THINK FAST

Competitive Programming

From Problem 2 Solution in O(1)

Graph Theory

System of Difference Constraints

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System of Difference Constraint

- We won't learn new algorithm
- Actually, we will show how Bellman Ford algorithm solves this system

System of Difference Constraint

- **Difference constraints**: a system of linear inequalities of the form $xj - xi \leq wij$
- Example
  - $x1 - x2 \leq 3$
  - $x2 - x3 \leq -2$
  - $x1 - x3 \leq 2$
  - Solution: **x1** = 3 **x2** = 0 **x3** = 2
- Another helpful format: $xi + wij \geq xj$
  - Inspire a graph construction / solves the problem
- If all wij >= 0, then trivial solution X = 0

Usage

- Parallel task scheduling with precedence constraints
  - If a job i needs to be finished before job j starts
  - **xj ≥ xi +duration(i)**
  - The time from start to finish should be at most t
  - **xj +duration(j) – xi ≤ t for all i,j**
- Cartesian Points Relationships
  - Find some points (x, y) with constraints e.g. A is above B with 5 units...etc

Preparing Equations

- $x1 - x2 < 5 \quad \Rightarrow \quad x2 - x1 \leq 4$
- $3 \leq x1 - x2 < 15$
  - Break to 2 statements
  - $x1 - x2 < 15 \quad \Rightarrow \quad x1 - x2 \leq 14$
  - $3 \leq x1 - x2 \quad \Rightarrow \quad x2 - x1 \leq -3$

Constraint Graph

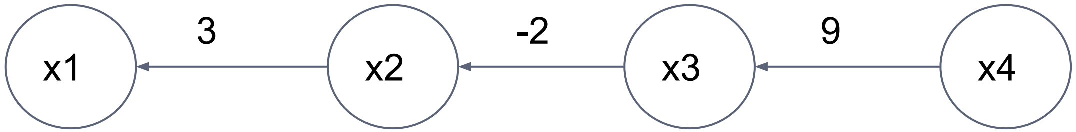
- $xi + wij \geq xj$
- Thinking in xi and xj as graph nodes
- Wij is edge cost from xi to xj
- Build graph
  - Every variable is node
  - Edge cost from xi to xj is wij

Constraint Graph

$x1 - x2 \leq 3 \quad \Rightarrow x2 + 3 \geq x1$

$x2 - x3 \leq -2$

$x3 - x4 \leq 9$



Set $x4 = 0 \quad \Rightarrow x3 - \mathbf{x4} \leq 9 \quad \Rightarrow x3 - 0 \leq 9 \quad \Rightarrow x3 \leq 9$

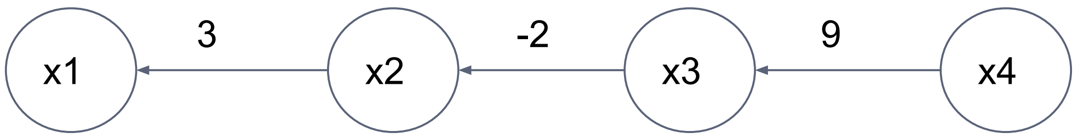
Let $x3 = 0 \quad \Rightarrow x2 - \mathbf{x3} \leq -2 \Rightarrow x2 - 0 \leq -2 \quad \Rightarrow x2 \leq -2$

Let $x2 = -2 \quad \Rightarrow x1 - \mathbf{x2} \leq 3 \quad \Rightarrow x1 + 2 \leq 3 \quad \Rightarrow x1 \leq 1$

On set of valid solutions: $x4 = 0 \quad x3 = 0 \quad x2 = -2 \quad x1 \leq 1$

Constraint Graph

$x1 - x2 \leq 3 \quad \Rightarrow x2 + 3 \geq x1$
$x2 - x3 \leq -2$
$x3 - x4 \leq 9$



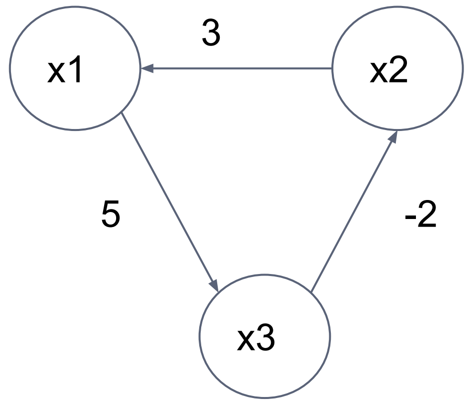
Set $x4 = 0 \quad \Rightarrow x3 - \mathbf{x4} \leq 9 \quad \Rightarrow x3 - 0 \leq 9 \quad \Rightarrow x3 \leq 9$
Let $x3 = 9 \quad \Rightarrow x2 - \mathbf{x3} \leq -2 \Rightarrow x2 - 9 \leq -2 \quad \Rightarrow x2 \leq 7$
Let $x2 = 7 \quad \Rightarrow x1 - \mathbf{x2} \leq 3 \quad \Rightarrow x1 - 7 \leq 3 \quad \Rightarrow x1 \leq 10$

On set of valid solutions: $x4 = 0 \quad x3 = 9 \quad x2 = 7 \quad x1 \leq 10$

Notice: solutions were **path cost** from x4 to xi

Constraint Graph

$x1 - x2 \leq 3$
$x2 - x3 \leq -2$
$x3 - x1 \leq 5$



Set $x3 = 0$

$\Rightarrow x2 - \mathbf{x3} \leq -2 \quad \Rightarrow x2 - 0 \leq -2 \Rightarrow \quad x2 \leq -2 \quad \Rightarrow$ Let $x2 = \mathbf{-}2$

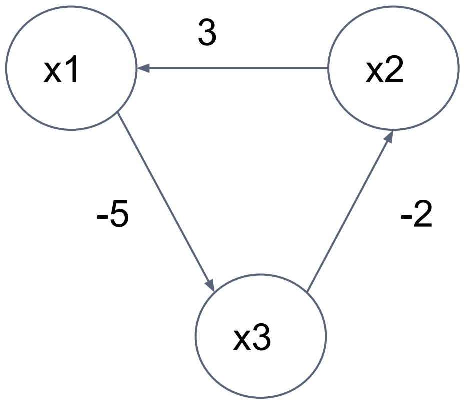
$\Rightarrow \mathbf{x3} - x1 \leq 5 \quad \Rightarrow 0 - x1 \leq 5 \quad \Rightarrow \quad x1 \geq \mathbf{-}5$

$\Rightarrow x1 - \mathbf{x2} \leq 3 \quad \Rightarrow x1 + 2 \leq 3 \quad \Rightarrow \quad x1 \leq 1$

On set of valid solutions: $x3 = 0 \qquad x2 = -2 \qquad 5 \leq x1 \leq 1$

Constraint Graph

$x1 - x2 \leq 3$
$x2 - x3 \leq -2$
$x3 - x1 \leq$ **-5**



Set $x3 = 0$

$\Rightarrow x2 - \mathbf{x3} \leq -2 \;\Rightarrow x2 - 0 \leq -2 \Rightarrow\; x2 \leq -2 \;\Rightarrow$ Let $x2 = \mathbf{-}2$

$\Rightarrow \mathbf{x3} - x1 \leq \mathbf{-}5 \;\Rightarrow 0 - x1 \leq \mathbf{-}5 \;\Rightarrow\; x1 \geq 5$

$\Rightarrow x1 - \mathbf{x2} \leq 3 \;\Rightarrow x1 + 2 \leq 3 \;\Rightarrow\; x1 \leq 1$

No intersection between $x1 \leq 1$ and $x1 \geq 5$! No solution!

Notice **negative cycle:** 3-2-5 = -4

Constraint Graph

- If the constraint graph contains a **negative weight cycle**, then the system of differences is **unsatisfiable**.
- Mathematically **add all equations**:

$$x1 - x2 \le 3$$
$$x2 - x3 \le -2 \qquad \Rightarrow \qquad x1 \underline{- x2 + x2} \; \underline{- x3 + x3} - x1 \le 3\text{-}2\text{-}5$$
$$x3 - x1 \le \mathbf{-5} \qquad\qquad 0 \le -4 \quad \textbf{impossible}$$

Finding solution

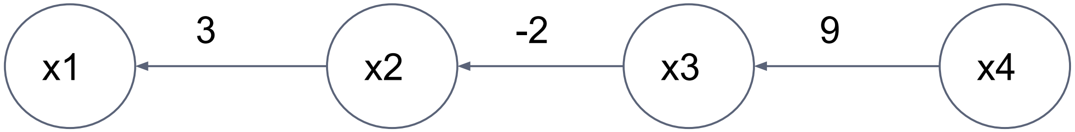
- Assume graph has 10 nodes
- Node X2 is reachable from 3 vertices
  - X5, X7, X9
  - X5 can reach X2 in several ways...shortest path is critical
  - Shortest path from X5 to X2 = 3 $\Rightarrow$ $X2 \leq 3$ is ok
  - Shortest path from X7 to X2 = 6 $\Rightarrow$ $X2 \leq 6$ is ok
  - Shortest path from X9 to X2 = 1 $\Rightarrow$ $X2 \leq 1$ is ok
  - To satisfy all of them: $X2 \leq \min(1, 3, 6) \leq 1$
- What if for every node, computed the min shortest path value for it from all other nodes?

Finding solution

$x1 - x2 \leq 3 \quad \Rightarrow x2 + 3 \geq x1$

$x2 - x3 \leq -2$

$x3 - x4 \leq 9$



x4 reachable with costs {} $\Rightarrow x4 = 0$

x3 reachable with costs {9} $\Rightarrow x3 = 9$

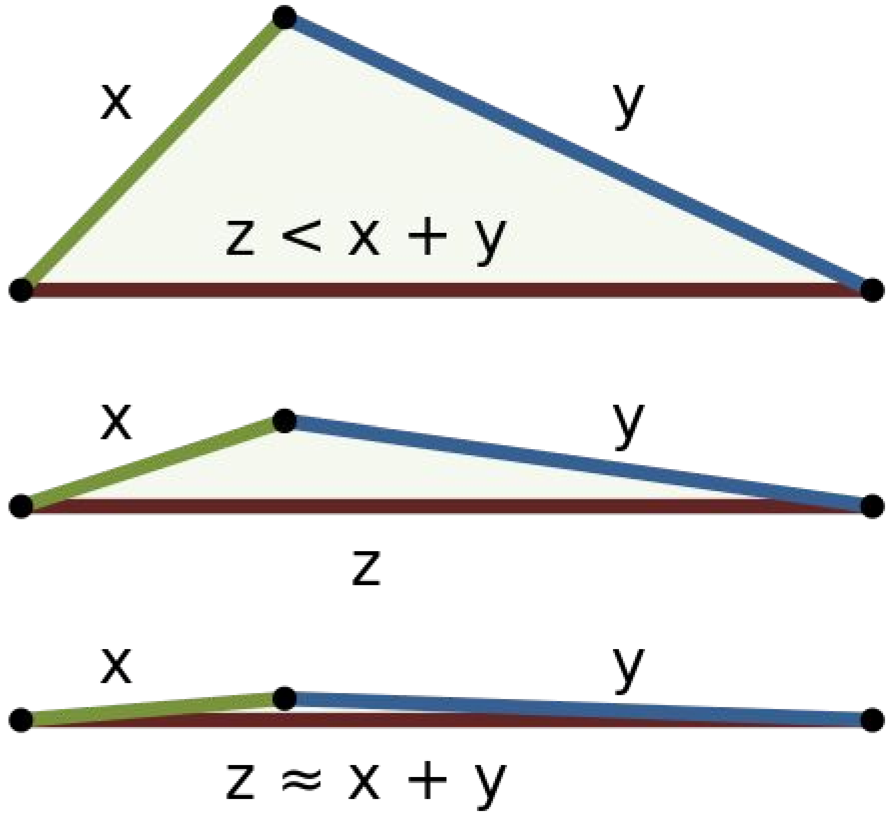
x2 reachable with costs {-2, -2+9} $\Rightarrow x2 = -2$

x1 reachable with costs {3, 3-2, 3-2+9} $\Rightarrow x1 = 1$

Finding solution

- How to get efficiently the minimum of all shortest paths to a node?
- Add a new vertex S, connect it to all nodes with cost 0
- Find shortest path from it to other nodes
- Then sp(S, xi) is the min of all shortest paths
- But why overall solutions to all xi is **correct**?
  - Triangular inequality property

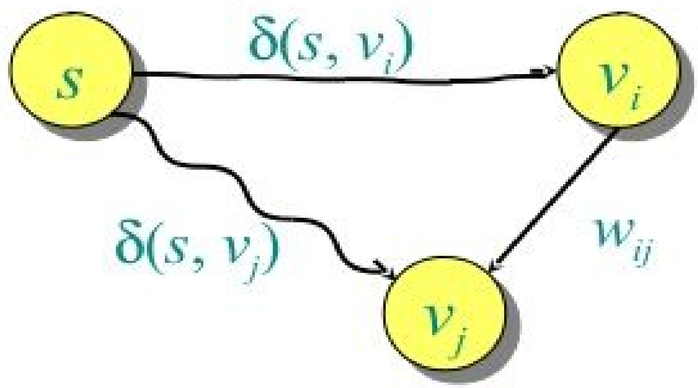
Triangular inequality



Src: https://en.wikipedia.org/wiki/Triangle_inequality

Triangular inequality

Consider any constraint $x_j - x_i \le w_{ij}$, and consider the shortest paths from s to v_j and v_i:



The triangle inequality gives us $\delta(s,v_j) \le \delta(s, v_i) + w_{ij}$. Since $x_i = \delta(s, v_i)$ and $x_j = \delta(s, v_j)$, the constraint $x_j - x_i \le w_{ij}$ is satisfied. $\square$

Src: http://image.slidesharecdn.com/shortest-paths-bf-140416091424-phpapp02/95/bellman-fords-algorithm-42-638.jpg?cb=1397640587

Solution overall

- Create constraints graph of N+1 nodes
  - For every constraint $\Rightarrow$ add edge
  - From vertex 0 add edge to N nodes with cost = 0
- Use Bellman Ford algorithm
- Check if any -ve cycle? If yes no solution
- No? **Shortest path values** from 0 to xi are the solution
- Note: Bellman-Ford also minimizes
  - maxi {xi} – mini {xi}

تم بحمد الله

علمكم الله ما ينفعكم

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