ECE 592 – Topics in Data Science

Test 3: Optimization – Fall 2022

October 24, 2022

Question 1 (Linear programming.)
Fruit Dude sells two types of fruit to NC State students, apples and oranges. It costs Fruit Dude \$9 and \$12 to buy cartons of apples and oranges, respectively. Suppose further that a carton of apples sells for \$12 while a carton of oranges sells for \$15, Fruit Dude's fruit stand can only stock 100 cartons of fruit, and he can spend \$1000 to buy cartons of fruit.

(a) Please help maximize Fruit Dude's profits by expressing a linear programming problem. (Note that we define the profit as how much Fruit Dude sells a carton for, minus its cost. And there is no need to provide numerical answers, just express it in canonical form.)

Solution: Denote the number of apples and oranges by X and Y, respectively. The profit is $(12-9)X+(15-12)Y = 3X+3Y$. We need X and Y to satisfy $X, Y \geq 0$. We also need $X+Y \leq 100$, because the stand can only stock 100 cartons, and $9X+12Y \leq 1000$, the daily budget. In summary, we want to find X, Y that maximize $3X+3Y$ subject to $X, Y \geq 0$, $X+Y \leq 100$, and $9X+12Y \leq 1000$.

(b) Suppose that Fruit Dude solved the linear programming problem on a fruity computer, and the answers were non-integer. Fruit cartons only come in integer numbers, of course. Help Fruit Dude stock his fruit stand in a way that maximizes profits while adhering to integer constraints. (Specify a new formulation that incorporates the integer constraints.)

Solution: The only extra requirement is that $X, Y \in \mathbb{Z}$.

Question 2 (Integer programming.)
Consider two variables, x and y, that must satisfy the following constraints: (i) they are both integers, i.e., $x, y \in \mathbb{Z}$; (ii) $x, y \geq 0$; (iii) $x + 2y \leq 4$; and (iv) $2x + y \leq 5$. Below, you will maximize the function

$$f(x, y) = x^2 - 2y^2 + 3xy,$$

subject to these constraints.

(a) Find the range of (x, y) pairs that satisfy the constraints. Make sure to justify your work, and express your answer as a list of ordered pairs, e.g., $(x, y) \in \{(-1, 1), (2, -3), (10, 8)\}$.

Solution: For $x = 0$, $y \in \{0, 1, 2\}$. For $x = 1$, $y \in \{0, 1\}$. For $x = 2$, $y \in \{0, 1\}$. Therefore, $(x, y) \in \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0), (2, 1)\}$.

(b) Among the answers you found in part (a), identify the locations that maximizes $f(x, y)$. Again, justify your work and express your answer as an ordered pair.

Solution: For these pairs from part (a), the function $f(x, y)$ is $0, -2, -8, 1, 2, 4, 8$. The maximum value, 8, is obtained for the pair $(2, 1)$.

Question 3 (Second order methods.)
We have discussed in class and seen through past test questions (e.g., 2019 Midterm Question 5 and 2020 Test3 Question 2) how Newton's method can often let us find roots of functions using linear (first order) approximations. To understand this principle, suppose that we form a first order Taylor approximation of $f(x)$ around our current iteration, x_t,

$$f(x) = f(x_t) + f'(x_t)(x - x_t) + \text{higher order.} \tag{1}$$

Solving for $f(x) = 0$ gives us $0 = x_t + f'(x_t)(x - x_t)$, meaning that $x_t = -f'(x_t)(x - x_t)$, hence the next iteration, x_{t+1}, satisfies

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}. \tag{2}$$

(a) One shortcoming of Newton's method appears when the derivative is zero or close to zero. To see why, consider the function,

$$f(x) = x^2 - 1, \tag{3}$$

where we want to find its root. For $x_0 = 0$ ($t = 0$), compute $f(x_t)$ and $f'(x_t)$. What is the first order approximation, (1)? Can you solve for x_{t+1} as in (2)?

Solution: $f(x_0) = f(0) = -1$, $f'(x_0) = 2x_0 = 0$, hence the first order approximation is -1 (constant), and it is never zero for any x_t. We cannot solve for x_{t+1}, because the derivative is 0, and dividing by zero in (2) is not defined.

(b) Suppose that we use a second order Taylor approximation,

$$f(x) = f(x_t) + f'(x_t)(x - x_t) + \frac{1}{2}f''(x_t)(x - x_t)^2 + \text{higher order.}$$

We can solve for x_{t+1} in a manner analogous to (2) by solving a quadratic equation. How does the second order approximation impact your answer from (a)? You do not need to find a root; instead, explain your reasoning how one could approach this problem.

Solution: The second derivative is now $f''(x_0) = 2$, and the second order approximation, $-1 + \frac{1}{2}2(x - 0)^2 = x^2 - 1$, is identical to the true function. We can find its root by finding the root of a standard quadratic function.

(c) The shortcoming in part (a) is exacerbated when we do not have access to $f(x)$ per (3). Instead, we estimate $f(x)$ by measuring the function; for some value x, we obtain a noisy measurement, $f(x) + \text{noise}$. Describe qualitatively how you could repeat part (a) by performing multiple measurements of the function. (Hint: you may want to measure the function for multiple values of x.)

Solution: Estimating $f(x_t)$ is performed by measuring the function at x_t. Estimating $f'(x_t)$ involves numerical derivation; we compute the function at 2 points close to x_t, for example $x_t \pm \delta$, and then estimate,

$$f'(x_t) \approx \frac{f(x_t + \delta) - f(x_t - \delta)}{(x_t + \delta) - (x_t - \delta)} = \frac{f(x_t + \delta) - f(x_t - \delta)}{2\delta}.$$

There are many ways to approach this question, of course. Some may want to improve the estimate by averaging over the function multiple times. Others will want to take δ to zero in the numerical derivation step. Other variations are also possible.