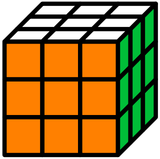
Rubik's Cubes

with a sprinkle of Combinatorics and Group Theory

Giorgio Grigolo

12 January, 2022



MALTA MATHEMATICAL SOCIETY

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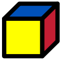
- $8 \times$  - Corner pieces

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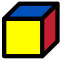
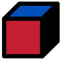
- $8 \times$  - Corner pieces
- $12 \times$  - Edge pieces

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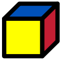
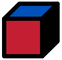
- $8 \times$  - Corner pieces
- $12 \times$  - Edge pieces
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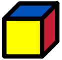
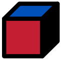
- $8 \times$  - Corner pieces
- $12 \times$  - Edge pieces
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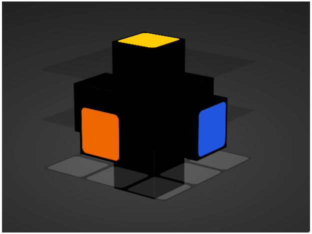

How many ways?

Suppose we now had to reconstruct it. In how many ways can we do so?

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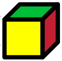
Consider just the **corner pieces** (),

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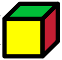
Consider just the **corner pieces** (),

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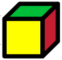
Consider just the **corner pieces** (),

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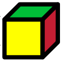
Consider just the **corner pieces** (),

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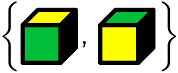

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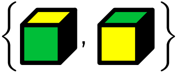

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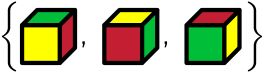

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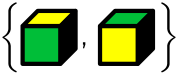

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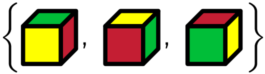

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$$\underbrace{12! \times 8!}_{\text{Permutations}} \times \underbrace{2^{12} \times 3^{8}}_{\text{Orientations}}$$

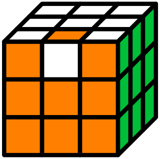
$$= 519,024,039,293,878,272,000$$

How many *valid* ways?

Previously we considered all possible reconstructions, but some of them are not solvable.

- Last **edge piece** must be oriented correctly.
- Last **corner piece** must be oriented correctly.
- Last two **corner pieces** must be placed correctly.

Invalid states



- wrong **edge** orientation.

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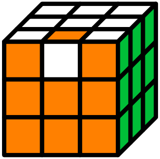

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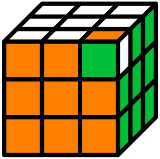

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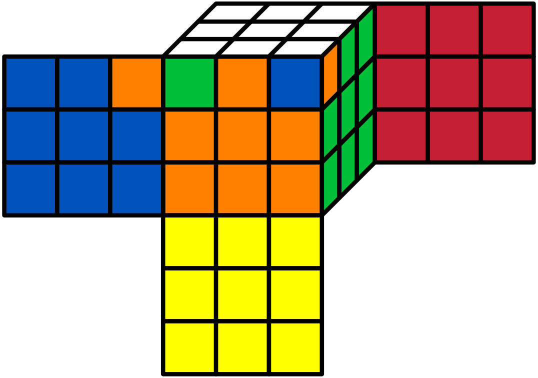

Wrong placement for last 2 corners (upper left/right on the front facing side).

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$$
\begin{aligned}
3^8 &= 3 \times 3 \times 3 \times \cdots \times 3 \times 3 \\
2^{12} &= 2 \times 2 \times 2 \times \cdots \times 2 \times 2 \\
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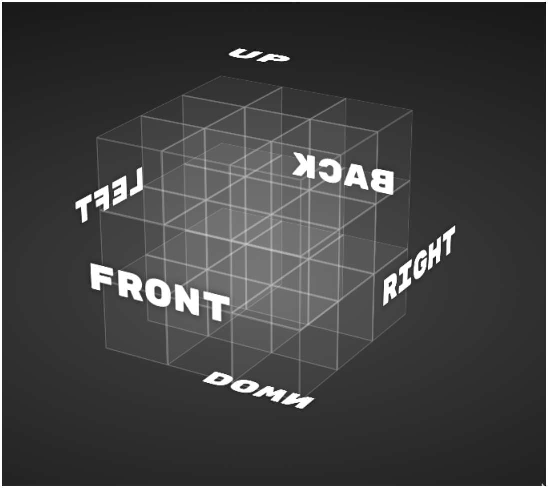
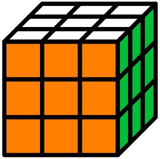
$$= 43,252,003,274,489,856,000$$

Some notation



F $\boxed{\curvearrowright}$ - A clockwise quarter turn of the front face



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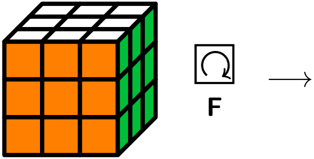

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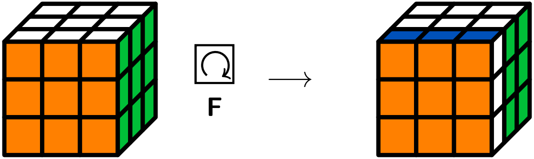
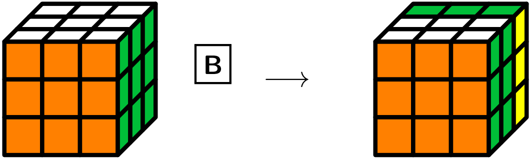
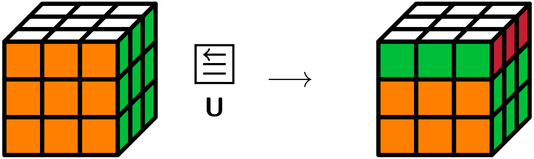


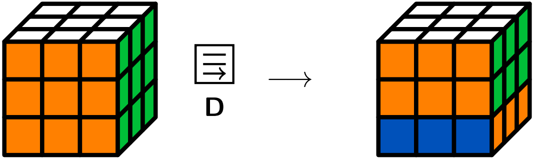
$\boxed{\text{B}}$ - A clockwise quarter turn of the back face



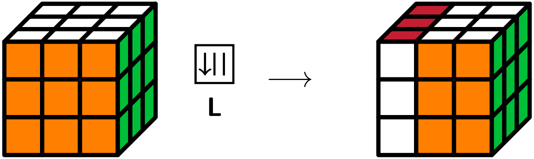
U - A clockwise quarter turn of the up face



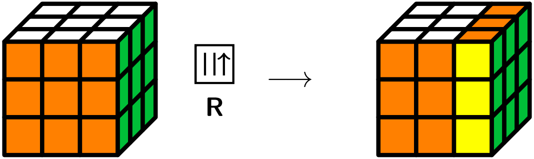
D - A clockwise quarter turn of the down face



L $\boxed{\downarrow||}$ - A clockwise quarter turn of the left face



R $\boxed{||\uparrow}$ - A clockwise quarter turn of the right face



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- **Identity:** $\exists\, e \in G$ such that $a * e = e * a = a$

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- **Identity:** $\exists\, e \in G$ such that $a * e = e * a = a$
- **Inverse:** $\forall\, a \in G,\ \exists\, a^{-1}$ such that $a * a^{-1} = e$

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- Let G denote the set of all possible moves that can be done on the cube i.e. **U L R**, **U**, **L R**, $\cdots$
- Two moves will be considered the same if they result in the same configuration of the cube.

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So, if $M_1 =$ **L R B U** and $M_2 =$ **D F L**,
then $M_1 * M_2 =$ **L R B U D F L**

Proving it! - Closure

This one is trivial - for any two moves M_1 and M_2 that we choose, we will surely obtain another cube state which is reachable by a move M_3.

Proving it! - Associativity

Associativity follows neatly from the physical nature of rotating the sides. Performing $M_1 * M_2$ first, then M_3 is equivalent to first doing M_1, followed by $M_2 * M_3$.

Proving it! - Inverse

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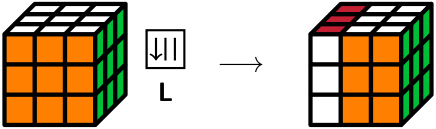

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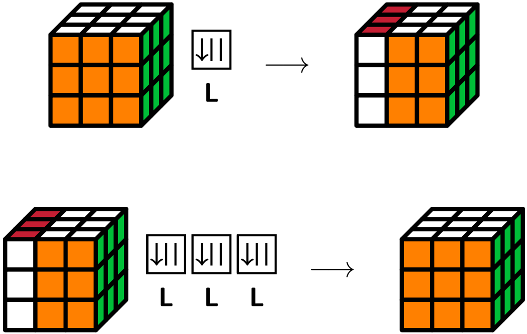

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L' $=$ **L L L**

R' $=$ **R R R**

U' $=$ **U U U**

D' $=$ **D D D**

F' $=$ **F F F**

B' $=$ **B B B**

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With this notation we can now conveniently define an inverse of any arbitrary move. For any move performed, we will reverse it and invert each single turn like so:

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$$M = \text{ } \textbf{U D' R L' F B' U D'}$$
$$M^{-1} = \text{ } \textbf{D U' B F' L R' D U'}$$

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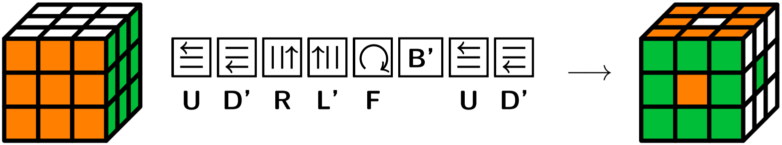

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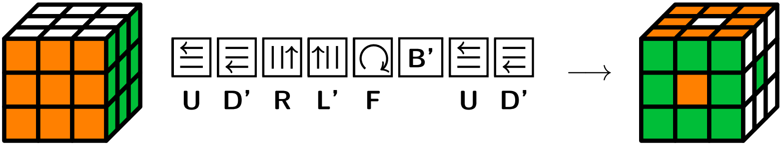

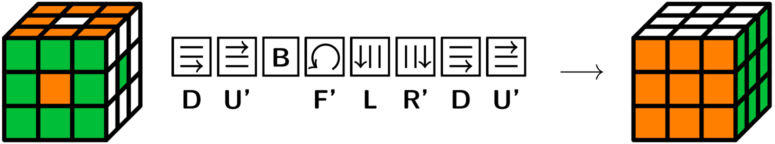

Proving it! - Identity

Finally, we need to show there exists a move, that when performed, leaves the cube untouched. This move, denoted by e is, well, do nothing.

For any move M we have:

$$M * e = M$$

Proposition:

There exists an infinite number of distinct moves that leave the cube untouched.

QED

After showing the 4 group axioms we can conclude that our Rubik's Cube set, or rather, our set of many Rubik's Cubes, is indeed a group.

It doesn't stop here!

Metric Spaces   Equivalence Classes   Group Generators

(not) Burnsides's Lemma   God's Algorithm   God's Number

Conjugation   Commutators   Group Actions

https://people.math.harvard.edu/~jjchen/docs/Group%20Theory%20and%20the%20Rubik's%20Cube.pdf

https://www.jaapsch.net/puzzles/theory.htm