Mathematical Methods

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Lecture 2: Gaussian-Jordan Elimination

A set of linear equations in the variables $x_1, x_2, \cdots, x_n$ is called a system of linear equations (linear system). A system of m linear equations in n unknowns can be written as:

$$\left.\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}\right\}$$

A linear system can be solved using the augmented matrix

$$\left(A \,\vdots\, B \right) = \left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)$$

A solution of a linear sequence is a sequence of numbers $s_1, s_2, \cdots, s_n$ s.t. $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ is a solution of every equation.

Solutions can be of 3 types:

- Infinite
- Unique
- Non-existent

An example of **non-existent** (*inconsistent*) system of equation would be

$$\left.\begin{array}{ll} x + y & = 2 \\ 2x + 2y & = 6 \end{array}\right\}$$

To solve consistent systems of linear equations one must use the **elementary row operations** to reduce the augmented matrix $\left(A \,\vdots\, B \right)$ into **row echelon form**.

The following algorithm (*also referred to as Gaussian Elimination*) must be followed to reduce an $m \times n$ matrix into a row echelon form one.

1. If a row has all entries 0, then it must be placed at the bottom.
2. If a row does contain an entry which is not 0, then the first non-zero entry must be 1 (*also referred to as the leading one*).
3. In any two successive rows, the bottom one must have the leading 1 furhter to the right than that of the higher.

If a reduced echelon form matrix is desired, it must be in row echelon form and have every entry of each column which contains a leading one (*except the leading one*), be 0. The process by which a reduced echelon form matrix is obtained is called *Gaussian-Jordan Elimination.*

Lecture 3: Rank of a Matrix

Definition 1. The rank of a matrix, denoted by rank($\mathbf{A}$), is equal to the number of non-zero rows in a row echelon form of $\mathbf{A}$

Theorem 1. Let $\mathbf{AX} = \mathbf{B}$ be a linear system of m linear equations in n unknowns with augmented matrix $\left(\begin{array}{c|c} A & B \end{array} \right)$, then

- the system has a solution if and only if $\text{rank}(\mathbf{A}) = \text{rank}\left(\begin{array}{c|c} A & B \end{array} \right)$
- the system has a uniue solution if and only if $\text{rank}(\mathbf{A}) = \text{rank}\left(\begin{array}{c|c} A & B \end{array} \right) = n$

Note. A rank of a matrix augmented with another cannot be smaller than the original non-augmented matrix and thus

Example. For which values of a does the following system have a unique solution? For which pairs of a, b does the system have more than one solution?

$$\begin{aligned} x - 2y &= 1 \\ x - y + az &= 2 \\ ay + 9z &= b \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & -1 & a & 2 \\ 0 & a & 9 & b \end{array}\right) \underset{\sim}{R} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & 9-a^2 & b-a \end{array}\right)$$

The solutions is unique if and only if $9 - a^2 \neq 0$ which is equivalent to $a \neq \pm 3 \Rightarrow \text{rank}(\mathbf{A}) = 3$. The solutions are infinite if and only if $a = \pm 3$ and $b - a = 0$.

$$(a, b) = (\pm 3, \pm 3) \Rightarrow \text{rank}\mathbf{A}) = 3$$

If a system of n equations in n unknowns (*also referred to as a square system*) has a unique solution, then the solution can be foudn by using the inverse of the coefficient matrix.

$$\left.\begin{array}{r} AX = B \\ A \text{ is } n \times n \\ \text{The system has 1 solution} \end{array}\right\} \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Note. An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\text{rank}(\mathbf{A}) = n$.