p-adic Group and Local Langlands Correspondence

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1 Introduction

p-adic number was first introduced by Kurt Hensel. It gives a completely different metric on $\mathbb{Q}$, leading to an 'extraordinary' completion of $\mathbb{Q}$. This completion field has some properties, like local compactness, which bring us to a wider picture: study (classify all local fields) them and representation of local fields, which highly relates to local Langlands correspondence(LLC). It is arduous to say why local Langlands conjecture is formulated in that way. To have some background for LLC, Artin's conjecture is where LLC originally generated. Despite its complicated historical background, we still give some quick introduction to this correspondence.

2 Some facts on p-adic groups/fields

There are two ways to construct the p-adic integers, $\mathbb{Z}_p$ or the p-adic numbers, $\mathbb{Q}_p$, algebraically and analytically.

2.1 Algebraic construction

Defition 2.1: p-adic number(algebraically)

Consider the inverse system $(\{\mathbb{Z}/p^k\mathbb{Z}\}_k, \{\pi_{kj}\})$ in the category of rings **Ring**. The morphisms π_{kj} exist for all pairs (k, j), where $k \geq j$, $\mathbb{Z}/p^k\mathbb{Z} \overset{\text{mod } j}{\longrightarrow} \mathbb{Z}/p^j\mathbb{Z}$ such that $\sum_{d=0}^{k-1} a_d p^d \mapsto \sum_{d=0}^{j-1} a_d p^d$, ignoring the terms j-th to $k-1$-th. So, the p-adic integer, $\mathbb{Z}_p$, is defined as the inverse limit of this inverse system

$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^k\mathbb{Z}$$

$\mathbb{Z}_p$ is called p-adic integer with respect to p. To define p-adic numbers, $\mathbb{Q}_p$. It can be either settled as a fraction field of $\mathbb{Z}_p$ or a polynomial ring of $\mathbb{Z}_p$:

$$\mathbb{Q}_p := \mathrm{Frac}(\mathbb{Z}_p) \text{ or } \mathbb{Q}_p := \mathbb{Z}_p\left[\frac{1}{p}\right]$$

Remark This definition actually tells more abundant information than itself:
(1) The inverse limit is abstract somehow. To make it clearer, here is also a concrete construction for it:
$\mathbb{Z}_p = \left\{(x_i)_i \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \middle| \forall n, \pi_{n,n-1}(x_n) = x_{n-1}\right\}$, where $\pi_{n,n-1}$ keeps the same as the prescribed definition.
In fact, this is not the unique way to think $\mathbb{Z}_p$, another construction uses equivalent classes:
$\mathbb{Z}_p = \left\{(x_i)_i \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \middle| \forall n, x_{n+1} \equiv x_n \mod p^n\right\}$
A third intuitive view of p-adic integers is to treat each of them a formal power series $\sum_{n=0}^{\infty} a_n p^n$.
(2) This is an example for profinite group.
(3) In general, this has a general form: I-adic completion. Here the ideals serving as I are $p^k\mathbb{Z}$ ranging over all natural numbers k and for a fixed prime p. The I-adic trick will be applied to the completion of function fields, presenting itself bunches of times when we are going to cope with non-Archimedean local fields with positive characteristics.
(4) Another way to view this stuff p-adic group (integer) comes from localization. Analogously, we zoom into regions getting 'smaller and smaller' (as k increases, $p^k\mathbb{Z}$ gets 'smaller'). So, we are looking at the local parts and this 'localization' tells us more information.
(5) To make the definition $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$ works, we have to make sure that $\mathbb{Z}_p$ is also an integral domain. Since we do not want dichotomy in the definition, we have to show the equivalency of two definitions. Or make one of them definition and equivalency of them as a corollary.

2.2 Analytic construction

Defition 2.2: p-adic absolute value

Let p be a fixed prime. We can define this on $\mathbb{Z}$ first: $\forall a \in \mathbb{Z}$, since a has a unique representation from fundamental theorem of arithmetic, so let n be the maximal order of p in a, i.e. $\exists n \in \mathbb{N}$, $a = p^n r$, such that $p^n | a$ but $p^{n+1} \not| a$. Then, the absolute value of a with respect to p is defined as

$$|a|_p := p^{-n}$$

Remark (1) Some history and motivations for this definition (c.f. [Neu91]) :
(2) This is indeed an absolute value since . This also answers why we cannot choose an arbitrary n and define an n-norm. Because it is not well-defined.
(3) It can be readily checked that such an absolute value is a norm, also a metric, on $\mathbb{Z}$ or on $\mathbb{Q}$.
Based on facts on metric spaces, either $\mathbb{Z}$ or $\mathbb{Q}$ can be completed with respect to $|\cdot|_p$. The process is standard [Gou20] :

- Define $\mathcal{C}_p(\mathbb{Q}) := \{(x_n) | (x_n) \text{ a Cauchy sequence w.r.t.} |\cdot|_p\}$

  △ Notice: not all non-zero elements has an inverse, so this is not a field.
- Define additions and multiplications on $\mathcal{C}_p(\mathbb{Q})$: $(x_n) + (y_n) := (x_n + y_n)$ and $(x_n) \cdot (y_n) := (x_n y_n)$

  △ Notice: Some sequences themsevles are very closed to each other and has the same limit but they are not the same element in $\mathcal{C}_p(\mathbb{Q})$, now we force them to be the same.
- Injects $\mathbb{Q}$ into $\mathcal{C}_p(\mathbb{Q})$ by $\iota : \mathbb{Q} \hookrightarrow \mathcal{C}_p(\mathbb{Q})$ $x \mapsto (x)$, where (x) is a constant sequence in $\mathcal{C}_p(\mathbb{Q})$ (quick to check).
- Define $\mathcal{N} := \{(x_n) \in \mathcal{C}_p(\mathbb{Q}) : \lim_{n\to\infty} |x_n|_p = 0\}$. See $\mathcal{N}$ is a maximal ideal of $\mathcal{C}_p(\mathbb{Q})$.
- Then $\mathcal{C}_p(\mathbb{Q})/\mathcal{N}$ is a field. Define $\mathbb{Q}_p := \mathcal{C}_p(\mathbb{Q})/\mathcal{N}$.
- There is a non-trivial ring homomorphism $\iota : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$, by $x \mapsto (x) + \mathcal{N}$, it is non-trivial field homomorphism hence injective, then $\mathbb{Q}_p/\iota(\mathbb{Q})$ is an extension of $\mathbb{Q}$ after identifying $\mathbb{Q}$ with $\iota(\mathbb{Q})$.

We have already had such a field, then we try to connect $\mathbb{Q}_p$ to $\mathbb{Q}$ more intimately. Here, 'connect' precisely means to extend properties of $\mathbb{Q}$ to $\mathbb{Q}_p$, such as the absolute value.
One could show the absolute value we have already defined could be uniquely extended to $\mathbb{Q}_p$. Just like how $\mathbb{Q}$ being embedded into $\mathbb{R}$, $\mathbb{Q}$ is a dense subset of $\mathbb{Q}_p$.
(4) So, we can define the respective complete metric space to be $\mathbb{Z}_p$ and $\mathbb{Q}_p$.
(5) It actually does not matter where we start, from $\mathbb{Z}$ or $\mathbb{Q}$. We can see under such analytic construction, $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$, and the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\text{comp}"] \arrow[d, "\text{Frac}"'] & \mathbb{Z}_p \arrow[d, "\text{Frac}"] \\ \mathbb{Q} \arrow[r, "\text{comp}"] & \mathbb{Q}_p \end{tikzcd}$$

2.3 Intertwining between two constructions

The two definitions interplay very well. The algebraic definition fits the absolute value (further, the topology induced by that absolute value) very well. Also, the analytic definition has the same algebraic structure. Using this connection,

$$|x - y| \leq p^{-n} \Leftrightarrow x - y \in p^n \mathbb{Z}_p$$

which helps convert the definition in analysis flavour into an algebraic one.
$\mathbb{Z}_p$ can be recovered from $\mathbb{Q}_p$ by considering the following valuation ring:

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p | |x| \leq 1\}$$

2.4 Properties on p-adic numbers

Lemma 2.1: A short exact sequence for $\mathbb{Z}_p$

The map $\mathbb{Z}_p \to \mathbb{Z}_p$ in the following sequence is $x \mapsto x \cdot p^n$, which is continuous map given the discrete topology. The second map $\mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ is the natural projection. The following sequence is exact.

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\cdot p^n} \mathbb{Z}_p \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0$$

In particular, $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$

Proof: (1) $i : \mathbb{Z}_p \to \mathbb{Z}_p \quad x \mapsto p^n \cdot x$ is injective. $\ker i = \{x | p^n \cdot x = 0\} = \{0\}$
(2) To handle the second map, it is convenient to use $\mathbb{Z}_p = \left\{(x_i)_i \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \middle| \forall n, \pi_{n,n-1}(x_n) = x_{n-1}\right\}$. Then $\pi : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ by $(x_i)_i \mapsto x_n$ is surjective because for every $x_n = \sum_{i=0}^{n-1} x_i p^i \in \mathbb{Z}/p^n\mathbb{Z}$, there are many ways to recover a sequence $(x_i) \in \mathbb{Z}_p$.
(3) $\ker\pi = \operatorname{im} i$ since

$$\ker\pi = \{(0, 0, \cdot, 0) \| (x_i)_{i\geq n} | \forall n \geq i, \pi_{n,n-1}(x_n) = x_{n-1}, \text{ and the first } n-1 \text{ term is } 0\}$$

Multiplying p^n to $\mathbb{Z}_p$ makes the first n terms in $\operatorname{im} i$ vanish as well.
In particular, π is surjective with kernel $\ker\pi = p^n\mathbb{Z}_p$, so $\mathbb{Z}/\ker\pi = \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$. ☐
Here we give properties of $\mathbb{Q}_p$ that distinguish it from $\mathbb{R}$.

Theorem 2.1: Connectedness

$\mathbb{Q}_p$ is a totally disconnected Hausdorff space.

Proof: Hausdorffness comes from the fact that $\mathbb{Q}_p$ is a metric space.
For total-disconnectedness, more generally, one can show that any non-Archimedean discrete valuation ring is totally disconnected.

☐

Theorem 2.2: Compactness

$\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact.

Proof: To prove $\mathbb{Z}_p$ is compact, notice that in metric space compact $\Leftrightarrow$ complete + totally bounded.
$\mathbb{Z}_p$ is complete: $\mathbb{Z}_p$ is closed in the complete field $\mathbb{Q}_p$
$\mathbb{Z}_p$ is totally bounded: It suffices to check that for every $\varepsilon = p^{-n}$, n being an integer, $\mathbb{Z}_p$ can be covered by finitely many open balls of radius ε. Lemma 2.1, saying $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$, gives us candidate of balls: $a + p^n\mathbb{Z}_p$, where a ranging over all $0, 1, \dots, p^n - 1$ $\forall a$, $a + p^n\mathbb{Z}_p$ is a ball since

$$a + p^n\mathbb{Z}_p = \{a + p^n x | x \in \mathbb{Z}_p\} \stackrel{\dagger}{=} \{y \in \mathbb{Z}_p \big| |y - a| \leq p^{-n}\} = \overline{B}_a(p^{-n})$$

†: This is from the definition of p-adic absolute value.
To prove local compactness of $\mathbb{Q}_p$, it is enough to prove that there is a compact neighbourhood of 0 in $\mathbb{Q}_p$. Because if this is true, let S be that compact neighbourhood of 0 in $\mathbb{Q}_p$. Since the translation by a map $T_a : x \mapsto x + a$ is continuous and image of a compact set under continuous map is compact. The image $T_a(S)$ for each a is a compact neighbourhood containing a. Ranging over all $a \in \mathbb{Q}_p$, we get a compact neighbourhood for every point. Since $\mathbb{Z}_p$ is a compact neighbourhood of 0 in $\mathbb{Q}_p$, $\mathbb{Q}_p$ is automatically locally compact. ☐
Ostrowski's theorem: Every non-trivial absolute value of $\mathbb{Q}$ is, either equivalent [1] to a p-adic absolute value for some prime p, or equivalent to the standard absolute value. In the first case, the completion of $\mathbb{Q}$ is $\mathbb{Q}_p$, and in the second case, the completion of $\mathbb{Q}$ is $\mathbb{R}$.

Now we have all the preliminary properties for $\mathbb{Q}_p$ and p-adic field, we are going to talk about the local Langlands correspondence.
Something further: Local field: From above properties, we saw $\mathbb{Q}_p$ is a field that is also locally compact and Hausdorff. This idea could be extended further to local fields. A **local field** is a locally compact Hausdorff(non-discrete) topological field. Topological field has the same meaning as we known for topological group. The addition, multiplication, taking additive and multiplicative inverse processes are continuous. We could also give a classification for local fields [Mil20]. Let K be a local field. Then K is in exactly one of the following cases:

(1) If K is complete with respect to an Archimedean absolute value $|\cdot|_\infty$, then K is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

(2.1) If K is a non-Archimedean local field of characteristic 0, then K is isomorphic to some finite extension of $\mathbb{Q}_p$.

(2.2) If K is a non-Archimedean local field of characteristic $p > 0$, K is isomorphic to a field $\mathbb{F}_q((T))$, where $\mathbb{F}_q$ is a finite field with $q = p^n$ for some n and $\mathbb{F}_q((T))$ is a completion of $\mathbb{F}_q(T)$ with respect to (T)-adic completion. $\mathbb{F}_q(T))$ can be thought of as the set of all formal Laurent series over $\mathbb{F}_q$.

[1]The definition of equivalent absolute value is similar to the equivalent norm. Two norms are equivalent if they induce the same topology.

Properties \ Field	$\mathbb{R}$	$\mathbb{Q}_p$, for some p
ordering	totally ordered	not ordered
Connectedness	connected	totally-disconnected
Compactness	locally compact	locally compact
Characteristic	0	0

Figure. Comparison between $\mathbb{R}$ and $\mathbb{Q}_p$

3 Local Langlands Correspondence

3.1 Statement of the conjecture

Local Langlands has many versions or say 'hierarchies', depending on the level of abstractness. I need to point out: The 'correspondence' itself is neither a bijection nor a categorical equivalence(or anti-equivalence). Instead, it is just a description of one side by the other side. Here we give some:
To avoid being lost in fancy words, here we just give a rough version of local Langlands and then a precise version follows. Roughly speaking, the local Langlands gives the correspondence:

$$\left\{\begin{matrix}\text{certain (typically } \infty\text{-dim)}\\ \mathbb{C}\text{-representation of } \mathrm{GL}_n(K)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{certain } n\text{-dim } \mathbb{C}\text{-representation}\\ \text{of a group related to } \mathrm{Gal}(\overline{K}/K)\end{matrix}\right\}$$

Complete version:

$$\left\{\begin{matrix}\text{smooth admissible}\\ \text{irreducible } \mathbb{C}-\text{representation}\\ \text{of } \mathrm{GL}_n(K)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}F\text{-semisimple Weil-Deligne}\mathbb{C}-\text{representation}\\ \text{of the Weil group } W_K \text{ related to } \mathrm{Gal}\,(\overline{K}/K)\end{matrix}\right\}$$

This could be more generalized, from $\mathrm{GL}_n(K)$ to a reductive group over a field K [2]:

$$\left\{\begin{matrix}\text{smooth admissible}\\ \text{irreducible } \mathbb{C}-\text{representation}\\ \text{of any reductive group } G(K)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Homomorphism from Weil group } W_K\\ \text{to the Langlands dual group } {}^LG,\ W_K \to {}^LG\end{matrix}\right\}$$

I will not give the definition of reductive group. Just keep in mind that $\mathrm{GL}_n(K)$ for arbitrary K is an example off reductive group. Here we choose to stay in the $\mathrm{GL}_n(K)$ cases rather than going to reductive groups.

3.2 Understanding the words in the conjecture

Defition 3.1: Smooth representation

Let (ρ, V) be a representation of G. (ρ, V) is a smooth representation $\Leftrightarrow \forall v \in V$,

$$\mathrm{Stab}_G(v) = \{g \in G : g \cdot v = v\}$$

is an open subgroup of G

Remark This is called algebraic representation somewhere [BZ76].

Defition 3.2: Admissible representation

Let G be a reductive group, a linear representation of G on a vector space V over K is **admissible** $\Leftrightarrow \forall$ compact open subgroup $U \subseteq G(F)$, $\dim_K V^U < \infty$

Remark Some authors refer 'admissible' to be smooth+admissible here.
For some irredeemable reason, here I did not give the definition of Weil group.

[2]This hierarchy was hinted by this paper [Har03] and the definition of $\mathrm{GL}_n(K)$ in conjunction with reductive group.

Remark Weil introduced this concept at 1951. It could be considered as a subgroup of absolute Galois group. For every Galois representation, it induces naturally a representation of Weil group by the morphism $W_K \to \mathrm{Gal}(\overline{K}/K)$, but not every representation of W_K is generated from this homomorphism. [Bum97].

Defition 3.3: Weil-Deligne representation

A **Weil-Deligne representation** is a triple (ρ, N, V) consisting of the following information:
(1) $\rho : W_K \to \mathrm{GL}(V)$ is a linear representation of the Weil group W_K for some field K. $\dim_{\mathbb{C}} V = n$.
(2) N is a nipotent monodromy operator satisfying $\forall\sigma \in W_K$, $\rho(\sigma)N\rho(\sigma)^{-1} = \|\sigma\|N$, where $\|\sigma\|$ is the valuation of the corresponding element of $K^\times$ under the isomorphism of local class field theory.

Defition 3.4: K-semisimple

Let (ρ, N) be a Weil-Deligne representation. K is the field associated to the Weil group W_K. (ρ, N, V) is K-**semisimple** $\Leftrightarrow$ ρ is semisimple.

3.3 Tentative classification in each case

Now let K be a non-Archimedean local field. We are trying to understand the 'left-hand side' of this conjecture, i.e. understand $\mathrm{GL}_n(K)$ for each n, and then we work with the 'right-hand side' of this conjecture. After understanding both sides, it is possible to find the correspondence.
First, here gives the classfication of all represetations of $\mathrm{GL}_n(K)$[3]:

3.3.1 Classfication of $\mathrm{GL}_n(K)$

Case : $n = 1$
The local Langlands correspondence here, for a non-Archimedean field, is just class field theory:

Given a 1-dimensional smooth representation χ, i.e. the quasicharacter of K, we can have the corresponding Weil-Deligne representation by the composition map:

$$W_K \longrightarrow W_K^{ab} \xrightarrow{\mathrm{rec}^{-1}} \mathrm{GL}(K) \xrightarrow{\chi} \mathrm{GL}(\mathbb{C})$$

where $\mathrm{rec} : \mathrm{GL}(K) \to W_K^{ab}$ is the Artin reciprocity map.
Case: $n = 2$
Some basic settings: For arbitrary field K, we can give the following definitions, which are some important subgroups of $\mathrm{GL}_2(K)$. They are:

- standard Borel subgroup of $\mathrm{GL}_2(K)$, $B := \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathrm{GL}_2(K) \right\}$, i.e. upper triangular matrices inside $\mathrm{GL}_2(K)$.
- unipotent radical of $\mathrm{GL}_2(K)$, $N := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(K) \right\}$
- standard split maximal torus in $\mathrm{GL}_2(K)$, $T := \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathrm{GL}_2(K) \right\}$

For arbrtrry field K, $B = N \rtimes T$ and **Bruhat decomposition** for $\mathrm{GL}_2(K)$, $\mathrm{GL}_2(K) = B \cup BwB$, where $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
From now on, we constrain the field K from arbitrary one to a non-Archimedean local field and $\mathcal{O}_K$ is the valuation ring of K Then, we have **Iwasawa decompostion**, $\mathrm{GL}_2(K) = B\,\mathrm{GL}_2(\mathcal{O})$. By the way, $\mathrm{GL}_2(\mathcal{O})$ is the maximal compact subgroup of $\mathrm{GL}_2(K)$, and like the case that every two maximal tori are conjugate in compact and connected Lie groups, every maximal subgroup of $\mathrm{GL}_2(K)$ is conjugate to $\mathrm{GL}_2(\mathcal{O})$.

Non-Archimedean case:
Let (π, V) be an irreducible representation of $\mathrm{GL}_2(K)$ and N is the unipotent radical of B Consider the subspace $V(N) := \mathrm{span}\{v - \pi(x)v | x \in N, v \in V\}$ and the quotient $V_N := V/V(N)$ Here V_N is called the **Jacquet module** of V at N.
Jacquet module is a smooth admissible representation of T of dimension ≤ 2.
So, there are three cases in total: $\dim V_N = 2, 1, 0$.

[3]The whole part is due to

Defition 3.5: Supercuspidal representation

Let (ρ, V) be an irreducible smooth representation of $\mathrm{GL}_2(K)$. (ρ, V) is **supercuspidal** $\Leftrightarrow$ the Jacquet module of V at N, V_N, is zero. Here N is the unipotent radical of B. Supercuspidal is called cuspidal or absolutely cuspidal somewhere.

So, consider the non-supercuspidal case first [4]:

Defition 3.6: Induced representations

(1) Definition of induced representations: There are many ways to write the induced representations:
Let G be a locally compact group and H be a subgroup of G, then for a continuous unitary representation into a Hilbert space V, (ρ, V), an induced representation of G with respect to H, $(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G V)$, is defined as

$$\mathrm{Ind}_H^G V := \left\{ f : G \to V \middle| \begin{array}{l} (1) \forall g \in G, \forall h \in H, f(hg) := \rho(h) f(g) \\ (2) \exists \text{ open subgroup } K_0 \subseteq G, \forall g \in G, \forall k \in K_0, f(gk) = f(g) \end{array} \right\}$$

$$\mathrm{Ind}_H^G \rho : G \to \mathrm{GL}(\mathrm{Ind}_H^G V) \quad g \mapsto \mathrm{Ind}_H^G \rho(g) := \forall x \in G, \forall f \in \mathrm{Ind}_H^G V, (\mathrm{Ind}_H^G \rho(g) f)(x) = f(gx)$$

There is a variation of induced representation, called compactly-supported induced representation, denoted $(c - \mathrm{Ind}_H^G \rho, c - \mathrm{Ind}_H^G V)$,

$$c - \mathrm{Ind}_H^G V := \left\{ f \in \mathrm{Ind}_H^G V | \overline{\mathrm{supp}\, f} \subseteq G/H \text{ is compact} \right\}$$

Remark The second condition is saying f is locally constant, which implies that f is smooth. As long as G/H is compact, condition (2) is equivalent to smoothness.
Frobenius Reciprocity: Like the case of finite groups, we also have a version of Frobenius reciprocity for locally compact subgroups and its closed subgroups:
Let G be a locally compact group and H be its closed subgroup. Let (ρ, V) and (σ, W) be smooth representations of G and H, respectively. Then,

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(W, \mathrm{Res}_H^G V)$$

In some other notations, this reciprocity is

$$\mathrm{Hom}_G(\rho, \mathrm{Ind}_H^G \sigma) \cong \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \rho)$$

where $\mathrm{Hom}_G(-,-)$ is the set of all G-equivariant maps from one space to the other.
Extending representation from T **to** B: For a representation of T, say $\rho : T \to \mathrm{GL}(V)$, one can always extend it to a (smooth) representation on B, denoted $\widetilde{\rho}$, by setting the action of T the same as ρ and trivial on N.
There is a handy gadget, called twisting, which can help us modify the character of some representation without 'destroying' the structure of induced representation.
Twisting: (ρ, V) is still a smooth representation and χ is a character of $\mathrm{GL}(K)$, there is another smooth representation of G, $(\chi \cdot \rho, V)$, called the twist of ρ by χ, defined by: $\forall g \in G$,

$$\chi \cdot \rho(g) := \chi(\det g) \rho(g)$$

Applying this to representation of T, $(\chi_1 \otimes \chi_2, V)$ by $\phi : \mathrm{GL}(K) \to \mathbb{C}^\times$ where $\chi_1, \chi_2 : \mathrm{GL}(K) \to \mathbb{C}^\times$, the twisting of $\chi_1 \otimes \chi_2$ is

$$\phi \cdot (\chi_1 \otimes \chi_2) := \phi \cdot \chi_1 \otimes \phi \cdot \chi_2$$

Continuing lifting this as a representation of B, $(\widetilde{\chi_1 \otimes \chi_2}, V)$

From now on, $\delta_B : B \to \mathbb{C}$ is a modular function, which could be used to represent Haar measures on N, μ_N on N by $\delta_B(t)\mu_N(S) = \mu_N(t^{-1}St)$ and to normalize the space $\mathrm{Ind}_B^G \mathbb{C}$. We are not going into details of that, just mentioning the form of it:

$$\delta_B : B \mapsto \mathbb{C} \quad b = tn \mapsto \left| \frac{t_2}{t_1} \right|, \text{ where } t = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$$

which comes from the Iwasawa decomposition mentioned above, so every element $b \in B$ has the unique decomposition. Hence, this δ_B is well-defined.

[4] This follows the books [06b] and [Bum97]

Theorem 3.1: Irreducible criterion for induced representations

Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. Consider the induced representation $(\mathrm{Ind}_B^G\widetilde{\chi}, \mathrm{Ind}_B^G\mathbb{C})$.
(1) $(\mathrm{Ind}_B^G\widetilde{\chi}, \mathrm{Ind}_B^G\mathbb{C})$ is reducible $\Leftrightarrow \chi_1\chi_2^{-1}$ is either trivial or $x \mapsto |x|^2$
(2) When $(\mathrm{Ind}_B^G\widetilde{\chi}, \mathrm{Ind}_B^G\mathbb{C})$ is reducible:
(2.1) G-composition length of $\mathrm{Ind}_B^G\mathbb{C}$ is 2.
(2.2) One composition factor in (2.1) of $\mathrm{Ind}_B^G\mathbb{C}$ has dimension 1, and the other factor has infinite dimension.
The two cases in (1) give some information on $\mathrm{Ind}_B^G\mathbb{C}$:
(2.3) $\mathrm{Ind}_B^G\mathbb{C}$ has 1-dimensional G-invariant subspace $\Leftrightarrow \chi_1\chi_2^{-1} = \mathbf{1}$, i.e. the trivial one.
(2.4) $\mathrm{Ind}_B^G\mathbb{C}$ has 1-dimensional G-invariant quotient subspace $\Leftrightarrow \chi_1\chi_2^{-1} : x \mapsto |x|^2$

Sketch of proof :
• First, look at all $f \in \mathrm{Ind}_B^G\widetilde{\chi}$,
• Then, notice that for $\pi : V \to V_N$, $\ker\pi$ is an irreducible representation of B. This uses some knowledge on representation of mirabolic group M of $\mathrm{GL}_2(K)$, and the following N-invariant isomorphism

$$V \to C_c^\infty(N) \quad f \mapsto \Big(f_N : n \mapsto f(wn)\Big)$$

This N-invariant isomorphism can be extended to an M-invariant isomorphism by noticing every $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in M$ can be written as a product of elements of N and S. Here the concrete elements are $\begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix} \in N$ and $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in S$.
Define the group action of S and N on $C_c^\infty(N)$ as:
$S \curvearrowright C_c^\infty(N)$ by $\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \cdot f\right)\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = \chi_2(a) f\left(\begin{bmatrix} 1 & a^{-1}x \\ 0 & 1 \end{bmatrix}\right)$
$N \curvearrowright C_c^\infty(N)$ by $(n \cdot f)(n') := f(nn')$, the canonical one
• Then, $\mathrm{Ind}_B^G\widetilde{\chi}$ has composition length 3. Two of the Jordan-Hölder factor has dimension 1 and one factor has dimension 2. [5].
• Based on the lemma that $\ker\pi$ is an irreducible subrepresentation of B, there is an equivalence can be utilized: $\chi_1 = \chi_2 \Leftrightarrow \mathrm{Ind}_B^G\mathbb{C}$, has a one dimensional N-invariant subspace. If the equivalence conditions hold, then:
$\mathrm{Ind}_B^G\mathbb{C}$ has a unique one-dimensional N-invariant subspace. Denote this invariant subspace $(\mathrm{Ind}_B^G\mathbb{C})_0$.
$(\mathrm{Ind}_B^G\mathbb{C})_0$ is also a G-invariant subspace of X and $(\mathrm{Ind}_B^G\mathbb{C})_0 \not\subseteq V$
we just omit the proof of equivalence in this entry.
• Now we are at the proof of the irreducible criterion. There are only two possibilities for the composition series. length 2(reducibility of $\mathrm{Ind}_B^G\mathbb{C}$ implies there is at least one G-invariant subspace) or length 3 (as described in a prescribed corollary, length ≤ 3). Then, paly with quotienting to get those irreducible factors and study their properties. The proof is also omitted.
Remark If $(\mathrm{Ind}_B^G\widetilde{\chi}, \mathrm{Ind}_B^G\mathbb{C})$ is irreducible, we are done and call this representation 'principal series representation'. But if $(\mathrm{Ind}_B^G\widetilde{\chi}, \mathrm{Ind}_B^G\mathbb{C})$ is not reducible, this criterion gives us somethings to motivate us finding its irreducible representations. How to find? Recall 'twisting'!

Assume now $(\mathrm{Ind}_B^G\widetilde{\chi}, \mathrm{Ind}_B^G\mathbb{C})$ is reducible, using the twisting trick, for any character on $T \subseteq \mathrm{GL}_2(K)$, there is a character $\phi : \mathrm{GL}(K) \to \mathbb{C}^\times$, such that $\chi = \phi \cdot \mathbf{1}_T$, where $\mathbf{1}_T$ is the trivial character of T or maybe in the normalized version $\chi = \phi \cdot \delta_B^{-1}$. We now deal with the simplest case and find the simplest irreducible representation and assign it a name:
For the simplest case of representation $(\mathrm{Ind}_B^G\widetilde{\mathbf{1}_T}, \mathrm{Ind}_B^G\mathbb{C})$, by Theorem 3.1, we know there is an irreducible representation for $(\mathrm{Ind}_B^G\widetilde{\mathbf{1}_T}, \mathrm{Ind}_B^G\mathbb{C})$, define the irreducible G-invariant quotient of $\mathrm{Ind}_B^G\mathbb{C}$ to be the Steinberg representation, denoted $(\mathrm{St}_G, \mathbb{C})$

[5]Let (ρ, V) be a smooth representation. A composition series of V is a descending chain of G-invariant subspaces:

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n \supseteq V_{n+1} = \{0\}$$

such that for each $i = 0, \ldots, n$, V_i/V_{i+1} is an irreducible representation of ρ after identifying each V_i/V_{i+1} with a subspace of V. Each factor V_i/V_{i+1} is called Jordan-Hölder factor of this series and the length of this composition series is the number of Jordan-Hölder factors. So, a series has length n has at least $n+1$ terms in this composition series. (c.f. [Mur09])

Lemma 3.1: Something

Let $\chi = \chi_1 \otimes \chi_2$ be a character for T.
The representation $\mathrm{Ind}_B^G(\delta_B^{-\frac{1}{2}} \otimes \chi)$ is reducible $\Leftrightarrow \chi_1\chi_2^{-1}$ is one of characters $\mathrm{GL}_n(K) \to \mathrm{GL}(\mathbb{C})$ $x \mapsto |x|^{\pm 1}$.
In other words, $\exists$ a character $\phi : F^\times \to \mathbb{C}^\times$, such that $\chi = \phi \cdot \delta_B^{\pm\frac{1}{2}}$.
Let χ, ξ be characters of T. $\mathrm{Hom}_G(\mathrm{Ind}_B^G(\delta_B^{-\frac{1}{2}} \otimes \chi), \mathrm{Ind}_B^G(\delta_B^{-\frac{1}{2}} \otimes \xi)) \neq 0 \Leftrightarrow \xi = \chi$ or $\xi = \chi^w$, where $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Proof: □

Our goal is to classify all irreducible smooth admissible representations. Without formally writing out, there is a theorem saying that every irreducible smooth non-supercuspidal representation is admissible. By virtue of this theorem, the question boils down to classifying all irreducible smooth representations.

Theorem 3.2: Classification for non-(super)cuspidal representation

Let K be a non-Archimedean local field here. Every isomorphism class of a irreducible smooth non-supercuspidal representation of $\mathrm{GL}_2(K)$ is:
(1) one-dimensional representation $\phi \circ \det$, ϕ ranging over all characters of $K^\times$.
(2) irreducible induced representation $\mathrm{Ind}_B^G(\delta_B^{-\frac{1}{2}} \otimes \chi)$, where $\forall$ character $\chi : K^\times \to \mathbb{C}^\times$, $\chi \neq \phi \cdot \delta_B^{\pm\frac{1}{2}}$. This representation is also called **principal series representation**.
(3) **Special representation** $\phi \cdot \mathrm{St}_G$, ranging over all characters of $K^\times$.

Some comments on classification of supercuspidal representations of [06a]:
The supercuspidal representations do exists [Bum97]: Kruzko (1978) established a theorem concerning the existence of supercuspidal one by constructing induced representation of subgroup of $\mathrm{GL}_2(K)$ in this case.
This is hard. In the language of Strata (stratum), there is a beautiful theorem to classify all supercuspidal representations of $\mathrm{GL}_2(K)$ (Just ignore the notation. They are not defined here. I just want to say it can be classify neatly):

$$(\mathfrak{A}, J, \Lambda) \mapsto \pi_\Lambda = c - \mathrm{Ind}_J^G \Lambda$$

induces a bijection

$$\left\{ \text{conjugacy classes of cuspidal type} \right\} \leftrightarrow \left\{ \substack{\text{isomorphism classes of} \\ \text{irreducible cuspidal representations}} \right\}$$

So, now we finished the classification of non-Archimedean cases.
Archimedean case: This part comes from [Kna94]
Case: for arbitrary n
For arbitrary n, we still deal with subgroups like the standard Borel subgroups, mirabolic sbugroups, etc. Because to understand complicated cases like $\mathrm{GL}_n(K)$, a universal way is to decompose it and understand how those subgroups act on vector space.

3.3.2 Weil-Deligne representation

Here we are not going to do this.

3.3.3 The correspondence

Correspondence for local field K and $n = 1$ is already given in **Section 3.3.1**
For the Archimedean field case and $n = 1, 2$, see [Kna94].
Here only a very specific case of correspondence is given. Now assume K is a p-adic field (not even a local field). Let (ρ, V) denote the irreducible smooth admissible representation of $\mathrm{GL}_2(K)$, and (τ, N, V) denote a 2-dimensional K-semisimple W-D representation of Weil group W_K. [HUA22] For each character $\chi_i : \mathrm{GL}(K) \to \mathrm{GL}(\mathbb{C})$, $i = 1, 2$, let $\tau_i : W_K \to \mathrm{GL}(\mathbb{C})$ be the WD representation associated to each χ_i under the 1-dimensional correspondence.

(1) For (ρ, V) being the one-dimensional principal series representation, $\rho = \chi \circ \det$ and $\dim_{\mathbb{C}} V = 1$, The associated W-D representation is:

$$(\tau_1 |\cdot|^{\frac{1}{2}} \oplus \tau_1 |\cdot|^{-\frac{1}{2}}, N = 0, V)$$

(2) For (ρ, V) being the irreducible principal series representation, $V = \mathrm{Ind}_B^G(\widetilde{\chi_1 \otimes \chi_2})$, here $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$. The associated W-D representation is

$$(\tau_1 \oplus \tau_2, N = 0, V)$$

(3) For (ρ, V) being the special representation, since $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$, we can merge two characters into one, and then $\rho = \mathrm{Ind}_B^G(\widetilde{\chi \otimes \chi|\cdot|})$ for some χ. The associated W-D representation is:

$$(\tau \oplus |\cdot|\tau, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, V)$$

(4) For (ρ, V) being the supercuspidal representation when $p \neq 2$, it can be viewed as 'base change representation' E the associated W-D representation is: the 2-dim representation

$$(\mathrm{Ind}_{W_L}^{W_K}(\sigma), N = 0, V), \quad \sigma \in \mathrm{Gal}\,(E/K) \text{ is the unique nontrivial element}$$

(5) For (ρ, V) being the supercuspidal representation but $p = 2$, this is hard and we do not say anything here.

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