Competitive Programming

From Problem 2 Solution in O(1)

Computational Geometry

Complex Number and 2D Point

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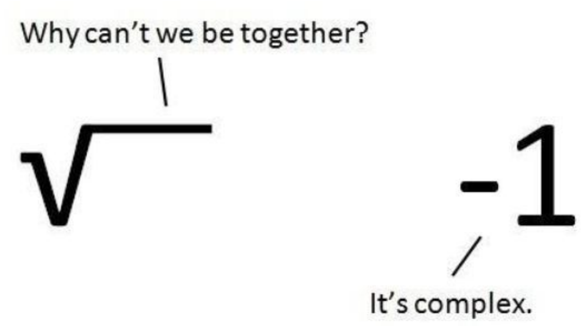
Complex Numbers



Imaginary Unit
Define the imaginary unit
$i = \sqrt{-1}$ so that $i^2 = -1$

Complex Number
Any number of the form
$a + bi$ where a and b
are real numbers.

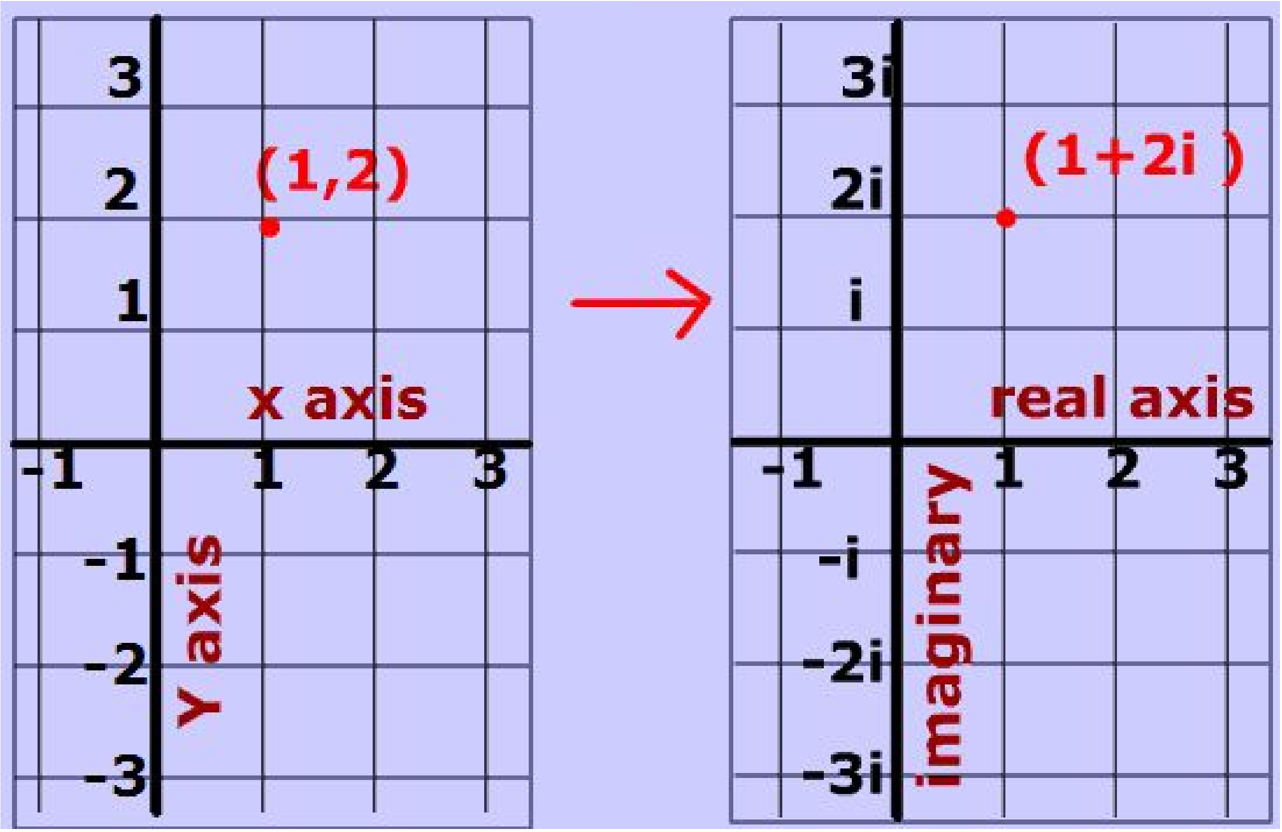
$$x^2 + 25 = 0$$
$$x^2 = -25$$
$$\sqrt{x^2} = \pm\sqrt{-25}$$
$$x = \pm 5i$$

$$(2-3i)(2+3i)$$
$$= 4 + 6i - 6i - 9i^2$$
$$= 4 - 9(-1)$$
$$= 4 + 9$$
$$= 13$$

complex number	conjugate
$3 + \frac{1}{2}i$	$3 - \frac{1}{2}i$
$12 - 5i$	$12 + 5i$
$1 - i$	$1 + i$
$45i$	$-45i$
101	101

Src: https://s-media-cache-ak0.pinimg.com/600x315/93/b0/6e/93b06e610801751518ac6f214e67b56c.jpg http://4.bp.blogspot.com/-AE0flAsDb2c/UJ0mz3rFK_I/AAAAAAAAF_U/ETr94p-7VMY/s1600/ex09-06001.PNG http://media.wiley.com/Lux/17/455317.image0.jpg

Complex Numbers: Cartesian



Src: http://www.mathwarehouse.com/algebra/complex-number/

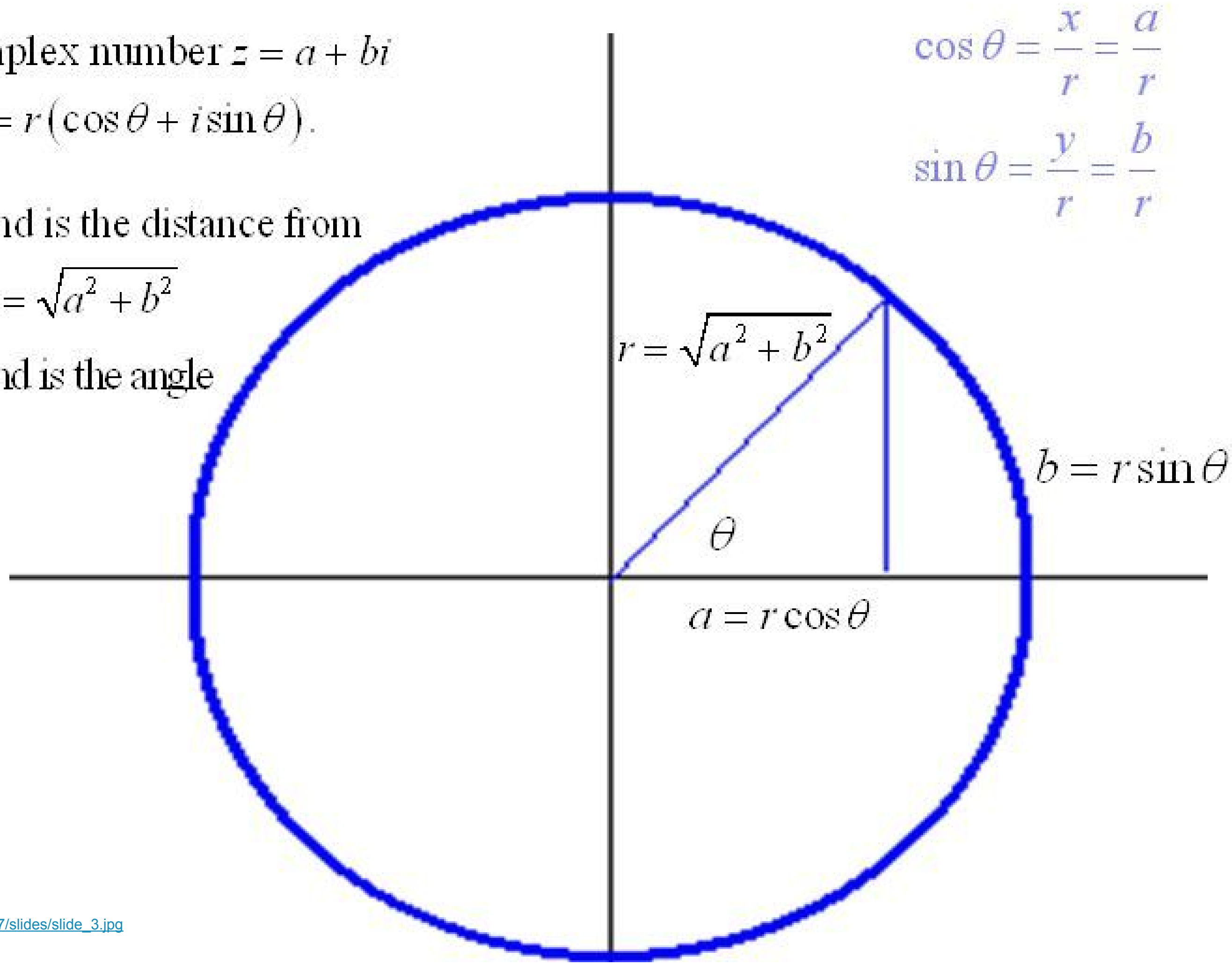
Complex Numbers: Polar

The trig form of the complex number $z = a + bi$ is $z = (r\cos\theta + ri\sin\theta) = r(\cos\theta + i\sin\theta)$.

r is called the modulus and is the distance from the origin to the point. $r = \sqrt{a^2 + b^2}$

θ is called the argument and is the angle formed with the x-axis.

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$



Src: http://images.slideplayer.com/7/1662837/slides/slide_3.jpg

Complex Numbers: Euler Formula

2.71828183

The polar form of a complex number can be rewritten as :

$$z = r(\cos\theta + i\sin\theta) = x + iy$$
$$= re^{i\theta}$$

This leads to the complex exponential function :

$$e^{z} = e^{x+iy} = e^{x}e^{iy}$$
$$= e^{x}\left(\cos y + i\sin y\right)$$

Further leads to :

$$\left.\begin{aligned}\cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-i\theta}\right)\\ \sin\theta &= \frac{1}{2j}\left(e^{j\theta} - e^{-i\theta}\right)\end{aligned}\right\}$$

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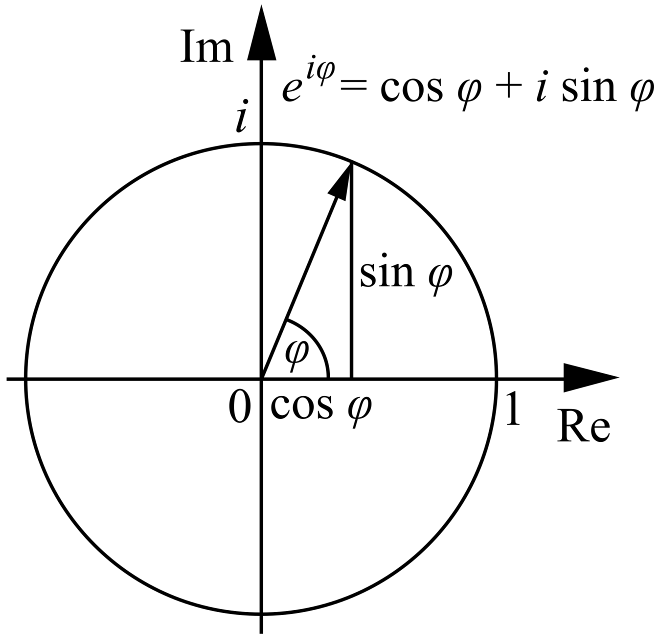


Src: http://image.slidesharecdn.com/complexnumbersadvance-130808012940-phpapp02/95/complex-numbers-advance-7-638.jpg?cb=1375926857

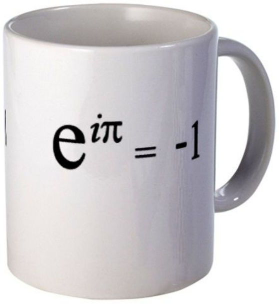
Complex Numbers: Euler Formula

$$\varphi = \frac{\pi}{2} \quad \Rightarrow \quad e^{i\cdot\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \quad \Rightarrow \quad e^{i\cdot\frac{\pi}{2}} = 0 + 1\cdot i = i,$$

$$\varphi = \pi \quad \Rightarrow \quad e^{i\cdot\pi} = \cos\pi + i\sin\pi \quad \Rightarrow \quad e^{i\cdot\pi} = -1 + 0\cdot i = -1,$$

$$\varphi = \frac{3\pi}{2} \quad \Rightarrow \quad e^{i\cdot\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} \quad \Rightarrow \quad e^{i\cdot\frac{3\pi}{2}} = 0 - 1\cdot i = -i,$$

$$\varphi = 2\pi \quad \Rightarrow \quad e^{i\cdot 2\pi} = \cos 2\pi + i\sin 2\pi \quad \Rightarrow \quad e^{i\cdot 2\pi} = 1 + 0\cdot i = 1.$$



Src: http://www.nabla.hr/UnitCircleF.gif

Complex Geometric interpretation

- Complex numbers form a **vector space** over the real numbers.
- Addition, subtraction, negation and scalar multiplication are exact same in both
- Conjugation = reflect the number in the real axis
- Complex multiplication
  - Not that direct
  - But decomposes to **dot and cross product**! Wow

Complex in C++

```
#include <complex>
#include <cmath>
const double PI = acos(-1.0);

complex<double> num1(2, 3); // 2 + 3i

std::cout << num1.real() << "+" << num1.imag() << "i\n";    // 2 + 3i

complex<double> num2(1, 1);

cout << "Modulus = " << abs(num2) << "\n";           // r = 1.41421
cout << "Argument = " << arg(num2) << "\n";          // theta radian = 0.785398
cout << "Angle = " << arg(num2) * 180 / PI << "\n"; // theta degree = 45
cout << "Norm = " << norm(num2) << "\n";             // 2 (1*1 + 1*1)

complex<double> num3 = std::polar(1.41421, 0.785398);
cout << "(x+iy) from polar are: " << num3 << "\n";  // (0.999998,0.999997)

complex<double> zero;
complex<double> x_part = 15;
cout << x_part << "\n";              // (15,0)

complex<double> a(1, 2);
complex<double> b(3, 4);

cout<<a+b<<"\n";    // (4,6)
cout<<a-b<<"\n";    // (-2,-2)
cout<<a*b<<"\n";    // (-5,10)
cout<<b*2.0<<"\n";  // (6,8)
cout<<b/2.0<<"\n";  // (1.5,2)
```

Complex in C++

```
cout << fixed << std::setprecision(1);

complex<double> i = -1;
cout<<sqrt(i)<<"\n";          // (0,1)

complex<double> c(2, 3);
cout<<conj(c)<<"\n";          // (2,-3)
cout<<pow(c, 2)<<"\n";        // (-5,12)

complex<double> i1 (0, -1);
cout<<exp(i1 * PI)<<"\n";   // (-1, 0)
```

2D point and Complex Numbers

- Complex numbers extremely fit with vectors
- We can use it as representation to (x, y)
- We can use it for operations (+, -, /, * scalar)
- We can do work around for dot/cross products
- We can have some polar representation and operations over it

Complex Numbers: Point Basics

```cpp
#include <complex>
const double PI  = acos(-1.0);
const double EPS = (1e-10);
typedef complex<double> point;

#define X real()
#define Y imag()

// We can rewrite any of following as functions
#define angle(a)                (atan2((a).imag(), (a).real()))
#define vec(a,b)                ((b)-(a))

#define length(a)               (hypot((a).imag(), (a).real()))
#define normalize(a)            (a)/length(a)
```

- Avoid sqrt, use hypot, it **avoids overflow**
- However, hypot is **slow** function
- Don’t compute hypot with actual distance if **squared** distance is enough. Use dot product

Complex Numbers: Dot/Cross

let $v_1 = x_1 + y_1 i \Rightarrow \text{Conj}(v_1) = x_1 - y_1 i$

let $v_2 = x_2 + y_2 i$

Then $v_1 * v_2 = \underbrace{(x_1 x_2 + y_1 y_2)}_{\text{dot product}} + \underbrace{(x_1 y_2 - x_2 y_1)}_{\text{cross product}} i$

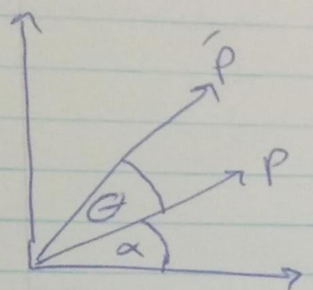
```
// dp = a*b cos(T), if zero -> prep
#define dp(a,b)                 ( (conj(a)*(b)).real() )
// cp = a*b sin(T), if zero -> parllel
#define cp(a,b)                 ( (conj(a)*(b)).imag() )
#define same(p1,p2)             (dp(vec(p1,p2),vec(p1,p2)) < EPS)
#define lengthSqr(p)            dp(p,p)
```

- Avoid using norm(x). Use dp(x, x) instead

Complex Numbers: Rotation

Let's rotate Point P with angle θ

let $P = r\,e^{\alpha i}$ $\Rightarrow$ rotation just increase angle



P' has same r

but angle $= \alpha + \theta$

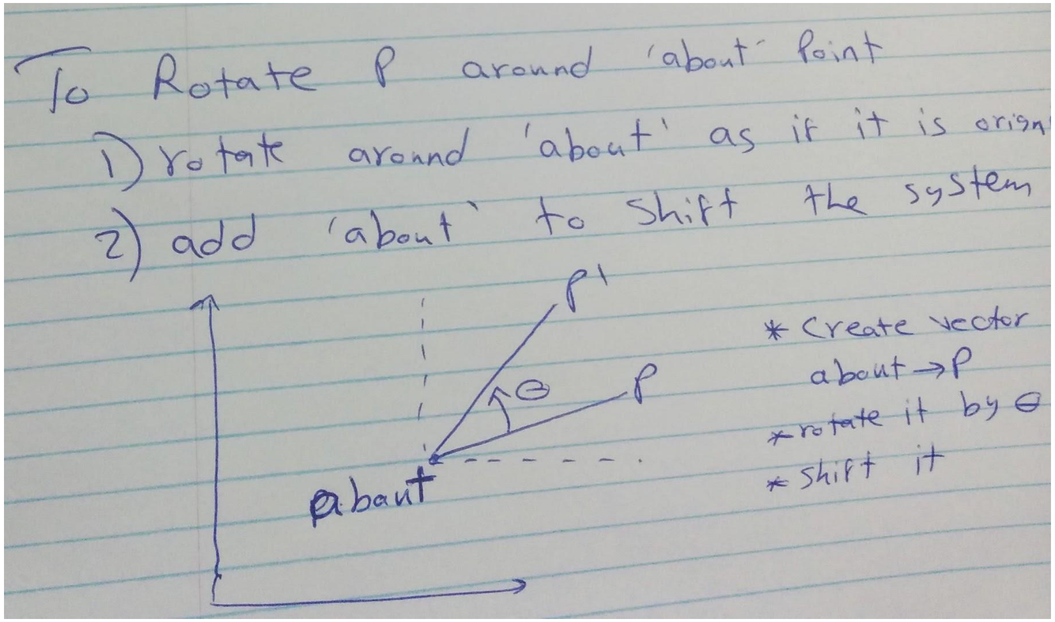
then $P * e^{\theta i} = r * e^{(\alpha+\theta)i}$

↳ = C++: exp(Point(0, θ))

This is rotation $\boxed{\text{around origin}}$

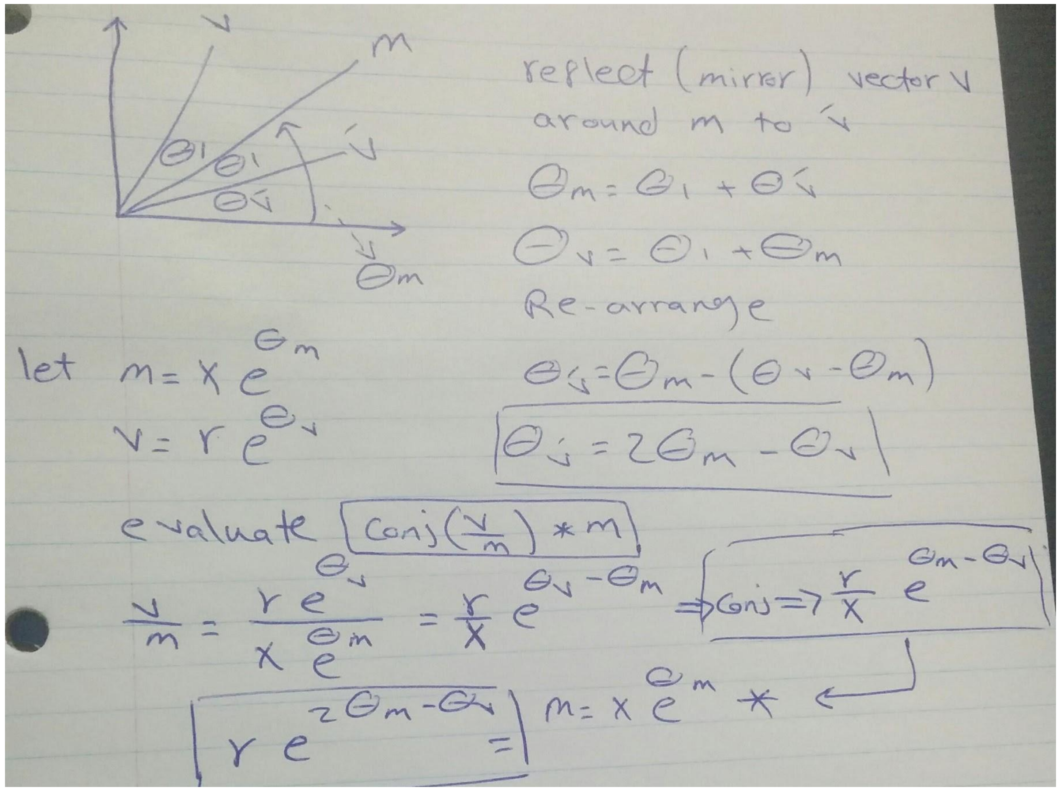
```
#define rotate0(p,ang)          ((p)*exp(point(0,ang)))
```

Complex Numbers: Rotation




```
#define rotateA(p,ang,about)  (rotate0(vec(about,p),ang)+about)
```

Complex Numbers: Reflection




```
#define reflect0(v,m)  (conj((v)/(m))*(m))
```

Complex Numbers: Reflection

```
// Refelect point p1 around p0p1
point reflect(point p, point p0, point p1) {
    point z = p-p0, w = p1-p0;
    return conj(z/w)*w + p0;
}
```

Java Guys

- No builtin complex class!
- But we can implement it
  - An implementation [code]
  - Same code one [ideone]
- Or you can just implement the needed geometry operations whatever way
- Java guys has strong geometry library for some operations (e.g. intersections of areas)
  - But nowadays, this is not advantage.

تم بحمد الله

علمكم الله ما ينفعكم

ونفعكم بما تعلمتم

وزادكم علماً