# Discrete Methods

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## Contents



# 1 Tuples, Permutations Sets and Multisets

**Proposition 1.1.** $|X \times Y| = |X| \cdot |Y|$

*Proof.* Let $x_1, x_2 \cdots x_m$ be the elements of $X$ and let $y_1, y_2, \cdots y_n$ be the elements of $Y$. $X \times Y$ consists of all the pairs with the choices for the first element from $X$ and the choices for the second element from $Y$. The number of elements of $X \times Y$ is the sum of $n$ $m$'s, i.e. $mn$. $\square$

**Proposition 1.2.** If $A_1, A_2, \cdots, A_k$ are mutually disjoint sets, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_k|$, i.e.:

$$\left| \bigcup_{k \in \mathbb{N}} A_k \right| = \sum_{k \in \mathbb{N}} |A_k|$$

*Proof.* By induction on k. This proposition is a special case of the **Inclusion-Exclusion** principle where the unions of all the sets are the empty set and thus have size 0. $\square$

## 1.1 Repetition and Order

Consider the set $[5] = \{ 1, 2, 3, 4, 5 \}$. Suppose we had to pick 4 numbers from this set, possibly with repetition. So the set of all choices would look like the following:

$$[5]^4 = \{ (1,1,1,1),\ (1,1,1,2),\ (1,1,2,1),\ \cdots,\ (5,5,5,5) \}$$

which is equivalent to $[5] \times [5] \times [5] \times [5]$. The following proposition will enlighten us on how to calculate the size of said set.

**Proposition 1.3.** For all $k \geq 2$ sets $X_1, X_2, \cdots, X_k$,

$$|X_1 \times X_2 \times \cdots \times X_k| = |X_1| \times |X_2| \times \cdots \times |X_k| .$$

*Proof.* By induction on $k$, the base case $k = 2$ is given by **Proposition 1.1**. Now, consider $k \geq 3$. Let $x_1, x_2, \cdots, x_k$ be the elements of $X_k$ and $A = X_1 \times X_2, \times \cdots \times X_k$, or rather

$$A = \{ x_{11}, x_{12}, \cdots, x_{1k} \} \times \{ x_{21}, x_{22}, \cdots, x_{2k} \} \times \cdots \times \{ x_{k1}, x_{k2}, \cdots, x_{kk} \} .$$

For $j \in \mathbb{N}$, let $A_j$ be the set of $k$-tuples ending in $x_j \in X_1$. Therefore, a $k-tuple$ is in $A_j$ if and only $\boxed{A_j \subseteq A.}$ if its first $k-1$ entries form a $(k-1)$-tuple in $X_1 \times X_2 \times \cdots \times X_{k-1}$ and its $k$-th entry is $x_j$. Thus, $|A_j|$, or the number of $k$-tuples in $A_j$ is $|X_1 \times X_2 \times \cdots \times X_{k-1}|$, which is $|X_1| \times |X_2| \times \cdots \times |X_{k-1}|$ by the inductive hypothesis.

Next, we show that $A = \bigcup_{n \in \mathbb{N}} A_n$. Let $a \in A$. Then, the $k$-th entry of $a$ is $x_i$ for some $i \in [n]$. So $a \in A_i$ and hence $a \in \bigcup_{n \in \mathbb{N}} A_n$ and therefore $A \subseteq \bigcup_{n \in \mathbb{N}}$. Now let $b \in \bigcup_{n \in \mathbb{N}} A_n$. Then, $b$ is an element of at least one set of $A_1, A_2, \cdots, A_n$. Since $\forall j \in \mathbb{N}$, $A_j \subseteq A$, we have $b \in A$ and therefore $\bigcup_{n \in \mathbb{N}} A_n \subseteq A$. Finally, $A = \bigcup_{n \in \mathbb{N}} A_n$.

Now, $A_1, A_2, \cdots, A_n$ are mutually disjoint, because for any two sets $A_i$ and $A_j$ with $i \neq j$, any $k$-tuple in $A_i$ ends with $x_i$, whilst any $k$-tuple in $A_j$ ends with $x_j$. This fact allows us to use **Proposition 1.2** as follows:

$$\begin{aligned} A &= A_1 \cup A_2 \cup \cdots \cup A_k \\ |A| &= |A_1| + |A_2| + \cdots + |A_n| \\ &= \underbrace{|X_1| \times |X_2| \times \cdots \times |X_{k-1}|}_{n \text{ times}} \\ &= (|X_1| \times |X_2| \times \cdots \times |X_{k-1}|)\, n \\ &= |X_1| \times |X_2| \times \cdots \times |X_{k-1}| \times |X_k| \end{aligned}$$

$\boxed{n = |X_k|}$

□

**Corollary 1.4.** For any $k \geq 2$ and any set $X$,

$$\left| X^k \right| = |X|^k .$$

*Proof.* This is given by **Proposition 1.3** with $X_1 = X_2 = \cdots = X_k = X$ □

## 1.2 ~~Repetition~~ and Order

Suppose we had to choose $n$ objects from $k$ distinct objects, that each object can only be chosen once, and that the order in which we picked the $n$ things mattered. We can label the set of $n$ objects $\{1, 2, \cdots, n\} = [n]$. Each unique possibility can be represented by a $k$-tuple $(a_1, a_2, \cdots, a_k)$, but this time we have $a_1 \neq a_2 \neq \cdots \neq a_k$, or rather, all $a_k$ are distinct elements from $[n]$. Let $P_n^k$ be the set of such $k$-tuples defined as follows:

$$P_n^k = \{ (a_1, a_2, \cdots, a_k) \in [n]^k : a_1 \neq a_2 \neq \cdots \neq a_k \}.$$

The problem is now reduced into finding the size of $P_n^k$, since any $k$-tuple in this set represents a valid possibility, and all possibilities must be in this form.

**Note.** Since we cannot choose more objects ($k$) than the amount we have ($n$) we have to assume $n$ is at least $k$. So we have:

$$P_n^k = \emptyset \Leftrightarrow k \leq n.$$

For example, if we had to pick 3 objects from a pool of 4, so $n = 4$ and $k = 3$, then the possibilities would be:

$$\begin{gathered}
(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), \\
(1, 2, 4), (1, 4, 2), (2, 1, 4), (2, 4, 1), (4, 1, 2), (4, 2, 1), \\
(1, 3, 4), (1, 4, 3), (3, 1, 4), (3, 4, 1), (4, 1, 3), (4, 3, 1), \\
(2, 3, 4), (2, 4, 3), (3, 2, 4), (3, 4, 2), (4, 2, 3), (4, 3, 2).
\end{gathered}$$

**Recall.** The product $n \times (n-1) \times (n-2) \cdots \times 2 \times 1$ is represented by $n!$. In general, $\prod_{i=1}^{n} a_i$ represents the product $a_1 \times a_2 \times \cdots \times a_n$. Thus, $n! = \prod_{i=1}^{m} i$. We define $0!$ as 1.

**Proposition 1.5.** For $1 \leq k \leq n$,

$$\left| P_n^k \right| = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

*Proof.* By induction on $k$. Consider the base case $k = 1$. The set $P_n^1$ is $\{ (1), (2), \cdots (n) \}$ and has size $n = \frac{n!}{(n-1)!}$ as required.

Now consider $k \geq 2$. For each $i \in [n]$, let $A_i$ be the set of tuples in $P_n^k$ which end with $i$. Then, a tuple $(a_1, a_2, \cdots, a_k) \in P_n^k$ is in $A_i$ if and only if $a_k = i$ and $a_1, a_2, \cdots, a_{k-1}$ are distinct elements of $[n] \setminus \{i\}$. So, a tuple in $A_i$ would look like the following:

$$A_i = \left\{ (a_1,\, a_2,\, a_3,\, \cdots,\, a_{k-1}, i) : a_1 \neq a_2 \neq \cdots \neq a_{k-1} \wedge\ \forall j \in \mathbb{N},\, a_j \in [n] \setminus \{i\} \right\}$$

Thus, size of $A_i$ is the number of $(k-1)$-tuples such that $a_1,\, a_2,\, \cdots,\, a_{k-1}$ are distinct elements of $[n] \setminus \{i\}$. By applying the inductive hypothesis

$$\begin{aligned}
|A_i| &= \left| P_{n-1}^{k-1} \right| \\
&= \frac{(n-1)!}{((n-1)-(k-1))!} = \frac{(n-1)!}{(k-1)!}
\end{aligned}$$

By an argument similar to that in **Proposition 1.3**, we have that $P_n^k = \bigcup_{n \in \mathbb{N}} A_n$ and that $A_1, A_2, \cdots, A_n$ are mutually disjoint. By **Proposition 1.2**,

$$\left| P_n^k \right| = |A_1| + |A_2| + \cdots + |A_n|$$

as required. $\square$

**Note.** $P_n^n$ is the set of all permutations of the elements of set $[n]$. Therefore, the number of permutations of a set of $n$ objects is $\frac{n!}{(n-n)!} = n!$. We can refer to $P_n^k$ as the set of permutations of the $k$-element subsets of $[n]$.

### 1.2.1 Stirling's Approximation

For two functions $f$ and $g$ we write that $f(n) \sim g(n)$ if

$$\frac{f(n)}{g(n)} \to 1 \text{ as } n \to \infty$$

A result using this notation is **Stirling's Approximation**, which approximates the value of $n!$, goes as follows:

$$n! \sim \sqrt{2n\pi} \left( \frac{n}{e} \right)^n$$

A direct consequence of **Stirling's Approximation** gives us this result, quotable without proof:

$$\left( \frac{n}{e} \right) \leq n! \leq \frac{(n+1)^{n+1}}{e^n}.$$