Homework Assignment: Robust MPC and CBF-Based Safety Filtering for the Dubins Car Model

MAE 248: Safety for Autonomous Systems
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1 Introduction and Problem Statement

In this assignment, you will implement several control strategies for the Dubins car model. The objectives are to:

(a) Implement a **Standard Model Predictive Control (MPC)** method for trajectory planning.

(b) Develop a **Robust MPC** formulation that accounts for bounded disturbances. You will consider different robust MPC methods (e.g., min-max formulation, tube-based MPC).

(c) Implement a **Safety Filter** using Control Barrier Functions (CBF) that modifies a nominal control input to ensure safety with respect to obstacles.

(d) Compare the performance of these controllers under various scenarios.

The dynamics of the Dubins car are given by:

$$x_{k+1} = x_k + v\cos(\theta_k)\,\Delta t, \tag{1}$$

$$y_{k+1} = y_k + v\sin(\theta_k)\,\Delta t, \tag{2}$$

$$\theta_{k+1} = \theta_k + u_k\,\Delta t, \tag{3}$$

where $x_k, y_k \in \mathbb{R}$ denote the position, $\theta_k \in \mathbb{R}$ the heading, $v > 0$ is a constant speed, and u_k is the steering control input subject to

$$u_k \in \mathcal{U} = \{u \in \mathbb{R} \mid -w_{\max} \le u \le w_{\max}\}.$$

Obstacles in the environment are modeled as circles with centers (o_x, o_y) and radii r. To guarantee safety, a margin $\delta > 0$ is added so that the safety condition for each obstacle is:

$$(x - o_x)^2 + (y - o_y)^2 \ge (r + \delta)^2.$$

2 Part I: Standard MPC for the Dubins Car

Objective: Develop an MPC formulation that minimizes a cost function over a finite horizon while ensuring dynamic feasibility and obstacle avoidance.

2.1 Decision Variables and Dynamics Constraints

Define the state trajectory over a prediction horizon N:

$$X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad \Theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_N \end{bmatrix},$$

and the control inputs:

$$U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}.$$

The dynamics (1)–(3) are enforced as equality constraints:

$$\begin{aligned} x_{k+1} - x_k - v\cos(\theta_k)\,\Delta t &= 0, \\ y_{k+1} - y_k - v\sin(\theta_k)\,\Delta t &= 0, \\ \theta_{k+1} - \theta_k - u_k\,\Delta t &= 0, \quad \forall k = 0, \ldots, N-1. \end{aligned}$$

2.2 Cost Function

Construct the cost function to penalize the final state error and control effort:

$$J = (x_N - x_{\text{goal}})^2 + (y_N - y_{\text{goal}})^2 + \alpha \sum_{k=0}^{N-1} u_k^2,$$

where $\alpha > 0$ is a small weight on the control effort.

2.3 Obstacle Avoidance Constraints

For each time step k and each obstacle (o_x, o_y, r), enforce:

$$(x_k - o_x)^2 + (y_k - o_y)^2 \geq (r + \delta)^2.$$

2.4 Task

1. Formulate the full nonlinear programming problem (NLP) for the standard MPC.
2. Use CasADi to define the decision variables, cost function, and constraints.
3. Solve the NLP and return the first control input u_0.

3 Part II: Robust MPC Formulations

Objective: Extend the standard MPC formulation to account for disturbances d_k that satisfy $\|d_k\| \leq d_{\max}$. Two common robust MPC approaches are:

(a) **Min-Max MPC:** Formulate a min-max optimization problem that minimizes the worst-case cost over all admissible disturbance sequences.

(b) **Tube-Based MPC:** Design a nominal MPC controller and an ancillary feedback controller to keep the system in a "tube" around the nominal trajectory.

3.1 Min-Max MPC

1. Write the disturbed dynamics:

$$\begin{aligned} x_{k+1} &= x_k + v\cos(\theta_k)\,\Delta t + d_k^x, \\ y_{k+1} &= y_k + v\sin(\theta_k)\,\Delta t + d_k^y, \\ \theta_{k+1} &= \theta_k + u_k\,\Delta t, \end{aligned}$$

   with $\|d_k\| \leq d_{\max}$.

2. Develop the robust constraints by ensuring that the obstacle avoidance constraint holds for all disturbances.

3. Formulate the min-max problem:

$$\min_{U} \max_{d_k \in \mathcal{D}} J(U, d),$$

   and discuss possible relaxations or approximations.

3.2 Tube-Based MPC

1. Define a nominal system and an error dynamics for the deviation due to disturbances.

2. Design a feedback controller to keep the error bounded, and tighten the constraints accordingly.

3. Formulate the nominal MPC with tightened constraints to guarantee that the actual state remains within a tube around the nominal trajectory.

3.3 Task

1. Implement a robust MPC controller using one of the approaches above (or both for comparison).

2. Analyze the performance of your robust MPC in simulation under different disturbance bounds $d_{\max}$.

3. Compare your results with the standard MPC from Part I.

4 Part III: CBF-Based Safety Filtering

Objective: Design a safety filter using Control Barrier Functions (CBF) that modifies a nominal control input u_{nom} to ensure safety with respect to obstacles.

4.1 Barrier Function and its Derivatives

For each obstacle (o_x, o_y, r), define the barrier function:

$$h(x, y) = (x - o_x)^2 + (y - o_y)^2 - (r + \delta)^2.$$

Its time derivative is:

$$\dot{h}(x, y, \theta) = 2v\left[(x - o_x)\cos(\theta) + (y - o_y)\sin(\theta)\right].$$

A simplified second derivative approximation can be written as:

$$L_f^2 h(x, y, \theta) = 2v^2 + 2\lambda\dot{h}(x, y, \theta) + \lambda^2 h(x, y),$$

and the control-dependent term:

$$L_g L_f h(x, y, \theta) = 2v\left[(y - o_y)\cos(\theta) - (x - o_x)\sin(\theta)\right].$$

4.2 CBF Constraint

The safety condition is enforced by:

$$L_f^2 h(x, y, \theta) + L_g L_f h(x, y, \theta)\, u \geq 0.$$

4.3 QP Formulation

Formulate the following Quadratic Program (QP) to obtain the safe control input u_{safe}:

$$\begin{aligned} \min_{u} \quad & \frac{1}{2}(u - u_{\text{nom}})^2 \\ \text{s.t.} \quad & L_f^2 h(x, y, \theta) + L_g L_f h(x, y, \theta)\, u \geq 0, \quad \forall \text{ obstacles}, \\ & u \in [-w_{\max}, w_{\max}]. \end{aligned}$$

You are required to solve this QP using CVXPY.

4.4 Task

1. Derive the expressions for h, $\dot{h}$, $L_f^2 h$, and $L_g L_f h$.
2. Implement the QP in CVXPY that outputs u_{safe}.
3. Validate your safety filter by comparing the trajectories generated using the nominal controller versus the safety-filtered control.

5 Part IV: Simulation and Comparative Analysis

Objective: Evaluate and compare the performance of the Standard MPC, Robust MPC, and CBF-based Safety Filter under various scenarios.

5.1 Simulation Setup

1. Define the simulation parameters: time step Δt, prediction horizon N, constant speed v, control bounds $w_{\max}$, and safety margin δ.
2. Set up initial conditions and goal positions.
3. Consider multiple scenarios by varying:
   - Disturbance bounds $d_{\max}$ for the robust MPC.
   - Obstacle configurations.
   - Presence or absence of disturbances in the simulation.

5.2 Performance Metrics

Compare the methods based on:

1. **Goal Convergence:** Distance to goal over time.
2. **Constraint Violation:** Minimum distance to obstacles.
3. **Control Effort:** Magnitude and smoothness of the control input.

5.3 Task

1. Simulate each control method and plot the resulting trajectories.
2. Provide a quantitative and qualitative comparison of the controllers.
3. Discuss the trade-offs between performance and robustness for each method.

6 Submission Instructions

- Submit your annotated code files (in Python) along with a brief report (PDF) summarizing your results and observations.
- Include plots comparing the trajectories and control inputs for the different methods.
- Provide detailed explanations of your implementation choices and any assumptions made.

7 Bonus

For extra credit, implement both min-max MPC and tube-based MPC, and compare their performance in environments with high disturbance levels.