P THINK FAST S

Competitive Programming

From Problem 2 Solution in O(1)

Computational Geometry

Point and Vector

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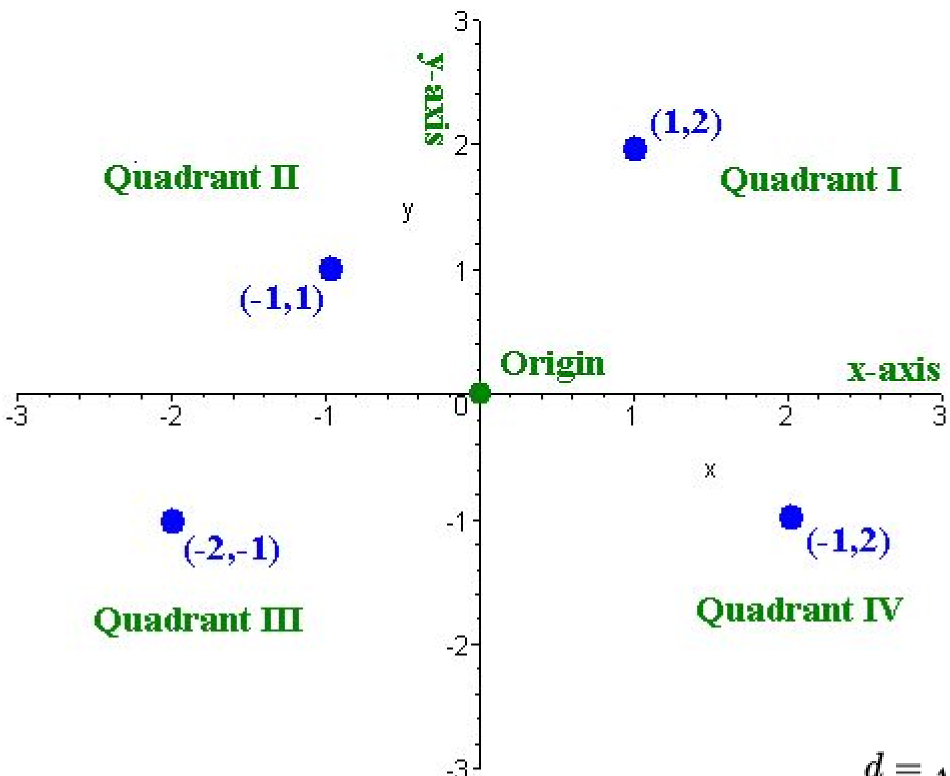
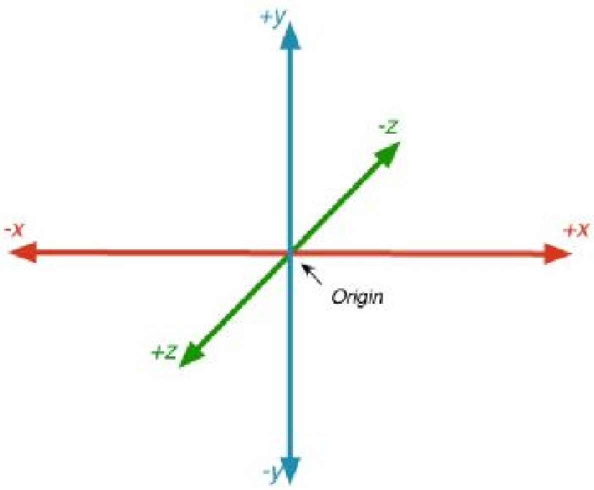
Cartesian coordinate system

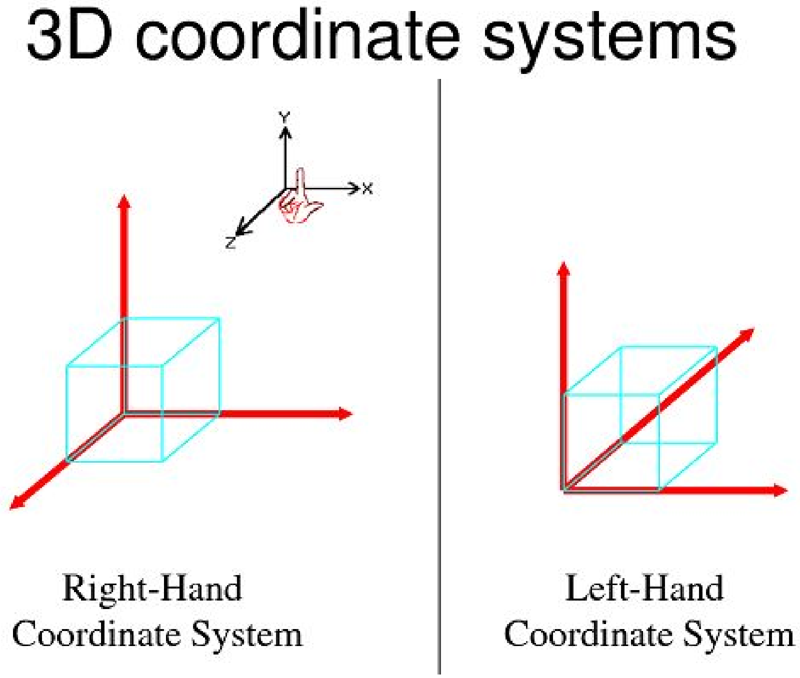


$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Src: https://www.ltcconline.net/greenl/courses/152a/functgraph/coord.8.gif

Cartesian coordinate system



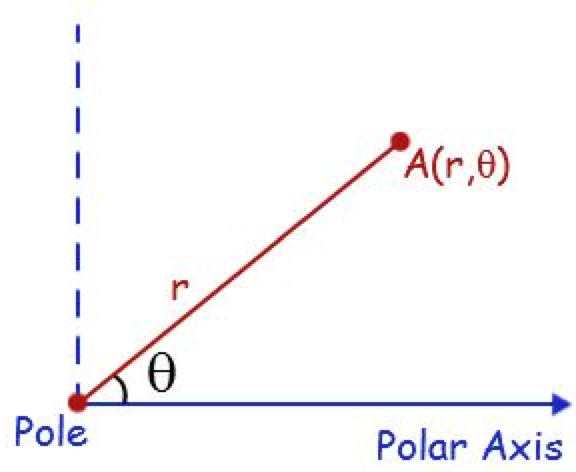


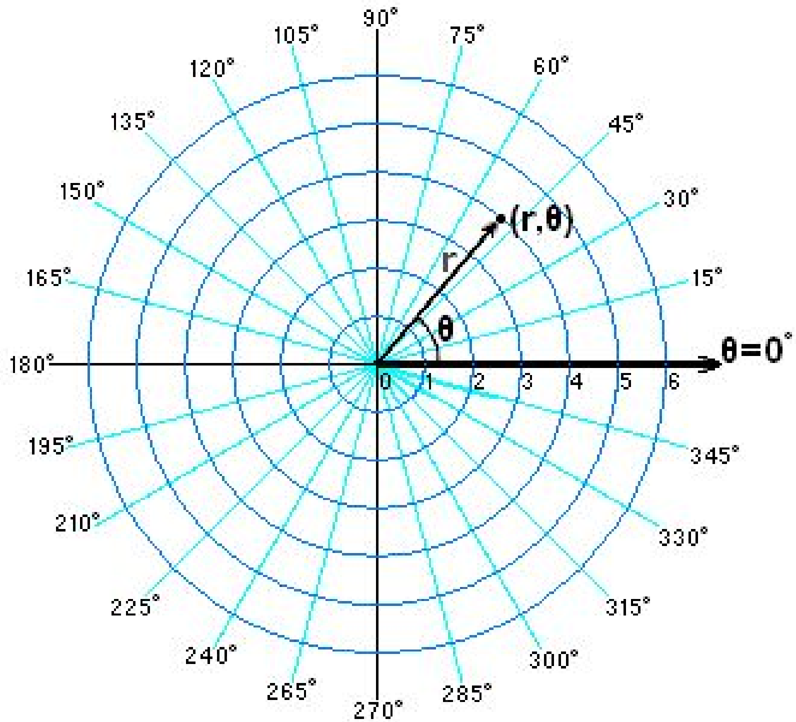
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

Src: http://www.c-jump.com/bcc/common/Talk3/Math/Matrices/const_images/3d_coord_systems.png

Polar coordinate system

The ***r*** and ***θ*** **coordinates** of a point P measure respectively the distance from P to **the origin** O and the angle between the line OP and the **polar axis**.



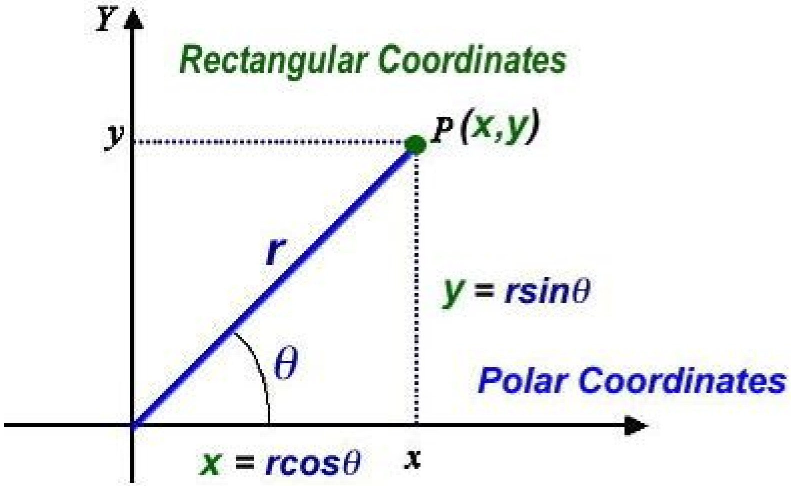


Src: http://images.tutorvista.com/cms/images/67/polar-coordinate1.png https://upload.wikimedia.org/wikipedia/commons/thumb/7/78/Polar_to_cartesian.svg/250px-Polar_to_cartesian.svg.png

Cartesian ⇔ Polar Conversions

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$r = \sqrt{x^2 + y^2}$$
$$\varphi = \operatorname{atan2}(y, x)$$

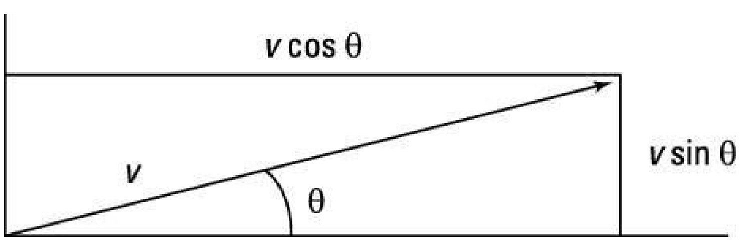


For examples: See

Src: http://www.kb6nu.com/wp-content/uploads/2012/08/rectangular-polar-coordinates.jpg

Vector

- Vector = Direction + Magnitude
  - For example, the line segment from a = (1,3) to b = (5,1) can be represented by the vector v = b - a = (4,-2)
  - Magnitude = norm of the vector
- Given (x, y), we can find angle / magnitude



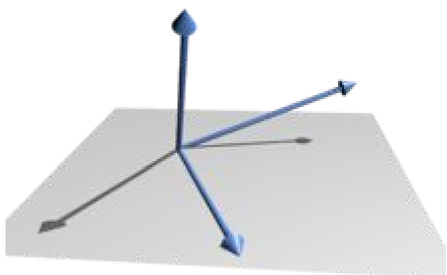
$$\frac{y}{x} = \frac{\cancel{v}\sin\theta}{\cancel{v}\cos\theta} = \tan\theta$$

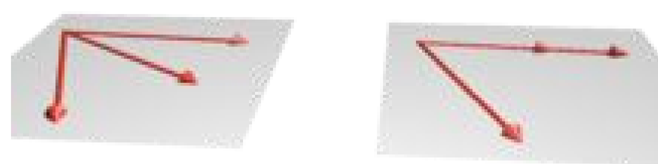
$$v = \sqrt{x^2 + y^2}$$

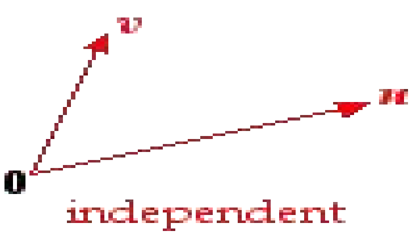
Src: http://www.dummies.com/how-to/content/how-to-find-a-vectors-magnitude-and-direction.html

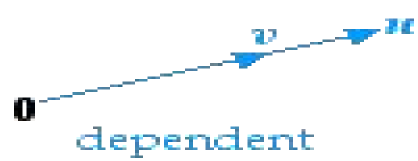
Linear independence of vectors

- a set of vectors is said to be **linearly dependent** if one of the vectors in the set can be defined as a **linear combination** of the others
- e.v. u(1, 3) and v = (2, 6)...notice: v = 2 u (dependent)
- Recall cos(angle = 0) = 1





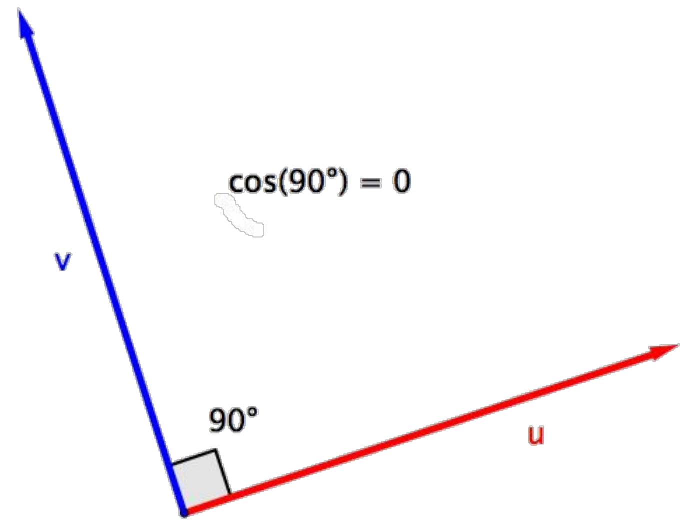




Src: https://en.wikipedia.org/wiki/Linear_independence http://i.stack.imgur.com/NKvYQ.gif

Perpendicular Vectors

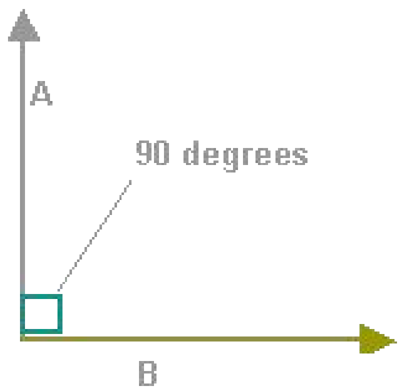
Two vectors are **perpendicular** if and only if their angle is a right angle

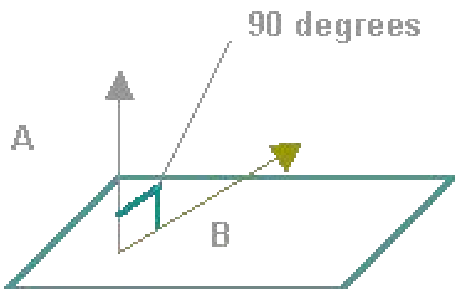


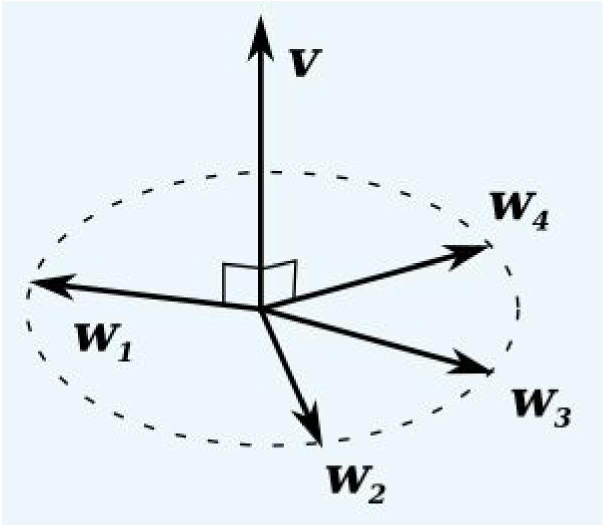
Src: https://dj1hlxw0wr920.cloudfront.net/userfiles/wyzfiles/3b4796bf-2911-46d8-8dad-9d1d96e6b389.gif

Orthogonal vectors

Set of vectors is **orthogonal** if and only if they are **pairwise perpendicular**



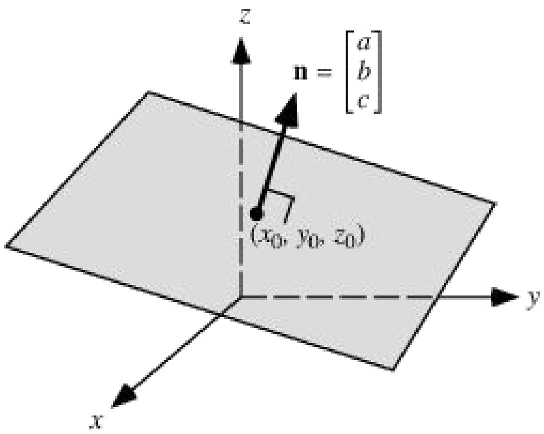


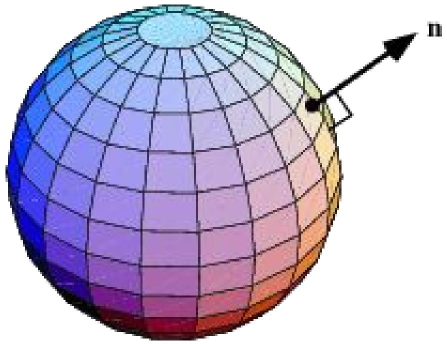


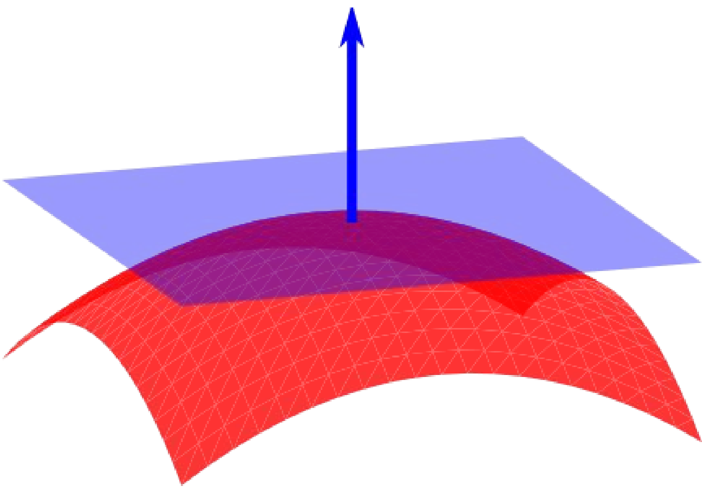
Src: https://dj1hlxw0wr920.cloudfront.net/userfiles/wyzfiles/3b4796bf-2911-46d8-8dad-9d1d96e6b389.gif http://lolengine.net/raw-attachment/blog/2013/09/21/picking-orthogonal-vector-combing-coconuts/math-vector-ortho.png

Normal Vector

The normal vector to a **surface** is a vector which is **perpendicular** to the surface at a given **point**



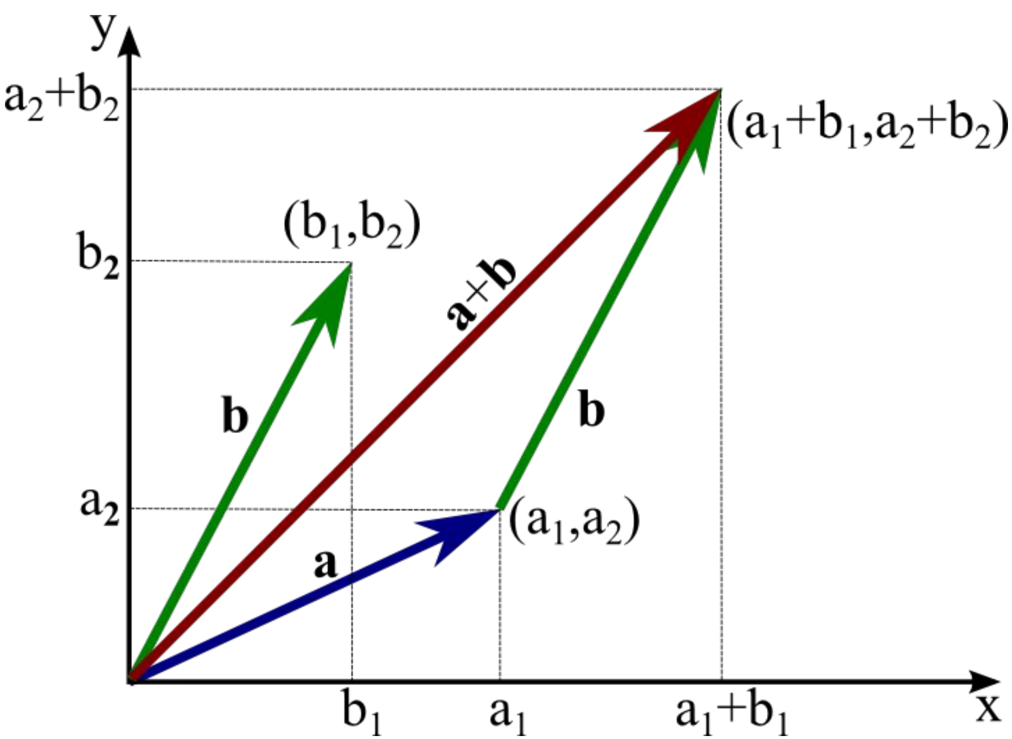


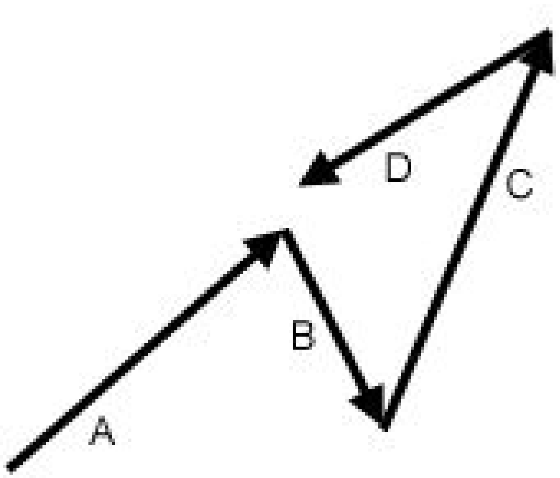


Src: http://mathworld.wolfram.com/NormalVector.html https://en.wikipedia.org/wiki/Normal_(geometry)#/media/File:Surface_normal_illustration.svg

Vector Addition

if a = (a1, a2) and b (b1, b2), then a + b = (a1+b1, a2+b2)

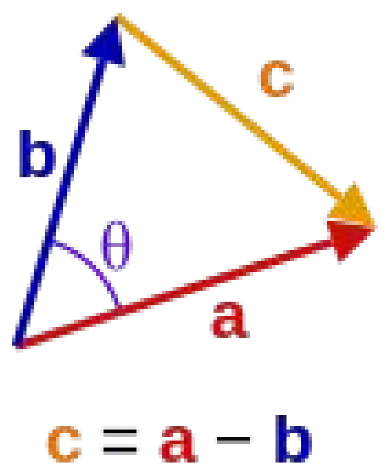


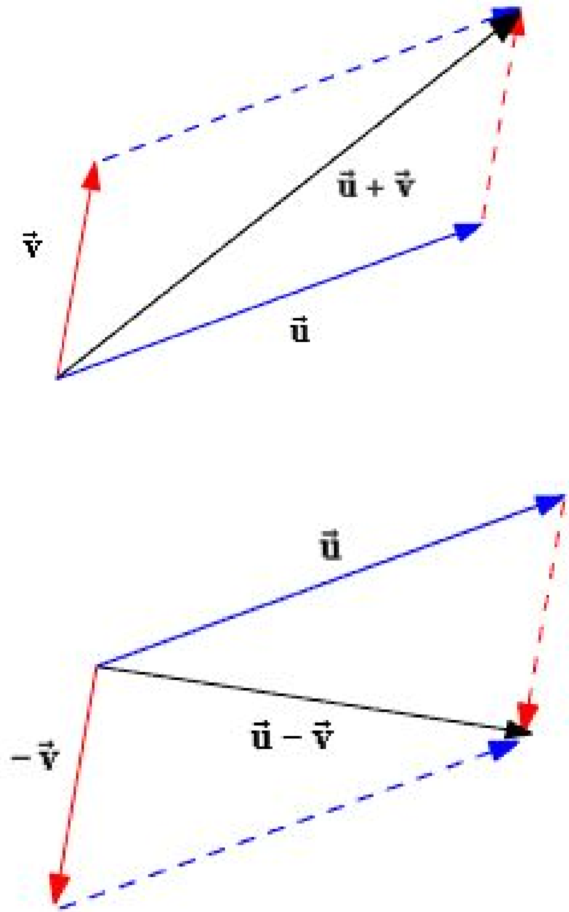


The sum of vectors A+B+C+D

Src: http://mathinsight.org/media/image/image/vector_2d_add.png http://community.topcoder.com/i/education/geometry01.png

Vector Subtraction





Src: http://hotmath.com/hotmath_help/topics/adding-and-subtracting-vectors.html

Vector Dot/Scalar/Inner Product

Algebraically, sum of the products of the corresponding entries

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{n} A_i B_i = A_1 B_1 + A_2 B_2 + \cdots + A_n B_n$$

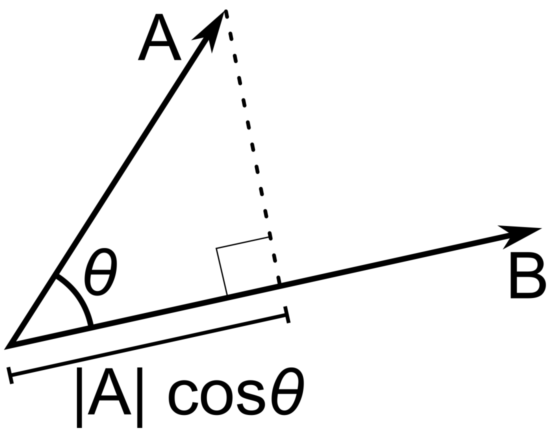
$$\begin{aligned} [1, 3, -5] \cdot [4, -2, -1] &= (1)(4) + (3)(-2) + (-5)(-1) \\ &= 4 - 6 + 5 \\ &= 3. \end{aligned}$$

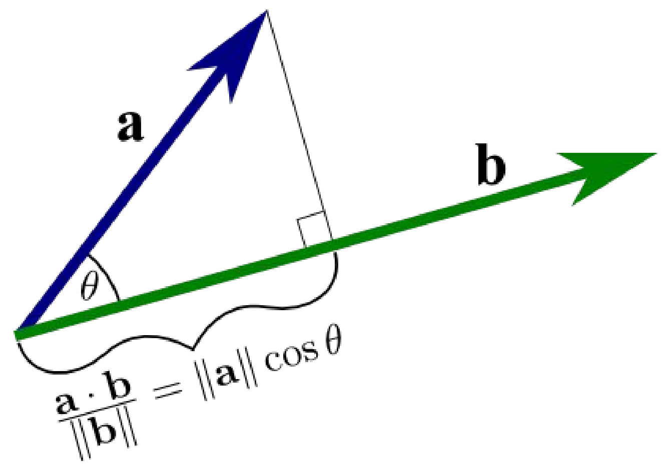
Geometrically, the product of the Euclidean magnitudes of the two vectors and the cosine of the angle between them

$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \, \|\mathbf{B}\| \cos(\theta),$$

Vector Dot/Scalar/Inner Product

Scalar projection of a vector A in the direction of a Euclidean vector B





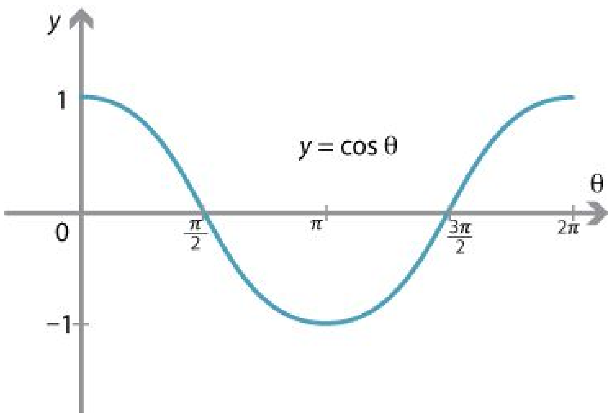
if A and B are **orthogonal**, then the angle between them is 90° $\mathbf{A}\cdot\mathbf{B}=0.$

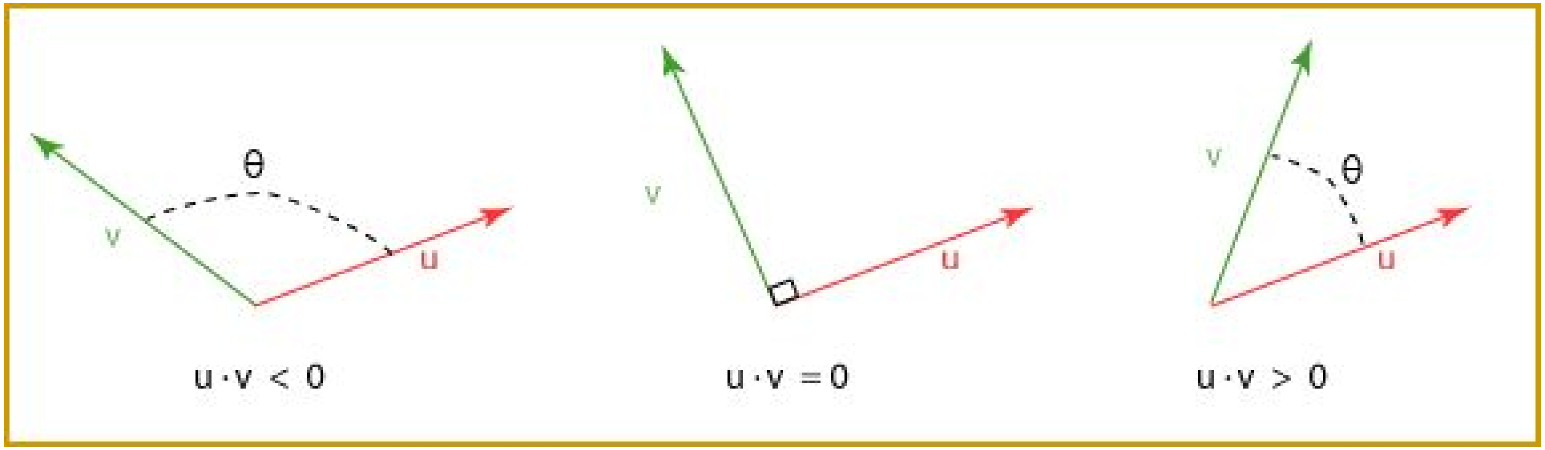
if they are **codirectional**, then the angle between them is 0° $\mathbf{A}\cdot\mathbf{B}=\|\mathbf{A}\|\ \|\mathbf{B}\|$

Src: https://upload.wikimedia.org/wikipedia/commons/thumb/3/3e/Dot_Product.svg/2000px-Dot_Product.svg.png http://mathinsight.org/media/image/image/dot_product_projection.png

Vector Dot/Scalar/Inner Product

$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\|\ \|\mathbf{B}\| \cos(\theta),$$





Src: https://chortle.ccsu.edu/VectorLessons/vch07/acuteORobtuse.gif http://amsi.org.au/ESA_Senior_Years/SeniorTopic2/2d/2d_2content_6.html

Vector Dot/Scalar/Inner Product

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a},$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

$$\mathbf{a} \cdot (r\mathbf{b} + \mathbf{c}) = r(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c}).$$

$$(c_1\mathbf{a}) \cdot (c_2\mathbf{b}) = c_1 c_2 (\mathbf{a} \cdot \mathbf{b}).$$

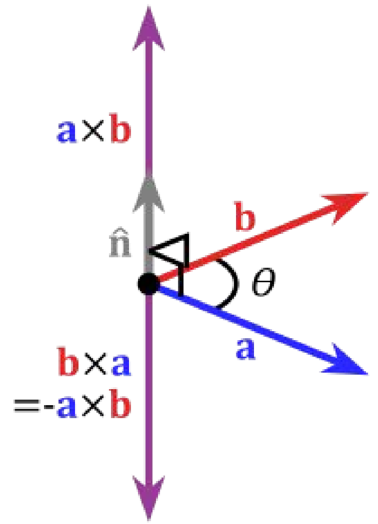
If a · b = a · c and a ≠ 0, then we can write: a · (b − c) = 0

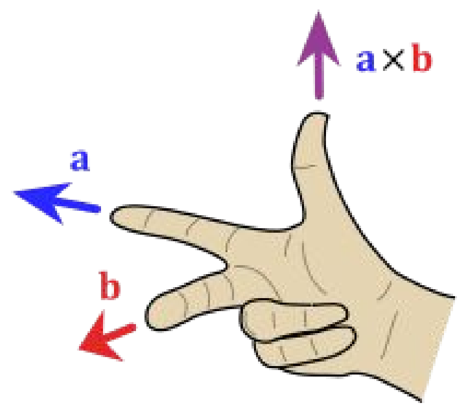
means that a is perpendicular to (b − c), and therefore b ≠ c.

Cross/Vector Product

The cross product, a × b, is a vector that is **perpendicular to both a and b** and therefore normal to the plane containing them.

Finding the direction of the cross product by the right-hand rule
Notice: A x B = vector…..But A.B = Scaler





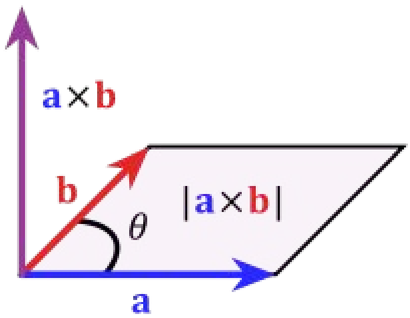
Src: http://mathworld.wolfram.com/NormalVector.html https://en.wikipedia.org/wiki/Normal_(geometry)#/media/File:Surface_normal_illustration.svg

Cross/Vector Product

The magnitude of the cross product can be interpreted as the positive area of the **parallelogram** having a and b as sides

The triangle formed by a, b has **half** of the **area** of the **parallelogram**, so we can calculate its area from the cross product

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta.$$



Given two unit vectors, their cross product has a **magnitude** of

- **one** if the two are **perpendicular** and a magnitude of **zero** if the two are **parallel**.
- The converse is true for the dot product of two unit vectors.

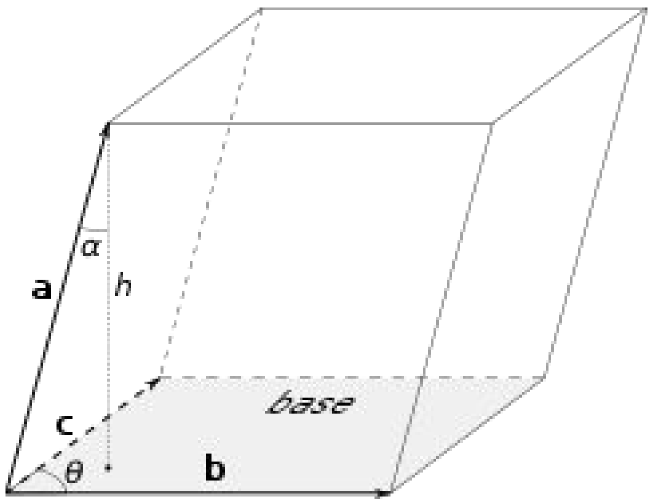
Src: https://en.wikipedia.org/wiki/Cross_product

Cross/Vector Product

Compute the volume V of a parallelepiped having a, b and c as edges by using a combination of a cross product and a dot product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

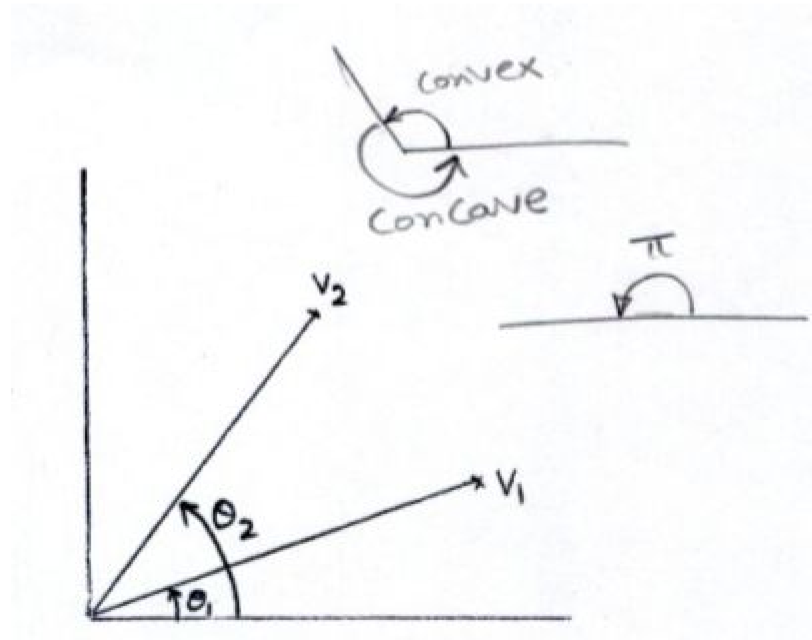
$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$



Cross/Vector Product

Cross product:

$$\text{cross product}(v_1, v_2) = x_1 y_2 - x_2 y_1$$
$$= r_1 \cos\theta_1 \, r_2 \sin\theta_2 - r_2 \cos\theta_2 \, r_1 \sin\theta_1$$
$$= r_1 r_2 (\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1)$$
$$= r_1 r_2 \sin(\theta_2 - \theta_1)$$



Cross/Vector Product

Test: Type of minor angle between two vectors (acute, Right, Obtuse)
* use dotproduct sign check

if cross product =
$$\begin{cases} +ve & \sin(\theta_2-\theta_1) > 0 \text{, angle between two vectors } V_1 V_2 \text{ is convex} \\ 0 & \sin(\theta_2-\theta_1) = 0 \text{, } \sim \sim \sim \sim \text{ is } 0 \text{ or } \pi \text{ (two v} \\ -ve & \sin(\theta_2-\theta_1) < 0 \text{, } \sim \sim \sim \sim \; V_1 V_2 \text{ is Concave} \end{cases}$$

cw

Cross/Vector Product

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}),$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$

$$(r\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (r\mathbf{b}) = r(\mathbf{a} \times \mathbf{b}).$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$$

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2.$$

Standard basis

Set of **unit** vectors pointing in the direction of the **axes** of a **Cartesian** coordinate system

$$\mathbf{e}_x = (1,0), \quad \mathbf{e}_y = (0,1),$$

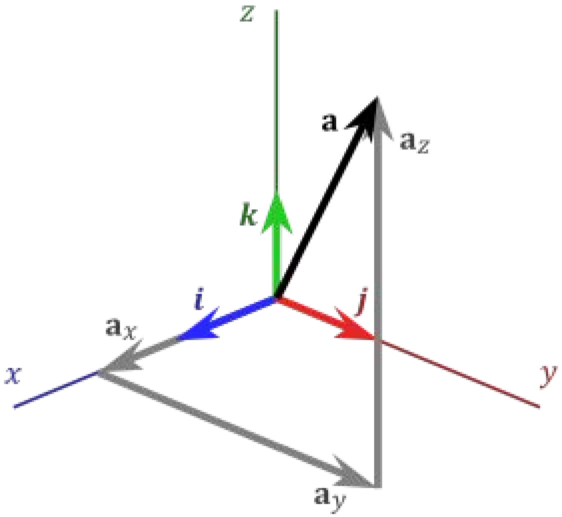
$$\mathbf{e}_x = (1,0,0), \quad \mathbf{e}_y = (0,1,0), \quad \mathbf{e}_z = (0,0,1).$$

$$\mathbf{i} = \mathbf{j} \times \mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{j} = \mathbf{k} \times \mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{k} = \mathbf{i} \times \mathbf{j} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$



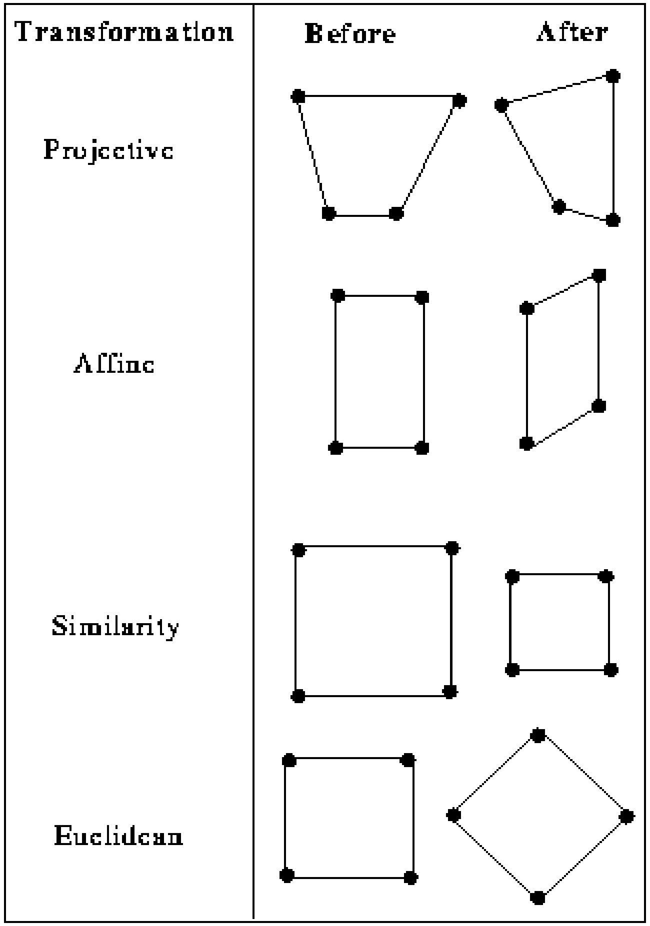
Cross Product and Standard basis

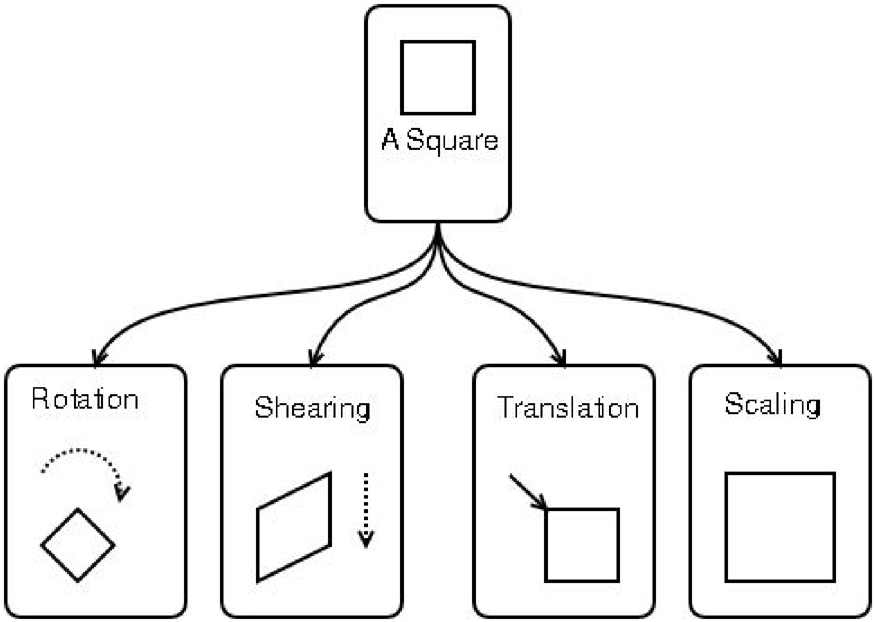
$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1(\mathbf{i} \times \mathbf{i}) + u_1v_2(\mathbf{i} \times \mathbf{j}) + u_1v_3(\mathbf{i} \times \mathbf{k}) + \\
&\quad\; u_2v_1(\mathbf{j} \times \mathbf{i}) + u_2v_2(\mathbf{j} \times \mathbf{j}) + u_2v_3(\mathbf{j} \times \mathbf{k}) + \\
&\quad\; u_3v_1(\mathbf{k} \times \mathbf{i}) + u_3v_2(\mathbf{k} \times \mathbf{j}) + u_3v_3(\mathbf{k} \times \mathbf{k})
\end{aligned}
$$

$$
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}
$$

$$
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}
$$

Geometric Operations





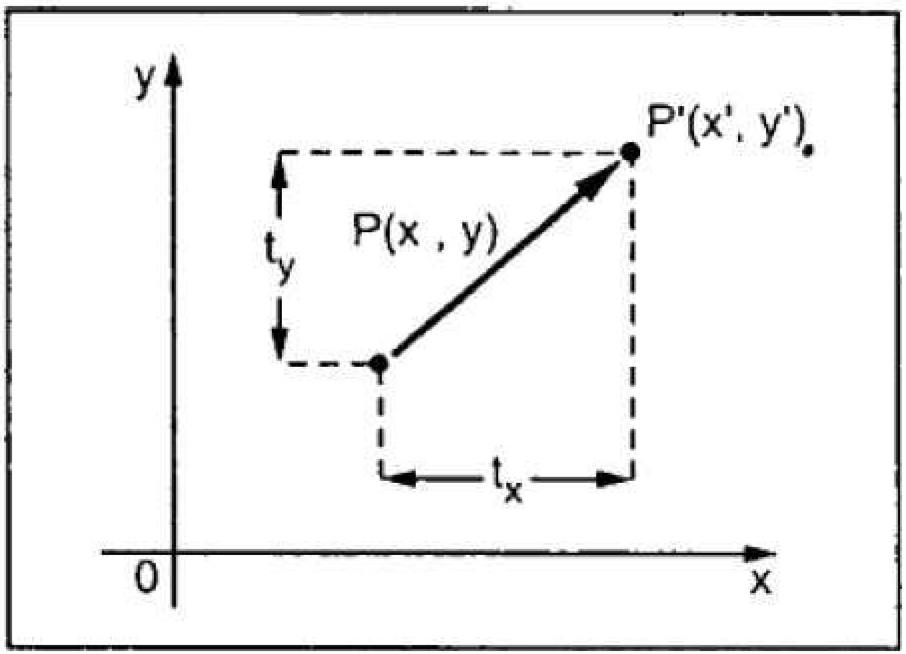
Src: http://homepages.inf.ed.ac.uk/rbf/HIPR2/figs/affhei.gif http://3.bp.blogspot.com/-wvk7fpfFUu0/U5NrFOYskEI/AAAAAAAAABg/PTJGVKOnEd4/s1600/art-affines.png

Euclidean Transformations

- A translation, a rotation, or a reflection
- Preserve length and angle measure.
- The shape of a geometric object will not change.
  - E.g. lines transform to lines, circles transform to circles
- See notes for affine
- Following notes from here

Euclidean: Translation

Add vector(h, k) to point (x, y)



$$(x', y') = (x + a, y + b).$$

Src: http://www.tutorialspoint.com/computer_graphics/2d_transformation.htm

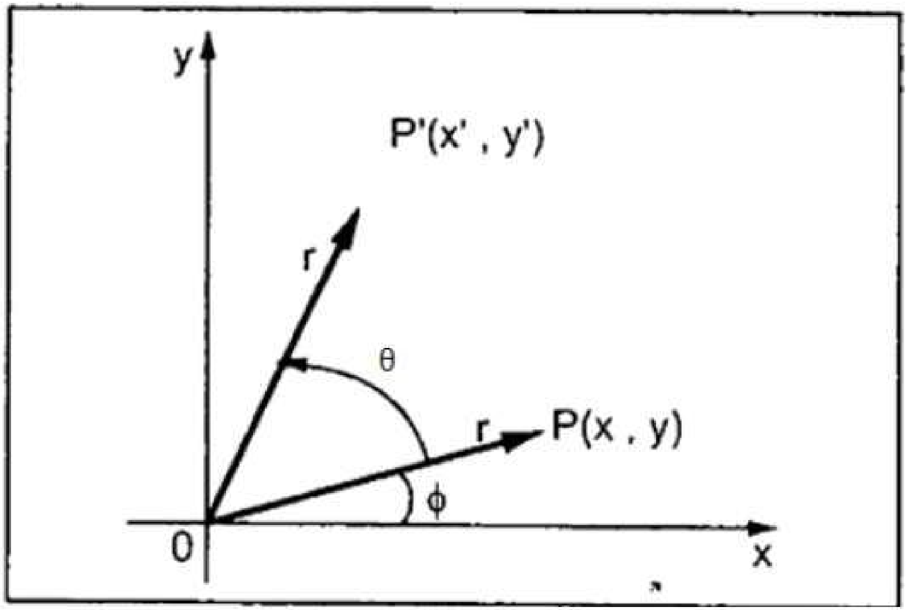
Euclidean: Translation

- We can represent using equation or matrix
- Matrix 1 for translation, matrix 2 for undo
- Multiply M1 * M2 = = Identity
- Line Ax + By + C = 0 $\Rightarrow$
  - Ax' + By' + (-Ah - Bk + C) = 0.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Euclidean: Rotation

- If a point (x, y) is rotated an **angle a** about the coordinate origin to become a new point (x', y')
- **Please** read how to get such equations



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta.$$

$$(x', y') = ((x\cos\theta - y\sin\theta), (x\sin\theta + y\cos\theta)).$$

Src: http://www.tutorialspoint.com/computer_graphics/2d_transformation.htm

Euclidean: Rotation

- Line Ax + By + C = 0 $\Rightarrow$
  - (Acosa - Bsina)x' + (Asina + Bcosa)y' + C = 0

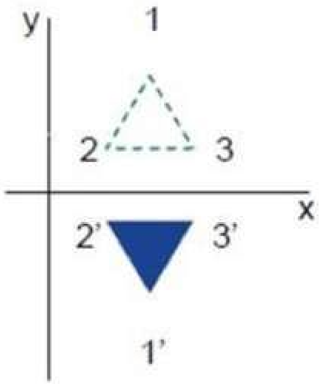
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Euclidean: Reflection - Special

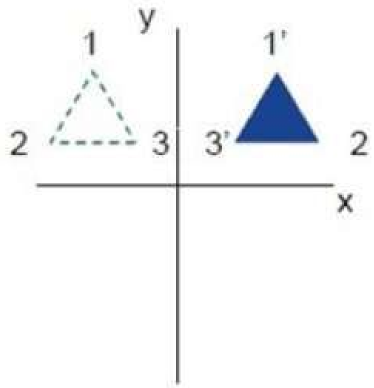
- Reflection across x – axis: $(x, y) \rightarrow (x, -y)$
- Reflection across y – axis: $(x, y) \rightarrow (-x, y)$
- Reflection over origin: $(x, y) \rightarrow (-x, -y)$
- Reflection over line $y = x$: $(x, y) \rightarrow (y, x)$

Src: http://slideplayer.com/slide/1398961/

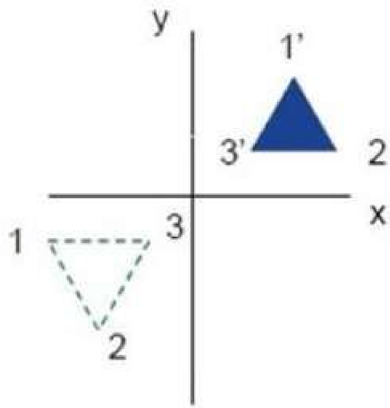
Euclidean: Reflection - Special



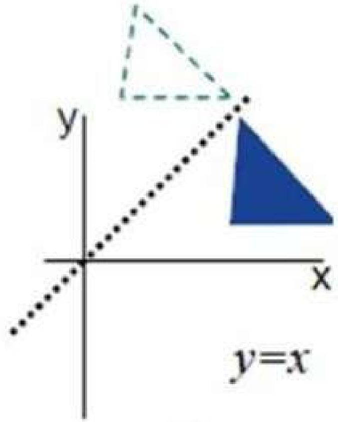
(a)



(b)



(c)



(d)

Src: http://www.tutorialspoint.com/computer_graphics/2d_transformation.htm

Euclidean: Reflection

Generally, reflection across a line through the origin making an angle theta with the x-axis, is equivalent to replacing every point with coordinates (x, y) by the point with coordinates (x′,y′), where

$$x' = x\cos 2\theta + y\sin 2\theta$$
$$y' = x\sin 2\theta - y\cos 2\theta.$$

$$(x', y') = ((x\cos 2\theta + y\sin 2\theta), (x\sin 2\theta - y\cos 2\theta)).$$

Euclidean: Composition

- We can do several operations together.
  - Just multiply their matrices
- Rotation around origin, then translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a & h \\ \sin a & \cos a & k \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos a & \sin a & -h\cos a - k\sin a \\ -\sin a & \cos a & h\sin a - k\cos a \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

تم بحمد الله

علمكم الله ما ينفعكم

ونفعكم بما تعلمتم

وزادكم علماً