1 OPERATIONS WITH MATRICES

Recommended reading: Chapter 1 (Sections 1.4 and 1.5)

1.1 DEFINITIONS

Definition An $m \times n$ **matrix** is a rectangular array of real numbers, arranged in m rows and n columns. The elements of a matrix are called the **entries**. The expression $m \times n$ denotes the **size** of the matrix.

An $m \times n$ matrix can be expressed in general as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \text{or} \qquad \left(a_{ij}\right)_{m\times n},$$

where a_{ij} denotes the entry that occurs in row i and column j, and i and j are the row and column indices, respectively.

A matrix A with n rows and n columns (i.e. $m = n$) is called a **square matrix of order n**, and the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are said to be on the **main diagonal** of A.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Examples

$$\begin{pmatrix} 2 & -7 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 0 & -3 \end{pmatrix}, \qquad \begin{pmatrix} -\sqrt{2} & \pi & e \\ 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 4 \end{pmatrix}.$$

1.2 MATRIX ALGEBRA

Equality
Two matrices are **equal** if they have the same size and corresponding entries are equal, i.e.

$$A = B \text{ if } a_{ij} = b_{ij} \text{ for all } i, j.$$

Example 1 State whether the following matrices are equal or not.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}.$$

Addition
If A and B are any two matrices of the same size, then the **sum** $A+B$ is a matrix obtained by adding together the corresponding entries in the two matrices. Matrices of different sizes cannot be added.

$$\left(a_{ij}\right)_{m\times n} + \left(b_{ij}\right)_{m\times n} = \left(a_{ij} + b_{ij}\right)_{m\times n}$$

Example 2 Consider the following matrices

$$A = \begin{pmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Compute $A+B$. What can you say about $A+C$ and $B+C$?

Multiplication by a scalar
If A is a matrix and k is any scalar (number), then the **product** kA is the matrix obtained by multiplying each entry of A by k.

$$k\left(a_{ij}\right)_{m\times n} = \left(ka_{ij}\right)_{m\times n}$$

Example 3 Consider the following matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ -1 & 0 \end{pmatrix}$. Calculate $2A$ and $-A$.

Subtraction
If A and B are two matrices of the same size, then $A-B=A+(-B)=A+(-1)B$.

Example 4 Consider the matrices $A=\begin{pmatrix}2 & 3 & 4\\ 1 & 2 & 1\end{pmatrix}$ and $B=\begin{pmatrix}0 & 2 & 7\\ 1 & -3 & 5\end{pmatrix}$. Compute $A-B$.

Matrix multiplication
If A is an $m\times r$ matrix and B is $r\times n$ matrix then the product AB is the $m\times n$ matrix whose entries are determined as follows. To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together and then add up the resulting products.

$$AB=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1r}\\ a_{21} & a_{22} & \cdots & a_{2r}\\ \vdots & \vdots & & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{ir}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mr}\end{pmatrix}\begin{pmatrix}b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n}\\ \vdots & \vdots & & \vdots & & \vdots\\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn}\end{pmatrix}$$

$$(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\cdots a_{ir}b_{rj}=\sum_{k=1}^{r}a_{ik}b_{kj}$$

Example 5 Given the matrices $A=\begin{pmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{pmatrix}$ and $B=\begin{pmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{pmatrix}$, compute AB.

In general, the multiplication of two matrices is not commutative, $AB\neq BA$. Equality can fail to hold for three reasons:

1. AB is defined and BA is undefined (e.g. if A is a 2×3 matrix and B is a 3×4 matrix).

2. AB and BA are both defined but have different sizes (e.g. if A is a 2×3 matrix and B is a 3×2 matrix).

3. AB and BA are both defined and have the same size but they are not equal.

Example 6 Consider the matrices $A=\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2\\ 3 & 0\end{pmatrix}$. Compute AB and BA, and check whether they are equal or not.

Rules of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A+B=B+A$

(b) $A+(B+C)=(A+B)+C$

(c) $(AB)C=A(BC)$

(d) $A(B+C)=AB+AC$

(e) $(B+C)A=BA+CA$

(f) $A(B-C)=AB-AC$

(g) $(B-C)A=BA-CA$

(h) $a(B+C)=aB+aC$

(i) $a(B-C)=aB-aC$

(j) $(a+b)C=aC+bC$

(k) $(a-b)C=aC-bC$

(l) $a(bC)=(ab)C$

(m) $a(BC)=(aB)C=B(aC)$

Transpose of a matrix

If A is an $m\times n$ matrix, then its **transpose** A^T is an $n\times m$ matrix given by interchanging the rows and columns of A.

$$\left(a_{ij}\right)_{m\times n}^{T}=\left(a_{ji}\right)_{n\times m}$$

Example 7 Compute the transpose of the following matrices

$$B=\begin{pmatrix}2 & 3\\ 1 & 4\\ 5 & 6\end{pmatrix},\quad C=\begin{pmatrix}1 & 3 & 5\end{pmatrix},\quad D=\begin{pmatrix}3 & 5 & -2\\ 5 & 4 & 1\\ -2 & 1 & 7\end{pmatrix}\quad \text{and}\quad E=\begin{pmatrix}4\end{pmatrix}.$$

Properties of the transpose:

1. $(kA)^T = kA^T$,
2. $(A+B)^T = A^T + B^T$,
3. $(A^T)^T = A$,
4. $(AB)^T = B^T A^T$.

Definition A matrix A is **symmetric** if and only if $A = A^T$. A matrix A is **skew-symmetric** if and only if $A = -A^T$.

Theorem 1 Every square matrix A can be decomposed uniquely as the sum of two matrices S and V, where S is symmetric and V is skew-symmetric.

Example 8 Decompose the matrix $A = \begin{pmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 0 \end{pmatrix}$ as a sum of a symmetric matrix S and a skew-symmetric matrix V.

1.2 PRACTICAL EXERCISES

1. Suppose that $A, B, C, D,$ and E are matrices with the following sizes:

$$\begin{array}{ccccc} A & B & C & D & E \\ (4\times5) & (4\times5) & (5\times2) & (4\times2) & (5\times4) \end{array}$$

Determine which of the following matrix expressions are defined. For those which are defined, give the size of the resulting matrix.

(a) BA (b) $AC+D$ (c) $AE+B$ (d) $AB+B$

(e) $E(A+B)$ (f) $E(AC)$ (g) E^TA (h) $(A^T+E)D$

2. Consider the matrices

$$A=\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix},\quad B=\begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix},\quad C=\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix},\quad D=\begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix},\quad E=\begin{pmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}.$$

Compute the following (where possible).

(1) $D+E$ (2) $D-E$ (3) $5A$ (4) $-7C$ (5) $2B-C$ (6) $4E-2D$

(7) $-3(D+2E)$ (8) D^T-E^T (9) $(D-E)^T$ (10) $\frac{1}{2}C^T-\frac{1}{4}A$ (11) $-B-B^T$ (12) $-7C$

(13) $2E^T-3D^T$ (14) $(2E^T-3D^T)^T$ (15) AB (16) BA (17) $(3E)D$ (18) $(AB)C$

(19) $A(BC)$ (20) CC^T (21) $(DA)^T$ (22) $(C^TB)A^T$ (23) $(2D^T-E)A$ (24) $(-AC)^T+5D^T$

(25) $(BA^T-2C)^T$ (26) $B^T(CC^T-A^TA)$ (27) $D^TE^T-(ED)^T$

3. Let $A=\begin{pmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{pmatrix}$ and $B=\begin{pmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{pmatrix}$. Compute the third row and the second column of AB without computing the entire product.

Answers

1. (a), (c), (d) and (g): not defined; (b) 4×2, (e) 5×5, (f) 5×2, (h) 5×2

2. (1) $\begin{pmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{pmatrix}$, (2) $\begin{pmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$, (3) $\begin{pmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{pmatrix}$, (4) $\begin{pmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{pmatrix}$,

(5) not defined, (6) $\begin{pmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{pmatrix}$, (7) $\begin{pmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{pmatrix}$, (8) $\begin{pmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$,

(9) $\begin{pmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$, (10) $\begin{pmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{9}{4} & 0 \\ \frac{3}{4} & \frac{9}{4} \end{pmatrix}$, (11) $\begin{pmatrix} -8 & 1 \\ 1 & -4 \end{pmatrix}$, (12) $\begin{pmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{pmatrix}$,

(13) $\begin{pmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{pmatrix}$, (14) $\begin{pmatrix} 9 & -13 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -6 \end{pmatrix}$, (15) $\begin{pmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{pmatrix}$, (16) not defined,

(17) $\begin{pmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{pmatrix}$, (18) $\begin{pmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{pmatrix}$, (19) $\begin{pmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{pmatrix}$, (20) $\begin{pmatrix} 21 & 17 \\ 17 & 35 \end{pmatrix}$,

(21) $\begin{pmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{pmatrix}$, (22) $\begin{pmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{pmatrix}$, (23) $\begin{pmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{pmatrix}$, (24) $\begin{pmatrix} 2 & -10 & 11 \\ 13 & 2 & 5 \\ 4 & -3 & 13 \end{pmatrix}$,

(25) $\begin{pmatrix} 10 & -6 \\ -14 & 2 \\ -1 & -8 \end{pmatrix}$, (26) $\begin{pmatrix} 40 & 72 \\ 26 & 42 \end{pmatrix}$, (27) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3. third row $\begin{pmatrix} 0\times 6+4\times 0+9\times 7 & 0\times(-2)+4\times 1+9\times 7 & 0\times 4+4\times 3+9\times 5 \end{pmatrix} = \begin{pmatrix} 63 & 67 & 57 \end{pmatrix}$

second column $\begin{pmatrix} 3\times(-2)+(-2)\times 1+7\times 7 \\ 6\times(-2)+5\times 1+4\times 7 \\ 0\times(-2)+4\times 1+9\times 7 \end{pmatrix} = \begin{pmatrix} 41 \\ 21 \\ 67 \end{pmatrix}$

1.3 SPECIAL TYPES OF MATRICES

The **zero** matrix $O_{m\times n}$

The **identity** matrix $I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$ or $I_n = \left(\delta_{ij}\right)_{n\times n}$,

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ is the **Kronecker delta**.

Diagonal matrix e.g. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

Triangular matrix e.g. $\begin{pmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

If A and is a $m\times n$ matrix then:

(a) $A + O_{m\times n} = O_{m\times n} + A = A$

(b) $A - A = O_{m\times n}$

(c) $O_{m\times n} - A = -A$

(d) $AO_{n\times p} = O_{m\times p}$; $O_{l\times m}A = O_{l\times n}$

(e) $AI_n = A$; $I_mA = A$

1.3 PRACTICAL EXERCISES

1. Find a matrix C such that the following sum is the zero matrix

$$3\begin{pmatrix} 1 & 2 & 0 \\ -2 & 3 & 4 \\ 6 & -2 & 1 \end{pmatrix} - 4\begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix} + 2C.$$

2. Show that the product of two diagonal matrices is again a diagonal matrix. Hence, without doing any calculations, find the inverse of the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Answers

1. $C = \begin{pmatrix} -\frac{3}{2} & 1 & 2 \\ 3 & -\frac{5}{2} & 8 \\ -3 & 5 & -\frac{3}{2} \end{pmatrix}$

2. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$

1.4 INVERSES OF MATRICES

If A is a square $n \times n$ matrix, and if a square $n \times n$ matrix A^{-1} can be found such that $AA^{-1} = A^{-1}A = I_n$, then A is said to be **invertible** and A^{-1} is the **inverse** of A.

Example 9 Given the matrices $A = \begin{pmatrix} 2 & -5 & 3 \\ 1 & 3 & -1 \\ 3 & -4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 3 & -4 \\ -6 & -3 & 5 \\ -13 & -7 & 11 \end{pmatrix}$, show that $B = A^{-1}$.

Theorem 2 If A and B are invertible matrices of the same size, then
(a) AB is invertible,
(b) $(AB)^{-1} = B^{-1}A^{-1}$.

Example 10 Show that if A is an invertible matrix then $\left(A^{-1}\right)^{-1} = A$.

Inverse of a 2×2 matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Example 11 Consider the matrices $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$. Compute A^{-1}, B^{-1}, $(AB)^{-1}$ and $B^{-1}A^{-1}$.

A method for inverting $n \times n$ matrices
- Use elementary row operations to reduce A to I.
- Then use the same operations on I to give A^{-1}

To accomplish this we shall adjoin the identity matrix to the right side of A, thereby producing an augmented matrix of the form

$$\left(A \middle| I\right)$$

then we shall apply elementary row operations to this matrix, so that the final matrix will have the form

$$\left(I \middle| A^{-1}\right).$$

The **elementary row operations** are:
1. Interchange any two rows
2. Add a multiple of one row to another
3. Multiply every element in a row by a fixed number

Example 12 Find the inverses of the following matrices:

$$A=\begin{pmatrix}1&2&3\\2&5&3\\1&0&8\end{pmatrix} \text{ and } B=\begin{pmatrix}0&0&-2&-1\\-2&0&-1&0\\-1&-2&-1&-5\\0&1&1&3\end{pmatrix}.$$

1.4 PRACTICAL EXERCISES

1. Compute the inverses of the following matrices.

$$A=\begin{pmatrix}3&1\\5&2\end{pmatrix},\ B=\begin{pmatrix}2&-3\\4&4\end{pmatrix},\ C=\begin{pmatrix}2&0\\0&3\end{pmatrix}$$

Verify that $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$.

2. Find the inverses of the following matrices.

(a) $\begin{pmatrix}-3&6\\4&5\end{pmatrix}$ (b) $\begin{pmatrix}3&4&-1\\1&0&3\\2&5&-4\end{pmatrix}$ (c) $\begin{pmatrix}1&0&0&0\\1&3&0&0\\1&3&5&0\\1&3&5&7\end{pmatrix}$ (d) $\begin{pmatrix}k_1&0&0&0\\0&k_2&0&0\\0&0&k_3&0\\0&0&0&k_4\end{pmatrix}$.

Answers

1. $(ABC)^{-1}=\begin{pmatrix}20&-15\\36&-21\end{pmatrix}^{-1}=\begin{pmatrix}-\frac{7}{40}&\frac{1}{8}\\-\frac{3}{10}&\frac{1}{6}\end{pmatrix}$

$$C^{-1}B^{-1}A^{-1}=\begin{pmatrix}\frac{1}{2}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{5}&\frac{3}{20}\\-\frac{1}{5}&\frac{1}{10}\end{pmatrix}\begin{pmatrix}2&-1\\-5&3\end{pmatrix}=\begin{pmatrix}-\frac{7}{40}&\frac{1}{8}\\-\frac{3}{10}&\frac{1}{6}\end{pmatrix}$$

2. (a) $\begin{pmatrix}-\frac{5}{39}&\frac{2}{13}\\\frac{4}{39}&\frac{1}{13}\end{pmatrix}$, (b) $\begin{pmatrix}\frac{3}{2}&-\frac{11}{10}&-\frac{6}{5}\\-1&1&1\\-\frac{1}{2}&\frac{7}{10}&\frac{2}{5}\end{pmatrix}$, (c) $\begin{pmatrix}1&0&0&0\\-\frac{1}{3}&\frac{1}{3}&0&0\\0&-\frac{1}{5}&\frac{1}{5}&0\\0&0&-\frac{1}{7}&\frac{1}{7}\end{pmatrix}$, (d) $\begin{pmatrix}\frac{1}{k_1}&0&0&0\\0&\frac{1}{k_2}&0&0\\0&0&\frac{1}{k_3}&0\\0&0&0&\frac{1}{k_4}\end{pmatrix}$

1.5 POWERS OF A MATRIX

If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I,\ A^n = \underbrace{AA\cdots A}_{n \text{ factors}} \quad (n > 0).$$

Moreover, if A is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ factors}} \quad (n > 0).$$

Example 13 If $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, compute A^3 and A^{-3}.

1.5 PRACTICAL EXERCISES

1. Let $A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$. Compute $A^2 - 2A + I$.

2. Show that if a square matrix A satisfies $A^2 - 3A + I = 0$, then $A^{-1} = 3I - A$.

3. Suppose we apply the following McLaurin series to matrices

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots,$$

$$(I + A)^{-1} = I - A + A^2 - A^3 + \cdots.$$

Consider the matrix $A = \begin{pmatrix} 0.1 & 0.2 \\ -0.1 & 0.3 \end{pmatrix}$. Find approximations for e^A, e^{-A} and $(I + A)^{-1}$ by adding the first three terms in the series. Hence, check whether

(a) $(I + A)^{-1}$ is the inverse of $I + A$,

(b) e^{-A} is the inverse of e^A.

Answers

1. $A^2 - 2A + I = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}$

3. $(I + A)^{-1} \approx \begin{pmatrix} 0.89 & -0.12 \\ 0.06 & 0.77 \end{pmatrix}$, $e^A \approx \begin{pmatrix} 1.095 & 0.240 \\ -0.120 & 1.335 \end{pmatrix}$, $e^{-A} \approx \begin{pmatrix} 0.895 & -0.160 \\ 0.080 & 0.735 \end{pmatrix}$

(a) $(I + A)(I + A)^{-1} = (I + A)^{-1}(I + A) \approx \begin{pmatrix} 0.991 & -0.022 \\ -0.011 & 1.0130 \end{pmatrix}$

(b) $e^A e^{-A} = e^{-A} e^A \approx \begin{pmatrix} 0.9992 & 0.0012 \\ -0.0006 & 1.004 \end{pmatrix}$

2 SYSTEMS OF LINEAR EQUATIONS

Recommended reading: Chapter 2 (Sections 2.1, 2.2 [up to page 103] and 2.4)

A **linear equation** in the n variables x_1, x_2, …, x_n can be expressed in the following form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b_n,$$

where a_1, a_2, …, a_n, and b are real constants. The variables in a linear equation sometimes are called **unknowns**.

A **solution** of a linear equation is a set of numbers s_1, s_2, …, s_n such that the equation is satisfied when we substitute $x_1 = s_1$, $x_2 = s_2$, ..., $x_n = s_n$. The set of all solutions is called its **solution set** or sometimes the **general solution** of the equation.

Example 14 Find the solution set of the following equations:

(i) $4x - 2y = 1$,

(ii) $x_1 - 4x_2 + 7x_3 = 5$.

A set of linear equations in the variables x_1, x_2, …, x_n is called a **system of linear equations** or a **linear system**. A system of m linear equations in n unknowns x_1, x_2, …, x_n can be expressed in general as:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{1n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}.$$

The subscripted a's and b's denote constants. The double subscripting on the coefficients of the unknowns x_1, x_2, …, x_n is used to specify the location of the coefficient in the system. These coefficients can be collected together in an $m \times n$ **coefficient matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

If we also let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

then the linear system is equivalent to the matrix equation $AX = B$.

An alternate way to express this system is to form the **augmented matrix**

$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

Each row of $(A|B)$ represents one equation in the original system and each column to the left of the vertical bar corresponds to one of the variables in the system. Hence, this augmented matrix contains all the information about the original system of linear equations.

A sequence of numbers $s_1,\ s_2,\ \ldots,\ s_n$ is called a **solution** of the system if $x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n$ is a solution of every equation in the system.

Solutions → unique
Solutions → infinitely many
Solutions → none

A system of equations that has no solutions is said to be **inconsistent**. If there is at least one solution, it is called **consistent**.

Example 15 Show that the following system of equations is inconsistent.

$$\begin{array}{ccccc} x & + & y & = & 4 \\ 2x & + & 2y & = & 6 \end{array}$$

The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but which is easier to solve by carrying out **elementary row operations**:
1. Interchange any two rows
2. Add a multiple of one row to another
3. Multiply every element in a row by a fixed number

Elementary row operations do not actually change the solutions to the equations.

The aim is to reduce the augmented matrix $(A|B)$ to row-echelon form:
1. If a row does not consist entirely of zeros, then the first nonzero number in the row (called **pivot**) is a 1 (called **leading 1**).
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right then the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else.

A matrix satisfying the properties 1, 2, and 3 (but not necessarily 4) is said to be in **row-echelon form**. A matrix satisfying all properties 1, 2, 3 and 4 is said to be in **reduced row-echelon form**.

Gaussian Elimination
Step-by-step procedure to reduce any matrix to row-echelon form.

Example 16 Reduce the following matrix to row-echelon form

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}.$$

Gauss-Jordan Elimination
Same as Gaussian elimination but includes a final step to get the reduced row-echelon form.

Example 17 Reduce the following matrix to reduced row-echelon form

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

After Gaussian elimination (see Example 16), the row-echelon form of the above matrix was

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Example 18 Solve by Gauss-Jordan elimination

$$\begin{array}{rcrcrcrcrcr} & & & - & 2x_3 & & & + & 7x_5 & = & 12 \\ 2x_1 & + & 4x_2 & - & 10x_3 & + & 6x_4 & + & 12x_5 & = & 28 \\ 2x_1 & + & 4x_2 & - & 5x_3 & + & 6x_4 & - & 5x_5 & = & -1 \end{array}$$

Back-substitution
It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form without continuing all the way to the reduced row-echelon form. The corresponding system of equations can be solved by a technique called **back-substitution**.

Example 19 Solve by Gaussian elimination

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & - & 7x_3 & = & 5 \\ -6x_1 & + & 5x_2 & + & 10x_3 & = & -11 \\ -2x_1 & + & 3x_2 & + & 4x_3 & = & -3 \\ -3x_1 & + & 2x_2 & + & 5x_3 & = & -5 \end{array}$$

Example 20 Use Gaussian Elimination to show that the following system is inconsistent

$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & + & 4x_3 & = & -54 \\ -x_1 & + & x_2 & - & 2x_3 & = & 20 \\ 5x_1 & - & 4x_2 & + & 8x_3 & = & -83 \end{array}$$

Rank of a matrix, existence and uniqueness theorem
Definition The rank of a matrix A, written $\operatorname{rank}(A)$, is equal to the number of nonzero rows (i.e. rows with nonzero pivot entries) in a row-echelon form of A.

Theorem 3 Let $AX = B$ be a system of m linear equations in n unknowns with augmented matrix $(A|B)$, then:

(a) The system has a solution if and only if $\operatorname{rank}(A) = \operatorname{rank}(A|B)$.

(b) The solution is unique if and only if $\operatorname{rank}(A) = \operatorname{rank}(A|B) = n$.

Example 21 For which of the values of a does the following system have a unique solution, and for which pairs (a,b) does the system have more than one solution? The value of b does not have any effect on whether the system has a unique solution. Why?

$$\begin{array}{rcrcrcr} x & - & 2y & & & = & 1 \\ x & - & y & + & az & = & 2 \\ & & ay & + & 9z & = & b \end{array}$$

Matrix equation of a square system of linear equations

Definition A system $AX = B$ of linear equations is square if and only if the matrix A of coefficients is square.

Theorem 4 A square system $AX = B$ of linear equations has a unique solution if and only if the matrix A is invertible. In such a case, $X = A^{-1}B$ is the unique solution of the system.

Remark An $n \times n$ matrix A is invertible if and only if $\operatorname{rank}(A) = n$.

Example 22 Consider the matrix $A = \begin{pmatrix} 2 & -5 & 3 \\ 1 & 3 & -1 \\ 3 & -4 & 3 \end{pmatrix}$. Is the matrix A invertible? If so find its inverse. Hence, does the following system of equations have a unique solution?

$$\begin{array}{rcrcrcl} 2x & - & 5y & + & 3z & = & 0 \\ x & + & 3y & - & z & = & 2. \\ 3x & - & 4y & + & 3z & = & 3 \end{array}$$

Homogeneous Linear Systems
A system of m linear equations in n variables $x_1, x_2, \dots, x_n$ is **homogeneous** if it is of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0' \end{array}$$

or, alternatively, under the matrix form $AX = O_{m\times 1}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Solutions
- Always at least one (**trivial** solution): $x_1 = x_2 = \cdots = x_n = 0$
- Can have infinitely many (**nontrivial** solution)

Theorem 4 Let $AX = O_{m\times 1}$ be a homogeneous system of m linear equations in n variables.

(1) If $\operatorname{rank}(A) < n$, then the system has a nontrivial solution.

(2) If $\operatorname{rank}(A) = n$, then the system has only the trivial solution.

Corollary 1 Let $AX = O_{m\times 1}$ be a homogeneous system of m linear equations in n variables. If $m < n$, then the system has a nontrivial solution.

Example 23 Solve the following homogeneous system of linear equations by Gauss-Jordan elimination

$$\begin{array}{rcrcrcrcrcl}
2x_1 & + & 2x_2 & - & x_3 & & & + & x_5 & = & 0 \\
-x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 0 \\
x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0 \\
 & & & & x_3 & + & x_4 & + & x_5 & = & 0
\end{array}$$

2 PRACTICAL EXERCISES

1. Reduce the following matrices to row-echelon form

(a) $\begin{pmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 4 & -7 & 5 & 2 & 8 \end{pmatrix}$.

2. Solve each of the following systems by Gauss-Jordan elimination.

a)
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

b)
$$\begin{aligned} w + 2x - y &= 4 \\ x - y &= 3 \\ w + 3x - 2y &= 7 \\ 2u + 4v + w + 7x &= 7 \end{aligned}$$

3. Solve each of the following systems by Gaussian elimination.

a)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -15 \\ 5x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 11 \\ -6x_1 - 4x_2 + 2x_3 &= 30 \end{aligned}$$

b)
$$\begin{aligned} 10y - 4z + w &= 1 \\ x + 4y - z + w &= 2 \\ 3x + 2y + z + 2w &= 5 \\ -2x - 8y + 2z - 2w &= -4 \\ x - 6y + 3z &= 1 \end{aligned}$$

4. Exercises for Section 2.1, pages 96-97: 1 (a), (c), (e) and (g); 2 (a) and (c); 3; 4; 6.

5. Exercises for Section 2.2, pages 107-108: 1; 2 (a), (b), (c) and (e); 7 (a).

6. For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2-14)z &= a+2 \end{aligned}.$$

7. Solve the following homogeneous system of equations by Gaussian elimination.

$$\begin{array}{rcrcrcrcl} & & v & + & 3w & - & 2x & = & 0 \\ 2u & + & v & - & 4w & + & 3x & = & 0 \\ 2u & + & 3v & + & 2w & - & x & = & 0 \\ -4u & - & 3v & + & 5w & - & 4x & = & 0 \end{array}.$$

8. Exercises for Section 2.2, pages 107-110: 4 (a) and (c); 5 (a) and (c); 12.

9. For which value(s) of λ does the following system of equations have nontrivial solutions?

$$\begin{array}{rcrcl} (\lambda-3)x & + & y & = & 0 \\ x & + & (\lambda-3)y & = & 0 \end{array}.$$

Answers

2. (a) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

3. (a) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix}$

6. If $a = -4$ the system is inconsistent and has no solutions.
   If $a = 4$ the system has infinitely many solutions.
   If $a \neq \pm 4$ the system has exactly one solution.

9. If $\lambda = 2$ or $\lambda = 4$ the system has nontrivial solutions.

3 DETERMINANTS

Recommended reading: Chapter 3 (Sections 3.1, 3.2 [up to page 160] and 3.3 [up to page 168])

Let A be a square matrix. The number $\det(A)$ is called the **determinant** of A.

The determinant of a square matrix associates a real number with a matrix and it can tell us whether a system of equations has solutions or not.

Determinant of a 2×2 matrix

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 24 Evaluate the determinant of the following matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix}$.

Determinant of a 3×3 matrix
Use a row or column to expand along. Multiply each entry in chosen row/column by the determinant of matrix remaining when you delete the row and column of that entry, remembering to use the following pattern of +/- signs.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Example 25 Evaluate the determinant of the following matrix $B = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{pmatrix}$.

Determinant of a $n \times n$ matrix
The determinant of a $n \times n$ matrix can be also calculated by expanding it along any row or any column as long as the following signs for the sub-determinants are used.

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \\ + & - & + & - & \\ - & + & - & + & \\ \vdots & & & & \end{pmatrix}$$

Note that it is easier to compute the determinant by expanding it along a row/column which has a number of zeros.

Example 26 Evaluate the determinant of the following matrix $C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix}$ by using row or column expansion.

Properties of determinants

The more zero entries there are in a matrix, the easier it is to find the determinant.

Example 27 Evaluate $\det(A)$, where $A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$.

Theorem 5 If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal; that is

$$\det(A) = a_{11}a_{22} \cdots a_{nn}.$$

Theorem 6 Let A be an $n \times n$ matrix. If B is formed by:

(a) multiplying any row or column of A by scalar k, then $\det(B) = k \det(A)$

(b) interchanging two rows or two columns of A, then $\det(B) = - \det(A)$

(c) adding a multiple of any row to another row, or a multiple of any column to any column, then $\det(B) = \det(A)$.

Remark Note that property (c) of Theorem 6 states that if $kR_i + R_j \to R_j$ then $\det(B) = \det(A)$. However, the following elementary row operation $kR_i - R_j \to R_j$ is a combination of two elementary row operations, i.e. $(-1)R_j \to R_j$ and $kR_i + R_j \to R_j$, and it changes the sign of the determinant: $\det(B) = -\det(A)$.

Evaluating determinants by row-reduction

The basic idea is to apply elementary row operations to reduce the given matrix A to a matrix R that is in row-echelon form. Since a row-echelon form of a square matrix is always upper triangular, $\det(R)$ can be evaluated easily. The value of $\det(A)$ can be obtained from $\det(R)$

Example 28 Evaluate $\det(A)$ by row-reduction, where $A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$.

Example 29 Evaluate the determinant of the matrix C in Example 26 by reducing it to row-echelon form.

More properties of determinants

1. If A is a square matrix, then $\det(A) = \det(A^T)$.

2. If a square matrix A has two proportional rows or two proportional columns, then $\det(A) = 0$

e.g $\begin{pmatrix} -1 & 4 \\ -2 & 8 \end{pmatrix}$, $\begin{pmatrix} 2 & 7 & 8 \\ 3 & 2 & 4 \\ 2 & 7 & 8 \end{pmatrix}$, $\begin{pmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{pmatrix}$

3. If A is an $n \times n$ matrix , then $\det(kA) = k^n \det(A)$.

Example 30 Consider the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. Compute $\det(A)$ and $\det(5A)$.

4. If A and B are both matrices of the same size, then $\det(AB) = \det(A)\ \det(B)$.

Example 31 Consider the matrices $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 3 \\ 5 & 8 \end{pmatrix}$. Verify that $\det(AB) = \det(A)\ \det(B)$.

Example 32 Show that $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} \\ a'_{21} & a'_{22} \end{pmatrix} = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21}+a'_{21} & a_{22}+a'_{22} \end{pmatrix}$.

5. Let A, A' and A'' be square matrices that differ only in a single row, say the rth, and assume that the rth row of A'' can be obtained by adding corresponding entries in the rth rows of A and A'. Then

$$\det\left(A''\right)=\det\left(A\right)+\det\left(A'\right).$$

The same result holds for columns.

Example 33 Without evaluating the determinants, show that

(a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11}+a_{21} & ka_{12}+a_{22} & ka_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11}-a_{21} & ka_{12}-a_{22} & ka_{13}-a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Example 34 Prove the following identity without evaluating the determinants.

$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

6. A square matrix A is invertible iff $\det(A)\neq 0$.

Example 35 Is the matrix $A=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}$ invertible?

7. If A is invertible, then $\det\left(A^{-1}\right)=\dfrac{1}{\det\left(A\right)}$.

3 PRACTICAL EXERCISES

1. Evaluate the determinants of the following matrices by reducing them to row-echelon form.

(a) $\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$.

2. Exercises for Section 3.2, page 162: 2 (a) and (e).

3. Evaluate the following determinants by inspection:

(a) $\begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ -8 & \sqrt{2} & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 9 & 5 & 6 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & -17 & 4 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 5 & -8 & 1 \end{vmatrix}$

4. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find

(a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ (b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$ (c) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$.

5. Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$.

6. Prove the following identities without evaluating the determinants

(a) $\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (b) $\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

7. For which value(s) of k does A fail to be invertible?

(a) $A=\begin{pmatrix} k-3 & -2 \\ -2 & k-2 \end{pmatrix}$ (b) $A=\begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{pmatrix}$.

8. Use properties of determinants or otherwise to prove the following identities:

(a) $\det\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}=(a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)$

(b) $\det\begin{pmatrix} \cos\phi & \sin\phi\cos\theta & \sin\phi\sin\theta \\ -\sin\phi & \cos\phi\cos\theta & \cos\phi\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}=1$

Answers

1. (a) 33 (b) 39 (c) 38

3. (a) -2 (b) -30 (c) 0

4. (a) -6 (b) 72 (c) 18

7. (a) $k=\dfrac{5\pm\sqrt{17}}{2}$ (b) $k=-1$

4 EIGENVALUES AND DIAGONALISATION

Eigenvectors and eigenvalues
Usually there is no relationship between vectors $\mathbf{x}$ and $A\mathbf{x}$. But sometimes $A\mathbf{x}$ is a multiple of $\mathbf{x}$.

Definition Let A be an $n \times n$ matrix. A non-zero n-vector $\mathbf{x}$ such that

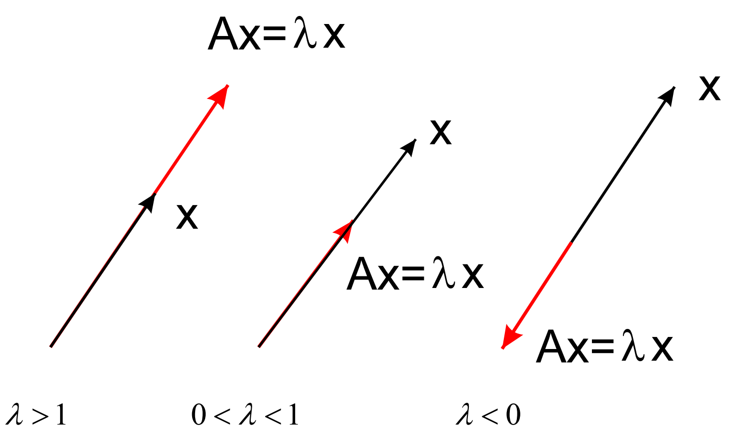
$$A\mathbf{x} = \lambda\mathbf{x}$$

is called an **eigenvector** of A. The multiple λ is called an **eigenvalue** and $\mathbf{x}$ is said to be the eigenvector **corresponding** to λ.

Example 36 The vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

Example 37 Show that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

Eigenvalues and eigenvectors have a useful geometrical interpretation in two and three dimensions, if $\mathbf{x}$ is an eigenvector of A, then matrix A acts by stretching or contracting the vector $\mathbf{x}$, changing or not changing its direction, e.g.



To find the eigenvalues of an $n \times n$ matrix A we can rewrite $A\mathbf{x} = \lambda\mathbf{x}$ as a matrix equation

$$A\mathbf{x} = \lambda I_n \mathbf{x}$$

or, equivalently,

$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}.$$

This is a homogeneous system, and only has non-trivial solutions if

$$\det(A - \lambda I_n) = 0.$$

So to find eigenvalues we look for the possible scalars λ which satisfy $\det(A-\lambda I_n)=0$.

Definition The equation

$$\det(A-\lambda I_n)=0$$

is called the **characteristic equation** of A. When it is expanded it becomes a polynomial of degree n in λ called the **characteristic polynomial** of A:

$$p_n(\lambda)=\det(A-\lambda I_n)=(-1)^n(\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+\cdots+c_{n-1}\lambda+c_n).$$

Remark Since the characteristic polynomial of an $n\times n$ matrix A is of degree n, we expect at most n eigenvalues.

The characteristic polynomial can be factored in the form

$$p_n(\lambda)=(-1)^n(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

where m_j is called the multiplicity of the eigenvalue λ_j. The sum of the multiplicities of all eigenvalues is n; that is

$$n=m_1+m_2+\cdots m_k.$$

Example 38 Find the eigenvalues of the matrix $A=\begin{pmatrix}3 & 2\\ -1 & 0\end{pmatrix}$.

Example 39 Find the eigenvalues of $A=\begin{pmatrix}5 & 0 & 1\\ 1 & 1 & 0\\ -7 & 1 & 0\end{pmatrix}$.

Theorem 7 If A is an $n\times n$ matrix, then the following are equivalent.

(a) λ is an eigenvalue of A.

(b) The system of equations $(A-\lambda I_n)\mathbf{x}=\mathbf{0}$ has nontrivial solutions.

(c) There is a nonzero vector $\mathbf{x}$ of dimension n such that $A\mathbf{x}=\lambda\mathbf{x}$.

(d) λ is the solution of the characteristic equation $\det(A-\lambda I_n)=0$.

The eigenvectors of A corresponding to an eigenvalue λ are the nonzero vectors that satisfy

$$(A-\lambda I_n)\mathbf{x}=\mathbf{0}.$$

Definition The complete solution set of the homogeneous system $(A-\lambda I_n)\mathbf{x}=\mathbf{0}$ is the **eigenspace** of A corresponding to λ and is denoted by E_λ. The set of particular solutions obtained by solving the homogeneous system $(A-\lambda I_n)\mathbf{x}=\mathbf{0}$ using Gaussian elimination and by setting each independent variable in turn equal to 1 and all the other independent variables equal to zero are called **fundamental eigenvectors**.

Example 40 Find the fundamental eigenvectors of the matrix $A=\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$.

Example 41 Find the eigenspaces and the fundamental eigenvectors of $A=\begin{pmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{pmatrix}$.

Diagonalisation

Definition An $n\times n$ matrix A is called **diagonalisable** if there exists an invertible matrix P such that $D=P^{-1}AP$ is a diagonal matrix. Any diagonalisable matrix A can be **decomposed** as $A=PDP^{-1}$.

Example 42 If $A=\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ and $P=\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$ then the matrix $D=P^{-1}AP$ is diagonal.

Theorem 8 An $n\times n$ matrix A is diagonalisable if and only if A has n fundamental eigenvectors.

Procedure for diagonalising a matrix:

(1) For each eigenvalue of A find the corresponding fundamental eigenvectors.

(2) If overall the number of fundamental eigenvectors of A is less than n, then the matrix A is not diagonalisable.

(3) Otherwise, form the matrix P whose columns are the n fundamental eigenvectors.

(4) Then $D=P^{-1}AP$ will be a diagonal matrix whose d_{ii} entry is the eigenvalue for the fundamental vector forming the i-th column of P.

Example 43 Consider the matrix $A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$. Obtain a matrix P, such that $D = P^{-1}AP$ is diagonal. State explicitly the matrix D.

Theorem 9 If an $n \times n$ matrix A possesses n distinct eigenvalues then A is diagonalisable.

Example 44

Let A be the matrix,

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{pmatrix}.$$

(i) Find the characteristic polynomial of A.
(ii) Determine the eigenvalue(s) of A.
(iii) For each eigenvalue of A, find the corresponding fundamental eigenvectors.
(iv) Obtain a matrix P, such that $D = P^{-1}AP$ is diagonal. State explicitly the matrix D.

Example 45 Explain why the matrix $A = \begin{pmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{pmatrix}$ is not diagonalizable.

Matrix diagonalisation has many applications in practice. It is much easier to work with diagonal matrices than with matrices in general as the result of addition and/or multiplication of diagonal matrices is a diagonal matrix, i.e.

$$\begin{pmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \\ \vdots & & \ddots \end{pmatrix} + \begin{pmatrix} b_1 & 0 & \cdots \\ 0 & b & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & 0 & \cdots \\ 0 & a_2 + b_2 & \\ \vdots & & \ddots \end{pmatrix},$$

$$\begin{pmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots \\ 0 & b & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 & \cdots \\ 0 & a_2 b_2 & \\ \vdots & & \ddots \end{pmatrix}.$$

For example, matrix diagonalisation can be used to compute powers of a matrix efficiently.

Example 46 Show that if a matrix A is diagonalisable, then $A^3 = PD^3P^{-1}$.

Remark If a matrix A is diagonalisable, then $A^n = PD^nP^{-1}$ for any $n \in \mathbb{N}$.

Example 47 Consider the matrix $A=\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$. Compute A^5.

This idea can be generalized even further to evaluate efficiently any algebraic expression which involves A only, e.g.

$A^3-2A^2+3A+I=$

4 PRACTICAL EXERCISES

In Questions 1-6, find the eigenvalues and eigenvectors of the given matrix, and determine whether the matrix is diagonalisable. When it is, find an invertible matrix P and diagonal matrix D such that $D=P^{-1}AP$.

1. $A=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$ 2. $A=\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ 3. $A=\begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

4. $A=\begin{pmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{pmatrix}$ 5. $A=\begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix}$ 6. $A=\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

7. Let $A=\begin{pmatrix} 5 & 7 & -2 \\ -4 & -6 & 2 \\ 0 & -1 & 1 \end{pmatrix}$. Find $A^{10}-5A$.

Answers

Note that in Questions 1-6 while the eigenvalues of matrices are unique, the entries of the corresponding fundamental eigenvectors are not as they are only defined up to a multiplicative real constant.

1. Eigenvalues: 3, 2, 1. $P=\begin{pmatrix}0&0&-2&0\\-5&5&0&1\\0&1&2&0\\1&0&1&0\end{pmatrix}$, $D=\begin{pmatrix}3&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$.

2. Eigenvalues: 4, -2. $P=\begin{pmatrix}1&0&-1\\1&0&1\\0&1&0\end{pmatrix}$, $D=\begin{pmatrix}4&0&0\\0&-2&0\\0&0&-2\end{pmatrix}$.

3. Eigenvalues: 2, -1. The matrix is not diagonalizable as there are only two fundamental eigenvectors.

4. Eigenvalues: 2, 0. $P=\begin{pmatrix}3&-2&-1\\0&1&3\\1&0&1\end{pmatrix}$, $D=\begin{pmatrix}2&0&0\\0&2&0\\0&0&0\end{pmatrix}$.

5. Eigenvalue: 5. The matrix is not diagonalizable as there is only one fundamental eigenvector.

6. Eigenvalues: 2, 1. $P=\begin{pmatrix}-1&0&-2\\0&1&1\\1&0&1\end{pmatrix}$, $D=\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}$.

7. $\begin{pmatrix}-28&-40&12\\24&36&-12\\4&10&-6\end{pmatrix}$