Pure Mathematics
Advanced Level

"Once your soul has been enlarged by a truth, it can never return to its original size."
-Blaise Pascal

Notes By

Giorgio Grigolo

Student of Mathematics and Computer Science
St.Aloysius College

2020

Contents

Chapter 1

Classification of numbers

- Natural numbers: $\mathbb{N}$; (1, 2, 3, 4 . . .)

  This set includes every number which is both positive and whole.

- Integer numbers: $\mathbb{Z}$; (-2, -1, 0, 1, 2, . . .)

  The integer number set includes every negative and positive whole numbers, similarly to $\mathbb{N}$

- Rational numbers: $\mathbb{Q}$; (-1, 2 , $\frac{1}{2}$)

  A number is s.t.b rational if expressed in the form $\frac{p}{q}; p, q \in \mathbb{Z}$.

- Irrational numbers: $\mathbb{Q}'$; $(\pi, e, \sqrt{2}, \sqrt{5}, \ldots))$

  If a number is not classified as any of the above, it is referred to as irrational.

- Real numbers: $\mathbb{R}$

  Anything mentioned above inclusively represent the set of Real numbers

We can additionally refer to positive or negative numbers in any set by using the notation:

$$\mathbb{R}^+ \text{ and } \mathbb{R}^-$$

Chapter 2

Surds

2.1 Introduction

Consider numbers $\sqrt{64}, \sqrt{16}$. These can be represented as exact quantities by writing 8 and 4. There are however other numbers which cannot be expressed as exact quantities using other symbols.

There is an option of expressing them as corrected decimals without however preserving their full value. Instead, we choose to keep the form $\sqrt{a}$ which preserves the full value of the numbers.

2.2 Examples

$a)$
$$\begin{aligned}\sqrt{2} &= \sqrt{16 \times 3}\\ &= \sqrt{16} \times \sqrt{3}\\ &= 4\sqrt{3}\end{aligned}$$

$b)$
$$\begin{aligned}\sqrt{72} &= \sqrt{8 \times 9}\\ &= \sqrt{9} \times \sqrt{8}\\ &= 3\sqrt{8}\end{aligned}$$

$$\begin{aligned}\sqrt{360} &= \sqrt{180} \times \sqrt{2}\\ &= \sqrt{36} \times \sqrt{10}\\ &= 6\sqrt{10}\end{aligned}$$

$d)$
$$\begin{aligned}&\left(1 + 2\sqrt{3}\right)\left(2 + 3\sqrt{5}\right)\\ &= 2 - \sqrt{3} - 10\sqrt{3}\\ &= -28 - \sqrt{3}\end{aligned}$$

$e)$[1]
$$\begin{aligned}&\left(2 - 3\sqrt{5}\right)\left(2 + 3\sqrt{5}\right)\\ &= 4 + 6\sqrt{5} - 6\sqrt{5} - 9(5)\\ &= -41\end{aligned}$$

[1] This expression shows that for products of the form $\left(a + b\sqrt{c}\right)\left(a - b\sqrt{c}\right)$ the surds will vanish.

2.3 Rationalizing the denominator

Consider the fraction:

$$\frac{1}{1+\sqrt{2}}$$

This fraction contains a surd, thus, making it irrational. To rationalize said fraction one should find the multiplicative operation of canceling the denominator *(see* [1]*)*.

Continuing...

$$\begin{aligned}
\frac{1}{1+\sqrt{2}} &= \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} \\
&= \frac{1-\sqrt{2}}{1} \\
&= 1-\sqrt{2}
\end{aligned}$$

Chapter 3

Partial Fractions

3.1 Introduction

Consider the expression and suppose it is simplified:

$$
\begin{aligned}
\frac{2}{x+1}+\frac{3}{2x-5} &= \frac{2(2x-5)+3(x+1)}{(x+1)(2x-5)} \\
&= \frac{4x-10+3x+3}{(x+1)(2x-5)} \\
&= \frac{4x-10+3x+3}{(x+1)(2x-5)} \\
&= \frac{7x-7}{(x+1)(2x-5)}
\end{aligned}
$$

In this chapter we reverse the approach above, hence decomposing one fraction to its corresponding partial fractions.

3.2 Types of partial fraction cases

Type 1: Linear Factors in denominator

Ex. 1. Decompose $\dfrac{7x-7}{(x+1)}$ into Partial Fractions

$$\frac{7x-7}{(x+1)} \equiv \frac{A}{(x+1)} + \frac{B}{(2x-5)}$$

$$\begin{aligned}
\implies \quad 7x-7 &= A\,(2x-5) + B\,(x+1)) \\
\implies 7(-1)-7 &= A(2(-1)-5) && x=-1 \\
\implies \quad -14 &= -7A \\
\implies \quad A &= 2 && (..1) \\
\implies 7\left(\frac{5}{2}\right)-7 &= B\left(\left(\frac{5}{2}\right)+1\right) && x=\frac{5}{2} \\
\implies \quad \frac{35}{2}-7 &= \frac{7B}{2} \\
\implies \quad B &= 3 && (..2)
\end{aligned}$$

$$\therefore \quad \frac{7x-7}{(x+1)} = \frac{2}{(x+1)} + \frac{3}{(2x-5)}$$

Type 2: Irreducible Quadratic Factor in Denominator

Ex. 2. Decompose $\dfrac{x^2+1}{(2x+1)(x^2+3)}$ into its corresponding partial fractions.

$$\frac{x^2+1}{(2x+1)(x^2+3)} \equiv \frac{A}{2x+1} + \frac{Bx+C}{x^2+3}$$

$$\begin{aligned}
\implies \quad x^2+1 &\equiv A(x^2+3) + (Bx+C)(2x+1) \\
\implies \quad x^2+1 &\equiv x^2(A+2B) + x(2B+2C) + (3A+C) && (*..)
\end{aligned}$$

At this stage, since both equations are identical, we analyse the different coefficients and constants to form a system of equations to solve.

Comparing coefficients of x^2:

$$1 = A + 2B \qquad (1..)$$

Comparing coefficients of x:

$$0 = B + C \tag{2..}$$

Comparing constants:

$$C = 1 - 3A \tag{3..}$$

Substituting 3.. in 2..

$$0 = B + 1 - 3A$$

$$\implies \quad B = 3A - 1 \tag{4..}$$

Substituting 4.. in 1..

$$1 = A + 2(3A - 1)$$

$$\implies \quad A = \frac{-1}{7} \tag{5..}$$

Substituting 5.. in 1..

$$1 = \frac{-1}{7} + 2B$$

$$\implies \quad -7 = 1 - 14B$$

$$\implies \quad B = \frac{4}{7} \tag{6..}$$

Substituting 5.. in 3..

$$C = 1 - 3\left(\frac{-1}{7}\right)$$

$$\implies \quad C = \frac{10}{7}$$

$$\therefore \quad \frac{x^2+1}{(2x+1)(x^2+3)} \equiv \frac{A}{2x+1} + \frac{Bx+C}{x^2+3}$$

Type 3: Repeated factor in the denominator

Ex. 3. Decompose $\dfrac{x+1}{(x+2)(x-3)^2}$ into its corresponding partial fractions.

$$\frac{x+1}{(x+2)(x-3)^2} \equiv \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$\implies \quad x+1 \equiv A(x-3)^2 + B(x-3)(x+2) + C(x+2)$$

$$\implies \quad x+1 \equiv Ax^2 - 6Ax + 9A + Bx^2 - Bx - 6B + Cx + 2C)$$

$$\implies \quad -2+1 = A(-2-3)^2 \qquad x=-2$$

$$\implies \quad A = \frac{-1}{25}$$

$$\implies \quad 3+1 = C(3+2) \qquad x=3$$

$$\implies \quad C = \frac{4}{5}$$

Comparing coefficients of x^2:

$$\implies \quad 0 = A + B$$

$$\implies \quad B = \frac{1}{25}$$

$$\therefore \quad \frac{x+1}{(x+2)(x-3)^2} \equiv \frac{1}{25(x-3)} + \frac{4}{5(x-3)^2} - \frac{-1}{25(x-3)}$$

The approach above, is similar to the previous one, with the addition of the fact that each repeated factor has to be listed in order of powers until its own.

Type 4: Improper fraction

Ex. 4. Decompose $\dfrac{2x^2-8x+11}{2x-5}$ into its corresponding partial fractions.

Since the fraction is improper, or top-heavy[2] it is required to perform a polynomial long division and acquire the proper terms.

$$
\begin{array}{r|l}
 & x - \dfrac{3}{2} \\
\hline
2x-5 & 2x^2 - 8x + 11 \\
 & \underline{-2x^2 + 5x} \\
 & - 3x + 11 \\
 & \underline{3x - \dfrac{15}{2}} \\
 & \dfrac{7}{2}
\end{array}
$$

$$\therefore \quad \frac{2x^2-8x+11}{2x-5} \equiv x - \frac{3}{2} + \frac{7}{2(2x-5)}$$

[2]Improper fractions containing a variable are recognized by the order of the exponent when the expression is expanded. i.e $\dfrac{x^2}{x+5}$ is regarded as improper

Chapter 4

Pascal's Triangle

Consider the following expansions:

$$(1+x)^0 = 1$$

$$(1+x)^1 = 1 + x$$

$$(1+x)^2 = 1 + 2x + x^2$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

$$\cdots$$

$n =$							1						
$n =$						1		1					
$n =$					1		2		1				
$n =$				1		3		3		1			
$n =$			1		4		6		4		1		
$n =$		1		5		10		10		5		1	
$n = 6$	1		6		15		20		15		6		1

The above array of numbers is called Pascal's Triangle. It can be used to expand any binomial. The following examples illustrate its use.

Ex. 1. Expand the following using Pascal's Triangle

$$\begin{aligned}(1+2x)^5 &\equiv 1(2x)^0 + 5(2x)^1 + 10(2x)^2 + 10(2x)^3 + 5(2x)^4 + 2x^5 \\ &\equiv 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5\end{aligned} \tag{a)}$$

$$\begin{aligned}\left(1-\frac{3x}{2}\right)^6 &\equiv 1\left(\frac{-3x}{2}\right)^0 + 6\left(\frac{-3x}{2}\right)^1 + 15\left(\frac{-3x}{2}\right)^2 + 20\left(\frac{-3x}{2}\right)^3 + \\ &\quad 15\left(\frac{-3x}{2}\right)^4 + 6\left(\frac{-3x}{2}\right)^5 + 1\left(\frac{-3x}{2}\right)^6 \\ &\equiv 1 - 9x + \frac{135x^2}{4} - \frac{135x^3}{2} + \frac{1215x^4}{16} - \frac{243x^5}{16} + \frac{728x^6}{64}\end{aligned} \tag{b)}$$

$$\begin{aligned}(p+q)^4 &\equiv (p(1+\frac{q}{p}))^4 \\ &\equiv p^4\left(1+\frac{q}{p}\right)^4 \\ &\equiv p^4\left(1+\frac{4q}{p}+\frac{6q^2}{p^2}+\frac{4q^3}{p^3}+\frac{q^4}{p^4}\right) \\ &\equiv p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4\end{aligned} \tag{c)}$$

In the above examples we observe that:

- The expansion contains the coefficients for Pascal's Triangle.
- The expansion is formed by descending exponents of the first term of the binomial & ascending exponents of the second.
- The sum of the exponents in each term is equal to the exponent by which the binomial was raised.

These observations can be applied to expand any binomials raised with positive integer exponents.

Chapter 5

The Remainder and Factor Theorems

5.1 The Remainder Theorem

Consider the polynomial $f(x)$. Suppose that this polynomial is to be divided by the linear expression $x - a$. This gives:

$$\frac{f(x)}{x-a}A \equiv Q(x) + \frac{R}{x-a}$$

$$\Longrightarrow \qquad f(x) \equiv Q(x) \cdot (x-a) + R$$

$$\text{Let } x - a = 0 \Longrightarrow \qquad x = a$$

$$\therefore \qquad f(a) = R; \qquad \text{Where R is the remainder of } \frac{f(x)}{x-a}$$

$$\text{and Q is the quotient of } \frac{f(x)}{x-a}$$

Ex. 1. Find the remainder when the cubic polynomial $f(x) = 2x^3 - 3x - 5$ is divided by $x - 2$

If $f(x)$ is to be divided by $x - 2$, then $f(2)$ is equal to the remainder of $\dfrac{2x^3 - 3x - 5}{x-2}$

$$2x^3 - 3x - 5$$

$$= \qquad 2(2)^3 - 3(2) - 5$$

$$\boxed{\therefore \quad R \quad = \quad 5}$$

5.2 The Factor theorem

The factor theorem states that:

- If the polynomial $f(x)$ is divided by $x - a$, then $f(a) = 0$ (i.e $R = 0$), therefore it can also be concluded that $x - a$ is a factor of $f(x)$

Ex. 1. Determine whether $2x + 3$ is a factor of $2x^3 + x^2 - 5x + 6$

Let $f(x) = 2x^3 + x^2 - 5x + 6$

If $2x + 3$ is a factor of $f(x)$: $\quad 0 = f\left(\frac{-3}{2}\right)$

However, $\quad 0 \neq 2\left(\frac{-3}{2}\right)^3 + \left(\frac{-3}{2}\right)^2 - 5\left(\frac{-3}{2}\right) + 6$

$\neq -3$

$\boxed{\therefore \quad 2x + 3 \text{ is } \textbf{not} \text{ a factor of } f(x)}$

Chapter 6

Quadratic Equations

6.1 Definition

A quadratic equation is of the form $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{R}, \quad a \neq 0$. These can be solved algebraically using one of the following methods:

- Fractions
- Completing the square
- The quadratic formula[3]

6.2 Nature of roots of the Quadratic Equation

Any quadratic equation has in general two roots, namely $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The quantity $b^2 - 4ac$ determines the nature of these roots.

- $b^2 - 4ac > 0$: Equation holds two real and distinct roots.
- $b^2 - 4ac = 0$: Equation holds two equal[4] roots.
- $b^2 - 4ac > 0$: Equation holds two complex roots.

Thus, the quantity $b^2 - 4ac$ discriminates among the type of roots that a quadratic equation may have. Therefore it is called the discriminant.

Ex. 1. Determine, without solving the nature of the following function.

$$\begin{aligned}
\textit{Let} \qquad f(x) &= 2x^2 + 3x - 17 \\
\implies \qquad b^2 - 4ac &= 3^2 - 4(2)(-17) \\
&= 145 \\
&> 0 \\
\therefore \qquad \text{Roots of } f(x) &\in \mathbb{R} \text{ and are distinct}
\end{aligned}$$

---

[3] $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

[4] It is implied that they are real.

Ex. 2. Determine the value of p if $px^2 - 10x + 1 = 0$ has two equal roots

Given that the equation has two equal roots:

$$b^2 - 4ac = 0$$

$$\implies \quad 100 - 4p = 0$$

$$\boxed{\therefore \quad p = 5}$$

6.3 Roots and Coefficients of a Quadratic Equation

6.3.1 Proof

Consider a general quadratic equation:

$$ax^2 + bx + c = 0 \tag{1..}$$

$$\implies \quad x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

Let α and β be the roots:

$$\implies \quad (x-\alpha)(x-\beta) = 0$$

$$\implies x^2 - \beta x - \alpha x + \alpha\beta = 0$$

$$\implies x^2 - (\alpha+\beta)x + \alpha\beta = 0 \tag{2..}$$

Since (1..) and (2..) are identical:

$$\implies \quad x^2 + \frac{bx}{a} + \frac{c}{a} \equiv x^2 - (\alpha+\beta)x + \alpha\beta$$

$$\therefore \quad \alpha + \beta = \frac{-b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

Ex. 1. Write down the quadratic equation whose roots have a sum of 7 & a product of 5.

$$x^2 - (\alpha+\beta) + (\alpha\beta)$$

$$= \quad x^2 - 7x + 7$$

Ex. 2. The roots of the equation $2x^2 + 5x - 1$ are α and β. Find the equation whose roots are $\frac{1}{\alpha}$ & $\frac{1}{\beta}$

$$\alpha + \beta = \frac{-5}{2}$$

$$\alpha\beta = \frac{-1}{2}$$

Sum of roots:

$$\frac{1}{\alpha}+\frac{1}{\beta}$$

$$=\frac{\alpha+\beta}{\alpha\beta}$$

$$=\frac{-5}{2}\div\frac{-1}{2}$$

$$=5$$

Product of roots:

$$\frac{1}{\alpha\beta}$$

$$=\frac{-2}{1}$$

$$=-2$$

$$\therefore \quad f(x)=x^2-5x-2$$

Chapter 7

Logarithms

7.1 Definition

In mathematics, the logarithm is the inverse function to exponentiation. That means the logarithm of a given number x is the exponent to which another fixed number, the base b, must be raised, to produce that number x.

Consider:

$$2^3 = 8$$

3 is the exponent by which 2 must be raised to obtain 8. This statement can also be reversed: 3 is the logarithm by which with a base of 2, results in 8. Thus:

$$3 = \log_2 8$$

In general:

$$a^b = c \iff \log_a c = b, \quad a \in \mathbb{R}^+$$

Furthermore, it is standard to represent $\log_{10}(x)$ as $\log(10)$ and $\log_e(x)$ as $\ln(x)$

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$\log_c(ab) \equiv \log_c a + log_c b$$

$$\log_c\left(\frac{a}{b}\right) \equiv \log_c a - log_c b$$

$$n \log_c a \equiv \log_c a^n$$

7.2 Proofs

Proof 1: $\log_a a = 1$

$$\text{Let} \log_a a = x$$

$$a^x = a$$

$$x = 1$$

Proof 2: $\log_a a = 1$

$$\text{Let} \log_a 1 = x$$

$$a^x = 1$$

$$x = 0$$

Proof 3: $\log_c ab = \log_c a + \log_c b$

$$\text{Let} \log_c a = x \text{ ; Let} \log_c b = y$$

$$\Rightarrow c^x = a \text{ ; } c^y = b$$

$$\Rightarrow ab = c$$

$$\Rightarrow \log_c(ab) = \log_c(x) + \log_c(y)$$

$$\therefore \quad \log_c(ab) = \log_c(a) + \log_c(b)$$

Proof 4: $\log_c \frac{a}{b} = \log_c a - \log_c b$

$$\text{Let} \log_c a = x \text{ ; Let} \log_c b = y$$

$$c^x = a \text{ ; } c^y = b$$

$$\frac{a}{b} = c^x \times c^{-y}$$

$$\log_c \frac{a}{b} = \log_c(x) - \log_c(y)$$

Proof 5: $\log_c a^n = n \log_c a$

$$\text{Let} \log_c a^n = x$$

$$c^x = a^n$$

$$c^{\frac{x}{n}} = a$$

$$\log_c a = \frac{x}{n}$$

$$x = n \log_c a$$

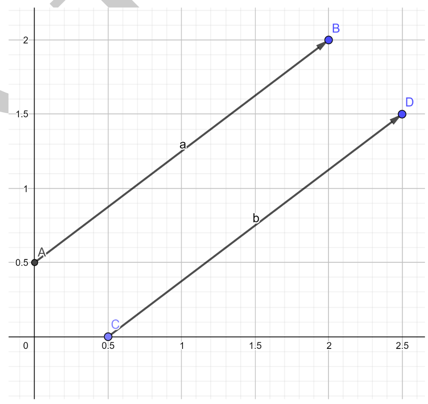
Chapter 8

Vectors

8.1 Definition

A vector quantity is one which has magnitude and direction. For example, length is defined only by size and therefore is a *scalar quantity*. However, acceleration due to gravity, while having a known magnitude, it is also acting in a particular direction. Hence, acceleration is a vector quantity.

Vectors are represented using line segments with arrows to denote their direction. Hence, we may consider vector $\overrightarrow{AB}$.



- Two vectors are s.t.b equal iff they have the same magnitude and act in the same direction.

- The modulus refers to the size of a vector $\underline{a}$ and is denoted by $|*|\, a$.

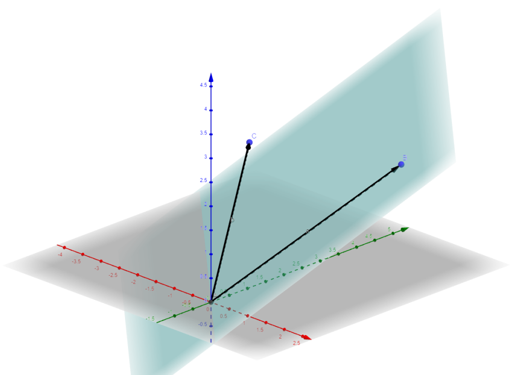
- Two vectors $\underline{a}$ and $-\underline{a}$ are equal in magnitude but opposite in direction. Hence the negative sign indicates opposing direction.

- When a vector is multiplied by a scalar, its magnitude changes. thus $\lambda\underline{a}$ is a vector in the same direction as $\underline{a}$ but has magnitude $\lambda|*|\,\underline{a}$

8.2 Position Vectors and Free Vectors

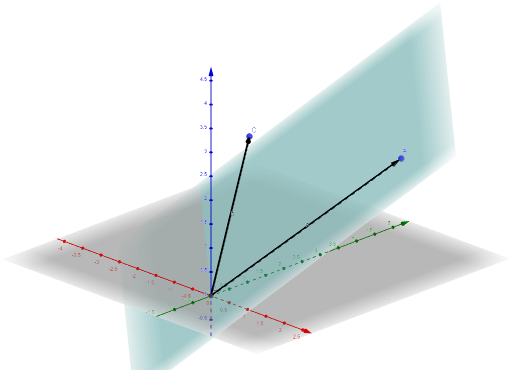
When we refer to a vector $\underline{a}$ we refer to a vector which is not confined to a specific position on the plane or in space. $\underline{a}$, as such, is a *free vector*.

When we refer to the position vector $\underline{b}$, we refer to a vector which is initially set to start from a specific location, usually, the origin.



Suppose that $\underline{a}$ and $\underline{b}$ are non-parallel free vectors, and that the origin is a fixed point. There exists only one plane which contains the point $(0, 0)$, $\underline{a}$ and $\underline{b}$.

Consider another point p on this plane.



- $O\vec{P}$ is the PV [1] of P
- $\overrightarrow{OC}$ is $\parallel$ to $\mathbf{b}$, thus $\overrightarrow{OC} = \lambda\mathbf{b}$
- $\overrightarrow{CP}$ is $\parallel$ to $\mathbf{a}$, thus $\overrightarrow{CP} = \gamma\mathbf{a}$
- $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{PC} = \gamma\mathbf{a} + \lambda\mathbf{b}$

Therefore, the point O and the vectors $\mathbf{a}$ and $\mathbf{b}$ form a frame of reference for the position of any point on the plane. The vectors $\mathbf{a}$ and $\mathbf{b}$ are known as the base vectors. Any vector parallel to the plane can be expressed in the form $\gamma\mathbf{a} + \lambda\mathbf{b}$ where λ and γ are scalar quantities.

This basis spans the usual x and y plane. The base vectors are $\mathbf{i}$ and $\mathbf{j}$ respectively denoting the unit base vector in the positive x direction and similarly in the positive y direction.

In general, given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, the vector $\overrightarrow{AB}$ is given by:

$$\overrightarrow{AB} = (x_2 - x_1)\mathbf{i} - (y_2 - y_1)\mathbf{j}$$

$$|*|\ \overrightarrow{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

[1] Position Vector

8.3 Unit vectors

A unit vector is a vector with magnitude 1. Consider vector $\mathbf{v}$ & let $|*|\,\mathbf{v}$ be its magnitude. A unit vector in the direction of $\mathbf{v}$, denoted by $\hat{\mathbf{v}}$ can be obtained by dividing $\mathbf{v}$ by its magnitude.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|*|\,\mathbf{v}}$$

8.4 Vectors in 3D space

The ideas developed in the previous section can be extended to the 3-dimensional space. Thus any point $A(x_1, y_1, z_1)$ in the 3D space has position vector $\overrightarrow{OA}$.

$$\overrightarrow{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$|*|\,\overrightarrow{OA} = \sqrt{x^2 + y^2 + z^2}$$

Also, given two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\overrightarrow{AB}$ is given by:

$$\overrightarrow{AB} = (x_2 - x_1)\mathbf{i} - (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

$$|*|\,\overrightarrow{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

8.4.1 Direction ratios of a Vector

Given the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, the ratios $a\colon b\colon c$ are called the direction ratios of the vector.

Since the numbers a, b and c define the inclination of the vector in space, it follows that parallel vectors have equivalent direction vectors.

In vectors, parallelism can be of two types:

- Like parallel

  Two like parallel vectors both travel in the same direction.

- Unlike parallel

  Two unlike parallel vectors are still positionally parallel but act in the opposite direction.

In both cases, the direction ratios, however, are the same. For instance, consider the two vectors:

$$\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{b} = -3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}$$

Both of them will simplify to an equivalent ratio when multiplied by -1.

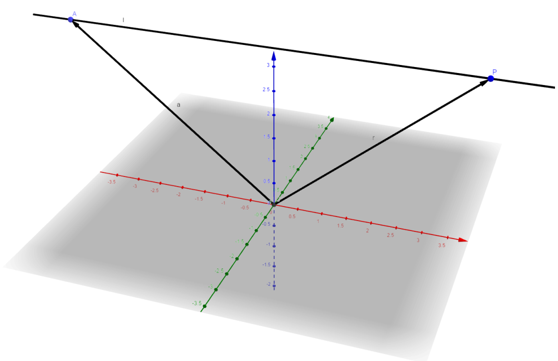
8.4.2 3-Dimensional Lines

Lines in 3D are better explained through vector geometry. Coordinate geometry is sufficient for the analysis of lines in 2D, however, when adding the third dimension, this becomes inefficient.

Equation of a straight line

The previous notion of the definition of a straight line, $y = mx + c$ or more particularly $y_2 - y_1 = m(x_2 - x_1)$ becomes insufficient to define a line on 3 planes of space.

A line in 3D space is defined by a point through which it passes, and a direction to which it is parallel. Consider the following diagram:



Let A be a fixed point on the line l with position vector $\mathbf{a}$ & let P represent a general point on the line. Let the position vector of P be $\mathbf{r}$. Consider also a vector $\mathbf{m}$ parallel to the line l.

We need to find the equation of line l, specifically, a rule defining the line l which governs all points P on this line. To generate the line l as described above, the point P must be such that $\overrightarrow{AP}$ is parallel to $\mathbf{m}$.

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{m}$$

$$x = x_1 + \lambda a$$

$$y = y_1 + \lambda b$$

$$z = z_1 + \lambda c$$

$$x = x_1 + \lambda a$$

$$y = y_1 + \lambda b$$

$$z = z_1 + \lambda c$$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

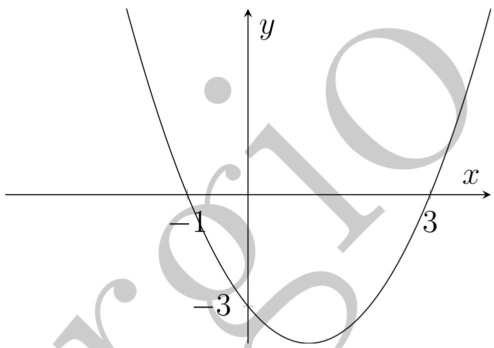
Chapter 9

Inequalities

9.1 Quadratic Inequalities

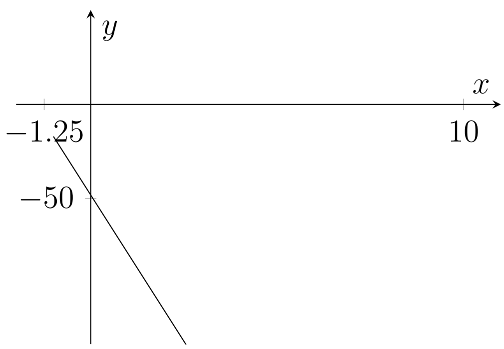
Ex. 1. Solve the inequality $x^2 - 2x > 3$

$$x^2 - 2x > 3$$
$$\implies x^2 - 2x - 3 > 0$$
$$\text{Let } x^2 - 2x - 3 = 0$$
$$\implies (x-3)(x+1) = 0$$



Ex. 2. Solve the inequality $\dfrac{2x^2}{5} \leq \dfrac{7x+10}{2}$

$$4x^2 \leq 35x + 50$$
$$\implies 4x^2 - 35x - 50 \leq 0$$
$$\text{Let } \quad 4x^2 - 35x - 50 = 0$$
$$(4x+5)(x-10) = 0$$



Chapter 10

Series

10.1 Maclaurin Series

10.1.1 Derivation

Let $f(x)$ be nay function of x and suppose that $f(x)$ can be expanded as a series of ascending powers of x and that this series can be differentiated $w.r.t.x$

$$f(x) \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ... + a_rx^r$$
$$\text{where } a_n \text{ are constants to be found}$$

Thus, inputting 0 into $f(x)$ returns:

$$\boxed{f(0) = a_0}$$

Differentiating $f(x)$ $w.r.t.x$:

$$f'(x) \equiv a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + ra_rx^{r-1} + \cdots$$

Inputting 0 into $f'(x)$:

$$\boxed{f'(0) = a_1}$$

Differentiating $f'(x)$ $w.r.t.x$:

$$f''(x) \equiv 2a_2 + 6a_3x + 12a_4x^2 + \cdots + (r-1)(r)a_rx^{r-2} + \cdots$$

Inputting 0 into $f''(x)$:

$$\boxed{f''(0) = 2a_2}$$

Differentiating $f''(x)$ $w.r.t.x$:

$$f'''(x) \equiv 6a_3 + 24a_4x + \cdots + (r-2)(r-1)(r)a_rx^{r-3} + \cdots$$

Inputting 0 into $f'''(x)$:

$$\boxed{f'''(0) = (2)(3)a_3}$$

$$\vdots$$

By the above calculation we can conclude that:

$$\boxed{a_r = \frac{f^r(x)}{r!}}$$

Considering all of the above:

$$f(x) \equiv f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^r(0)x^r}{r!} + \cdots$$

$$\therefore \quad f(x) \equiv \sum_{r=1}^{\infty} \frac{f^r(x)}{r!}$$

This is known as Maclaurin's Theorem, and can be obtained if and only if $f^r(0) \in \mathbb{R}$. In the following examples we use Maclaurin's Theorem to obtain the series expansion of some standard equations. The range of validity of each expansion is left as an exercise to the reader.

10.1.2 Examples

Ex. 1. *Express e^x as a series expansion using the Maclaurin theorem.*

Let $f(x) = e^x$

$$\begin{aligned} f(x) =& e^x \quad \Rightarrow \quad f(0) = 1 \\ f'(x) =& e^x \quad \Rightarrow \quad f'(0) = 1 \\ f''(x) =& e^x \quad \Rightarrow \quad f''(0) = 1 \\ f'''(x) =& e^x \quad \Rightarrow \quad f'''(0) = 1 \end{aligned}$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \ldots$$

Ex. 2. *Express* $\cos x$ *as a series expansion using the Maclaurin theorem.*

$$\begin{aligned} f(x) =& \ \cos(x) \quad \Rightarrow \quad f(0) = 1 \\ f'(x) =& \ \text{-}\sin(x) \quad \Rightarrow \quad f'(0) = 1 \\ f''(x) =& \ \text{-}\cos(x) \quad \Rightarrow \quad f''(0) = 1 \\ f'''(x) =& \ \sin(x) \quad \Rightarrow \quad f'''(0) = 1 \end{aligned}$$

$$\therefore \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r \times \frac{x^{2r}}{2r!} + \ldots$$

The above expansion justifies the fact that when x is very small and thus high powers of x may be neglected, then: $\boxed{\cos x \approx 1 - \frac{x^2}{2}}$

Ex. 3. *Express* $\ln(1+x)$ *as a series expansion using the Maclaurin theorem.*

$$
\begin{aligned}
f(x) &= \ln(1+x) &&\Rightarrow\quad f(0) = 0\\
f'(x) &= (x+1)^{-1} &&\Rightarrow\quad f'(0) = 1\\
f''(x) &= -(1+x)^{-2} &&\Rightarrow\quad f''(0) = -1\\
f'''(x) &= 2(1+x)^{-3} &&\Rightarrow\quad f'''(0) = 2
\end{aligned}
$$

$$\therefore \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \times \frac{x^r}{r} + \ldots$$

Ex. 4. *Expand* $\arcsin(x)$ *up to the term in* x^3*. By putting* $x = \frac{1}{2}$*, find an approximate value for* π

$$
\begin{aligned}
f(x) &= \arcsin(x) &&\Rightarrow\quad f(0) = 0\\
f'(x) &= (1-x^2)^{\frac{-1}{2}} &&\Rightarrow\quad f'(0) = 1\\
f''(x) &= x(1-x^2)^{\frac{-3}{2}} &&\Rightarrow\quad f''(0) = 0\\
f'''(x) &= 3x(1-x^2)^{\frac{-5}{2}} + (1-x^2)^{\frac{-3}{2}} &&\Rightarrow\quad f'''(0) = 1
\end{aligned}
$$

$$\therefore \quad \arcsin(x) = x + \frac{x^3}{3!} + \ldots$$

Putting $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\Longrightarrow \qquad \pi \approx 6\left(\frac{1}{2} + \frac{1}{81}\right)$$

$$\Longrightarrow \qquad \pi \approx \frac{83}{27}$$

10.1.3 Expanding compound functions using standard functions

Ex. 1. Expand *a)* $\dfrac{e^{2x}+e^{-3x}}{e^x}$ *b)* $\ln\left(\dfrac{1-2x}{(1+2x)^2}\right)$ as series of ascending powers of x up to the term in x^4. Give the general term in each case and the range of values of x for which each expansion is valid.

a)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-3x} = 1 + (-3)x + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!} + \dots$$

$$\therefore \quad e^x + e^{-3x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) + \left(1 + (-3)x + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!}\right)$$

$$= 2 - 2x + \frac{10x^2}{2!} - \frac{26x^3}{3!} + \frac{89x^4}{4!}$$

b) $$\ln\left(\frac{1-2x}{(1+2x)^2}\right) = \ln(1-2x) - 2\big(\ln(1+2x)\big)$$

Consider $\ln(1-2x)$:

$$\ln\big(1+(-2x)\big) = -2x + -2x^2 - \frac{8x^3}{3} - 4x^4 + \dots + \frac{(-1)^{r-1}(-2x)^r}{r} + \dots$$

Consider $\ln(1+2x)$:

$$\ln(1+2x) = 2x + -2x^2 + \frac{8x^3}{3} - 4x^4 + \dots + \frac{-2(-1)^{r-1}(2x)^r}{r} + \dots$$

$$\therefore \quad \ln\left(\frac{1-2x}{(1+2x)^2}\right) = \left(-2x + -2x^2 - \frac{8x^3}{3} - 4x^4\right) - 2\left(2x + -2x^2 + \frac{8x^3}{3} - 4x^4\right)$$

$$= -6x + 2x^2 - 8x^3 + 4x^4$$

Rage of Validity:

$$\frac{(-1)^{r-1}(-2x)^r}{r} - \frac{2(-1)^{r-1}(2x)^r}{r}$$

$$= \frac{(-1)^{r-1}(-1)^r(2x)^r + 2(-1)^r(2x)^r}{r}$$

$$= \frac{(-1)^{2r-1}(2x) + 2(-1)^r(2x)^r}{r}$$

$$= \frac{\left((-1)^{2r-1} + 2(-1)^r\right)(2x)^r}{r}$$

$$= \frac{\left(-1 + 2(-1)^r \left(2x)^r\right)\right.}{r}$$

$$= \frac{2^r (2(-1)^r - 1)x^r}{r}$$

Ex. 2. Expand $\ln\left(\dfrac{x+1}{x}\right)$ as series of ascending powers of x up to the term in x^4. Give the general term in each case and the range of values of x for which each expansion is valid.

$$f(x)\ln\left(\frac{x+1}{x}\right) = \ln\left(1 + \frac{1}{x}\right)$$

$$\begin{aligned}
f(x) &= \ln(1 + \frac{1}{x}) & \Rightarrow \quad & f(0) = 0 \\
f'(x) &= (x+1)^{-1} & \Rightarrow \quad & f'(0) = 1 \\
f''(x) &= -(1+x)^{-2} & \Rightarrow \quad & f''(0) = -1 \\
f'''(x) &= 2(1+x)^{-3} & \Rightarrow \quad & f'''(0) = 2
\end{aligned}$$

$$= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \ldots + \frac{(-1)^{r+1}}{r}$$

Ex. 3. Expand $\sin^2 x$ using Maclaurin's series up to x^4

$$sin^2(x) \equiv \frac{1 - \cos(2x)}{2}$$

Consider $\cos(2x)$:

$$= 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + \frac{(-1)^r (2x)^{2r}}{(2r)!} + \cdots$$

$$= 1 - 2x^2 + \frac{2x^4}{3} - \cdots + \frac{(-1)^r (2x)^{2r}}{(2r)!} + \cdots$$

$$\therefore \quad \sin^2(x) \equiv \frac{1}{2}\left(1 - (1 - 2x^2 + \frac{2x^4}{3} - \cdots + \frac{(-1)^r (2x)^{2r}}{(2r)!} + \cdots)\right)$$

$$= \frac{1}{2}\left(1 - 1 + 2x^2 - \frac{2x^4}{3} + \cdots + \frac{(-1)^r (2x)^{2r}}{(2r)!} + \cdots\right)$$

$$= x^2 - \frac{x^4}{3} + \cdots + \frac{(-1)^{r+1} (2x)^{2r}}{(2r)!} + \cdots$$

Ex. 4. Given $e^{2x} \cdot \ln 1 + ax$ find possible values for p and q.

Consider e^{2x}:

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

Consider $\ln(1 + ax)$:

$$\ln(1 + ax) = ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \cdots + \frac{(-1)^{r+1}x^r}{r} + \cdots$$

$$\therefore \quad e^{2x} \cdot \ln(1 + ax) = \left(1 + 2x + 2x^2 + \frac{4x^3}{3}\right)\left(ax - \frac{a^2x^2}{2} + \frac{a^3x^3}{3}\right)$$

$$= ax - \frac{a^2x^2}{2} + \frac{a^3x^3}{3} + 2ax^2 - 2a^2x^3$$

$$= ax - \left(\frac{a^2}{2} + 2a\right)x^2 + \left(\frac{a^3}{3} - 2a^2\right)x^3$$

$$\therefore \quad \left.\begin{aligned} p &= a \\ \frac{a^2}{2} + 2a &= \frac{-3}{2} \\ \frac{a^3}{3} - 2a^2 &= q \end{aligned}\right\}$$

$$p = -3, -1$$

$$q = -27, -\frac{7}{3}$$

10.2 Summation of Series

10.2.1 Method 1: Generating differences

Ex. 1. Simplify $f(r) - f(r+1)$, when $f(x) = \dfrac{1}{r^2}$. Hence, find the sum up to n terms of:

$$\sigma_1 = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots$$

Simplifying $f(r) - f(r+1)$:

$$\begin{aligned} f(r) - f(r+1) &= \frac{1}{r^2} - \frac{1}{(r+1)^2} \\ &= \frac{(r+1)^2 - r^2}{r^2(r+1)^2} \\ &= \frac{2r+1}{r^2(r+1)^2} \end{aligned}$$

Generating series and adding quantitatively equivalent terms:

$$\begin{array}{c} \dfrac{1}{1^2} - \cancel{\dfrac{1}{2^2}} \\ \cancel{\dfrac{1}{2^2}} - \cancel{\dfrac{1}{3^2}} \\ \cancel{\dfrac{1}{3^2}} - \cancel{\dfrac{1}{4^2}} \\ \vdots \\ \cancel{\dfrac{1}{n^2}} - \dfrac{1}{n+1^2} \end{array}$$

$$\boxed{\therefore \quad \sigma_1 = 1 - \frac{1}{n+1^2}}$$

Ex. 2. If $f(r) = r(r+1)!$ simplify $f(r) - f(r-1)$. Hence sum the series:

$$\sigma_1 = 5 \cdot 2! + 10 \cdot 3! + 17 \cdot 4! + ... + (n^2 = 1)n!$$

$$\begin{aligned} f(r) - f(r-1) &= r(r+1)! - (r-1)r! \\ &= r(r+1)r! - (r-1)r! \\ &= r!(r^2 + r - r + 1) \\ &= r!(r^2 + 1) \end{aligned}$$

Generating series and adding quantitatively equivalent terms:

$$\cancel{f(2)} - f(1)$$

$$\cancel{f(3)} - \cancel{f(2)}$$

$$\cancel{f(4)} - \cancel{f(3)}$$

$$\vdots$$

$$\cancel{f(n-1)} - \cancel{f(n-2)}$$

$$f(n) - \cancel{f(n-1)}$$

Ex. 3. If $f(r) = \cos 2r\theta$, simplify $f(r) - f(r+1)$. Hence find $\sin 3\theta + \sin 5\theta + \sin 7\theta$ ________

$$\begin{aligned} f(r) - f(r+1) &= \cos(2r\theta) - \cos(2(r+1)\theta) \\ &= -2\sin\left(\frac{2r\theta + (2r+2)\theta}{2}\right)\cdot\sin\left(\frac{2r\theta - 2(r+1)\theta}{2}\right) \\ &= -2\sin(2r\theta + \theta)\sin(-\theta) \\ &= 2\sin(\theta[2r+1])\sin\theta \end{aligned}$$

Generating series and adding quantitatively equivalent terms:

$$\begin{array}{cccc} r=1 & 2\sin(3\theta)\sin(\theta) & = & f(1)\cancel{-f(2)} \\ r=2 & 2\sin(5\theta)\sin(\theta) & = & \cancel{f(2)-f(3)} \\ r=3 & 2\sin(7\theta)\sin(\theta) & = & \cancel{f(3)-f(4)} \\ \vdots & \vdots & = & \vdots \\ r=n & 2\sin(2n+1)\sin(\theta) & = & \cancel{f(n)} - f(n+1) \end{array}$$

$$\begin{aligned} f(1) - f(n+1) &= 2\sin(\theta)\sin(2n+1) \\ &= \frac{\cos(2\theta) - \cos(2\theta(n+1))}{2\sin(\theta)} \\ &= \frac{2\sin(\theta(2r+1))\sin\theta}{2\sin(\theta)} \\ &= \frac{sin((n=1)\theta)\sin(n\theta)}{\sin(\theta)} \end{aligned}$$

10.2.2 Method 2: Using partial fractions

A special case of the previous method can happen to imply a partial fraction decomposition.

Ex. 1. Decompose $\dfrac{1}{r(r+1)}$. Hence find the sum of

$$\sigma_1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

Decomposing:

$$\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r-1}$$

Generating series and adding quantitatively equivalent terms:

$$\begin{array}{lccc}
r=1 & \dfrac{1}{1} & - & \cancel{\dfrac{1}{2}} \\[2ex]
r=2 & \cancel{\dfrac{1}{2}} & - & \cancel{\dfrac{1}{3}} \\[2ex]
r=3 & \cancel{\dfrac{1}{3}} & - & \cancel{\dfrac{1}{4}} \\[2ex]
\vdots & \cancel{\vdots} & - & \cancel{\vdots} \\[2ex]
r=n & \cancel{\dfrac{1}{n}} & - & \dfrac{1}{n+1}
\end{array}$$

$$\therefore \quad \sigma_1 = 1 - \frac{1}{n+1}$$

Finding convergent value:

$$1 = \lim_{x\to\infty}\left(1 - \frac{1}{n+1}\right)$$

Ex. 2. Find $\displaystyle\sum_{r=3}^{n} \frac{2}{(r-1)(r+1)}$

Consider $\dfrac{2}{(r-1)(r+1)}$:

$$\frac{2}{(r-1)(r+1)} \equiv \frac{1}{r-1} - \frac{1}{r+1}$$

$$\therefore \quad \sum_{r=3}^{n} \frac{2}{(r-1)(r+1)} \quad \equiv \quad \sum_{r=3}^{n} \frac{1}{(r-1)} - \frac{1}{(r+1)}$$

Generating series and adding
quantitatively equivalent terms:

10.2.3 Method 3: Using standard results

10.2.4 Method 4: Comparing to standard results

Chapter 11

Complex Numbers

11.1 Loci

Let $z = x + yi$ where the complex number z is represented on the Argand diagram by the line OP where P is the point (x, y). In general, as x and y are variable P can be anywhere on the Argand diagram. Suppose however that a condition is imposed on z. Consider the case where $|*|\, z = 4$. In this case the position of P is restricted such that the line OP is of a constant length of 4 units. Thus, P can lie anywhere on the circumference of a circle centre $(0, 0)$ with radius 4.

Thus, the locus of P is the circle with equation:

$$x^2 + y^2 = 4^2 \qquad \text{Cartesian form}$$

$$|*|\, z = 4 \qquad \text{Complex form}$$

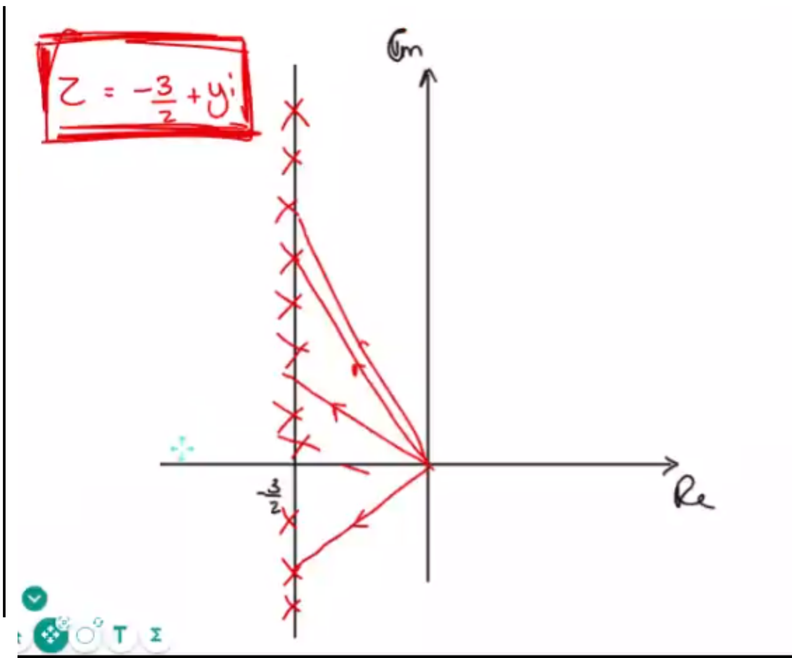
Alternatively, we can say that if $|*|\, z = r^2$

Ex. 1. If $z = x + yi$ and P is the point (x, y) , find the locus of P such that:
a) $|*|\, x - 4 = 5$
b) $|*|\, x + 2 - i = 7$

Ex. 2. Find the locus of z if $|*|\, z - 1 = k|*|\, z + 4$ when $k = 1, k = 3$

Let $z = x + yi$

$$\begin{aligned}
& |*|\, x + yi - 1 = |*|\, x + yi + 4 \\
\implies \quad & |*|\, (x - 1) + yi = |*|\, (x + 4) + yi \\
\implies \quad & \sqrt{(x-1)^2 + y^2} = \sqrt{(x+4)^2 + y^2} \\
\implies \quad & (x-1)^2 + y^2 = (x+4)^2 + y^2 \\
\implies \quad & x^2 - 2x + 1 = x^2 + 8x + 16 \\
\therefore \quad & x = \frac{-3}{2}
\end{aligned}$$



Ex. 3. Given $|*| \dfrac{z-1}{z+4i} = 2$, find the locus of z.

Remembering: $|*| \dfrac{z_1}{z_2} = \dfrac{|*| \, z_1}{|*| \, z_2}$

Let $z = x + yi$

$$|*| \frac{z-1}{z+4i} = 2$$

$$\implies \quad |*| \, z - 1 = 2|*| \, z + 4i$$

$$\implies \quad |*| \, (x-1) + yi = 2|*| \, x + (y+4)i$$

$$\implies \quad \sqrt{(x-1)^2 + y^2} = 2\sqrt{x^2 + (y+4)^2}$$

$$\implies \quad \sqrt{(x-1)^2 + y^2} = 2\sqrt{x^2 + (y+4)^2}$$

Ex. 4. Given that $z_A = \dfrac{1}{10}(-1+i)$ and $z_B = -\dfrac{1}{500}(11+127i)$, find in the form a+bi the complex numbers $\dfrac{z_A}{z_B}$. If $P(x,y)$ is the point on the Argand diagram representing $z = x + yi$, determine the equation of the locus of P where $|*| \, z - z_A = |*| \, z - z_B$.

$$\frac{z_A}{z_B} = \frac{1}{10}(-1+i) \div -\frac{1}{500}(11+127i)$$

$$= -5 \cdot \frac{-1+i}{11+127i}$$

$$= \frac{5-5i}{11+127i} \cdot \frac{11-127i}{11-127i}$$

$$= a + bi$$

$$\therefore \quad a = -\frac{58}{1625}, \quad b = -\frac{69}{1625}$$

$$|*| \, z - z_A = |*| \, z - z_B$$

Ex. 5. If the real part of $\frac{z+2}{z+2i}$ is equal to 1, show that the point z lies on a straight line. Hence find the point z_0 on this line such that $|*|\, z_0 = \sqrt{2}$. Find also the quadratic equation with real coefficients which has z_0 as one of the roots.

Let $z = x + yi$

Showing that z lies on a straight line:

$$Re(\frac{z+2}{z+2i}) = 1$$

$$\implies \quad 1 = Re(\frac{x+2+yi}{x+2i+yi})$$

$$\implies \quad 1 = Re\left(\frac{(x+2)+yi}{x+(y+1)i} \cdot \frac{x-(y+1)i}{x-(y+1)i}\right)$$

$$\implies \quad 1 = Re\left(\frac{x(x+2) - i(y+2)(x+2) + iyx + y(2+y}{x^2+(y+2)^2}\right)$$

$$\implies \quad 1 = \frac{x(x+2)+y(y+2)}{x^2+(y+2)^2}$$

$$\implies \quad x(x+2)+y(y+2) = x^2+y^2+4y+4$$

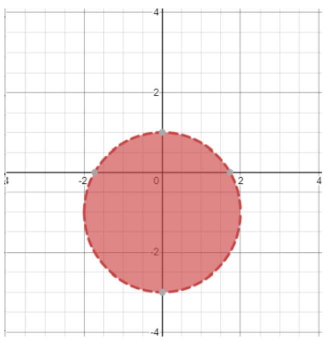
$$\implies \quad 2y = 2x-4$$

$$\therefore \quad y = x-2 \quad \square$$

Point z lies on the straight line $y = x - 2$

Ex. 6. Shade on an Argand diagram the area represented by $|*|\, z + i < 4$.

Finding the loci:



$$|*|\, z+i = |*|\, x+(y+1)i$$

$$= \sqrt{x^2+(y+1)^2}$$

$$= \sqrt{(x-0)^2+(y-(-1))^2}$$

is the distance from point (0,-1) to (x,y)

Finding $z_0 = (a,b) = a + bi$:

$$z_0 = a + bi$$

$$\implies \quad |*|\, z_0 = \sqrt{a^2+b^2}$$

Given: $|*|\, z_0 = \sqrt{2}$

$$\implies \qquad \sqrt{a^+b^2} = \sqrt{2}$$

$$\implies \qquad a^2 + b^2 = 2 \tag{1}$$

We also know z_0 is on the line $y = x - 2$

$$\therefore b = a - 2 \tag{2}$$

Substituting 2 in 1:

$$a^2 + (a-2)^2 = 2$$

$$\implies \quad 2a^2 - 4a + 4 = 2$$

$$\implies \quad a^2 - 2a + 1 = 0$$

$$\therefore \quad z_0 = 1 - i \quad \square$$

Finding quadratic equation with roots: $z_0, \bar{z_0}$

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\implies x^2 - (1 - i + 1 + i)x + (1 - i)(1 + i) = 0$$

$$\implies x^2 - 2x + 2 = 0 \quad \square$$

Chapter 12

Integration

12.1 Reduction Formulæ

Integrating using a reduction formula is in essence repeating integration by parts over and over again.

We can think of the process of finding a reduction formula for a given integral as a *recursive approach* to integration by parts. By listing all the iterations of $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$, more specifically the $\int v\frac{du}{dx}\,dx$ part in terms of I_n where n is the *iterative index*, or the *step number*, if you will.

As expected, finding this recursively valid form is not as direct, and thus, the exponent has to be *split* in such a way that trigonometric identities can be used.

Ex. 1. If $I_n = \displaystyle\int \cos^n x\,dx$ show that $I_n = \dfrac{1}{n}\sin\cos^{n-1} x + \dfrac{n-1}{n}\cdot I_{n-2}$. Hence find $\displaystyle\int \cos^5 x\,dx$.

$$\begin{aligned}
I_n &= \int \cos^n x\,dx \\
&= \int \cos x \cdot \cos^{n-1} x\,dx \\
\therefore\quad I_n &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx \\
&= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx \\
&= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx \;-\; (n-1)\int \cos^n x\,dx \\
&= \cos^{n-1} x\sin x + (n-1)I_{n-2} - (n-1)I_n \\
I_n + (n-1)I_n &= \cos^{n-1} x\sin x + (n-1)I_{n-2} \\
\Longrightarrow\; nI_n &= \cos^{n-1} x\sin x + (n-1)I_{n-2} \\
\Longrightarrow\; I_n &= \frac{1}{n}\cos^{n-1} x\sin x + \left(\frac{n-1}{n}\right) I_{n-2}
\end{aligned}$$

$$\int \cos^5 x\,dx = I_5$$

$$I_5 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_3$$

$$I_3 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_1$$

$$I_1 = \int \cos x\,dx = \sin x + k$$

$$\therefore \quad \int \cos^5 x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}(\sin x + k)\right)$$

$$= \frac{1}{5}\cos^2 x \cdot \sin x + \frac{4}{15}\cos^2 x \cdot \sin x + \frac{8}{15}\sin x + c \quad \square$$

Ex. 2. If $I_n = \displaystyle\int \tan^n \theta\,d\theta$, find a reduction formula for I_n and use it to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6 \theta\,d\theta$.

$$I_n = \int \tan^n \theta\,d\theta$$

$$= \int \tan^2 \theta \tan^{n-2} \theta\,d\theta$$

$$= \int (\sec^2 \theta - 1) \tan^{n-2} \theta\,d\theta$$

$$= \int \sec^2 \theta \tan^{n-2} \theta\,d\theta - \int \underbrace{\tan^{n-2} \theta\,d\theta}$$

$$= \frac{\tan^{n-1} \theta}{n-1} - \underbrace{I_{n-2}}$$

$$\int_0^{\frac{\pi}{4}} \tan^6 \theta\,d\theta = I_6 \Big|_0^{\frac{\pi}{4}}$$

$$I_6 = \frac{tan^5 \theta}{5} - I_4$$

$$I_4 = \frac{\tan^3 \theta}{3} - I_2$$

$$I_2 = \tan \theta - I_0$$

$$I_0 = \int 1\,d\theta = \theta + k$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^6 \theta\,d\theta = \frac{\tan^6 \theta}{5} - \frac{\tan^3 \theta}{3} + \tan \theta - \theta \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$$

$$= \frac{13}{15} - \frac{\pi}{4} \quad \square$$

Ex. 3. Establish a reduction formula that could be used to find $\int x^n e^x \, dx$ and use it to find $\int x^4 e^4$.

$$\text{Let } I_n = \int x^n e^x \, dx$$

$$\text{Let } u = x^n \qquad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = e^x$$

$$\therefore \quad I_n = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - n \, I_{n-1}$$

$$\int x^4 e^x = I_4$$

$$I_4 = x^4 e^x - 4I_3$$

$$I_3 = x^3 e^x - 3I_2$$

$$I_2 = x^2 e^x - 2I_1$$

$$I_1 = xe^x - I_0$$

$$I_0 = e^x + k$$

$$\therefore \quad I_4 = x^4 e^x - 4(x^3 e^x - 3(x^2 e^x - 2(xe^x - e^x + k)))$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + c \quad \square$$

Ex. 4. Establish a reduction formula which can be used to evaluate $\int x^n \sin x \, dx$.

$$\text{Let } I_n = \int x^n \cdot \sin x$$

$$= -x^n \cos x + \int nx^{n-1} \cos x \, dx$$

$$= -x^n \cos x + n \left(x^{n-1} \sin x - \int (n-1) x^{n-2} \sin x \, dx \right)$$

$$= -x^n \cos x + n \left(x^{n-1} \sin x - (n-1) \int x^{-2} \cdot x^n \sin x \, dx \right)$$

$$= -x^n \cos x + n \left(x^{n-1} \sin x - (n-1) \underbrace{x^{n-2} \sin x \, dx} \right)$$

$$\therefore \quad I_n = -x^n \cos x + n \left(x^{n-1} \sin x - (n-1) I_{n-2} \right)$$

$$= -x^n \cos x + nx^{n-1} \sin x - n(n-1) I_{n-2} \quad \square$$

Ex. 5. Establish a reduction formula to find $\int \csc^n x\,dx$. Hence find $\int csc^5 x\,dx$

$$\begin{aligned}
\text{Let } I_n &= \int \csc^n x\,dx \\
&= \int \csc^2 x \cdot \csc^{n-2} x\,dx \\
\text{Let } u &= \csc^{x-2} x \qquad \frac{dv}{dx} = \csc^2 x\,dx \\
\frac{du}{dx} &= -(n-2)\csc^{n-2}\cot x v \qquad = -\cot x \\
\therefore \int \csc^n x\,dx &= -\cot x \cdot \csc^{n-2} x - (n-2)\int \csc^{n-2} x \cot^2 x\,dx \\
I_n &= -\cot x \cdot \csc^{n-2} x - (n-2)\int \csc^{n-2} x \left(\csc^2 x - 1\right)\,dx \\
&= -\cot x \cdot \csc^{n-2} x - (n-2)\int \csc^n x\,dx + (n-2)\int \csc^{n-2} x dx \\
&= -\cot x \cdot \csc^{n-2} x - (n-2)\,I_n + (n-2)I_{n-2} \\
I_n + nI_n - 2I_n &= -\cot x \cdot \csc^{n-2} x + (n-2)I_{n-2} \\
(n-1)I_n &= -\cot x \cdot \csc^{n-2} x + (n-2)I_{n-2} \\
I_n &= \frac{-1}{n-1} - \cot x \cdot \csc^{n-2} x + \frac{n-2}{n-1} I_{n-2} \\
&= \left(1 - \frac{1}{n-1}\right) I_{n-2} - \frac{\cot x \csc^{n-2} x}{n-1} \quad \square
\end{aligned}$$

Ex. 6. Show that if $I_n - \int_0^\pi x^n \sin x\,dx$, then $I_n = \pi^n - n(n-1)\,I_n - 1$. Hence evaluate $\int_0^\pi \sin x\,dx$

$$\begin{aligned}
I_n &= [-x^n \cos x]_0^\pi + n\int_0^\pi x^{n-1}\cos x\,dx \\
&= \pi^n + n\int_0^\pi x^{n-1}\cos x\,dx \\
&= \pi^n \int_0^\pi
\end{aligned}$$

$$\text{Let } u =$$

Ex. 7. Show that, if $I_n = \int_0^1 x^n e^{x^3}\,dx$, then $I_n = \frac{e}{3} - \frac{n-2}{3} \cdot I_{n-3}$

$$\begin{aligned}
I_n &= \int_0^1 x^n e^{x^3}\,dx \\
&= \int_0^1 x^{n-2} x^2 e^{x^3}\,dx \\
\therefore \quad I_n &= \left[\frac{x^{n-2}e^{x^3}}{3}\right]_0^1 - \frac{n-2}{3}\int_0^1 x^{n-3}e^{x^3}\,dx \\
&= \frac{e}{3} - \frac{n-2}{3} \cdot I_{n-3}
\end{aligned}$$

Ex. 8. Show that, if $I_n = \int_0^1 x^n (1+x^5)^4\,dx$, then $I_n = \frac{1}{n+21}\left[32 - (n-4) \cdot I_{n-5}\right]$

$$\begin{aligned}
I_n &= \int_0^1 x^n (1+x^5)^4\,dx \\
&= x^{n-4}x^4(1+x^5)^4\,dx \\
&= \left[x^{n-4}\frac{(1+x^5)^5}{25}\right]_0^1 - \frac{n-4}{25}\int x^{n-5}(1+x^5)^5\,dx \\
&= \frac{32}{25} - \frac{n-4}{25}\int_0^1 x^n - 5(1+x^5)(1+x^5)^4\,dx \\
&= \frac{32}{25} - \frac{n-5}{25}\int_0^1 x^{n-5}(1+x^5)^4\,dx - \frac{n-4}{25}\int_0^1 x^n(1+x^5)^4\,dx \\
&= \frac{32}{25} - \left(\frac{n-4}{25}\right) I_{n-5} - \left(\frac{n-4}{25}\right) I_n \\
25I_n &= 32 - (n-4)I_{n-5} - \left(\frac{n-4}{25}\right) I_n \\
25I_n + nI_n - 4I_n &= 32 - (n-4)\,I_{n-5} \\
(n+21)I_n &= 32 - (n-4)I_{n-5} \\
I_n &= \frac{1}{n+21}\left(32 - (n-4)I_{n-5}\right)
\end{aligned}$$

Ex. 9. Given $I_n = \displaystyle\int_0^1 (1+x^2)^{-n}\,dx$, show that $2n\,I_{n+1} = 2^{-n} + (2n-1)\,I_n$

$$
\begin{aligned}
I_n &= \int_0^1 (1+x^2)^{-n}\,dx \\
&= \int_0^1 (1+x^2)^{-n}\cdot 1\,dx \\
\therefore \quad I_n &= -2nx^2(1+x^2)^{-(n+1)}\Big|_0^1 + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,dx \\
&= 2^{-n} + 2n\int_0^1 (x^2+1-1)(1+x^2)^{-(n+1)}\,dx \\
&= 2^{-n} + 2n\int_0^1 (1+x^2)^{-2}\,dx - 2n\int_0^1 (1+x^2)^{-(n+1)}\,dx \\
&= 2^{-n} + 2n\,I_n - 2n\,I_{n+1} \\
2n\,I_{n+1} &= 2^{-n} + (2n-1)\,I_n
\end{aligned}
$$

Chapter 13

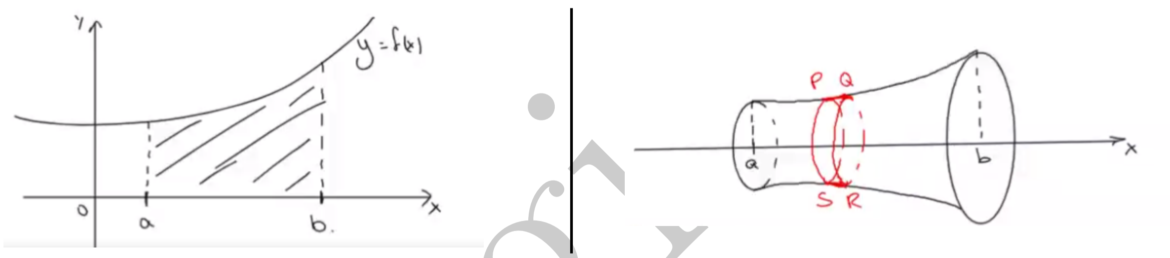
Applications of Calculus

13.1 Integrals

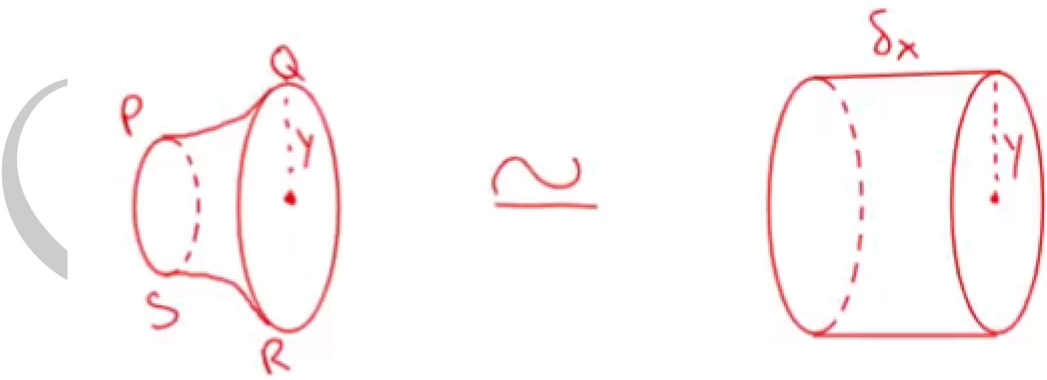
13.1.1 Volume of Revolution

If part of a curve and the area underneath is rotated about a straight line, the solid formed is called a solid of revolution.

Consider the graph $y = f(x)$ and suppose we rotate the part of the curve from $x = a$ to $x = b$ about the x-axis. If the shaded area of $f(x)$ is notated about the x-axis the following shape is formed:



Suppose that the solid formed is cut into sections as shown. Let $PQRS$ be a typical section. If the cuts are reasonably close to each other, $PQRS$ approximates a cylinder with height δx and radius y as shown below:



The volume, δv, of $PQRS$ is given by:

$$\delta V \simeq \pi y^2 \delta x$$

Thus, the volume V of solid $PQRS$ is given by:

$$V \simeq \sum_{x=a}^{b} \pi y^2 \delta x$$

This summation approaches V as $\delta x \to 0$

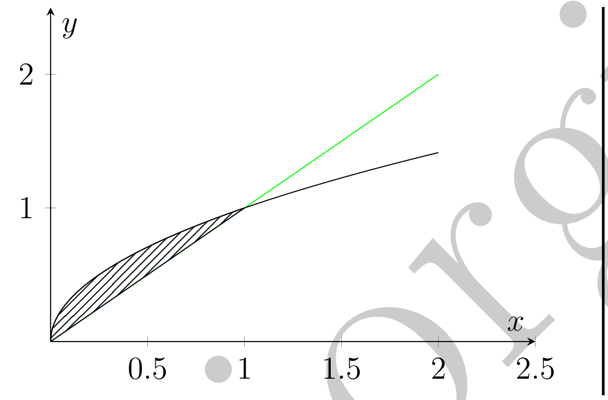
$$V = \lim_{\delta x \to 0} \sum_{x=a}^{b} \pi y^2 \delta x$$

$$\boxed{V = \int_a^b \pi y^2 \, dx}$$

Ex. 1. Find the volume generated when the area between $y = e^x$, the x-axis

$$V = \int_0^1 \pi y \, dx$$

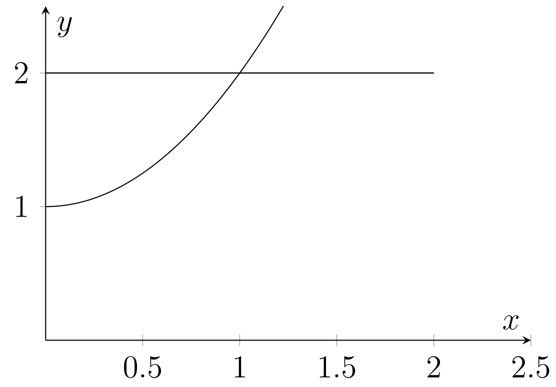
Ex. 2. Find the volume generated when the area defined by the inequalities $y \leq x^2$ and $y \geq x$ is rotated about the x-axis.



$$\begin{aligned} V_{e-c} &= V_e - V_c \\ &= \pi \int_0^1 \sqrt{x}^2 \, dx - \pi \int_0^1 x^2 \, dx \end{aligned}$$

Ex. 3. Find the volume generated when the area in the first quadrant bounded by the circle $x = 4\cos\theta, y = 4\sin\theta$ rotates completely about the x-axis.

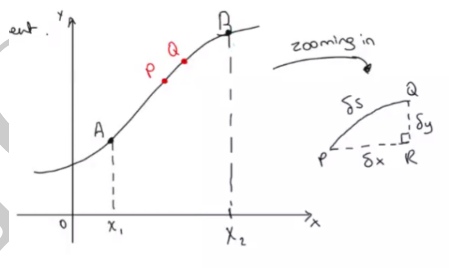
Ex. 4. Find the volume when the region defined by $y \geq x^2 + 1, \quad x \geq 0$ and $y \leq 2$ is rotated about the y-axis.



13.1.2 Length of an arc of a curve

To find the length of an arc of a curve we use the method of summing small elements of one length.
Suppose that arc PQ, of length δs, is such an element. Then, the length S of the curve AB is given by:

$$\sum_{x=x_1}^{x_2} \delta S$$



As δs is very small, it can be approximated to the hypothenuse of the $\triangle PQR$. Thus:

$$(\delta x)^2 + (\delta y)^2 \simeq (\delta s)^2$$

$$\implies \quad 1 + \left(\frac{\delta y}{\delta x}\right)^2 \simeq \left(\frac{\delta S}{\delta x}\right)^2$$

$$\implies \qquad \delta S \simeq \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

$$\therefore \qquad S \simeq \sum_{x=x_1}^{x_2} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)}\,\delta x$$

As $\delta x \leftarrow 0, (\dfrac{\delta y}{\delta x}) \leftarrow \dfrac{dy}{dx}$ and $s = \lim\limits_{\delta x \to 0} \sum\limits_{x=x_1}^{x_2} \sqrt{1 + \left(\dfrac{\delta y}{\delta x}\right)^2}\,\delta x$

$$\boxed{\therefore \quad S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Knowing the Cartesian equation of the curve, the necessary integration can be carried out.

Let us now consider a curve S parametrically in terms of t.
We can use again :

$$(\delta s)^2 \simeq (\delta x)^2 + (\delta y)^2$$

$$\Longrightarrow \left(\frac{\delta s}{\delta t}\right)^2 \simeq \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

$$\Longrightarrow \qquad \delta s \simeq \sqrt{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2}\,\delta t$$

as $\delta t \to 0$; $\quad \dfrac{\delta x}{\delta t} \to \dfrac{dx}{dt}$ and $\left(\dfrac{\delta y}{\delta t}\right) \to \dfrac{dy}{dt}$

$$\therefore S = \lim_{\delta t \to 0} \sum_{t=t_1}^{t_2} \sqrt{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2}\,\delta t$$

$$\boxed{S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

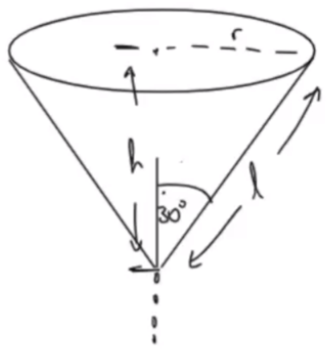
13.2 Rates of change

The notation $\dfrac{dy}{dx}$ denotes the rate of change of y w.r.t.x. Suppose that x and y are quantities a such as length and volume respectively. Then $\dfrac{dy}{dx}$ denotes the rate of change of volume w.r.t some length. In this section we use differentiation to deal with practical problems involving rates of change.

Ex. 1. A spherical balloon is blown up so that its volume increases at a constant rate of $2\text{cm}^3/s$. Find the rate of increase of the radius when the volume of the balloon is 50cm^3

$$\frac{dV}{dt} = 2$$

Ex. 2. A container with water is in the form of an inverted hollow cone with a iaerrentical angle of 30°. Water drips out from the vertex at the rate $3\text{cm}^2/s$. Find the rate at which the surface area in contact with the water is changing when there are $8\pi\text{cm}^3$ of water remaining in the cone.



$$\frac{dV}{dt} = -3$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi r^2 \sqrt{3} r$$

$$V = \frac{\sqrt{3}}{3}\pi r^3$$

$$\frac{dV}{dt} = \sqrt{3}\pi r^2$$

Required derivative: $\frac{dA}{dt}$

$$A = \pi r l$$