Competitive Programming

From Problem 2 Solution in O(1)

Combinatorics

Permutations and Combinations - 2

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Combinations

- Binomial Coefficients based on C(n, k)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(a+b)^0 = 1$$

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2\,a\,b + b^2$$

$$(a+b)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3$$

$$(a+b)^4 = a^4 + 4\,a^3\,b + 6\,a^2\,b^2 + 4\,a\,b^3 + b^4$$

$$(a+b)^5 = a^5 + 5\,a^4\,b + 10\,a^3\,b^2 + 10\,a^2\,b^3 + 5\,a\,b^4 + b^5$$

Combinations

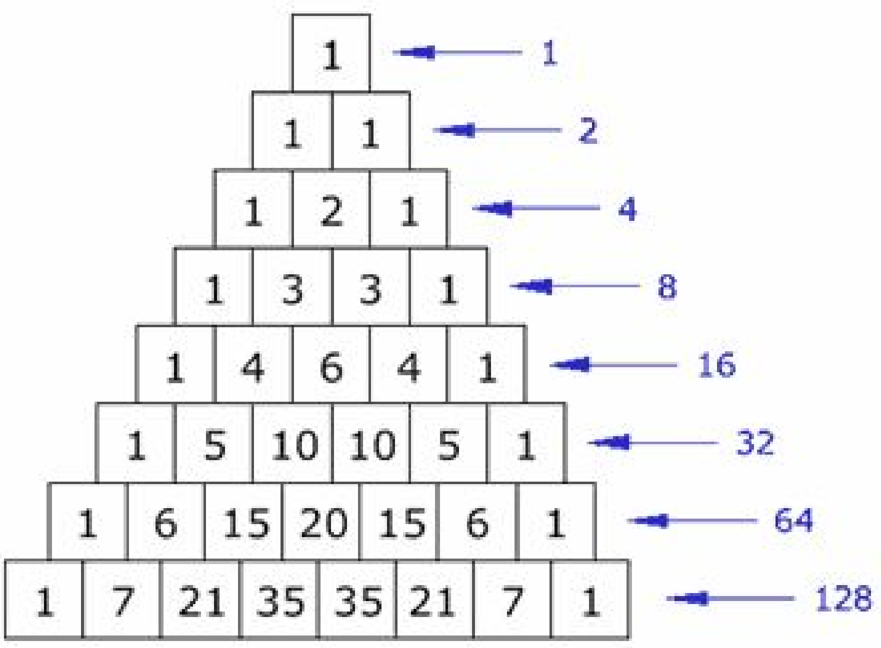
$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Hence

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Combinations

- We can compute combinatorics from **pascal triangle**. This triangle has many patterns



Combinations

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

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Combinations

$$C_n^k = \frac{n}{k} C_{n-1}^{k-1}$$

$$\sum_{k=0}^{n} C_n^k = 2^n$$

$$\sum_{m=0}^{n} C_m^k = C_{n+1}^{k+1}$$

$$\sum_{m=0}^{n} C_m^k = C_{n+1}^{k+1}$$

$$\sum_{k=0}^{m} C_{n+k}^k = C_{n+m+1}^m$$

$$(C_n^0)^2 + (C_n^1)^2 + \ldots + (C_n^n)^2 = C_{2n}^n$$

$$1C_n^1 + 2C_n^2 + \ldots + nC_n^n = n2^{n-1}$$

$$C_n^0 + C_{n-1}^1 + \ldots + C_{n-k}^k + \ldots + C_0^n = F_{n+1}$$

Fn = Fibonacci

Combinations

- Can we find recurrence for C(n, k)
  - If we have 6 students and want to pick 3
  - let them be s1, s2, s3, s4, s5, s6 and 3 needed
  - for s1, what if we selected him with us?
    - then we now have s2, s3, s4, s5 and 3 needed persons
    - C(n-1, k-1)
  - for s1, what if we did not take him with us?
    - Then we now have s2, s3, s4, s5 and 3 needed persons
    - C(n-1, k)
  - base cases: (n ==k, 1) (k == 1, 1), (n == 0, 0)
- Code it top down or bottom up

Combinations

```
long long C[50][100] = {0}; // whatever needed

void build_nCk() {  // Bottom up approach
    for(int i = 0; i < 50; ++i)
        for(int j = 0; j < 100; ++j)
            C[i][j] = (j == 0) ? 1 : ( (i == 0) ? 0 : C[i-1][j-1]+C[i-1][j]);
}
```

Combinations with repetition

- What if we have groups, each of items. We need k items from them and repeat per group?
- E.g. pick 15 cans from the following 6 groups
  - Below 3 different ways to pick the 15 categories

	Coke	Pepsi	Diet Coke	Root Beer	Sprite	
A:	111	111	111	111	111	=15
B:	11		111111	111111	1	=15
C:		1111	1111111	1111		=15

- Imagine putting 0 after each group except last
  - Row will be 15 ones + 4 zeros
  - B = 11 0 0 111111 0 111111 0 1 = 1100111111011111101

Combinations with repetition

- New problem: Given string of 19 ones, how many 4 bits to choose to flip to 0? C(19, 4)
- Theorem: The number of ways to fill r slots from n categories with repetition allowed is:
  - $C(r+n-1, r) = C(r+n-1, n-1)$
  - $n = 5$, $r = 15 \Rightarrow C(19, 15) = C(19, 4)$
- How many non-negative integer solutions are there to the equation $a + b + c + d = 100$.
  - $n = 4$, $r = 100 \Rightarrow C(103, 100) = C(103, 3)$

Combinations with repetition

- How many integer solutions are there to:
  - $a + b + c + d = 100$,
  - where $a \geq 3$, $b \geq 0$, $c \geq 2$ and $d \geq 1$?
  - $(A+3) + (B+0) + (C+2) + (D+1) = 100$
  - We now have 6 values already reserved, remaining are free? so $n = 4$, $r = 100-6$
- What if: $a \geq -3$, $b \geq 5$, $c \geq -2$ and $d \geq -1$?
  - $(A-3) + (B+5) + (C-2) + (D-1) = 100$
  - $A + B + C + D = 100 + 3 - 5 + 2 + 1 = 101$
- Back to theorem: if we must select one from each, then $C(r-1, r-n)$, by removing n 1s

Combinations with repetition

```
long long cnt = 0;

for (int i1 = 1; i1 <= 20; ++i1) {
    for (int i2 = i1; i2 <= 20; ++i2) {
        for (int i3 = i2; i3 <=20; ++i3) {
            for (int i4 = i3; i4 <= 20; ++i4) {
                cnt++;
            }
        }
    }
}

cout<<cnt<<"\n";     // 8855 = C(20+4-1,4)
// {i1, i2, i3, i4} 4 values selected with repetition
// from set {1, 2, 3.....20}
```

Partitions

- Partition group of 6 friends into 3 groups each with size {3, 2, 1} friends
  - Pick 2 for 1st group C(6, 3) => remains 3
  - Pick 2 for 2nd group C(3, 2) => remains 1
  - Pick 2 for 3rd group C(1, 1) => remains 0
  - C(6, 3) * C(3, 2) * C(2, 1) = 6! / (3!*2!*1!)
- Partition our group of 18 friends into 5 groups each with size {5, 5, 5, 2, 1} friends
  - We have problem in the groups of **same size**.
  - If we take order into account, then 18!/(5!*5!*5!*2!*1!)
  - If unordered of same size => 18!/(5!*5!*5!*2!*1! * **3!**)

Partitions

- A set with n elements can be partitioned into k **ordered** subsets of r1, r2, ….rk elements (where r 1 + r 2 + + r k = n) in the following number of ways

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}$$

Partitions

- A set of n elements can be partitioned n into k **unordered** subsets of same width in the following number of ways:

$$\frac{1}{k!}\binom{n}{r, r, \ldots, r} = \frac{n!}{k!\, r!\, r! \ \ldots \ r!} = \frac{n!}{k!(r!)^k}.$$

- That is for each group of same value, divide by k! where k is number of them

Partitions

- What if we want to partition n persons to k **non empty** groups whatever groups sizes?
  - {1, 2, 3, 4, 5} to 2 groups can be {1, 3, 5} {2, 4}
- Let's find its recurrence
  - If we have 6 students and want to split to 3 groups: g1..g3
  - For s1, either put it in g1, g2, g3 and keep the groups
    - Then we have 3 ways * Solution (5 students, 3 groups)
  - Or put s1 in first group and nothing more for g1
    - Then we have solutions (5 students, 2 groups)
    - Note, they must be non-empty groups
  - F(n, k) = k * F(n-1, k) + F(n-1, k);
  - Bases: (n == k, 1) and (k == 1, 1)

Partitions

- This is called **Stirling** numbers of the **second** kind.
- Let's code it as top down instead of bottom up. Run this using (50, 50)

```
long long stirling2(long long n, long long k) {
    if(n == k || k == 1)
        return 1;
    return      k * stirling2(n-1, k) + stirling2(n-1, k-1);
}
```

Partitions

- What if we want to partition n persons to whatever number of groups: $\sum_k$ stirling2(n, k)
- This is called Bell Number
  - Its sequence: 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147
  - To partition 2 numbers => 2 ways {{1}, {2}}, {1,2}
  - To partition 3 numbers => 5 numbers
  - Note, unordered: {1, 2} same as {2, 1}
- You can code Bell numbers using Bell Triangle or using Binomial Coefficients or **simply** using sum of stirling2

Partitions

- If n is a positive integer, then a partition of n is a non-increasing sequence of positive integers p1, p2 ... pk whose sum is n
- 5 has 7 partitions
- F(n) = 1, 1, 2, 3, 5, 7, 11,
  15, 22, 30, 42, 56,
  77, 101, 135, 176
- See sequence

The partitions of 5 are

5

$4 + 1$

$3 + 3$

$3 + 1 + 1$

$2 + 2 + 1$

$2 + 1 + 1 + 1$

$1 + 1 + 1 + 1 + 1.$

Partitions

- Can we find recurrence for F(n)? Yes, in terms of summation...we can push the sum as index
- Here is a trick: n should have some maximum value in its summation, the first one
- Let $F(n, k)$ = the number of integer partitions of n with largest part at most k
- $F(n, k) = F(n - k, k) + F(n, k - 1)$.
  - Basis: are $F(1, 1) = 1$ and $F(n, k) = 0$ whenever $k > n$.
- Initial call: $F(n, n)$

Your Turn

- Watch this series (All of it)
- Summations and binomial coefficients
- Read **Counting derangements**
- Read **Vandermonde's identity**
- Read **Double Counting / Bijection**
- Read proves in Kenneth Rosen - Ch 6
- Write code to get "**next_permutation**" and "**next_combination**" … See Rosen book
- Express permutation with repetition **formula** in terms of combinations formula..why useful?

تم بحمد الله

علمكم الله ما ينفعكم

ونفعكم بما تعلمتم

وزادكم علماً

- CF131-D2-C, CF124B-D2
- UVA (326, 369, **153,**, 485, 530, 911, 10105, 10219, 10375, 10532, 10541)
- http://ideone.com/ixb6O9