Pure Mathematics
Advanced Level

"Once your soul has been enlarged by a truth, it can never return to its original size."
-Blaise Pascal

Notes By

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2020

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Chapter 1

Integration

1.1 Reduction Formulæ

Ex. 1. If $I_n = \displaystyle\int \cos^n x\,dx$ show that $I_n = \dfrac{1}{n}\sin\cos^{n-1} x + \dfrac{n-1}{n}\cdot I_{n-2}$. Hence find $\displaystyle\int \cos^5 x\,dx$.

$$\begin{aligned}
I_n &= \int \cos^n x\,dx \\
&= \int \cos x \cdot \cos^{n-1} x\,dx \\
\therefore \quad I_n &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \;-\; (n-1)\int \cos^n x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n \\
I_n + (n-1)I_n &= \cos^{n-1} x \sin x + (n-1)I_{n-2} \\
\implies nI_n &= \cos^{n-1} x \sin x + (n-1)I_{n-2} \\
\implies I_n &= \frac{1}{n}\cos^{n-1} x \sin x + \left(\frac{n-1}{n}\right) I_{n-2}
\end{aligned}$$

$$\int \cos^5 x\,dx = I_5$$

$$I_5 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_3$$

$$I_3 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_1$$

$$I_1 = \int \cos x\,dx = \sin x + k$$

$$\therefore \quad \int \cos^5 x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}(\sin x + k)\right)$$

$$= \frac{1}{5}\cos^2 x \cdot \sin x + \frac{4}{15}\cos^2 x \cdot \sin x + \frac{8}{15}\sin x + c \quad \square$$

Ex. 2. If $I_n = \displaystyle\int \tan^n \theta\,d\theta$, find a reduction formula for I_n and use it to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6 \theta\,d\theta$.

$$\begin{aligned} I_n &= \int \tan^n \theta\,d\theta \\ &= \int \tan^2 \theta \tan^{n-2}\theta\,d\theta \\ &= \int (\sec^2\theta - 1)\tan^{n-2}\theta\,d\theta \\ &= \int \sec^2\theta \tan^{n-2}\theta\,d\theta - \int \underbrace{\tan^{n-2}\theta\,d\theta}_{} \\ &= \frac{\tan^{n-1}\theta}{n-1} - \underbrace{I_{n-2}}_{} \end{aligned}$$

$$\int_0^{\frac{\pi}{4}} \tan^6\theta\,d\theta = I_6\Big|_0^{\frac{\pi}{4}}$$

$$I_6 = \frac{tan^5\theta}{5} - I_4$$

$$I_4 = \frac{\tan^3\theta}{3} - I_2$$

$$I_2 = \tan\theta - I_0$$

$$I_0 = \int 1\,d\theta = \theta + k$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^6\theta\,d\theta = \frac{\tan^6\theta}{5} - \frac{\tan^3\theta}{3} + \tan\theta - \theta\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$$

$$= \frac{13}{15} - \frac{\pi}{4}$$

Ex. 3. Establish a reduction formula that could be used to find $\displaystyle\int x^n e^x\,dx$ and use it to find $\displaystyle\int x^4 e^4$.

$$\text{Let } I_n = \int x^n e^x \, dx$$

$$\text{Let } u = x^n \qquad \frac{dv}{dx} = e^x$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = e^x$$

$$\therefore I = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - n I_{n_1}$$

Ex. 4. Establish a reduction formula which can be used to evaluate $\int x^n \sin x \, dx$.

$$\text{Let } I_n = \int x^n \cdot \sin x$$

$$\text{Let } u = x^n \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = nx^{n-1} \qquad v = -\cos x$$

Ex. 5. Establish a reduction formula to find $\int \csc^n x \, dx$. Hence find $\int csc^5 x \, dx$

$$\text{Let } I_n = \int \csc^n x \, dx$$

$$= \int \csc^2 x \cdot \csc^{n-2} x \, dx$$

$$\text{Let } u = \csc^{x-2} x \qquad \frac{dv}{dx} = \csc^2 x \, dx$$

$$\frac{du}{dx} = -(n-2) \csc^{n-2} \cot x v \qquad = -\cot x$$

$$\therefore \int \csc^n x \, dx = -\cot x \cdot \csc^{n-2} x - (n-2) \int \csc^{n-2} x \cot^2 x \, dx$$

$$I_n = -\cot x \cdot \csc^{n-2} x - (n-2) \int \csc^{n-2} x \left(\csc^2 x - 1\right) \, dx$$

$$= -\cot x \cdot \csc^{n-2} x - (n-2) \int \csc^{n-2} x \left(\csc^2 x - 1\right) \, dx$$