THINK FAST

Competitive Programming

From Problem 2 Solution in O(1)

Algebra

Patterns in Sequences

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Sequences

- **Sequence** (progression) is a list of numbers [in order]. **Series** refers to sequence addition
  - Can be increasing, decreasing, **strictly** increasing
  - Can increase then decrease (or reverse), positive, negative
  - May have closed formula
- What are these sequences?
  - 1 2 3 4 5 6 7 ….
  - 0 2 4 6 8 ….
  - 2 3 5 7 11 13 17 19
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, ….
  - 1, 1, 2, 3, 5, 7, 12, 19, ….
  - 15 27 39 1 3 5 7 9 **1 3 5 7 9** 1 3 5 7 9 …

Sequences Growth

- 1 1 1 1 1 1 .. Constant => f(n) = 1
- 1 2 3 4 5 6 .. Linear f(n) = n
- 1 4 9 16 25 36 .. Quadratic => f(n) = n*n
- 1 8 27 64 125 216 .. Cubic => f(n) = n*n*n
- 1 2 4 8 16 32 64 128 .. Exponential => f(n) = 2^n
- 1 3 9 27 81 243 .. Exponential => f(n) = 3^n
- 1 2 6 24 120 720 5040 .. Factorial => f(n) = n!
- f(20): 1, 20, 400, 8000, 1048576, 3486784401, 2432902008176640000
- Accumulation: $f1(n) = \sum_i f2(i)$
  - 1 2 3 4 5 6 7 … (linear)
  - 1 3 6 10 15 21 … (quadratic)
  - 1 4 10 20 35 56 … (cubic)
  - 1 5 15 35 70 126 … See Pascal Triangle

Sequences Patterns

- Sometimes, we find the answer to function is some sequence of numbers that looks **regular**
- If we know **formula/function** => done
- You need to **memorize** some basic sequences, and have some skills to guess others
- Finding the pattern is challenging..needs your **intuition**...**skills**...**memorization** of known functions

Guess Sequence

- 0, 1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, 13, 7, 15, 8, 17, 9, 19
  - Seems increase...decrease! Let's compare i vs f(i)
- 0, **1**, 2, **3**, 4, **5**, 6, 7, 8, **9**, 10, 11, 12, 13, 14, 15, 16, **17**, 18
- 0, **1**, 1, **3**, 2, **5**, 3, 7, 4, **9**, 5, 11, 6, 13, 7, 15, 8, **17**, 9
  - clearly f(n) = n if n is odd … n/2 if n is even
- 5, 8, 11, 14, 17, 20, 23….
  - f(n) = f(n-1) + 3. f(0) = 5 [This is a **Recurrence**]
  - f(n) = f(n-2) + 2*3 [let's expand]
  - f(n) = f(n-3) + 3*3 [let's expand]
  - f(n) = f(n-k) + k*3
  - when n = k => f(n) = f(n-n) + n*3 = 5 + n*3 = a + bn
- What about: 1 5 13 29 61 … ? Can we reach a formula?

Guess Sequence

- 0, 1, 3, 6, 10, 15, 21 ….
  - accumulation of 0+1+2+3….n
  - sum of 1, 2, 3….100 = (1, 100)+(2, 99)+....(50, 51)
  - = 101 * 50 terms => rule: (n * (n+1)) /2
- f(n) = 5, 13, 24, 38, 55, 75, 98, …
  - Hmm...big gaps...what if it is accumulated? Let's **go back**
  - 5, 8, 11, 14, 17, 20, 23….fx(n) = 5 + n*3 … cool!
  - Then f(n) = $\sum_i$ fx(i) => accumulation of linear = quadratic
  - f(4) = 5+0*3 + 5+1*3 + 5+2*3 + 5+3*3 + 5+4*3
  - f(4) = 5*5 + 3 * (0+1+2+3+4)
  - f(n) = (n+1)*5 + 3 * n * (n+1) / 2 => (n+1)(3n+10)/2
- Your turn: 5 50 137 278 485 770...

Guess Sequence

- In previous $f(n) = \sum_k fx(i)$ where $fx(n) = 5 + n*3 = a + bn$
- $f(n) = \sum_{k=0} a + bk$ => **replace** k => n-k
- $f(n) = \sum_{k=0} a + b(n-k)$ => think reverse order
- Add both
- $2 f(n) = \sum_{k=0} 2a + b(k+n-k)$ [Notice k is eliminated]
- $2 f(n) = \sum_{k=0} 2a + bn$
- $2 f(n) = (2a + bn) \sum_{k=0} 1 = 2(a + bn)(n+1)$
- $f(n) = (2a + bn)(n+1) / 2 = (10 + 3n)(n+1) / 2$
- f(n) terms are called a series, sum of previous elements
  - Special case: $(a = 0, b = 1) \Rightarrow (n * (n+1))/2$
- **Replacements** can help us **remove** some parameters

Guess Sequence

- In previous method, we keep de-accumulating till reached well known sequence, then accumulate sequences f(n)
- A more systematic way for accumulated sequences is **system of linear equations**
  - If you know it is quadratic equation => $f(n) = an^2+bn+c$
  - 3 unknown. Evaluate f(1), f(2), f(3) => 3 equations
  - Solve them: $a = 3/2$, $b = 7/2$, $c = 0$ => $(3n^2+7n)/2$
  - See Difference Method

Guess Sequence

- 0 1 3 7 15 31 63 127 …. See Mersenne Number
  - **Let's think** about last term/idx
  - 31 vs (15 value and 4 its index)? 31 = 15 + 2^4
  - $F(n) = F(n-1)+2^{n-1}$
  - $F(n) = F(n-2)+ 2^{n-1} + 2^{n-2}$
  - $F(n) = F(n-3)+ 2^{n-1} + 2^{n-2} + 2^{n-3}$
  - $F(n) = 1 + 2 + 4 \ldots 2^{n-1} = \sum_{k[0 - n-1]} 2^k = 2^n - 1$
  - **Let's think** about term and its previous term
  - 31 vs 15? 127 vs 63? => $F(n) = 2F(n-1)+1$
  - 2 views => 2 different recurrences
- 1 1 2 3 5 8 13 21 34 ..
  - $F(n) = F(n-1) + F(n-2)$ => Fibonacci sequence (later)

Guess Sequence

- Back to $F(n) = 2F(n-1)+1$..Expand and solve? No way
- General Recurrences form $A(n)F(n) = B(n)F(n-1) + C(n)$
  - Where $A(n)$, $B(n)$, $C(n)$ are the constants in the nth term
  - E.g. In above, $A(n) = 1$. $B(n) = 2$. $C(n) = 1$
  - If $A(n-1) = B(n)$, we can expand easily and find sum: **Examples**:
  - $n\,F(n) = (n-1)\,F(n-1) + 1$
  - $F(n)/n = F(n-1)/(n-1) + 4$
  - $F(n)/2^n = F(n-1)/2^{(n-1)}$ => $A(n) = 1/2^n$, $A(n-1) = 1/2^{(n-1)}$, $B(n-1) = 1/2^{(n-1)}$
- If it is not, we need to multiply equation with factor, to switch
  - Multiply equation by **factor D** $= 1/2^n$
  - $F(n)/2^n = F(n-1)/2^{n-1} + 1/2^n$ => Systematic in decreasing
  - Let $T(n) = F(n)/2^n$ => $T(n) = T(n-1) + 1/2^n = \sum_{k=1} 1/2^k$
  - $T(n) = ½ + ¼ + ⅛ + \ldots 1/2^n = 1 - 1 / 2^n$ => $F(n) = 2^n - 1$

Guess Sequence

- To compute D for any recurrence of this form, See Concrete Mathematics book, **Sums** Chapter
- What if right term has sum of F(k)
  - Idea: compute F(n-1), then F(n) - F(n-1) removes sum

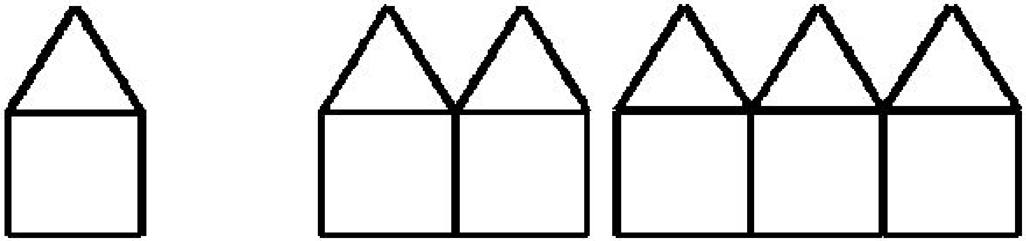
$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$ Multiply by n $$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k$$

Let n = n-1 $$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

Subtract $$nC_n - (n-1)C_{n-1} = 2n + 2C_{n-1}$$

Rearrange $$nC_n = (n+1)C_{n-1} + 2n$$ A(n) = n, B(n) = n+1, C(n) = 2n. D = ?

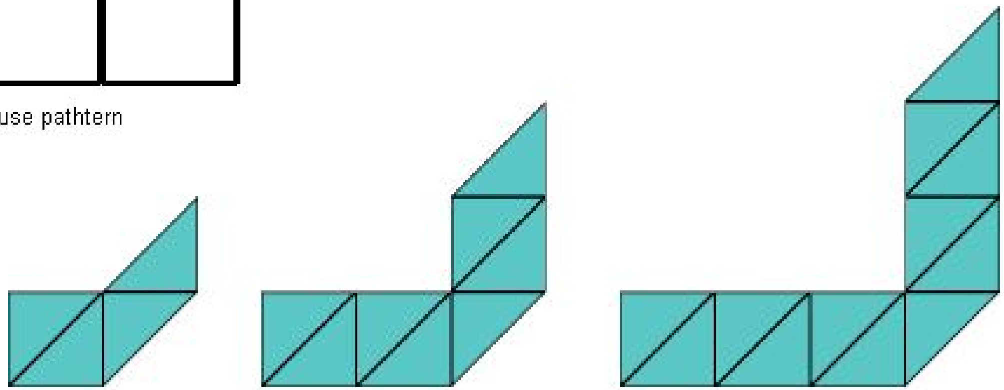
Guess Sequence: shapes



1 - house path

2 - house path

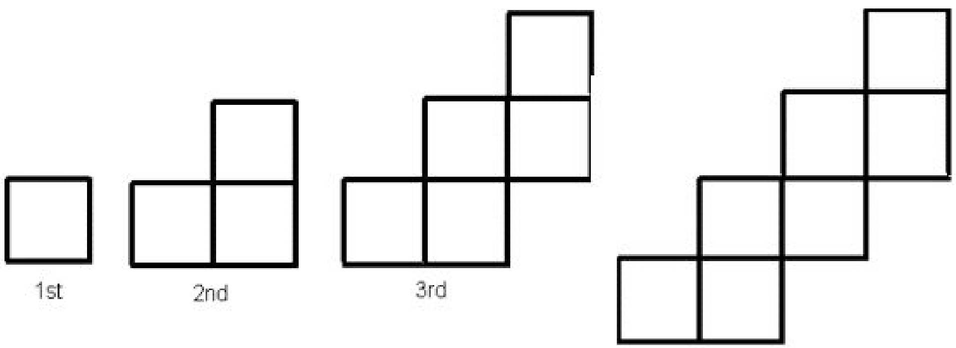
3 - house pathtern



Pattern 1

Pattern 2

Pattern 3



1st

2nd

3rd

4th

Guess Sequence: 2D

- Little times you may face 2D sequence. E.g. given (i, j), compute sequence based on that
- Think in row, column and the 2 diagonals. Many times it can be converted to many 1D sequences
- Popular shapes: Pascal, Bell, Floyd triangles

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

1
0 1
1 1 2
1 2 3 5
4 5 7 10 15
11 15 20 27 37 52
41 52 67 87 114 151 203
162 203 255 322 409 523 674 877

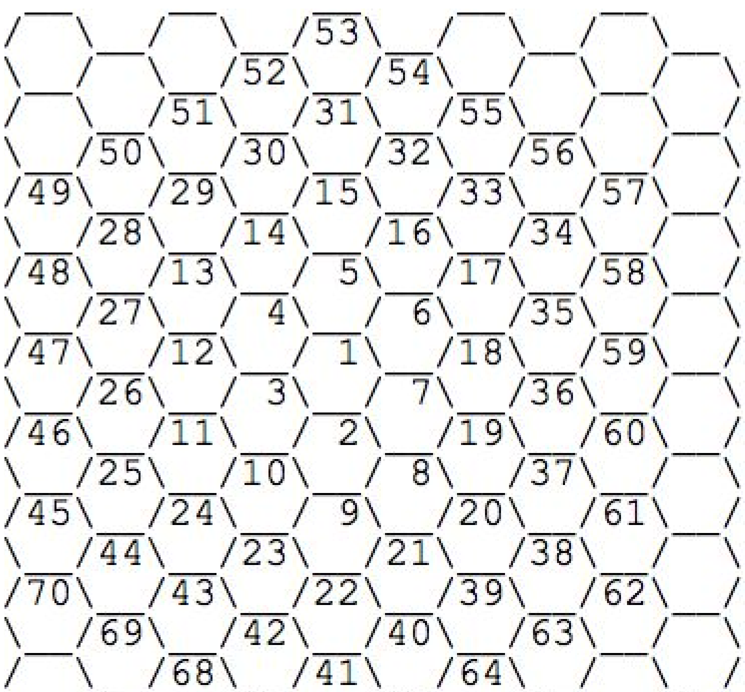
1
2 3
4 5 6
7 8 9 10
11 12 13 14 15

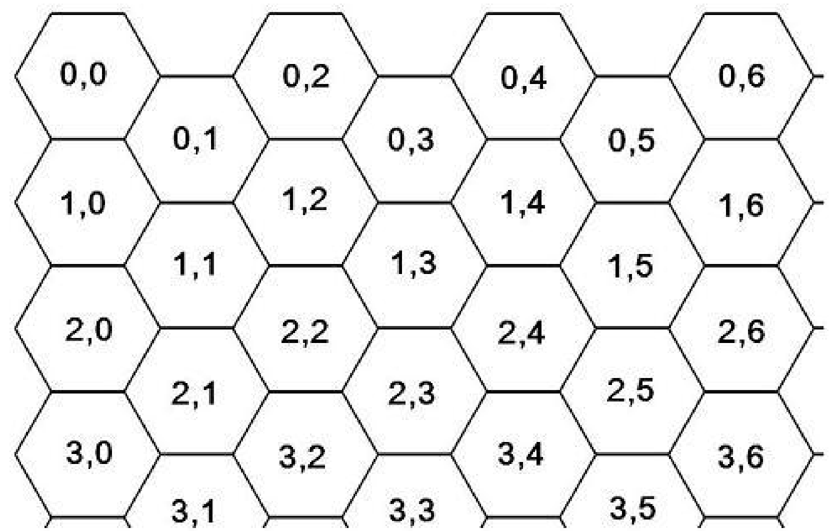
Guess Sequence: Problem

- Given X [< 10^{18}], how many numbers < X in the infinite list: 1, 4, 9, 16, 25, 36. . . n^2
- **Intuition**: n^2 drops many numbers...**sqrt** may be answer
  - Work on your intuition/guessing skills
- F(X) = 0, 1, 1, 1, **2**, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4…
  - All numbers from 5-9 will have 2
  - All numbers from 10-16 will have 3
  - All numbers from 17-25 will have 4...and so on
- Notice 5 = (4+1) => first to get 2
- Notice 10 = (9+1) => first to get 3
- Notice 17 = (16+1) => first to get 4
- Clearly answer is sqrt(X-1)

Guess Sequence: Problem

- Investigate the different sequences (e.g. diagonal sequence) in hexagon grid?
- Is it feasible to find minimum steps between 2 numbers?





Guess Sequence

- Problem that needs pattern, may have other solution (e.g. DP)
- You need to be clever..consider accumulation...consider f(n) versus last terms (1, 2, 3)..or terms and indices…
- Try to guess sequence nature..is it quadratic? exponential?
- Sometimes, we **guess** sequence (not derive it), and then try to **validate** if this is correct guess. You can do that by **trying** some small values...or **proving** it
- Sometimes enumerating sequence elements is hard/buggy. We may write simple code to **generate** its first elements.
- Be careful..and validate your formula versus 5+ terms...you can have wrong formula that works for the first terms only
- One way to prove, is **prove by induction**: See 1. See 2. See 3.

Arithmetic & geometric sequence

- Arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant
  - $a_n = a_1 + (n-1)d,$
  - $a_n = a_m + (n-m)d.$
  - 2, 5, 8, 11.....
- Geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous one by fixed value
  - $a_n = a\,r^{n-1}.$
  - 5, 10, 20, 40, … (start 5 with r = 2)
  - -11, 22, -44, 88, ... (start -11 with r = -2)
  - $4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \frac{64}{81}, \ldots$ (start 4 with r = 2/3)

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}.$$

Online encyclopedia of sequences

- Site can help you know about sequences
- https://oeis.org and seems wiki

Search: **seq:1,3,6,10,15,21**

Displaying 1-10 of 125 results found. page 1 2 3 4

Sort: relevance | references | number | modified | created Format: long | short | data

A000217 Triangular numbers: a(n) = C(n+1,2) = n(n+1)/2 = 0+1+2+...+n.
(Formerly M2535 N1002)

0, **1**, **3**, **6**, **10**, **15**, **21**, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 21
276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 82
946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431 (list; graph; refs; listen; hi
internal format)

OFFSET 0,3

COMMENTS Also referred to as T(n) but C(n+1,2) or binomial(n+1,2) are preferred (
favored slightly over the former).

تم بحمد الله

علمكم الله ما ينفعكم

ونفعكم بما تعلمتم

وزادكم علماً

problems

- 10783, 10509, 11805, 12149, 10644, 10976, 10958, 10025