Pure Mathematics Advanced Level

"Once your soul has been enlarged by a truth, it can never return to its original size."
-Blaise Pascal

Notes By

Giorgio Grigolo

Student of Mathematics and Computer Science
St.Aloysius College

2020

Contents

Chapter 1

Integration

1.1 Reduction Formulæ

Example 1. If $I_n = \int \cos^n x\, dx$ show that $I_n = \dfrac{1}{n}\sin\cos^{n-1} x + \dfrac{n-1}{n} \cdot I_{n-2}$. Hence find $\int \cos^5 x\, dx$.

$$
\begin{aligned}
I_n &= \int \cos^n x\, dx \\
&= \int \cos x \cdot \cos^{n-1} x\, dx
\end{aligned}
$$

Integrating by parts:

$$
\begin{aligned}
\therefore \quad I_n &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x\, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x)\, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x)\, dx
\end{aligned}
$$

$$\int \cos^5 x\, dx = I_5$$

$$
\begin{aligned}
I_5 &= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_3 \\
I_3 &= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_3 \\
I_1 &= \int \cos x\, dx = \sin x + k
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int \cos^5 x &= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}(\sin x + k)\right) \\
&= \frac{1}{5}\cos^2 x \cdot \sin x + \frac{4}{15}\cos^2 x \cdot \sin x + \frac{8}{15}\sin x + c
\end{aligned}
$$

Example 2. If $I_n = \displaystyle\int \tan^n \theta \, d\theta$

Example 3. Establish a reduction formula that could be used to find $\displaystyle\int x^n e^x \, dx$ and use it to find $\displaystyle\int x^4 e^4$.

$$\text{Let } I_n = \int x^n e^x \, dx$$

$$\text{Let } u = x^n \qquad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = e^x$$

$$\therefore I = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n_1}$$

Example 4. Establish a reduction formula which can be used to evaluate $\displaystyle\int x^n \sin x \, dx$.

$$\text{Let } I_n = \int x^n \cdot \sin x$$

$$\text{Let } u = x^n \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = -\cos x$$

Example 5. Establish a reduction formula to find $\displaystyle\int \csc^n x \, dx$. Hence find $\displaystyle\int csc^5 x \, dx$

$$\begin{aligned}
\text{Let } I_n &= \int \csc^n x\,dx \\
&= \int \csc^2 x \cdot \csc^{n-2} x\,dx
\end{aligned}$$

$$\text{Let } u = \csc^{x-2} x \qquad \frac{dv}{dx} = \csc^2 x\,dx$$

$$\frac{du}{dx} = -(n-2)\csc^{n-2}\cot xv \qquad = -\cot x$$

$$\begin{aligned}
\therefore \int \csc^n x\,dx &= -\cot x \cdot \csc^{n-2} x - (n-2)\int \csc^{n-2} x \cot^2 x\,dx \\
I_n &= -\cot x \cdot \csc^{n-2} x - (n-2)\int \csc^{n-2} x \left(\csc^2 x - 1\right)\,dx \\
&= -\cot x \cdot \csc^{n-2} x - (n-2)\int \csc^{n-2} x \left(\csc^2 x - 1\right)\,dx
\end{aligned}$$