Mathematical Methods

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Lecture 3: Rank of a Matrix

Definition 1. The rank of a matrix, denoted by $\operatorname{rank}(\mathbf{A})$, is equal to the number of non-zero rows in a row echelon form of $\mathbf{A}$

Theorem 1. Let $\mathbf{AX} = \mathbf{B}$ be a linear system of m linear equations in n unknowns with augmented matrix $\left(\mathbf{A} \,\vdots\, \mathbf{B} \right)$, then

- the system has a solution if and only if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}\left(\mathbf{A} \,\vdots\, \mathbf{B} \right)$.
- the system has a unique solution if and only if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}\left(\mathbf{A} \,\vdots\, \mathbf{B} \right) = n$.

Note. The rank of a matrix augmented with another cannot be smaller than the original non-augmented matrix and thus

$$\operatorname{rank}(\mathbf{A}) \neq \operatorname{rank}\left(\mathbf{A} \,\vdots\, \mathbf{B} \right) \Leftrightarrow \operatorname{rank}(\mathbf{A}) < \operatorname{rank}\left(\mathbf{A} \,\vdots\, \mathbf{B} \right)$$

Example. For which values of a does the following system have a unique solution? For which pairs of a, b does the system have more than one solution?

$$\left.\begin{aligned} x - 2y &= 1 \\ x - y + az &= 2 \\ ay + 9z &= b \end{aligned}\right\}$$

$$\mathbf{A} = \left(\begin{array}{ccc:c} 1 & -2 & 0 & 1 \\ 1 & -1 & a & 2 \\ 0 & a & 9 & b \end{array}\right) \underset{\sim}{R} \left(\begin{array}{ccc:c} 1 & -2 & 0 & 1 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & 9 - a^2 & b - a \end{array}\right)$$

The solution is unique if and only if $a \neq \pm 3$. Thus, $\operatorname{rank}(\mathbf{A}) = 3$. The solutions are infinite if and only if $a = \pm 3$ and $b - a = 0$.

$$(a, b) = (\pm 3, \pm 3) \Rightarrow \operatorname{rank}(\mathbf{A}) = 3$$

If a system of n equations in n unknowns (*also referred to as a square system*) has a unique solution, then the solution can be found by using the inverse of the coefficient matrix.

$$\left.\begin{array}{r}\mathbf{AX}=\mathbf{B}\\ A\text{ is }n\times n\\ \text{The system has 1 solution}\end{array}\right\}\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$$

Note. An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\operatorname{rank}(\mathbf{A}) = n$.