Analysis I

Course Notes

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1 The Real Numbers ($\mathbb{R}$)

Basic Properties of $\mathbb{R}$

The set of real numbers forms an *ordered field* $(\mathbb{R}, +, \cdot)$ under real addition and multiplication. By this we mean it satisfies a list of axioms which make it a field, and furthermore, axioms which make it an ordered field.

Field Axioms

1. Closure: $\forall\, a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$.
2. Associativity: $\forall\, a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$
   and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
3. Commutativity: $\forall\, a, b \in \mathbb{R}$, $a + b = b + a$
   and $a \cdot b = b \cdot a$.
4. Distributivity: $\forall\, a, b, c \in \mathbb{R}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
5. Identities: $\exists\, 0, 1 \in \mathbb{R} : \forall\, a \in \mathbb{R}$, $a + 0 = a$
   and $a \cdot 1 = a$.
6. Inverses: $\forall\, a \in \mathbb{R}$, $\exists\, (-a) \in \mathbb{R} : a + (-a) = 0$
   and $\forall\, a \in \mathbb{R} \setminus \{0\}, \exists\, a^{-1} : a \cdot a^{-1} = 1$.

Ordered Field Axioms

A field is said to be *ordered* if it has total order; i.e. $\forall\, a, b, c \in \mathbb{R}$;

$$a \leq b \Longrightarrow a + c \leq b + c$$

and

$$0 \leq a \text{ and } 0 \leq b \Longrightarrow 0 \leq a \cdot b$$

Additionally, the set of real numbers satisfies the *completeness axiom.* But to state this axiom, we must define 'supremum'.

Definition 1.1 (Supremum) An upper-bound of a set $S \subseteq \mathbb{R}$ is a value $x \in \mathbb{R}$ such that $\forall\, s \in S$, $x \geq s$. The *supremum* of S, denoted $\sup S$, is the least possible value x such that x is an upper-bound to S.

If a set S has an upper-bound, then it is said to be *bounded above.*

The Completeness Axiom Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

Definition 1.2 (Absolute Value) The absolute value (or modulus) of $x \in \mathbb{R}$, denoted $|x|$, is defined

$$|x| = \begin{cases} x, & \forall\, x \geq 0 \\ -x, & \forall\, x < 0 \end{cases}$$

Theorem 1.3 For every $a, b \in \mathbb{R}$:

(i) $|a \cdot b| = |a||b|$

(ii) $|a + b| \leq |a| + |b|$ (The triangle inequality)

Proof. To prove (i) and (ii), we will consider the cases where a and b are either both or individually positive or negative. Four such cases arise.

Case 1: $a \geq 0$ and $b \geq 0$.
$|a \cdot b| = a \cdot b = |a||b|$ and $|a + b| = a + b = |a| + |b|$, as required.

Case 2: $a \geq 0$ and $b < 0$.
Note that if $a = 0$, then $|a \cdot b| = |0| = 0 = |0||b| = |a||b|$ and $|a+b| = |0+b| = |b| = |0|+|b| = |a|+|b|$.
Now if $a \neq 0$, then $a \cdot b \leq 0$. Thus $|a \cdot b| = -(a \cdot b) = a \cdot (-b) = |a||b|$.
If $a + b = 0$, then $|a + b| = |0| \leq |a| + |b|$
$a + b > 0$, then $|a + b| = a + b < a < |a| + |b|$
$a + b < 0$, then $|a + b| = -(a + b) < -b = |b| < |a| + |b|$

Hence we have considered every possible case with $a \geq 0$ and $b \leq 0$ and (i) and (ii) both hold.

Case 3: $a < 0$ and $b \geq 0$.
This follows from case 2 with a and b interchanged.

Case 4: $a < 0$ and $b < 0$.
In this case, $a \cdot b > 0$ and $a + b < 0$, so $|a \cdot b| = |a||b|$, since $|a \cdot b| = a \cdot b = (-a) \cdot (-b)$ whereas $|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|$. □

Note The triangle inequality $|a + b| \leq |a| + |b|$ is used extensively throughout further proofs.

Remark Let us define the distance between two numbers $a, b \in \mathbb{R}$ as $|a - b|$. Then $\forall\, a, b, c \in \mathbb{R}$:

(i) $|a - b| \geq 0$, and $|a - b| = 0 \Leftrightarrow a = b$

(ii) $|a - b| = |b - a|$

(iii) $|a - b| \leq |a - c| + |b - c|$

Proof. For (i), we have $|a - b| \geq 0$ by definition, and $|a - b| = 0 \iff a - b = 0 \iff a = b$.

For (ii), By theorem 1.3 we have that $|a \cdot b| = |a||b|$ for any $a, b \in \mathbb{R}$, therefore $|a - b| = 1 \cdot |a - b| = |-1||a - b| = |(-1)(a - b)| = |b - a|$, as required.

Now (iii) follows immediately by the triangle inequality since $|a-b| = |a-c+c-b| \leq |a-c|+|c-b|$, which is equal to $|a - c| + |b - c|$ by (ii). □

Theorem 1.4 For any $a, b \in \mathbb{R}$, $a = b$ iff $|a - b| < \epsilon \ \ \forall\, \epsilon > 0$.

Proof. If $a = b$, then $|a - b| = 0 < \epsilon$ for any $\epsilon > 0$.

Conversely, for contradiction, assume $a \neq b$ and let $\epsilon_0 = |a - b| > 0$. If $|a - b| < \epsilon$ for any $\epsilon > 0$, then it must also be true for $\epsilon = \epsilon_0$. Therefore we have both $\epsilon_0 = |a - b|$ and $|a - b| < \epsilon_0$ ※. □

Definition 1.5 (Infimum) A lower-bound of a set $S \subseteq \mathbb{R}$ is a value $x \in \mathbb{R}$ such that $\forall\, s \in S$, $x \leq s$. the *infimum* of S, denoted $\inf S$, is the largest possible value x such that x is a lower-bound to S.

If a set S has a lower-bound, the it is said to be *bounded below.*

Definition 1.6 (Boundedness) A set $S \subseteq \mathbb{R}$ is said to be *bounded* if it is both bounded above and below; i.e. it has both a lower- and an upper-bound.

Remark 1.7 Although a set $S \subseteq \mathbb{R}$ can have an infinite number of upper/lower-bounds, it can only have one supremum/infimum.

This can easy be shown by contradiction. Assume $S \subseteq \mathbb{R}$ and s_1 and s_2 are both suprema of S where $s_1 \neq s_2$. Since they are not equal, then one of them, say s_1, is the larger one; i.e. $s_1 > s_2$. But this contradicts the definition of supremum, since s_1 is no longer the least upper-bound.

Hence for any $S \subseteq \mathbb{R}$, $\sup S$ is unique. A similar argument shows the uniqueness of $\inf S$. $\square$

Although the completeness axiom mentions only suprema, it may be stated in terms of infima as well. Define $-A$ to be the set $\{-a : a \in A\}$ for any set $A \subseteq \mathbb{R}$. This gives the following theorem.

Theorem 1.8 (Completeness Axiom Variant) Every nonempty set $S \subseteq \mathbb{R}$ which is bounded below has an infimum, where $\inf S = -\sup(-S)$.

Proof. Let $S \subseteq \mathbb{R}$ be bounded below. Then the set $-S = \{-s : s \in S\}$ is obviously bounded above. By the completeness axiom, $\exists \sup(-S)$ where

$$\sup(-S) > \alpha \quad \forall \alpha \in -S$$

which rearranges to $-\alpha < -\sup(-S)$ which holds for all $(-\alpha) \in -S$, and hence for all $\alpha \in S$. Therefore $\inf S = -\sup(-S)$. $\square$

Example 1.9 Consider the set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. Then $-A$ is the set $\left\{-1, -\frac{1}{2}, \cdots\right\}$.

The infimum of A is 0, which is also $-\sup(-A)$.

Definition 1.10 (Max and Min) Let $A \subseteq \mathbb{R}$. Then $m \in \mathbb{R}$ is said to be the *maximum* of A, denoted $\max A$, if

(i) $m \in A$ and

(ii) $m \geq a$, $\forall a \in A$

We similarly define the *minimum* of A, denoted $\min A$.

Example 1.11 Let $a, b \in \mathbb{R}$ where $a \leq b$. Let $A = \{x \in \mathbb{R} : a \leq x \leq b\} = [a, b]$.

Then $\inf A = \min A = a$, and $\sup A = \max A = b$.

Now let $B = \{x \in \mathbb{R} : a < x < b\} = (a, b)$. In this case, $\nexists \max B$ or $\min B$; however we still have $\inf B = a$ and $\sup B = b$.

Note The supremum or infimum of a set can exist and not its the maximum or minimum. But when the maximum or minimum exist, then they are always the supremum and infimum respectively.

Lemma 1.12 Let $s \in \mathbb{R}$ be an upper-bound of $A \subseteq \mathbb{R}$. Then $s = \sup A$ iff for every $\epsilon > 0$, $\exists a \in A$ such that $s - \epsilon < a$.

Proof. Let $s = \sup A$. Since $s - \epsilon < s$ and s is the least upper-bound, then $s - \epsilon$ is no longer an upper-bound of A, and therefore there is an $a \in A$ where $s - \epsilon < a$.

Now we prove the converse by contradiction. Suppose s is an upper-bound of A but *not* the supremum, and that for every $\epsilon > 0$, we can find an $a \in A$ such that $s - \epsilon < a$. Since s is not the supremum, then $\exists\, b \in \mathbb{R}$ where $b < s$ such that b is also an upper-bound of A.

Now let $\epsilon_0 = s - b > 0$. By the hypothesis, we have an $a \in A$ where $s - \epsilon_0 < a$. Hence $s - (s - b) = b < a$. But b is an upper-bound of A ※. $\square$

Let $t \in \mathbb{R}$ be a lower-bound of $B \subseteq \mathbb{R}$. A similar proof to the above can be constructed to show that $t = \inf A$ iff for every $\epsilon > 0$, $\exists\, a \in A$ such that $a < t + \epsilon$.

The Density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$

If the set $\mathbb{Q}$ and $\mathbb{I}$ denote the rational and irrational numbers respectively, then

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

We will proceed to prove two theorems about $\mathbb{Q}$ and $\mathbb{I}$ known as their *density* in $\mathbb{R}$. This means that between any two real numbers, we can find a rational number as well as an irrational number. But to prove these, we must first show that the set of positive integers has the Archimedean property in $\mathbb{R}$.

Lemma 1.13 (The Archimedean Property of $\mathbb{R}$) The set of natural numbers $\mathbb{N}$ has no upper-bound in $\mathbb{R}$.

Proof. We prove this by contradiction. If $\mathbb{N}$ were bounded above in $\mathbb{R}$, then by the completeness axiom, $\mathbb{N}$ would have a supremum; i.e. we would have $m \in \mathbb{R}$ such that $m \geq n$ for all $n \in \mathbb{N}$. Since m is the *least* upper-bound, then $m - 1$ is not an upper-bound, and we have some integer k with $m - 1 < k$. But then $m < k + 1 \in \mathbb{N}$, which contradicts that m is an upper-bound for $\mathbb{N}$ ※. $\square$

Theorem 1.14 (Density of $\mathbb{Q}$ in $\mathbb{R}$) For every $x, y \in \mathbb{R}$ where $x < y$, we have $q \in \mathbb{Q}$ where $x < q < y$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. Since $y - x > 0$, then $\dfrac{1}{x - y} \in \mathbb{R}$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $N > \dfrac{1}{x - y}$, i.e. such that $x - y > \dfrac{1}{N}$. Now, let us define $A \subseteq \mathbb{Q}$ by

$$A = \left\{ \frac{m}{N} : m \in \mathbb{N} \right\}$$

We proceed to show that $A \cap (x, y) \neq \varnothing$ by contradiction, i.e. that there must always be a number in $A \subseteq \mathbb{Q}$ which is between x and y.

Suppose $A \cap (x, y) = \varnothing$, and let m_0 be the largest integer such that $\dfrac{m_0}{N} < x$. Then $y < \dfrac{m_0 + 1}{N}$, since otherwise we would have a rational number in (x, y). This gives

$$x - y < \frac{m_0 + 1}{N} - \frac{m_0}{N} = \frac{1}{N} < x - y \quad ※$$

Therefore for any two $x, y \in \mathbb{R}$ with $x < y$, $A \cap (x, y)$ is nonempty and thus there exists a rational number between x and y. $\square$

Now we proceed to show the density of $\mathbb{I}$ in $\mathbb{R}$. However first we must show that $\mathbb{I}$ *exists*; i.e. that it is not $= \varnothing$. Consider $\sqrt{2}$. we can show that $\sqrt{2}$ is in $\mathbb{I}$ by contradiction.

Assume that $\sqrt{2}$ is not irrational; i.e. that $\sqrt{2} \in \mathbb{Q}$. Then we can find $p, q \in \mathbb{Z}$ where

$$\sqrt{2} = \frac{p}{q}$$

and we can assume that p and q have no common factors (if they do, they cancel out in the numerator and the denominator).

$$2 = \frac{p^2}{q^2}$$

which implies

$$2q^2 = p^2$$

Thus p^2 is even. The only way this can happen is if p itself is even. But then p^2 is actually divisible by 4. Hence q^2 and therefore q must also be even. So p and q are both even, which contradicts the fact that they have no common factors ※. Hence $\sqrt{2}$ is irrational, i.e. $\sqrt{2} \in \mathbb{I}$.

Theorem 1.15 (Density of $\mathbb{I}$ in $\mathbb{R}$) For every $x, y \in \mathbb{R}$ where $x < y$, we have $i \in \mathbb{I}$ where $x < i < y$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we have a rational number p between $\dfrac{x}{\sqrt{2}}$ and $\dfrac{y}{\sqrt{2}}$. Hence $\sqrt{2}p \in (x, y)$.

Now if $\sqrt{2}p = q$ is rational, then $\sqrt{2} = \dfrac{q}{p}$, which is also rational since both q and p are rational ※. Thus $\sqrt{2}p \in (x, y)$ must be irrational. □

Countable Sets

Recall[1] A function $f : A \to B$ is said to be:

(i) *Injective* (or one-to-one) if $f(\alpha_1) = f(\alpha_2)$ implies that $\alpha_1 = \alpha_2$.

(ii) *Surjective* (or onto) if for all $\beta \in B$ there exists $\alpha \in A$ such that $f(\alpha) = \beta$.

(iii) *Bijective* (or a one-to-one correspondence) if it is both injective and surjective.

Definition 1.16 (Same Cardinality) Two sets A and B are said to have the same *cardinality*, denoted

$$A \sim B$$

if there exists a bijection between A and B.

Note If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection. Hence $A \sim B$ iff $B \sim A$.

The empty set $\varnothing$ is said to have cardinality zero, and for $n \in \mathbb{N}$, the set $[n] = \{1, 2, 3, \cdots, n\}$ has cardinality n.

Definition 1.17 (Finite and Infinite Sets) A set is said to be *finite* if it is empty or has the same cardinality as $[n]$ for some $n \in \mathbb{N}$. It is said to be *infinite* if it is not finite.

Remark 1.18 One can easily prove that $[n] \nsim [m]$ if $n \neq m$. Therefore the cardinality of a finite set is uniquely determined by its size.

[1]See Introductory Mathematics notes.

Definition 1.19 (Countability) A set S is said to be *countably infinite* if $S \sim \mathbb{N}$. A set is said to be *countable* if it is finite or countably infinite.

Theorem 1.20 A set A is countable iff there exists a bijection between A and a subset of $\mathbb{N}$.

Proof. A set is countable if it is finite or countably infinite. If it is finite, then it is either empty or has the same cardinality as $[k]$ for some $k \in \mathbb{N}$. In either case, we have a bijection between A and a subset of $\mathbb{N}$. If A is countably infinite, then by definition it we have a bijection from A to $\mathbb{N}$.

Conversely, suppose we have a bijection between A and $S \subseteq \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $S \subseteq [n]$, then S is finite, which implies that A is finite (due to the bijection), hence countable.

Now suppose there does not exist $n \in \mathbb{N}$ for which $S \subseteq [n]$. Then for all $n \in \mathbb{N}$, $S \cap (\mathbb{N} \setminus [n]) \neq \varnothing$.

Let us define an injection $f : \mathbb{N} \to S$ where

$$\begin{aligned}
f(1) &= \min S \\
f(2) &= \min S \setminus \{f(1)\} \\
f(3) &= \min S \setminus \{f(1), f(2)\} \\
&\vdots \\
f(n) &= \min S \setminus \{f(1), f(2), f(3), \cdots, f(n-1)\}
\end{aligned}$$

i.e. this function recursively removes the smallest element in S, and once it has removed $n-1$ minimum elements it returns the nth minimum. Now, if $f(n) = \varnothing$ for some $n \in \mathbb{N}$, then $S \subseteq [n-1]$, which contradicts the hypothesis. Therefore our function f is defined for all $n \in \mathbb{N}$.

It is clear that if $f(n_1) = f(n_2)$ then n_1 must be n_2, hence f is injective. It is also clear that for any $s \in S$, we can find $n \in \mathbb{N}$ which gives us $f(n) = s$. Hence we have a well-defined bijection f, giving us that $S \sim \mathbb{N}$. $\square$

Corollary 1.21 If a set A is countable and $B \subseteq A$, then B is also countable.

Theorem 1.22 A nonempty set A is countable iff there exists a surjection f from $\mathbb{N}$ onto A.

Proof. If A is countably infinite, we have a bijection $g : \mathbb{N} \to A$. If A is finite, we have a bijection $h : [n] \to A$ for some $n \in \mathbb{N}$. Let us define $f : \mathbb{N} \to A$ by

$$\begin{aligned}
f : \ & \mathbb{N} \to A \\
& k \longmapsto h(k) \quad \text{for } k \in [1, n] \\
& k \longmapsto h(n) \quad \text{for } k > n
\end{aligned}$$

Given any $a \in A$, we know that f maps some value(s) to it, hence it is surjective.

Conversely, suppose we have a surjection ϕ from $\mathbb{N}$ onto A, and for every $a \in A$, we define another function $\psi(a)$ by $\phi^{-1}(a)$. This maps every element $a \in A$ to an element in some subset of $\mathbb{N}$. Now if $\psi(a) = \psi(b)$, we know that $a = b$, hence ψ is an injection. Furthermore, for any element n in the subset of $\mathbb{N}$, we have $\psi(n)$.

Hence there exists a bijection ψ between A and some subset of $\mathbb{N}$, thus A is countable. $\square$

Corollary 1.23 If A is countable and $f : A \to B$ is surjective then B is also countable.

Theorem 1.24 $\mathbb{N}^2$ is countably infinite.

Proof. Let us define $f : \mathbb{N}^2 \to \mathbb{N}$ by

$$\begin{aligned} f :\ \mathbb{N}^2 &\longrightarrow \mathbb{N} \\ (n, m) &\longmapsto 2^n\, 3^m \end{aligned}$$

By the fundamental theorem of arithmetic[2] we immediately realise that f is one-to-one, since if $2^n\, 3^m = 2^a\, 3^b$ then we must have $n = a$ and $m = b$. The codomain is obviously a subset of $\mathbb{N}$, hence $\mathbb{N}^2$ is countable.

Now consider

$$\begin{aligned} g :\ \mathbb{N} \times \{1\} &\longrightarrow \mathbb{N} \\ (n, 1) &\longmapsto n \end{aligned}$$

This is obviously a bijection, hence $\mathbb{N} \times \{1\}$ is countably infinite. But $\mathbb{N} \times \{1\} \subseteq \mathbb{N}^2$, and an infinite set cannot be contained in a finite set; hence $\mathbb{N}^2$ is also countably infinite. $\square$

It follows from the definition of infinite countability that $\mathbb{N}$ is countably infinite, hence countable. The set $\mathbb{Q}$ is also countably infinite. By the bijection

$$\begin{aligned} f :\ \mathbb{N}^2 &\longrightarrow \mathbb{Q}^+ \\ (n, m) &\longmapsto \frac{n}{m} \end{aligned}$$

we have that $\mathbb{Q}^+$ is countably infinite; and a similar one can be constructed for $\mathbb{Q}^-$. Since

$$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$$

then it follows by the upcoming theorem 1.25 that the union of countable sets is also countable, so $\mathbb{Q}$ is countable.

$\mathbb{Z}$, the integers, is also countable by mapping the positives (and 0) to the evens, and the negatives to the odds; i.e. by the bijection

$$\begin{aligned} f :\ \mathbb{Z} &\longrightarrow \mathbb{N} \\ n &\longmapsto 2n && \text{for } n \geq 0 \\ n &\longmapsto 2(-n) + 1 && \text{for } n < 0 \end{aligned}$$

Theorem 1.25 If $A_1, A_2, \ldots, A_n$ are countable sets, then $A_1 \cup A_2 \cup \cdots \cup A_n$ is also countable.

Proof. This theorem follows directly from the proof of theorem 1.26. $\square$

Theorem 1.26 If A_n is countable for all $n \in \mathbb{N}$, then the countable union $\displaystyle\bigcup_{n\in\mathbb{N}} A_n$ is also countable.

Proof. Since each A_n is countable, we have a surjection $g_n : \mathbb{N} \to A_n$ from $\mathbb{N}$ onto A_n, for each $n \in \mathbb{N}$.

[2] i.e. that every number may be written as a *unique* product of primes, see Introductory Mathematics notes.

Now consider

$$\begin{aligned} f:\ \mathbb{N}^2 &\longrightarrow \bigcup_{n\in\mathbb{N}} A_n \\ (n,m) &\longmapsto g_n(m) \end{aligned}$$

The function f is onto since:

(i) it is defined for all $n \in \mathbb{N}$, so for any $g_n(m)$ in the codomain we have $n \in \mathbb{N}$; and

(ii) for every $g_n(m)$ is the codomain, we have m in the domain since g_n itself is a surjection.

Thus given any element in the codomain we have an ordered pair in $\mathbb{N}^2$ associated with it, making f onto.

It follows from corollary 1.23 that $\displaystyle\bigcup_{n\in\mathbb{N}} A_n$ is countable since $\mathbb{N}^2$ is countable. □

Theorem 1.27 A finite product of countable sets is countable.

Proof. We prove this theorem by induction. For two countable sets A and B, if any of A or $B = \varnothing$, then $A \times B = \varnothing$. So assume $A \neq \varnothing \neq B$.

Being countable, we have two surjections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ by theorem 1.22.

Now, we define the surjection

$$\begin{aligned} h:\ \mathbb{N}^2 &\longrightarrow A \times B \\ (m,n) &\longmapsto (f(m), g(n)) \end{aligned}$$

and by corollary 1.23, $A \times B$ is countable.

For arbitrarily finite products, we proceed by induction. Assume the result is true for all $k \leq n$, i.e. that $A_1 \times A_2 \times \cdots \times A_k$ is countable. We must show that $A_1 \times A_2 \times \cdots \times A_{n+1}$ is countable, where all of $A_1, A_2, \cdots, A_{n+1}$ are countable.

Consider the function ϕ, where

$$\begin{aligned} \phi:\ A_1 \times A_2 \times \cdots \times A_k \times A_{n+1} &\longrightarrow (A_1 \times A_2 \times \cdots \times A_n) \times A_{n+1} \\ (a_1, a_2, \cdots, a_n, a_{n+1}) &\longmapsto ((a_1, a_2, \cdots, a_n), a_{n+1}) \end{aligned}$$

which is obviously a bijection. Now by the inductive hypothesis, $A_1 \times A_2 \times \cdots \times A_n$ is countable and therefore $(A_1 \times A_2 \times \cdots \times A_n) \times A_{n+1}$ is also countable as a product of two countable sets (by the base case above). Since ϕ is bijective, $A_1 \times A_2 \times \cdots \times A_{n+1}$ is countable. □

As in the case of unions, one might think that a countable product of countable sets is countable, however this is untrue, as we see in the upcoming example 1.29.

Uncountable Sets

Definition 1.28 A set which is not countable is said to be *uncountable.*

Example 1.29 (Cantor's Diagonal Argument) Let $D = \{0, 1\}$. We will show that the set

$$D^\omega = D \times D \times D \times \cdots \times D \times \cdots$$

is uncountable; i.e. that there is no surjection $f : \mathbb{N} \to D^\omega$.

The elements of D^ω are all the infinite sequences of binary digits (i.e. each digit is a zero or a one). Let $d_1, d_2, \dots, d_n, \dots$ represent elements of D^ω arranged in some arbitrary order. So we have

$$\begin{aligned}
d_1 &= (0,0,0,0,0,0,0,0,\cdots)\\
d_2 &= (1,1,1,1,1,1,1,1,\cdots)\\
d_3 &= (0,1,0,1,0,1,0,1,\cdots)\\
d_4 &= (0,1,1,1,0,0,1,0,\cdots)\\
d_5 &= (0,0,1,0,1,0,1,0,\cdots)\\
d_6 &= (0,0,1,1,0,0,1,0,\cdots)\\
d_7 &= (0,1,0,0,1,1,1,0,\cdots)\\
d_8 &= (0,1,0,0,0,1,0,1,\cdots)\\
&\vdots
\end{aligned}$$

where the dth element in our list is denoted $(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \cdots, a_{n_k}, \cdots)$ where we have $a_{n_k} = 0$ or 1 for all $n \in \mathbb{N}$. We will now define an element $d_\Omega = (\Omega_1, \Omega_2, \Omega_3, \cdots, \Omega_k, \cdots)$ in D^ω where

$$\Omega_n = \begin{cases} 0 \text{ if } a_{n_n} = 1 \\ 1 \text{ if } a_{n_n} = 0 \end{cases}$$

i.e. we are constructing an element d_Ω which is different from d_1 in the first position, different from d_2 in the second position, different from d_3 in the third position,. . . , different from d_k in the kth position, and so on. With respect to the example above,

$$\begin{aligned}
d_1 &= (\underline{\mathbf{0}},0,0,0,0,0,0,0,\cdots)\\
d_2 &= (1,\underline{\mathbf{1}},1,1,1,1,1,1,\cdots)\\
d_3 &= (0,1,\underline{\mathbf{0}},1,0,1,0,1,\cdots)\\
d_4 &= (0,1,1,\underline{\mathbf{1}},0,0,1,0,\cdots)\\
d_5 &= (0,0,1,0,\underline{\mathbf{1}},0,1,0,\cdots)\\
d_6 &= (0,0,1,1,0,\underline{\mathbf{0}},1,0,\cdots)\\
d_7 &= (0,1,0,0,1,1,\underline{\mathbf{1}},0,\cdots)\\
d_8 &= (0,1,0,0,0,1,0,\underline{\mathbf{1}},\cdots)\\
&\vdots
\end{aligned}$$

then $d_\Omega = (1,0,1,0,0,1,0,0,\cdots)$. It is dot difficult to see that d_Ω is not on our list, and therefore if $f(n) = d_n$ for all $n \in \mathbb{N}$,

$$d_\Omega \notin f(\mathbb{N})$$

In fact, given any $n \in \mathbb{N}$, d_Ω differs from d_n in at least one position.

Therefore f is not surjective and D^ω is uncountable.

Theorem 1.30 Let A be a set. There is no injective map $f : \wp(A) \to A$ and no surjective map $g : A \to \wp(A)$.

Proof. First of all, note that the second assertion is enough to prove the first.

If there exists an injective map $f : \wp(A) \to A$, then one can define a surjective map $g : A \to \wp(A)$ in the following way.

$$\begin{aligned} g:\ & A \longrightarrow \wp(A) \\ & a \longmapsto f^{-1}(a) \qquad \text{for all } a \in f(\wp(A)) \\ & a \longmapsto \text{arbitrarily for all } a \notin f(\wp(A)) \end{aligned}$$

Thus g is well-defined and surjective.

Now, consider any map $g : A \to \wp(A)$. Given any $a \in A$, then $g(A)$ is a subset of A, and we can have either $a \in g(a)$ or $a \notin g(a)$.

Let $B = \{a \in A : a \notin g(a)\}$. Let us show that B (which is a subset of A and hence an element of $\wp(A)$, the codomain) is not in $g(a)$. This will prove that g is not onto.

We prove this by contradiction. Assume there is $a \in A$ which gives us $g(a) = B$. But then

$$b \in B \iff b \in g(b) \iff b \notin B \ ※$$

Hence there can be no surjective map $g : A \to \wp(A)$, and no injective map $f : \wp(A) \to A$. □

Note We have shown that there is no function from $\mathbb{N}$ onto $\wp(\mathbb{N})$ (theorem 1.22), hence $\wp(\mathbb{N})$ is uncountable.

We will proceed to show that $\mathbb{R}$ is uncountable, and therefore that $\mathbb{I}$ is uncountable; since

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

where $\mathbb{Q}$ is countable; so if $\mathbb{I}$ were also countable, then $\mathbb{R}$ must be countable by theorem 1.25. To prove the uncountability of $\mathbb{R}$ will make use of the nested interval property.

Theorem 1.31 (The Nested Interval Property) Let there be given a decreasing sequence[3] of nonempty closed intervals of $\mathbb{R}$:

$$(I_n = [a_n, b_n])_{n\in\mathbb{N}}$$

such that every $I_{n+1} \subseteq I_n$. Then $\displaystyle\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$.

Proof. Consider $A = \{a_n : n \in \mathbb{N}\}$. Note that given any b_n and any $m \in \mathbb{N}$, we have that:

(i) if $m \leq n$ then $a_m \leq a_n \leq b_n$; and

(ii) if $m > n$ then $a_m \leq b_m \leq b_n$

So given any b_n we have $a_m \leq b_n$. In particular, this means that A is bounded above and hence, there exists $s = \sup A$ by the completeness axiom.

Now we will show that $s \in I_n$ for every $n \in \mathbb{N}$. Take a particular $I_n = [a_n, b_n]$. Since s is an upper-bound for A, then $a_n \in A$ and $a_n \leq s$. Also, b_n is an upper-bound for A, so $s \leq b_n$. Hence

$$a_n \leq s \leq b_n \Longrightarrow s \in I_n$$

but we never specified a value of n, so this holds for every $n \in \mathbb{N}$ hence $s \in I_n$ for all n and therefore

$$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$$

as required. □

[3] See definition 2.1.

Remark 1.32 Note that the intervals must be closed in order for the result to hold. In fact, consider the sequence of intervals $I_n = \left[0, \dfrac{1}{n}\right)$, whose $\displaystyle\bigcap_{n\in\mathbb{N}} I_n$ is empty.

Theorem 1.33 (Uncountability of $\mathbb{R}$) The set $\mathbb{R}$ is uncountable.

Proof. We proceed by contradiction. Suppose $\mathbb{R}$ is countable. Evidently, $\mathbb{R}$ is infinite (in particular, countably infinite). So there must exist a bijection $f : \mathbb{N} \to \mathbb{R}$ such that $x_1 = f(1)$, $x_2 = f(2)$, $\dots x_n = f(n)$, $\dots$. Since f is onto, for any $x \in \mathbb{R}$ we have $k \in \mathbb{N}$ such that $x_k = f(k)$. Hence one can list the elements of $\mathbb{R}$ as

$$\mathbb{R} = \{x_1, x_2, x_3, \cdots, x_n, \cdots\}$$

Now we will use the nested interval property to find a real number which is not in the above list. Let $I_1 \subseteq \mathbb{R}$ be a closed interval such that $x_1 \notin I_1$.
Let $I_2 \in I_1$ be a closed interval such that $x_2 \notin I_2$.
Let $I_3 \in I_2$ be a closed interval such that $x_3 \notin I_3$.

$$\vdots$$

Let $I_{n+1} \in I_n$ be a closed interval such that $x_{n+1} \notin I_{n+1}$.

The existence of such a closed interval is not difficult to verify. In fact, given I_n, take any two disjoint closed intervals of I_n; then x_n can be in at most one of those closed intervals.

So if we have constructed a decreasing sequence of closed intervals I_n,

$$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$$

by the nested interval property. On the other hand, as noted above, for all $x \in \mathbb{R}$ we have $k \in \mathbb{N}$ such that $x = x_k$ and $k \notin I_k$, and hence

$$x_k \notin I_1 \cup I_2 \cup I_3 \cup \cdots \cup I_k$$

thus

$$\bigcup_{n\in\mathbb{N}} I_n = \varnothing \quad ※$$

Hence $\mathbb{R}$ is uncountable. □

Summary

- The supremum of a set $S \subseteq \mathbb{R}$, denoted $\sup S$ is the least upper-bound of S, in other words, $\sup S = \min\{x : \forall s \in S, x \geq s\}$. (1.1)
- The infimum of a set $S \subseteq \mathbb{R}$, denoted $\inf S$ is the greatest lower-bound of S, in other words, $\inf S = \max\{x : \forall s \in S, x \leq s\}$. (1.5)
- Density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$: Between any two real numbers, there is both a rational number and an irrational number. (1.14 and 1.15)
- Two sets have the same cardinality if there exists a bijection between them. (1.16)
- A set S is said to be finite if there is a bijection between S and $[n] \subseteq \mathbb{N}$. (1.17)
- A set is said to be infinite if it is not finite. (1.17)
- A set S is said to be countably infinite if there is a bijection between S and $\mathbb{N}$. (1.19)

- A set is countable if it is finite or countably infinite. (1.19)
- The following are some useful results about countable sets:
  - A set A is countable iff there exists a bijection between A and a subset of $\mathbb{N}$. (1.20)
  - A subset of a countable set is countable. (1.21)
  - A nonempty set A is countable iff there exists a surjection from f onto A. (1.22)
  - If A is countable and there exists a surjection from A onto some set B, then B is also countable. (1.23)
  - The countable union of countable sets is countable, in other words, $\bigcup_{n\in\mathbb{N}} A_n$ is countable if A_i is countable $\forall\, i \in \mathbb{N}$. (1.26)
  - A finite product of countable sets is finite, in other words, $A_1 \times A_2 \times \cdots \times A_k$ is finite if A_i is countable $\forall\, i \in [k]$. (1.27)
- The sets $\mathbb{N}$, $\mathbb{N}^2$, $\mathbb{Z}$ and $\mathbb{Q}$ are all countably infinite. (1.24)
- A set is uncountable if it is not countable. (1.28)
- For any set A, there is no injection $f : \wp(A) \to A$ and no surjection $g : A \to \wp(A)$. (1.30)
- Nested Interval Property: The countable intersection of nested closed intervals is nonempty. (1.31)
- The set of real numbers ($\mathbb{R}$) is uncountable. (1.33)

2 Sequences and Limits

Introduction to Sequences

Definition 2.1 (Sequence) A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$.

Remark 2.2 We have the notation $a_n = f(n)$ for the general term of the sequence, and $(a_n)_{n\in\mathbb{N}}$ or $\langle a_n \rangle$ for the whole sequence. Some examples of sequences include

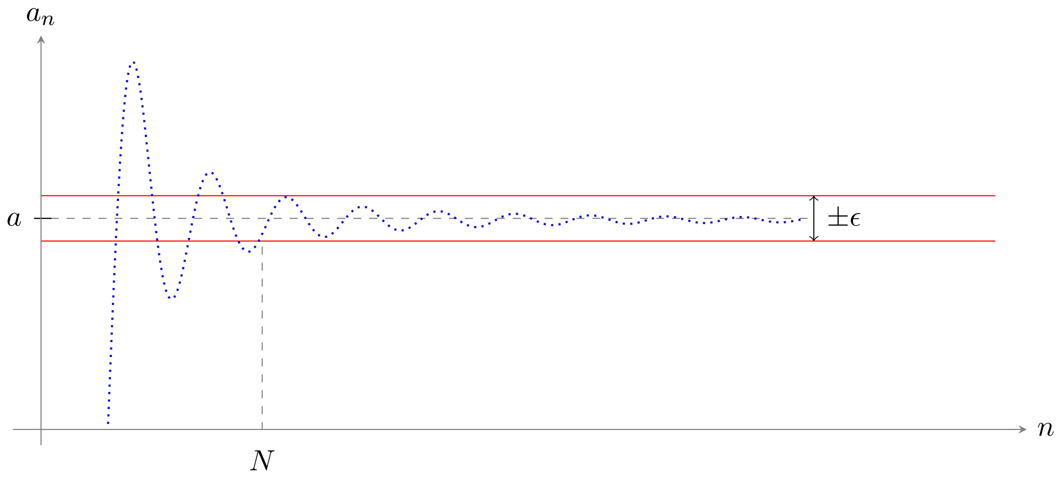
$$\begin{aligned}
(n)_{n\in\mathbb{N}} &= 1, 2, 3, 4, \cdots \\
(1)_{n\in\mathbb{N}} &= 1, 1, 1, 1, \cdots \\
\left(\frac{1}{n}\right)_{n\in\mathbb{N}} &= 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \\
(2^n)_{n\in\mathbb{N}} &= 2, 4, 8, 16, \cdots \\
\left(\frac{(-1)^n}{n!}\right)_{n\in\mathbb{N}} &= -1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, \cdots
\end{aligned}$$

Definition 2.3 (Convergence) A sequence $(a_n)_{n\in\mathbb{N}}$ is said to *converge* to the number $a \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$. Formally:

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} : \forall n \geq N, |a_n - a| < \epsilon$$

The real number a is called the *limit* of the sequence $(a_n)_{n\in\mathbb{N}}$.

Remark 2.4 We write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ as $n \to \infty$ to denote that the limit of the sequence $(a_n)_{n\in\mathbb{N}}$ is a. The following plot graphically represents the notion of convergence.



Intuitively, we see that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to a if given any $\epsilon > 0$, we can find an N beyond which all values a_n are enveloped in the range $(a+\epsilon, a-\epsilon)$. It is important to note that the choice of $N \in \mathbb{N}$ depends on ϵ.

Definition 2.5 (ϵ-Neighbourhood) Let $a \in \mathbb{R}$ and $\epsilon > 0$. The set denoted $B(a;\epsilon)$ or $B_\epsilon(a)$, defined

$$B(a;\epsilon) = \{x \in \mathbb{R} : |x-a| < \epsilon\} = (a-\epsilon, a+\epsilon)$$

is called the ϵ-*neighbourhood* with centre a (or sometimes the open ball with centre a, radius ϵ).

One can define the convergence of $(a_n)_{n\in\mathbb{N}}$ in terms of ϵ-neighbourhoods as follows:

$$\lim_{n\to\infty} a_n = a \iff \forall\epsilon > 0, \exists\, N \in \mathbb{N} : \forall n \geq N, a_n \in B(a;\epsilon)$$

Definition 2.6 (Null Sequence) If $\lim_{n\to\infty} a_n = 0$, the sequence $(a_n)_{n\in\mathbb{N}}$ is called a null sequence; i.e.

$$\forall\epsilon > 0, \exists\, N \in \mathbb{N} : \forall n \geq N, |a_n| < \epsilon$$

It is clear that $\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} (a_n - a) = 0$.

Example 2.7 $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ is a null sequence.

Proof. Given $\epsilon > 0$, by the Archimedean property of $\mathbb{R}$ there exists $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. Then for all $n \geq N$ we get

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Hence $\lim_{n\to\infty} \dfrac{1}{n} = 0$, as required. □

Theorem 2.8 A sequence $(a_n)_{n\in\mathbb{N}}$ is null iff the sequence $(|a_n|)_{n\in\mathbb{N}}$ is null.

Proof. The theorem follows trivially from the fact that $||a_n|| = |a_n|$. □

Examples 2.9 We give two examples which apply the theorems introduced so far.

(i) The sequence $\left(\dfrac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ is null since

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

which is null (i.e. we have combined example 2.7 with theorem 2.8).

(ii) The sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = \dfrac{n}{n+1}$ has $\lim\limits_{n\to\infty} a_n = 1$.

Proof. Given some $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left|\frac{n}{n+1} - 1\right| < 0.$$

But

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1}$$

so we need to find some $N \in \mathbb{N}$ such that $\dfrac{1}{n+1} < \epsilon$ for all $n \geq N$.

Now $\dfrac{1}{n+1} < \epsilon \iff \dfrac{1}{\epsilon} < n+1 \iff \dfrac{1}{\epsilon} - 1 < n$. Hence if we choose any $N > \dfrac{1}{\epsilon} - 1$ (which must exist by the Archimedean property) then we would have $n > \dfrac{1}{\epsilon} - 1$ for any $n \geq N$ and therefore

$$\left|\frac{1}{n+1}\right| < \epsilon$$

for every $n \geq N$; which in turn implies $\left|\dfrac{n}{n+1} - 1\right| < \epsilon$ for every $n \geq N$.

Therefore we have $\lim\limits_{n\to\infty} a_n = 1$, as required. □

Definition 2.10 (Divergence) A sequence that does not converge is said to *diverge*.

Example 2.11 Consider the sequence $-3, -2, -1, 0, 1, 0, 1, 0, 1, \ldots$. We can show that this sequence diverges by contradiction.

Suppose the sequence converges to $a \in \mathbb{R}$. By definition, $\forall \epsilon > 0, \exists\, N \in \mathbb{N} : \forall n \geq N$,

$$|a_n - a| < \epsilon$$

Take $\epsilon = \dfrac{1}{3}$. According to the above, there exists some $N \in \mathbb{N}$ such that

$$|a_n - a| < \frac{1}{3}$$

for any $n \geq N$. But whatever N is, a_n for $n \geq N$ contains both zeros and ones. Hence

$$|0 - a| < \frac{1}{3} \qquad \text{and} \qquad |1 - a| < \frac{1}{3}$$

Therefore $1 = |1 - 0| = |1 - a + a - 0| \leq |1 - a| + |a - 0| < \dfrac{1}{3} + \dfrac{1}{3} = \dfrac{2}{3}$ ※

Remark 2.12 Note the use of the triangle inequality and its manipulation. It will be used often throughout such proofs.

Theorem 2.13 A sequence $(a_n)_{n\in\mathbb{N}}$ can have at most one limit; i.e. if $(a_n)_{n\in\mathbb{N}}$ converges, the limit is unique.

Proof. We proceed by contradiction. Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$ with $a \neq b$. Then

$$\begin{aligned} \forall\epsilon > 0,\ \exists\, N_1 \in \mathbb{N} : \forall n \geq N_1, |a_n - a| < \epsilon \\ \exists\, N_2 \in \mathbb{N} : \forall n \geq N_2, |a_n - b| < \epsilon \end{aligned}$$

In particular, these would be true for $\epsilon = \dfrac{1}{3}|a-b| > 0$. Now we let $N = \max\{N_1, N_2\}$; so if $n \geq N$,

$$|a-b| = |a + a_n - a_n - b| \leq |a_n - a| + |a_n - b| < \frac{1}{3}|a-b| + \frac{1}{3}|a-b| = \frac{2}{3}|a-b| \quad ※$$

Hence the limit must be unique. □

Bounded Sequences and Limit Theorems

Definition 2.14 (Boundedness) A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be *bounded* if $\exists\, M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$; i.e. $-M \leq a_n \leq M$ or $a_n \in [-M, M]$.

Theorem 2.15 Every convergent sequence is bounded.

Proof. Consider a convergent sequence $(a_n)_{n\in\mathbb{N}}$ and let $\lim_{n\to\infty} a_n = a$. Taking $\epsilon = 1$, by definition of convergence,

$$\exists\, N \in \mathbb{N} \text{ such that } \forall n \geq N, |a_n - a| < 1$$

But

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|$$

so the sequence from N onwards is bounded by $1 + |a|$.

For all the previous terms, of which we have finitely many, it shouldn't be difficult to show that they are bounded. If we let $M = \max\{|a_1|, |a_2|, |a_3|, \cdots, |a_N|, 1 + |a|\}$, then $|a_N| \leq M$ for every $n \in \mathbb{N}$. □

Note The contrapositive[4] to this theorem tells us that every unbounded sequence is divergent. In fact, boundedness is a necessary condition but *not a sufficient one* to show convergence.

Theorem 2.16 If $\lim_{n\to\infty} a_n = a \neq 0$, then $\exists\, N \in \mathbb{N}$ such that

$$|a_n| > \frac{|a|}{2}$$

for all $n \geq N$. Moreover, if $a > 0$, then one has $a_n > \dfrac{a}{2}$ for all $n \geq N$ whereas if $a < 0$, then one has $a_n < \dfrac{a}{2}$ for all $n \geq N$.

[4] The contrapositive of $a \Rightarrow b$ is $\neg b \Rightarrow \neg a$. The contrapositive of a true statement is always true (see Introductory Mathematics notes).

Proof. Consider
$$|a| = |a - a_n + a_n| \leq |a - a_n| + |a_n|$$
hence $|a| - |a_n| \leq |a - a_n|$.

Now let $\epsilon = \dfrac{|a|}{2} > 0$. Since $\lim\limits_{n\to\infty} a_n = a \neq 0$, then $\exists\, N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \dfrac{|a|}{2}$. Therefore
$$|a| - |a_n| \leq |a - a_n| < \frac{|a|}{2}$$
so
$$|a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}$$
proving the first part of our theorem. Next, consider
$$|a_n - a| < \frac{|a|}{2} \iff -\frac{|a|}{2} < a_n - a < \frac{|a|}{2} \iff a - \frac{|a|}{2} < a_n < a + \frac{|a|}{2}$$
So if $a > 0$ we have
$$a_n > a - \frac{|a|}{2} = a - \frac{a}{2} = \frac{a}{2}$$
whereas with $a < 0$ we have
$$a_n < a + \frac{|a|}{2} = a - \frac{a}{2} = \frac{a}{2}$$
both for all n. □

Theorem 2.17 (Algebraic Limit Theorem) Let $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Then

(i) $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$.

(ii) $\lim\limits_{n\to\infty} (a_n b_n) = ab$.

(iii) If in addition, $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

Proof. We will prove each part of the theorem individually.

(i) We aim to show that $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall n \geq N$, $|(a_n + b_n) - (a + b)| < \epsilon$. But note that
$$|(a_n + b_n) - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b|$$
Now
$$\lim_{n\to\infty} a_n = a \implies \exists\, N_1 \in \mathbb{N} \text{ such that } \forall n \geq N_1, |a_n - a| < \frac{\epsilon}{2}$$
$$\lim_{n\to\infty} b_n = b \implies \exists\, N_2 \in \mathbb{N} \text{ such that } \forall n \geq N_2, |b_n - b| < \frac{\epsilon}{2}$$
so if we let $N = \max\{N_1, N_2\}$, for all $n \geq N$ we have $|a_n - a| + |b_n - n| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$.

(ii) We aim to show that $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n b_n - ab| < \epsilon$. But note that
$$|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab| = |(a_n - a)b_n + (b_n - b)a| \leq |a_n - a||b_n| + |b_n - b||a|$$
Since $(b_n)_{n\in\mathbb{N}}$ is convergent, then it is bounded. This means that $|b_n| \leq M_1$ where $M_1 > 0$. Let $M = \max\{|a|, M_1\}$. Now
$$\lim_{n\to\infty} a_n = a \implies \exists\, N_1 \in \mathbb{N} \text{ such that } \forall n \geq N_1, |a_n - a| < \frac{\epsilon}{2M}$$
$$\lim_{n\to\infty} b_n = b \implies \exists\, N_2 \in \mathbb{N} \text{ such that } \forall n \geq N_2, |b_n - b| < \frac{\epsilon}{2M}$$
Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ we have $|a_n - a||b_n| + |b_n - b||a| < \dfrac{\epsilon}{2M}M + \dfrac{\epsilon}{2M}M = \epsilon$.

(iii) We aim to show that $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall n \geq N$, $\left|\dfrac{a_n}{b_n} - \dfrac{a}{b}\right| < \epsilon$. But note that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - a b_n}{b_n b}\right| = \frac{|a_n b - a b_n|}{|b_n||b|}$$

Also, $|a_n b - a b_n| = |a_n b - ab + ab - a b_n| = |(a_n - a)b + a(b - b_n)| \leq |a_n - a||b| + |a||b - b_n|$, therefore

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{|a_n - a||b| + |a||b - b_n|}{|b_n||b|}$$

Now

$$\lim_{n\to\infty} b_n = b \implies \exists\, N_1 \in \mathbb{N} \text{ such that } \forall n \geq N_1, |b_n| < \frac{|b|}{2} \quad \text{(by theorem 2.16)}$$

$$\lim_{n\to\infty} a_n = a \implies \exists\, N_2 \in \mathbb{N} \text{ such that } \forall n \geq N_2, |a_n - a| < \frac{\epsilon|b|}{4}$$

$$\lim_{n\to\infty} b_n = b \implies \exists\, N_3 \in \mathbb{N} \text{ such that } \forall n \geq N_3, |b_n - b| < \frac{\epsilon|b|^2}{4|a|}$$

Letting $N = \max\{N_1, N_2, N_3\}$, then for all $n \geq N$ we have

$$\frac{|a_n - a||b| + |a||b - b_n|}{|b_n||b|} < \frac{\frac{\epsilon|b|^2}{4} + \frac{|a|\epsilon|b|^2}{4|a|}}{\frac{|b|}{2}|b|} = \epsilon$$

□

Corollary 2.18 If $\lim\limits_{n\to\infty} a_n = a$ then $\lim\limits_{n\to\infty} (c a_n) = ca$ for all $c \in \mathbb{R}$.

Proof. This result follows from the Algebraic Limit Theorem part (ii) with $b_n = c$ for all $n \in \mathbb{N}$. □

Theorem 2.19 (Order Limit Theorem) Let $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$ and $\lim\limits_{n\to\infty} c_n = c$ then

(i) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(ii) If $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $a = b$, then $a = c = b$.

Proof. (i) We proceed by way of contradiction. Suppose $b < a$ and let $0 < \epsilon < \dfrac{a-b}{2}$.

$$\lim_{n\to\infty} a_n = a \implies \exists\, N_1 \in \mathbb{N} \text{ such that } \forall n \geq N_1, |a_n - a| < \epsilon$$

i.e. that $-\epsilon < a_n - a < \epsilon$ and hence that $-\epsilon + a < a_n < \epsilon + a$. Similarly;

$$\lim_{n\to\infty} b_n = b \implies \exists\, N_2 \in \mathbb{N} \text{ such that } \forall n \geq N_2, |b_n - b| < \epsilon$$

i.e. that $-\epsilon + b < b_n < \epsilon + b$. Now let $N = \max\{N_1, N_2\}$. Recall that $\epsilon < \dfrac{a-b}{2}$, so $2\epsilon < a - b$ and hence $b + \epsilon < a - \epsilon$. Therefore we have $b_n < \epsilon + b < a - \epsilon < a_n$ which contradicts the hypothesis that $a_n \leq b_n$ for all n ※.

(ii) If $a_n \leq b_n \leq c_n$, then $a_n \leq c_n$ and $c_n \leq b_n$. Hence by (i) we have that $a \leq c$ and $c \leq b$, thus $a \leq c \leq b$. But $a = b$, so $a = c = b$.

□

Corollary 2.20 Let $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. If $\exists\, c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$; then $c \leq b$. Similarly if $a_n \leq c$ for all $n \in \mathbb{N}$; then $a \leq c$.

The Monotone Convergence Theorem

We have already seen that boundedness of a sequence is not enough for convergence. Let us now show that if a bounded sequence is also monotone, then it converges.

Recall that a sequence is bounded if $\exists\, M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. In particular, this means that a sequence is bounded above if there exists $K > 0$ such that $a_n \leq K$ and a sequence is bounded below if there exists $T > 0$ such that $a_n \geq -T$, for all $n \in \mathbb{N}$.

It is clear that a sequence is bounded iff it is bounded above and below.

Definition 2.21 (Monotonicity) A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. Similarly, it is said to be *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. It is said to be *monotone* if it is either increasing or decreasing.

Examples 2.22 We give a few examples of sequences, and state whether they are monotone.

(i) $(c)_{n\in\mathbb{N}}$ where $c \in \mathbb{R}$ is both increasing and decreasing, hence monotone.

(ii) $\left(n^2\right)_{n\in\mathbb{N}}$ is increasing, hence monotone.

(iii) $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is decreasing, hence monotone.

(iv) $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ is neither increasing nor decreasing, hence it is not monotone.

Theorem 2.23 (Monotone Convergence Theorem) If a sequence is monotone and bounded, then it converges. In fact, if a sequence is bounded above and increasing or bounded below and decreasing, then it converges.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence which is bounded above. This means that $A = \{a_n : n \in \mathbb{N}\}$ is bounded above and has a supremum by the completeness axiom. Let $a = \sup A$. We show that $\lim_{n\to\infty} a_n = a$.

By the definition of supremum, $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $a - \epsilon < a_N$. But $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence, so $a_n \leq a_N$ for all $n \geq N$. So together with the fact that a is an upper-bound, we get

$$a - \epsilon < a_N \leq a_n \leq a < a + \epsilon \iff -\epsilon < a_n - a < \epsilon \iff |a_n - a| < \epsilon$$

for all $n \in \mathbb{N}$, and hence $(a_n)_{n\in\mathbb{N}}$ converges. $\square$

Exercise 2.24 Prove the above result if the sequence is decreasing and bounded below.

Example 2.25 Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined recursively by

$$a_1 = \sqrt{2} \qquad \text{and} \qquad a_n = \sqrt{a_{n-1} + 2} \quad \forall n > 1$$

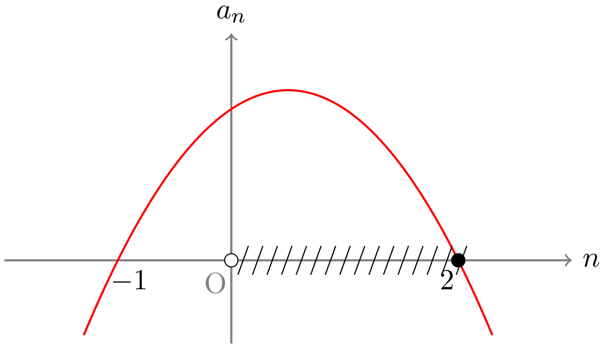
We wish to show that $(a_n)_{n\in\mathbb{N}}$ converges.

First we show that $(a_n)_{n\in\mathbb{N}}$ is bounded above by induction. For $n = 1$, $a_1 = \sqrt{2} \leq 2$ (we hypothesise that the limit is 2). Now assume that $a_k \leq 2$. For $n = k+1$, $a_{k+1} = \sqrt{a_k + 2} \leq \sqrt{2+2} = 2$. Therefore $(a_n)_{n\in\mathbb{N}}$ is bounded above. Also note that $a_n > 0$ for all $n \in \mathbb{N}$ (i.e. it is is bounded below as well).

Next we show it is increasing; i.e. $a_{k+1} \geq a_k$ for all $n \in \mathbb{N}$. But this is the same as saying that

$$\begin{aligned} &\sqrt{a_n + 2} \geq a_n \\ \iff &a_n + 2 \geq {a_n}^2 \\ \iff &2 + a_n - {a_n}^2 \geq 0 \\ \iff &(2 - a_n)(1 + a_n) \geq 0 \end{aligned}$$

Which is true since $0 < a_n \leq 2$, and graphically:



Therefore by the monotone convergence theorem, $(a_n)_{n\in\mathbb{N}}$ converges. Now we verify that its limit is actually 2.

If $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} a_{n-1} = a$ as well. Since $a_n = \sqrt{a_{n-1}+2}$ then

$${a_n}^2 = a_{n-1} + 2$$

and by the algebraic limit theorem, $\lim_{n\to\infty} {a_n}^2 = a^2$ and $\lim_{n\to\infty}(a_{n-1}+2) = a + 2$. Hence by the same theorem,

$$\begin{aligned} &a^2 = a + 2 \\ &\Rightarrow a^2 - a - 2 = 0 \\ &\Rightarrow (a+1)(a-2) = 0 \\ &\Rightarrow a = -1 \quad \veebar \quad a = 2 \end{aligned}$$

But $a_n \geq 0$ for all n, so $a = -1$ is contradictory by the order limit theorem ※. Therefore

$$\lim_{n\to\infty} a_n = 2.$$

Now we use the monotone convergence theorem to elaborate further on the nested interval property.

Theorem 2.26 (Stronger Nested Interval Property) Let there be given a decreasing sequence of nonempty closed intervals

$$(I_n = [a_n, b_n])_{n\in\mathbb{N}}$$

with the property that $\lim_{n\to\infty}(b_n - a_n) = 0$, i.e. that the length of the intervals tends to zero. Then there is a unique point which belongs to all the intervals.

Proof. We have already show in theorem 1.31 that

$$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$$

Let us now show that it contains exactly one point.

It is not difficult to see that the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing and bounded above (say by b_1) and similarly that $(b_n)_{n\in\mathbb{N}}$ is increasing and bounded below (by a_1, for example). Thus the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge.

$$\text{Say } \lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b$$

By the order limit theorem, since $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$. Moreover, by the algebraic limit theorem, $\lim\limits_{n\to\infty} b_n - \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}(b_n - a_n) = 0$. Hence $a = b$.

Since we therefore have that $a_n \leq a = b \leq b_n$ for all $n \in \mathbb{N}$, then $a = b \bigcap\limits_{n\in\mathbb{N}} I_n$.

Finally we show that $a = b$ is the only element in the intersection. Suppose $c \in \bigcap\limits_{n\in\mathbb{N}} I_n$. Then $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$. By the order limit theorem (part ii) with $c_n = c$, we have that $a = c = b$ and hence the intersection contains exactly one element. □

The Bolzano-Weierstrass Theorem

Definition 2.27 (Subsequence) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let $n_1 < n_2 < \cdots < n_k < \cdots$ be in $\mathbb{N}$. The sequence $a_{n_1}, a_{n_2}, a_{n_3}, \cdots, a_{n_k}, \cdots$ is called a *subsequence* of $(a_n)_{n\in\mathbb{N}}$ and is denoted by $(a_{n_k})_{k\in\mathbb{N}}$, where $k \in \mathbb{N}$ is the index of the subsequence.

Remark 2.28 The order of the terms in a subsequence is always preserved and repetitions are not allowed.

For example, given the sequence $\left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$, we have

(i) $\left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}, \cdots\right\} = \left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$ which is trivially a subsequence (i.e. the sequence itself).

(ii) $\left\{\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}, \dfrac{1}{11}, \cdots\right\}$ is a subsequence.

(iii) $\left\{\dfrac{1}{10}, \dfrac{1}{100}, \dfrac{1}{1000}, \dfrac{1}{1000}, \cdots\right\}$ is a subsequence.

(iv) $\left\{1, 1, \dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{3}, \cdots\right\}$ is not a subsequence since it contains repetitions of elements in the original sequence.

(v) $\left\{\dfrac{1}{2}, \dfrac{1}{7}, \dfrac{1}{3}, 1, \cdots\right\}$ is not a subsequence since it does not respect the order of the original sequence.

Remark 2.29 One may also note that the indices of the subsequence $(n_k)_{k\in\mathbb{N}}$ form an increasing sequence, and we must have $k \leq n_k$ for all $k \in \mathbb{N}$.

Lemma 2.30 Let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$ where $\lim\limits_{n\to\infty} a_n = a$. Then $\lim\limits_{k\to\infty} a_{n_k} = a$.

Proof. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $|a_{n_k} - a| < \epsilon$. Now $\lim_{n\to\infty} a_n = a$ implies that

$$\exists\, N \in \mathbb{N} \text{ such that } |a_k - a| < \epsilon$$

for all $k \in \mathbb{N}$. Hence if $k \geq N$, $n_k \geq k \geq N$, and therefore $|a_{n_k} - a| < \epsilon$. □

One can use this result to prove the divergence of a sequence. For instance, consider the sequence

$$-3, -2, -1, 0, 1, 0, 1, 0, 1, \dots$$

It is clear that the sequences 0,0,0,0,... and 1,1,1,1,... are subsequences of the original where the first converges to 0 and the second converges to 1. Hence we can conclude that the original sequence cannot be convergent.

Example 2.31 Let $-1 < b < 1$ and consider the sequence

$$(b^n)_{n\in\mathbb{N}} = \{b^1, b^2, b^3, \dots\}$$

We will show that this sequence converges to zero.

Case 1: $0 < b < -1$. Since $b^1 > b^2 > b^3 > b^5 > \cdots > 0$ we get that $(b^n)_{n\in\mathbb{N}}$ is a decreasing sequence bounded below by zero. Therefore by the monotone convergence theorem, $\lim_{n\to\infty} b_n = a$ for some $a \in \mathbb{R}$. Since $b^n > 0$, then by corollary 2.20, $a \geq 0$.

Note that $\left(b^{2n}\right)_{k\in\mathbb{N}} = \{b^2, b^4, b^6, \dots\}$ is a subsequence of $(b^n)_{n\in\mathbb{N}}$. This means that $\lim_{n\to\infty} b^{2n} = a$ by lemma 2.30.

Also, $b^{2n} = b^n b^n$ so by the algebraic limit theorem we have that $\lim_{n\to\infty} b^{2n} = \lim_{n\to\infty} b^n \cdot \lim_{n\to\infty} b^n$. Therefore we have

$$a^2 = a$$

and since $a < 1$, then we must have $a = 0$.

Case 2: $b = 0$. Evidently, $\lim_{n\to\infty} b^n = 0$.

Case 3: $-1 < b < 0$. In this case, $0 < |b| < 1$ so the sequence $(|b|^n)_{n\in\mathbb{N}}$ converges to 0 by case 1. Hence, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n > N$,

$$||b^n| - 0| = ||b|^n| = |b^n| < \epsilon$$

Hence $|b^n - 0| < \epsilon$ for all $n \geq N$, so $\lim_{n\to\infty} b^n = 0$.

Theorem 2.32 (Bolzano-Weierstrass Theorem) Every bounded sequence has a subsequence that converges.

Proof. We shall make use of the stronger version of the nested interval property (2.26) to find a convergent subsequence to a given bounded sequence.

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then

$$\exists\, M > 0 : |a_N| \leq M \ \forall n \in \mathbb{N}$$

Consider the interval $[-M, M]$.

We divide this interval into two closed intervals, $[-M, 0]$ and $[0, M]$. At least one of these two intervals contains an infinite number of elements of the sequence $(a_n)_{n\in\mathbb{N}}$ Let one for which this is the case be denoted by I_1.

Let $a_{n_1} \in I_1$.

We now repeat the procedure to obtain I_2 by dividing I_1 into two closed intervals of equal length, denoting the half which contains an infinite number of elements from $(a_n)_{n\in\mathbb{N}}$ as I_2.

Let $a_{n_2} \in I_2$.

In general, we obtain a closed interval I_k by taking half I_{k-1} containing an infinite number of elements from $(a_n)_{n\in\mathbb{N}}$, and then selecting $n_k > n_{k-1}$ satisfying $a_{n_k} \in I_k$.

Let us now show that the subsequence $(a_{n_k})_{k\in\mathbb{N}}$ converges.

Note that the intervals I_k for $k \in \mathbb{N}$ form a decreasing sequence of closed intervals in $\mathbb{R}$ whose lengths tend to zero. In fact, by construction, the length of I_k is $\dfrac{M}{2^{k-1}}$ (which converges to zero). It follows from the stronger nested interval property that there is a unique point which belongs to all intervals I_k.

Let $\{a\} = \displaystyle\bigcap_{n\in\mathbb{N}} I_n$.

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $k \geq N$, which means that the length I_k is less than ϵ (any $N > 1 + \log_2 \frac{M}{\epsilon}$ will do). Since a and a_{n_k} are both in I_k, it follows that $|a_{n_k} - a| < \epsilon$ for all $k \geq N$. $\square$

The Cauchy Criterion

In this section we derive an intrinsic criterion for convergence, i.e. one that does not depend on the limit. We have already encountered such a criterion in the monotone convergence theorem.

Definition 2.33 (Cauchy Sequence) A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be a *Cauchy sequence* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon$ for every $m, n \geq N$. Formally:

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} : \forall m, n \geq N, |a_m - a_n| < \epsilon$$

Lemma 2.34 Every convergent sequence is a Cauchy sequence.

Proof. Given $\epsilon > 0$, we need to show that

$$\exists\, N \in \mathbb{N} : \forall m, n \geq N, |a_m - a_n| < \epsilon$$

Now $|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a_n - a|$. Given that $(a_n)_{n\in\mathbb{N}}$ converges, $\exists\, N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| \leq \dfrac{\epsilon}{2}$. Therefore for all $n, m \geq N$, we have

$$|a_m - a_n| \leq |a_m - a| + |a_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as required. $\square$

Lemma 2.35 Every Cauchy Sequence is bounded.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence and let $\epsilon = 1$. Then $\exists\ N \in \mathbb{N}$ such that $\forall\, m, n \geq N$, $|a_n - a_m| < 1$. Now

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N|$$

so we have $|a_n| < 1 + |a_N|$ for all $n \geq N$. Now let $M = \max\{|a_1|, |a_2|, |a_3|, \cdots, |a_{N-1}|, 1 + |a_N|\}$. Therefore $|a_n| \leq M$ for all $n \in \mathbb{N}$, and thus $(a_n)_{n\in\mathbb{N}}$ is bounded. $\square$

Lemma 2.36 A Cauchy sequence with a convergent subsequence converges to the same limit as the subsequence.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} a_{n_k} = a$.

Let $\epsilon > 0$ be given. We need to show that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$. Now

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a|$$

Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy, then there exists $N_1 \in \mathbb{N}$ such that $\forall m, n \geq N_1$, $|a_m - a_n| < \dfrac{\epsilon}{2}$.

Also, since $(a_{n_k})_{k\in\mathbb{N}}$ is convergent and $\lim\limits_{k\to\infty} a_{n_k} = a$, there exists $N_2 \in \mathbb{N}$ such that $\forall k \geq N_2$, $|a_{n_k} - a| < \dfrac{\epsilon}{2}$.

Now let $N = \max\{N_1, N_2\}$. If $k \geq N$, then $n_k \geq k \geq N$, so that we have

$$|a_k - a| \leq |a_k - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as required. □

Theorem 2.37 (Cauchy Criterion) A sequence $(a_n)_{n\in\mathbb{N}}$ converges iff it is Cauchy.

Proof. The proof that every convergent sequence is Cauchy is given by lemma 2.34.

Conversely, if $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then by lemma 2.35, $(a_n)_{n\in\mathbb{N}}$ is bounded; and by the Bolzano-Weierstrass theorem, it has a convergent subsequence. Finally by lemma 2.36, it converges to the same limit as the subsequence.

Therefore $(a_n)_{n\in\mathbb{N}}$ is convergent. □

Remark 2.38 It follows from the order limit theorem that if $\exists\, N \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq N$, then $\lim\limits_{n\to\infty} a_n = 0$ if $\lim\limits_{n\to\infty} b_n = 0$.

Examples 2.39 We give a few examples of convergent sequences.

(i) We show that $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$ for all $p > 0$.

Indeed by the Archimedean property of $\mathbb{R}$, given $\epsilon > 0$

$$\exists\, N \in \mathbb{N} \text{ such that } N > \sqrt[p]{\frac{1}{\epsilon}}$$

hence for all $n \geq N$, $\dfrac{1}{n^p} < \epsilon$, and therefore

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

(ii) We show that $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$ for all $p > 0$.

If $p > 1$, let $a_n = \sqrt[n]{p} = 1$. Thus

$$\begin{aligned} \sqrt[n]{p} &= 1 + a_n \\ \Rightarrow p &= (1 + a_n)^n \end{aligned}$$

By the binomial theorem, $1+na_n \leq (1+a_n)^n$ (by just taking the first two terms of the expansion). Therefore $1 + na_n \leq p$ and

$$0 < a_n \leq \frac{p-1}{n}$$

Hence $\lim\limits_{n\to\infty} a_n = 0$ by remark 2.38, and thus $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$ for $p > 1$.

If $p = 1$, $\sqrt[n]{1} = 1$ for all $n \in \mathbb{N}$.

Finally, if $0 < p < 1$, we take the case $\dfrac{1}{p} > 1$ so by the first case we have

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{p}} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1$$

(iii) We show that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Let $a_n = \sqrt[n]{n} - 1$. Then $(a_n + 1)^n = n$ and by the binomial theorem (similar to ii) we have

$$\frac{n(n-1)}{2}\, {a_n}^2 \leq (a_n + 1)^n$$

therefore ${a_n}^2 \leq \dfrac{2}{n-1}$ and therefore

$$a_n \leq \sqrt{\frac{2}{n-1}} \quad \forall n \geq 2$$

Hence $\lim\limits_{n\to\infty} a_n = 0$ by remark 2.38, and thus $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Limit Superior and Limit Inferior

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence, i.e. we have $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. We define the notation $\sup\limits_{k\geq n} a_k$ as

$$\sup_{k\geq n} a_k := \sup\,\{a_k\}_{k\geq n} = \sup\{a_k : k \geq n\} = \{a_n, a_{n+1}, a_{n+2}, \dots\}$$

and similarly we have $\inf\limits_{k\geq n} a_k$. Now the sequence $\big(\sup\limits_{k\geq n} a_k\big)_{n\in\mathbb{N}}$ is bounded (since $(a_n)_{n\in\mathbb{N}}$ is bounded) and since, by definition, the supremum is the *least* upper-bound, then the sequence in question must be decreasing. Therefore we have that $\big(\sup\limits_{k\geq n} a_k\big)_{n\in\mathbb{N}}$ is both bounded below (since it is bounded) and decreasing, and hence by the monotone convergence theorem, it must converge; i.e.

$$\lim_{n\to\infty} \big(\sup_{k\geq n} a_k\big)_{n\in\mathbb{N}}$$

exists. Similarly, $\big(\inf\limits_{k\geq n} a_k\big)_{n\in\mathbb{N}}$ is bounded above and increasing, hence $\lim\limits_{n\to\infty} \big(\inf\limits_{k\geq n} a_k\big)_{n\in\mathbb{N}}$ exists.

Definition 2.40 (Limit Inferior and Superior) The *limit superior* of a bounded sequence $(a_n)_{n\in\mathbb{N}}$, denoted $\limsup\limits_{n\to\infty} a_n$, is the limit

$$\lim_{n\to\infty} \big(\sup_{k\geq n} a_k\big)_{n\in\mathbb{N}}$$

Similarly, the *limit inferior* of a bounded sequence $(a_n)_{n\in\mathbb{N}}$, denoted $\liminf\limits_{n\to\infty} a_n$, is the limit

$$\lim_{n\to\infty}\big(\inf_{k\geq n} a_k\big)_{n\in\mathbb{N}}$$

Both exist by the monotone convergence theorem.

Theorem 2.41 If $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence, then $\limsup\limits_{n\to\infty} a_n = a$ iff

(i) $\forall\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N$, $a_n < a + \epsilon$; and

(ii) $\forall\epsilon > 0$ and $\forall\, N \in \mathbb{N}$, $\exists\, n \geq N$ such that $a - \epsilon < a_n$.

Proof. If $\lim\limits_{n\to\infty}\big(\sup\limits_{k\geq n} a_k\big)_{n\in\mathbb{N}} = a$, then $\forall\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N$, $\big|\sup\limits_{k\geq n} a_k - a\big| < \epsilon$.

For (i), we have $\sup\limits_{k\geq n} a_k < a + \epsilon \quad \forall\, n \geq N$ which gives us that $a_n < a + \epsilon \quad \forall\, n \geq N$, as required. Now for (ii), we have $a - \epsilon < \sup\limits_{k\geq n} a_k$ which means that $a - \epsilon$ is not an upper-bound for $\{a_k\}_{k\geq n}$, i.e. given any $N' \in \mathbb{N}$, $\exists\, n \geq N'$ such that $a - \epsilon < a_n$.

Conversely, given $\epsilon > 0$ with $N \in \mathbb{N}$ such that $\forall\, n \geq N$, $a_n < a + \frac{\epsilon}{2}$[5], then $\sup\limits_{k\geq n} a_k \leq a + \frac{\epsilon}{2} < a + \epsilon$. Also, given any $N \in \mathbb{N}$ with $n \geq N$ such that $a - \epsilon < a_n$, then $\sup\limits_{k\geq N} a_k > a - \epsilon \quad \forall, N \in \mathbb{N}$.

Therefore, $\forall n \geq N$, $\big|\sup\limits_{k\geq n} a_k - a\big| < \epsilon$. □

A similar proof can be constructed to show the following analogous result for $\liminf\limits_{n\to\infty} a_n$.

Theorem 2.42 If $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence, then $\liminf\limits_{n\to\infty} a_n = a$ iff

(i) $\forall\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N$, $a_n > a - \epsilon$; and

(ii) $\forall\epsilon > 0$ and $\forall\, N \in \mathbb{N}$, $\exists\, n \geq N$ such that $a_n < a + \epsilon$.

Theorem 2.43 For any bounded sequence $(a_n)_{n\in\mathbb{N}}$, we have:

(i) $\liminf\limits_{n\to\infty} a_n \leq \limsup\limits_{n\to\infty} a_n$

(ii) $\lim\limits_{n\to\infty} a_n = a \iff \liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n = a$

Proof. For any $n \in \mathbb{N}$, we must have $\inf\limits_{k\geq n} a_k \leq \sup\limits_{k\geq n} a_k$, and therefore by the order limit theorem, it follows that $\lim\limits_{n\to\infty}\big(\inf\limits_{k\geq n} a_k\big) \leq \lim\limits_{n\to\infty}\big(\sup\limits_{k\geq n} a_k\big)$, i.e. that $\liminf\limits_{n\to\infty} a_n \leq \limsup\limits_{n\to\infty} a_n$.

Now for (ii), we give the following argument:

$$\begin{aligned}
&\lim_{n\to\infty} a_n = a \\
\iff &\forall\epsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, n \geq N,\ |a_n - a| < \epsilon
\end{aligned}$$

[5]Note that we take $\frac{\epsilon}{2}$ here since the supremum of $(a_n)_{n\in\mathbb{N}}$ is not necessarily in $(a_n)_{n\in\mathbb{N}}$. Therefore using ϵ would have given us $\sup(a_n) \leq a + \epsilon$, but $\frac{\epsilon}{2}$ guarantees a strict inequality ($<$).

$$\begin{aligned}
&\iff \begin{array}{l} \forall \epsilon > 0,\ \exists\, N_1 \in \mathbb{N} \text{ such that } \forall\, n \geq N_1,\ a_n < a + \epsilon \\ \text{and } \exists\, N_2 \in \mathbb{N} \text{ such that } \forall\, n \geq N_2,\ a - \epsilon < a_n \end{array} \\
&\iff \begin{array}{l} \forall \epsilon > 0,\ \exists\, N_1 \in \mathbb{N} \text{ such that } \forall\, n \geq N_1,\ a_n < a + \epsilon \\ \text{and } \forall\, N_0 \in \mathbb{N},\ \exists\, n \geq N_0 \text{ such that } a - \epsilon < a_n \\ \qquad\qquad\qquad\qquad \text{and} \\ \exists\, N_2 \in \mathbb{N} \text{ such that } \forall\, n \geq N_2,\ a - \epsilon < a_n \\ \text{and } \forall N_0 \in \mathbb{N},\ \exists\, n \geq N_0 \text{ such that } a_n < a + \epsilon \end{array} \\
&\iff \lim_{n\to\infty} a_n = a \iff \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a
\end{aligned}$$

by theorems 2.41 and 2.42 respectively. □

Remark 2.44 If $\liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n$, then $(a_n)_{n\in\mathbb{N}}$ does not converge by theorem 2.43(ii). For example, take the sequence $1, -1, 1, -1, \dots$. This has unequal limit superior/inferior, and hence does not converge.

Remark 2.45 By a reasoning similar to that in theorem 1.8, one can easily show that for every bounded sequence $(a_n)_{n\in\mathbb{N}}$, we have $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$.

Theorem 2.46 If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two bounded sequences, then:

(i) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

(ii) $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$

(iii) $\limsup_{n\to\infty}(c\,a_n) = c \limsup_{n\to\infty} a_n$ for $c \geq 0$.

(iv) $\liminf_{n\to\infty}(c\,a_n) = c \liminf_{n\to\infty} a_n$ for $c \geq 0$.

This result is straightforward to prove using the algebraic limit theorem and is left as an exercise.

Summary

- A sequence $(a_n)_{n\in\mathbb{N}}$ is a function $f : \mathbb{N} \to \mathbb{R}$. (2.1)
- A sequence $(a_n)_{n\in\mathbb{N}}$ converges to a if $\forall \epsilon > 0,\ \exists\, N \in \mathbb{N} : \forall\, n \geq N, |a_n - a| < \epsilon$. We write $\lim_{n\to\infty} a_n = a$. (2.3)
- The ϵ-neighbourhood of a, denoted $B(a; \epsilon)$ or $B_\epsilon(a)$, is the open interval $(a - \epsilon, a + \epsilon)$. (2.5)
- A null sequence is a sequence which vanishes; i.e. has $\lim_{n\to\infty} a_n = 0$. (2.6)
- A sequence which does not converge is said to diverge. (2.10)
- The limit of a sequence is unique. (2.13)
- A sequence is bounded if its magnitude does not exceed some value M, i.e. $(a_n)_{n\in\mathbb{N}}$ is bounded by M if $|a_n| \leq M$ for all $n \in \mathbb{N}$. (2.14)
- The following are some useful results about bounded sequences:
  - Every convergent sequence is bounded. (2.15)

  - If $\lim_{n\to\infty} a_n = a \neq 0$, then $\exists\, N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n| > \dfrac{|a|}{2}$. (2.16)

- Algebraic Limit Theorem: If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then
  (i) $\lim_{n\to\infty}(a_n + b_n) = a + b$.
  (ii) $\lim_{n\to\infty}(a_n b_n) = ab$.
  (iii) If in addition, $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$. (2.17)
- Order Limit Theorem: If $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $\lim_{n\to\infty} c_n = c$ then
  (i) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
  (ii) If $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$ and $a = b$, then $a = c = b$. (2.19)
- A sequence is increasing if each successive term is larger than its previous, or decreasing if each successive term is smaller than its previous. In either case, such a sequence is called monotone. (2.21)
- Monotone Convergence Theorem: If a sequence is bounded above and increasing, or bounded below and decreasing, then it converges. (2.23)
- Stronger nested interval property: The countable intersection of nested closed intervals consists of a unique point. (2.26)
- A subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ is a sequence with indices $n_1 < n_2 < \cdots < n_k < \cdots$ in $\mathbb{N}$. (2.27)
- Bolzano-Weierstrass Theorem: Every bounded sequence has a subsequence which converges. (2.32)
- Cauchy Sequence: A sequence $(a_n)_{n\in\mathbb{N}}$ is Cauchy if $\forall \epsilon > 0, \exists\, N \in \mathbb{N} : \forall m, n \geq N, |a_m - a_n| < \epsilon$. (2.33)
- Cauchy Criterion: A sequence converges iff it is Cauchy. (2.37)
- The limit superior and inferior of a bounded sequence $(a_n)_{n\in\mathbb{N}}$ exist by the monotone convergence theorem, and are given by $\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\big(\sup_{k\geq n} a_k\big)$ and $\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\big(\inf_{k\geq n} a_k\big)$. (2.40)
- For any bounded sequence $(a_n)_{n\in\mathbb{N}}$,
  (i) $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$
  (ii) $\lim_{n\to\infty} a_n = a \iff \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$ (2.43)

3 Infinite Series

Introduction to Series

Consider the infinite sum
$$S = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$
whose terms are generated by a geometric progression. We will define the nth partial sum as the sum of the first n terms; i.e.
$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

To define the infinite sum S, we simply let $S = \lim_{n\to\infty} S_n$, i.e. S is the limit of the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$.

Definition 3.1 (Convergence of Series) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and let $S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$. Then the series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is said to *converge* if the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ converges.

The limit of $(S_n)_{n\in\mathbb{N}}$ is called the sum of the series. If the series is not convergent, it is said to be divergent.

Theorem 3.2 If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $S_n = \sum_{k=1}^{n} a_k$ be convergent. Then by definition, $S = \lim_{n\to\infty} S_n$ exists. Furthermore, $\lim_{n\to\infty} S_{n-1} = S$ as well. The partial sum S_n is given by $S_n = S_{n-1} + a_n$, hence we have

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ \Rightarrow \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= S - S = 0 \end{aligned}$$

□

Similarly, we have the following theorem.

Theorem 3.3 If $\sum_{n=1}^{\infty} a_n$ is convergent, then $S_{2n} - S_n = a_{n+1} + a_{n+2} + \cdots + a_{2n}$ goes to zero as $n \to \infty$.

Proof. Let $S_n = \sum_{k=1}^{n} a_k$ be convergent. Then by definition, $S = \lim_{n\to\infty} S_n$ exists. But $(S_{2n})_{n\in\mathbb{N}}$ is a subsequence of $(S_n)_{n\in\mathbb{N}}$ and hence $\lim_{n\to\infty} S_{2n} = S$ as well. Therefore

$$\lim_{n\to\infty} (S_{2n} - S_n) = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} S_n = S - S = 0$$

□

Remark 3.4 These results are the simplest necessary conditions for the convergence of a series, e.g. the following series

$$1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$$
$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

diverge since they do not satisfy either condition.

Example 3.5 (The Harmonic Series) The series $H_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since

$$S_{2n} - S_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

so that $\lim_{n\to\infty}(S_{2n} - S_n) \geq \frac{1}{2} \neq 0$, thus dissatisfying the condition for convergence in theorem 3.3.

Also, ${H_n}^p = \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ converges if $p > 1$ but not if $p \leq 1$ (examples 3.12 and 3.17).

Theorem 3.6 (Algebraic Theorem for Series) Let $\sum_{n=1}^{\infty} a_n = S$ and $\sum_{n=1}^{\infty} b_n = R$. Then

(i) $\sum_{n=1}^{\infty} ca_n = cS$.

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = S + R$.

Proof. (i) Consider the partial sums of $\sum_{n=1}^{\infty} ca_n$:

$$t_n = \sum_{k=1}^{n} ca_k = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = cS_n$$

Now by the algebraic limit theorem, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} cS_n = \lim_{n\to\infty} c \lim_{n\to\infty} S_n = c \lim_{n\to\infty} S_n = cS$.

(ii) Consider the partial sums of $\sum_{n=1}^{\infty}(a_n + b + n)$:

$$u_n = \sum_{k=1}^{n}(a_k+b+k) = (a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n) = (a_1+\cdots+a_n)+(b_1+\cdots+b_n) = S_n+R_n$$

Now by the algebraic limit theorem, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(S_n + R_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} R_n = S + R$.

$\square$

Theorem 3.7 (Cauchy Criterion for Series) The series $\sum_{n=1}^{\infty} a_n$ converges iff for every $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$$

for all $n > m \geq N$.

Proof. By definition, $\sum_{n=1}^{\infty} a_n$ converges iff the sequence of partial sums converges. By the Cauchy criterion for sequences (2.37), $(S_n)_{n\in\mathbb{N}}$ converges iff it is a Cauchy sequence. i.e.

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} : \forall m, n \geq N, |S_m - S_n| < \epsilon$$

Now $|S_m - S_n| = |S_n - S_m|$, therefore it suffices to show that $|S_n - S_m| < \epsilon$ for all $n > m \geq N$.

But $|S_n - S_m| = |a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_n|$, thus proving the theorem. □

Note Theorem 3.7 also gives an alternative proof for theorem 3.2. If one takes $n \in \mathbb{N}$ such that $n > n - 1 \geq N$ in theorem 3.7, one observes that if $\sum_{n=1}^{\infty} a_n$ converges, then $|a_n| < \epsilon$ for all $n \geq N$, which gives

$$\lim_{n\to\infty} a_n = 0$$

Series of Nonnegative Terms

Let us now consider series whose terms are nonnegative, i.e.

$$\sum_{n=1}^{\infty} a_n \quad \text{with } a_n \in \mathbb{R}^+ \quad \forall n \in \mathbb{N}$$

The main point is that the sequence of partial sums is increasing. In fact, $S_{n+1} - S_n = a_{n+1} \in \mathbb{R}^+$, implying that $S_{n+1} \geq S_n$ for all $n \in \mathbb{N}$. Thus the monotone convergence theorem for sequences allows us to formulate simple yet useful criteria for convergence of series.

Theorem 3.8 A series of nonnegative terms converges iff the sequence of partial sums is bounded above.

Proof. If the series converges, by definition, the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ converges, and is hence bounded (by theorem 2.15).

Conversely, if $(S_n)_{n\in\mathbb{N}}$ is bounded above, it follows from the hypothesis that it is an increasing bounded sequence in $\mathbb{R}$ and hence by the monotone convergence theorem it converges. □

Theorem 3.9 (The Comparison Test) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with nonnegative terms satisfying the inequality $a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Let $S_n = \sum_{k=1}^{n} a_k$ and $R_n = \sum_{k=1}^{n} b_k$ where $a_k \leq b_k$ for all k. Then we must have $S_n \leq R_n$, $\forall\, n$.

Now if R_∞ converges, then by theorem 3.8 it is bounded above and hence $(S_n)_{n\in\mathbb{N}}$ is bounded above. Therefore again by theorem 3.8, $\sum_{n=1}^{\infty} a_n$ converges. □

Remark 3.10 Note the contrapositive of the comparison test – if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note also that the test also holds if instead of the inequality $a_n \leq b_n$ we had $a_n \leq kb_n$ with some $k \in \mathbb{R}^+$. Indeed $S_n \leq kR_n$ for all n, and the result follows from theorem 3.8 exactly as before.

Remark 3.11 It is important to note that from the definition of convergence of a series that the fact that a series converges/diverges remains unaffected if a finite number of terms is added, removed or altered. Then it follows that the comparison test holds even if $a_n \leq b_n$ for $n \geq N$ for some $N \in \mathbb{N}$. The same goes for remark 3.10.

Similarly, if $\sum_{n=1}^{\infty} a_n$ is a series with nonnegative terms, then by deleting those terms which are equal to zero we will not affect the convergence/divergence of the series. We can therefore assume that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

Example 3.12 We have noted that the generalised harmonic series $H_n^p = 1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots\frac{1}{n^p}+\cdots$ converges for $p > 1$ and diverges for $p \leq 1$. Let us use the comparison test to show that it diverges for $p \leq 1$. In fact if $p \leq 1$, then $\frac{1}{n} \leq \frac{1}{n^p}$ for all $n \in \mathbb{N}$, and since H_n diverges, then H_n^p diverges by the comparison test.

Theorem 3.13 Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.

Proof. If $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ converges, then $\forall \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall n \geq N$, $\left|\frac{a_n}{b_n} - L\right| < \epsilon$.

In particular, this holds for $\epsilon = \frac{L}{2} > 0$; so

$$\begin{aligned} & \left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \\ \Rightarrow \quad & -\frac{L}{2} < \frac{a_n}{b_n} - L < \frac{L}{2} \\ \Rightarrow \quad & \frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2} \\ \Rightarrow \quad & b_n < \frac{2}{L}a_n \qquad \text{and} \qquad a_n < \frac{3L}{2}b_n \end{aligned}$$

By the comparison test, if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges and if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. □

Examples 3.14 We give two examples which use the test from theorem 3.13.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$ diverges since

$$\lim_{n\to\infty} \left(\frac{\left(\frac{2n}{n^2+1}\right)}{\frac{1}{n}} \right) = \lim_{n\to\infty} \frac{2n^2}{n^2+1} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{n^2}} = 2$$

and since the limit exists and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then so does $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3-2}$ converges since

$$\lim_{n\to\infty} \left(\frac{\left(\frac{n}{2n^3+2}\right)}{\frac{1}{n^2}} \right) = \lim_{n\to\infty} \frac{n^3}{2n^3+2} = \lim_{n\to\infty} \frac{1}{2+\frac{2}{n^3}} = \frac{1}{2}$$

and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then so does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3-2}$.

Example 3.15 (Geometric Series) We prove the result that $\displaystyle\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$ and diverges otherwise. If $r \neq 1$, then

$$\begin{aligned}
(1-r)\sum_{k=0}^{n-1} r^k = (1-r)S_n &= (1-r)\left(1+r^2+r^3+\cdots+r^{n-1}\right) \\
&= (1+r+r^2+r^3+\cdots+r^{n-1}) - (r+r^2+r^3+\cdots+r^n) \\
&= 1-r^n \\
\therefore S_n &= \frac{1-r^n}{1-r}
\end{aligned}$$

By the algebraic limit theorem (2.17) and example 2.31, $\displaystyle\lim_{n\to\infty} S_n = \frac{1}{1-r}$ for $|r| < 1$. Now if $|r| \geq 1$, then $|r|^n \geq 1$ for all $n \in \mathbb{N}$, so $\displaystyle\lim_{n\to\infty} r^n \neq 0$, and therefore $\displaystyle\sum_{r=0}^{\infty} r^n$ diverges by theorem 3.2.

Therefore we can conclude that $\displaystyle\sum_{r=0}^{\infty} r^n = \frac{1}{1-r}$ iff $|r| < 1$ and more generally,

$$\sum_{r=0}^{\infty} ar^n = \frac{a}{1-r} \iff |r| < 1$$

The upcoming theorem is particularly interesting because of the striking feature that a rather *thin* subsequence of $(a_n)_{n\in\mathbb{N}}$ is determining whether or not $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Theorem 3.16 (Cauchy's Condensation Test) Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0$. Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges iff the *condensed* series $\displaystyle\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$ converges.

Proof. Let us denote the nth partial sums by $S_n = a_1 + a_2 + \cdots + a_n$ and $T_n = a_1 + 2a_2 + \cdots + 2^{n-1}a_{2^{n-1}}$. By theorem 3.8, it is sufficient to consider the boundedness of $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$. Let $k \in \mathbb{N}$.

For every $n < 2^k - 1$,

$$\begin{aligned} S_n &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k - 1}) \\ &\leq a_1 + \quad 2a_2 \quad + \qquad 4a_4 \qquad + \cdots + \qquad 2^{k-1}a_{2^{k-1}} \\ &= T_k \end{aligned}$$

On the other hand, if $n > 2^{k-1}$ then

$$\begin{aligned} S_n &\geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-2}+1} + \cdots + a_{2^{k-1}}) \\ &\geq \frac{1}{2}a_1 + a_2 + \quad 2a_4 \quad + \qquad 4a_8 \qquad + \cdots + \qquad 2^{k-2}a_{2^{k-1}} \\ &= \frac{1}{2}T_k \\ \Rightarrow 2S_n &\geq T_k \end{aligned}$$

Now assume $(T_k)_{k\in\mathbb{N}}$ is convergent and let $T = \lim_{k\to\infty} T_k$. Then for any fixed n, taking the limit as $k \to \infty$ in the first inequality (which holds since we have $n < 2^k - 1$ for k large enough) gives

$$S_n \leq \lim_{k\to\infty} T_k = T$$

hence $(S_n)_{n\in\mathbb{N}}$ is a series of nonnegative terms bounded above by T, and therefore it converges by theorem 3.8.

Conversely, assume $(S_n)_{k\in\mathbb{N}}$ converges and let $S = \lim_{n\to\infty} S_n$. Then for any fixed k, taking the limit as $n \to \infty$ in the second inequality (which holds since we have $n > 2^{k-1}$ for n large enough) gives

$$T_k \leq \lim_{n\to\infty} 2S_n = 2S$$

hence $(T_k)_{k\in\mathbb{N}}$ is a series of nonnegative terms bounded above by $2S$, and is convergent by theorem 3.8. $\square$

Example 3.17 Consider again the generalised harmonic series ${H_n}^p$. We have shown in previous examples (3.5 and 3.12) that this diverges for $p \leq 1$. We will now show its convergence for $p > 1$.

If $p > 0$, then the Cauchy condensation test is applicable and we are lead to the series

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{(2^n)^p}\right) = \sum_{n=0}^{\infty} 2^{(1-p)n}$$

Now $2^{1-p} < 1$ iff $1 - p < 0$ and the result of convergence follows by comparison with the geometric series (example 3.15) with $r = 2^{1-p}$.

Example 3.18 We show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^p n}$ if $p > 1$ and diverges if $p \leq 1$.

We shall use the comparison test (3.9) to prove the divergence for $p \leq 0$.

First of all, if $p \leq 0$ then $\dfrac{1}{n} \leq \dfrac{1}{n \ln^p n}$ for all $n \leq 2$, since $\ln 3 > 1$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then divergence of the series for $p \leq 0$ above follows from the comparison test.

We use Cauchy's condensation test to prove the rest. Let $p > 0$. The monotonicity of the 'ln' function means that $(\ln n)_{n\in\mathbb{N}}$ is increasing and hence that $\left(\dfrac{1}{n \ln^p n}\right)_{n \geq 2}$ is decreasing. Hence the condensation test leads us to the series

$$\sum_{n=1}^{\infty} 2^n \frac{1}{2^n \ln^p 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^p \ln^p 2} = \frac{1}{\ln^p 2} \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{H_n{}^p}{\ln^p 2}$$

which converges if $p > 1$ and diverges if $p \leq 1$ (example 3.17).

Theorem 3.19 (The Root Test) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. Suppose $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L$. Then

(i) if $L < 1$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(ii) if $L > 1$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(iii) if $L = 1$ then the test is inconclusive.

Proof. Note that $L \geq 0$ since $a_n > 0$ for all $n \in \mathbb{N}$.

Suppose $L < 1$, and let $r \in \mathbb{R}$ such that $0 \leq L < r < 1$. Then by definition of convergence, we have $N \in \mathbb{N}$ such that $\forall n \geq N$, $\sqrt[n]{a_n} < r$. This rearranges to $a_n < r^n$, and hence convergence follows from comparison (3.9) with the series

$$\sum_{n=1}^{\infty} r^n$$

which converges for $|r| < 1$. Now we consider the case $L > 1$. Again, by definition of convergence we have $N \in \mathbb{N}$ such that $\forall n \leq N$, $\sqrt[n]{a_n} > 1$; and therefore $a_n > 1$ for all $n \geq N$. Then by theorem 3.2, the series diverges. □

Theorem 3.20 (The Ratio Test) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. Let $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$. Then

(i) If $L < 1$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $L > 1$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $L = 1$ then the test is inconclusive.

Proof. Note that $L \geq 0$ since $a_n \geq 0$ for all $n \in \mathbb{N}$.

Suppose $L < 1$, and let $r \in \mathbb{R}$ such that $0 \leq L < r < 1$. Then by definition of convergence, we have $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r$$

$$\begin{aligned}
\Rightarrow a_n &= \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_{N+2}}{a_{N+1}} \cdot \frac{a_{N+1}}{a_N} \cdot a_N \\
&< r^{n-N} a_N \quad \text{(since we have } n - N \text{ terms above excluding } a_N\text{)} \\
&= \frac{a_N}{r^N} r^n \quad \forall n \geq N
\end{aligned}$$

Hence the convergence of $\sum_{n=1}^{\infty} a_n$ follows from comparison (3.9) with the series $\sum_{n=1}^{\infty} r^n$ for $|r| < 1$. Now we consider $L > 1$. Again, by definition of convergence we have $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} > 1$$

so that $a_{n+1} > a_n > 0$ for all $n \geq N$. Hence $\sum_{n=1}^{\infty} a_n$ diverges by theorem 3.2. ☐

Example 3.21 We determine whether or not the series $\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$ converges by the root test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^{3n}}{3^{2n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{(2^3)^n}}{\sqrt[n]{(3^2)^n}} = \frac{2^3}{3^2} = \frac{8}{9} < 1$$

Hence the series converges.

Example 3.22 We determine whether or not the series $\sum_{n=1}^{\infty} \frac{2^n + n}{3^n - n}$ converges by the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2^{n+1} + n + 1}{2^n + n} \cdot \frac{3^n - n}{3^{n+1} - n - 1} \right) = \lim_{n\to\infty} \left(\frac{2 + \frac{n}{2^n} + \frac{1}{2^n}}{1 + \frac{n}{2^n}} \cdot \frac{1 - \frac{n}{3^n}}{3 - \frac{n}{3^n} - \frac{1}{3^n}} \right) = \frac{2}{3} < 1$$

Hence the series converges.

Absolute and Conditional Convergence

Although the tests introduced in the previous section require the terms of the series to be positive (or nonnegative), they are often used together with the following notion and theorem to handle series containing negative terms.

Definition 3.23 (Absolute Convergence) A series $\sum_{n=1}^{\infty} a_n$ is said to converge *absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 3.24 If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Proof. We will use the Cauchy criterion for series to prove this theorem. Since $\sum_{n=1}^{\infty}|a_n|$ converges, given $\epsilon > 0$ we have $N \in \mathbb{N}$ such that $\forall n, m \leq N$, $||a_{m+1}| + |a_{m+2}| + |a_{m+3}| + \cdots + |a_n|| < \epsilon$ (theorem 3.7). Now by the triangle inequality we have

$$|a_{m+1} + a_{m+2} + a_{m+3} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + |a_{m+3}| + \cdots + |a_n| < \epsilon$$

which again by theorem 3.7 gives us that $\sum_{n=1}^{\infty} a_n$ converges. □

Definition 3.25 (Conditional Convergence) A convergent series $\sum_{n=1}^{\infty} a_n$ is said to converge *conditionally* if it does not converge absolutely.

Lemma 3.26 Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ and let $S_n = b_1 + b_2 + b_3 + \cdots + b_n$. If $1 \leq m < n$, then

$$\sum_{k=m}^{n} b_k a_k = \sum_{k=m}^{n-1} S_k(a_k - a_{k+1}) + S_n a_n - S_{m-1}a_m$$

Proof. Since $b_k = S_k - S_{k-1}$, then

$$\begin{aligned}
\sum_{k=m}^{n} b_k a_k = \sum_{k=m}^{n}(S_k - S_{k-1})a_k &= \sum_{k=m}^{n} S_k a_k - \sum_{k=m}^{n} S_{k-1}a_k \\
&= \sum_{k=m}^{n} S_k a_k - \sum_{k=m-1}^{n-1} S_k a_{k+1} \\
&= \sum_{k=m}^{n-1} S_k a_k + S_n a_n - \sum_{k=m}^{n-1} S_k a_{k+1} - S_{m-1}a_m \\
&= \sum_{k=m}^{n-1} S_k(a_k - a_{k+1}) + S_n a_n - S_{m-1}a_m
\end{aligned}$$

□

Theorem 3.27 Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ and let $S_n = b_1 + b_2 + b_3 + \cdots + b_n$. Suppose that

(i) $(S_n)_{n\in\mathbb{N}}$ is bounded,

(ii) $a_1 \geq a_2 \geq a_3 \geq \cdots$,

(iii) and $\lim_{n\to\infty} a_n = 0$

Then the series $\sum_{n=1}^{\infty} b_n a_n$ converges.

Proof. We show that $\sum_{n=1}^{\infty} b_n a_n$ converges in Cauchy criterion terms. This means that given any $\epsilon > 0$, we have $N \in \mathbb{N}$ such that $\forall n \geq m \geq N$,

$$\left| \sum_{k=m}^{n} b_k a_k \right| < \epsilon$$

which by the previous lemma is equivalent to

$$\left| \sum_{k=m}^{n-1} S_k(a_k - a_{k+1}) + S_n a_n - S_{m-1} a_m \right| < \epsilon$$

Now since $(S_n)_{n\in\mathbb{N}}$ is bounded (by (i)), there exists $M > 0$ such that $|S_n| \leq M$ for all $n \in \mathbb{N}$, and therefore $S_k, S_n, -S_{m-1} \leq M$.

Also, since $\lim_{n\to\infty} a_n = 0$, given any $\epsilon > 0$ we have $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n| < \dfrac{\epsilon}{2M}$, which in particular means that $|a_N| < \dfrac{\epsilon}{2}$ and hence $|a_n| < |a_N| < \dfrac{\epsilon}{2M}$ (by (ii)).

Therefore we have

$$\begin{aligned}
\left| \sum_{k=m}^{n-1} S_k(a_k - a_{k+1}) + S_n a_n - S_{m-1} a_m \right| &\leq M \left| \sum_{k=m}^{n-1} (a_k - a_{k+1}) + a_n + a_m \right| \\
&= M|a_m - a_n + a_n + a_m| \\
&= M|2a_m| \\
&\leq 2Ma_N \\
&< 2M\left(\frac{\epsilon}{2M}\right) \\
&= \epsilon
\end{aligned}$$

□

Theorem 3.28 (The Alternating Series Test) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $a_1 \geq a_2 \geq a_3 \geq \cdots$ and $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Proof. Note that $(a_n)_{n\in\mathbb{N}} \geq 0$ for all $n \in \mathbb{N}$. Now let S_n denote the partial sums of $\sum_{k=1}^{\infty} b_k$ where $b_k = (-1)^{k+1}$ for all $k \in \mathbb{N}$. Thus

$$S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and hence $(S_n)_{n\in\mathbb{N}}$ is bounded. Consequentially, $\sum_{n=1}^{\infty} b_n a_n = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges by theorem 3.27. □

Remark 3.29 If $a_n \geq 0$, then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is called an alternating series.

Example 3.30 We show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

Since $1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \frac{1}{4} \geq \cdots$ and $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$, then it follows by the alternating series test that this series converges. However note that $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence the alternating harmonic series converges conditionally.

Rearrangements

Consider the alternating harmonic series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

We already know by example 3.30 that this converges. Consider the following:

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\ \Rightarrow \frac{1}{2}S &= \quad\;\; \frac{1}{2} \qquad\; - \frac{1}{4} \qquad\; + \frac{1}{6} - \cdots \end{aligned}$$

so adding them gives

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

We observe that the resulting series $\frac{3}{2}S$ consists of two positive terms followed by a negative term. In fact, these are the exact same terms that we had in the original series S. Such a series is called a rearrangement of the original (obtained by permuting the elements of the original series into a different order).

Definition 3.31 (Rearrangement) Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series. A series $\displaystyle\sum_{n=1}^{\infty} b_n$ is said to be a *rearrangement* of $\displaystyle\sum_{n=1}^{\infty} a$ if there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(n)} = a_n$ for all $n \in \mathbb{N}$.

The above example shows that a rearrangement of a convergent series can have a different sum. One can note that the above example is conditionally convergent. We will continue to show that rearrangements of absolute convergent series always converge to the same limit.

Theorem 3.32 If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely, then every rearrangement $\displaystyle\sum_{n=1}^{\infty} b_n$ of $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to the same limit.

Proof. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be absolutely convergent and let S be its sum. If $\displaystyle\sum_{n=1}^{\infty} b_n$ is a rearrangement of $\displaystyle\sum_{n=1}^{\infty} a_n$, then we have a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(n)} = a_n$ for all $n \in \mathbb{N}$.

Let $S_n = \sum_{k=0}^{n} a_k$ and $T_n = \sum_{k=1}^{n} b_k$. Thus $\lim_{n\to\infty} S_n = S$, and we need to show that $\lim_{n\to\infty} T_n = S$, i.e. that

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} : |T_n - S| < \epsilon$$

Now by definition of convergence of $(S_n)_{n\in\mathbb{N}}$, $\exists\, N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|S_n - S| < \dfrac{\epsilon}{2}$.

But $\sum_{n=1}^{\infty} |a_n|$ converges as well (absolute convergence) so by the Cauchy criterion for series we have $N_2 \in \mathbb{N}$ such that $\forall n > m \geq N_2$, $|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \dfrac{\epsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. The finite set of terms $\{a_1, a_2, a_3, \cdots, a_N\}$ appear in the rearranged series as the terms $\{b_{f(1)}, b_{f(2)}, b_{f(3)}, \cdots, b_{f(N)}\}$ and we want to go far enough through the rearranged series so that we have included all of these terms. Let $M = \max\{f(1), f(2), f(3), \cdots, f(N)\}$. Now it is clear that if $m \geq M$, the terms $a_1, a_2, a_3, \cdots, a_N$ will cancel in the difference $T_m - S_N$. Thus $T_m - S_N$, $m \in \mathbb{N}$, is a finite set of terms whose absolute value appears in the tail of $\sum_{n=N+1}^{\infty} |a_n|$.

By way of the triangle inequality, our choice of N guarantees that $|T_m - S_N| < \epsilon$.

Hence $|T_m - S| \leq |T_m - S_N| + |S_N - S| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$. □

Summary

- A series converges if its sequence of partial sums converges. The limit of the series is the limit of the sequence of partial sums. (3.1)
- If a series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. (3.2)
- The Harmonic Series $H_n = \sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, but the series $H_n{}^p = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges for $p > 1$. (3.5, 3.12, 3.17)
- Algebraic Limit Theorem for Series: $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (3.6)
- Cauchy Criterion for Series: The series $\sum_{n=1}^{\infty} a_n$ converges iff for every $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$ for all $n > m > N$. (3.7)
- A series of nonnegative terms converges iff the sequence of partial sums is bounded above. (3.8)
- Tests for the convergence of Series with nonnegative terms:
  - Comparison Test: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with nonnegative terms such that $a_n \leq b_n$ for all n, $\sum_{n=1}^{\infty} a_n$ converges if b_n converges. (3.9)

– If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of positive numbers and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges. (3.12)

– Cauchy's Condensation Test: Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff the condensed series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. (3.16)

– Root Test: If $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers and $\lim_{n\to\infty} \sqrt[n]{a_n} = L$, then $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$, or diverges if $L > 1$. (3.19)

– Ratio Test: If $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, then $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$, or diverges if $L > 1$. (3.20)

- A sequence $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. If it converges but does not converge absolutely, then it is said to converge conditionally. (3.23, 3.25)
- If a series converges absolutely, then it converges. (3.24)
- The Alternating Series Test: If $(a_n)_{n\in\mathbb{N}}$ is a sequence with $a_1 \geq a_2 \geq a_3 \geq \cdots$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. (3.28)
- The Alternating Harmonic Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. (3.30)
- A rearrangement of a series $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} b_n$ such that there exists a bijection $f : \mathbb{N} \to \mathbb{N}$, where $b_{f(n)} = a_n$ for all $n \in \mathbb{N}$. (3.31)
- Every rearrangement of an absolutely convergent series converges to the same limit. (3.32)

4 Topology of $\mathbb{R}$

Open and Closed Sets

Recall For $a \in \mathbb{R}$ and $\epsilon > 0$, the set

$$B(a;\epsilon) = B_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

is called the ϵ-neighbourhood of a or the open ball with centre a and radius ϵ. This terminology allowed us to write the definition for convergence of a sequence $(a_n)_{n\in\mathbb{N}}$ as follows:

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} : \forall n \geq N, a_n \in B(a;\epsilon)$$

Definition 4.1 (Openness) A set $O \subseteq \mathbb{R}$ is said to be *open* if for every $a \in O$, $\exists\, \epsilon > 0$ such that $B(a;\epsilon) \subseteq O$.

Examples 4.2 We give some examples of open sets.

(i) The real line $\mathbb{R}$ itself is an open set. Indeed given any $a \in \mathbb{R}$, $B(a;\epsilon) \subseteq \mathbb{R}$ for all $\epsilon > 0$.

(ii) The empty set $\varnothing$ is open. In fact, suppose $\varnothing$ is not open. Then $\exists\, a \in \varnothing$ such that $\forall \epsilon > 0$, $B(a;\epsilon) \nsubseteq \varnothing$. This is obviously a contradiction.

(iii) Every open interval is an open set.

$$(c,d) \in \mathbb{R} = \{x \in \mathbb{R} : c < x < d\}$$

Indeed given $a \in (c,d)$, let $\epsilon = \min\{a-c, d-a\}$. Then $B(a;\epsilon) \subseteq (c,d)$.

It follows from (iii) that every ϵ-neighbourhood is an open set.

Note The argument used in 4.2(iii) will not work for half-open intervals $(c,d]$ or $[c,d)$ nor for closed intervals $[c,d]$. The problem is caused by the endpoints: given any ϵ it's impossible to find an ϵ-neighbourhood in the interval.

Theorem 4.3 The arbitrary union of open sets is open.

Proof. Let $a \in \bigcup_{\alpha \in \mathcal{A}} A_\alpha$, where $\bigcup_{\alpha \in \mathcal{A}} A_\alpha$ is the arbitrary union of open sets $\{A_\alpha\}_{\alpha \in \mathcal{A}}$.
So $\exists\, \alpha_0 \in \mathcal{A}$ such that $a \in A_{\alpha_0}$. But A_{α_0} is open, which means $\exists\, \epsilon > 0$ such that

$$B(a;\epsilon) \subseteq A_{\alpha_0} \subseteq \bigcup_{\alpha \in \mathcal{A}} A_\alpha$$

proving the openness of the union. □

Theorem 4.4 The intersection of a finite collection of open sets is open.

Proof. Let $\{O_1, O_2, O_3, \cdots, O_n\}$ be a finite collection of open sets, and let $a \in \bigcap_{k=1}^{n} O_k$. Since each O_k is open, then $\exists\, \epsilon_k > 0$ such that $B(a;\epsilon_k) \subseteq O_k$ for all $k \in [1,n]$. So if we take $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_n\}$ then $B(a;\epsilon) \subseteq O_k$ for all $k \in [1,n]$ and therefore

$$B(a;\epsilon) \subseteq \bigcap_{k=1}^{n} O_k$$

proving the openness of the intersection. □

Remark 4.5 Note that the intersection of arbitrary/countable collection of open sets need not be open. In fact, let $O_n = \left(-\dfrac{1}{n}, \dfrac{1}{n}\right)$ for all $n \in \mathbb{N}$. This is an open interval and hence an open set for all $n \in \mathbb{N}$. But then the intersection $\bigcap_{n \in \mathbb{N}} O_n = \{0\}$, which is not open. Suppose it is. Hence $\exists\, \epsilon > 0$ such that $B(0;\epsilon) \subseteq \{0\}$ ※. Note that this applies to any singleton (set with only one element).

We will now address the dual notion of closed sets.

Definition 4.6 (Limit point) Let $A \subseteq \mathbb{R}$. A point $a \in \mathbb{R}$ is said to be a *limit point* (or accumulation point) of A if for every $\epsilon > 0$, the ϵ-neighbourhood $B(a;\epsilon)$ contains points of A other than a itself, i.e.

$$[B(a;\epsilon) \cap A] \setminus \{a\} \neq \varnothing$$

Theorem 4.7 Let $A \subseteq \mathbb{R}$. The point $a \in \mathbb{R}$ is a limit point of A iff there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A satisfying

$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad a_n \neq a \ \ \forall n \in \mathbb{N}$$

Proof. Let $a \in \mathbb{R}$ be a limit point of A. Then by definition, taking $\epsilon = \dfrac{1}{n}$ we have

$$\left[B\left(a;\tfrac{1}{n}\right) \cap A\right] \setminus \{a\} \neq \varnothing$$

Therefore one can choose a point $a_n \in B\left(a;\frac{1}{n}\right) \cap A$ with $a_n \neq a$ for all $n \in \mathbb{N}$. Now we show $\lim\limits_{n\to\infty} a_n = a$. Given $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. Therefore

$$|a_n - a| < \frac{1}{n} < \frac{1}{N} < \epsilon$$

for all $n \geq N$.

Conversely, suppose there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that $\lim\limits_{n\to\infty} a_n = a$ and $a_n \neq a$ for all $a \in \mathbb{N}$. Then by the definition of convergence, for every $\epsilon > 0$ we have $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Hence $a_n \in B\left(a;\frac{1}{n}\right)$ for all $n \geq N$. Since $a_n \neq a$ for all $n \in \mathbb{N}$, we have

$$\left[B\left(a;\tfrac{1}{n}\right) \cap A\right] \setminus \{a\} \neq \varnothing$$

and since this holds for all $\epsilon > 0$, then a is a limit point of A. $\square$

Remark 4.8 We make a few observations.

(i) A limit point of a set $A \subseteq \mathbb{R}$ does *not* have to belong to A.

For example, take $(c,d) \in \mathbb{R}$. In this case both c and d are limit points although $c, d \notin A$, since $(c+\epsilon, c-\epsilon) \cap (c,d) \neq \varnothing$ for all $\epsilon > 0$; and the same applies for d.

(ii) A point $a \in A \subseteq \mathbb{R}$ does not have to be a limit point of A.

In this case, $\exists\,\epsilon > 0$ such that $B(a;\epsilon) \cap A = \{a\}$. For example, take the set $[0,1] \cup \{2\}$. Here $2 \in A$ but it is not a limit point of A.

(iii) If $a \in \mathbb{R}$ is a limit point of A, then $B(a;\epsilon)$ contains infinitely many points of A.

We can show this is the case by contradiction. Suppose $\exists\,\epsilon > 0$ such that $B(a;\epsilon)$ is finite, and all elements are distinct from a. Say $[B(a;\epsilon) \cap A] \setminus \{a\} = \{a_1, a_2, a_3, \cdots, a_n\}$.
Now let $\epsilon' = \min\{(a_1 - a), (a_2 - a), (a_3 - a), \cdots, (a_n - a)\}$. Thus $[B(a;\epsilon') \cap A] \setminus \{a\} = \varnothing$ ※.

Definition 4.9 (Isolated point) Let $A \subseteq \mathbb{R}$. A point $a \in A$ is said to be an *isolated point* if $\exists\,\epsilon > 0$ such that

$$B(a;\epsilon) \cap A = \{a\}$$

i.e. it is not a limit point of A.

Remark 4.10 Note that an isolated point of A is always an element of A, whereas a limit point of A may or may not be an element of A.

Definition 4.11 (Closedness) A set $A \subseteq \mathbb{R}$ is said to be *closed* if it contains all its limit points.

Examples 4.12 We give some examples of closed sets.

(i) The set of real numbers $\mathbb{R}$ is closed.

(ii) The empty set $\varnothing$ is closed.

(iii) Singletons, i.e. sets of the form $\{a\}$, are closed.

(iv) Any finite set is closed.

(v) The set of natural numbers $\mathbb{N}$ is closed.

(vi) The set of rational numbers $\mathbb{Q}$ is not closed.

By the density of $\mathbb{Q}$ in $\mathbb{R}$ (1.14), we can find a rational number between any two real numbers. Therefore given $x \in \mathbb{R}$, for all $\epsilon > 0$, the intervals $(x - \epsilon, x)$ and $(x, x + \epsilon)$ contain rational numbers;

i.e. $[B(x; \epsilon) \cap \mathbb{Q}] \setminus \{x\} \neq \varnothing$. Therefore x is a limit point of $\mathbb{Q}$.

Since $x \in \mathbb{R}$ was arbitrary, we could have had $x \in \mathbb{I}$, i.e. $\mathbb{Q}$ does not contain all its limit points and thus is not closed.

(vii) The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}$ is not closed.

We can start by stating that every element in A is an isolated point. In fact, given any $\frac{1}{n} \in A$, the nearest point to it is $\frac{1}{n}$. Hence by taking $\epsilon = \frac{1}{n} - \frac{1}{n+1}$ we have

$$B(a; \epsilon) \cap A = \left\{\frac{1}{n}\right\}$$

Now we show that $0 \in \mathbb{R}$ is a limit point of A. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$, and since $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is in A, whose points are all isolated points, then by theorem 4.7 we can conclude that $\{0\}$ is a limit point of A. Since $0 \notin A$, then we conclude that A is not closed.

(viii) The closed interval $[c, d] \subseteq \mathbb{R}$ is closed.

$$[c, d] = \{x \in \mathbb{R} : c \leq x \leq d\}$$

We show that $[c, d]$ contains all of its limit points.

Suppose $a \notin [c, d]$, i.e. either $a < c$ or $a > d$. In the first case, take $\epsilon = c - a > 0$, so we have

$$[B(a; \epsilon) \cap [c, d]] \setminus \{a\} = \varnothing$$

and hence a is not a limit point of $[c, d]$. A similar procedure shows that $a > d$ is not a limit point of $[c, d]$.

Hence every limit point of $[c, d]$ must be in $[c, d]$, i.e. $[c, d]$ is closed.

Alternatively, we could have shown this using sequences. Let $a \in \mathbb{R}$ be a limit point of $[c, d]$. By theorem 4.7, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $[c, d]$ satisfying $\lim_{n\to\infty} a_n = a$. Since for all

$n \in \mathbb{N}$ we have $c \leq a_n \leq d$ then by corollary 2.20 to the order limit theorem, $c \leq a \leq d$ and thus $a \in [c, d]$, making the interval $[c, d]$ closed.

The following theorems follow from theorem 4.7 and example 4.12(vi).

Theorem 4.13 For every $x \in \mathbb{R}$, there exists a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ such that $\lim_{n\to\infty} q_n = x$. One can assume that $q_n \neq x$ for all $n \in \mathbb{N}$.

Theorem 4.14 For every $x \in \mathbb{R}$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{I}$ such that $\lim_{n\to\infty} t_n = x$. One can assume that $t_n \neq x$ for all $n \in \mathbb{N}$.

What makes theorem 4.13 more interesting is that the set $\mathbb{Q}$ is countable. As already mentioned in remark 4.8(i), a set $A \subseteq \mathbb{R}$ can have limit points outside A. Let us denote the set of all the limit points of A by L.

Definition 4.15 (Closure in $\mathbb{R}$) The set $\bar{A} = A \cup L$, where L contains all the limit points of A, is called the *closure* of A in $\mathbb{R}$.

Examples 4.16 We give a few example of closures of sets in $\mathbb{R}$.

(i) $\overline{\mathbb{Q}} = \bar{\mathbb{I}} = \mathbb{R}$

(ii) $A = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$, $\bar{A} = A \cup \{0\}$

(iii) $\overline{[c, d]} = [c, d]$

(iv) $\overline{(c, d)} = \overline{(c, d]} = \overline{[c, d)} = [c, d]$

One can observe that the closure of each set is closed. In fact we have the following result:

Theorem 4.17 The closure $\bar{A}$ of every set $A \subseteq \mathbb{R}$ is a closed set, and it is the smallest closed set containing A, i.e. if $F \subseteq \mathbb{R}$ is closed and $A \subseteq F$, then $\bar{A} \subseteq F$.

Proof. Let $A \subseteq \mathbb{R}$ and let L be the set of all limit points of A. We wish to show that $\bar{A} = A \cup L$ is closed; i.e. that $\bar{A}$ contains all its limit points (we proceed by taking $x \notin \bar{A}$ and showing it cannot be a limit point of $\bar{A}$).

So let $x \in \mathbb{R} \setminus \bar{A}$. Then $x \notin A$ and $x \notin L$. Since $x \notin L$, i.e. x is not a limit point of A, then $\exists\, \epsilon > 0$ such that $B(a; \epsilon) \cap A = \varnothing$.

We now show that $B(x; \epsilon) \cap \bar{A} = \varnothing$ by showing that $B(x; \epsilon) \cap A = \varnothing$ and that $B(x; \epsilon) \cap L = \varnothing$. The first appears above; the second we show by contradiction.

Suppose $B(x; \epsilon) \cap L \neq \varnothing$, i.e. that $\exists\, y \in B(x; \epsilon) \cap L$. Then $y \in B(x; \epsilon)$, and hence $|x - y| < \epsilon$. Let $\epsilon' = \epsilon - |x - y| > 0$. Now $y \in L$, so $B(y; \epsilon') \cap A \neq \varnothing$. (in fact, $B(y; \epsilon')$ contains points of A other than y, $y \notin A$). So now let $z \in B(y; \epsilon') \cap A$. Thus we have

$$|z - x| = |z - y + y - x| \leq |z - y| + |y - x| < \epsilon' + |y - x| < \epsilon$$

Hence $z \in B(x; \epsilon) \cap A$ ※ since $B(x; \epsilon) \cap A = \varnothing$.

Finally, if F is closed and $A \subseteq F$, then every limit point of A is a limit point of F (the proof of this is left as an exercise). Since F is closed each limit point is in F. Hence $\bar{A} = A \cup L \subseteq F$. $\square$

Exercise Show that F is closed iff $F = \bar{F}$.

Remark 4.18 It is important to note that $x \notin \bar{A} \iff \exists\, \epsilon > 0$ such that $B(x;\epsilon) \cap A = \varnothing$.

The forward implication follows from theorem 4.17.

For the converse: if $\exists\, \epsilon > 0$ such that $B(x;\epsilon) \cap A = \varnothing$, then $x \notin A$ and $x \notin L$. Hence $x \notin \bar{A}$.

Thus if A is closed, then $x \notin A$ iff $\exists\, \epsilon > 0$ such that $B(x;\epsilon) \cap A = \varnothing$.

We have already mentioned that the notion of closed sets is dual to that of open sets. This is clearly seen in the following theorem.

Theorem 4.19 A subset $O \subseteq \mathbb{R}$ is open iff $\mathbb{R} \setminus O$ is closed.

Proof. Let $O \subseteq \mathbb{R}$ be open, and let $F = \mathbb{R} \setminus O$. We show that F is closed, i.e. that it contains all its limit points.

If $x \notin F$, then $x \in O$ which is open. Therefore there exists $\epsilon > 0$ such that $B(x;\epsilon) \subseteq O$. Hence $B(x;\epsilon) \cap \mathbb{R} \setminus O = \varnothing$, and thus $B(x;\epsilon) \cap F = \varnothing$. Therefore x is not a limit point of F, and F is closed.

Conversely, assume $F = \mathbb{R} \setminus O$ is closed and let $x \notin \mathbb{R} \setminus O$. Since $x \notin F$, by remark 4.18, $\exists\, \epsilon > 0$ such that $B(x;\epsilon) \cap F = \varnothing$. Then $B(x;\epsilon) \subseteq \mathbb{R} \setminus F = O$, $\forall x \in O$. Hence O is open. $\square$

Corollary 4.20 A subset $F \subseteq \mathbb{R}$ is closed iff $\mathbb{R} \setminus F$ is open.

We proceed to use De Morgan's Laws[6] to prove the properties of closed sets which are dual to those of open sets. Firstly, note that since $\varnothing$ and $\mathbb{R}$ are open, then $\mathbb{R} = \mathbb{R} \setminus \varnothing$ and $\varnothing = \mathbb{R} \setminus \mathbb{R}$ are closed.

Theorem 4.21 The union of a finite collection of closed sets is closed.

Proof. Let $\{F_1, F_2, F_3, \dots, F_n\}$ be a finite collection of closed sets and let $F = \displaystyle\bigcup_{k=1}^{n} F_k$.

Let $O_k = \mathbb{R} \setminus F_k$, $\forall k \in [n]$. By theorem 4.4, $O = \displaystyle\bigcap_{k=1}^{n} O_k$ is open, and hence theorem 4.19 above gives us that $\mathbb{R} \setminus O$ is closed. But

$$\mathbb{R} \setminus O = \mathbb{R} \setminus \bigcap_{k=1}^{n} O_k = \bigcup_{k=1}^{n} \mathbb{R} \setminus O_k = \bigcup_{k=1}^{n} F_k = F$$

hence F is closed. $\square$

Theorem 4.22 The intersection of an arbitrary collection of closed sets is closed.

Proof. Let $\{F_\alpha\}_{\alpha \in \mathcal{A}}$ be an arbitrary collection of closed sets, and let $F = \displaystyle\bigcap_{\alpha \in \mathcal{A}} F_\alpha$.

[6] See Introductory Mathematics notes

Now let $O_\alpha = \mathbb{R} \setminus F_\alpha$, which is open $\forall \alpha \in \mathcal{A}$ by theorem 4.19. By theorem 4.3, we get that $O = \bigcup_{\alpha \in \mathcal{A}} O_\alpha$ is open, and by corollary 4.20 that $\mathbb{R} \setminus O$ is closed. But

$$\mathbb{R} \setminus O = \mathbb{R} \setminus \bigcup_{\alpha \in \mathcal{A}} O_\alpha = \bigcap_{\alpha \in \mathcal{A}} \mathbb{R} \setminus O_\alpha = \bigcap_{\alpha \in \mathcal{A}} F_\alpha = F$$

hence F is closed. $\square$

One can see from remark 4.5 that the union of an arbitrary collection of closed sets need not be closed. It is enough to take $F_n = \mathbb{R} \setminus O_n$ where $O_n = \left(-\dfrac{1}{n}, \dfrac{1}{n}\right)$ for all $n \in \mathbb{N}$. Then the union $\bigcup_{n \in \mathbb{N}} F_n = \mathbb{R} \setminus \{0\}$ which is not closed. Indeed $\lim_{n \to \infty} \dfrac{1}{n} = 0$, and the sequence $\left(\dfrac{1}{n}\right)_{n \in \mathbb{N}}$ is in $\mathbb{R} \setminus \{0\}$.

Another example is obtained by taking $F_n = \left[\dfrac{1}{n}, 1\right]$ for all $n \in \mathbb{N}$. We get $\bigcup_{n \in \mathbb{N}} F_n = \bigcup_{n \in \mathbb{N}} \left[\dfrac{1}{n}, 1\right] = (0, 1)$, which is not closed.

Remark 4.23 The set $\mathbb{R}$ and $\varnothing$ are both open and closed. Such sets are called *clopen*. One can show that these are the only subsets of $\mathbb{R}$ with this property. On the other hand, there are infinitely many subsets of $\mathbb{R}$ which are neither open nor closed, such as the half-open interval $(c, d]$.

Compact Sets

Definition 4.24 (Compactness) A set $K \subseteq \mathbb{R}$ is said to be *compact* if every convergent sequence in K has a convergent subsequence whose limit is in K.

Examples 4.25 We give some examples of compact sets.

(i) The half-open interval $(0, 1]$ is not compact. In fact, $\left(\dfrac{1}{n}\right)_{n \in \mathbb{N}}$ is in $(0, 1]$, whose every subsequence converges to 0 (since $\lim_{n \to \infty} \dfrac{1}{n} = 0$), and $0 \notin (0, 1]$.

(ii) The closed interval $[c, d]$ is compact. In fact, every sequence in $[c, d]$ is bounded, so that by the Bolzano-Weierstrass theorem, every bounded sequence has a convergent subsequence. But $[c, d]$ is a closed subset of $\mathbb{R}$, i.e. it contains all its limit points. Therefore the limits of the convergent subsequences in $[c, d]$ must belong to $[c, d]$.

Theorem 4.26 A subset $K \subseteq \mathbb{R}$ is compact iff every infinite subset of K has a limit point in K.

Proof. Let $K \subseteq \mathbb{R}$ be compact and let A be an infinite subset of K. Since A is infinite, one can choose $(a_n)_{n \in \mathbb{N}}$ in A such that all the a_n are distinct. Now $(a_n)_{n \in \mathbb{N}}$ is a sequence in K which is compact, and therefore it must have a convergent subsequence whose limit is in K. Let a be this limit. It follows that given any $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall\, k \geq N$, $|a_{n_k} - a| < \epsilon$. Since all the elements of the sequence are distinct, we can conclude that $B(a; \epsilon) \cap A \setminus \{a\} \neq \varnothing$, i.e. that a is a limit point of A.

Conversely, suppose that every infinite subset $A \subseteq K$ has a limit point in K, and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in K. If there is an element x in $(x_n)_{n \in \mathbb{N}}$ which appears infinitely often, then $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ where $x_{n_k} = x$ for all $k \in \mathbb{N}$. Clearly, $\lim_{k \to \infty} x_{n_k} = x \in K$.

If no such element exists, then one can assume that all the elements of $(x_n)_{n\in\mathbb{N}}$ are distinct. Therefore the set $A = \{x_n : n \in \mathbb{N}\}$ is an infinite subset of K, and by the hypothesis A has a limit point x in K. Let $\epsilon = 1$. Then choose x_{n_1} such that $|x_{n_1} - x| < 1$ and $x_{n_1} \neq x$. Similarly, choose x_{n_2} such that

$$|x_{n_2} - x| < \frac{1}{2} \quad \text{and} \quad x_{n_2} \neq x$$

Similarly, for every $k \in \mathbb{N}$ one can choose $x_{n_k} \in A$ such that $|x_{n_k} - x| < \frac{1}{k}$ and $x_{n_k} \neq x$. Following this procedure, one constructs a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ whose limit x is in K.

Indeed, for every $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\frac{1}{N} \leq \epsilon$ and hence $|x_{n_k} - x| < \epsilon$ for all $k \geq N$. □

Definition 4.27 (Bounded set) A set $A \subseteq \mathbb{R}$ is *bounded* if $\exists\, M > 0$ such that $|a| \leq M$, $\forall a \in A$.

Note that we already defined boundedness in definition 1.6, the above is simply a paraphrasing.

Remark 4.28 Note that if $A \subseteq \mathbb{R}$ is bounded and $(a_n)_{n\in\mathbb{N}}$ is a sequence in A, then the sequence is bounded and hence has a convergent subsequence by the Bolzano-Weierstrass theorem.

Theorem 4.29 (Heine-Borel Theorem) A subset $K \subseteq \mathbb{R}$ is compact iff it is closed and bounded.

Proof. Let $K \subseteq \mathbb{R}$ be a compact set and suppose it is not bounded. Let $M = 1$ in the definition above. Since K is not bounded, then we have $x_1 \in K$ such that $|x_1| > 1$, $x_2 \in K$ such that $|x_2| > 2$, and in general some $x_n \in K$ for which $|x_n| > n$. In this way we are constructing a sequence $(x_k)_{n\in\mathbb{N}}$ satisfying $x_n > n$ for all $n \in \mathbb{N}$. If $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$, then $|x_{n_k}| > n_k$ for all $k \in \mathbb{N}$, and since $n_k \geq k$ (remark 2.29), we see that $(x_{n_k})_{k\in\mathbb{N}}$ is an unbounded sequence which by theorem 2.15 must diverge, meaning that K cannot be compact ※.

We have hence shown that K must be bounded. Next we show K is closed.

Let $x \in \mathbb{R}$ be a limit point of K. By theorem 4.7 we have a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $\lim_{n\to\infty} x_n = x$ with $x_n \neq x$ $\forall n \in \mathbb{N}$. By definition of compactness, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_k} = y \in K$. Also by theorem 2.30, $\lim_{k\to\infty} x_{n_k} = x$, and by the uniqueness of the limit (theorem 2.13) we must have $x = y \in K$. Hence K contains all its limit points, i.e. K is closed.

Conversely, let $K \subseteq \mathbb{R}$ be closed and bounded. As mentioned above, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in K then $(x_n)_{n\in\mathbb{N}}$ is bounded and therefore has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ (Bolzano-Weierstrass). Now say $\lim_{k\to\infty} x_{n_k} = x$ for some $k \in \mathbb{N}$. Then we immediately have $x \in K$. Otherwise, if $x \neq x_{n_k}$ for all $k \in \mathbb{N}$, then by theorem 4.7, x is a limit point of K. But K, being closed, contains all its limit points, meaning that $x \in K$ regardless. Therefore K is compact. □

Remark 4.30 Observe that the Heine-Borel theorem confirms that the closed interval is compact, being closed and bounded. In fact, compact sets and closed intervals behave in a similar manner, as we shall see in the upcoming theorem 4.31.

Exercise If F is closed and K is compact, show that $F \cap K$ is compact.

Theorem 4.31 (Cantor's Intersection Theorem) Let there be given an decreasing sequence of nonempty compact subsets $(K_n)_{n\in\mathbb{N}}$ such that every $K_{n+1} \subseteq K_n$. Then $\bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$ and is compact.

Proof. Since each K_n is nonempty, one can choose a sequence $(x_n)_{n\in\mathbb{N}}$ such that each $x_n \in K_n$. Now since the compact sets are nested, then $x_n \in K_1$ for all $n \in \mathbb{N}$, and since K_1 is compact, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} x_{n_k} = x \in K_1$.

We will show that $x \in K_n$ $\forall n \in \mathbb{N}$, and hence in $K = \bigcap\limits_{n\in\mathbb{N}} K_n$.

Since the compact sets are nested $x_2, x_3, \ldots, x_n, \cdots \in K_2$ and hence $x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \cdots \in K_2$ (remark 2.29). But K_2 is closed and $\lim\limits_{\substack{k\geq 2\\ k\to\infty}} x_{n_k} = x \in K_2$, since K_2 is closed (by theorem 4.7).

Similarly, given any $m \in \mathbb{N}$, $x_m, x_{m+1}, x_{m+2}, \cdots \in K_m$ and hence $x_{n_m}, x_{n_{m+1}}, \cdots \in K_m$ (by remark 2.29). But K_m is closed and we have $\lim\limits_{\substack{k\geq m\\ k\to\infty}} x_{n_k} = x \in K_m$. Due to the arbitrary selection of m, we have $x \in K_n$ for all $n \in \mathbb{N}$, giving us that

$$\bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$$

Finally, K is the intersection of compact (and hence closed) sets, therefore K is closed. Also,

$$K = \bigcap_{n\in\mathbb{N}} K_n \subseteq K_1$$

which is bounded (due to compactness), therefore K is bounded as well. Thus K is closed and bounded, hence compact. $\square$

Definition 4.32 (Open Cover) Let $A \subseteq \mathbb{R}$. A collection of open sets $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be an *open cover* of A if

$$A \subseteq \bigcup_{\alpha\in\mathcal{A}} O_\alpha$$

Definition 4.33 (Finite Subcover) Let $A \subseteq \mathbb{R}$, and $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of A. The subcollection $\{O_\alpha\}_{\alpha\in\mathcal{A}_0}$ is said to be a *finite subcover* of $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ if $\mathcal{A}_0$ is finite and we still have

$$A \subseteq \bigcup_{\alpha\in\mathcal{A}_0} O_\alpha$$

i.e. the finite collection $\{O_\alpha\}_{\alpha\in\mathcal{A}_0}$ is still an open cover of A.

Example 4.34 Consider the interval $[0, 1)$ and the collection $\{O_n\}_{n\in\mathbb{N}}$, where each $O_n = \left(\frac{1}{n}, 2\right)$. This is an open cover of $[0, 1)$, in fact $(0, 1] \subseteq \bigcup\limits_{n\in\mathbb{N}} O_n$.

However $\{O_n\}_{n\in\mathbb{N}}$ does not have a finite subcover. Indeed, if $\{O_{n_1}, O_{n_2}, \ldots, O_{n_N}\}$ were a finite subcover of $(0, 1]$, let $M = \max\{n_k : k \in [N]\}$. Then $\left(\frac{1}{n_k}, 2\right) \subseteq \left(\frac{1}{M}, 2\right)$ for all $k \in [N]$. So $O_{n_k} \subseteq O_M$ and therefore $(0, 1] \subseteq O_M = \left(\frac{1}{M}, 2\right)$ ※.

Now consider the interval $[0, 1]$ instead. The collection $\{O_n\}_{n\in\mathbb{N}}$ still does not cover this interval, since $0 \notin \bigcup\limits_{n\in\mathbb{N}} O_n$. Hence we add another set to the collection. Let $O_0 = (-\epsilon, \epsilon) = B(0; \epsilon)$, $\epsilon > 0$. Then the collection $\{O_n\}_{n=0}^{\infty}$ is an open cover of $[0,1]$, which also has a finite subcover: take any $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. Then $[0,1] \subseteq O_0 \cup O_n = (-\epsilon, \epsilon) \cup \left(\dfrac{1}{N}, 2\right)$. In other words, the collection $\{O_0, O_N\}$ gives us a finite subcovering of $[0,1]$.

Lemma 4.35 Every open cover of a closed interval $[c,d]$ has a finite subcover.

Proof. We proceed by contradiction. Suppose that there exists an open cover $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ of $I = [c,d]$ which does not have a finite subcover. Let a be the midpoint of $[c,d]$ (i.e. $a = \frac{c+d}{2}$). Then

$$[c,a] \cup [a,d] = [c,d]$$

and at least one of them, say I_1, cannot be covered by a finite subcollection of $\{O_\alpha\}_{\alpha\in\mathcal{A}}$. We repeat the procedure on I_1, to obtain a closed interval I_2 which cannot be covered by a finite subcollection of $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ and $I_2 \subseteq I_1 \subseteq I = [c,d]$. In this way, we construct a sequence $(I_n)_{n\in\mathbb{N}}$ of closed intervals whereby:

(i) each $I_{n+1} \subset I_n$,

(ii) each I_n cannot be covered by a finite subcollection of $\{O_\alpha\}_{\alpha\in\mathcal{A}}$,

(iii) and the length of each I_n is $\dfrac{d-c}{2^n}$.

Therefore by the nested interval property, it follows that

$$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$$

and furthermore, by the stronger nested interval property, since $\lim\limits_{n\to\infty} \dfrac{d-c}{2^n} = 0$, there is exactly one point $x \in \bigcap\limits_{n\in\mathbb{N}} I_n$. Now

$$x \in I \subseteq \bigcup_{\alpha\in\mathcal{A}} O_\alpha \Rightarrow \exists\, \alpha \in \mathcal{A} \text{ such that } x \in O_\alpha.$$

and since O_α is open, $\exists\, \epsilon > 0$ such that $B(x;\epsilon) \subseteq O_\alpha$. Finally, by the Archimedean property in ℝ, we have $N \in \mathbb{N}$ such that

$$\frac{d-c}{2^N} < \epsilon \Rightarrow I_n \subseteq O_\alpha,$$

contradiction (ii) above ※. In fact, if $y \in I_n$, then $x - y < \dfrac{d-c}{2^N} < \epsilon$ (due to the fact that the difference $x-y$ of two points in I_n cannot be outside I_n), giving us that $y \in B(x;\epsilon) \subseteq O_\alpha$. □

Theorem 4.36 The following statements are equivalent for $K \subseteq \mathbb{R}$.

(i) K is compact, i.e. every sequence in K has a convergent subsequence whose limit is in K.

(ii) Every infinite subset $A \subseteq K$ has a limit point in K.

(iii) K is closed and bounded.

(iv) Every open cover of K has a finite subcover.

Proof. We have already shown (i) $\iff$ (ii) (theorem 4.26) and (i) $\iff$ (iii) (Heine-Borel). Hence by combining the two theorems we have that (i), (ii) and (iii) are equivalent. There remains to show that (iv) is equivalent to the first three.

We show that (iii) $\implies$ (iv).

Let $K \subseteq \mathbb{R}$ be closed and bounded. Since K is bounded, we have $M > 0$ for which $|x| \leq M\ \forall x \in K$, i.e. $K \subseteq [-M, M]$. Now $K \subseteq \mathbb{R}$ is closed means that $Q = \mathbb{R} \setminus K$ is open by corollary 4.20. Let $\{Q_\alpha\}_{\alpha\in\mathbb{N}}$ be an open cover of K, i.e. $K \subseteq \bigcup_{\alpha\in\mathcal{A}} Q_\alpha$. Then

$$Q \cup \Big(\bigcup_{\alpha\in\mathcal{A}} Q_\alpha\Big) = \mathbb{R}$$

hence $Q \cup \big(\bigcup_{\alpha\in\mathcal{A}} Q_\alpha\big)$ covers $[-M, M]$. By lemma 4.35, we have a finite subcover, i.e. $[-M, M] \subseteq Q \cup Q_{\alpha_1} \cup Q_{\alpha_2} \cup \cdots \cup Q_{\alpha_n}$, and therefore $\{Q \cup Q_{\alpha_1} \cup Q_{\alpha_2} \cup \cdots \cup Q_{\alpha_n}\}$ is a finite subcovering of K.

Now we show that (iv) $\implies$ (ii) by contradiction.

Let $A \subseteq K$ be an infinite subset which does not have a limit point in K. Then $\forall\, x \in K$, we have $\epsilon(x) > 0$ such that $B\,(x; \epsilon(x)) \cap A$ is empty, except for possibly x itself if $x \in A$. Then $K \subseteq \bigcap_{x\in K} B\,(x; \epsilon(x))$, so $\{B\,(x; \epsilon(x))\}_{x\in K}$ is an open cover of K. By the hypothesis, this cover has a finite subcover, say $\{B\,(x_1; \epsilon(x_1))\,, B\,(x_2; \epsilon(x_2))\,, \ldots, B\,(x_n; \epsilon(x_n))\}$. Since each $B(x_k; \epsilon(x_k))$ contains at most one element of A for $k \in [n]$, then A must be finite. ※ □

Perfect Sets

Definition 4.37 (Perfect set) A set $P \subseteq \mathbb{R}$ is said to be *perfect* if it is closed and contains no isolated points; i.e. all its points are limit points.

Remark 4.38 If $P \subseteq \mathbb{R}$ is perfect, then by definition it contains all its limit points, i.e. P consists *exactly* of its limit points.

Examples 4.39 We give some examples of perfect sets.

(i) The empty set $\varnothing$ is perfect.

(ii) The singleton $\{a\}$ is not perfect.

(iii) $[n] = \{1, 2, \ldots, n\}$ is not perfect.

(iv) $\mathbb{N}$ is not perfect.

(v) $\mathbb{Q}$ is not perfect (not even closed).

(vi) $\mathbb{I}$ is not perfect.

(vii) (c, d), $(c, d]$ and $[c, d)$ are not perfect.

(viii) $[c, d]$ is perfect.

(ix) $\mathbb{R}$ is perfect.

Theorem 4.40 Every nonempty perfect set is uncountable.

Proof. Since P is perfect and $\neq \varnothing$, it has limit points and by remark 4.8(iii) it follows that it must be infinite.

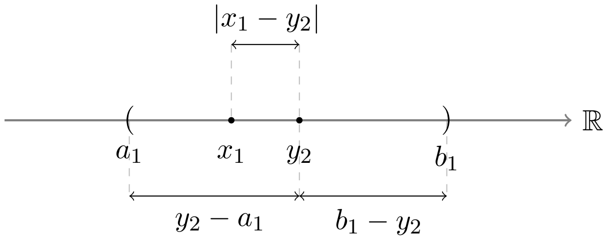
For contradiction, suppose that P is countably infinite, i.e. that $P = \{x_1, x_2, \ldots, x_n, \ldots\}$. Using induction, we shall construct a sequence of closed intervals $(I_n)_{n\in\mathbb{N}}$ where $I_n = [a_n, b_n]$, and

(i) $I_{n+1} \subseteq I_n$,

(ii) $x_n \notin I_{n+1}$,

(iii) and $(a_n, b_n) \cap P \neq \varnothing$ for all $n \in \mathbb{N}$.

Let $I_1 = [a_1, b_1]$ such that $x_1 \in (a_1, b_1)$. Since $x_1 \in P$, x_1 is not an isolated point, and therefore (a_1, b_1) contains points of P other than x_1. Say that $y_2 \in (a_1, b_1) \cap P$ such that $y_2 \neq x_1$ and $\epsilon = \min\{y_2 - a_1, b_1 - y_2, |x_1 - y_2|\}$:



Then $I_2 = \left[y_2 - \frac{\epsilon}{2}, y_2 + \frac{\epsilon}{2}\right] = [a_2, b_2]$. Note that this choice of I_2 adheres to (i), (ii) and (iii) above. Now suppose we have constructed such a sequence $I_1, I_2, I_3, \ldots, I_n = [a_n, b_n]$ satisfying (i), (ii) and (iii) and let $y_n \in (a_n, b_n) \cap P$. Then y_n is not an isolated point of P, and (a_n, b_n) contains points of P other than y_n. Now suppose $y_{n+1} \in (a_n, b_n) \cap P$ such that $y_{n+1} \neq y_n$ (we can insist $y_{n+1} \neq x_n$ as well), and let $\epsilon = \min\{y_{n+1} - a_n, b_n - y_{n+1}, |x_n - y_{n+1}|\}$. Then $I_{n+1} = \left[y_{n+1} - \frac{\epsilon}{2}, y_{n+1} + \frac{\epsilon}{2}\right]$ has the desired points.

Let $K_n = I_n \cap P$ $(*)$. Each K_n is closed since both I_n and P are closed (theorem 4.22). Since $K_n \subseteq I_n$, then I_n being a closed interval is bounded, giving us that K_n is also bounded. Hence by Heine-Borel, K_n is compact $\forall n \in \mathbb{N}$.

By (i) and (iii) we have that $K_n \neq \varnothing$ and $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$, and it follows from Cantor's Intersection theorem that $K = \bigcap\limits_{n\in\mathbb{N}} K_n \neq \varnothing$.

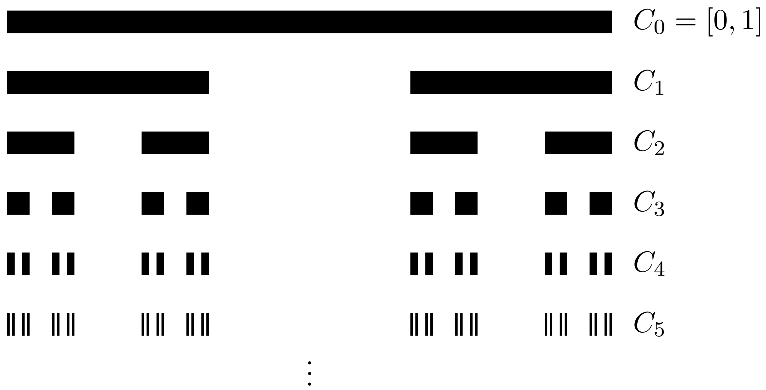
Now by (ii), we have $x_n \notin I_{n+1}$, which means that x_n is not in K_{n+1} by $(*)$, and by the above we have that $x_n \notin K$ for all $n \in \mathbb{N}$. But since $K \subseteq P$, which is perfect, means that K must be empty. ※. $\square$

The Cantor Set

We will construct a perfect set which is compact and contains no intervals.

$$\begin{aligned}
\text{Let } C_0 &= [0,1] \\
C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
&\vdots \\
C_n &= \left[0, \frac{1}{3^n}\right] \cup \left[\frac{2}{3^n}, \frac{3}{3^n}\right] \cup \cdots \cup \left[\frac{3^{n-1}}{3^n}, \frac{3^n}{3^n}\right]
\end{aligned}$$

Visually, we have the following.



Note that $[0,1] \supseteq C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$, so each C_n is bounded. Also, C_n is the union of 2^n closed intervals, each of length $\frac{1}{3^n}$. By theorem 4.21, we have that each C_n is closed. Hence each C_n is both closed and bounded, and therefore compact.

Definition 4.41 (The Cantor Set) The set $\mathcal{C}$, known as the *Cantor Set*, is the set given by

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

Note that since each C_n is compact, then $\mathcal{C}$ is compact by theorem 4.22. Furthermore, by Cantor's Intersection theorem, it follows that $\mathcal{C} \neq \varnothing$. We claim that $\mathcal{C}$ is a perfect set. We already know that $\mathcal{C}$ is compact, so all there remains to show is that it contains no isolated points.

Theorem 4.42 The set $\mathcal{C}$ contains no isolated points.

Proof. Let $x \in \mathcal{C}$ and consider $B(x;\epsilon)$. Since $x \in \mathcal{C}$, then $x \in C_n$ for all $n \in \mathbb{N}$. Let I_n be one of the closed intervals of C_n which contains x. The length of I_n is $\frac{1}{3^n}$. Choose $n \in \mathbb{N}$ such that $\frac{1}{3^n} < \epsilon$, hence we get $I_n \subseteq B(x;\epsilon)$. Now since I_n has two end points, at least one of them is $\neq x$. Let this endpoint be denoted x_n. Hence, it follows from the construction of $\mathcal{C}$ that

$$x_n \in [B(x;\epsilon) \cap \mathcal{C}] \setminus \{x\}$$

and this holds in general for any $x \in \mathcal{C}$, meaning that $\mathcal{C}$ has no isolated points, and is therefore perfect. $\square$

Furthermore, by theorem 4.40, $\mathcal{C}$ is uncountable. Now we proceed to show that $\mathcal{C}$ has no intervals.

Theorem 4.43 The set $\mathcal{C}$ contains no intervals.

Proof. It is sufficient to show that $\mathcal{C}$ contains no open intervals.

Suppose we have $(a,b) \subseteq \mathcal{C}$. Then there is an $n \in \mathbb{N}$ such that $|a-b| > \frac{1}{3^n}$. But when considering C_n, all its intervals have length $\frac{1}{3^n}$. Hence there is a middle third taken out of (a,b), which means that there are points in (a,b) which are not in $\mathcal{C}$. Thus $(a,b) \setminus \mathcal{C} \neq \varnothing$, contradicting that $(a,b) \subseteq \mathcal{C}$ ※. $\square$

Remark 4.44 (Measure of $\mathcal{C}$) Let us use the term *measure* to refer to the length of an interval. Furthermore, suppose two intervals I_1 and I_2 do not intersect and each have a measure of i_1 and i_2, then we take the measure of $I_1 \cup I_2$ to be $i_1 + i_2$. We beg the question: what is the measure of $\mathcal{C}$?

Well, each C_n is made up of 2^n closed intervals with no overlap, each of length $\frac{1}{3^n}$. Hence the measure of each C_n is $c_n = \frac{2^n}{3^n}$. Thus the measure of $\mathcal{C}$ must be the limit of the sequence $(c_n)_{n\in\mathbb{N}}$:

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{2^n}{3^n} = 0$$

This is a very surprising result. We have that $\mathcal{C}$ is an uncountable infinite set with *zero* measure. Alternatively, we can find the measure of $\mathcal{C}$ by observing that we obtain C_{n+1} by removing the middle third of C_n, i.e. removing a set with measure $\frac{1}{3}\left(\frac{2^n}{3^n}\right) = \frac{2^n}{3^{n+1}}$. Thus in obtaining $\mathcal{C}$, we are removing the following total length:

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1$$

But initially C_0 has measure 1, so we have a measure left of $1 - 1 = 0$.

Summary

- A set $O \subseteq \mathbb{R}$ is said to be open if for every element $a \in O$, we have $\epsilon > 0$ such that $B(a;\epsilon) \subseteq O$. (4.1)
- The arbitrary union of open sets is open. (4.3)
- The intersection of a finite collection of open sets is open. (4.4)
- A limit point of a set $A \subseteq \mathbb{R}$ is a point $a \in \mathbb{R}$ such that $[B(a;\epsilon) \cap A] \setminus \{a\} \neq \varnothing$. (4.6)
- A point $a \in \mathbb{R}$ is a limit point of a set $A \subseteq \mathbb{R}$ iff there is a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$ and $a_n \neq a$ for all $n \in \mathbb{N}$. (4.7)
- An isolated point of A is a point $a \in A$ which is not a limit point; i.e. $B(a;\epsilon) \cap A = \{a\}$. (4.9)
- A set $A \subseteq \mathbb{R}$ is said to be closed if it contains all its limit points. (4.11)
- For every $x \in \mathbb{R}$, we have sequences $(q_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ and $\mathbb{I}$ respectively such that $\lim_{n\to\infty} q_n = x$ and $\lim_{n\to\infty} t_n = x$. (4.13, 4.14)
- The closure of a set $A \subseteq \mathbb{R}$, denoted $\bar{A}$, is the union of A with the set containing all its limit points. (4.15)
- The closure of a set $A \subseteq \mathbb{R}$ is always closed, and it is the smallest closed set containing A. (4.17)
- A subset $O \subseteq \mathbb{R}$ is open iff $R \setminus O$ is closed, and similarly, A subset $F \subseteq \mathbb{R}$ is closed iff $R \setminus F$ is open. (4.19, 4.20)
- The union of a finite collection of closed sets is closed. (4.21)
- The intersection of an arbitrary collection of closed sets is closed. (4.22)
- $\mathbb{R}$ and $\varnothing$ are the only clopen subsets of $\mathbb{R}$. (4.23)
- A set $K \subseteq \mathbb{R}$ is said to be compact if every convergent subsequence in K has a convergent subsequence whose limit is in K. (4.24)
- Cantor's Intersection Theorem: A decreasing sequence $(K_n)_{n\in\mathbb{N}}$ of nonempty compact sets with $K_{n+1} \subseteq K_n$ has a nonempty intersection which is also compact; i.e. $\bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$ and is compact. (4.31)

- An open cover of a set $A \subseteq \mathbb{R}$ is a collection of open sets $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ whose union over $\mathcal{A}$ covers A, i.e. $A \subseteq \bigcup_{\alpha\in\mathcal{A}} O_\alpha$. (4.32)
- A finite subcover of an open cover $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ of $A \subseteq \mathbb{R}$ is a finite subcollection of sets in $\{O_\alpha\}_{\alpha\in\mathcal{A}}$ which still covers A. (4.33)
- The following statements are equivalent:
  - (i) K is compact, i.e. every sequence in K has a convergent subsequence whose limit is in K.
  - (ii) Every infinite subset $A \subseteq K$ has a limit point in K.
  - (iii) K is closed and bounded (Heine-Borel).
  - (iv) Every open cover of K has a finite subcover. (4.36)
- A set is said to be perfect if it is closed and contains no isolated points; i.e. it consists solely of its limit points. (4.37)
- Every nonempty perfect set is uncountable. (4.40)
- The Cantor set $\mathcal{C}$ is obtained by recursively removing the middle third of an interval, starting with $C_0 = [0,1]$. It is perfect, contains no intervals and has zero measure. (4.41, 4.42, 4.43)