THINK FAST

Competitive Programming

From Problem 2 Solution in O(1)

Combinatorial Game Theory

Wythoff Game

Mostafa Saad Ibrahim

PhD Student @ Simon Fraser University

Wythoff's game

- Given 2 piles of stone with 2 moves:
  - move 1: choose 1 pile, remove whatever
  - move 2: choose from the 2 piles SAME # of items
  - Example (10, 20) => (5, 20) or (10, 12) or (5, 15)
  - Loser: No moves
- A different rephrasing
  - 2D grid with coin in position (N, M)
    - Either move up, left or diagonal (up left direction)
  - Or chess queen travel south, west or southwest
- Wythoff found formula for losing positions
  - It depends on Golden Ratio

Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$$

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1\,.$$

$$\varphi + 1 = \varphi^2$$

$$1 + \frac{1}{\varphi} = \varphi.$$

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}.$$

$$\varphi = [1;1,1,1,\ldots] = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$$

$$F\left(n\right) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$

Wythoff's game

- The formula for losing positions
  - Let k be for the kth term (n, m), a losing position
  - There are 2 ways to compute sequence
  - Computing n, m using k only
  - Computing n, m using relationships between them

$$n_k = \lfloor k\phi \rfloor = \lfloor m_k \phi \rfloor - m_k$$

$$m_k = \lfloor k\phi^2 \rfloor = \lceil n_k \phi \rceil = n_k + k$$

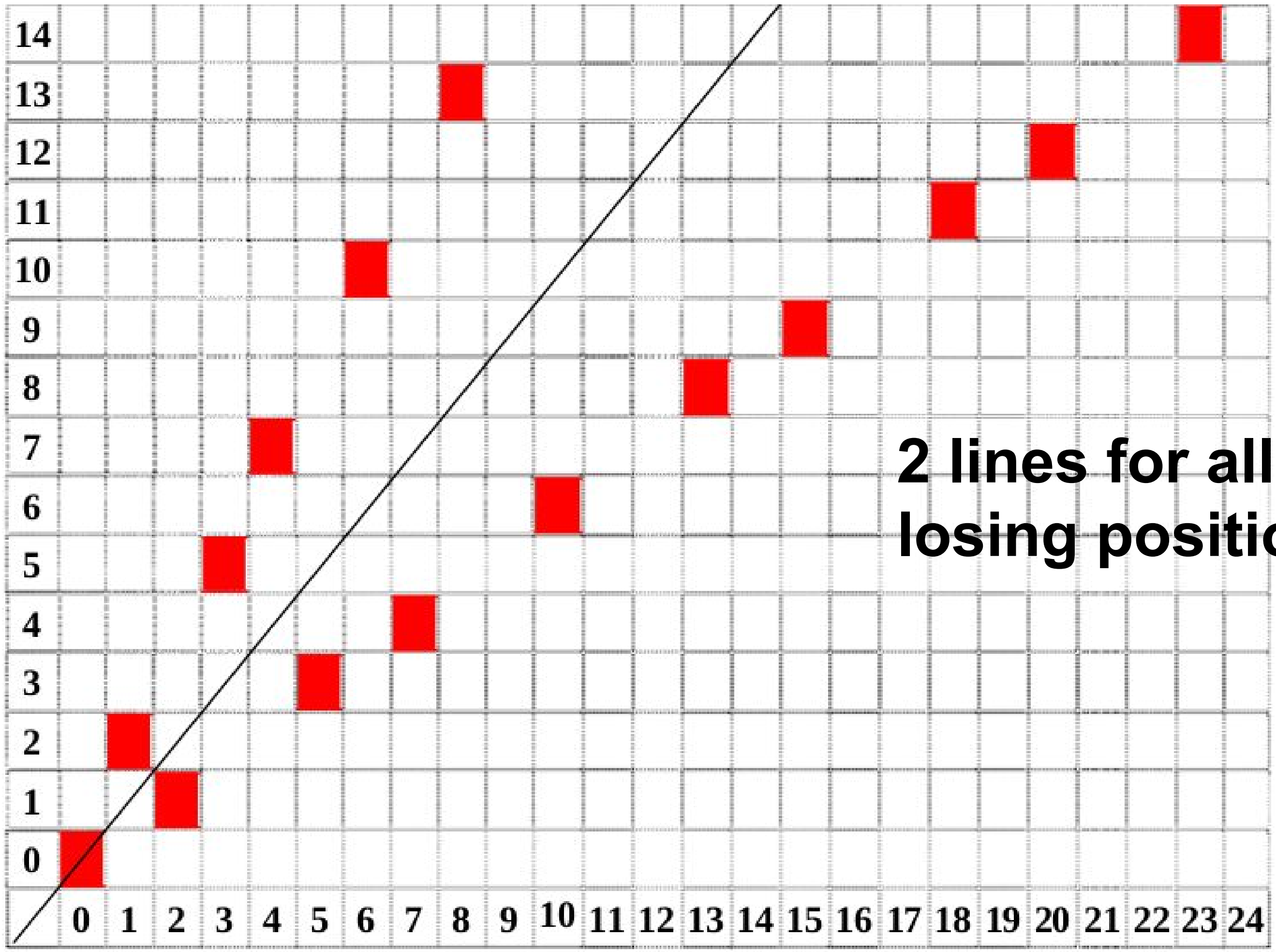
Wythoff's game

```
const double GOLDEN_RATIO = (1 + sqrt(5.0)) / 2;

pair<int, int> getKthWythoffLosingPos(int k) {
  int a = k * GOLDEN_RATIO;
  int b = k * GOLDEN_RATIO * GOLDEN_RATIO;

  // important relationship: then k = b-a
  assert(b == a+k);
  return make_pair(a, b);
}

bool isWythoffLosingPosition(int a, int b) {
  if(a > b)
    swap(a, b);
  int k = b-a;
  return a == getKthWythoffLosingPos(k).first;
}
```

Wythoff's game: Sequences

- N = {1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, 33, 35, 37, 38, 40}
- M = {2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, 31, 34, 36, 39, 41}
- N ∪ M = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, ...} **Cover** all values (e.g. rows/cols)
- These sequences have **no terms in common**
  - Beatty's Theorem [(0, 0) is the losing terminal position]

Wythoff's game: Losing positions



2 lines for all losing positions

Src: http://math.rice.edu/~michael/teaching/2012Fall/Wythoff.pdf

Wythoff's game: Losing positions

- As mentioned, every row/col is covered
  - Actually, every row/col has exactly 1 losing position
  - This is a very sparse grid!
- Your turn
  - Given Q queries: Each query is a rectangle (0,0,X,Y)
  - What is the percentage of losing positions?
  - Hint: Think in generation style (not formula)
  - Hint: Use a data structure for efficient processing
  - Solution

Your turn: 3 piles Nim

- Given 3 piles
  - Either move from 1 pile
  - Or same amount from the 3 piles
- Prove that this game is **equivalent to normal Nim**
  - Solution: See discussion here or here
- In fact, for any N piles where N is odd
  - Wythoff game is equal to Normal Nim.
  - For formal proof see Theorem 9

Wythoff game in small inputs

- The game can have many **extensions**, **restrictions and** **generalizations**
  - Some of these variants are for the 2 piles case
  - Others variants for N piles
    - e.g. pick same amount from a **subset** of piles
    - Such as dividing N by power of squarefree numbers
  - Many of them are hard or unsolvable
  - Not in competitions so far
- If the **constraints are small** on a variant
  - Just think in **dynamic programming** solution
  - Example

Final notes

- The problem came little times in competitions
  - Hard to play with this game without making it complex
  - Most of the problems are about the direct game
    - Generate Sequence
    - Is winning/losing position ?
  - Few more challenging
    - The first move for the winner? **Your turn (or, or)**
    - Simulate the game? **Your turn**
- Lesson: Think about patterns/formula
- Optional readings: See1, See2, See3, See4

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