SHOELACE ALGORITHM

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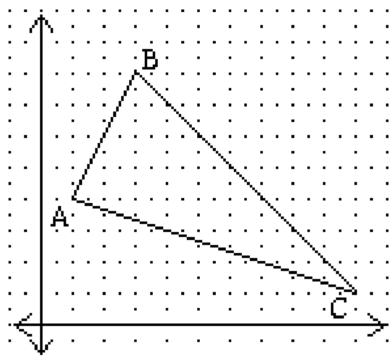
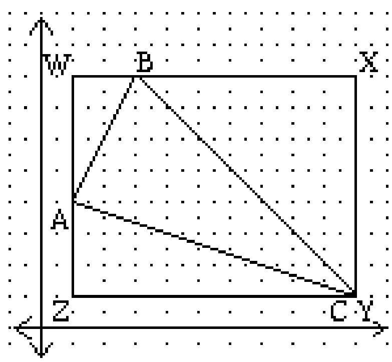
A favorite mathematical contest problem involves the determining of the area of a polygon whose vertices are described by ordered pairs in the plane. For example, ΔABC has vertices A(1,4), B(3,8) and C(9,1).



The area of ΔABC can be determined by encasing ΔABC within rectangle WXYZ.



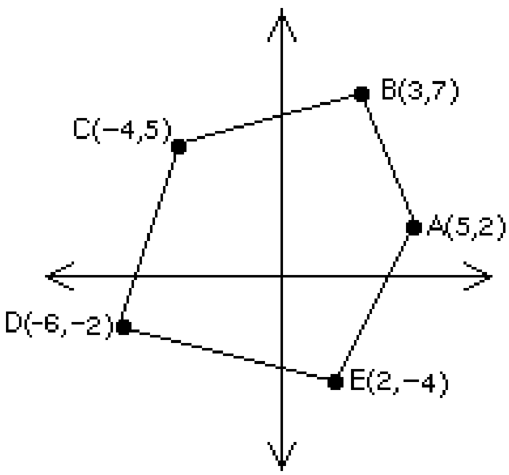
We then compute the areas of the following figures:

Figure	Area
Rectangle WXYZ	56
area ΔACZ	12
area ΔABW	4
area ΔBXY	21

The area of ΔABC is: 56 – (12 + 4 + 21) = 19.

It is possible to describe an algorithmic process to determine the enclosed area of any polygonal region. The process we will use is called the SHOELACE METHOD .
The process works for any polygonal region regardless of the number of sides.

Our example will be for a pentagon:



Find the area of pentagon ABCDE.

The area, K, can be found by following the procedure:

1) $$\begin{bmatrix} 2 & -4 \\ 5 & 2 \\ 3 & 7 \\ -4 & 5 \\ -6 & -2 \\ 2 & -4 \end{bmatrix}$$

Select a vertex and travel around the pentagon ending with the starting point. Write a matrix of the coordinates of the path including both the starting and ending coordinates.

2) $$\begin{bmatrix} 2 & -4 \\ 5 & 2 \\ 3 & 7 \\ -4 & 5 \\ -6 & -2 \\ 2 & -4 \end{bmatrix}$$

Determine the sum of all "\" products:
(2)(2) + (5)(7) + (3)(5) + (–4)(–2) + (–6)(–4)
= 4 + 35 + 15 + 8 + 24
= 86

3) $$\begin{bmatrix} 2 & -4 \\ 5 & 2 \\ 3 & 7 \\ -4 & 5 \\ -6 & -2 \\ 2 & -4 \end{bmatrix}$$

Determine the sum of all "/" products:
(5)(–4) + (3)(2) + (–4)(7) + (–6)(5) + (2)(–2)
= –20 + 6 – 28 – 30 – 4
= –76

4) Determine the absolute value of the difference of the "\" products and "/" products:

abs ((86) – (–76)) = abs (162)
= 162

5) Take $\frac{1}{2}$ of (4).

$$\frac{1}{2}(162) = 81 \;\boxed{\text{area of pentagon}}$$

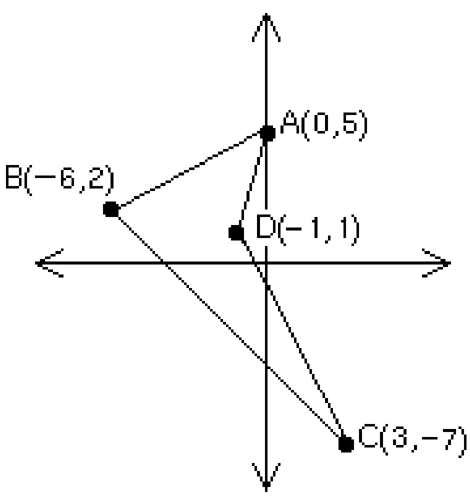
The formula for the area is displayed:

$$K = \frac{1}{2}\ \text{abs}\begin{bmatrix} 2 & -4 \\ 5 & 2 \\ 3 & 7 \\ -4 & 5 \\ -6 & -2 \\ 2 & -4 \end{bmatrix} = \frac{1}{2}\ \text{abs}\,((86) - (-76)) = \frac{1}{2}(162) = 81$$

<u>Note</u>: Do you see the "shoelaces"?

<u>Example</u>:

Determine the area of quadrilateral ABCD.



$$K = \frac{1}{2}\ \text{abs}\begin{bmatrix} -6 & 2 \\ 0 & 5 \\ -1 & 1 \\ 3 & -7 \\ -6 & 2 \end{bmatrix}$$

Note: this time we travelled in a clockwise direction.

$K = \frac{1}{2}$ abs $[(-30 + 0 + 7 + 6) - (0 - 5 + 3 + 42)]$

$K = \frac{1}{2}$ abs $[(-17) - (40)]$

$K = \frac{1}{2}$ (57)

$K = 28\frac{1}{2}$ square units.

In general, let $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, . . . , $P_n(x_n,y_n)$ be the vertices of an n-gon, $P_1P_2P_3 \ldots P_n$. The "shoelace" formula for determining the area of the n-gon is:

$$K = \frac{1}{2} \text{ abs } \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{bmatrix}$$

$$K = \frac{1}{2} \text{ abs } \left[(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (x_2y_1 + x_3y_2 + \ldots + x_1y_n) \right] .$$

SHOELACE PROGRAMS
FOR THE TI-82 & TI-81

Notes for running the program:
N = the number of points
X,Y = the coordinates of each point
A counter outputs which point is to be input.

TI-82

```
Prompt N
For (C,1,N)
Disp C
Prompt X
X –> L1(C)
Prompt Y
Y –> L2(C)
End
L1(N)L2(1) –> S
L1(1)L2(N) –> T
For (C,2,N)
S + L1(C – 1)L2(c) –> S
T + L1(C)L2(C – 1) –> T
End
Abs(S – T)/2 –> A
Disp "AREA"
Disp A
Stop
```

TI-81

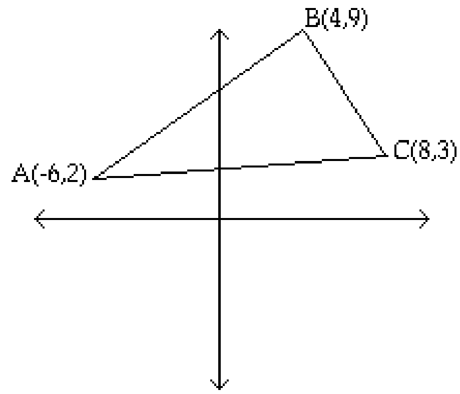
```
Disp "N"
Input N
1 –> C
Lbl 1
Disp C
Disp "X"
Input X
X –> {x}(C)
Disp "Y"
Input Y
Y –> {y}(C)
C + 1 –> C
If C ≤ N
Goto 1
{x}(N){y}(1) –> S
{x}(1){y}(N) –> T
2 –> C
Lbl 2
S + {x}(C – 1){y}(C) –> S
T + {x}(C){y}(C – 1) –> T
C + 1 –> C
If C ≤ N
Goto 2
abs (S – T)/2 –> A
Disp "AREA"
Disp A
 End
```

SHOELACE EXPLORATIONS

Mathematical Investigations I
Fall, 1993

EXPLORATION 1:

1. a. Determine the area of ΔABC.



b. Find $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(x_3,y_3)$:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 4 & 9 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ 8 & 3 \\ -6 & 2 \end{bmatrix}$$

c. Find the area of the triangle whose vertices are $P_1(x_1,y_1)$, $P(x_2,y_2)$, $P_3,(x_3,y_3)$.

d. Compare the areas of ΔABC and $\Delta P_1P_2P_3$.

e. Calculate the centroids of ΔABC and $\Delta P_1P_2P_3$ by taking the averages of the x-coordinates and the average of the y-coordinates.

2. Find the set of ordered pairs, (a,b), so that $P_1(7 + a,9)$, $P_2(b,7)$, $P_3(a + b, 5)$ are collinear.

EXPLORATION 2:

1. Determine the area of quadrilateral $A_1A_2A_3A_4$ whose vertices are $A_1(6,0)$, $A_2(8,3)$, $A_3(1,9)$ and $A_4(-10,-1)$.

2. Find:
$P_1(x_1,y_1)$ = ___________ $P_2(x_2,y_2)$ = ___________

$P_3(x_3,y_3)$ = ___________ $P_4(x_4,y_4)$ = ___________

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 8 & 3 \\ 1 & 9 \\ -10 & -1 \end{bmatrix} - \begin{bmatrix} -10 & -1 \\ 6 & 0 \\ 8 & 3 \\ 1 & 9 \end{bmatrix}$$

3. Determine the area of quadrilateral $P_1P_2P_3P_4$.

4. Compare the area of quadrilateral $A_1A_2A_3A_4$ with the area of quadrilateral $P_1P_2P_3P_4$.

5. Find (a,b) = (average of the x-coordinates, average of the y-coordinates) for quadrilateral $P_1P_2P_3P_4$.