

Research



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Boltzmann-type control of opinion consensus through leaders

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The study of formations and dynamics of opinions leading to the so-called opinion consensus is one of the most important areas in mathematical modelling of social sciences. Following the Boltzmann-type control approach recently introduced by the first two authors, we consider a group of opinion leaders who modify their strategy accordingly to an objective functional with the aim of achieving opinion consensus. The main feature of the Boltzmann-type control is that, owing to an instantaneous binary control formulation, it permits the minimization of the cost functional to be embedded into the microscopic leaders' interactions of the corresponding Boltzmann equation. The related Fokker–Planck asymptotic limits are also derived, which allow one to give explicit expressions of stationary solutions. The results demonstrate the validity of the Boltzmann-type control approach and the capability of the leaders' control to strategically lead the followers' opinion.

1. Introduction

Mean-field games and mean-field-type control theory have created a lot of interest in recent years (see, for example, [1–5] and references therein). The general setting consists in a control problem involving a very large number of agents where both the evolution of the state and the objective functional of each agent are influenced by the collective behaviour of all other agents. Typical examples in socio-economical sciences, biology and engineering are represented by the problems of persuading voters to vote for a specific candidate, influencing buyers towards a given good or asset, forcing human crowds or a group of animals to follow a specific path or to reach a desired zone, or optimizing the traffic flow in road networks and supply chains [2,6–14].

In this paper, we focus on control problems where the collective behaviour corresponds to the formation process of opinion consensus [15–23]. In particular, we consider models where the control strategy is based on hierarchical leadership. This hierarchical leadership concept is discussed in [10], where a population of leaders is considered giving rise to aggregate opinions and convergence towards specific patterns. Opinion dynamics in the presence of different populations has been previously been introduced in [24–26]. We mention here that control through leaders in self-organized flocking systems has been studied in [6,8,27].

We introduce a hierarchical opinion formation dynamics where the leaders aim at controlling the followers through a suitable cost function which characterizes the leaders' strategy in trying to influence the followers' opinion. Based on this microscopic model, we develop a Boltzmann-type optimal control approach following the ideas recently presented in [28]. The approach is closely related to model predictive and instantaneous control techniques [29–32]. We derive an explicit controller for the leaders' dynamic using an instantaneous binary control framework on the microscopic level and, similar to the mean field control, study the related kinetic description for a large number of agents. Owing to this formulation, the minimization of the cost functional is embedded into the microscopic leaders' interactions of the corresponding Boltzmann equation.

The rest of the manuscript is organized as follows. First, in §2, we introduce the microscopic model of the leaders' strategy in the leader–follower interactions and derive the corresponding Boltzmann-type control formulation. The main properties of the kinetic model are studied in §3; in particular, we show that the leaders' control strategy may lead the followers' opinion towards the desired state. Explicit asymptotic opinion distributions are computed in §4 using an approximated Fokker–Planck description derived in the so-called quasi-invariant opinion limit. Several numerical results confirm the theoretical analysis in §5.

2. Microscopic models of opinion control through leaders

A rather common assumption in opinion formations is that interactions are formed mainly by binary exchange of information (e.g. [16,20,33,34]). Similarly to [10], we are interested in the opinion formation process of a followers' population steered by the action of a leaders' group. The major novelty here is that the leaders' behaviour is driven by a suitable control strategy based on the interplay between the desire to force followers towards a given state and the necessity to keep a position close to the mean opinion of the followers in order to influence them. In the following, we first generalize the approach of Albi *et al.* [28], starting from a differential system describing the evolution of the two populations of leaders and followers. In the second part, we present a binary interaction model for the same dynamic showing how the two descriptions are related.

(a) Microscopic modelling

Let us assume that we have two populations, one of followers and one of leaders. Each follower is mutually influenced by the other followers and by the leaders, whose target is to steer the followers' opinion to a desired configuration of consensus following some prescribed strategy. We consider the evolution of a population of N_L leaders and N_F followers, with opinions $w_i, \tilde{w}_k \in I = [-1, 1]$, for $i = 1, \dots, N_F$ and $k = 1, \dots, N_L$, evolving according to

$$\dot{w}_i = \frac{1}{N_F} \sum_{j=1}^{N_F} P(w_i, w_j)(w_j - w_i) + \frac{1}{N} \sum_{h=1}^{N_L} S(w_i, \tilde{w}_h)(\tilde{w}_h - w_i), \quad w_i(0) = w_{i,0} \quad (2.1)$$

and

$$\dot{\tilde{w}}_k = \frac{1}{N_L} \sum_{h=1}^{N_L} R(\tilde{w}_k, \tilde{w}_h)(\tilde{w}_h - \tilde{w}_k) + u, \quad \tilde{w}_k(0) = \tilde{w}_{k,0}, \quad (2.2)$$

where $P(\cdot, \cdot)$, $S(\cdot, \cdot)$ and $R(\cdot, \cdot)$ are given *compromise functions*, typically taking values in $[0, 1]$, measuring the relative importance of the interacting agent in the consensus dynamic. The control

term u characterizes the strategy of the leaders and is given by the solution of the following optimal control problem:

$$u = \operatorname{argmin}\{J(u, w, \tilde{w})\}, \quad (2.3)$$

where

$$J(u, w, \tilde{w}) = \frac{1}{2} \int_0^T \left\{ \frac{\psi}{N_L} \sum_{h=1}^{N_L} (\tilde{w}_h - w_d)^2 + \frac{\mu}{N_L} \sum_{h=1}^{N_L} (\tilde{w}_h - m_F)^2 \right\} ds + \int_0^T \frac{\nu}{2} u^2 ds. \quad (2.4)$$

In the latter equation, w and \tilde{w} are the vectors with the followers' and leaders' opinions, T represents the final time horizon, w_d is the desired opinion and m_F is the average opinion of the followers' group at time $t \geq 0$ defined as

$$m_F = \frac{1}{N_F} \sum_{j=1}^{N_F} w_j.$$

The parameter $\nu > 0$, as usual, is a regularization term representing the importance of the control u in the overall dynamic. More precisely, ν penalizes the action of the control u in such a way that for large values of ν the control action vanishes and vice versa.

The problem may also be formulated as a constrained minimization problem for u^n, w^n and \tilde{w}^n in the form

$$\begin{aligned} \min \quad & J(u^n, w^n, \tilde{w}^n) \\ \text{subject to} \quad & (2.1) \text{ and } (2.2). \end{aligned} \quad (2.5)$$

In general, the solution of this problem is a difficult task, in particular for nonlinear constraints and non-convex functionals. In the following, we assume sufficient regularity on the constraints of (2.5), in such a way that the minimizer fulfils the necessary first-order optimality conditions. We refer to [35] for a detailed discussion of the necessary and sufficient optimality conditions.

Thus, the control strategy of the leaders' population is based on an interplay of two behaviours weighted by the non-negative constants ψ and μ such that $\psi + \mu = 1$. On the one hand, they aim at minimizing the distance with respect to the desired state w_d (*radical behaviour*); and on the other hand, they aim at minimizing the distance with respect to the followers' mean opinion (*populistic behaviour*). Therefore, the leaders influence the followers' opinion through the function $S(\cdot, \cdot)$, and the followers influence the leaders' strategy through their mean opinion in the cost functional (2.4).

The above optimization problem is approximated using the Boltzmann-type optimal control approach recently presented in [28] which corresponds to a binary model predictive control of (2.1)–(2.3) in the case of a very large number of agents [29,30].

(b) Instantaneous binary control

The main idea is to avoid the solution of the dynamics on the whole time interval and to consider a closed-loop strategy for the opinion model in the case of binary interactions. Hence, we split the time interval $[0, T]$ in M time intervals of length Δt and let $t^n = \Delta t \sim n$ and solve sequentially the optimal control problem in each time interval. This approach is related to the receding horizon strategy, or instantaneous control in the engineering literature, which allows us to express the control as a feedback of the state variables. In general, with respect to the associated optimal control problem (2.1)–(2.3) this technique furnishes a suboptimal solution. Rigorous results on the properties of u for a constrained quadratic cost functional are discussed, for example, in [29,30].

More precisely, we approximate both (2.1) and (2.2) by the following discretized binary dynamics:

$$\begin{aligned} w_i^{n+1} &= w_i^n + \alpha P(w_i^n, w_j^n)(w_j^n - w_i^n) + \alpha S(w_i^n, \tilde{w}_l^n)(\tilde{w}_l^n - w_i^n) \\ w_j^{n+1} &= w_j^n + \alpha P(w_j^n, w_i^n)(w_i^n - w_j^n) + \alpha S(w_j^n, \tilde{w}_l^n)(\tilde{w}_l^n - w_j^n) \end{aligned} \quad (2.6)$$

and

$$\left. \begin{aligned} \tilde{w}_k^{n+1} &= \tilde{w}_k^n + \alpha R(\tilde{w}_k^n, \tilde{w}_h^n)(\tilde{w}_h^n - \tilde{w}_k^n) + 2\alpha u^n \\ \tilde{w}_h^{n+1} &= \tilde{w}_h^n + \alpha R(\tilde{w}_h^n, \tilde{w}_k^n)(\tilde{w}_k^n - \tilde{w}_h^n) + 2\alpha u^n \end{aligned} \right\} \quad (2.7)$$

where $\alpha = \Delta t/2$, i and j are the indices of the two interacting followers, l is the index of an arbitrary leader, and h and k are the indices of the two interacting leaders. The control variable u is given by the solution of the following optimization problem:

$$u^n = \operatorname{argmin}\{J(u^n, w^n, \tilde{w}^n)\} \quad (2.8)$$

and

$$J(u^n, w^n, \tilde{w}^n) = \alpha \left(\frac{\psi}{2} \sum_{p=\{k,h\}} (\tilde{w}_p^n - w_d)^2 + \frac{\mu}{2} \sum_{p=\{k,h\}} (\tilde{w}_p^n - m_F^n)^2 + \nu(u^n)^2 \right). \quad (2.9)$$

In order to solve the minimization problem introduced in (2.8), we can proceed as in [28] using a standard Lagrange multipliers approach to compute explicitly u^n . In this way, we obtain the feedback control

$$2\alpha u^n = - \sum_{p=\{k,h\}} \frac{2\alpha^2}{\nu} [\psi(\tilde{w}_p^{n+1} - w_d) + \mu(\tilde{w}_p^{n+1} - m_F^{n+1})]. \quad (2.10)$$

Note that since the feedback control u^n in (2.10) depends on the post-interaction opinion the constrained binary interaction (2.7) is implicitly defined but it can be easily inverted. The explicit version of the control reads

$$2\alpha u^n = - \sum_{p=\{k,h\}} \frac{\beta}{2} [\psi(\tilde{w}_p^n - w_d) + \mu(\tilde{w}_p^n - m_F^n)] - \frac{\alpha\beta}{2} (R(\tilde{w}_k^n, \tilde{w}_h^n) - R(\tilde{w}_h^n, \tilde{w}_k^n))(\tilde{w}_h^n - \tilde{w}_k^n), \quad (2.11)$$

where we further approximated m_F^{n+1} with m_F^n to have a fully explicit expression and introduced the parameter β defined as

$$\beta = \frac{4\alpha^2}{\nu + 4\alpha^2}. \quad (2.12)$$

3. Boltzmann-type control

In this section, we consider a Boltzmann dynamic corresponding to the above instantaneous control formulation. In order to derive a kinetic equation, we introduce a density distribution of followers $f_F(w, t)$ and leaders $f_L(\tilde{w}, t)$ depending on the opinion variables $w, \tilde{w} \in I$ and time $t \geq 0$. It is assumed that the followers' density is normalized to one, that is,

$$\int_I f_F(w, t) dw = 1,$$

whereas

$$\int_I f_L(\tilde{w}, t) d\tilde{w} = \rho \leq 1.$$

The kinetic model can be derived by considering the change in time of $f_F(w, t)$ and $f_L(\tilde{w}, t)$ depending on the interactions with the other individuals and the leaders' strategy. This change depends on the balance between the gain and loss due to the binary interactions.

(a) Binary constrained interactions dynamic

Let us consider the pairwise opinions (w, v) and (\tilde{w}, \tilde{v}) , respectively, of two followers and two leaders; the corresponding post-interaction opinions are computed according to three dynamics, the interaction between two followers, the interaction between follower and leader and, finally, the interaction between two leaders.

The post-interaction opinions $(\tilde{w}^*, \tilde{v}^*)$ of two leaders are given by

$$\left. \begin{aligned} \tilde{w}^* &= \tilde{w} + \alpha R(\tilde{w}, \tilde{v})(\tilde{v} - \tilde{w}) + 2\alpha u + \tilde{\theta}_1 \tilde{D}(\tilde{w}) \\ \tilde{v}^* &= \tilde{v} + \alpha R(\tilde{v}, \tilde{w})(\tilde{w} - \tilde{v}) + 2\alpha u + \tilde{\theta}_2 \tilde{D}(\tilde{v}), \end{aligned} \right\} \quad (3.1)$$

where the feedback control is defined as

$$\begin{aligned} 2\alpha u &= -\frac{\beta}{2} [\psi((\tilde{w} - w_d) + (\tilde{v} - v_d)) + \mu((\tilde{w} - m_F) + (\tilde{v} - m_F))] \\ &\quad - \frac{\alpha\beta}{2} (R(\tilde{w}, \tilde{v}) - R(\tilde{v}, \tilde{w}))(\tilde{v} - \tilde{w}) \end{aligned} \quad (3.2)$$

and

$$m_F(t) = \int_I f_F(w, t) w dw. \quad (3.3)$$

Note that the control term is now embedded into the binary interaction and that we considered an additional noise component such that the diffusion variables $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are realizations of a random variable with zero mean and finite variance $\tilde{\sigma}^2$. Moreover, the noise influence is weighted by the function $\tilde{D}(\cdot)$, representing the local relevance of diffusion for a given opinion, and such that $0 \leq \tilde{D}(\cdot) \leq 1$.

We assume that the opinions (w^*, v^*) in the follower-follower interactions are derived according to

$$\left. \begin{aligned} w^* &= w + \alpha P(w, v)(v - w) + \theta_1 D(w) \\ v^* &= v + \alpha P(v, w)(w - v) + \theta_2 D(v), \end{aligned} \right\} \quad (3.4)$$

where the diffusion variables θ_1 and θ_2 are again realizations of a random variable with zero mean, finite variance σ^2 and $0 \leq D(\cdot) \leq 1$. Finally, the leader-follower interaction is described for every agent from the leaders' group, thus in general we have

$$\left. \begin{aligned} w^{**} &= w + \alpha S(w, \tilde{v})(\tilde{v} - w) + \hat{\theta} \hat{D}(w) \\ \tilde{v}^{**} &= \tilde{v}, \end{aligned} \right\} \quad (3.5)$$

where, similar to the previous dynamic, $\hat{\theta}$ is a random variable with zero mean and finite variance $\hat{\sigma}^2$ and $0 \leq \hat{D}(\cdot) \leq 1$.

Since we are dealing with a kinetic problem in which the variable belongs to a bounded domain, namely $I = [-1, 1]$, we must deal with additional mathematical difficulties in the definition of the agents' interactions. In fact, it is essential to consider only interactions that do not produce values outside the finite interval.

For the leaders' interaction if we consider the constrained binary interactions system (3.1)–(2.11), without diffusion we obtain that $|\tilde{w}^* - \tilde{v}^*|$ is a contraction if $\alpha \leq \frac{1}{2}$

$$|\tilde{w}^* - \tilde{v}^*| = |(\tilde{w} - \tilde{v}) - \alpha(\tilde{w} - \tilde{v})(R(\tilde{w}, \tilde{v}) + R(\tilde{v}, \tilde{w}))| \leq |1 - 2\alpha||\tilde{w} - \tilde{v}|.$$

The following proposition gives sufficient conditions to preserve the bounds for the leaders' interactions (3.1).

Proposition 3.1. *Let r, d_+ and d_- be defined as follows:*

$$r = \min_{\tilde{v}, \tilde{w} \in I} [R(\tilde{v}, \tilde{w})] \quad \text{and} \quad d_{\pm} = \min_{\tilde{w} \in I} \left[\frac{1 \mp \tilde{w}}{\tilde{D}(\tilde{w})}, \tilde{D}(\tilde{w}) \neq 0 \right]. \quad (3.6)$$

If $\tilde{v}, \tilde{w} \in I$ then $\tilde{v}^*, \tilde{w}^* \in I$ if the following conditions hold:

$$\alpha r \geq \frac{\beta}{2} \quad \text{and} \quad d_- \left(1 - \frac{\beta}{2} \right) \leq \tilde{\theta}_i \leq d_+ \left(1 - \frac{\beta}{2} \right), \quad i = 1, 2. \quad (3.7)$$

The proof follows by the same arguments used in [10,28] and we omit the details. On the other hand, from the definition of binary interaction between followers (3.4), in the absence of diffusion, the boundaries are never violated. Indeed since $|w| \leq 1$ it follows that $|v - w| \leq 1$ and, as $0 \leq P(\cdot, \cdot) \leq 1$, it is easily seen that $w^*, v^* \in I$.

Finally, as shown in [10], the post-interaction opinion of followers w^{**} , in the leader–follower interaction (3.5), takes values in the reference interval I if the hypothesis of the following proposition is satisfied.

Proposition 3.2. *Let K_- and K_+ be defined as follows:*

$$K_{\pm} = \min_{w \in I} \left[\frac{1 \mp w}{\hat{D}(w)}, \hat{D}(w) \neq 0 \right]. \quad (3.8)$$

If $w \in I$ then $w^{**} \in I$ if the following conditions hold:

$$(1 - \alpha)K_- \leq \hat{\theta} \leq (1 - \alpha)K_+, \quad i = 1, 2. \quad (3.9)$$

(i) Main properties

Following the derivation in [34], for a suitable choice of test functions φ we can describe the evolution of $f_F(w, t)$ owing to the integro-differential equation of Boltzmann-type

$$\frac{d}{dt} \int_I \varphi(w) f_F(w, t) dw = (Q_F(f_F, f_F), \varphi) + (Q_{FL}(f_L, f_F), \varphi), \quad (3.10)$$

where

$$(Q_F(f_F, f_F), \varphi) = \left\langle \int_{I^2} B_{\text{int}}^F(\varphi(w^*) - \varphi(w)) f_F(w, t) f_F(v, t) dw dv \right\rangle \quad (3.11)$$

and

$$(Q_{FL}(f_F, f_L), \varphi) = \left\langle \int_{I^2} B_{\text{int}}^{FL}(\varphi(w^{**}) - \varphi(w)) f_F(w, t) f_L(\tilde{v}, t) dw d\tilde{v} \right\rangle. \quad (3.12)$$

In (3.11) and (3.12), we used the notation $\langle \cdot \rangle$ to indicate the expectations with respect to the random variables, respectively θ_i , $i = 1, 2$ and $\hat{\theta}$, and the non-negative interaction kernels B_{int}^F and B_{int}^{FL} are related to the probability of the microscopic interactions. The simplest choice of interaction kernels which guarantees that the post-interaction opinions never violate the bounds is given by

$$\begin{aligned} B_{\text{int}}^F &= B_{\text{int}}^F(w, v, \theta_1, \theta_2) = \eta_F \chi(|w^*| \leq 1) \chi(|v^*| \leq 1) \\ B_{\text{int}}^{FL} &= B_{\text{int}}^{FL}(w, \tilde{v}, \hat{\theta}) = \eta_{FL} \chi(|w^{**}| \leq 1) \chi(|\hat{v}| \leq 1), \end{aligned} \quad \left. \right\} \quad (3.13)$$

where $\eta_F, \eta_{FL} > 0$ are constant relaxation rates and $\chi(\cdot)$ is the indicator function. If we now assume that the interaction parameters are such that $|w^*|, |v^*| \leq 1$ the Boltzmann operators can be written as

$$(Q_F(f_F, f_F), \varphi) = \eta_F \left\langle (\varphi(w^*) - \varphi(w)) f_F(w, t) f_F(v, t) dw dv \right\rangle \quad (3.14)$$

and

$$(Q_{FL}(f_F, f_L), \varphi) = \eta_{FL} \left\langle (\varphi(w^{**}) - \varphi(w)) f_F(w, t) f_L(\tilde{v}, t) dw d\tilde{v} \right\rangle. \quad (3.15)$$

In order to study the evolution of the average opinion $m_F(t)$, we take $\varphi(w) = w$ in (3.10). We have that the evolution of the average opinion of followers is

$$\begin{aligned} \frac{d}{dt} m_F(t) &= \frac{\eta_F}{2} \left[\int_{I^2} (w^* + v^* - w - v) f_F(w, t) f_F(v, t) dw dv \right] \\ &\quad + \eta_{FL} \int_{I^2} (w^{**} - w) f_F(w, t) f_L(\tilde{v}, t) dw d\tilde{v}, \end{aligned} \quad (3.16)$$

since the noise term in (3.4) has zero mean. From the definition of binary interactions between followers (3.4) and the definition of leader–follower interactions (3.5), we have

$$\begin{aligned} \frac{d}{dt} m_F(t) &= \frac{\eta_F}{2} \alpha \int_{I^2} (v - w)(P(w, v) - P(v, w)) f_F(w, t) f_F(v, t) dw dv \\ &\quad + \eta_{FL} \alpha \int_{I^2} S(w, \tilde{v})(\tilde{v} - w) f_F(w, t) f_L(\tilde{v}, t) dw d\tilde{v}. \end{aligned} \quad (3.17)$$

Remark 3.3. If we suppose P symmetric, that is, $P(w, v) = P(v, w)$, and $S \equiv 1$ we obtain a simplified equation for the time evolution of m_F

$$\frac{d}{dt}m_F(t) = \tilde{\eta}_{FL}\alpha(m_L(t) - m_F(t)), \quad (3.18)$$

where we introduced the notations $\tilde{\eta}_{FL} = \rho \eta_{FL}$ and $m_L(t) = (1/\rho) \int_I \tilde{w} f_L(\tilde{w}, t) d\tilde{w}$.

The evolution equation for $m_L(t)$ can be found owing to similar arguments. We can describe the dynamics of $f_L(\tilde{w}, t)$ owing to the following integro-differential equation of Boltzmann-type in a weak form:

$$\frac{d}{dt} \int_I \varphi(\tilde{w}) f_L(\tilde{w}, t) d\tilde{w} = (Q_L(f_L, f_L), \varphi), \quad (3.19)$$

where

$$(Q_L(f_L, f_L), \varphi) = \left\langle \int_{I^2} B_{\text{int}}(\varphi(\tilde{w}^*) - \varphi(\tilde{w})) f_L(\tilde{w}, t) f_L(\tilde{v}, t) d\tilde{w} d\tilde{v} \right\rangle. \quad (3.20)$$

As before $\langle \cdot \rangle$ denotes the expectation taken with respect to the random variables $\tilde{\theta}_i$, $i = 1, 2$ and B_{int} is related to the probability of the microscopic interactions. A choice which preserves post-interaction opinion bounds is

$$B_{\text{int}} = B_{\text{int}}(\tilde{w}, \tilde{v}, \tilde{\theta}_1, \tilde{\theta}_2) = \eta_L \chi(|\tilde{w}^*| \leq 1) \chi(|\tilde{v}^*| \leq 1), \quad (3.21)$$

where $\eta_L > 0$ is a constant rate and $\chi(\cdot)$ is the indicator function. Let us consider as a test function $\varphi(\tilde{w}) = \tilde{w}$. Then equation (3.19) assumes the form

$$\frac{d}{dt} \int_I \tilde{w} f_L(\tilde{w}, t) d\tilde{w} = \eta_L \left\langle \int_{I^2} (\tilde{w}^* - \tilde{w}) f_L(\tilde{w}, t) f_L(\tilde{v}, t) d\tilde{w} d\tilde{v} \right\rangle, \quad (3.22)$$

which is equivalent to

$$\frac{d}{dt} \int_I \tilde{w} f_L(\tilde{w}, t) d\tilde{w} = \frac{\eta_L}{2} \left\langle \int_{I^2} (\tilde{v}^* + \tilde{w}^* - \tilde{v} - \tilde{w}) f_L(\tilde{w}, t) f_L(\tilde{v}, t) d\tilde{w} d\tilde{v} \right\rangle.$$

Then being the noise term in (3.1) with zero mean we have

$$\begin{aligned} \frac{d}{dt} m_L(t) &= \eta_L \alpha (1 - \beta) \frac{1}{\rho} \int_{I^2} (R(\tilde{w}, \tilde{v}) - R(\tilde{v}, \tilde{w})) \tilde{v} f_L(\tilde{w}, t) f_L(\tilde{v}, t) d\tilde{w} d\tilde{v} \\ &\quad + \tilde{\eta}_L \psi \beta (w_d - m_L(t)) + \tilde{\eta}_L \beta \mu (m_F(t) - m_L(t)), \end{aligned} \quad (3.23)$$

where $\tilde{\eta}_L = \rho \sim \eta_L$.

Remark 3.4. If $R(\tilde{w}, \tilde{v}) = R(\tilde{v}, \tilde{w})$ equation (3.23) becomes

$$\frac{d}{dt} m_L(t) = \tilde{\eta}_L \psi \beta (w_d - m_L(t)) + \tilde{\eta}_L \mu \beta (m_F(t) - m_L(t)). \quad (3.24)$$

Moreover, if the assumptions on P and S in remark 3.3 hold we obtain the following closed system of differential equations for the mean opinions m_L and m_F :

$$\left. \begin{aligned} \frac{d}{dt} m_L(t) &= \tilde{\eta}_L \psi \beta (w_d - m_L(t)) + \tilde{\eta}_L \mu \beta (m_F(t) - m_L(t)) \\ \frac{d}{dt} m_F(t) &= \tilde{\eta}_{FL} \alpha (m_L(t) - m_F(t)). \end{aligned} \right\} \quad (3.25)$$

Straightforward computations show that the exact solution of the above system has the following structure:

$$\left. \begin{aligned} m_L(t) &= C_1 \exp\{-|\lambda_1|t\} + C_2 \exp\{-|\lambda_2|t\} + w_d \\ m_F(t) &= C_1 \left(1 + \frac{\lambda_1}{\beta \mu \tilde{\eta}_L}\right) \exp\{-|\lambda_1|t\} + C_2 \left(1 + \frac{\lambda_2}{\beta \mu \tilde{\eta}_L}\right) \exp\{-|\lambda_2|t\} + w_d, \end{aligned} \right\} \quad (3.26)$$

where C_1 and C_2 depend on the initial data $m_F(0)$ and $m_L(0)$ in the following way:

$$C_1 = -\frac{1}{\lambda_1 - \lambda_2} ((\beta \tilde{\eta}_L m_L(0) + \lambda_2)m_L(0) - \mu \beta \tilde{\eta}_L m_F(0) - (\lambda_2 + \beta \tilde{\eta}_L \psi) w_d) \quad (3.27)$$

and

$$C_2 = \frac{1}{\lambda_1 - \lambda_2} ((\beta \tilde{\eta}_L m_L(0) + \lambda_1)m_L(0) - \mu \beta \tilde{\eta}_L m_F(0) - (\lambda_1 + \beta \tilde{\eta}_L \psi) w_d) \quad (3.28)$$

with

$$\lambda_{1,2} = -\frac{1}{2}(\alpha \tilde{\eta}_{FL} + \beta \tilde{\eta}_L) \pm \frac{1}{2}\sqrt{(\alpha \tilde{\eta}_{FL} + \beta \tilde{\eta}_L)^2 - 4\psi\alpha\beta\tilde{\eta}_L\tilde{\eta}_{FL}}. \quad (3.29)$$

Note that $\lambda_{1,2}$ are always negative; this ensures that the contribution of the initial averages, $m_L(0)$ and $m_F(0)$, vanishes as soon as time increases and the mean opinions of leaders and followers converge towards the desired state w_d .

We now take into account the evolution of the second-order moments

$$E_F(t) = \int_I w^2 f_F(w, t) dw, \quad E_L(t) = \frac{1}{\rho} \int_I \tilde{w}^2 f_L(\tilde{w}, t) d\tilde{w}.$$

First we analyse the followers' group from equation (3.10) with test functions $\varphi(w) = w^2$; we have

$$\begin{aligned} \frac{d}{dt} E_F(t) &= \frac{\eta_F}{2} \left\langle \int_{I^2} ((w^*)^2 + (v^*)^2 - w^2 - v^2) f_F(w, t) f_F(v, t) dw dv \right\rangle \\ &\quad + \eta_{FL} \left\langle \int_{I^2} ((w^{**})^2 - w^2) f_F(w, t) f_L(\tilde{v}, t) dw d\tilde{v} \right\rangle. \end{aligned} \quad (3.30)$$

Owing to (3.4) and (3.5), in the simplified case $P \equiv S \equiv 1$, we obtain

$$\begin{aligned} \frac{d}{dt} E_F(t) &= 2\eta_F \alpha(\alpha - 1)(E_F(t) - m_F^2(t)) + \tilde{\eta}_{FL} \alpha^2(E_L + E_F - 2m_L(t)m_F(t)) \\ &\quad + 2\alpha \tilde{\eta}_{FL}(m_F(t)m_L(t) - E_F(t)) + \eta_F \sigma^2 \int_I D^2(w) f_F(w, t) dw \\ &\quad + \tilde{\eta}_{FL} \hat{\sigma}^2 \int_I \hat{D}^2(w) f_F(w, t) dw. \end{aligned} \quad (3.31)$$

Finally, for the leaders' group let us consider the function $\varphi(\tilde{w}) = \tilde{w}^2$ in (3.19) and the case $R \equiv 1$. Then owing to equation (3.1), we obtain

$$\begin{aligned} \frac{d}{dt} E_L(t) &= \frac{\eta_L}{2} \frac{1}{\rho} \left\langle \int_{I^2} ((\tilde{w}^*)^2 + (\tilde{v}^*)^2 - \tilde{w}^2 - \tilde{v}^2) f_L(\tilde{w}, t) f_L(\tilde{v}, t) d\tilde{w} d\tilde{v} \right\rangle \\ &= \tilde{\eta}_L \left[2\alpha(\alpha - 1)(E_L(t) - m_L^2(t)) - \frac{\beta}{2}(2 - \beta)(E_L(t) + m_L^2(t)) \right. \\ &\quad \left. + 2\beta(1 - \beta)(\psi w_d + \mu m_F(t))m_L(t) + \beta^2(\psi w_d + \mu m_F(t))^2 \right. \\ &\quad \left. + \tilde{\sigma}^2 \int_I \tilde{D}^2(\tilde{w}) f_L(\tilde{w}, t) d\tilde{w} \right]. \end{aligned} \quad (3.32)$$

In the absence of diffusion, since $m_F(t), m_L(t) \rightarrow w_d$ as $t \rightarrow \infty$, it follows that $E_F(t), E_L(t)$ converge towards w_d^2 . Then the quantities

$$\left. \begin{aligned} \int_I f_F(w, t)(w - w_d)^2 dw &= E_F(t) + w_d^2 - 2m_F(t)w_d \\ \frac{1}{\rho} \int_I f_L(\tilde{w}, t)(\tilde{w} - w_d)^2 d\tilde{w} &= E_L(t) + w_d^2 - 2m_L(t)w_d \end{aligned} \right\} \quad (3.33)$$

go to zero as $t \rightarrow \infty$, i.e. under the above assumptions the steady-state solutions have the form of a Dirac delta centred in the target opinion w_d .

4. Fokker–Planck modelling

In the general case, it is quite difficult to obtain analytic results on the large time behaviour of the kinetic equation (3.10). A step towards the simplification of the analysis is the derivation of asymptotic states of the Boltzmann model resulting in simplified Fokker–Planck-type models, for which the study of the asymptotic properties is easier [34]. In order to obtain such simplification we will follow the approach usually referred to as the quasi-invariant opinion limit [20,34], which is closely related to the so-called grazing collision limit of the Boltzmann equation [36,37].

(a) Quasi-invariant opinion limit

The main idea is to rescale the interaction frequencies η_L, η_F and η_{FL} , the propensity strength α , the diffusion variances $\tilde{\sigma}^2, \sigma^2$ and $\hat{\sigma}^2$ and the action of the control v at the same time, in order to maintain, at the asymptotic level, memory of the microscopic interactions.

Let us introduce the parameter $\varepsilon > 0$, and consider the rescaling

$$\left. \begin{aligned} \alpha = \varepsilon, \quad v = \varepsilon\kappa, \quad \sigma^2 = \varepsilon\zeta^2, \quad \hat{\sigma}^2 = \varepsilon\tilde{\zeta}^2, \quad \tilde{\sigma}^2 = \varepsilon\tilde{\zeta}^2 \\ \text{and} \quad \eta_F = \frac{1}{c_F\varepsilon}, \quad \eta_{FL} = \frac{1}{c_{FL}\varepsilon}, \quad \eta_L = \frac{1}{c_L\varepsilon}, \quad \beta = \frac{4\varepsilon}{\kappa + 4\varepsilon}. \end{aligned} \right\} \quad (4.1)$$

This corresponds to the situation where the interaction operator concentrates on binary interactions which produce a very small change in the opinion of the agents. From a modelling viewpoint, we require that the scaling (4.1), in the limit $\varepsilon \rightarrow 0$, preserves the main macroscopic properties of the kinetic system. To this extent, let us consider the evolution of the scaled first two moments under the simplifying hypothesis P, R symmetric and $S \equiv 1$.

The evolution of the mean opinions described in system (3.25) rescales as

$$\left. \begin{aligned} \frac{d}{dt}m_F(t) &= \varepsilon \frac{1}{c_{FL}\varepsilon} (m_L(t) - m_F(t)) \\ \frac{d}{dt}m_L(t) &= \frac{\psi}{c_L\varepsilon} \frac{4\varepsilon}{\kappa + 4\varepsilon} (w_d - m_L(t)) + \frac{\mu}{c_L\varepsilon} \frac{4\varepsilon}{\kappa + 4\varepsilon} (m_F(t) - m_L(t)), \end{aligned} \right\} \quad (4.2)$$

which as $\varepsilon \rightarrow 0$ yields

$$\left. \begin{aligned} \frac{d}{dt}m_F(t) &= \frac{\rho}{c_{FL}} (m_L(t) - m_F(t)) \\ \frac{d}{dt}m_L(t) &= \frac{4\rho}{c_L\kappa} [\psi(w_d - m_L(t)) + \mu(m_F(t) - m_L(t))]. \end{aligned} \right\} \quad (4.3)$$

The second moment equations (3.31) and (3.32) are then scaled as follows:

$$\begin{aligned} \frac{d}{dt}E_F(t) &= (\varepsilon - 1) \frac{2}{c_F} (E_F(t) - m_F^2(t)) + \frac{\varepsilon\rho}{c_{FL}} (E_L(t) + E_F(t) - 2m_L(t)m_F(t)) \\ &\quad + \frac{2\rho}{c_{FL}} (m_F(t)m_L(t) - E_F(t)) + \frac{\zeta^2}{c_F} \int_I D^2(w)f_F(w, t) dw \\ &\quad + \frac{\tilde{\zeta}^2\rho}{c_{FL}} \int_I \hat{D}^2(w)f_F(w, t) dw \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \frac{d}{dt}E_L(t) &= \frac{\rho}{c_L\varepsilon} \left[2\varepsilon(\varepsilon - 1)(E_L(t) - m_L^2(t)) - \frac{2\varepsilon}{\kappa + 4\varepsilon} \left(2 - \frac{4\varepsilon}{\kappa + 4\varepsilon} \right) (E_L(t) + m_L^2(t)) \right. \\ &\quad + \frac{8\varepsilon}{\kappa + 4\varepsilon} \left(1 - \frac{4\varepsilon}{\kappa + 4\varepsilon} \right) (\psi w_d + \mu m_F(t))m_L(t) \\ &\quad \left. + \left(\frac{4\varepsilon}{\kappa + 4\varepsilon} \right)^2 (\psi w_d + \mu m_F(t))^2 + \tilde{\sigma}^2 \int_I \tilde{D}^2(\tilde{w})f_L(\tilde{w}, t) d\tilde{w} \right] \end{aligned} \quad (4.5)$$

and as $\varepsilon \rightarrow 0$ we obtain

$$\left. \begin{aligned} \frac{d}{dt} E_F(t) &= -\frac{2}{c_F}(E_F(t) - m_F^2(t)) + \frac{2\rho}{c_{FL}}(m_F(t)m_L(t) - E_F(t)) \\ &\quad + \frac{\xi^2}{c_F} \int_I D^2(w)f_F(w,t)dw + \frac{\hat{\xi}^2\rho}{c_{FL}} \int_I \hat{D}^2(w)f_F(w,t)dw \\ \frac{d}{dt} E_L(t) &= -\frac{2\rho}{c_L}(E_L(t) - m_L^2(t)) - \frac{4\rho}{c_{LK}}(E_L(t) + m_L^2(t)) \\ &\quad + \frac{8\rho}{c_{LK}}(\psi w_d + \mu m_F(t))m_L(t) + \frac{\xi^2\rho}{c_L} \int_I \tilde{D}^2(\tilde{w})f_L(\tilde{w},t)d\tilde{w}. \end{aligned} \right\} \quad (4.6)$$

Therefore, the asymptotic scaling preserves the behaviour of the first two moments of the solution. We show how this approach leads to a constrained Fokker–Planck system for the description of the opinion distribution of leaders and followers. Even if the computations presented here are formal following the same arguments as in [20,34] it is possible to give a rigorous mathematical basis of our derivation. Here we omit the details for the sake of brevity.

(b) Fokker–Plank equations

The scaled equation (3.10) reads

$$\begin{aligned} \frac{d}{dt} \int_I \varphi(w)f_F(w,t)dw &= \frac{1}{c_F\varepsilon} \left\langle \int_{I^2} (\varphi(w^*) - \varphi(w))f_F(w,t)f_F(v,t)dw dv \right\rangle \\ &\quad + \frac{1}{c_{FL}\varepsilon} \left\langle \int_{I^2} (\varphi(w^{**}) - \varphi(w))f_F(w,t)f_L(\tilde{w},t)dw d\tilde{w} \right\rangle. \end{aligned} \quad (4.7)$$

Considering the second-order Taylor expansion of φ around w , we obtain

$$\left. \begin{aligned} \varphi(w^*) - \varphi(w) &= (w^* - w)\varphi'(w) + \frac{1}{2}(w^* - w)^2\varphi''(\bar{w}) \\ \varphi(w^{**}) - \varphi(w) &= (w^{**} - w)\varphi'(w) + \frac{1}{2}(w^{**} - w)^2\varphi''(\hat{w}), \end{aligned} \right\} \quad (4.8)$$

where for some $0 \leq \vartheta_1, \vartheta_2 \leq 1$

$$\bar{w} = \vartheta_1 w^* + (1 - \vartheta_1)w \quad \text{and} \quad \hat{w} = \vartheta_2 w^{**} + (1 - \vartheta_2)w.$$

Taking into account the binary interactions (3.4) and (3.5) in (4.8), and substituting in (4.7), we obtain a second-order approximation of the dynamics. In the limit $\varepsilon \rightarrow 0$, the leading order is given by

$$\begin{aligned} \frac{d}{dt} \int_I \varphi(w)f_F(w)dw &= \frac{1}{c_F} \left[\int_{I^2} P(w,v)(v-w)\varphi'(w)f_F(w,t)f_F(v,t)dw dv \right] \\ &\quad + \frac{1}{c_{FL}} \left[\int_{I^2} S(w,\tilde{w})(\tilde{w}-w)\varphi'(w)f_F(w,t)f_L(\tilde{w})dw d\tilde{w} \right] \\ &\quad + \frac{1}{2} \frac{\xi^2}{c_F} \int_I \varphi''(w)D^2(w)f_F(w,t)dw \\ &\quad + \frac{1}{2} \frac{\hat{\xi}^2\rho}{c_{FL}} \int_I \varphi''(w)\hat{D}^2(w)f_F(w,t)dw. \end{aligned} \quad (4.9)$$

Integrating backwards by parts the last expression, we obtain the Fokker–Planck equation for the followers' opinion distribution

$$\begin{aligned} \frac{\partial f_F}{\partial t} + \frac{\partial}{\partial w} \left(\frac{1}{c_F} K_F[f_F](w) + \frac{1}{c_{FL}} K_{FL}[f_L](w) \right) f_F(w) \\ = \frac{1}{2} \frac{\partial^2}{\partial \tilde{w}^2} \left(\frac{\xi^2}{c_F} \tilde{D}^2(\tilde{w}) + \frac{\hat{\xi}^2\rho}{c_{FL}} \right) f_F(w), \end{aligned} \quad (4.10)$$

where

$$\left. \begin{aligned} K_F[f_F](w) &= \int_I P(w, v)(v - w)f_F(v, t) dv \\ K_{FL}[f_L](w) &= \int_I S(w, \tilde{w})(\tilde{w} - w)f_L(\tilde{w}) d\tilde{w}. \end{aligned} \right\} \quad (4.11)$$

Following the same strategy, we obtain the analogous result for the leaders' opinion distribution

$$\frac{\partial f_L}{\partial t} + \frac{\partial}{\partial \tilde{w}} \left(\frac{\rho}{c_L} H[f_L](\tilde{w}) + \frac{1}{c_L} K_L[f_L](\tilde{w}) \right) f_L(\tilde{w}) = \frac{1}{2} \frac{\tilde{\zeta}^2 \rho}{c_L} \frac{\partial^2}{\partial \tilde{w}^2} \tilde{D}^2(\tilde{w}) f_L(\tilde{w}), \quad (4.12)$$

where

$$K[f_L](\tilde{w}) = \int_I R(\tilde{w}, \tilde{v})(\tilde{v} - \tilde{w})f_L(\tilde{v}, t) d\tilde{v} \quad (4.13)$$

and

$$H[f_L](\tilde{w}) = \frac{2\psi}{\kappa} (\tilde{w} + m_L(t) - 2w_d) + \frac{2\mu}{\kappa} (\tilde{w} + m_L(t) - 2m_F(t)). \quad (4.14)$$

(c) Steady-state solutions

In this section, we show that in some cases it is possible to find explicit stationary states of the Fokker–Planck system, described in (4.10) and (4.12). Here, we restrict ourselves to the simplified situation where every interaction function is constant and unitary, i.e. $P \equiv S \equiv R \equiv 1$, and

$$D(w) = \tilde{D}(w) = \hat{D}(w) = 1 - w^2. \quad (4.15)$$

The steady state of equations (4.10) and (4.12) is a solution of the following equations:

$$\left. \begin{aligned} \left(\frac{1}{c_F} (m_F - w) + \frac{\rho}{c_{FL}} (m_L - w) \right) f_{F,\infty} &= \frac{1}{2} \left(\frac{\zeta^2}{c_F} + \frac{\hat{\zeta}^2 \rho}{c_{FL}} \right) \frac{\partial}{\partial w} D^2(w) f_{F,\infty} \\ \left(\frac{2\psi}{\kappa} [\tilde{w} - 2w_d - m_L] + \frac{2\mu}{\kappa} [\tilde{w} - 2m_F + m_L] \right) f_{L,\infty} &= \frac{1}{2} \frac{\tilde{\zeta}^2 \rho}{c_L} \frac{\partial}{\partial \tilde{w}} \tilde{D}^2(\tilde{w}) f_{L,\infty}. \end{aligned} \right\} \quad (4.16)$$

As soon as $t \rightarrow \infty$, owing to equation (4.3), the followers' and leaders' mean opinions m_F and m_L relax to the desired opinion w_d . Then

$$\left(\frac{1}{c_F} + \frac{\rho}{c_{FL}} \right) (w_d - w) f_{F,\infty} = \frac{1}{2} \left(\frac{\zeta^2}{c_F} + \frac{\hat{\zeta}^2 \rho}{c_{FL}} \right) \frac{\partial}{\partial w} D^2(w) f_{F,\infty}, \quad (4.17)$$

that is,

$$\left(\frac{1}{c_F} + \frac{\rho}{c_{FL}} \right) (w_d - w) \frac{g_F}{D^2(w)} = \frac{1}{2} \left(\frac{\zeta^2}{c_F} + \frac{\hat{\zeta}^2 \rho}{c_{FL}} \right) \frac{\partial}{\partial w} g_F, \quad (4.18)$$

where $g_F = D^2(w) f_{F,\infty}$. This implies

$$g_{F,\infty} = a_F \exp \left\{ -\frac{2}{b_F} \int_0^w \frac{z - w_d}{(1 - z^2)^2} dz \right\} \quad \text{and} \quad b_F = \frac{\zeta^2 c_{FL} + \hat{\zeta}^2 c_F \rho}{c_{FL} + c_F \rho}, \quad (4.19)$$

and a_F is a normalization constant such that $\int_I g_{F,\infty} dw = 1$. Finally, we have

$$f_{F,\infty} = \frac{a_F}{(1 - w^2)^2} \exp \left\{ -\frac{2}{b_F} \int_0^w \frac{z - w_d}{(1 - z^2)^2} dz \right\}. \quad (4.20)$$

Similarly, we can find the steady-state $f_{L,\infty}$ as a solution of the equation

$$-\left(\frac{2\psi}{\kappa} + \frac{2\mu}{\kappa} \right) (w_d - \tilde{w}) \frac{g_{L,\infty}}{D^2(\tilde{w})} = \frac{1}{2} \frac{\tilde{\zeta}^2 \rho}{c_L} \frac{\partial}{\partial \tilde{w}} g_{L,\infty}, \quad (4.21)$$

where $g_{L,\infty} = f_{L,\infty} D^2(w)$. The solution of the differential equation (4.21) is given by

$$g_{L,\infty} = a_L \exp \left\{ -\frac{2}{b_L} \int_0^{\tilde{w}} \left(\frac{z - w_d}{(1 - z^2)^2} \right) dz \right\}, \quad b_L = \frac{\tilde{\zeta} \rho \kappa}{2c_L(\psi + \mu)}, \quad (4.22)$$

Table 1. Computational parameters for the different test cases.

test	$S(\cdot, \cdot)$	c_F	\hat{c}_{FL}	\hat{c}_L	ρ	ψ	w_d
1a	1	1	0.1	0.1	0.05	0.5	0.5
1b	equation (5.3)	1	0.1	0.1	0.05	0.5	0.5
	$S(\cdot, \cdot)$	c_F	\hat{c}_{FL_1}	\hat{c}_{L_1}	ρ_1	ψ_1	w_{d_1}
2	1	1	0.1	0.1	0.05	0.5	0.5
3	1	1	0.1	0.1	0.05	equation (5.7)	0.5
	$S(\cdot, \cdot)$	c_F	\hat{c}_{FL_2}	\hat{c}_{L_2}	ρ_2	ψ_2	w_{d_2}
							-0.5
							-0.5

and a_L is chosen such that the mass of $g_{L,\infty}$ is equal to ρ . Then the steady state is

$$f_{L,\infty} = \frac{a_L}{(1 - \tilde{w}^2)^2} \exp \left\{ -\frac{2}{b_L} \int_0^{\tilde{w}} \left(\frac{z - w_d}{(1 - z^2)^2} \right) dz \right\}. \quad (4.23)$$

5. Numerical simulation

In this section, we present several numerical results concerning the numerical simulation of the Boltzmann-type control model introduced in the previous paragraphs. All the results have been computed by a Monte Carlo method for the Boltzmann model (see [34,38] for more details) in the Fokker–Planck regime $\varepsilon = 0.01$ under the scaling (4.1). In the numerical tests, we assume that 5% of the population is composed of opinion leaders (e.g. [10]). Note that, for clarity, in all figures the leaders’ profiles have been magnified by a factor 10. The regularization term in the control is fixed to $v = 1$. The random diffusion effects have been computed in the case (4.15) for a uniform random variable with scaled variance $\zeta^2 = \tilde{\zeta}^2 = \hat{\zeta}^2 = 0.01$. It is easy to check that the above choices preserve the bounds in the numerical simulations. First, we present some test cases with a single population of leaders as discussed in our theoretical analysis. Then we consider the case of multiple leaders’ populations with different time-dependent strategies. This leads to more realistic applications of our arguments, introducing the concept of competition between leaders’ populations. For the sake of simplicity, we fix constant interaction functions $P(\cdot, \cdot) \equiv 1$ and $R(\cdot, \cdot) \equiv 1$ and the remaining scaled computational parameters have been summarized in table 1.

(a) Test 1. Leaders driving followers

In the first test case, we consider a single population of leaders driving followers, described by the system of Boltzmann equations:

$$\left. \begin{aligned} \frac{d}{dt} \int_I \varphi(w) f_F(w, t) dw &= (Q_F(f_F, f_F), \varphi) + (Q_{FL}(f_F, f_L), \varphi) \\ \frac{d}{dt} \int_I \varphi(\tilde{w}) f_L(\tilde{w}, t) d\tilde{w} &= (Q_L(f_L, f_L), \varphi). \end{aligned} \right\} \quad (5.1)$$

Numerical experiments show that the instantaneous control problem is capable of introducing a non-monotone behaviour of $m_L(t)$. We report the evolution, over the time interval $[0, 1]$, of the kinetic densities $f_F(w, t)$ and $f_L(\tilde{w}, t)$ in figure 1 for constant interaction functions P, R and S . We take the initial distributions $f_F \sim U([-1, -0.5])$ and $f_L \sim N(w_d, 0.05)$ where $U(\cdot)$ and $N(\cdot, \cdot)$ denote, as usual, the uniform and the normal distributions. We use the compact notations

$$\hat{c}_{FL} = \frac{c_{FL}}{\rho} \quad \text{and} \quad \hat{c}_L = \frac{c_L}{\rho}. \quad (5.2)$$

This non-monotone behaviour shows that the leaders use a combination of populistic and radical strategy to drive the followers towards their desired state. In an electoral context, this is a characteristic which can be found in populistic radical parties, which typically

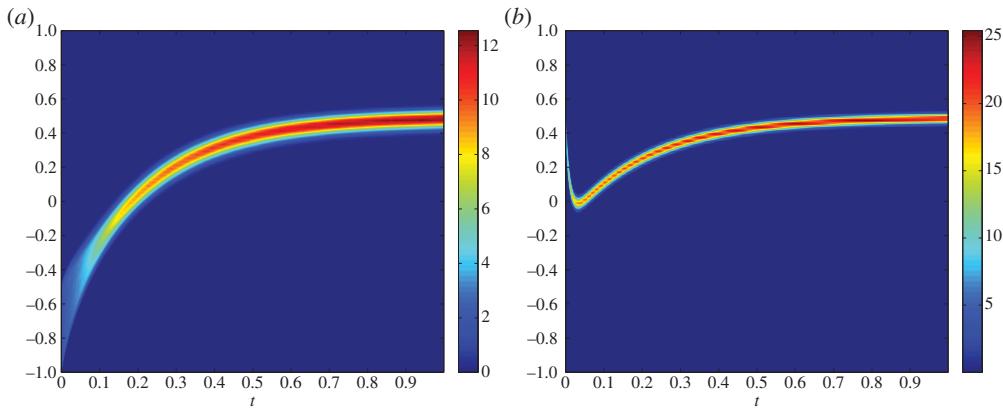


Figure 1. Test 1a: kinetic density evolution over the time interval $[0, 1]$ for a single population of leaders. (a) Kinetic density of followers and (b) kinetic density of leaders. (Online version in colour.)

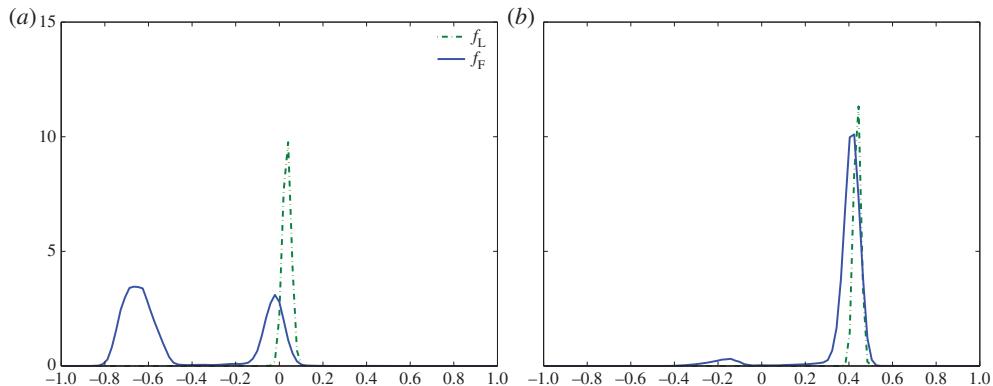


Figure 2. Test 1b: kinetic densities at different times for a single population of leaders with bounded confidence interaction. (a) $t = 1$ and (b) $t = 3$. (Online version in colour.)

include non-populist ideas, and their leadership generates through a dense network of radical movements [39].

Next we consider a bounded confidence model for the leader–follower interaction with

$$S(w, \tilde{w}) = \chi(|w - \tilde{w}| \leq \Delta), \quad (5.3)$$

where $0 \leq \Delta \leq 2$. In the simulation, we assume $\Delta = 0.5$ and use the same initial data as the previous case. It is interesting to observe how the model is capable of reproducing a realistic behaviour where the leaders first are able to attract a small group of followers which subsequently are capable of driving the whole majority towards the desired state (figures 2 and 3).

(b) Test 2. The case of multiple leaders' populations

Similarly, if more than one population of leaders occurs, each one with a different strategy, we can describe the evolution of the kinetic density of the system through a Boltzmann approach. Let $M > 0$ be the number of families of leaders, each of them described by the density f_{L_p} , $p = 1, \dots, M$ such that

$$\int_I f_{L_p}(\tilde{w}) d\tilde{w} = \rho_p. \quad (5.4)$$

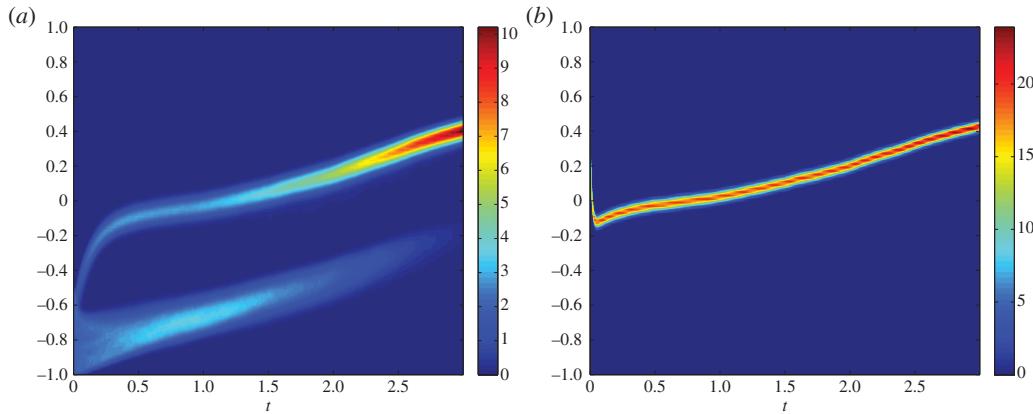


Figure 3. Test 1b: kinetic density evolution over the time interval $[0, 3]$ for a single population of leaders with bounded confidence interaction. (a) Kinetic density of followers and (b) kinetic density of leaders. (Online version in colour.)

If we suppose that a unique population of followers does exist, with density f_F , and that every follower interacts both with the other agents from the same population and with every leader of each p th family, for a suitable test function φ we obtain the following system of Boltzmann equations:

$$\left. \begin{aligned} \frac{d}{dt} \int_I \varphi(w) f_F(w, t) dw &= (Q_F(f_F, f_F), \varphi) + \sum_{p=1}^M (Q_{FL}(f_{L_p}, f_F), \varphi) \\ \frac{d}{dt} \int_I \varphi(\tilde{w}) f_{L_p}(\tilde{w}, t) d\tilde{w} &= (Q_L(f_{L_p}, f_{L_p}), \varphi), \quad p = 1, \dots, M. \end{aligned} \right\} \quad (5.5)$$

We assume that the leaders aim at minimizing cost functionals of the type (2.8) and therefore the differences consist in two factors: the target opinions w_{d_p} and the leaders' attitude towards a radical ($\psi_p \approx 1$) or populistic strategy ($\mu_p \approx 1$). We therefore introduce the analogous rescaling (4.1) and we define

$$\hat{c}_{FL_p} = \frac{c_{FL_p}}{\rho_p} \quad \text{and} \quad \hat{c}_{L_p} = \frac{c_{L_p}}{\rho_p}, \quad p = 1, \dots, M. \quad (5.6)$$

In the numerical test, we establish a link between our arguments and a Hotelling-type model [40]. The model describes how two shop owners, who sell the same product at the same price in the same street, must locate their shops in order to reach the maximum number of customers, uniformly distributed along the street (in other words, in order to maximize their profits). Paradoxically, the model yields that the equilibrium, without changing prices, is reached if they get closer. In the cited original paper, electoral dynamics are placed in this context and it can be regarded as the reason why political parties' programmes are often perceived as similar. We consider the case of two populations of leaders, described by the densities f_{L_1} and f_{L_2} , exercising different controls over a population of followers uniformly distributed within the interval $I = [-1, 1]$. Initially, the leaders are distributed as $f_{L_p} \sim N(w_{d_p}, 0.05)$, $p = 1, 2$. We can observe that the model leads to a centrist population of followers, whose opinion spreads in a range between leaders' mean opinions (figures 4 and 5).

(c) Test 3. Two leaders' populations with time-dependent strategies

Finally, we introduce a multi-population model for opinion formation with time-dependent coefficients. This approach leads to the concept of adaptive strategy for every family of leaders

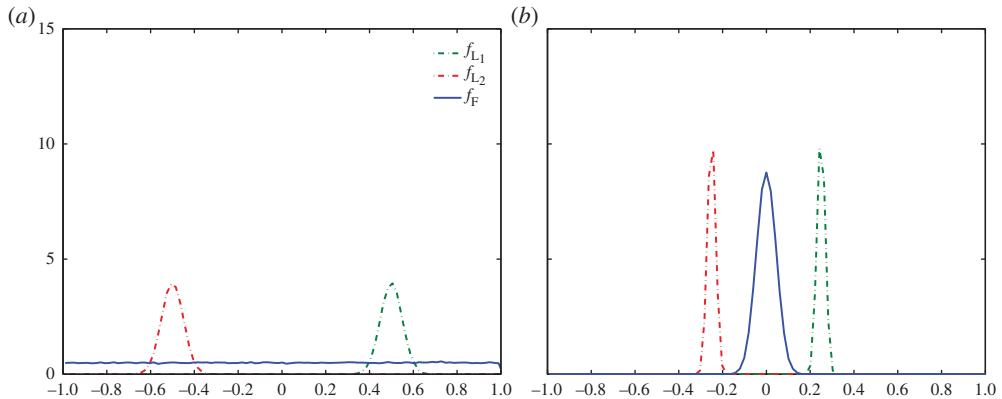


Figure 4. Test 2: kinetic densities at different times reproducing a Hotelling-like model behaviour for two populations of leaders. (a) $t = 0$ and (b) $t = 0.25$. (Online version in colour.)

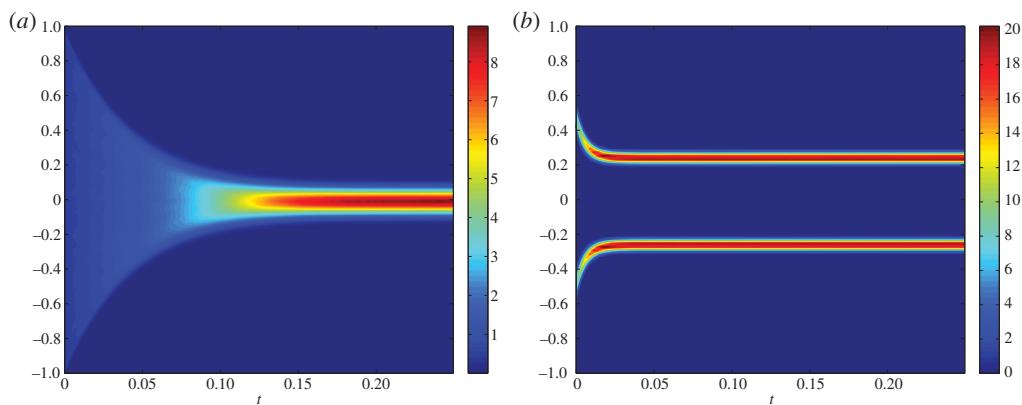


Figure 5. Test 2: kinetic density evolution over the time interval $[0, 0.25]$ reproducing a Hotelling-like model behaviour for two populations of leaders. (a) Kinetic density of followers and (b) kinetic density of leaders. (Online version in colour.)

$p = 1, \dots, M$. The coefficients ψ and μ which appear in the functional now evolve in time and are defined for every $t \in [0, T]$ as

$$\left. \begin{aligned} \psi_p(t) &= \frac{1}{2} \int_{w_{dp}-\delta}^{w_{dp}+\delta} f_F(w) dw + \frac{1}{2} \int_{m_{L_p}-\bar{\delta}}^{m_{L_p}+\bar{\delta}} f_F(w) dw \\ \mu_p(t) &= 1 - \psi_p(t), \end{aligned} \right\} \quad (5.7)$$

and

where both $\delta, \bar{\delta} \in [0, 1]$ are fixed and m_{L_p} is the average opinion of the p th leader. This choice of coefficients is equivalent to introducing a competition between the populations of leaders, where each leader tries to adapt its populistic or radical attitude according to the success of the strategy. Note also that the success of the strategy is based on the local perception of the followers.

In the numerical experiments reported in figures 6 and 7, we take into account two populations of leaders, initially normally distributed with mean values w_{d_1} and w_{d_2} and parameters $\delta = \bar{\delta} = 0.5$, respectively, and a single population of followers, represented by a skewed distribution $f_F \sim \Gamma(2, \frac{1}{4})$ over the interval $[-1, 1]$, where $\Gamma(\cdot, \cdot)$ is the Gamma distribution. Here, the frequencies of interactions are assumed to be unbalanced since $\hat{c}_{FL_1} = 0.1$ and $\hat{c}_{FL_2} = 1$. In the test case, we assume that the followers' group has an initial natural inclination for a position represented by one leader but, owing to communication strategies pursued by the minority's

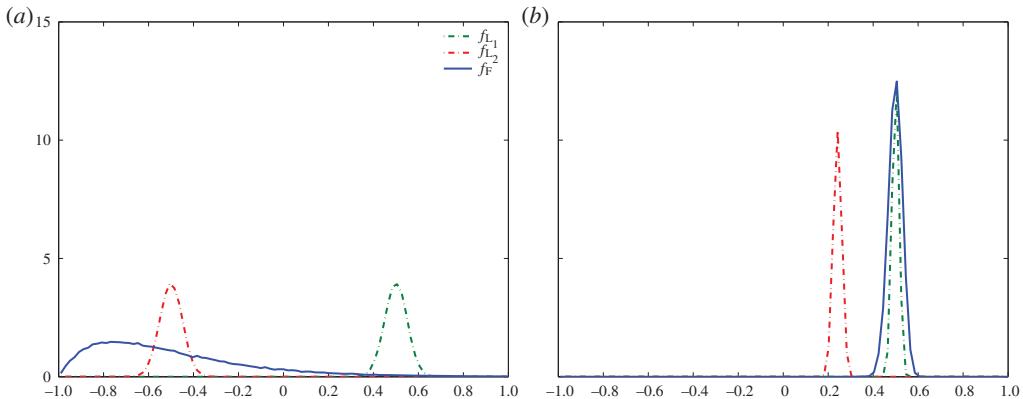


Figure 6. Test 3: kinetic densities at different times for a two populations of leaders model with time-dependent strategies. (a) $t = 0$ and (b) $t = 1$. (Online version in colour.)

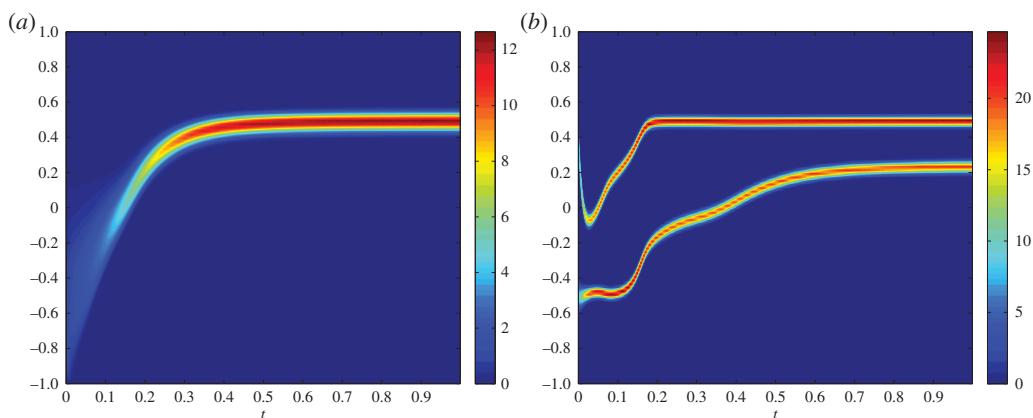


Figure 7. Test 3: kinetic density evolution over the time interval $[0, 1]$ for a two populations of leaders model with time-dependent strategies. (a) Kinetic density of followers and (b) kinetic density of leaders. (Online version in colour.)

leader, it is driven to different positions (figures 6 and 7). In a bipolar electoral context, an example of the described behaviour would be a better use of the media in a coalition with respect to the opponents.

6. Conclusion

We introduced a Boltzmann-type control for a hierarchical model of opinion formation where the leaders' behaviour is influenced both by the desire to achieve a prescribed opinion consensus and by the mean opinion of the followers. The main novelty of the method is that, owing to an instantaneous binary control approximation, the control is explicitly incorporated in the resulting leaders' dynamic. The use of instantaneous control and the kinetic description permit us to pass from an $O(N^2)$ dynamic, which must be solved forward-backward in time, to a much simpler forward $O(N)$ stochastic simulation. This is of paramount importance in view of possible applications of this kind of constrained opinion modelling. In the so-called quasi-invariant opinion limit, the corresponding Fokker–Planck descriptions have been derived and

explicit expressions of their steady states computed. Several numerical examples illustrate the robustness of the controlled dynamics using various leaders' strategies even in the presence of different groups of competing leaders.

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