Introduction to Vector Spaces

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Lecture 1: Linear In/dependence

Definition. A field is a set F whose elements are scalars with 2 binary operations such that $(F, +)$ is a group, and so is $(F \setminus \{0\}, \cdot)$

Definition. A linear combination of the vectors $v_1, v_2, \cdots, v_r \in V$ with $\alpha_1, \alpha_2, \cdots, \alpha_r \in F$

$$\sum_{i=1}^{r} \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r$$

Definition. If $v = \sum_{j=0}^{r} \beta_j w_j$, then v is said to be **spanned** / **generated** by $w_1, w_2, \cdots, w_r$

Definition. In $V(F)$, the vectors $v_1, v_2, \cdots, v_r \in V$ are linearly dependant if $\exists\ \alpha_1, \alpha_2, \cdots, \alpha_r \in F$, that are not all 0, such that $\sum_{i=1}^{r} \alpha_i v_i = 0$. In English, we say that a vectors $v_1, v_2, \cdots, v_r$ are linearly dependent if their linear combination can be 0.

Note. In $\mathbb{R}^2$, any two vectors which aren't scalar multiples of each other are linearly independent.

Proposition. The elements of a set of vectors containing $\mathbf{0}$, are linearly dependent.

Proof. Let $\{\, 0, v_1, v_2, \cdots, v_k, \cdots, v_n \,\}$ be the set of vectors. Now, we set the linear combination of said vectors to be 0, i.e.

$$\alpha_1 0 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_r v_r = 0.$$

For the above statement to be true, we only require $\alpha_2, \alpha_3, \cdots, \alpha_r = 0$, whilst α_1 can be whatever. This means that there is a case where not all α's are 0 and thus the vectors $\{\, 0, v_1, v_2, \cdots, v_k, \cdots, v_n \,\}$ are linearly dependent. $\square$

Lecture 2: Linear dependence in matrices

Theorem. Let S be an ordered set of linearly dependent vectors in $V(F)$. Then, some vector is a linear combination of its predecessors.

Proof. asd $\square$