Pure Mathematics
Advanced Level

"Once your soul has been enlarged by a truth, it can never return to its original size."
-Blaise Pascal

Notes By

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1 | Matrices II

Determinant of a 3x3 Matrix

The determinant of a 3x3 matrix is calculated by extracting a row of 2x2 determinants from the given 3x3 matrix. These 2x2 determinants are referred to as minors.

Ex. 1. Find the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & -3 \\ 4 & 5 & 6 \end{pmatrix}$

(Parentheses about vector product)

Consider the vectors $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and b $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$.

Since vector product is distributive across addition:

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \times (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\
&+ \cancel{x_1x_2(\mathbf{i} \times \mathbf{i})} + x_1y_2(\mathbf{i} \times \mathbf{j}) + x_1z_2(\mathbf{i} \times \mathbf{k}) \\
&+ y_1x_2(\mathbf{j} \times \mathbf{i}) + \cancel{y_1y_2(\mathbf{j} \times \mathbf{j})} + y_1z_2(\mathbf{j} \times \mathbf{k}) \\
&+ z_1x_2(\mathbf{k} \times \mathbf{i}) + z_1y_2(\mathbf{k} \times \mathbf{j}) + \cancel{z_1z_2(\mathbf{k} \times \mathbf{k})} \\
&= x_1y_2\mathbf{k} - x_1z_2\mathbf{j} - y_1x_2\mathbf{k} + y_1z_2\mathbf{i} + z_1x_2\mathbf{j} - z_1y_2\mathbf{i} \\
&= (y_1z_2 - z_1y_2)\mathbf{i} - (x_1z_2 - z_1x_2)\mathbf{j} + (x_1y_2 - y_1x_2)\mathbf{k}
\end{aligned}
$$

Some properties of determinant

The value of the determinant is unaltered if all the rows and columns of a given matrix are interchanged.
If the above happens, the resulting determinant will be of opposite sign.
If one row/column of a determinant D is multiplied by λ, the resulting determinant is equal to λD

The Inverse of a 3x3 matrix

Matrix of cofactors

Consider the matrix $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

For each element of any given matrix $\mathbf{A}$:

- Ignore the values of the current row and column.
- Calculate the determinant of the remaining values.
- Apply alternating signs starting from $+$.
- Input this determinant into a new matrix

1 The result would be the below matrix referred to as *the matrix of co-factors*.

$$\mathbf{C} = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & g \\ f & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \\ -\begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} d & g \\ f & i \end{vmatrix} & -\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \\ \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & g \\ f & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \end{bmatrix}$$

Adjugate matrix

The next step of finding the inverse matrix of $\mathbf{A}$ is finding the *adjugate* matrix of $\mathbf{A}$. This is done by obtaining the *transpose* of the matrix of cofactors, in our case, $\mathbf{C}$.

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} e & f \\ h & i \end{vmatrix} \\ \\ -\begin{vmatrix} d & g \\ f & i \end{vmatrix} & \begin{vmatrix} d & g \\ f & i \end{vmatrix} & -\begin{vmatrix} d & g \\ f & i \end{vmatrix} \\ \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \end{bmatrix}$$

Inverse matrix

Let the inverse of a matrix A be A^{-1}. It is defined as a matrix of the same size of A such that

$$AA^{-1} = A^{-1}A = I$$

This same matrix A^{-1} is defined more particularly as

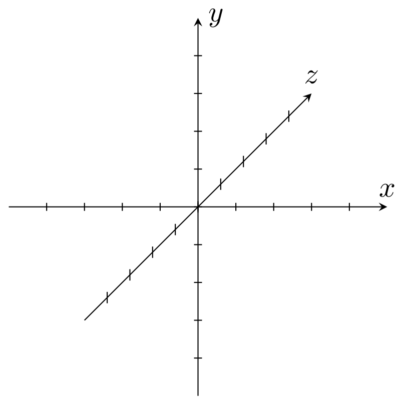
$$A^{-1} = \frac{1}{\det A} \quad \text{adj } A$$

Ex. 1. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix}$

Ex. 2. Solve using the inverse matrix method the system of equations:
$x + y + z = 7$
$x - y + 2z = 9$
$2x + y - z = 1$

Transformation Matrices in 3D

Reflection along the xy plane



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Rotation along the y-axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotation along the z-axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation along the z-axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Enlargement

When we enlarge (or conversely, reduce) by scale factor n, the unit base vector is multiplied by n. Thus the matrix representing an enlargement by scale factor n is given by:

$$n\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}n & 0 & 0\\0 & n & 0\\0 & 0 & n\end{bmatrix}$$

Geometric Interpretation of the Determinant

Consider an object with volume V transformed by the matrix $\mathbf{A}$ with determinant $|\mathbf{A}|$. The volume of the image is given by $|\mathbf{A}|V$. Thus, the determinant of the transformation matrix denotes the number of times by which th volume of the object increases or decreases.

Ex. 1. Describe the effect on volume in 3D space of the transformation given by $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\5 & 0 & -1\\2 & 4 & -3\end{bmatrix}$

Ex. 2. Find the images of $P(4,5,1)$, $Q(3,-1,-2)$, $R(6,-2,0)$ under the transformation given by $\mathbf{A}=\left(\begin{smallmatrix}4 & 3 & 2\\-1 & 5 & 0\\6 & 2 & -3\end{smallmatrix}\right)$

Ex. 3. Find the equation of the line which is the image of the line $\mathbf{r}=3\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}-5\mathbf{k})$ under the transformation given by $\mathbf{A}=\begin{bmatrix}2 & 3 & 1\\-1 & 2 & 4\\0 & 6 & 1\end{bmatrix}$

Find two points on the line. Find images of points. Find equation of the image line

Ex. 4. Find the image of the plane $x+2y-7z=2$ under the transformation defined by $\mathbf{A}=\begin{smallmatrix}-1 & 2 & 1\\-3 & 1 & 4\\0 & 1 & 2\end{smallmatrix}$

2 | Permutations and Combinations

Consider n objects from which r are to be arranged in a particular order, where $r \leq n$. The number of *permutations* of which r objects from a total of n refers to the number of ways in which these r objects can be arranged, where the order of arrangement matters.

Let us consider 3 letters {A, B, C}. There are 6 possible arrangements, or *permutations*, of these letters, namely:

$$\{\{A, B, C\}, \{A, C, B\}, \{B, A, C\}, \{B, C, A\}, \{C, B, A\}, \{C, A, B\}\}$$

In general, the number of permutations of r objects from a total of n is denoted by ${}^{n}P_{r}$ defined as

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Ex. 1. Consider the set of letters $\{A, B, C, D, E\}$.
(a) How many of these arrangements start with a vowel?

Ex. 2. Consider the set of numbers $\{1, 2, 3, 4, 5, 6\}$.
(a) In how many ways can a 4 digit number be formed from the above set?
(b) How many of these numbers are even?
(c) How many of these numbers are greater than 3000?

Ex. 3. Consider the set of numbers $\{1, 2, 3, 4, 5, 0\}$.
(a) How many 3 digit numbers can be formed?
(b) How many of these numbers are even?
(c) How many of these numbers are greater than 400?
(d) How many even numbers can be formed?

a. $5 \times 5 \times 4 = 100$

b. $5 \times 4 \times 1 = 20$

c. $5 \times 5 \times 4 = 100$

d. $5 \times 5 \times 4 = 100$

Permutations with Identical Objects

The above method, however, does not suffice in the case that we have identical objects. Suppose we have the set $\{S, E_1, E_2\}$. For every time t the repeated element is present in the given set, we have to divide the total we have to divide by t!

Ex. 1. In how many ways can the letters of the word 'MALTA' be arranged?

$$\frac{5!}{2!}$$

Ex. 2. In how many ways can the letters of the word 'ILLUSTRATIONS' be arranged?

Since the letters $\{I, L, S, T\}$ are repeated twice, the total possibilities have to be divided by the factorial of the number of each recurring letter. (*i.e., divide by* 2! *for the two 'I's, by* 2! *for the two 'L's ...*)

$$\frac{13!}{2!\,2!\,2!\,2!}$$

Circular Permutations

In a particular field of mathematics referred to as group theory, a cyclic permutation is a permutation of the elements of some set X which maps the elements of some subset S of X to each other in a cyclic fashion, while fixing all other elements of X. In other words, this is the number of ways in which a set can be permuted whilst omitting identical cycles.

Ex. 1. In how many ways can 6 people be seated at a round table?

Ex. 2. In how many ways can 4 couples be arranged around a table?
 (a) In how many of these arrangements are all the males separated? (*i.e., no male sits next to another*)