THINK FAST

Competitive Programming

From Problem 2 Solution in O(1)

Combinatorial Game Theory

Game of Nim - Examples

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Nim with skip move

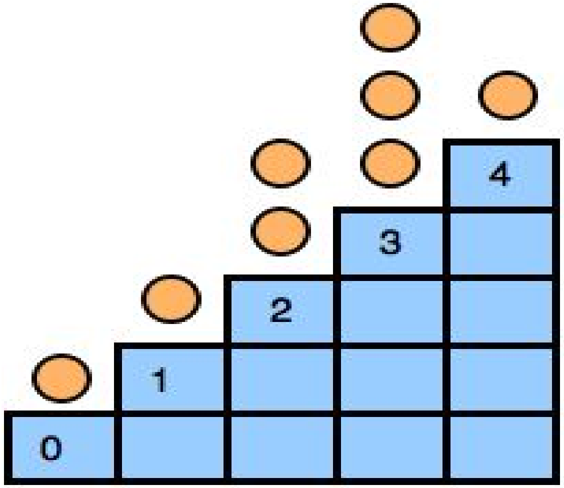
- Given the original nim game with extra rule
  - Skip turn: A player is allowed to say I will skip turn
  - Player 1 is allowed up to A skippings
  - Player 2 is allowed up to B skippings
  - Given the N piles, A, B who is the winner?
  - If A = B? Then whoever the winner can **cancel** the other player move to insure winning (**Move Cancellation)**
  - If A > B and 1st wins in normal nim, then he just plays normally and cancels moves for 2nd if he skipped
  - If A > B and 1st loses in normal nim, then he **skip the first time**, and then play as previous case to win too :)
  - So if A > B, 1st always win. If A < B, 2nd always win.

Dividing a number

- Given an integer N
  - Move: divide N by a prime power > 1
    - e.g. 3, 3^2, 3^3…
    - primes: 2, 3, 5, 7, 11….
  - Loser: N = 1
  - **Solution**: Represent N using its prime powers
  - $$n = 2^{a_1} 3^{a_2} 5^{a_3} 7^{a_4} \cdots p_k^{a_k}$$
  - Then we have k piles, each has ai stones
  - N = 1440600 = 2*2*2*3*5*5*7*7*7*7
  - So piles are = {3, 1, 2, 4}
    - E.g. we have 4 7s

Staircase Nim

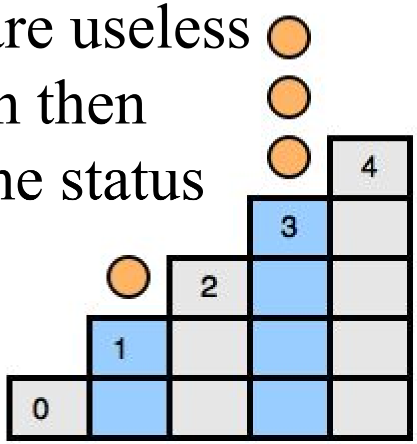
- Staircase Nim
  - Staircase with n steps, each step has some coins
  - Move: move some coins to the left step (except first step)
  - Loser: Can't make a move (e.g. all coins at arr[0])
  - Intuition: Every step is a pile? No only odd positions



Src: http://codeforces.com/blog/entry/44651

Staircase Nim

- Notes
  - Movement from step 1 to step 0 => remove action
  - If opponent moved 5 coins from step 2 to step 1, you can just move them to step 0, hence removed them too
  - Recall: this is move **cancellation strategy**
  - In general, movements on even position are useless
  - Movements from odd position to even can then be considered as **removed**, but affect game status
  - **Solution**: xor odd positions (the piles)



Even steps are useless!

Turning Turtles Game

- Given a horizontal line of N coins: Head/Tail
  - 1 2 3 4 5 6 7 8 9 10
  - T H T T H H T T T H
- Moves
  - Pick any head, and flip it to tail
  - Optionally, flip any coin on left of your chosen coin
  - Loser: No more heads to flip
- Example: THHH (let's make action on 4)
  - **THHT** (flip 4th only)
  - **THTT** (flip 4th and 3rd) **TTHT** (flip 4th and 2nd)
    HHHT(flip 4th and 1st) So Position 4 => 4 moves

Turning Turtles Game

- Observations
  - Action only on Heads. So may be heads ~= Nim piles
  - Head in kth position has k moves. May be Pile size = k
  - E.g. THHTTH => {2, 3, 6} as Nim pile sizes!
  - We need to verify 3 game moves as 2 nim possibilities
    - We can take whole nim pile
    - We can take ANY thing < whole size
- Verification for:
  - Flip Head only
  - Flip Head and Flip a left coin: Tail to Head
  - Flip Head and Flip a left coin: Head to Tail

Turning Turtles Game

- Verification
  - Flip k-th Head only
    - If we did that, we cancelled the k moves from position
    - Seems as if we just removed while pile content!
  - Flip k-th Head and t-th tail (recall k > t)
    - Head at k has k moves cancelled
    - But a tail at is now H and has t moves
    - As if pile of k stones reduced to t stones only
  - Flip k-th Head and t-th head (recall k > t)
    - This is tricky. We already satisfied nim moves
    - Intuition: If this game is nim => this move is not a new case or a useless move

Turning Turtles Game

- Flip k-th Head and t-th head
  - This move cancels: my k moves and his t moves!!
  - As if it means, the other head will never change the final status situation (e.g. I have a winning for it)
  - Let's try example: THTTH => Nim {2, 5}
  - Recall the **move duplication strategy** to always win
    - Make them equal piles => (2, 2) => (0, 0) => 1st win
  - So flipping kth head means reduce k to t (e.g. 5 to 2)
  - Then his second pile equal to mine (2, 2)
  - So I will win anyway => e.g. the 2 H's are cancelled
- Overall: 2 equivalent games.

Your turn: Twins game

- Same as Turning Turtles Game, but
  - You must flip 2 coins, not optionally
  - Then kth position has k-1 move NOT k
  - Also, it means we can't take whole pile move!
    - Actually both previous notes lead to same thing
    - As in terms of nim, pile size = 1 is losing condition
    - So F(n) in this game = **F(n-1) in normal nim**
  - So kth position is pile of size n-1
  - So THHTTH => {1, 2, 5} NOT {2, 3, 6} sizes
  - If you **indexed** game as 0-based, then F(n) = n
    - Sometimes one of the 2 indexings makes computations/patterns easier (later)

Silver Dollar Game

- From judge
  - Array of N cells, that has coins (at most 1 coin in a cell)
  - Move: Pick a coin and move to **any left square: BUT**
    - **Can't** jump over other coins
    - **Can't** put two coins in one cell
  - Loser: Can't do a move
  - Example. for 4 coins in position {2, 5, 7, 10}
  - 1 2 3 4 5 6 7 8 9 10 11

    0 1 0 0 1 0 1 0 0 0 1

Silver Dollar Game

- Thoughts
  - Issue: each coin is **not independent** game any more
  - **Intuition**: Each coin is constrained with its left one.
    - Distance between every 2 consecutive is pile size
    - E.g. $\{2, 5, 7, 11\} \Rightarrow$ piles $\{1, 2, 1, 3\}$
      - E.g. between 7 and 11 = 3 steps
    - **Wrong equivalence**: Moving 7 to 6: decrease in pile and add in another (e.g. generates piles: $\{1, 2, \mathbf{0, 4}\}$

Silver Dollar Game

- Let's understand the game more
  - Input={5}: First player will win directly
  - Input={5, 6}: Second player will always win directly
    - 1st will have to move 5 to 4
    - 2nd can just return them consecutive: e.g. from 6 to 5
    - So overall even # of steps for any input {x, x+1}
    - Clearly this input is always like {1, 2}: pile size = 0
  - Input={5, 8}
    - In similar manner, a move from 5 to 4 can be **canceled** by move from 8 to 7 to have same difference (poker)
    - So, the 2 players should **focus** on moving 2nd not 1st
    - Clearly this input is always like {1, 4}: pile size = 2

Silver Dollar Game

- Let's understand the game more
  - Input={2, 4, 6}
  - One can force (4, 6) as a pile
  - And (0, 2) as a pile (e.g. 0 from left boundary)
  - And handle each pair with same winning strategy for 2 values. so total piles {1, 1} => xor = 0 (2nd can win)
  - Given that we identified a winning strategy that we can definitely apply (either first if will win or 2nd), we can safely consider it
  - You don't need to think of what other ways of winning: having one way to force winning is enough
  - Game using our grouping/cancellation tricks = Nim

Silver Dollar Game

- Let’s understand the game more
  - Input={2, 4, 6}
  - What about grouping (2, 4) => pile size = 1
  - Now 6 is free: it doesn’t corresponds to something in Nim
  - Let’s restrict it with its max possible moves to left
  - 6 has 2 numbers before it: so its pile size = 6-2-1=3
  - E.g. piles (1, 3) => xor != 0 => 1st win => where mistake?
  - Although 6 has up to 3 moves to left, if one directly moved 6 to 5, now it corresponds to empty pile
  - When 4 moves to 3: 6 pile grows up again
  - So our mapping is not nim equivalent = wrong approach

Silver Dollar Game

- Overall solution
  - From most right number to left:
  - Group each 2 consecutive numbers
  - If they are odd, most left number will be grouped with 0
  - {2, 5, 7, 11} => {2, 5}, {7, 11} => (2, 3)
  - {2, 5, 7, 11, 20} => {0, 2}, {5, 7}, {11, 20} => (1, 1, 8)
- For other ways of explaining it: See1, See2
  - Mainly, they map faster to nim and consider it poker nim
  - I cancelled a move in the game ⇒ Direct nim
  - Think **much**: Duplication, Cancellation, Actual game, Win strategy, Correct equivalence

Pots of gold game

- Given set of coins (e.g. 10, 7, 8, 12).
  - Move: Take coins in a cell from one of ends (e.g. 10 | 12)
  - The **winner** is the player who collected a **higher number of coins** at the end of game.
- Solution:
  - Don't be cheated. This is NOT nim-related game
  - Winning is not about **LAST** move, but **overall** choices
  - This is a classical search/minimax/dp problem
- **Lesson**: Remember properties of Nim

Final Notes

- Make sure game is impartial
  - If it is not, think in search/backtrack directions
- Remember Nim game nature
  - With exception, the independance feature of sub-games
- Dependent games are tricky/hard to analyze
  - Understand & analyze the game
  - Try to decouple to independent sub-games
  - Suggest mappings and verify it carefully
  - Let duplication/cancellation strategies be your friends

تم بحمد الله

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