ECE 592 – Topics in Data Science

Test 3: Optimization – Fall 2022

October 24, 2022

Please remember to justify your answers carefully.

Last name: _____ First name: _____

Please recall the course academic integrity policy (from the syllabus):
When working on tests, no cooperation or "collaboration" between students
is allowed. While it could be tempting to text or email a friend during a test
that is administered electronically, this is not allowed. You will be allowed
to use your notes, books, a browser, and software such as Matlab and/or
Python.[1] However, while working on the test you should not text, email, or
communicate with other people (certainly not other students) in any way,
unless you are consulting with the course staff. By submitting the test, you
will be acknowledging that you completed the work on your own without the
help of others in any capacity. Any such aid would be unauthorized and a
violation of the academic integrity policy.

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[1]You can use the browser to access Moodle, the course webpage, and look up technical topics. Similar to
a normal test, you must not communicate with other people.

Question 1 (Linear programming.)

Fruit Dude sells two types of fruit to NC State students, apples and oranges. It costs Fruit Dude $9 and $12 to buy cartons of apples and oranges, respectively. Suppose further that a carton of apples sells for $12 while a carton of oranges sells for $15, Fruit Dude's fruit stand can only stock 100 cartons of fruit, and he can spend $1000 to buy cartons of fruit.

(a) Please help maximize Fruit Dude's profits by expressing a linear programming problem. (Note that we define the profit as how much Fruit Dude sells a carton for, minus its cost. And there is no need to provide numerical answers, just express it in canonical form.)

(b) Suppose that Fruit Dude solved the linear programming problem on a fruity computer, and the answers were non-integer. Fruit cartons only come in integer numbers, of course. Help Fruit Dude stock his fruit stand in a way that maximizes profits while adhering to integer constraints. (Specify a new formulation that incorporates the integer constraints.)

Question 2 (Integer programming.)

Consider two variables, x and y, that must satisfy the following constraints: (i) they are both integers, i.e., $x, y \in \mathbb{Z}$; (ii) $x, y \geq 0$; (iii) $x + 2y \leq 4$; and (iv) $2x + y \leq 5$. Below, you will maximize the function

$$f(x, y) = x^2 - 2y^2 + 3xy,$$

subject to these constraints.

(a) Find the range of (x, y) pairs that satisfy the constraints. Make sure to justify your work, and express your answer as a list of ordered pairs, e.g., $(x, y) \in \{(-1, 1), (2, -3), (10, 8)\}$.

(b) Among the answers you found in part (a), identify the locations that maximizes $f(x, y)$. Again, justify your work and express your answer as an ordered pair.

Question 3 (Second order methods.)
We have discussed in class and seen through past test questions (e.g., 2019 Midterm Question 5 and 2020 Test3 Question 2) how Newton's method can often let us find roots of functions using linear (first order) approximations. To understand this principle, suppose that we form a first order Taylor approximation of $f(x)$ around our current iteration, x_t,

$$f(x) = x_t + f'(x_t)(x - x_t) + \text{higher order.} \tag{1}$$

Solving for $f(x) = 0$ gives us $0 = x_t + f'(x_t)(x - x_t)$, meaning that $x_t = -f'(x_t)(x - x_t)$, hence the next iteration, x_{t+1}, satisfies

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}. \tag{2}$$

(a) One shortcoming of Newton's method appears when the derivative is zero or close to zero. To see why, consider the function,

$$f(x) = x^2 - 1, \tag{3}$$

where we want to find its root. For $x_0 = 0$ ($t = 0$), compute $f(x_t)$ and $f'(x_t)$. What is the first order approximation, (1)? Can you solve for x_{t+1} as in (2)?

(b) Suppose that we use a second order Taylor approximation,

$$f(x) = x_t + f'(x_t)(x - x_t) + \frac{1}{2}f''(x_t)(x - x_t)^2 + \text{higher order.}$$

We can solve for x_{t+1} in a manner analogous to (2) by solving a quadratic equation. How does the second order approximation impact your answer from (a)? You do not need to find a root; instead, explain your reasoning how one could approach this problem.

(c) The shortcoming in part (a) is exacerbated when we do not have access to $f(x)$ per (3). Instead, we estimate $f(x)$ by measuring the function; for some value x, we obtain a noisy measurement, $f(x) + \text{noise}$. Describe qualitatively how you could repeat part (a) by performing multiple measurements of the function. (Hint: you may want to measure the function for multiple values of x.)