Pure Mathematics
Advanced Level

"Once your soul has been enlarged by a truth, it can never return to its original size."
-Blaise Pascal

Notes By

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Chapter 1

Integration

1.1 Reduction Formulæ

Integrating using a reduction formula is in essence repeating integration by parts over and over again.

We can think of the process of finding a reduction formula for a given integral as a *recursive approach* to integration by parts. By listing all the iterations of $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, more specifically the $\int v\frac{du}{dx}\,dx$ part in terms of I_n where n is the *iterative index*, or the *step number*, if you will.

As expected, finding this recursively valid form is not as direct, and thus, the exponent has to be *split* in such a way that trigonometric identities can be used.

Ex. 1. If $I_n = \displaystyle\int \cos^n x\,dx$ show that $I_n = \dfrac{1}{n}\sin \cos^{n-1} x + \dfrac{n-1}{n} \cdot I_{n-2}$. Hence find $\displaystyle\int \cos^5 x\,dx$.

$$\begin{aligned}
I_n &= \int \cos^n x\,dx \\
&= \int \cos x \cdot \cos^{n-1} x\,dx \\
\therefore\quad I_n &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \;-\; (n-1)\int \cos^n x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n \\
I_n + (n-1)I_n &= \cos^{n-1} x \sin x + (n-1)I_{n-2} \\
\implies nI_n &= \cos^{n-1} x \sin x + (n-1)I_{n-2} \\
\implies I_n &= \frac{1}{n}\cos^{n-1} x \sin x + \left(\frac{n-1}{n}\right) I_{n-2}
\end{aligned}$$

$$\int \cos^5 x \, dx = I_5$$

$$I_5 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_3$$

$$I_3 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_1$$

$$I_1 = \int \cos x \, dx = \sin x + k$$

$$\therefore \quad \int \cos^5 x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}(\sin x + k)\right)$$

$$= \frac{1}{5}\cos^2 x \cdot \sin x + \frac{4}{15}\cos^2 x \cdot \sin x + \frac{8}{15}\sin x + c \quad \square$$

Ex. 2. If $I_n = \int \tan^n \theta \, d\theta$, find a reduction formula for I_n and use it to evaluate $\int_0^{\frac{\pi}{4}} \tan^6 \theta \, d\theta$.

$$I_n = \int \tan^n \theta \, d\theta$$

$$= \int \tan^2 \theta \tan^{n-2} \theta \, d\theta$$

$$= \int (\sec^2 \theta - 1) \tan^{n-2} \theta \, d\theta$$

$$= \int \sec^2 \theta \tan^{n-2} \theta \, d\theta - \int \underbrace{\tan^{n-2} \theta \, d\theta}$$

$$= \frac{\tan^{n-1} \theta}{n-1} - \underbrace{I_{n-2}}$$

$$\int_0^{\frac{\pi}{4}} \tan^6 \theta \, d\theta = I_6 \Big|_0^{\frac{\pi}{4}}$$

$$I_6 = \frac{tan^5\theta}{5} - I_4$$

$$I_4 = \frac{\tan^3 \theta}{3} - I_2$$

$$I_2 = \tan \theta - I_0$$

$$I_0 = \int 1 \, d\theta = \theta + k$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^6 \theta \, d\theta = \frac{\tan^6 \theta}{5} - \frac{\tan^3 \theta}{3} + \tan \theta - \theta \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$$

$$= \frac{13}{15} - \frac{\pi}{4} \quad \square$$

Ex. 3. Establish a reduction formula that could be used to find $\int x^n e^x \, dx$ and use it to find $\int x^4 e^4$.

$$\text{Let } I_n = \int x^n e^x \, dx \qquad \int x^4 e^x = I_4$$

$$\text{Let } u = x^n \qquad \frac{dv}{dx} = e^x \qquad I_4 = x^4 e^x - 4I_3$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = e^x \qquad I_3 = x^3 e^x - 3I_2$$

$$I_2 = x^2 e^x - 2I_1$$

$$\therefore \quad I_n = x^n e^x - n \int x^{n-1} e^x \, dx \qquad I_1 = xe^x - I_0$$

$$= x^n e^x - n\, I_{n-1} \qquad I_0 = e^x + k$$

$$\therefore \quad I_4 = x^4 e^x - 4(x^3 e^x - 3(x^2 e^x - 2(xe^x - e^x + k)))$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + c \quad \square$$

Ex. 4. Establish a reduction formula which can be used to evaluate $\int x^n \sin x \, dx$.

$$\text{Let } I_n = \int x^n \cdot \sin x$$

$$\text{Let } u = x^n \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = -\cos x$$

Ex. 5. Establish a reduction formula to find $\int \csc^n x \, dx$. Hence find $\int csc^5 x \, dx$

$$
\begin{aligned}
\text{Let } I_n &= \int \csc^n x \, dx \\
&= \int \csc^2 x \cdot \csc^{n-2} x \, dx \\
\text{Let } u &= \csc^{x-2} x \qquad \frac{dv}{dx} = \csc^2 x \, dx \\
\frac{du}{dx} &= -(n-2) \csc^{n-2} \cot x v \qquad = -\cot x \\
\therefore \int \csc^n x \, dx &= -\cot x \cdot \csc^{n-2} x - (n-2) \int \csc^{n-2} x \cot^2 x \, dx \\
I_n &= -\cot x \cdot \csc^{n-2} x - (n-2) \int \csc^{n-2} x \left(\csc^2 x - 1\right) \, dx \\
&= -\cot x \cdot \csc^{n-2} x - (n-2) \int \csc^n x \, dx + (n-2) \int \csc^{n-2} x dx \\
&= -\cot x \cdot \csc^{n-2} x - (n-2) \, I_n + (n-2) I_{n-2} \\
I_n + n I_n - 2 I_n &= -\cot x \cdot \csc^{n-2} x + (n-2) I_{n-2} \\
(n-1) I_n &= -\cot x \cdot \csc^{n-2} x + (n-2) I_{n-2} \\
I_n &= \frac{-1}{n-1} - \cot x \cdot \csc^{n-2} x + \frac{n-2}{n-1} I_{n-2} \\
&= \left(1 - \frac{1}{n-1}\right) I_{n-2} - \frac{\cot x \csc^{n-2} x}{n-1} \quad \square
\end{aligned}
$$

Ex. 6. Show that if $I_n - \int_0^\pi x^n \sin x \, dx$, then $I_n = \pi^n - n(n-1) \, I_n - 1$. Hence evaluate $\int_0^\pi \sin x \, dx$

Let $u = x^n$ $\qquad \dfrac{dv}{dx} = \sin x$

$\dfrac{du}{dx} = nx^{n-1}$ $\qquad v = -\cos x$

$$\therefore \quad I_n = \left[-x^n \cos x\right]_0^{\pi} + n \int_0^{\pi} x^{n-1} \cos x \, dx$$

$$= \pi^n + n \int_0^{\pi} x^{n-1} \cos x \, dx$$

Consider: $\displaystyle\int_0^{\pi} x^{n-1} \, dx$

Let $u =$

Ex. 7. Show that, if $I_n = \displaystyle\int_0^1 x^n e^{x^3} \, dx$, then $I_n = \dfrac{e}{3} - \dfrac{n-2}{3} \cdot I_{n-3}$

$$I_n = \int_0^1 x^n e^{x^3} \, dx$$

$$= \int_0^1 x^{n-2} x^2 e^{x^3} \, dx$$

Let $u = x^{n-2}$ $\qquad \dfrac{dv}{dx} = \dfrac{1}{3}\left(3x \cdot e^{x^3}\right)$

$\dfrac{du}{dx} = (n-2)x^{n-3}$ $\qquad v = \dfrac{1}{3} e^{x^3}$

$$\therefore \quad I_n = \left[\frac{x^{n-2} e^{x^3}}{3}\right]_0^1 - \frac{n-2}{3} \int_0^1 x^{n-3} e^{x^3} \, dx$$

$$= \frac{e}{3} - \frac{n-2}{3} \cdot I_{n-3}$$

Ex. 8. Show that, if $I_n = \displaystyle\int_0^1 x^n (1 + x^5)^4 \, dx$, then $I_n = \dfrac{1}{n+21}\left[32 - (n-4) \cdot I_{n-5}\right]$

$$
\begin{aligned}
I_n &= \int_0^1 x^n (1+x^5)^4 \, dx \\
&= x^{n-4} x^4 (1+x^5)^4 \, dx \\
&= \left[x^{n-4} \frac{(1+x^5)^5}{25} \right]_0^1 - \frac{n-4}{25} \int x^{n-5} (1+x^5)^5 \, dx \\
&= \frac{32}{25} - \frac{n-4}{25} \int_0^1 x^n - 5(1+x^5)(1+x^5)^4 \, dx \\
&= \frac{32}{25} - \frac{n-5}{25} \int_0^1 x^{n-5} (1+x^5)^4 \, dx - \frac{n-4}{25} \int_0^1 x^n (1+x^5)^4 \, dx \\
&= \frac{32}{25} - \left(\frac{n-4}{25} \right) I_{n-5} - \left(\frac{n-4}{25} \right) I_n \\
25 I_n &= 32 - (n-4) I_{n-5} - \left(\frac{n-4}{25} \right) I_n \\
25 I_n + n I_n - 4 I_n &= 32 - (n-4)\, I_{n-5}
\end{aligned}
$$

Ex. 9. Given $I_n = \int_0^1 (1+x^2)^{-n} \, dx$, show that $2n\, I_{n+1} = 2^{-n} + (2n-1)\, I_n$.

$$
\begin{aligned}
I_n &= \int_0^1 (1+x^2)^{-n} \, dx \\
&= \int_0^1 (1+x^2)^{-n} \cdot 1 \, dx \\
\therefore \quad I_n &= -2nx^2 (1+x^2)^{-(n+1)} \Big|_0^1 + 2n \int_0^1 x^2 (1+x^2)^{-n-1} \, dx \\
&= 2^{-n} + 2n \int_0^1 (x^2 + 1 - 1)(1+x^2)^{-(n+1)} \, dx \\
&= 2^{-n} + 2n \int_0^1 (1+x^2)^{-2} \, dx - 2n \int_0^1 (1+x^2)^{-(n+1)} \, dx \\
&= 2^{-n} + 2n\, I_n - 2n\, I_{n+1} \\
2n\, I_{n+1} &= 2^{-n} + (2n-1)\, I_n
\end{aligned}
$$