the boundary, taking $h = 1$.

Solution: For the given Poisson equation $\nabla^2 u = -10\left(x^2 + y^2 + 10\right)$, the square matrix 3×3 matrix is



Standard five - point formula(SFPF) for Poisson equation :

$$
\begin{aligned}
u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} &= h^2 \cdot f\,(ih, jh) \\
&= -10\left[(ih)^2 + (jh)^2 + 10\right] \\
&= -10\left[i^2 + j^2 + 10\right] && [\because h = 1]
\end{aligned}
$$

At $A(i = 1, j = 2)$:

$$
\begin{aligned}
u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} &= -10\left[1^2 + 2^2 + 10\right] = -150 \\
0 + u_2 + u_3 + 0 - 4u_1 &= -150 \\
u_1 &= \frac{1}{4}\left(150 + u_2 + u_3\right) \qquad (1)
\end{aligned}
$$

At $B(i = 2, j = 2)$:

$$
\begin{aligned}
u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} &= -10\left[2^2 + 2^2 + 10\right] = -180 \\
u_1 + 0 + u_4 + 0 - 4u_2 &= -180 \\
u_2 &= \frac{1}{4}\left(180 + u_1 + u_4\right) \qquad (2)
\end{aligned}
$$

At $C(i = 1, j = 1)$:

$$
\begin{aligned}
0 + u_1 + 0 + u_4 - 4u_3 &= -10\,[12] = -120 \\
u_3 &= \frac{1}{4}\left(120 + u_1 + u_4\right) \qquad (3)
\end{aligned}
$$

At $D(i = 2, j = 1)$:

$$
\begin{aligned}
u_3 + 0 + u_2 + 0 - 4u_4 &= -10\,[15] = -150 \\
u_4 &= \frac{1}{4}\left(150 + u_2 + u_3\right) \qquad (4)
\end{aligned}
$$

From (1) and (4) we get $u_1 = u_4$

Use any method to find $u_1 = u_4, u_2, u_3$

Now we are going to use Gauss - Seidel method:

[$\because$ Equations (1), (2), (3), (4) are diagonally dominant]

Initially find u_1 by putting $u_2 = u_3 = 0$

Iterations become

$$\begin{array}{c|c|c} u_1^{(1)} = 37.5 & u_2^{(1)} = 63.75 & u_3^{(1)} = 48.5 \\ u_1^{(2)} = 65 & u_2^{(2)} = 80 & u_3^{(2)} = 60 \\ u_1^{(3)} = 70 & u_2^{(3)} = 80 & u_3^{(3)} = 65 \\ u_1^{(4)} = 75 & u_2^{(4)} = 82.5 & u_3^{(4)} = 67.5 \\ u_1^{(5)} = 75 & u_2^{(5)} = 82.5 & u_3^{(5)} = 67.5 \end{array}$$

$\therefore u_1 = u_4 = 75, u_2 = 82.5, u_3 = 67.5$

Example 5.13. Solve the Poisson equation $\nabla^2 u = -81xy$, $0 < x < 1$, $0 < y < 1$, given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ & $h = \frac{1}{3}$.

Solution: [Ans : $u_1 = u_4 = 51.0, u_2 = 76.5, u_3 = 25.8$]

Example 5.14. Solve $\nabla^2 u = 8x^2y^2$ in the square mesh given $u = 0$ on the four boundaries dividing the square into 16 sub squares of length 1 unit.

Solution: [Ans : $u_1 = u_3 = u_7 = u_9 = -3, u_2 = u_4 = u_6 = u_8 = -2, u_5 = -2$]

5.5.3 Anna University Questions

1. Solve $\nabla^2 u = 8x^2y^2$ in the square region $-2 \le x, y \le 2$ with $u = 0$ on the boundaries after dividing the region into 16 sub intervals of length 1 unit. (ND10)

   Solution: [$u_1 = 51.0833, u_2 = 76.5417, u_3 = 25.7915, u_4 = 51.0833$]

2. Solve the equation $\nabla^2 u = -10\left(x^2 + y^2 - 10\right)$ over the square mesh with sides $x = 0, y = 0, x = 3, y = 3$ with $u = 0$ on the boundary with mesh length 1 unit. (AM11)

   Solution: [Ref : Example 5.20]

3. Solve $\nabla^2 u = 8x^2y^2$ for square mesh given $u = 0$ on the four boundaries dividing the square into 16 sub-squares of length 1 unit. (ND11)

   Solution: [$u_1 = 51.0833, u_2 = 76.5417, u_3 = 25.7915, u_4 = 51.0833$]

4. Solve the Poisson equation $\nabla^2 u = -10\left(x^2 + y^2 + 10\right)$ over the square mesh with sides $x = 0, y = 0, x = 3$ and $y = 3$ with u = 0 on the four boundary and mesh length 1 unit. (ND12)

   Solution: [Ref : Example 5.20]

5. Solve $\nabla^2 u = 8x^2y^2$ over the square $x = -2, x = 2, y = -2, y = 2$ with $u = 0$ on the boundary and mesh length = 1. (MJ13)

   Solution: [$u_1 = 51.0833, u_2 = 76.5417, u_3 = 25.7915, u_4 = 51.0833$]

6. Solve the Poisson equation $\nabla^2 u = -10\left(x^2 + y^2 + 10\right), 0 \le x \le 3, 0 \le y \le 3, u = 0$ on the boundary. (AM14)

   Solution: [Ref : Example 5.20]

5.6 One dimensional heat flow equation by explicit method(Bender-Schmidt's method)

One dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$,
where $\alpha^2 = \frac{1}{a}$

$$\frac{\partial u}{\partial t} = \frac{1}{a}\frac{\partial^2 u}{\partial x^2}$$
$$\text{i.e., } a\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(i) Bender-Schmidt's difference equation is

$$u_{i,j+1} = \lambda\left(u_{i-1,j} + u_{i+1,j}\right) + (1-2\lambda)\,u_{i,j} \tag{1}$$

where $\lambda = \frac{k}{ah^2}$ $\left(\text{Here } \lambda \neq \frac{1}{2}\right)$.

Equation (1) is called explicit formula. This is valid if $0 < \lambda < \frac{1}{2}$.
Bender-Schmidt's difference equation is

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \tag{2}$$

where $\lambda = \frac{1}{2} = \frac{k}{ah^2}$ i.e., $k = \frac{a}{2}h^2$.
Equation (2) is called Bender-Schmidt's recurrence equation.

5.6.1 Part A

1. Write down Bender-Schmidt's difference scheme in general form and using suitable value of λ, write the scheme in simplified form. (ND2012)

5.6.2 Part B

Example 5.15. Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given $u(0,t) = 0, u(4,t) = 0$, $u(x,0) = x(4-x)$. Assume $h = 1$. Find the values of u upto $t = 5$.

Solution: Given $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$ is of the form $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t} \Rightarrow a = 2$.

By Bender-Schmidt's relation, $k = \frac{a}{2}h^2 \Rightarrow k = \frac{2}{2}(1)^2 = 1$

Step size in time $(t) = k = 1 = \Delta t$

Step size in $x = h = 1 = \Delta x$

The values of u_{ij} are calculated by Bender-Schmidt's recurrence equation formula $u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$ as follows:

		⟵	⟵	'x' direction	⟶	⟶
	$t \backslash x$	0	1	2	3	4
↑	0	$u(0,t)=0$	$u(1,0)=1(4-1)$ $=3$	$u(2,0)=2(4-2)$ $=4$	$u(3,0)=3(4-3)$ $=3$	$u(4,t)=0$
↑	1	0	$\mathbf{\frac{0+4}{2}=2}$	$\mathbf{\frac{3+3}{2}=3}$	$\mathbf{\frac{4+0}{2}=2}$	0
't'	2	0	**1.5**	**2**	**1.5**	0
dir.	3	0	**1**	**1.5**	**1**	0
↓	4	0	**0.75**	**1**	**0.75**	0
↓	5	0	**0.5**	**0.75**	**0.5**	0

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Example 5.16. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$, $f(0,t) = f(5,t) = 0$, $f(x,0) = x^2\left(25 - x^2\right)$, find f in the range takingn $h = 1$ and upto 5 seconds.

Solution: Given $\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$ is of the form $\frac{\partial^2 f}{\partial x^2} = a\frac{\partial f}{\partial t} \Rightarrow a = 1, h = 1$

By Bender-Schmidt's relation, $k = \frac{a}{2}h^2 \Rightarrow k = \frac{1}{2}(1)^2 \Rightarrow k = \frac{1}{2}$

Step size in time $(t) = k = \frac{1}{2} = \Delta t$

Step size in $x = h = 1 = \Delta x$

Given $f(0,0) = 0$, $f(1,0) = 1^2\left(25 - 1^2\right) = 24$, $f(2,0) = 2^2\left(25 - 2^2\right) = 84$,

$f(3,0) = 3^2\left(25 - 3^2\right) = 144$, $f(4,0) = 4^2\left(25 - 4^2\right) = 144$, $f(5,0) = 0$

The values of $u_{i,j}$ are calculated by $u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$ as follows:

		⟵	⟵	'x' direction	⟶	⟶	⟶
	$t \backslash x$	0	1	2	3	4	5
↑	0	0	24	84	144	144	0
't'	0.5	0	**42**	**84**	**114**	**72**	0
d	1	0	**42**	**78**	**78**	**57**	0
i	1.5	0	**39**	**60**	**67.5**	**39**	0
r	2	0	**30**	**53.25**	**49.5**	**33.75**	0
e	2.5	0	**26.625**	**39.75**	**43.5**	**24.75**	0
c	3	0	**19.875**	**35.0625**	**32.25**	**21.75**	0
t	3.5	0	**17.5312**	**26.0625**	**28.4062**	**16.125**	0
i	4	0	**13.0312**	**22.9687**	**21.0938**	**14.2031**	0
o	4.5	0	**11.4843**	**17.0625**	**18.5859**	**10.5469**	0
n	5	0	**8.5312**	**15.0351**	**13.8047**	**9.2930**	0

Example 5.17. Solve $u_t = u_{xx}$ subject to $u(0,t) = 0$, $u(1,t) = 0$ and $u(x,0) = \sin \pi x$, $0 < x < 1$.

Solution: Given Here $0 < x < 1$, $a = 1$, take $h = 0.2$

$k = \frac{a}{2}h^2 \Rightarrow k = \frac{1}{2}(0.2)^2 \Rightarrow k = 0.02$. Form table.

Example 5.18. **Solve $u_t = u_{xx}$, given $u(0,t) = 0, u(4,t) = 0$ and $u(x,0) = x(4-x)$ assuming $h = k = 1$. Find the values of u upto $t = 5$.**

Solution:

Example 5.19. $\left\{\textbf{Here } \lambda \neq \frac{1}{2} \textbf{ [Otherwise, the values of } h \& k \textbf{ are given]}\right\}$ **Solve $u_t = u_{xx}, 0 \leq x \leq 1, t > 0, u(x,0) = 100\left(x - x^2\right), u(0,t) = u(1,t) = 0, h = \frac{1}{4}, k = \frac{1}{16}$.**

Solution: Given $u_t = u_{xx}, a = 1, h = \frac{1}{4}, k = \frac{1}{16}$.

Let $\lambda = \frac{k}{ah^2} = \frac{1/16}{(1)(1/16)} = 1 \Rightarrow \lambda = 1\left(\neq \frac{1}{2}\right)$.

We need to use formula $u_{i,j+1} = \lambda\left(u_{i-1,j} + u_{i+1,j}\right) + (1-2\lambda)\,u_{i,j}$ (1)

When $\lambda = 1, (1) \Rightarrow u_{i,j+1} = \left(u_{i-1,j} + u_{i+1,j} - u_{i,j}\right)$.

The values of $u_{i,j}$ are calculated by $u_{i,j+1} = \left(u_{i-1,j} + u_{i+1,j} - u_{i,j}\right)$ as follows:

		⟵	⟵	'x' direction	⟶	⟶
	t \ x	0	0.25	0.5	0.75	1
↑	0	0	$u(0.25,0)$ $= 100\left(0.25 - 0.25^2\right)$ $= 18.75$	$u(0.5,0)$ $= 100\left(0.5 - 0.5^2\right)$ $= 25$	$u(0.75,0)$ $= 100\left(0.75 - 0.75^2\right)$ $= 18.75$	0
↑	$\frac{1}{16}$	0	**6.25**	**12.5**	**6.25**	0
't'	$\frac{2}{16}$	0	**6.25**	**0**	**6.25**	0
dir.	$\frac{3}{16}$	0	**−6.25**	**12.5**	**−6.25**	0
↓	$\frac{4}{16}$	0	**18.75**	**− 25**	**18.75**	0

5.6.3 Anna University Questions

1. Solve $u_t = u_{xx}$ in $0 < x < t, t > 0$ given that $u(0,t) = 0, u(5,t) = 0, u(x,0) = x^2\left(25 - x^2\right)$. Compute u upto $t = 2$ with $\Delta x = 1$, by using Bender-Schmidt formula. (ND10)

   Solution: [Ref : Example 5.6.]

2. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, given $u(0,t) = 0, u(5,t) = 0, u(x,0) = x^2\left(25 - x^2\right)$, fijnd u in the range taking $h = 1$ upto 3 seconds using Bender-Schmidt recurrence equation. (AM11)

   Solution: [Ref : Example 5.6.]

3. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, subject to $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin \pi x, 0 < x < 1$, using Bender-Schmidt method. (MJ12)

   Solution: [Ref : Example 5.7.]

4. Solve $u_{xx} = 32u_t, h = 0.25$ for $t \geq 0, 0 < x < 1, u(0,t) = 0, u(x,0) = 0, u(1,t) = t$. (MJ13)

   Solution: [Ref : Similar Example 5.5.]

5. Using Bender-Schmidt's method solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(0,t) = 0, u(1,t) = 0, u(x,0) = \sin \pi x, 0 < x < 1$ and $h = 0.2$. Find the value of u upto $t = 0.1$. (AM14)

   Solution: [Ref : Example 5.7.]

5.7 One dimensional heat flow equation by implicit method (Crank-Nicolson's method)

Crank-Nicolson's formula is

$$u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right)$$

5.7.1 Part A

1. Write down the Crank - Nicolson's formula to solve parabolic equation. (ND2010)
   (Or) State Crank-Nicholson's difference scheme. (ND2012)

   Solution :

$$\frac{1}{2}\lambda u_{i+1,j+1} + \frac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1)\,u_{i,j+1} = -\frac{1}{2}\lambda u_{i+1,j} - \frac{1}{2}\lambda u_{i-1,j} + (\lambda-1)\,u_{i,j}$$

$$\text{i.e., } \lambda\left(u_{i+1,j+1} + u_{i-1,j+1}\right) - 2(\lambda+1)\,u_{i,j+1} = 2(\lambda-1)\,u_{i,j} - \lambda\left(u_{i+1,j} + u_{i-1,j}\right)$$

2. State whether the Crank Nicholson's scheme is an explicit or implicit scheme. Justify. (AM2014)

5.7.2 Part B

Example 5.20. Solve $u_t = u_{xx}$, $0 < x < 5, t \geq 0$, given that $u(x,0) = 20, u(0,t) = 0$, $u(5,t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicolson's method.

Solution: Given $u_t = u_{xx}, a = 1, h = 1$,
By Crank-Nicolson's formula,
$\lambda = \frac{k}{ah^2} = 1 \Rightarrow k = ah^2 = 1(1)^2 \Rightarrow k = 1$
Crank-Nicolson's formula is

$$u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right)$$

The values of $u_{i,j}$ are calculated as follows:

		⟵ ⟵ 'x' direction ⟶ ⟶ ⟶					
	$t \backslash x$	0	1	2	3	4	5
↑	0	0	20	20	20	20	0
't'	1	0	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	$\boldsymbol{u_3}$	$\boldsymbol{u_4}$	0

Use Crank-Nicolson's formula $u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right)$, we get

$$u_1 = \frac{1}{4}(0 + 0 + 20 + u_2) \Rightarrow 4u_1 = 20 + u_2 \Rightarrow 4u_1 - u_2 = 20 \quad (1)$$

$$u_2 = \frac{1}{4}(u_1 + 20 + 20 + u_3) \Rightarrow 4u_2 = 40 + u_1 + u_3 \Rightarrow u_1 - 4u_2 + u_3 = -40 \quad (2)$$

$$u_3 = \frac{1}{4}(u_2 + 20 + 20 + u_4) \Rightarrow 4u_3 = 40 + u_2 + u_4 \Rightarrow u_2 - 4u_3 + u_4 = -40 \quad (3)$$

$$u_4 = \frac{1}{4}(u_3 + 20 + 100 + 100) \Rightarrow 4u_4 = 220 + u_3 \Rightarrow u_3 - 4u_4 = -220 \quad (4)$$

Solving (1) and (2), we get

$$15u_2 - 4u_3 = 180 \tag{5}$$

Solving (3) and (4), we get

$$4u_2 - 15u_3 = -380 \tag{6}$$

Solving (5) and (6), we get

$$209u_3 = 6420$$
$$\Rightarrow u_3 = 30.72$$
$$\therefore (4) \Rightarrow 30.72 - 4u_4 = -220$$
$$\Rightarrow u_4 = 62.68$$
$$\therefore (3) \Rightarrow u_2 - 4(30.72) + 62.68 = -40$$
$$\Rightarrow u_2 = 20.2$$
$$\therefore (1) \Rightarrow 4u_1 - u_2 = 20$$
$$\Rightarrow u_1 = 10.05$$

$\therefore$ The values of u are 10.05,20.2,30.72,62.68.

Example 5.21. Solve by Crank-Nicolson's method, the equation $u_{xx} = u_t$ subject to $u(x,0) = 0$, $u(0,t) = 0$ & $u(1,t) = t$ for two time steps.

Solution: Given $u_t = u_{xx}, a = 1, x = 0$ to $x = 1$, take $h = \frac{1}{4}$.

$\therefore k = ah^2 = \frac{1}{16}$

Crank-Nicolson's formula is

$$u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right)$$

The values of $u_{i,j}$ are calculated as follows:

	t \\ x	0	1	2	3	4
		⟵	⟵	'x' direction	⟶	⟶
↑	0	0	0	0	0	0
't'	$\frac{1}{16}$	0	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	$\boldsymbol{u_3}$	$\frac{1}{16}$
↓	$\frac{2}{16}$	0	$\boldsymbol{u_4}$	$\boldsymbol{u_5}$	$\boldsymbol{u_6}$	$\frac{2}{16}$

Use Crank-Nicolson's formula $u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right)$, we get

$$u_1 = \frac{1}{4}(0 + 0 + 0 + u_2) \Rightarrow 4u_1 = u_2 \tag{1}$$

$$u_2 = \frac{1}{4}(u_1 + 0 + 0 + u_3) \Rightarrow 4u_2 = u_1 + u_3 \tag{2}$$

$$u_3 = \frac{1}{4}\left(u_2 + 0 + 0 + \frac{1}{16}\right) \Rightarrow 4u_3 = u_2 + \frac{1}{16} \tag{3}$$

$$\therefore (2) \Rightarrow u_2 = \left(\frac{u_1}{4} + \frac{u_2}{4} + \frac{1}{64}\right) \Rightarrow u_2 = \frac{1}{224} = 0.0045$$

$$\therefore (1) \Rightarrow u_1 = \frac{1}{896} = 0.0011$$

$$\therefore (3) \Rightarrow u_3 = 0.0168$$

&

$$u_4 = \frac{1}{4}(0 + 0 + u_2 + u_5) \Rightarrow u_4 = \frac{1}{4}(0.0045 + u_5) \tag{4}$$

$$u_5 = \frac{1}{4}(u_4 + u_1 + u_3 + u_6) \Rightarrow u_5 = \frac{1}{4}(u_4 + 0.0011 + 0.0168 + u_6) \tag{5}$$

$$u_6 = \frac{1}{4}\left(u_5 + u_2 + \frac{1}{16} + \frac{2}{16}\right) \Rightarrow u_6 = \frac{1}{4}\left(u_5 + 0.0045 + \frac{3}{16}\right) \tag{6}$$

$$u_4 = 0.005899, u_5 = 0.01913, u_6 = 0.05277$$

The values of u are 0.0011, 0.0045, 0.0168, 0.005899, 0.01913, 0.05277

Example 5.22. Solve $u_t = u_{xx}$, $0 < x < 2, t \geq 0$, given that $u(0,t) = u(2,t) = 0, t > 0$ and $u(x,0) = \sin\frac{\pi x}{2}, 0 \leq x \leq 2$. Using $\Delta x = h = 0.5, \Delta t = 0.25$ for 2 steps by Crank-Nicolson's implicit finite difference method.

Solution: The u values are 0.3864, 0.5469, 0.3867, 0.2115, 0.2991, 0.2115.

5.7.3 Anna University Questions

1. Obtain the Crank - Nicholson finite difference method by taking $\lambda = \frac{kc^2}{h^2} = 1$. Hence, find $u(x,t)$ in the rod for two times steps for the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, given $u(x,0) = \sin(\pi x), u(0,t) = 0, u(1,t) = 0$. Take $h = 0.2$. (AM10)

   Solution:

		$\longleftarrow$	$\longleftarrow$	'x' direction	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$
	$t \backslash x$	0	0.588	0.951	0.951	0.588	0
$\uparrow$	0	0	0.399	0.646	0.646	0.399	0
't'	1	0	0.271	0.439	0.439	0.271	0

2. Solve $u_{xx} = 32u_t, h = 0.25$ for $t \geq 0, 0 < x < 1, u(0,t) = 0, u(x,0) = 0, u(1,t) = t$. (MJ13)

   Solution:

		$\longleftarrow$	$\longleftarrow$	'x' direction	$\longrightarrow$	$\longrightarrow$
	$t \backslash x$	0	0.25	0.5	0.75	1
$\uparrow$	0	0	0	0	0	0
$\uparrow$	1	0	0	0	0	1
't'	2	0	0	0	0.5	2
$\downarrow$	3	0	0	0.25	1	3
$\downarrow$	4	0	0.125	0.5	1.625	4
$\downarrow$	5	0	0.25	0.875	2.25	5

3. Using Crank-Nicolson's scheme, solve $16\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0$ subject to $u(x,0) = 0, u(0,t) = 0, u(1,t) = 100t$. Compute u for one step in t direction taking $h = \frac{1}{4}$. (ND13)

Solution:

		⟵	⟵	'x' direction	⟶	⟶
	t \ x	0	0.25	0.5	0.75	1
↑	0	0	0	0	0	0
't'	1	0	u_1	u_2	u_3	100

$\therefore u_1 = 1.7857, u_2 = 7.1429, u_3 = 26.7857$

4. Solve by Crank-Nicolson's method $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ for $0 < x < 1, t > 0$ given that $u(0,t) = 0$, $u(1,t) = 0$ and $u(x,0) = 100x(1-x)$. Compute u for some time step with $h = \frac{1}{4}$ and $k = \frac{1}{64}$. (ND14)

5.8 One dimensional wave equation by explicit method

$$\text{One dimensional wave equation is } \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$
$$\text{valid for } k = \frac{h}{a}$$

After filling given boundary conditions

(i.e., after filling first column, first row, last column by given conditions)

To find unknown 2^{nd} row values alone by

$$\frac{\text{sum of diagonals}}{2}$$

To find unknown 3^{rd} row and onwards by

$$\text{sum of diagonals} - 2^{nd} \text{ previous value in same column}$$

5.8.1 Part A

1. Write down the explicit finite difference method for solving one dimensional wave equation. (MJ2010)

Solution : The formula to solve numerically the wave equation $a^2 u_{xx} - u_{tt} = 0$ is

$$u_{i,j+1} = 2\left(1 - \lambda^2 a^2\right) u_{i,j} + \lambda^2 a^2 \left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1}$$

2. In one dimensional wave equation, write down the equation of explicit scheme. (MJ2011)

Solution : The general form of the difference equation to solve the equation $u_{tt} = a^2 u_{xx}$ is

$$u_{i,j+1} = 2\left(1 - \lambda^2 a^2\right) u_{i,j} + \lambda^2 a^2 \left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1} \quad (1)$$

If $\lambda^2 a^2 = 1$, the coefficient of $u_{i,j}$ in (1) is $= 0$

The recurrence equation (1) takes the simplified form

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

5.8.2 Part B

Example 5.23. Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0,t) = 0, y(1,t) = 0, y_t(x,0) = 0$ & $y(x,0) = 10 + x(1-x)$.

Solution: Given differential equation $y_{tt} = y_{xx}$ is of one dim. wave equation form.

i.e., One dimensional wave equation is $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$

$$\text{valid for } k = \frac{h}{a}$$

Here coefficient of y_{xx} in $y_{tt} = a^2 y_{xx}$ is $1 \Rightarrow a^2 = 1 \Rightarrow a = 1$ Given $k = t$ difference $= 0.1 = k$

WKT, $k = \dfrac{h}{a} \Rightarrow 0.1 = \dfrac{h}{1} \Rightarrow h = 0.1$

Now draw table as follows:

t \ x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	10+0.1(0.9)=10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
0.1	0	$\dfrac{0+10.16}{2} = 5.08$	**10.15**	**10.20**	**10.23**	**10.24**	**10.23**	**10.20**	**10.15**	**5.08**	0
0.2	0	**0+10.15−10.09=0.06**	**5.12**	**10.17**	**10.20**	**10.21**	**10.20**	**10.17**	**5.12**	**0.06**	0
0.3	0	**0+5.12−5.08=0.04**	**0.08**	**5.12**	**10.15**	**10.16**	**10.15**	**5.12**	**0.08**	**0.04**	0
0.4	0	**0+0.08−0.06=0.02**	**0.04**	**0.06**	**5.08**	**10.09**	**5.08**	**0.06**	**0.04**	**0.02**	0
0.5	0	**0+0.04 − 0.04 = 0**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	0

$y(0,t) = 0 \rightarrow$ $\qquad$ $y(1,t) = 0 \rightarrow$

Example 5.24. Solve numerically, $4u_{xx} = u_{tt}$ with the boundary conditions $u(0,t) = 0$, $u(4,t) = 0$ and initial conditions $u_t(x,0) = 0$ and $u(x,0) = x(4-x)$, taking $h = 1$(for 4 time steps).

Solution:

Example 5.25. Approximate the solution to the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $0 < x < 1$, $t > 0$, $u(0,t) = u(1,t) = 0$, $t > 0$, $u(x,0) = \sin 2\pi x$, $0 \le x \le 1$ and $\dfrac{\partial u(x,0)}{\partial t} = 0$, $0 \le x \le 1$ with $\Delta x = 0.25$ and $\Delta t = 0.25$.

Solution:

5.8.3 Anna University Questions

1. Solve $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0$ given $u(x,0) = 0$, $\dfrac{\partial u}{\partial t}(x,0) = 0, u(0,t) = 0$ and $u(1,t) = 100\sin \pi t$. Compute $u(x,t)$ for four times steps with $h = 0.25$. (ND10)

   Solution:

		$\longleftarrow$	$\longleftarrow$	'x' direction	$\longrightarrow$	$\longrightarrow$
	$t \backslash x$	0	0.25	0.5	0.75	1
$\uparrow$	0	0	0	0	0	0
't'	0.25	0	0	0	0	70.7106
	0.50	0	0	0	70.7106	100
	0.75	0	0	70.7106	100	70.7106
	1.0	0	70.7106	100	70.7106	0

2. Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, 0 < x < 1, t > 0$ satisfying the conditions $u(x,0) = 0$, $\dfrac{\partial u}{\partial t}(x,0) = 0, u(0,t) = 0$ and $u(1,t) = \dfrac{1}{2}\sin \pi t$. Compute $u(x,t)$ for 4 time-steps by taking $h = \dfrac{1}{4}$. (ND12)

Solution:

		$\longleftarrow$	$\longleftarrow$	'x' direction	$\longrightarrow$	$\longrightarrow$
	$t \backslash x$	0	0.25	0.5	0.75	1
$\uparrow$	0	0	0	0	0	0
$\uparrow$	0.25	0	0	0	0	0.3536
't'	0.50	0	0	0	0.3536	100
$\downarrow$	0.75	0	0	0.3536	0.5	0.3536
$\downarrow$	1.0	0	0.3536	0.5	0.3536	0

3. Solve $4u_{tt} = u_{xx}, u(0,t) = 0, u(4,t) = 0, u(x,0) = x(4-x), u_t(x,0) = 0, h = 1$ upto $t = 4$. (MJ13)

4. Solve $u_{tt} = u_{xx}, 0 < x < 2, t > 0$ subject to $u(x,0) = 0, u_t(x,0) = 100\left(2x - x^2\right), u(0,t) = 0, u(2,t) = 0$, choosing $h = \dfrac{1}{2}$ compute u for four times steps. (ND13)

5. Solve the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $0 < x < 1, t > 0$, $u(0,t) = u(1,t) = 0$, $t > 0$, $u(x,0) = \begin{cases} 1 & 0 \le x \le 0.5 \\ -1 & 0.5 \le x \le 1 \end{cases}$ and $\dfrac{\partial u}{\partial t}(x,0) = 0$, using $h = k = 0.1$, find u for three steps. (AM14)

5.8.4 Assignment problems

1. Solve the boundary value problem $y'' = xy$ subject to the condition $y(0) = y'(0) = 1, y(1) = 1$, taking $h = 1/3$, by finite difference method. (ND10)

2. Using the finite difference method, compute $y(0.5)$, given $y'' - 64y + 10 = 0, x \in (0,1), y(0) = y(1) = 0$, subdividing the interval into (i) 4 equal parts (ii) 2 equal parts. (ND11)

3. Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$. (MJ12)

4. Solve $u_t = u_{xx}$ in $0 < x < t, t > 0$ given that $u(0,t) = 0, u(5,t) = 0, u(x,0) = x^2\left(25 - x^2\right)$. Compute u upto $t = 2$ with $\Delta x = 1$, by using Bender-Schmidt formula. (ND10)

5. Solve $u_{xx} = 32u_t, h = 0.25$ for $t \ge 0, 0 < x < 1, u(0,t) = 0, u(x,0) = 0, u(1,t) = t$. (MJ13)

6. Using Bender-Schmidt's method solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(0,t) = 0, u(1,t) = 0, u(x,0) = \sin \pi x, 0 < x < 1$ and $h = 0.2$. Find the value of u upto $t = 0.1$. (AM14)

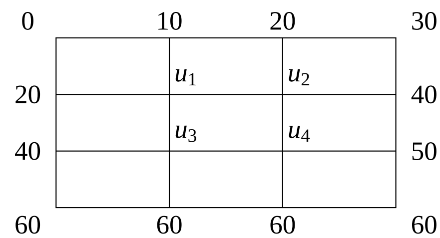
7. Obtain the Crank - Nicholson finite difference method by taking $\lambda = \frac{kc^2}{h^2} = 1$. Hence, find $u(x,t)$ in the rod for two times steps for the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, given $u(x,0) = sin(x), u(0,t) = 0, u(1,t) = 0$. Take $h = 0.2$. (AM10)

8. Solve $u_{xx} = 32u_t, h = 0.25$ for $t \geq 0, 0 < x < 1, u(0,t) = 0, u(x,0) = 0, u(1,t) = t$. (MJ13)

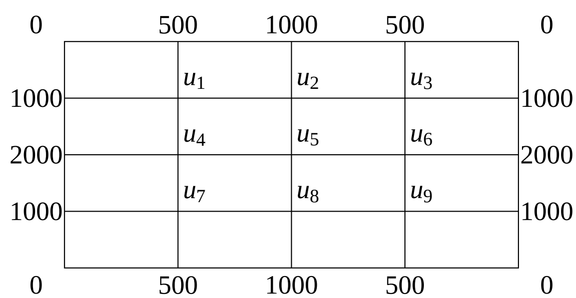
9. Using Crank-Nicolson's scheme, solve $16\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0$ subject to $u(x,0) = 0, u(0,t) = 0, u(1,t) = 100t$. Compute u for one step in t direction taking $h = \frac{1}{4}$. (ND13)

10. Solve by Crank-Nicolson's method $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ for $0 < x < 1, t > 0$ given that $u(0,t) = 0$, $u(1,t) = 0$ and $u(x,0) = 100x(1-x)$. Compute u for some time step with $h = \frac{1}{4}$ and $k = \frac{1}{64}$. (ND14)

11. Deduce the standard five point formula for $\nabla^2 u = 0$. Hence, solve it over the square region given by the boundary conditions as in the figure below: (AM10)



12. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown: (MJ12)



13. By iteration method solve the elliptic equation $u_{xx} + u_{yy} = 0$ over the square region of side 4, satifying the boundary conditions.

(i) $u(0,y) = 0, 0 \leq y \leq 4$	(iii) $u(x,0) = \frac{x^2}{2}, 0 \leq x \leq 4$
(ii) $u(4,y) = 8 + 2y, 0 \leq y \leq 4$	(iv) $u(x,4) = x^2, 0 \leq x \leq 4$

Compute the values at the interior points correct to one decimal with $h = k = 1$. (ND13)

14. By iteratin method, solve the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ over a square region of side 4, satisfying the boundary conditions.

(i) $u(0, y) = 0, 0 \le y \le 4$	(iii) $u(x, 0) = 3x, 0 \le x \le 4$
(ii) $u(4, y) = 12 + y, 0 \le y \le 4$	(iv) $u(x, 4) = x^2, 0 \le x \le 4$

By dividing the square into 16 square meshes of side 1 and always correcting the computed values to two places of decimals, obtain the values of u at 9 interior pivotal points. (ND14)

15. Solve $\nabla^2 u = 8x^2y^2$ for square mesh given $u = 0$ on the four boundaries dividing the square into 16 sub-squares of length 1 unit. (ND11)

16. Solve the Poisson equation $\nabla^2 u = -10\left(x^2 + y^2 + 10\right), 0 \le x \le 3, 0 \le y \le 3, u = 0$ on the boundary. (AM14)

17. Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < 1, t > 0$ satisfying the conditions $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = 0, u(0, t) = 0$ and $u(1, t) = \frac{1}{2}\sin \pi t$. Compute $u(x, t)$ for 4 time-steps by taking $h = \frac{1}{4}$. (ND12)

18. Solve $4u_{tt} = u_{xx}, u(0, t) = 0, u(4, t) = 0, u(x, 0) = x(4 - x), u_t(x, 0) = 0, h = 1$ upto $t = 4$. (MJ13)

19. Solve $u_{tt} = u_{xx}, 0 < x < 2, t > 0$ subject to $u(x, 0) = 0, u_t(x, 0) = 100\left(2x - x^2\right), u(0, t) = 0, u(2, t) = 0$, choosing $h = \frac{1}{2}$ compute u for four times steps. (ND13)

20. Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0 < x < 1, t > 0$, $u(0, t) = u(1, t) = 0$, $t > 0$, $u(x, 0) = \begin{cases} 1 & 0 \le x \le 0.5 \\ -1 & 0.5 \le x \le 1 \end{cases}$ and $\frac{\partial u}{\partial t}(x, 0) = 0$, using $h = k = 0.1$, find u for three steps. (AM14)

6 Anna University Question Papers

6.1 Apr/May 2015(Regulation - 2013)

B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2015

Fourth Semester

Civil Engineering

MA 6469 - NUMERICAL METHODS

(Common to Aeronautical Engineering, Electrical and Electronics Engineering, Instrumentation and Control Engineering, Electronics and Instrumentation Engineering , Instrumentation and Control Engineering, Geo Informatics Engineering, Petrochemical Technology, Production Engineering, Chemical and Electrochemical Engineering, Textile Chemistry and Textile Technology)

(Regulation 2013)

Time : Three hours　　　　Maximum : 100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Interpret Newton Raphson method geometrically.

2. Which of the iterative methods for solving linear system of equations converge faster? Why?

3. Given $y_0 = 3, y_1 = 12, y_2 = 81, y_4 = 100$. Find $\Delta^4 y_0$.

4. Distinguish between Newton divided difference interpolation and Lagrange's interpolation

5. Find $y'(0)$ from the following table.

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

6. Using two point Gaussina quadrature formula evaluate $I = \frac{\pi}{4}\int_{-1}^{1} \sin\left(\frac{\pi t + \pi}{4}\right) dt$.

7. Find by Taylor's series method, the value of y at $x = 0.1$ from $\frac{dy}{dx} = y^2 + x, y(0) = 1$.

8. Distinguish between single step methods and multi-step methods.

9. Classify the follwoign equation: $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$.

10. Express $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ in terms of difference approximation.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Using Newton Raphson method find the real root of $f(x) = 3x + \sin(x) - e^x = 0$ by choosing initial approximation $x_0 = 0.5$. (8)

(ii) Determine the largest eigen value and the corresponding eigen vector of the matrix

$$A = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$ (8)

Or

(b) (i) Apply Graeffe's method to find all the roots of the equation $x^3-2x^2-5x+6 = 0$ by squaring thrice. (8)

(ii) Solve the following system of equations, starting with the initial vector of $[0, 0, 0]$ using Gauss-Seidel method. (8)

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

12. (a) (i) Using Lagrange's interpolation find the interpolated value for $x = 8$ of the table (8)

x :	3.2	2.7	1.0	4.8
y :	22.0	17.8	14.2	38.2

(ii) The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x =height:	100	150	200	250	300	350	400
y =distance:	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Find the values of y when $x = 218$ ft using Newton's forward interpolation formula. (8)

Or

(b) (i) Employ a third order Newton polynomial to estimate l_{n2} with the four points given in table. (8)

x :	1	4	6	5
$f(x)$:	0	1.386294	1.791759	1.609438

(ii) The following values of x and y are given in table : (8)

x :	1	2	3	4
y :	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$.

13. (a) (i) The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:

t :	0	2	4	6	8	10	12
v :	0	10	18	25	29	32	30

A. Estimate approximately the distance covered in 12 minutes, by Simpson's $1/3^{rd}$ rule. (8)

B. Estimate the acceleration at $t = 2$ seconds. (8)

Or

(b) (i) Given that

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\dfrac{dy}{dx}$ at $x = 1.1$. (8)

(ii) Use the Romberg method to get an improved estimate of the integral from $x = 1.8$ to $x = 3.4$ from the data in table with $h = 0.4$. (8)

t :	1.6	1.8	2.0	2.2	2.4	2.6
$f(x)$:	4.953	6.050	7.389	9.025	110.23	13.464
t :	2.8	3	3.2	3.4	3.6	3.8
$f(x)$:	16.445	20.056	24.533	29.964	36.598	44.701

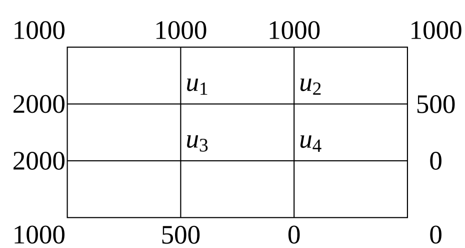
14. (a) (i) Solve the initial value problem $\dfrac{dy}{dx} = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's Bashforth predictor corrector method and for starting solutions, use the information below. $y(0.1) = 0.9117$, $y(0.2) = 0.8494$. Compute $y(0.3)$ using Runge Kutta method of fourth order. (16)

Or

(b) (i) Emply the classical fourth order Runge-Kutta method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to $t = 1$ using a stepsize of 1 with $y(0) = 2$. (8)

(ii) Given $\dfrac{dy}{dx} = xy + y^2$ and $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 0.2267$, evaluate $y(0.4)$ by Milne's predictor corrector method. (8)

15. (a) (i) Given the values of $u(x, y)$ on the boundary of the square in fig. evaluate the function $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this fig by Gauss Seidel method. (8)

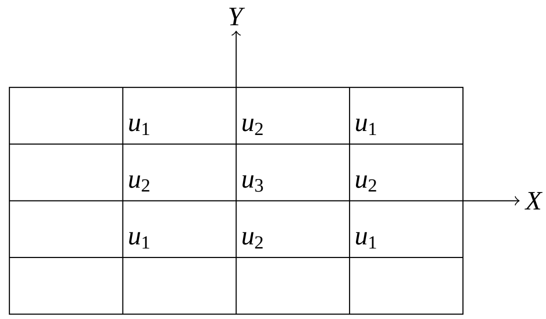


(8)

(ii) Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, subject tot he condition $u(x, 0) = \sin \pi x$, $0 \le x \le 1$; $u(0, t) = u(1, t) = 0$ using Crank-Nicolson method. (8)

Or

(b) (i) Solve the Poisson's equation $\nabla^2 u = 8x^2y^2$ for the square mesh of fig. with $u(x, y) = 0$ on the boundary and mesh length= 1 (8)



(8)

(ii) Evaluate the Pivotal values of the equation $u_{tt} = 16u_{xx}$ taking $\Delta x = 1$ upto $t = 1.25$. The boundary conditions are $u(0, t) = u(5, t) = u_t(x, 0) = 0$ and $u(x, 0) = x^2(5 - x)$. (8)

6.2 Nov/Dec 2015(Regulation - 2013)

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECMBER 2015

Fourth Semester

Civil Engineering

MA 6469 - NUMERICAL METHODS

(Common to Aeronautical Engineering, Electrical and Electronics Engineering, Instrumentation and Control Engineering, Electronics and Instrumentation Engineering , Instrumentation and Control Engineering, Geo Informatics Engineering, Petrochemical Technology, Production Engineering, Chemical and Electrochemical Engineering, Textile Chemistry and Textile Technology)

(Regulation 2013)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. What is the criterion for the convergence of Newton-Raphson method?
2. Give two direct methods to solve a system of linear equations.
3. For cubic splines, what are the $4n$ conditions required to evaluate the unknowns.
4. Construct the divided difference table for the data $(0, 1)$, $(1, 4)$, $(3, 40)$ and $(4, 85)$.
5. Apply two point Gaussian quadrature formula to evaluate $\int_0^2 e^{-x^2} dx$.
6. Under what condition Simpson's $\frac{3}{8}$ rule can be applied and state the formula.
7. Using Euler's method, find $y(0,1)$ given that $\frac{dy}{dx} = x + y$, $y(0) = 1$.
8. State Adam's Predictor-Corrector formulae.
9. What is the central difference approximation for y''?
10. Write down the difference scheme for solving the equation $y_{tt} = \alpha y_{xx}$.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find the largest eigenvalue and the corresponding eigenvector of a matrix $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$. (8)

(ii) Using Gauss Jordan method find the inverse of a matrix $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$. (8)

Or

(b) (i) Apply Gauss-Seidal method to solve the equations (8)

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35.$$

(ii) Find the root of $4x - e^x = 0$ that lies betwen 2 and 3 by Newton-Raphson method. (8)

12. (a) (i) Using Lagrage's interpolation formula calculate the profit in the year 200 from the following data: (8)

Year:	1997	1999	2001	2002
Profit in lakhs of Rs. :	43	65	159	248

(ii) Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values : (8)

x :	0	1	2	3
y :	1	2	1	10

Or

(b) The following values of x and y are given: (16)

x :	1	2	3	4
y :	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$.

13. (a) (i) Using Trapezoidal rule evaluate $\int_0^1 \int_0^1 \frac{dxdy}{x+y+1}$ with $h = 0.5$ along x−direction and $k = 0.25$ along y−direction. (8)

(ii) Find $f'(10)$ from the following data: (8)

x :	3	5	11	27	34
y :	−13	23	899	17315	35606

Or

(b) Use Romberg's method to evaluate $\int_0^1 \frac{dx}{1+x^2}$ correct to 4 decimal places. Also compute the same integral using three point Gaussian quadrature formula. Comment on the obtained values by comparing with the exact values of the integral which is equal to $\frac{\pi}{4}$. (16)

14. (a) Determine the value of $y(0.4)$ using Milne's method given $y' = xy + y^2$, $y(0) = 1$. Use Taylor's series method to get the values of $y(0.1)$, $y(0.2)$ and $y(0.3)$. (16)

Or

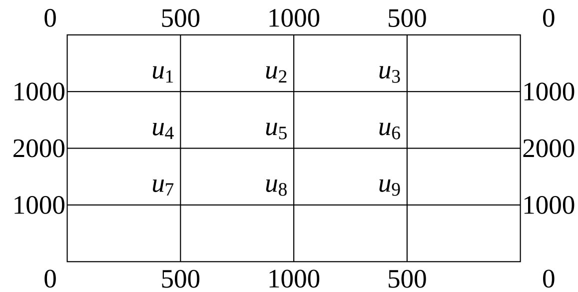
(b) Find $y(0.1)$, $y(0.2)$ and $y(0.3)$ from $y' = x + y^2$, $y(0) = 1$ by using Runge-Kutta method of Fourth order and then find $y(0.4)$ by Adam's method. (16)

15. (a) (i) Solve $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$. (6)

(ii) Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ subject tot he conditions $u(x, 0) = \sin \pi x$, $0 < x < 1$ $u(0, t) = u(1, t) = 0$ using Bender-Schemidt method. (10)

Or

(b) Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown. (16)



6.3 May/Jun 2016(Regulation - 2013)

B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2016

Fourth Semester

Civil Engineering

MA 6469 - NUMERICAL METHODS

(Common to Aeronautical Engineering, Electrical and Electronics Engineering, Instrumentation and Control Engineering, Electronics and Instrumentation Engineering , Instrumentation and Control Engineering, Geo Informatics Engineering, Petrochemical Technology, Production Engineering, Chemical and Electrochemical Engineering, Textile Chemistry and Textile Technology Also common to Petrochemical Technology, Polymer Technology, Plastic Technology & Chemical Engineering and Also Sixth Semester Manufacturing Engineerings)

(Regulation 2013)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. What is the condition for convergence in Fixed point Iteration method?
2. Name the two methods to solve a system of linear simultaneous equations.
3. Construct a table of divided for the given data:

x :	654	658	659	661
y :	2.8156	2.8182	2.8189	2.8202

4. Write down the Newton's forward difference interpolation formula for equal intervals.
5. Write down the general quadrature formula for equidistance ordinates.
6. Write down the forward difference formulae to compute the first two derivatives at $x = x_0$.
7. Write down the improve Euler's formula to first order differential equation.
8. How many values are needed to use Milne's predictor-corrector formula prior to the required value?
9. Write down the diagonal five point formula in the solution of elliptic equations.
10. Classify the partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find the approximate root of $xe^x = 3$ by Newton's method correct to three decimal places. (8)

(ii) Using Gauss-Jordan method solve the given system of equations: (8)

$$10x + y + z = 12,$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

Or

(b) (i) Solve the following system of equations using Jacobi's iteration method. (8)

$$20x + y - 2z = 17,$$
$$3x + 20y - z = -18,$$
$$2x - 3y + 20z = 25$$

(ii) Using power method find the dominant eigen value and the corresponding eigen vector for the given matrix. (8)

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

12. (a) (i) From the given table compute the value of $\sin 38°$. (8)

$x:$	0	10	20	30	40
$\sin x:$	0	0.17365	0.34202	0.5	0.64279

(ii) Using Lagrange's formula find the value of $\log_{10} 323.5$ for the given data: (8)

$x:$	321.8	322.8	324.2	325.0
$\log_{10} x:$	2.50651	2.50893	2.51081	2.51188

Or

(b) (i) Find the cubic polynomial from the following table using Newton's divided dirrerence formula and hence find $f(4)$. (8)

$x:$	0	1	2	3
$y = f(x)$	2	3	12	147

(ii) Find the cubic splines for the following table: (8)

$x:$	1	2	3
$y:$	-6	-1	16

Hence evaluate $y(1.5)$ and $y'(2)$.

13. (a) (i) Find the first and second derivatives of the function tabulated below at $x = 1.5$. (8)

$x:$	1.5	2.0	2.5	3.0	3.5	4.0
$f(x):$	3.375	7.0	13.625	24.0	38.875	59.0

(ii) Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3}dx$ using Simpson's $\frac{1}{3}$ rule with $h = 0.25$. (8)

Or

(b) (i) Evaluate $\int_1^2 \frac{1}{1+x^3} dx$ using Gauss 3 point formula. (8)

(ii) Evaluate $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y) dx\, dy$ by using Trapezoidal rule by taking $h = k = \frac{\pi}{4}$. (8)

14. (a) (i) Using Taylor series method, compute the value of $y(0.2)$ correct to 3 decimal places from $\frac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$. (8)

(ii) Using modified Euler's method, find $y(0.1)$ and $y(0.2)$ for the given equation $\frac{dy}{dx} = x^2 + y^2$, given that $y(0)1$. (8)

Or

(b) (i) Find the value of $y(1.1)$ using Runge-Kutta method of $4^t h$ order for the given equation $\frac{dy}{dx} = y^2 + xy$; $y(1) = 1$. (8)

(ii) Using Adam's method find $y(0.4)$ given that $\frac{dy}{dx} = \frac{xy}{2}$, $y(0) = 1$, $y(0.1) = 1.01$, $y(0.2) = 1.022$, $y(0.3) = 1.023$. (8)

15. (a) Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the interior points of the square region given as below: (16)

0	11.1	17.0	19.7	18.6
	41	42	43	
0				21.9
	44	45	46	
0				21.2
	47	48	49	
0				17.0
0	8.7	12.1	12.8	9.0

Or

(b) Given that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(0,t) = 0$, $u(4,t) = 0$, and $u(x,0) = \frac{x}{3}\left(16 - x^3\right)$. Find u_{ij} : $1, 2, 3, 4$ and $j = 1, 2$ by using Crank-Nicholson method. (16)

6.4 Nov/Dec 2016(Regulation - 2013)

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2016

Fourth Semester

Civil Engineering

MA 6469 - NUMERICAL METHODS

(Common to Aeronautical Engineering, Electrical and Electronics Engineering, Instrumentation and Control Engineering, Electronics and Instrumentation Engineering , Instrumentation and Control Engineering, Geo Informatics Engineering, Petrochemical Technology, Production Engineering, Chemical and Electrochemical Engineering, Textile Chemistry and Textile Technology)

(Regulation 2013)

Time : Three hours Maximum : 100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Derive a formula to find the value of $\sqrt{N}$, where N is a real number, by Newton's method.
2. Which of the iteration method for solving linear system of equation converges faster? Why?
3. Using Lagrange's interpolation formula find y value when $x = 1$ from the following data:

x :	0	−1	2	3
y :	−8	3	1	12

4. State Newton's forward formula and Backward formula.
5. Compare Trapezoidal rule and Simpson's 1/3 rule for evaluating numerical integration.
6. Change the limits of $\int_0^{\pi/2} \sin x \, dx$ into $(-1, 1)$.
7. Compare Single-step method and Multi-step method.
8. Write down the Milne's predictor and corrector formulas.
9. Classify the following equation $u_{xx} + 4u_{xy} + 4u_{yy} - y_x + 2u_y = 0$.
10. Write down the standard five point formula.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find a root of $x \log_{10} x - 1.2 = 0$ using Newton Raphson method correct to three decimal places. (8)

(ii) Solve by Gauss Seidal method, the following system:
$20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$. (8)

Or

(b) (i) Find the dominant Eigen values of $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ using power method. (8)

(ii) Apply Gauss Jordan method, find the solution of the following system:
$2x - y + 3z = 8,\ -x + 2y + z = 4,\ 3x + y - 4z = 0.$ (8)

12. (a) (i) Find an approximate polynomial for $f(x)$ using Lagrange's interpolation for the following data: (8)

$x:$	0	1	2	5
$y = f(x):$	2	3	12	147

(ii) Find the value of y at $x = 21$ from the data given below: (8)

$x:$	20	23	26	29
$y:$	0.3420	0.3907	0.4384	0.4848

Or

(b) (i) Given the tables:

$x:$	5	7	11	13	17
$y = f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's divided difference formula. (8)

(ii) Fit a cubic spline from the given table:

$x:$	1	2	3
$f(x):$	-8	-1	18

Compute $y(1.5)$ and $y'(1)$ using cubic spline. . (8)

13. (a) (i) The population of a certain town is shown in the following table.

Year:	1931	1941	1951	1961	1971
Population (in thousands):	40.6	60.8	79.9	103.6	132.7

Find the rate of growth of the population in the year 1945. (8)

(ii) Evaluate $\int_0^1 \frac{1}{1+x}dx$ using Romberg's method and hence find the value of $\log 2$. (8)

Or

(b) (i) The velocity V of a particle at a distance S from a point on its path is given by the table.

S (ft):	0	10	20	30	40	50	60
V (ft./sec):	47	58	64	65	61	52	38

Estimate the time taken to travel to travel 60 feet by using Simpson's $\frac{1}{3}$ rule. (8)

(ii) Evaluate $\int_1^1 .4 \int_2^2 .4 \frac{1}{xy} dx\, dy$ using Trapezoidal rule by taking $h = k = 0.1$ and verify with actual integration. (8)

14. (a) (i) Find the value of y at $x = 0.1$ from $\frac{dy}{dx} = x^2 y - 1,\ y(0) = 1$ by Taylor's series method. (8)

(ii) Solve $(1 + x)\frac{dy}{dx} = -y^2,\ y(0) = 1$ by Modified Euler's method by choosing $h = 0.1$, find $y(0.1)$ and $y(0.2)$. (8)

Or

(b) (i) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$ at $x = 0.2$. (8)

(ii) Given $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. Compute $y(0.8)$ using Milne's method . (8)

15. (a) (i) Using Bender Schmidt's method solve $u_t = u_{xx}$ subject to the condition, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \sin \pi x$, $0 < x < 1$ and $h = 0.2$. Find the value of u up to $t = 0.1$ (8)

(ii) Evaluate the pivotal values of the equation $u_{tt} = 16u_{xx}$ taking $h = 1$ upto $t = 1.25$. The boundary conditions are $u(0,t) = u(5,t) = u_t(x,0) = 0$ and $u(x,0) = x^2(5 - x)$. (8)

Or

(b) By Iteration method solve the elliptic equation $u_{xx} + u_{yy} = 0$ over the square region of side 4, satisfying the boundary conditions $u(0,y) = 0$, $0 \leq y \leq 4$, $u(4,y) = 12 + y$, $0 \leq y \leq 4$, $u(x,0) = 3x$, $0 \leq x \leq 4$, $u(x,4) = x^2$, $0 \leq x \leq 4$. By dividing the square into 16 square meshes of side 1 and always correcting the computed values to two places to decimals. Obtain the values of u at 9 interior pivotal points. (16)

6.5 Apr/May 2010

B.E./B.Tech. DEGREE EXAMINATION, Apr/May 2010

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Write sufficient condition for convergence of an iterative method for $f(x) = 0$; written as $x = g(x)$.
2. Write down the procedure to find the numerically smallest eigen value of a matrix by power method.
3. Form the divided difference table for the data (0,1) , (1,4) , (3,40) and (4,85).
4. Define a cubic spline $S(x)$ which is commonly used for interpolation.
5. State the Romberg's integration formula with h_1 and h_2. Further, obtain the formula when $h_1 = h$ and $h_2 = h/2$.
6. Use two - point Gaussian quadrature formula to solve $\int\limits_{-1}^{1} \frac{dx}{1+x^2}$.
7. Use Euler's method to find $y(0.2)$ and $y(0.4)$ given $\frac{dy}{dx} = x + y, y(0) = 1$.
8. Write the Adam-Bashforth predictor and corrector formulae.
9. Write down the explicit finite difference method for solving one dimensional wave equation.
10. Write down the standard five point formula to find the numerical solution of Laplace equation.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton - Raphson method. (8)

(ii) Use Gauss - Seidal iterative method to obtain the solution of the equations: $9x - y + 2z = 9; x + 10y - 2z = 15; 2x - 2y - 13z = -17$. (8)

Or

(b) (i) Find the inverse of the matrix by Gauss - Jordan method: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$. (8)

(ii) Find the dominant eigen value and the corresponding eigen vector of the matrix $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. (8)

12. (a) (i) Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12$, $f(1) = 0$, $f(3) = 6$ and $f(4) = 12$. Hence find $f(2)$. (8)

(ii) Find the function $f(x)$ from the following table using Newton's divided difference formula: (8)

x :	0	1	2	4	5	7
$f(x)$:	0	0	–12	0	600	7308

Or

(b) (i) If $f(0) = 1$, $f(1) = 2$, $f(2) = 33$ and $f(3) = 244$, find a cubic spline approximation, assuming $M(0) = M(3) = 0$. Also, find $f(2.5)$. (8)

(ii) Given the following table, find the number of students whose weight is between 60 and 70 lbs: (8)

Weight (in lbs) x :	0 - 40	40 - 60	60 - 80	80 - 100	100 - 120
No. of students y :	250	120	100	70	50

13. (a) (i) Given the following data, find $y'(6)$ and the maximum value of y (if it exists). (8)

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922

(ii) Evaluate $\int_1^{1.2}\int_1^{1.4} \frac{dxdy}{x+y}$ by trapezoidal formula by taking $h = k = 0.1$. (8)

Or

(b) (i) Using Romberg's integration to evaluate $\int_0^1 \frac{dx}{1+x^2}$. (8)

(ii) The velocity v of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
v (m / sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule. (8)

14. (a) (i) Use Taylor series method to find $y(0.1)$ and $y(0.2)$ given that $\frac{dy}{dx} = 3e^x + 2y, y(0) = 0$, correct to 4 decimal accuracy. (8)

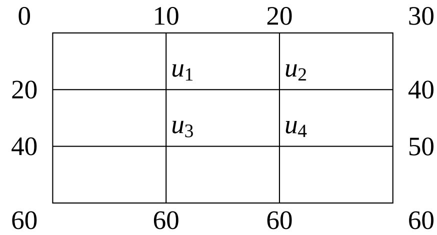
(ii) Use Milne's predictor-corrector formula to find $y(0.4)$, given $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}, y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12$ and $y(0.3) = 1.21$. (8)

Or

(b) (i) Use Rung - Kutta method of fourth order to find $y(0.2)$, given $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, y(0) = 1$, taking $h = 0.2$. (8)

(ii) Given $\frac{dy}{dx} = xy + y^2, y(0) = 1, y(0.1) = 1.1169, y(0.2) = 1.2774, y(0.3) = 1.5041$. Use Adam's method to estimate $y(0.4)$. (8)

15. (a) Deduce the standard five point formula for $\nabla^2 u = 0$. Hence, solve it over the square region given by the boundary conditions as in the figure below: (16)



Or

(b) Obtain the Crank - Nicholson finite difference method by taking $\lambda = \frac{kc^2}{h^2} = 1$. Hence, find $u(x,t)$ in the rod for two times steps for the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, given $u(x,0) = \sin(\pi x), u(0,t) = 0, u(1,t) = 0$. Take $h = 0.2$. (16)

6.6 Nov/Dec 2010

B.E./B.Tech. DEGREE EXAMINATION, Nov/Dec 2010

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. What is Newton's algorithm to solve the equayion $x^2 = 12$?

2. To what kind of a matrix, can the Jacobi's method be applied to obtain the eigenvalues of a matrix?

3. When to use Newton's forward interpolation and when to use Newton's backward interpolation?

4. Find the first and second divided differences with arguments a, b, c of the function $f(x) = \frac{1}{x}$.

5. Write down the formulae for finding the first derivative using Newton's forward difference at $x = x_0$ and Newton's backward difference at $x = x_n$.

6. Evaluate $\int_0^2 e^{-x^2} dx$ by two point Gaussian quadrature formula.

7. Find $y(0.1)$ by using Euler's method given that $\frac{dy}{dx} = x + y, y(0) = 1$.

8. What are multi-step methods? How are they better than single step methods?

9. What is the error for solving Laplace and Poisson's equations by finite difference method?

10. Write down the Crank-Nicolson formula to solve parabolic equation.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find, by power method, the largest eigenvalue and the corresponding eigenvector of a matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ with initial vector $(1 \quad 1 \quad 1)^T$. (8)

(ii) Solve the given system of equations by using Gauss-Seidal iteration method.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$ (8)

Or

(b) (i) Find the inverse of $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ by using Gauss-Jordan method. (8)

(ii) Find the Smallest positive root of $3x = \sqrt{1+\sin x}$ correct to three decimal places by iterative method. (8)

12. (a) (i) Using Lagrange's interpolation formula to fit a polynomial to the given data $f(-1) = -8$, $f(0) = 3$, $f(2) = 1$ and $f(3) = 12$. Hence find the value of $f(1)$. (8)

(ii) Find the value of $\tan 45°15'$ by using Newton's forward difference interpolation formula for (8)

$x°$:	45	46	47	48	49	50
$\tan x°$:	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

Or

(b) Find the natural cubic spline approximation for the function $f(x)$ defined by the following data: (16)

x :	0	1	2	3
$f(x)$:	1	2	33	244

13. (a) (i) From the following table of values of x and y, obtain $y'(x)$ and $y''(x)$ for $x = 16$ (8)

x :	15	17	19	21	23	25
y :	3.873	4.123	4.359	4.583	4.796	5

(ii) Using Romberg's rule evaluate $\int_0^1 \frac{1}{1+x}dx$ correct to three decimal places by taking $h = 0.5, 0.25$, and 0.125. (8)

Or

(b) (i) Find the first derivatives of $f(x)$ at $x = 2$ for the data $f(-1) = -21$, $f(1) = 15$, $f(2) = 12$ and $f(3) = 3$, using Newton's divided difference formula. (8)

(ii) Evaluate $\int_1^5 \left[\int_1^4 \frac{1}{x+y}dx\right] dy$ by Trapezoidal rule in x–direction with $h = 1$ and Simpson's one-third rule in y–direction with $k = 1$. (8)

14. (a) (i) Evaluate the value of y at $x = 0.1, 0.2$ given $\frac{dy}{dx} = x^2y - 1, y(0) = 1$, by Taylor's series method upto four terms. (8)

(ii) Find the value of $y(0.1)$ by Runge-Kutta method of fourth order given $y'' + xy' + y = 0, y(0) = 1$ and $y'(0) = 0$. (8)

Or

(b) Given $\frac{dy}{dx} = xy + y^2, y(0) = 1, y(0.1) = 1.1169$ and $y(0.2) = 1.2774$, find (i) $y(0.3)$ by Runge-Kutta method of fourth order and (ii) $y(0.4)$ by Milne's method. (16)

15. (a) (i) Solve the boundary value problem $y'' = xy$ subject to the condition $y(0) + y'(0) = 1, y(1) = 1$, taking $h = 1/3$, by finite difference method. (8)

(ii) Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0$ given $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0, u(0,t) = 0$ and $u(1,t) = 100\sin \pi t$. Compute $u(x,t)$ for four times steps with $h = 0.25$. (8)

Or

(b) (i) Solve $\nabla^2 u = 8x^2y^2$ in the square region $-2 \le x, y \le 2$ with $u = 0$ on the boundaries after dividing the region into 16 sub intervals of length 1 unit. (8)

(ii) Solve $u_t = u_{xx}$ in $0 < x < 5, t > 0$ given that $u(0,t) = 0, u(5,t) = 0, u(x,0) = x^2\left(25 - x^2\right)$. Compute u upto $t = 2$ with $\Delta x = 1$, by using Bender-Schmidt formula. (8)

6.7 Apr/May 2011

B.E./B.Tech. DEGREE EXAMINATION, Apr/May 2011

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. What is the criterion for the convergence in Newton's method?
2. By Gauss elimination method solve $x + y = 2, 2x + 3y = 5$.
3. Find the divided differences of $f(x) = x^3 - x + 2$ for the arguments $1, 3, 6, 11$.
4. Prove that $\Delta \log(f(x)) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$.
5. Write down the Newton-Cote's formula for equidistant ordinates.
6. When do you apply Simpson's $\frac{1}{3}$ rule, and what is the order of the error in Simpson's $\frac{1}{3}$ rule.
7. Using Taylor series method find $y(1.1)$ given that $y' = x + y, y(1) = 0$.
8. Find $y(0.2)$ for the equation $y' = y + e^x$, given that $y(0) = 0$ by using Euler's method.
9. Classify the partial differential equation $u_{xx} - 2u_{xy} + 4u_{yy} = 0, x, y > 0$.
10. In one dimensional wave equation, write down the equation of explicit scheme.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find the approximate root of $xe^x = 3$ by Newton's method correct to 3 decimal places. (8)

(ii) Apply Gauss-Jordan method to solve the following system of equations $x - y - z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40$. (8)

Or

(b) (i) Solve the following system by Gauss-Seidel method, $x + y + 54z = 110, 27x + 6y + z = 85, 6x + 15y + 2z = 72$. (8)

(ii) Using Jacobi's method, find all the eigenvalues and eigenvectors for the given matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$. (8)

12. (a) (i) Find the expression of $f(x)$ using Lagrange's formula for the following data. (8)

$x:$	0	1	4	5
$f(x):$	4	3	24	39

(ii) Find the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0' = y_3' = 0$. (8)

x:	−1	0	1	2
y:	−1	1	3	35

Or

(b) (i) Find the value of x when $y = 20$ using Lagrange's formula from the following table. (8)

x :	1	2	3	4
$y = f(x)$:	1	8	27	64

(ii) Given the table

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's divided difference formula. (8)

13. (a) (i) Find $f'(10)$ from the following data. (8)

x :	3	5	11	27	34
$f(x)$:	−13	23	899	17135	35606

(ii) Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Romberg's method. (8)

Or

(b) (i) Evaluate $\int_1^2 \frac{1}{1+x^3}dx$ using Gauss three point formula. (8)

(ii) Evaluate $\int_0^1\int_0^1 \frac{1}{x+y+1}dx\,dy$ by using Trapezoidal rule taking $h = 0.5$ and $k = 0.25$. (8)

14. (a) Using Runge-Kutta method of fourth order, find y for $x = 0.1, 0.2, 0.3$ given that $y' = xy + y^2, y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method. (16)

Or

(b) Solve $y' = x - y^2, y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Taylor's method using the value $h = 0.1$. (16)

15. (a) Solve the equation $\nabla^2 u = -10\left(x^2 + y^2 - 10\right)$ over the square mesh with sides $x = 0, y = 0, x = 3, y = 3$ with $u = 0$ on the boundary with mesh length 1 unit. (16)

Or

(b) Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, given $u(0,t) = 0, u(5,t) = 0, u(x,0) = x^2\left(25 - x^2\right)$, find u in the range taking $h = 1$ upto 3 seconds using Bender-Schmidt recurrence equation. (16)

6.8 Nov/Dec 2011

B.E./B.Tech. DEGREE EXAMINATION, Nov/Dec 2011

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Solve $e^x - 3x = 0$ by the method of iteration.
2. Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$.
3. State Lagrange's interpolation formula for unequal intervals.
4. Define cubic spline function.
5. State Simpson's one-third rule.
6. Write down two point Gaussian quadrature formula.
7. State Euler's method to solve $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$.
8. State Adam's predictor-corrector formulae.
9. Classify the PDE $y(x_0) = y_0$.
10. State Standard Five Point formula with relevant diagram.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find an iterative formula to find the reciprocal of a given number N and hence find the value of $\frac{1}{19}$. (8)

(ii) Apply Gauss-Jordan method to find the solution of the following system: (8)

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

Or

(b) (i) Solve, by Gauss-Seidel method, the following system: (8)

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

(ii) Find the largest eigenvalue of $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ by using Power method. (8)

12. (a) The population of a town is as follows:

x Year:	1941	1951	1961	1971	1981	1991
y Population in thousands:	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976. (16)

Or

(b) Determine $f(x)$ as a polynomial in x for the following data, using Newton's divided difference formulae. Also find $f(2)$.

x:	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

(16)

13. (a) Find the first two derivatives of $x^{1/3}$ at $x = 50$ and $x = 56$, for the given table:

x :	50	51	52	53	54	55	56
$y = x^{1/3}$:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

(16)

Or

(b) Evaluate $I = \int_0^6 \frac{1}{1+x} dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) Simpson's one-third rule (iv) Simpson's three-eighth rule. (16)

14. (a) Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$. Find the value of $y(0.1)$ by using Runge-Kutta method of fourth order. (16)

Or

(b) Given that $\frac{dy}{dx} = \frac{1}{2}\left(1 + x^2\right)y^2; y(0) = 1; y(0.1) = 1.06; y(0.2) = 1.12$ and $y(0.3) = 1.21$, evaluate $y(0.4)$ and $y(0.5)$ by Milne's predictor corrector method. (16)

15. (a) Using the finite difference method, compute $y(0.5)$, given $y'' - 64y + 10 = 0, x \in (0, 1), y(0) = y(1) = 0$, subdividing the interval into (i) 4 equal parts (ii) 2 equal parts. (16)

Or

(b) Solve $\nabla^2 u = 8x^2y^2$ for square mesh given $u = 0$ on the four boundaries dividing the square into 16 sub-squares of length 1 unit. (16)

6.9 May/Jun 2012

B.E./B.Tech. DEGREE EXAMINATION, May/Jun 2012

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. State the order of convergence and the criterion for the convergence in Newton's method.
2. Give two direct methods to solve a system of linear equations.
3. For cubic splines, what are the $4n$ conditions required to evaluate the unknowns.
4. Construct the divided difference table for the following data:

x :	0	1	2	5
$f(x)$:	2	3	12	147

5. Apply two point Gaussian quadrature formula to evaluate $\int_{-1}^{1} \frac{1}{1+x^2} dx$.
6. Under what condition, Simpson's 3/8 rule can be applied and state the formula.
7. What is the major drawback of Taylor series method?
8. Using Euler's method, find the solution of the initial value problem $\frac{dy}{dx} = \log(x+y)$, $y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$.
9. What is the central difference approximation for y''.
10. Write the difference scheme for solving the Poisson equation $\nabla^2 u = f(x,y)$.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Using Gauss Jordan method, find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{pmatrix}$. (8)

(ii) Solve $e^x - 3x = 0$ by the method of fixed point iteration. (8)

Or

(b) (i) Apply Gauss-Seidal method to solve the equations $20x + y - 2z = 17; 3x + 20y - z = -18; 2x - 3y + 20z = 25$. (8)

(ii) Determine the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. (8)

12. (a) (i) Find the cubic polynomial which takes the following values: (8)

x :	0	1	2	3
$f(x)$:	1	2	1	10

(ii) Derive Newton's backward difference formula by using operator method. (8)

Or

(b) (i) The following values of x and y are given:

x :	1	2	3	4
y :	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$. (8)

(ii) Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data: (8)

Year:	1997	1999	2001	2002
Profit in Lakhs Rs. :	43	65	159	248

13. (a) (i) Using Trapezoidal rule, evaluate $\int_1^2\int_1^2 \frac{dxdy}{x^2+y^2}$ numerically with $h = 0.2$ along x–direction and $k = 0.25$ along y–direction. (8)

(ii) A slider in a machine moves along a fixed straight rod. Its distance x cm alogn the rod is given below for various values of the time $'t'$ seconds. Find the velocity of the slider when $t = 1.1$ second. (8)

t :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
x :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Or

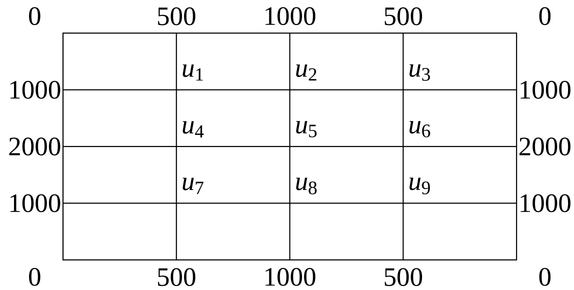
(b) Use Romberg's method to compute $\int_0^1 \frac{1}{1+x^2}dx$ correct to 4 decimal places. Also evaluate the same integral using tree-point Gaussian quadrature formula. Comment on the obtained values by comparing with the exact value of the integral which is equal to $\frac{\pi}{4}$. (16)

14. (a) Given that $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ obtain y for $x = 0.1, 0.2$ and 0.3 by Taylor's series method and find the solution for $y(0.4)$ by Milne's method. (16)

Or

(b) Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using Fourth order Runge Kutta algorithm, find $y(0.2)$. (16)

15. (a) Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown: (16)



Or

(b) (i) Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$. (6)

(ii) Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, subject to $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin \pi x, 0 < x < 1$, using Bender-Schmidt method. (10)

6.10 Nov/Dec 2012

B.E./B.Tech. DEGREE EXAMINATION, Nov/Dec 2012

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Write down the order of convergence and the condition for convergence of fixed point iteration method.
2. What are the advantages of iterative methods over direct methods for solving a system of linear equations.
3. Define cubic spline.
4. State Newton's backward difference formula.
5. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule.
6. Write down the three point Gaussian quadrature formula to evaluate $\int_{-1}^{1} f(x)dx$.
7. Find $y(0.1)$ if $\frac{dy}{dx} = 1 + y, y(0) = 1$ using Taylor series method.
8. State the fourth order Runge - Kutta algorithm.
9. State Crank-Nicholson's difference scheme.
10. Write down Bender-Schmidt's difference scheme in general form and using suitable value of λ, write the scheme in simplified form.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find the Newton's iterative formula to calculate the reciprocal of N and hence find the value of $\frac{1}{23}$. (8)

(ii) Using Gauss-Jordan method, find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$. (8)

Or

(b) (i) Solve the following system of equations using Gauss-Seidel method:

$$10x + 2y + z = 9$$
$$x + 10y - z = -22$$
$$-2x + 3y + 10z = 22. \quad (8)$$

(ii) Find all the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$ using Jacobi method. (8)

12. (a) (i) Using Newton's divided difference formula, find $f(x)$ from the following data and hence find $f(4)$. (8)

$x:$	0	1	2	5
$f(x):$	2	3	12	147

(ii) Find the value of y when $x = 5$ using Newton's interpolation formula from the following table: (8)

$x:$	4	6	8	10
$y:$	1	3	8	16

Or

(b) (i) Use Lagrange's method to find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, and $\log_{10} 661 = 2.8202$. (8)

(ii) Obtain the cubic spline for the following data to find $y(0.5)$. (8)

$x:$	-1	0	1	2
$y:$	-1	1	3	35

13. (a) (i) Find $f'(x)$ at $x = 1.5$ and $x = 4.0$ from the following data using Newton's formulae for differentiation. (8)

$x:$	1.5	2.0	2.5	3.0	3.5	4.0
$y = f(x):$	3.375	7.0	13.625	24.0	38.875	59.0

(ii) Compute $\int_0^{\pi/2} \sin x dx$ using Simpson's 3/8 rule. (8)

Or

(b) Evaluate $\int_0^2 \int_0^1 4xy dx dy$ using Simpson's rule by taking $h = \frac{1}{4}$ and $k = \frac{1}{2}$. (16)

14. (a) (i) Using Modified Euler's method, find $y(4.1)$ and $y(4.2)$ if $5x\frac{dy}{dx} + y^2 - 2 = 0; y(4) = 1$. (8)

(ii) Given that $\frac{dy}{dx} = 1 + y^2; y(0.6) = 0.6841, y(0.4) = 0.4228, y(0.2) = 0.2027, y(0) = 0$, find $y(-0.2)$ using Milne's method. (8)

Or

(b) Solve $y(0.1)$ and $z(0.1)$ from the simultaneous differential equations $\frac{dy}{dx} = 2y + z; \frac{dz}{dx} = y - 3z; y(0) = 0, z(0) = 0.5$ using Runge-Kutta method of the fourth order. (16)

15. (a) (i) Solve $y'' - y = 0$ with the boundary condition $y(0) = 0$ and $y(1) = 1$. (8)

(ii) Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < 1, t > 0$ satisfying the conditions $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0, u(0,t) = 0$ and $u(1,t) = \frac{1}{2}\sin \pi t$. Compute $u(x,t)$ for 4 time-steps by taking $h = \frac{1}{4}$. (8)

Or

(b) Solve the Poisson equation $\nabla^2 u = -10\left(x^2 + y^2 + 10\right)$ over the square mesh with sides $x = 0, y = 0, x = 3$ and $y = 3$ with u = 0 on the four boundary and mesh length 1 unit. (16)

6.11 May/Jun 2013

B.E./B.Tech. DEGREE EXAMINATION, May/Jun 2013

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Find an iterative formula to find the reciprocal of a given number $N(N \neq 0)$.
2. What is the use of Power method?
3. State Newton's forward interpolation formula.
4. Using Lagrange's formula, find the polynomial to the given data.

X :	0	1	3
Y :	5	6	50

5. State Simposon's one-third rule.
6. Evaluate $\int\limits_0^{\pi} \sin x dx$ by Trapezoidal rule by dividing ten equal parts.
7. Find $y(1.1)$ if $y' = x + y, y(1) = 0$ by Taylor series method.
8. State Euler's formula.
9. Obtain the finite difference scheme for the differential equation $2y'' + y = 5$.
10. Write Liebmann's iteration process.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find a positive root of the equation $\cos x - 3x + 1 = 0$ by using iteration method. (8)

(ii) Solve by Gauss-Seidel method, the equations
$27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110.$ (8)

Or

(b) (i) Find, by Gauss-Jordan method, the inverse of the matrix $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$. (8)

(ii) Using Jacobi method find the all eigen values and their corresponding eigen vectors of the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. (8)

12. (a) (i) Apply Lagrange's formula, to find $y(27)$ to the data given below. (8)

x :	14	17	31	35
y :	68.8	64	44	39.1

(ii) Fit a polynomial, by using Newton's forward interpolation formula, to the data given below. (8)

x :	0	1	2	3
y :	1	2	1	10

Or

(b) (i) Use Newton's divided difference formula to find $f(x)$ from the following data. (8)

x :	1	2	7	8
y :	1	5	5	4

(ii) Using cubic spline, compute $y(1.5)$ from the given data. (8)

x:	1	2	3
y:	−8	−1	18

13. (a) (i) Find the first three derivatives of $f(x)$ at $x = 1.5$ by using Newton's forward interpolation formula to the data given below. (8)

x :	1.5	2	2.5	3	3.5	4
y :	3.375	7	13.625	24	38.875	59

(ii) Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{1}{(1+x^2)} dx$ by taking eight equal intervals. (8)

Or

(b) (i) Evaluate $\int_{0}^{2} \frac{x^2+2x+1}{1+(x+1)^2} dx$ by Gaussian three point formula. (8)

(ii) Evaluate $\int_{1}^{1.4} \int_{2}^{2.4} \frac{1}{xy} dxdy$ using Simpson's one-third rule. (8)

14. (a) (i) Using Taylor series method to find $y(0.1)$ if $y' = x^2 + y^2, y(0) = 1$. (8)

(ii) Using Runge-Kutta method find $y(0.2)$ if $y'' = xy'^2 - y^2, y(0) = 1, y'(0) = 0, h = 0.2$. (8)

Or

(b) (i) Solve $y' = \frac{y-x}{y+x}, y(0) = 1$ at $x = 0.1$ by taking $h = 0.02$ by using Euler's method. (8)

(ii) Using Adam's method to find $y(2)$ if $y' = (x+y)/2, y(0) = 2, y(0.5) = 2.636, y(1) = 3.595, y(1.5) = 4.968$. (8)

15. (a) Solve $\nabla^2 u = 8x^2y^2$ over the square $x = -2, x = 2, y = -2, y = 2$ with $u = 0$ on the boundary and mesh length = 1. (16)

Or

(b) (i) Solve $u_{xx} = 32u_t, h = 0.25$ for $t \geq 0, 0 < x < 1, u(0,t) = 0, u(x,0) = 0, u(1,t) = t$. (8)

(ii) Solve $4u_{tt} = u_{xx}, u(0,t) = 0, u(4,t) = 0, u(x,0) = x(4-x), u_t(x,0) = 0, h = 1$ upto $t = 4$. (8)

6.12 Nov/Dec 2013

B.E./B.Tech. DEGREE EXAMINATION, Nov/Dec 2013

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. What do you mean by the order of convergence of an iterative method for finding the root of the equation $f(x) = 0$?

2. Solve the equations $x + 2y = 1$ and $3x - 2y = 7$ by Gauss-Elimination method.

3. State Newton's forward difference formula for equal intervals.

4. Find the divided differences of $f(x) = x^3 - x^2 + 3x + 8$ for arguments $0, 1, 4, 5$.

5. Evaluate $\int_{-2}^{2} e^{-x/2} dx$ by Gauss two point formula.

6. Evaluate $\int_{0}^{6} \frac{dx}{1 + x^2}$ using Trapezoidal rule.

7. State Adam's Predictor-Corrector formula.

8. Using Euler's method find the solution of the initial value problem $y' = y - x^2 + 1, y(0) = 0.5$ at $x = 0.2$ taking $h = 0.2$.

9. Write the diagonal five point five point formula for solving the two dimensional Laplace equation $\nabla^2 u = 0$.

10. Using finite difference solve $y'' - y = 0$ given $y(0) = 0, y(1) = 1, n = 2$.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Solve the equations by Gauss-Seidel method of iteration. $10x + 2y + z = 9, x + 10y - z = -22, -2x + 3y + 10z = 22$. (8)

(ii) Determine the largest eigenvalue and the corresponding eigenvector of a matrix $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ with $(1 \quad 1 \quad 1)^T$ as initial vector by power method. (8)

Or

(b) (i) Find the inverse of the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ using Gauss-Jordan method. (8)

(ii) Using Newton's method, find the real root of $x\log_{10}x = 1.2$ correct to five decimal places. (8)

12. (a) Find the natural cubic spline to fit the data:

x :	0	1	2
$f(x)$:	−1	3	29

Hence find $f(0.5)$ and $f(1.5)$. (16)

Or

(b) (i) The following table gives the values of density of saturated water for various temperatures of the saturated steam. (8)

Temperture C° :	100	150	200	250	300
Density hg/m^3 :	958	917	865	799	712

Find by interpolation, the density when the temperature is 275°.

(ii) Use Lagrange's formula to find the value of y at $x = 6$ from the following data: (8)

x :	3	7	9	10
y :	168	120	72	63

13. (a) (i) Apply three point Gaussian quadrature formula to evaluate $\int_0^1 \frac{\sin x}{x}dx$. (8)

(ii) Find the first and second order derivative of $f(x)$ at $x = 1.5$ for the following data: (8)

x :	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

Or

(b) (i) The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time(min):	0	2	4	6	8	10	12
Velocity(km/hr):	0	22	30	27	18	7	0

Using Simpson's $\frac{1}{3}$rd rule find the distance covered by the car. (8)

(ii) Evaluate $\int_2^{2.4}\int_4^{4.4} xy\,dx\,dy$ by Trapezoidal rule taking $h = k = 0.1$. (8)

14. (a) (i) Obtain y by Taylor series method, given that $y' = xy+1, y(0) = 1$, for $x = 0.1$ and 0.2 correct to four decimal places. (8)

(ii) Use Milne's method to find $y(0.8)$, given $y' = \frac{1}{x+y}, y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755$, $y(0.6) = 2.2493$. (8)

Or

(b) Using Runge-Kutta method of order four, find y when $x = 1.2$ in steps of 0.1 given that $y' = x^2 + y^2$ and $y(1) = 0.5$. (16)

15. (a) By iteration method solve the elliptic equation $u_{xx} + u_{yy} = 0$ over the square region of side 4, satisfying the boundary conditions.

(i) $u(0, y) = 0, 0 \le y \le 4$

(ii) $u(4, y) = 8 + 2y, 0 \le y \le 4$

(iii) $u(x, 0) = \dfrac{x^2}{2}, 0 \le x \le 4$

(iv) $u(x, 4) = x^2, 0 \le x \le 4$

Compute the values at the interior points correct to one decimal with $h = k = 1$. (16)

Or

(b) (i) Using Crank-Nicolson's scheme, solve $16\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0$ subject to $u(x, 0) = 0, u(0, t) = 0, u(1, t) = 100t$. Compute u for one step in t direction taking $h = \dfrac{1}{4}$. (8)

(ii) Solve $u_{tt} = u_{xx}, 0 < x < 2, t > 0$ subject to $u(x, 0) = 0, u_t(x, 0) = 100\left(2x - x^2\right), u(0, t) = 0, u(2, t) = 0$, choosing $h = \dfrac{1}{2}$ compute u for four times steps. (8)

6.13 May/Jun 2014

B.E./B.Tech. DEGREE EXAMINATION, May/Jun 2014

Regulation 2008

Fourth Semester

MA 2264-NUMERICAL METHODS

Time:Three hours | Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Evaluate $\sqrt{15}$ using Newton-Raphson's formula.
2. Using Gauss elimination method solve : $5x + 4y = 15, 3x + 7y = 12$.
3. Find the second divided difference with arguments a, b, c, if $f(x) = \frac{1}{x}$.
4. Define cubic spine.
5. Write down the expression for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_n$ by Newton's backward difference formula.
6. Taking $h = 0.5$, evaluate $\int_1^2 \frac{dx}{1+x^2}$ using Trapezoidal rule.
7. State the advantages and disadvantages of the Taylor's series method.
8. State the Milne's predictor and corrector formulae.
9. Obtain the finite difference scheme for the differential equation $2y''(x) + y(x) = 5$.
10. State whether the Crank Nicholson's scheme is an explicit or implicit scheme. Justify.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Find the numerically largest eigen value of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ and its corresponding eigen vector by power method, taking the initial eigen vector as $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ (upto three decimal places) . (8)

(ii) Using Gauss-Jordan method, find the inverse of $\begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & 8 \end{pmatrix}$. (8)

Or

(b) (i) Solve the system of equations by Gauss-Jordan method: $5x_1 - x_2 = 9; -x_1 + 5x_2 - x_3 = 4; -x_2 + 5x_3 = -6$. (8)

(ii) Using Gauss-Seidel method, solve the following system of linear equation s $4x + 2y + z = 14; x + 5y - z = 10; x + y + 8z = 20$. (8)

12. (a) (i) Find $f(3)$ by Newton's divided difference formula for the following data: (8)

$x:$	-4	-1	1	2	5
$y:$	1245	33	5	9	1335

(ii) Using Lagrange's interpolation formula, find $y(2)$ from the following data:
$y(0) = 0; y(1) = 1; y(3) = 81; y(4) = 256; y(5) = 625.$ (8)

Or

(b) Fit the cubic splines for the following data. (16)

$x:$	1	2	3	4	5
$y:$	1	0	1	0	1

13. (a) (i) For the given data, find the first two derivatives at $x = 1.1$. (8)

$x:$	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$y:$	7.989	8.4.3	8.781	9.129	9.451	9.750	10.031

(ii) Evaluate $\int_0^{1/2} \frac{x}{\sin x} dx$ correct to three decimal places using Romberg's method. (8)

Or

(b) (i) Taking $h = 0.05$ evaluate $\int_1^{1.3} \sqrt{x} dx$ using Trapezoidal rule and Simpson's three-eighth rule. (8)

(ii) Taking $h = k = \frac{1}{4}$, evaluate $\int_0^{1/2}\int_0^{1/2} \frac{\sin(xy)}{1+xy} dxdy$ using Simpson's rule. (8)

14. (a) (i) Using Adam's Bashforth method, find $y(4.4)$ given that $5xy' + y^2 = 2, y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097$ and $y(4.3) = 1.0143$. (8)

(ii) Using Taylor's series method, find y at $x = 1.1$ by solving the equation $\frac{dy}{dx} = x^2+y^2; y(1) = 2$. Carryout the computations upto fourth order derivative. (8)

Or

(b) Using Runge Kutta method of fourth order, find the value of y at $x = 0.2, 0.4, 0.6$ given $\frac{dy}{dx} = x^3 + y, y(0) = 2$. Also find the value of y at $x = 0.8$ using Milne's predictor and corrector method. (16)

15. (a) (i) Using Bender-Schmidt's method solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(0,t) = 0, u(1,t) = 0, u(x,0) = \sin \pi x$, $0 < x < 1$ and $h = 0.2$. Find the value of u upto $t = 0.1$. (8)

(ii) Solve $y'' - y = x, x \in (0,1)$ given $y(0) = y(1) = 0$ using finite differences by dividing the interval into four equal parts. (8)

Or

(b) (i) Solve the Poisson equation $\nabla^2 u = -10\left(x^2+y^2+10\right), 0 \le x \le 3, 0 \le y \le 3, u = 0$ on the boundary. (8)

(ii) Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < 1, t > 0, u(0,t) = u(1,t) = 0, t > 0,$

$u(x,0) = \begin{cases} 1 & 0 \le x \le 0.5 \\ -1 & 0.5 \le x \le 1 \end{cases}$ and $\frac{\partial u}{\partial t}(x,0) = 0$, using $h = k = 0.1$, find u for three steps. (8)

6.14 Nov/Dec 2014

B.E./B.Tech. DEGREE EXAMINATION, Nov/Dec 2014

Regulation 2008

Fourth Semester

Common to ECE & Bio Medical Engineering

MA 2264 Numerical Methods

Time:Three hours Maximum:100 marks

Answer ALL questions.

PART A - (10 × 2 = 20 marks)

1. Write down the condition for convergence of Newton-Raphson method for $f(x) = 0$.
2. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ by Gauss-Jordan method.
3. Find the second degree polynomial through the points $(0,2)$, $(2,1)$,$(1,0)$ using Lagrange's formula.
4. State Newton's backward formula for interpolation.
5. State the local error term in Simpson's $\frac{1}{3}$ rule.
6. State Romberg's integration formula to find the value of $I = \int_a^b f(x)dx$ for first two intervals.
7. State the Milne's predictor-corrector formulae.
8. Given $y' = x + y, y(0) = 1$, find $y(0.1)$ by Euler's method.
9. What is the central difference approximations for y''?
10. Write down the standard five-point formula to find the numerical solution of Laplace equation.

PART B - (5 × 16 = 80 marks)

11. (a) (i) Apply Gauss-Seidal method to solve the system of equations $20x+y-2z = 17$, $3x+20y-z = -18$, $2x - 3y + 20z = 25$. (8)

(ii) Find by Newton-Raphson method a positive root of the equation $3x - \cos x - 1 = 0$. (8)

Or

(b) (i) Find the numerically largest eigenvalue if $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and the corresponding eigenvector. (8)

(ii) Using Gauss-Jordan method to solve $2x - y + 3z = 8$; $-x + 2y + z = 4$, $3x + y - 4z = 0$. (8)

12. (a) (i) Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values: (8)

$x:$	0	1	2	3
$f(x):$	1	2	1	10

(ii) Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$. (8)

$x:$	-1	0	1	2
$y:$	-1	1	3	35

Or

(b) (i) By using Newton's divided difference formula find $f(8)$, gives (8)

$x:$	4	5	7	10	11	13
$f(x):$	48	100	294	900	1210	2028

(ii) Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for the following values of x and y: (8)

$x:$	0	1	2	5
$y:$	2	3	12	147

13. (a) (i) Evaluate $\int_1^2 \frac{dx}{1+x^2}$ using 3 point Gaussian formula. (8)

(ii) The velocity υ of a particle at a distance α from a point on its path is given by the table: (8)

$s(ft):$	0	10	20	30	40	50	60
$\upsilon:$	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet by using Simpson's $\frac{3}{8}$ rule.

Or

(b) (i) Evaluate $\int_0^1\int_0^1 \frac{1}{1+x+y}dxdy$ by Trapezoidal rule. (8)

(ii) Evaluate $\int_0^1 \frac{dx}{1+x}$ and correct to 3 decimal places using Romberg's method and hence find the value of $\log_e 2$. (8)

14. (a) (i) Using Taylor's series method, find y at $x = 0$ if $\frac{dy}{dx} = x^2y - 1, y(0) = 1$. (6)

(ii) Given $5xy' + y^2 = 2, y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$. Compute $y(4.4)$ using Milne's method. (10)

Or

(b) (i) Apply modified Euler's method to find $y(0.2)$ and $y(0.4)$ given $y' = x^2+y^2$, $y(0) = 1$ by taking $h = 0.2$. (6)

(ii) Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ find the value of $y(0.1)$ by Runge-Kutta's method of fourth order. (10)

15. (a) By iteratin method, solve the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ over a square region of side 4, satisfying the boundary conditions.

(i). $u(0, y) = 0, 0 \leq y \leq 4$

(ii). $u(4, y) = 12 + y, 0 \leq y \leq 4$

(iii). $u(x, 0) = 3x, 0 \leq x \leq 4$

(iv). $u(x, 4) = x^2, 0 \leq x \leq 4$

By dividing the square into 16 square meshes of side 1 and always correcting the computed values to two places of decimals, obtain the values of u at 9 interior pivotal points. (16)

Or

(b) Solve by Crank-Nicolson's method $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ for $0 < x < 1, t > 0$ given that $u(0, t) = 0$, $u(1, t) = 0$ and $u(x, 0) = 100x(1 - x)$. Compute u for some time step with $h = \frac{1}{4}$ and $k = \frac{1}{64}$. (16)

List of Formulae

I. Solution of Equations and Eigenvalue Problems

1. Fixed point iteration method

Write the given equation as $f(x) = 0$.
Find x_1, x_2 with opposite signs of $f(x)$. Find an relation as $x = g(x)$ such that

$$\left|g'(x_1)\right| < 1$$
$$\left|g'(x_2)\right| < 1$$

$$X_0 = \begin{cases} x_1, |f(x_1)| < |f(x_2)| \Rightarrow f(x_1) \text{ is nearer to zero.} \\ x_2, |f(x_1)| > |f(x_2)| \Rightarrow f(x_2) \text{ is nearer to zero.} \end{cases}$$

$$\begin{aligned} X_1 &= g(X_0) \\ X_2 &= g(X_1) \\ &\vdots \\ X_n &= g(X_{n-1}) \\ X_{n+1} &= g(X_n) \end{aligned}$$

Note: Stop method if the consecutive values of $X_n \& X_{n+1}$ are equal upto required place of decimal. i.e., $X_n = X_n + 1$.

2. Newton Raphson method

(Newton's method or method of tangents)
Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = \phi(x_n)$, where $n = 0, 1, 2, \cdots$

3. Gauss elimination method

Step 1: Write the augmented matrix for the given system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & b_{33} & b_3 \end{pmatrix}$$

Step 2: Using elementary row operations reduce the given matrix into an upper-triangular matrix say

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

Step 3: By back substitution we get the values for unknowns.

4. Pivoting - Gauss Jordan method

Step 1: Write the augmented matrix for the given system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & b_{33} & b_3 \end{pmatrix}$$

Step 2: Using elementary row operations reduce the given matrix into a diagonal matrix say

$$\begin{pmatrix} c_{11} & 0 & 0 & d_1 \\ 0 & c_{22} & 0 & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

$$\left[\text{Here} \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \text{is a} \begin{pmatrix} \text{diagonal matrix} \\ \text{or} \\ \text{unit matrix} \end{pmatrix}\right]$$

Step 2: By direct substitution we get the values for unknowns.

5. Iterative methods of Gauss-Jacobi

Step 1: Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

with diagonally dominant.

Step 2: The above system should write into the form

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \qquad (1)$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \qquad (2)$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \qquad (3)$$

Step 3: Start with the initial values $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ for x, y, z and get $x^{(1)}, y^{(1)}, z^{(1)}$

$\therefore$ (1),(2),(3) become

$$x^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)}\right)$$
$$y^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(0)} - a_{23}z^{(0)}\right)$$
$$z^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(0)} - a_{32}y^{(0)}\right)$$

Step 4: Using this $x^{(1)}$ for x, $y^{(1)}$ for y, $z^{(1)}$ for z in (1),(2),(3) respectively, we get

$$\begin{aligned}
x^{(2)} &= \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(1)} - a_{13}z^{(1)}\right)\\
y^{(2)} &= \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(1)} - a_{23}z^{(1)}\right)\\
z^{(2)} &= \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)}\right)
\end{aligned}$$

Continuing in the same procedure until the convergence is confirmed.

The general iterative formula of Gauss-Jacobi is

$$\begin{aligned}
x_1^{(j+1)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
x_2^{(j+1)} &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(j)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
x_3^{(j+1)} &= \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(j)} + a_{32}x_2^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
&\vdots\\
x_n^{(j+1)} &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(j)} + a_{n2}x_2^{(j)} + \cdots + a_{n,n-1}x_{n-1}^{(j)}\right)\right]
\end{aligned}$$

Note : Suppose n equations with n unknown variables $x_1, x_2, \cdots, x_n$, then

In Gauss Jacobi iteration method, for $x_n^{(j+1)}$, use $x_n^{(j)}$ values only. [Start with $x_1 = x_2 = \cdots = 0$]

In Gauss Seidel iteration method, for $x_n^{(j+1)}$, use latest values of $x_n^{(j)}$ or $x_n^{(j+1)}$. [Start with $x_2 = x_3 = \cdots = 0$]

6. Iterative methods of Gauss-Seidel

Step 1: Let the system of equations be

$$\begin{aligned}
a_{11}x + a_{12}y + a_{13}z &= b_1\\
a_{21}x + a_{22}y + a_{23}z &= b_2\\
a_{31}x + a_{32}y + a_{33}z &= b_3
\end{aligned}$$

with diagonally dominant.

Step 2: The above system should write into the form

$$x = \frac{1}{a_{11}}\left(b_1 - a_{12}y - a_{13}z\right) \quad (1)$$
$$y = \frac{1}{a_{22}}\left(b_2 - a_{21}x - a_{23}z\right) \quad (2)$$
$$z = \frac{1}{a_{33}}\left(b_3 - a_{31}x - a_{32}y\right) \quad (3)$$

Step 3: Start with the initial values $y^{(0)} = 0, z^{(0)} = 0$ for y, z and get $x^{(1)}$ from the first equation.

$\therefore$ (1) becomes $x^{(1)} = \dfrac{1}{a_{11}}\left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)}\right)$

Step 4: Using this $x^{(1)}$ in (2), we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of $x^{(0)}$, we get

$\therefore$ (2) becomes $y^{(1)} = \dfrac{1}{a_{22}}\left(b_2 - a_{21}x^{(1)} - a_{23}z^{(0)}\right)$

Step 5: Substitute $x^{(1)}, y^{(1)}$ for x, y in the third equation.

$\therefore$ (3) becomes $z^{(1)} = \dfrac{1}{a_{33}}\left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)}\right)$

Step 6: To find the values of unknowns, use the latest available values on the right side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterate values, then the next iteration will be

$$x^{(r+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(r)} - a_{13}z^{(r)}\right)$$
$$y^{(r+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(r+1)} - a_{23}z^{(r)}\right)$$
$$z^{(r+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(r+1)} - a_{32}y^{(r+1)}\right)$$

Step 7: This process of continued until the convergence is confirmed.

The general iterative formula of Gauss-Seidel is

$$\begin{aligned}
x_1^{(j+1)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
x_2^{(j+1)} &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(j+1)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
x_3^{(j+1)} &= \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(j+1)} + a_{32}x_2^{(j+1)} + \cdots + a_{2n}x_n^{(j)}\right)\right]\\
&\vdots\\
x_n^{(j+1)} &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(j+1)} + a_{n2}x_2^{(j+1)} + \cdots + a_{n,n-1}x_{n-1}^{(j+1)}\right)\right]
\end{aligned}$$

Note : If either method converges, Gauss-Seidel converges faster than Jacobi.

7. Matrix Inversion by Gauss Jordan method

Given Matrix $A_{n\times n}$,

Form

$$[A_{n\times n} \mid I_{n\times n}] \quad \underset{\text{Operations}}{\overset{\text{Row}}{\sim}} \quad \left[I_{n\times n} \mid A^{-1}_{n\times n}\right]$$

8. Eigen values of a matrix by Power method

Suppose a given square matrix is A.

Let $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ be an arbitrary initial eigen vector of the matrix A.

$$AX_1 = \begin{bmatrix}\text{value 1}\\ \text{value 2}\\ \text{value 3}\end{bmatrix} = \begin{pmatrix}\text{numerically largest value}\\ \text{of matrix } A \text{ say}\\ \text{value 1}\end{pmatrix}\begin{bmatrix}1\\ \text{value 2/value 1}\\ \text{value 3/value 1}\end{bmatrix} = \lambda_1 [X_2]$$

$$AX_2 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \begin{pmatrix} \text{numerically largest value} \\ \text{of matrix } A \text{ say} \\ \text{value 2} \end{pmatrix} \begin{bmatrix} \text{value 1/value 2} \\ 1 \\ \text{value 3/value 2} \end{bmatrix} = \lambda_2 [X_3]$$

$$\vdots$$

$$\left.\begin{aligned} AX_n &= \lambda_n [X_{n+1}] \\ AX_{n+1} &= \lambda_{n+1} [X_{n+2}] \end{aligned}\right\} \begin{array}{c} \text{If } \lambda_n = \lambda_{n+1} \text{ and } X_{n+1} = X_{n+2} \text{ are same} \\ \text{upto required decimals,then stop iteration.} \end{array}$$

$$\begin{aligned} \therefore \text{ Dominant eigen value } &= \lambda_n \, (or) \, \lambda_{n+1} \text{and the corresponding} \\ \text{eigen vector } &= X_{n+1} \, (or) \, X_{n+2} \end{aligned}$$

To find the smallest eigen value of A

Form $B = A - \lambda I$ & $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary eigen vector of the matrix B.

$$BY_1 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \begin{pmatrix} \text{numerically largest value} \\ \text{of matrix } B \text{ say} \\ \text{value 1} \end{pmatrix} \begin{bmatrix} 1 \\ \text{value 2/value 1} \\ \text{value 3/value 1} \end{bmatrix} = \lambda_1 [Y_2]$$

$$BY_2 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \begin{pmatrix} \text{numerically largest value} \\ \text{of matrix } B \text{ say} \\ \text{value 2} \end{pmatrix} \begin{bmatrix} \text{value 1/value 2} \\ 1 \\ \text{value 3/value 2} \end{bmatrix} = \lambda_2 [Y_3]$$

$$\vdots$$

$$\left.\begin{aligned} BY_n &= \lambda_n [Y_{n+1}] \\ BY_{n+1} &= \lambda_{n+1} [Y_{n+2}] \end{aligned}\right\} \begin{array}{c} \text{If } \lambda_n = \lambda_{n+1} \text{ and } Y_{n+1} = Y_{n+2} \text{ are same} \\ \text{upto required decimals,then stop iteration.} \end{array}$$

$$\therefore \text{The smallest eigen value of } A = \begin{cases} \text{Dominant eigen value of } B \\ + \\ \text{Dominant eigen value of } A \end{cases}$$

To find third eigen value of A:

$\lambda_1 + \lambda_2 + \lambda_3$ = Sum of the main diagonal elements.

II. Interpolation and Approximation

1. Lagrange's interpolation

Lagrangian Polynomials(Equal and unequal intervals):

Let $y = f(x)$ be a function which takes the values $y = y_0, y_1, \cdots, y_n$ corresponding to $x = x_0, x_1, \cdots, x_n$.

Lagrange's interpolation formula(x given, finding y in terms of x)

$$y = y(x) = f(x) = \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}y_0 + \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)}y_1 + \dots + \frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_2 - x_0)(x_2 - x_1)\dots(x - x_{n-1})}y_n$$

Inverse Lagrange's interpolation formula(y given, finding x in terms of y)

$$x = x(y) = f(y) = \frac{(y - y_1)(y - y_2)\dots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\dots(y_0 - y_n)}x_0 + \frac{(y - y_0)(y - y_2)\dots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\dots(y_1 - y_n)}x_1 + \dots + \frac{(y - y_0)(y - y_1)\dots(y - y_{n-1})}{(y_2 - y_0)(y_2 - y_1)\dots(y - y_{n-1})}x_n$$

Note: Lagrange's interpolation formula can be used for equal and unequal intervals.

2. Newton's divided difference interpolation

First divided difference for arguments x_0, x_1 :

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1) = \underset{x_1}{\Delta} f(x_0) = [x_0, x_1](or)[x_1, x_0] = \underset{x_0}{\Delta} f(x_1)$$

First divided difference for arguments x_1, x_2 :

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = [x_1, x_2](or)[x_2, x_1] = \underset{x_1}{\Delta} f(x_2)$$

Second divided difference for arguments x_1, x_2, x_3 :

$$\frac{\underset{x_3}{\Delta} f(x_2) - \underset{x_2}{\Delta} f(x_1)}{x_3 - x_1} = f(x_1, x_2, x_3) = \underset{x_3 x_2}{\Delta^2} f(x_1) = [x_1, x_2, x_3]$$

Third divided difference for arguments x_0, x_1, x_2, x_3 :

$$\frac{\underset{x_3 x_2}{\Delta^2} f(x_1) - \underset{x_2 x_1}{\Delta^2} f(x_0)}{x_3 - x_0} = f(x_0, x_1, x_2, x_3) = \underset{x_3 x_2 x_1}{\Delta^3} f(x_0) = [x_0, x_1, x_2, x_3]$$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

3. Cubic Splines

Interpolating with a cubic spline

The cubic spline interpolation formula is

$$S(x) = y(x) = y = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}(x_i - x)[y_{i-1} - \frac{h^2}{6}M_{i-1}] + \frac{1}{h}(x - x_{i-1})[y_i - \frac{h^2}{6}M_i]$$

where

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

n = number of data

i = number of intervals [i.e., $i = 1, 2, 3, , (n-1)$]

h = length of interval = interval length.

Note : If M_i and y_i'' values are not given, then assume $M_0 = M_n = 0$ $\left[\text{or } y_0'' = y_n'' = 0\right]$, and find $M_1, M_2, \cdots, M_{n-1}$ in 1st interval, 2nd interval, $\cdots$, $(n-1)$th interval value.

Note : Order of convergence of the cubic spline is 4.

4. Newton's forward and backward difference formulae

Newton's forward and backward difference formulae for Uniform (or) equal intervals only.

Newton's forward interpolation difference formula:

[If y (required x near to x_0) =? and use Δ]

$$\begin{aligned} y(x) = f(x) &= f(x_0 + uh) \\ &= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \end{aligned}$$

where $u = \dfrac{x - x_0}{h}$, h = length of interval.

5. Newton's backward interpolation difference formula

[If y (required x near to x_n) =? and use ∇]

$$\begin{aligned} y(x) = f(x) &= f(x_n + vh) \\ &= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots \end{aligned}$$

where $v = \dfrac{x - x_n}{h}$, h = length of interval.

III. Numerical Differentiation and Integration

1. Newton's forward difference formula to compute derivative

WKT, Newton's forward difference interpolation formula is

$$\begin{aligned} y(x) = f(x_0+uh) &= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \\ &= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \frac{u^4-6u^3+11u^2-6u}{4!}\Delta^4 y_0 + \cdots \end{aligned}$$

where $u = \dfrac{x - x_0}{h}$

First derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{du}$$

$$\text{i.e., } \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\left[\begin{array}{c}\text{at } x = x_0 \\ \Rightarrow u = 0\end{array}\right]} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{du}\frac{du}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{du}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h}\frac{d}{du}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h^2}\frac{d^2y}{du^2}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{12u^2 - 36u + 22}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{\left[\begin{array}{c}\text{at } x = x_0 \\ \Rightarrow u = 0\end{array}\right]} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{22}{24}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

Third derivative

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \left(\frac{24u - 36}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{\left[\begin{array}{c}\text{at } x = x_0 \\ \Rightarrow u = 0\end{array}\right]} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{36}{24}\Delta^4 y_0 + \cdots\right]$$

2. Newton's backward difference formula to compute derivatives

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

$$= y_n + \frac{v}{1!}\nabla y_n + \left(\frac{v^2+v}{2!}\right)\nabla^2 y_n + \left(\frac{v^3 + 3v^2 + 2v}{3!}\right)\nabla^3 y_n + \left(\frac{v^4 + 6v^3 + 11v^2 + 6v}{4!}\right)\nabla^4 y_n + \cdots$$

$$\text{where } v = \frac{x - x_n}{h}$$

First derivative

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{dv}$$

$$= \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2 + 6v + 2}{3!}\right)\nabla^3 y_n + \left(\frac{4v^3 + 18v^2 + 22v + 6}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\left[\begin{array}{c}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}\right]} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{dv}\frac{dv}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{d}{dv}\frac{dv}{dx}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{d}{dv}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{1}{h^2}\left[\frac{d^2y}{dv^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \left(\frac{6v+6}{3!}\right)\nabla^3 y_n + \left(\frac{12v^2+36v+22}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{\left[\begin{array}{c}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}\right]} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6}{3!}\nabla^3 y_n + \frac{22}{4!}\nabla^4 y_n + \cdots\right]$$

Third derivative

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \left(\frac{24v+36}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{\left[\begin{array}{c}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}\right]} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \frac{36}{4!}\nabla^4 y_n + \cdots\right]$$

3. Maxima and Minima of a tabulated function

(If the intervals are same)

WKT, Newton's forward difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

First derivative

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots\right] \tag{1}$$

substitute $h, \Delta y_0, \Delta^3 y_0, \Delta^4 y_0, \cdots$ gives the equation

$$\frac{dy}{dx} = \text{ an equation in } u \tag{2}$$

For maxima or minima is obtained by the equation by

$$\frac{dy}{dx} = 0 \text{ (or) } y'(x) = 0$$

Put RHS of (1) = 0, find $u = u_1, u_2, \cdots$

Find $\frac{d^2y}{dx^2}$ from (2):

Suppose $\left(\frac{d^2y}{dx^2}\right)_{\text{at } u_1} = -\text{ve} \Rightarrow u_1$ is maximum point

$\Rightarrow y(u_1) =$ maximum value of y

Suppose $\left(\frac{d^2y}{dx^2}\right)_{\text{at } u_2} = +\text{ve} \Rightarrow u_2$ is minimum point

$\Rightarrow y(u_2) =$ minimum value of y

4. Numerical integration using Trapezoidal

Trapezoidal rule($n = 1$ in quadrature formula)

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \\ \text{(or)} \\ \frac{h}{2}\left[(\text{first term} + \text{last term}) + 2(\text{remaining terms})\right] \end{cases}$$

5. Numerical integration using Simpson's 1/3 rule

Simpson's one third rule (Simpson's 1/3 rule) ($n = 2$ in quadrature form)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\right] \\ \text{(or)} \\ \frac{h}{3}\begin{bmatrix} (\text{first term} + \text{last term}) \\ +4(\text{odd suffices}) \\ +2(\text{even suffices}) \end{bmatrix} \end{cases}$$

6. Numerical integration using Simpson's 3/8 rule

Simpson's three eighth rule (Simpson's 3/8 rule) ($n = 3$ in quadratic form)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \frac{3h}{8}\begin{bmatrix} (y_0 + y_n) \\ +2(y_3 + y_6 + \cdots) \\ +3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) \end{bmatrix} \\ \text{(or)} \\ \frac{3h}{8}\begin{bmatrix} (\text{first term} + \text{last term}) \\ +2(\text{suffices with a multiple of 3}) \\ +3(\text{remaining terms}) \end{bmatrix} \end{cases}$$

7. Romberg's method

Romberg's method for a given interval $\left(I = \int_a^b f(x)dx\right)$

when $h = \frac{b-a}{2}$, by trapezoidal rule, we get I_1

when $h = \frac{b-a}{4}$, by trapezoidal rule, we get I_2

when $h = \frac{b-a}{8}$, by trapezoidal rule, we get I_3

$$\text{Romberg's formula for } I_1 \& I_2 = I_{RM_{1,2}} = I_2 + \frac{(I_2 - I_1)}{3}$$

$$\text{Romberg's formula for } I_2 \& I_3 = I_{RM_{2,3}} = I_3 + \frac{(I_3 - I_2)}{3}$$

$$\text{If } I_{RM_{1,2}} = I_{RM_{2,3}}, \text{ then we can equal } I = I_{RM_{1,2}} = I_{RM_{2,3}}$$

Note : Check, use actual integration, we get $I_{AI} = I = I_{RM_{1,2}} = I_{RM_{2,3}}$

8. Two point Gaussian quadrature formula

Two point Gaussian quadrature formula : (Guass two point formula)

Given $I = \int_a^b f(x)dx$

Case (i) If $a = -1, b = +1$, then

$$I = \int_{-1}^{1} f(x)dx = f\left[-\frac{1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right]$$

Case (ii) If $a = 0, b = 1$, then

$$I = \int_0^1 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f(x)\,dx, \text{ if } f(x) \text{ is an even function}$$

Case (iii) If $(a \neq -1 \& b \neq 1)$, then $x = \frac{b-a}{2}z + \frac{b+a}{2} = mz + c \Rightarrow dx = mdz$

$$\begin{aligned} I &= \int_a^b f(x)dx = \int_{-1}^{1} f(z)\,mdz = m\int_{-1}^{1} f(z)\,dz \\ &= m\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right] \end{aligned}$$

9. Three point Gaussian quadrature formula

Case (i) $\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$

Case (ii) $\int_0^1 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f(x)dx$ [for even function $f(x)$]

$$= -\frac{1}{2}\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right\}$$

Case (iii) $(a \neq -1 \& b \neq 1)$, then $x = \frac{b-a}{2}z + \frac{b+a}{2} = mz + c \Rightarrow dx = m\,dz$

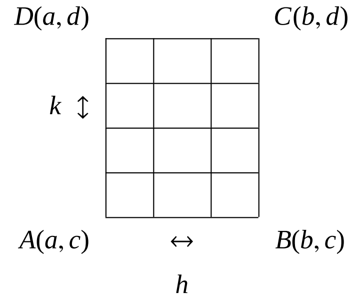
$$I = \int_a^b f(x)dx = \int_{-1}^{1} f(z)\,m\,dz = m\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right\}$$

10. Double integrals by Trapezoidal

Double integration by trapezoidal rule is

Given $\int_c^d \int_a^b f(x)\,dx\,dy[= I(Say)]$

$$\text{i.e., } I = \frac{hk}{4}\left\{\begin{array}{l} [\text{sum of values of } f \text{ at the four corners}] \\ +2\left[\begin{array}{l}\text{sum of values of } f \text{ at the nodes} \\ \text{on the boundary except the corners}\end{array}\right] \\ +4\,[\text{sum of the values at the interior nodes}] \end{array}\right\}$$



where $h = \dfrac{b-a}{n}$, $k = \dfrac{d-c}{m}$

where $n =$ is number of equal intervals in (a, b).

where $m =$ is number of equal intervals in (c, d).

11. Double integrals by Simpson's 1/3 rules

Double integration by Simpson's 1/3 rule is Given $\int_c^d \int_a^b f(x)\, dx\, dy [= I (Say)]$

$$\text{i.e., } I = \frac{hk}{9}\left\{\begin{array}{l} [\text{sum of values of } f \text{ at the four corners}] \\ +2\left[\begin{array}{l}\text{sum of values of } f \text{ at the odd positions} \\ \text{on the boundary except the corners}\end{array}\right] \\ +4\left[\begin{array}{l}\text{sum of the values of } f \text{ at the even positions} \\ \text{on the boudary except the corners}\end{array}\right] \\ +4\left[\begin{array}{l}\text{sum of the values of } f \text{ at odd positions} \\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right] \\ +8\left[\begin{array}{l}\text{sum of the values of } f \text{ at even positions} \\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right] \\ +8\left[\begin{array}{l}\text{sum of the values of } f \text{ at odd positions} \\ \text{on the even rows of the matrix except boundary rows}\end{array}\right] \\ +16\left[\begin{array}{l}\text{sum of the values of } f \text{ at even positions} \\ \text{on the even rows of the matrix except boundary rows}\end{array}\right] \end{array}\right\}$$

IV. Initial Value Problems for Ordinary Differential Equations

1. Taylor's series method

$$\text{Given } \frac{dy}{dx} = y' = f(x, y) \text{ with } y(x_0) = y_0$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\text{i.e., } y'' = f_x + f_y (y')$$

lll^{ly} find $y''', y'''', \cdots$

Taylor's series expansion of $y(x)$ above $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

$$= y_0 + \frac{(x-x_0)}{1!}y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \cdots$$

(or)

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots, \quad \text{where } x_1 = x_0 + h, h = x_1 - x_0$$

$$y(x_2) = y_2 = y_1 + \frac{h}{1!}y'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \cdots, \quad \text{where } x_2 = x_1 + h, h = x_2 - x_1$$

2. Euler's method

Given $y' = f(x,y), x_0, y_0, h$

Euler algorithm is

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$\vdots$$

In general, $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

3. Modified Euler's method

Modified Euler's method formula is

In general, $y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right]$

When $n = 0, y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right]$

When $n = 1, y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x, y)\right]$

When $n = 2, y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2}f(x_2, y_2)\right]$

4. Fourth order Runge-Kutta method for solving first order equations

Fourth order Runge-Kutta method for solving I order Differential Equations [Single step method]

Given $\frac{dy}{dx} = y' = f(x,y)$

& $y(x_0) = y_0$

We have to find $y(x_1) =? = y_1, y(x_2) =? = y_2, y(x_3) =? = y_3, \cdots$

To find $f(x_{n+1}) = y(x_{n+1}) = y_{n+1}$:

$$y_{n+1} = y(x_n + h) = y(x_n) + \Delta y = y_n + \Delta y$$

$$\text{where } \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\text{where } k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

5. Milne's forth predictor corrector methods for solving first order equations

Milne's predictor and corrector methods (multistep method)

Milne's predictor formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

Milne's corrector formula

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

6. Adams-Bash forth predictor corrector methods for solving first order equations

Adams predictor and corrector methods(efficient multistep method) [Adam's-Bashforth method]

Adam's predictor formula :

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]$$

Adam's corrector formula :

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$

V. Boundary Value Problems in Ordinary and Partial Differential Equations

1. Finite difference methods for solving two-point linear boundary value problems

Consider the problem

$$y''(x) + f(x)g'(x) + g(x)y(x) = r(x)$$

with the boundary conditions $y(x_0) = a$ and $y(x_n) = b$.

For solving the equation, replace y'' and y' by the formulae

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$
$$y' = \frac{y_{i+1} - y_{i-1}}{2h}$$

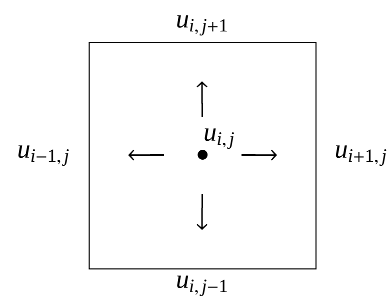
2. Finite difference technique for the solution of 2D Laplace's equations

Two dimensional Laplace equation is

$$\nabla^2 u = 0 \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

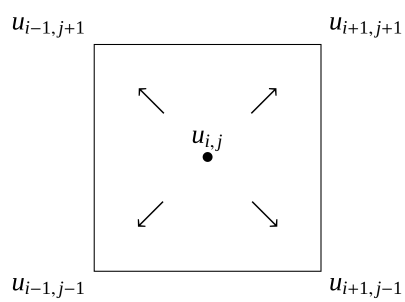
Standard five - point formula(SFPF):

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

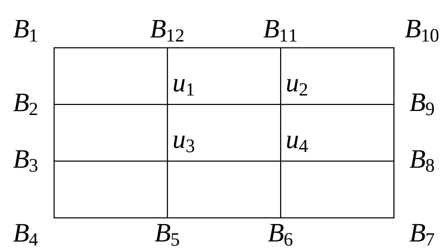


Diagonal five - point formula(DFPF):

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$



For 3×3 matrix form:



To find u_1, u_2, u_3, u_4 :

Assume $u_i = 0$, for any one i

Initially take $u_4 = 0$ or take $u_4 =$ some finite value

Find Rough Values [Initial values (or) Starting values] :

$$u_1 = \frac{1}{4}[B_1 + B_3 + u_4 + B_{11}] \qquad \text{(by DFPF)}$$

$$u_2 = \frac{1}{4}[u_1 + u_4 + B_9 + B_{11}] \qquad \text{(by SFPF)}$$

$$u_3 = \frac{1}{4}[u_1 + B_3 + B_5 + u_4] \qquad \text{(by SFPF)}$$

$$u_4 = \frac{1}{4}[u_2 + u_3 + B_6 + B_8] \qquad \text{(by SFPF)}$$

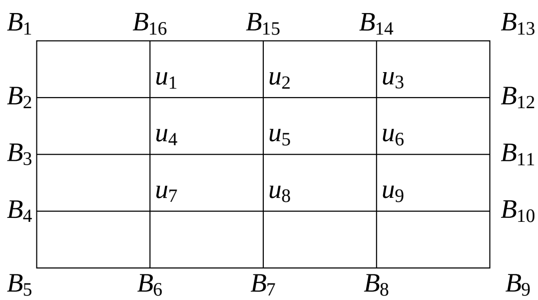
First Iteration and onwards (apply only SFPF)

$$\left.\begin{array}{l}\text{The Liebmann's iterative}\\ \text{formula is}\end{array}\right\} \quad u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

Stop procedure until two consecutive iteration values are same.

$$\text{i.e., stop if } u_i^{(n)} = u_i^{(n+1)}, \forall i \text{ with error} < 0.1$$

<u>For 4×4 matrix form:</u>



To find $u_i, i = 1, 2, \cdots 9$:

Find Rough Values [Initial values (or) Starting values] :

$$u_5 = \frac{1}{4}[B_3 + B_7 + B_{11} + B_{15}] \qquad \left(\text{by SFPF, extreme boundary values only}\right)$$

$$u_1 = \frac{1}{4}[B_1 + B_3 + u_5 + B_{15}] \qquad \text{(by DFPF)}$$

$$u_3 = \frac{1}{4}[B_{15} + u_5 + B_{11} + B_{13}] \qquad \text{(by DFPF)}$$

$$u_7 = \frac{1}{4}[B_3 + B_5 + B_7 + u_5] \qquad \text{(by DFPF)}$$

$$u_9 = \frac{1}{4}[u_5 + B_7 + B_9 + B_{11}] \qquad \text{(by DFPF)}$$

$$u_2 = \frac{1}{4}[u_1 + u_5 + u_3 + B_{15}] \qquad \text{(by SFPF)}$$

$$u_4 = \frac{1}{4}[u_1 + B_3 + u_7 + u_5] \qquad \text{(by SFPF)}$$

$$u_6 = \frac{1}{4}[u_5 + u_9 + B_{11} + u_3] \qquad \text{(by SFPF)}$$

$$u_8 = \frac{1}{4}[u_7 + B_7 + u_9 + u_5] \qquad \text{(by SFPF)}$$

First Iteration and onwards (apply only SFPF)

$$\left.\begin{array}{l}\text{The Liebmann's iterative}\\ \text{formula is}\end{array}\right\} \quad u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

Stop procedure until two consecutive iteration values are same.

$$\text{i.e., stop if } u_i^{(n)} = u_i^{(n+1)}, \forall i \text{ with error} < 0.1$$

3. Finite difference technique for the solution of 2D Poisson's equations

Two dimensional Poisson equation is

$$\nabla^2 u = f(x,y) \Rightarrow u_{xx} + u_{yy} = f(x,y) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

where $x = ih, y = jk = jh$ $\quad (\because k = h)$

Standard five - point formula(SFPF) for Poisson equation :

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 \cdot f(ih, jh)$$

4. One dimensional heat flow equation by explicit method (Bender-Schmidt's method)

One dimensional heat equation is $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$,

where $\alpha^2 = \dfrac{1}{a}$

$$\frac{\partial u}{\partial t} = \frac{1}{a}\frac{\partial^2 u}{\partial x^2}$$

$$\text{i.e., } a\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(i) Bender-Schmidt's difference equation is

$$u_{i,j+1} = \lambda\left(u_{i-1,j} + u_{i+1,j}\right) + (1 - 2\lambda)\, u_{i,j} \tag{1}$$

where $\lambda = \dfrac{k}{ah^2}$ $\left(\text{Here } \lambda \neq \dfrac{1}{2}\right)$.

Equation (1) is called explicit formula. This is valid if $0 < \lambda < \dfrac{1}{2}$.

Bender-Schmidt's difference equation is

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \tag{2}$$

where $\lambda = \dfrac{1}{2} = \dfrac{k}{ah^2}$ i.e., $k = \dfrac{a}{2}h^2$.

Equation (2) is called Bender-Schmidt's recurrence equation.

5. One dimensional heat flow equation by implicit method (Crank-Nicolson's method)

Crank-Nicolson's formula is

$$u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right)$$

6. One dimensional wave equation by explicit method

$$\text{One dimensional wave equation is } \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$
$$\text{valid for } k = \frac{h}{a}$$

After filling given boundary conditions

(i.e., after filling first column, first row, last column by given conditions)

To find unknown 2^{nd} row values alone by

$$\frac{\text{sum of diagonals}}{2}$$

To find unknown 3^{rd} row and onwards by

$$\text{sum of diagonals} - 2^{nd} \text{ previous value in same column}$$

Abbreviation

Adv : Advantage,
C : Condition,
CG : Convergence,
P : Problem,
E : Error,
PC : Procedure,
F : Formula,
EV : Eigenvalue,
ER : Eigenvector,
D : Direct Methods,
d : definition,
S : State,
W : What,
U : Use,
Pf : Proof,
DB : Drawback,

UNIT I : SEEP :- Solution of Equations and Eigenvalue Problems:

FP : Fixed point iteration method,
I : Iteration method,
NR : Newton Raphson method,
GE : Gauss elimination method,
PI : Pivoting,
GJO : Gauss Jordan method,
GJA : Gauss Jacobi Iterative method,
GS : Gauss Seidel Iterative method,
IGJ : Matrix Inversion by Gauss Jordan method,
PM : Power method(Eigen values of a matrix)

UNIT II : IA : Interpolation and Approximation

CP : Cubic polynomial,
L : Lagrange's interpolation method,
IL : Inverse Lagrange's interpolation method,
ND : Newton's divided difference interpolation method,
CS : Cubic Splines,
NF : Newton's forward difference formula,
NB : Newton's backward difference formula

UNIT III NDI : Numerical Differentiation and Integration

DIP : Approximation of derivatives using interpolation polynomials,
DER : Derivative,
NI : Numerical integration,
NC : Newton-Cote's formula,
T : Trapezoidal,
S1/3 : Simpson's 1/3 rule,
S3/8 : Simpson's 3/8 rule,
RM : Romberg's method,
G2P : Gaussian quadrature Two point formula,
G3P : Gaussian quadrature Three point formula,
DIT : Double integrals by Trapezoidal rule,
DIS : Double integrals by Simpson's 1/3 rule

UNIT IV : IVPODE : Initial Value Problems for Ordinary Differential Equations

SSM : Single Step methods,
TS : Taylor's series method,
EM : Euler's method,
ME : Modified Euler's method,
RK1 : Fourth order Runge-Kutta method for solving I order eqn.,
RK2 : Fourth order Runge-Kutta method for solving II order eqn.,
MSM : Multi step methods,
MP : Milne's predictor corrector method for solving first order equations,
AP : Adams-Bash forth predictor corrector methods for solving first order equations

UNIT V : BVP O&PDE:Boundary Value Problems in Ordinary and Partial Differential Equations

CPDE : Classification of PDE,

SFP : Standard Five Point Formula,

DFP : Diagonal Five Point Formula,

CD : Central difference,

LP : Liebman's Iterative process,

FD : Finite difference methods for solving two-point linear boundary value problems,

2dLE : Finite difference techniques for the solution of two dimensional Laplace's equation on rectangular domain,

2dPE : Finite difference techniques for the solution of two dimensional Poisson's equation on rectangular domain,

1dHE : One dimensional heat flow equation by explicit (Bender-Schmidt's)method,

1dHI : One dimensional heat flow equation by implicit (Crank Nicholson) method,

CN : Crank Nicholson method,

1dW : One dimensional wave equation by explicit method

6.15 AU Unit-topics/Semester wise

Unit	Q.No.	AM10	ND10	AM11	ND11	MJ12	ND12	MJ13	ND13	MJ14	ND14	AM17
SEE	1	C(IM)	P(NR)	CG(CR)	P(IM)	CG(NR)	CG(FP)	F(IM)	CG(IM)	P(NR)	C(NR)	?
	2	PC(SEL)	PC(GJA)	P(GE)	P(NR)	2dM	Adv(I,D)	U(PM)	P(GE)	P(GE)	P(IGJ)	
	11a(i)	P(NR)	P(PM)	P(NR)	P(IM)	P(IGJ)	P(NR)	P(IM)	P(GS)	P(EVER)	P(GS)	
	11a(ii)	P(GS)	P(GS)	P(GJ)	P(GJO)	P(FPIM)	P(IGJ)	P(GS)	P(PM)	P(IGJ)	P(NR)	
	11b(i)	P(IGJ)	P(IGJ)	P(GS)	P(GS)	P(GS)	P(GS)	P(IGJ)	P(IGJ)	P(GJO)	P(EVER)	
	11b(ii)	P(PM)	P(IM)	P(GJA)	P(PM)	P(EVER)	P(GJA)	P(GJA)	P(NR)	P(GS)	P(GJO)	
IA	3	P(ND)	U(NF,NB)	P(ND)	F(L)	C(CS)	d(CS)	F(NF)	F(NF)	P(ND)	P(L)	
	4	D(CS)	P(ND)	Pf	d(CS)	P(ND)	F(NB)	P(L)	P(ND)	d(CS)	F(NB)	
	12a(i)	P(L)	P(L)	P(L)	P(NF,NB)	P(CP)	P(ND)	P(L)	P(CS)	P(ND)	P(NF)	
	12a(ii)	P(NDD)	P(NF)	P(CS)	–	P(NB)	P(NF)	P(NF)	–	P(L)	P(CS)	
	12b(i)	P(CS)	P(CS)	P(IL)	P(ND)	P(CS)	P(L)	P(ND)	P(NB)	P(CS)	P(NDD)	
	12b(ii)	P(NFF)	–	P(ND)	–	P(L)	P(CS)	P(CS)	P(L)	–	P(L)	
NDI	5	S(RM)	F(DER)	F(NC)	F(S1/3)	P(G2P)	P(T)	S(S1/3)	P(G2P)	F(DER)	E(S1/3)	
	6	P(G2P)	P(G2P)	U(S1/3)	F(G2P)	C(S3/8)	P(G3P)	P(T)	P(T)	P(T)	F(RM)	
	13a(i)	P(DER)	P(RM)	P(DER)	P(DER)	P(DIT)	P(DER)	P(DER)	P(G3P)	P(DER)	P(G3P)	
	13a(ii)	P(DIT)	P(DER)	P(RM)	–	P(DER)	–	P(T)	P(DER)	P(RM)	P(S3/8)	
	13b(i)	P(RM)	P(DER)	P(G3P)	P(T,S1/3,3/8)	P(RM)	P(S3/8)	P(G3P)	P(S1/3)	P(T,S1/3)	P(DIT)	
	13b(ii)	P(T,S1/3,3/8)	P(DIT,DIS)	P(DIT)	–	–	–	P(DIS)	P(DIT)	P(DIS)	P(RM)	

Unit	Q.No.	AM10	ND10	AM11	ND11	MJ12	ND12	MJ13	ND13	MJ14	ND14	AM15
ODE	7	P(EM)	P(EM)	P(TS)	S(EM)	DB(TS)	P(TS)	P(TS)	S(AP)	Adv(TS)	F(MP)	
	8	F(AP)	W(SSM,MSM)	P(EM)	S(AP)	P(EM)	F(RK)	S(EM)	P(EM)	S(MP)	P(EM)	
	14a(i)	P(TS)	P(TS)	P(RK)	P(RK,II)	P(TS,MP)	P(EM)	P(TS)	P(TS)	P(AP)	P(TS)	
	14a(ii)	P(MP)	P(RK,II)	–	–	–	P(MP)	P(RK,II)	P(MP)	P(TS)	P(MP)	
	14b(i)	P(RK)	P(RK,MP)	P(TS,AP)	P(MP)	P(RK,II)	P(RK,SE)	P(EM)	P(RK)	P(RK)	P(ME)	
	14b(ii)	P(AP)	–	–	–	–	–	P(AP)	–	–	P(RK2)	
BVP	9	F(1dW)	W(2dPE)	P(CP)	P(CP)	P(CD)	S(1dHI)	P(FD)	F(DFP)	P(FD)	F(CD)	
	10	F(2dLE)	F(IDHI)	F(1dWE)	F(SFP)	F(2dPE)	F(1dHE)	S(LP)	P(FD)	S(CN)	F(SFP)	
	15a(i)	P(2dLE)	P(FD)	P(2dLE)	P(FD)	P(2dLE)	P(CD)	P(2dPE)	P(2dLE)	P(1dHE)	P(2dLE)	
	15a(ii)	–	P(1dWE)	–	–	–	P(1dWE)	–	–	P(FD)	–	
	15b(i)	P(1dHI)	P(2dPE)	P(1dHE)	P(2dLE)	P(FD)	P(2dPE)	P(1dHE)	P(1dHI)	P(2dPE)	P(1dHI)	
	15b(ii)	–	P(1dHE)	–	–	P(1dHE)	–	P(1dWE)	P(1dHE)	P(1dWE)	–	

6.16 Preparation for Anna University Exams[Units in Sets (Priority wise)]

#	Unit	I Set (Compulsory)	II Set (Compulsory)	III Set	IV Set
1.	S.A.T.	Gauss Seidel (8) (Solution of system of equations)	Gauss Jordan (8) (Inverse of matrix)	Power Method (8) (Eigenvalue & Eigenvector)	Newton's Method & (8) Fixed Point Iteration Method (8)
2.	I.A.	Lagrange & (8) Inverse Lagrange (8)	Cubic spline (8-16)	Newton Forward & Backward(8-16)	Newton Div. Diff. Formula(8-16)
3.	N.D.I.	Double Integration by Trapezoidal & Simpson (8-16)	Derivatives (8)	Trapezoidal,S$\left(\frac{1}{3}\right)$,S$\left(\frac{3}{8}\right)$ (8-16)	Romberg (8-16) Gaussian 2 & 3 point Formula
4.	I.V.P.	Runge-Kutta (8-16) (I,II order, simultaneous)	Milne & Adam (8-16)	Taylor Series (8-16)	Euler's Method (8) Modified Euler's Method (8)
5.	B.V.P.	2DHF:Laplace Equation (8-16) 2DHF:Poisson Equation (8-16)	1DHF: Explicit(B-S Method) (8-16) 1DHF: Implicit(C-N Method) (8-16)	1DWE(Explicit) (8)	Finite difference (8-16)
	Min%	(40-56)%	(40-64)%	(40-56)%	(40-64)%

Type	#(Q.) × M. = T.M.	#(Q.) $\times\overline{T}$ = T.T.(Min)	*****Important Note :*****	Approach	Best Selection(Well Known) in Q.P.
Part A	10×2=20	10×3=30	*START Practice any one A.U. Q.P. until to get 80%*	MM=MM	Start (Statement or Formula)
Part B	5×16=80	5×30=150	Repeat it atleast 3Q.P. before I.A.&A.U. Exams	Cont.Upd.Prac.	Flow (Starting to Conclusion)
Total	Marks = 100	Time = 180 Min(3Hrs)	* Success to get CONFIDENT Pass = 100% *	Smart Work	Conclusion (Put in Box)

Important Note :

*START Practice full A.U.Q.P. until 80%++
*Repeat it atleast 3Q.P. before I.A.&A.U.
This practice gives us
*[MM = MM]
⇒[Minimize Mistakes=Maximize Marks],
*so Continuous updated practice with
smart work is must.

Best Selection of the questions in Exams are based on all 3 together as follows:

(1) starting lines of the problem

(2) flow of the problem

(3) upto conclusion of the problem.