Pure Mathematics
Advanced Level

"Once your soul has been enlarged by a truth, it can never return to its original size."
-Blaise Pascal

Notes By

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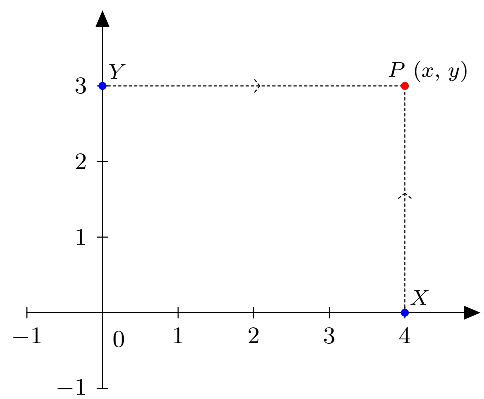
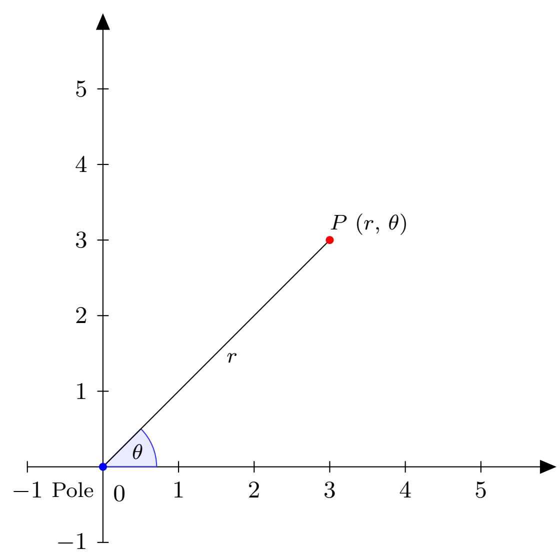
Introduction

The position of a point on a plane can be described in sever ways. With respect to some origin O we can locate a point P on a plane by noting a horizontal distance followed by a vertical distance.



However the point P can be located on a plane with respect to the origin O, a horizontal line and the distance of P from O.



This is the polar coordinate system where we refer to the origin as the pole and the horizontal line as the initial line. The anti clockwise angle is usually measured in the principal range $-\pi < \theta \leq \pi$.

Relationship between Polar and Cartesian Coordinates

Consider the following diagram showing the point P on the plane, both in Cartesian and Polar coordinates
The above relationship can be used to convert from one form to another.

Ex. 1. Find the polar coordinates of the curve given by $y = x^2 + y^2 = 2x$

$$\begin{aligned} x^2 + y^2 &= 2x \\ \implies 2r\cos\theta &= r^2 \\ \implies 2\cos\theta &= r \end{aligned}$$

Ex. 2. Find the Cartesian equation corresponding to the curves **a)** $r = 4(1+\cos\theta)$ and **b)** $3 = r\sin 2\theta$.

$$r = 4(1+\cos\theta)$$

$$\begin{aligned} 3 &= r\sin 2\theta \\ 3 &= 2r\sin\theta\cos\theta \\ 3 &= 2x\sin\frac{y}{r} \end{aligned}$$

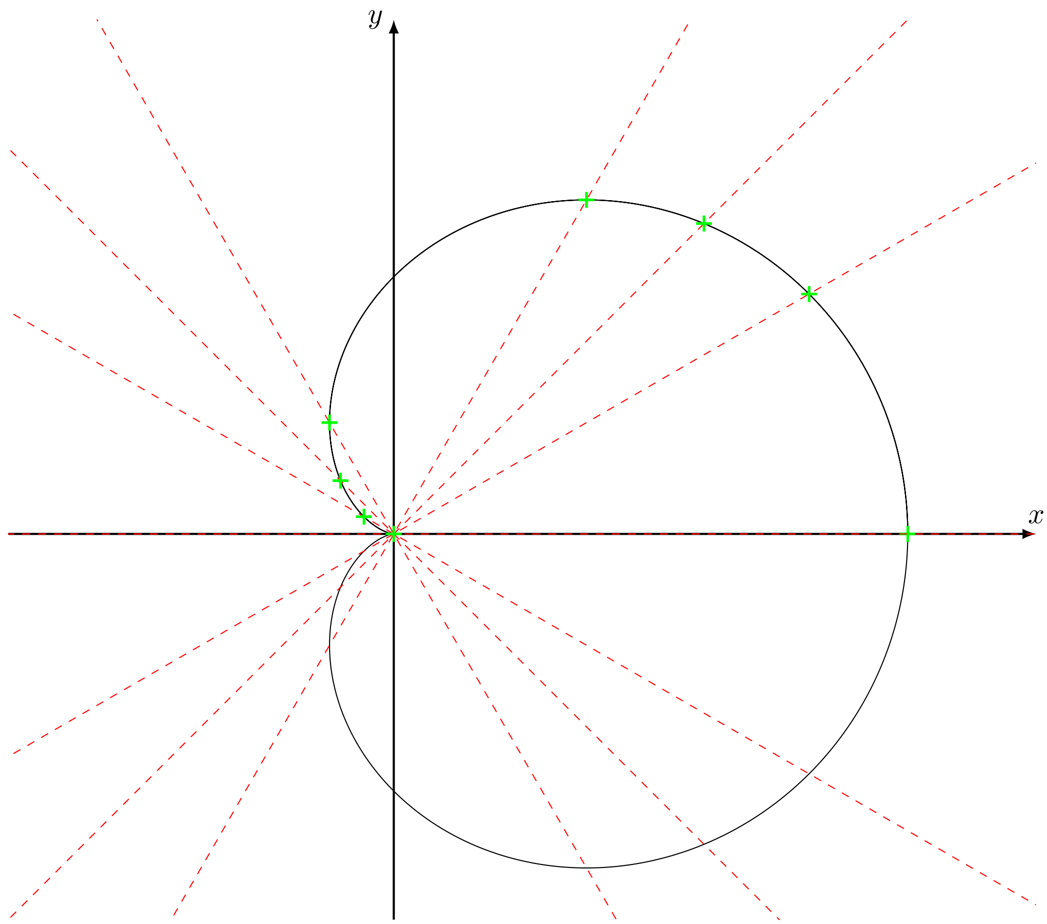
$$i^2 + 1^2 = 0$$

Sketching Polar Curves

Ex. 1. Sketch the polar curve of $2 + 2\cos\theta$.

Observing the following sketch, it is shown that a line of symmetry is present in the x-axis for the *cosine* function family.

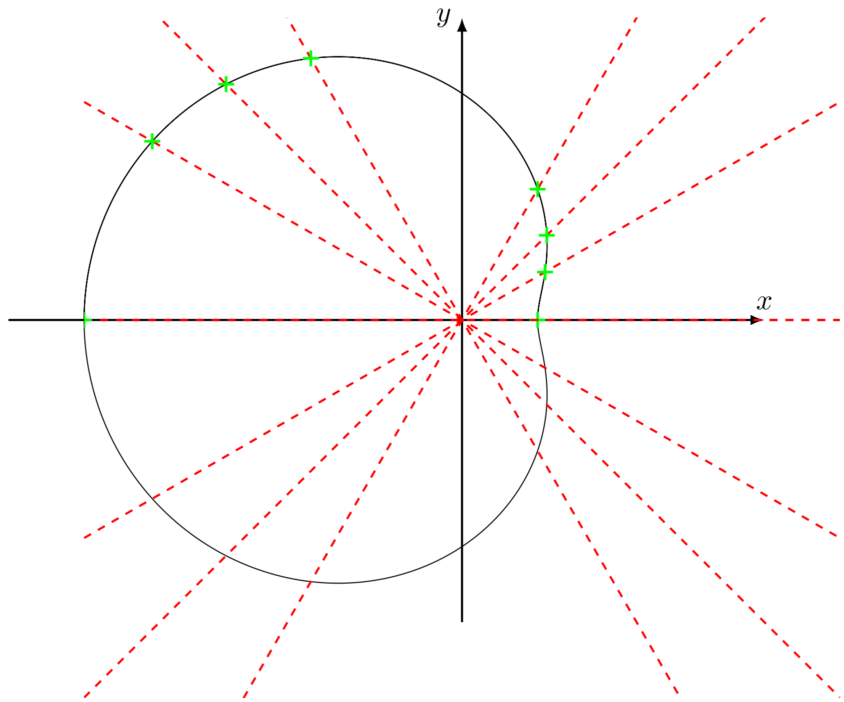
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.7	3.4	3	2	1	0.6	0.3	0



Ex. 2. Sketch the graph with polar equation $r = 3 - 2\cos\theta$.

Similarly to the above example, a line of symmetry is present in the x-axis.

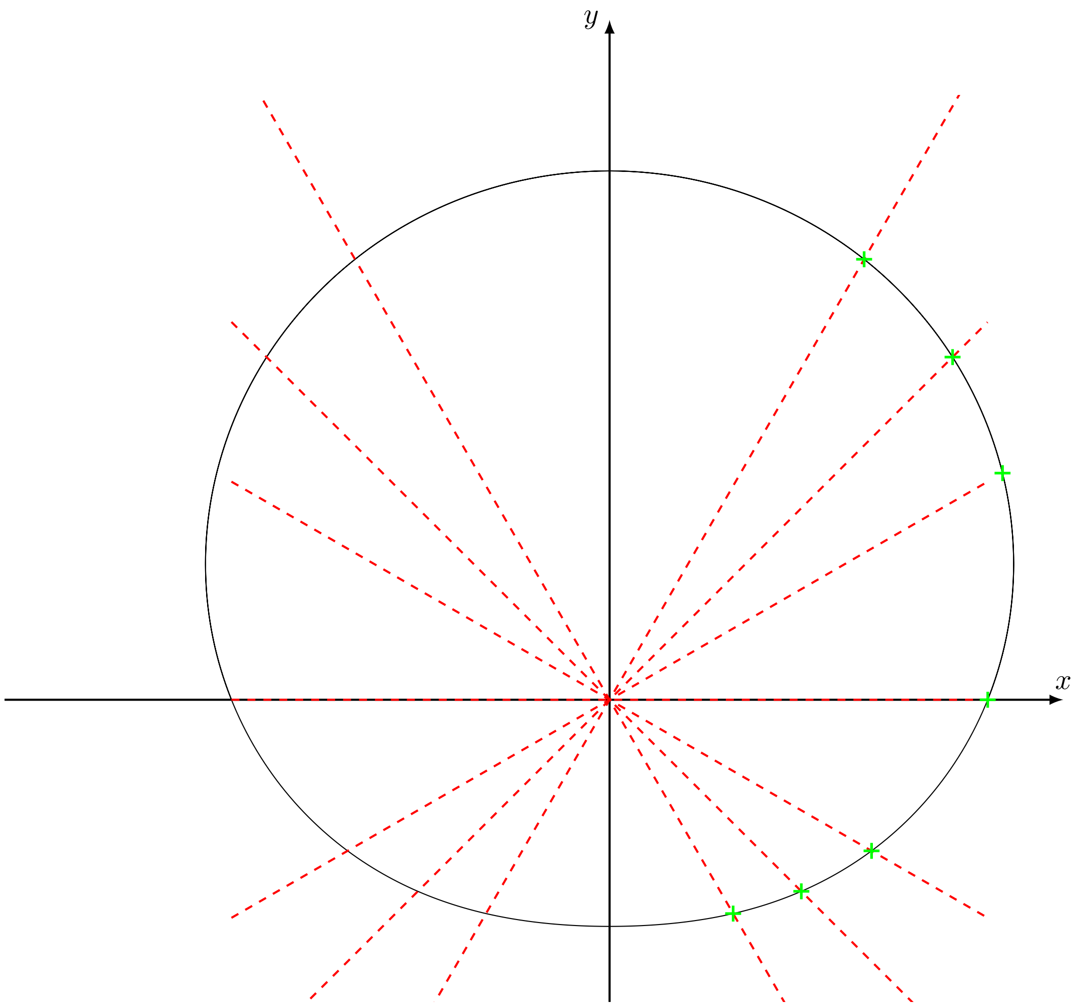
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	1	1.3	1.6	2	3	4	4.4	4.7	5



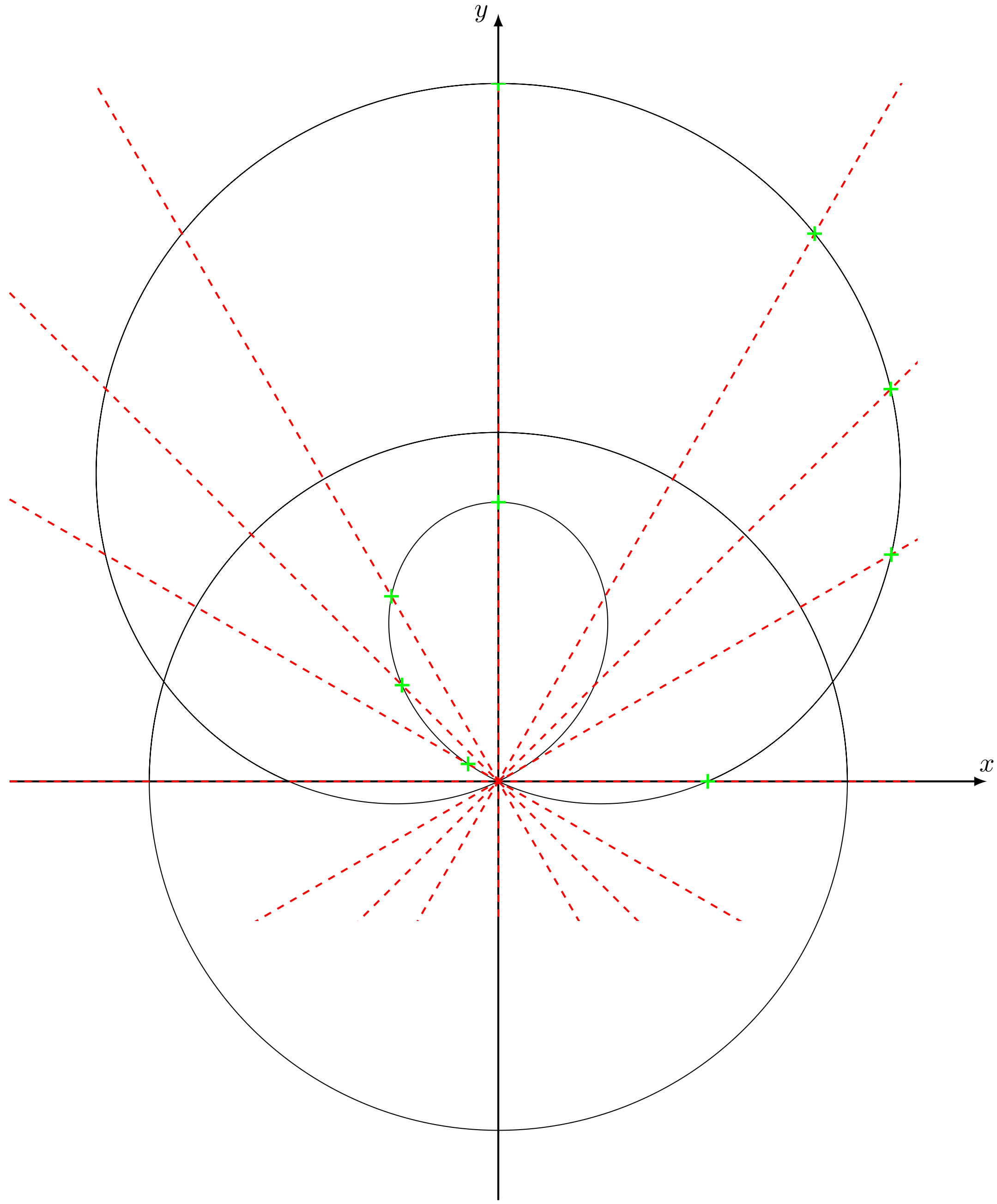
Ex. 3. Sketch the graph with polar equation $r = 5 + 2\sin\theta$.

It is noted that since the graph is part of the *sine* function family,the line of symmetry is now present in the y-axis.

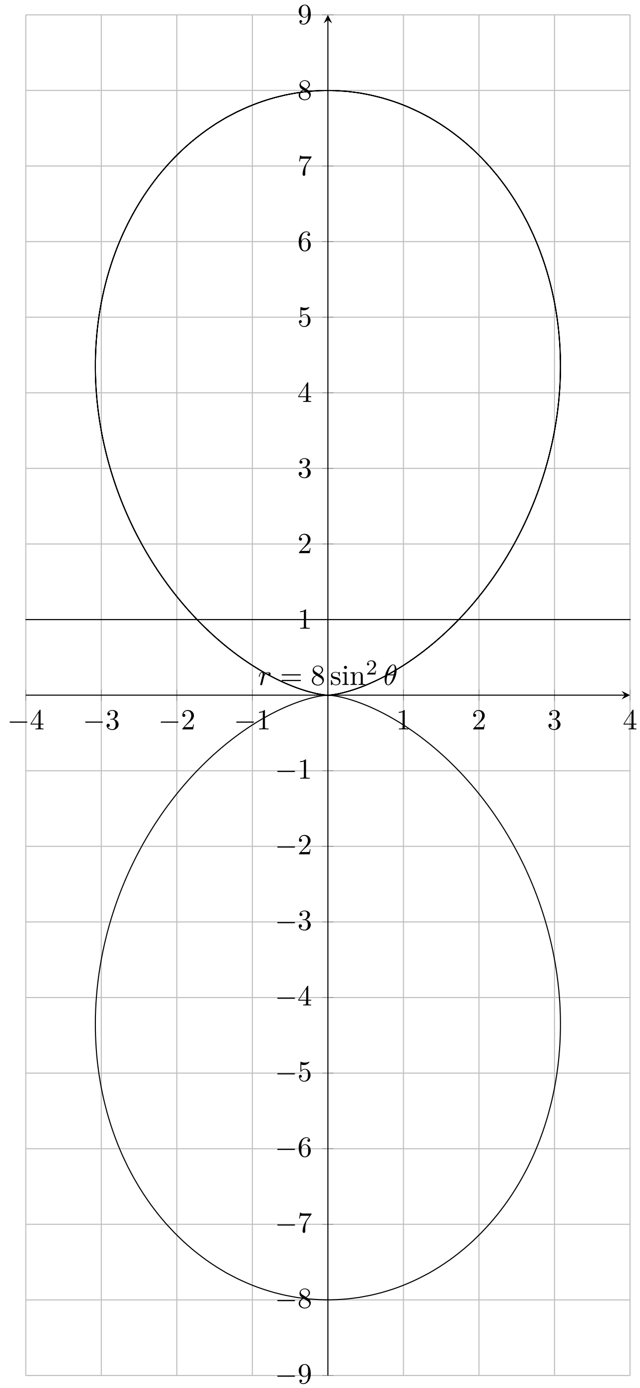
θ	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	3	3.3	3.6	4	5	6	6.4	6.7	7



Ex. 4. Sketch the polar curve $r = 3 + 7\sin\theta$ and the circle $r = 5$. Find the polar coordinates of their points of intersection.

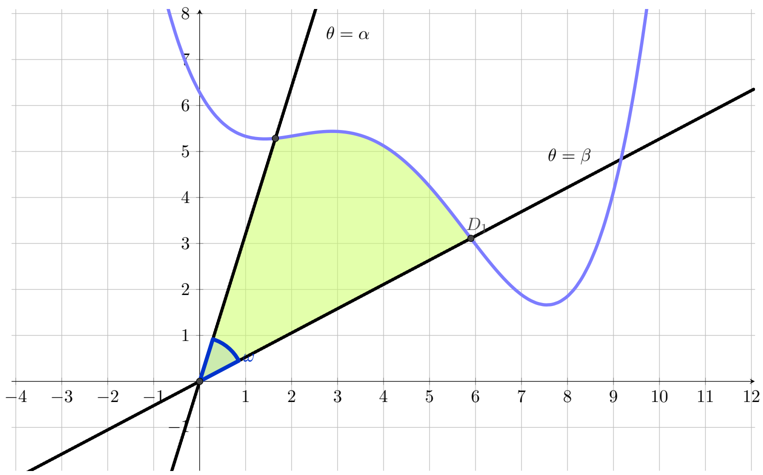


Ex. 5. Sketch the polar curve $r = 8\sin^2\theta$ and the line $r = \csc\theta$. Find the polar coordinates of their points of intersection.



Area partly bounded by a polar curve

Consider the following diagram showing the graph of $r = f(\theta)$.

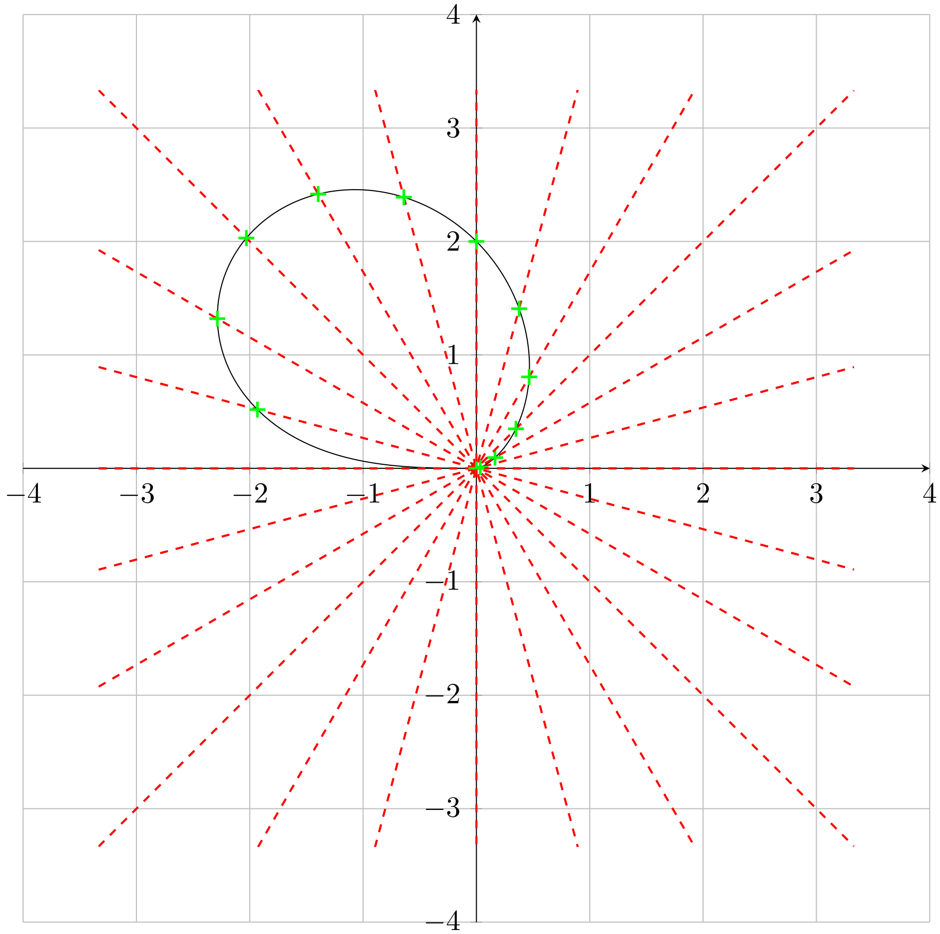


It can be shown that the area bounded by the curve $r = f(\theta)$, from $\theta = \alpha$ to $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Ex. 1. Sketch the curve with polar equation $r = 2(1 - \cos\theta)\sqrt{\sin\theta}$ for $0 \le \theta \le \pi$ and find the area it encloses.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
r	0	0.03	0.2	0.5	0.9	1.5	2	2.5	11	2.9	2.6	2	0



$$A = \frac{1}{2}\int_0^{\pi} \left(2(1-\cos\theta)\sqrt{\sin\theta}\right)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} 4(1-\cos\theta)^2 \sin(\theta)\, d\theta$$

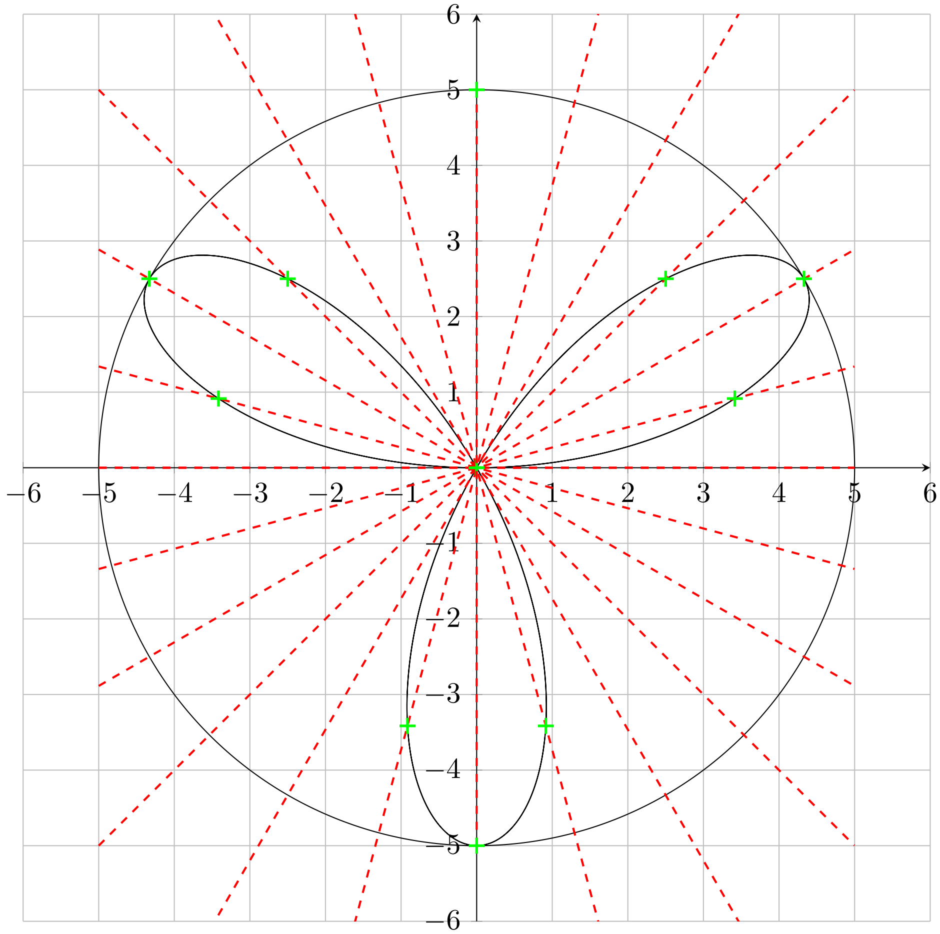
$$= 2\int_0^{\pi} \sin(\theta)(1 - 2\cos(\theta) + \cos^2\theta)\, d\theta$$

$$= 2\int_0^{\pi} \sin\theta - 2\sin\theta\cos\theta + \sin\theta\cos^2\theta\, d\theta$$

$$= 2\left(-\cos\theta + \cos^2\theta - \frac{\cos^3\theta}{3}\Big|_0^{\pi}\right)$$

$$= \frac{7}{3} - (-1+)$$

Ex. 2. Sketch the curve $r = 5\sin 3\theta$ and the circle $r = 5$. Find the area of the region which lies inside the circle but outside the curve.



$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} (5\sin(3\theta))^2\,d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\,d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{3}} \frac{1-\cos(6\theta)}{2}$$

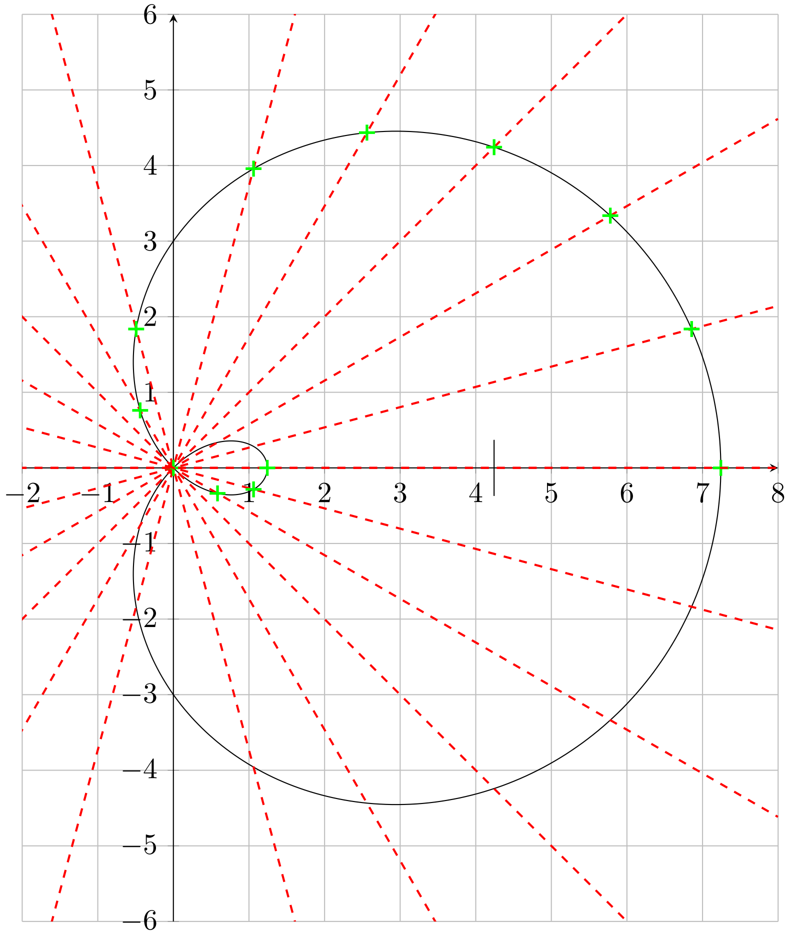
$$= \frac{25}{4} \quad \theta - \frac{\cos(6\theta)}{6}\Big|_0^{\frac{\pi}{3}}$$

$$= \frac{25}{24} \quad 6\theta - \cos 6\theta|_\alpha^\beta$$

$$A = \pi r^2$$

$$= 25\pi$$

Ex. 3. Sketch the curve $r = 3(1+\sqrt{2}\cos\theta)$ and the line $r = 3\sqrt{2}\sec\theta$. Find the polar coordinates of their points of intersection. Find the area of the region which lies inside the circle but outside of the cardioid.



Ex. 4. Sketch the curve $r = 5 + \sin\theta$ and the line $r\sin\theta = 8$.