ODE Short notes

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1 First Order ODEs

1.1 Separable Equation

A separable differential equation is one of the form:

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

This is solved by collecting like variables to both sides and integrating:

$$dy = f(x) \cdot g(y)dx$$
$$\therefore \int \frac{dy}{g(y)} = \int f(x)dx + C$$

Where C is a constant of integration.

1.2 Linear Equations

A Linear First Order ODE is of the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

These can be reduced to standard form by dividing by a_1 achieving:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Theorem 1. *For Equations of this form there exists an integrating factor of the form:*

$$I(x) = e^{\int P(x)dx}$$

Proof. Consider some Integrating factor $I(x)$ of the above differential equation.This is then multiplied through out to achieve the following ODE:

$$I(x)\frac{dy}{dx} + P(x) \cdot I(x)y = Q(x) \cdot P(X)$$

By the Product rule, If $I'(x) = P(x) \cdot I(X)$, the LHS is just the derivative of the function $y \cdot I(x)$. This gives:

$$\frac{d[y \cdot I(x)]}{dx} = I(x) \cdot Q(x)$$

This equation is now separable, and can be solved as so. Since $I(x)$ has a condition it must satisfy, a function can be found by solving:

$$\begin{aligned} \frac{dI}{dx} &= P(x) \cdot I(x) \\ \therefore \frac{1}{I}dI &= p(x)dx \\ \therefore ln(I) &= \int P(x)dx \\ \therefore I &= \exp\left[\int P(x)dx\right] \end{aligned}$$

□

1.3 Exact Equations

Exact ODEs are of the form:

$$M(x,y)dx + N(x,y)dy = 0$$

If and only if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the equation satisfies this condition then there exists a function F such that:

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \qquad\qquad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

This is then solved by integrating both sides of one of these equations and then taking the partial derivative w.r.t the other variable to find the constant of integration. Take The one on the left for example:

$$F(x,y) = \int M(x,y)dx + g(y)$$
$$\therefore \frac{\partial F}{\partial y} = N(x,y)$$
$$\therefore \frac{\partial}{\partial y}\left[\int M dx\right] + g'(y) = N(x,y)$$

The partial derivative w.r.t to y of the integral of M w.r.t x should cancel out any terms in x leaving only:

$$g'(y) = h(y)$$

This function is then integrated w.r.t y as to find $g(x)$ giving a final function $F(x,y)$. The general solution of this exact equation is given by:

$$F(x,y) = C$$

Where C is an integrating factor.

1.4 Special Integrating Factors

When an equation of the form: $Mdx + Ndy = 0$ is not exact, separable or linear, There exists an integrating factor which is a function of x and y such that when applied it gives an exact equation.

Theorem 2. *If* $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ *is a function of only x, then the integrating factor is given by:*

$$I(x) = \exp\left[\int \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] dx\right]$$

Similarly If $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$ *is a function of only y, then the integrating factor is given by:*

$$I(x) = \exp\left[\int \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dy\right]$$

Proof. Let $I(x)$ be an integrating factor of the given ODE, and is only a function of x. Therefore the following equation is exact:

$$I(x)M(x,y)dx + I(x)N(x,y) = 0$$

Since the above equation is exact, It must satisfy the following condition:

$$\frac{\partial}{\partial y}[I(x)M(x,y)] = \frac{\partial}{\partial x}[I(x)N(x,y)]$$

$$\implies \cancelto{0}{\frac{\partial I}{\partial y}}M + \frac{\partial M}{\partial y}N = \frac{\partial I}{\partial x}N + \frac{\partial N}{\partial x}I$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]\partial x = \frac{\partial I}{I}$$

$$\therefore I(x) = \exp\left[\int \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]dx\right]$$

□

1.5 Homogeneous First Order ODEs

Consider the Equation:

$$\frac{dy}{dx} = f(x,y)$$

If $f(x,y)$ is a function of y/x then the ODE is a Homogeneous ODE and can be solved by substituting $y = vx$ This then forms a Separable Equation. Substitute into:

$$P(x,y)dx + Q(x,y)dy = 0$$

N.B. If $f(x,y)$ is a function of a linear combination of x and y, i.e. $f(x,y) = g(ax+by)$, then the substitution $z = ax+by$ can be used to solve the ODE.

1.6 Bernoulli Equations

A Differential equation of the form:

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)y^n$$

is known as Bernoulli Equation. These equations can be solved by making the substitution $v = y^{n-1}$ and will then result in a linear differential equation.

1.7 Differential Equations with linear Coefficients

In general, a DE with linear coefficients is of the form:

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

There are three cases that may arise in a linear coefficients equation.

Case 1: If $a_1 b_2 = a_2 b_1$ (i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$):
The Differential equation can be put in the form:

$$\frac{dy}{dx} = G(ax + by)$$

Case 2 : If $c_1 = c_2 = 0$, the equation is homogeneous.

Case 3 : The else case:
Use the substitutions $x = u+h$ and $y = v+k$, where u and v are the variables and h and k are constants.