Lie Groups and Lie Algebras

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Contents

Our contents will be entailed in three main parts:
1. Elementary Lie theory
In this part, basic notions of Lie groups and Lie algebras are entrenched. As we do for many math structures, we study the structure itself, the substructure and the morphisms between two such structures. There are three correspondences: Lie groups and Lie algebra, Lie subgroup and Lie subalgebra, morphism of Lie groups and morphism of Lie algebras. After the work of these levels, it can be packed into the more condensed categorical languages. We finally end up this part with some main theorems that relates certain categories.
2. Compact Lie group (Weyl and Cartan)
Structure, classification and representation of compact Lie group (complex semisimple Lie algebra).

$$\text{Rep of compact Lie group (Weyl)} \longleftrightarrow \text{Rep of complex semisimple Lie algebra (Cartan)}$$

Root system, highest weight, Peter-Weyl, Weyl character formula, dimension formula and integration.
3. Semisimple Lie group (Harish-Chandra)
Structure, representation of semisimple Lie group. Different decompositions: Iwasawa, Bruhat, Cartan decomposition. discrete series, main serial series, $\mathfrak{g} - K$ module, Plancherel formula.

1 Elementary Lie theory

1.1 Lie groups

1.1.1 Abstract Lie groups

Here the Lie groups are real Lie groups. By 'real Lie groups', we mean finite dimensional $\mathbb{R}$-smooth manifolds.
Related concepts: complex Lie groups, p-adic Lie groups

1.1.2 Matrix Lie groups

Theorem 1.1: Cartan's closed group theorem

Every matrix Lie group is an abstract Lie group.

1.2 Lie algebras

Lie algebra, from the word algebra, it is a certain algebra. Starting from a vector space, an algebra requires the third operation, multiplication. In Lie algebra, the multiplication is not the usual one. Instead, it is defined to be a 'bracket'.

Definition 1.1: Lie algebra

A $\mathbb{K}$-Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{K}$ together with a binary operation, called **Lie bracket**, $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying bilinearity, alternating property and Jacobi identity:
• Bilinearity: $\forall a, b \in \mathbb{F}$ and $\forall x, y, z \in \mathfrak{g}$,

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

• Alternating property: $\forall x \in \mathfrak{g}$,

$$[x, x] = 0$$

• Jacobi identity: $\forall x, y, z \in \mathfrak{g}$,

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$$

Example (1) k-Lie algebras and associative k-algebras. From a k-Lie algebra $\mathfrak{g}$, how to get a Lie algebra? There is a universal one $U(\mathfrak{g})$ called the enveloping algebra of $\mathfrak{g}$.
(2)

1.2.1 Abstract Lie algebras

Theorem 1.2: Induced Lie algebra homomorphism

Let G, H be Lie groups. Let $f : G \to H$ be a Lie group homomorphism. Then, the induced tangent map at $e \in G$

$$\mathrm{d}f_e : T_{e_G}G \to T_{e_H}H$$

is a Lie algebra homomorphism.

Proof: Need to show: $\forall v_{e_G}, w_{e_G} \in T_{e_G}G$,

$$\mathrm{d}f_{e_G}([v_{e_G}, w_{e_G}]_{T_{e_G}G}) = \left[\mathrm{d}f_{e_G}(v_{e_G}), \mathrm{d}f_{e_G}(w_{e_G})\right]_{T_{e_H}H} \tag{1}$$

$$\Leftrightarrow \mathrm{d}f_{e_G}\left(\left[V_{v_{e_G}}, V_{w_{e_G}}\right]_{T_{e_G}G}(e_G)\right) = \left[V_{\mathrm{d}f_{e_G}(v_{e_G})}, V_{\mathrm{d}f_{e_G}(w_{e_G})}\right]_{T_{e_H}H}(e_H) \tag{2}$$

$\forall v_{e_G} \in T_{e_G}G$, it could be extended to a left invariant vector field $V : G \to TG \in \mathfrak{X}^L(G)$, such that $V_{e_G} = v_{e_G}$. Since $f : G \to H$ is a smooth map between two manifolds, then we have an induced map

$$\mathrm{d}f_e : T_{e_G}G \to T_{e_H}H$$

Denote $\widetilde{v}_{e_H} := \mathrm{d}f_{e_G}(v_{e_G})$. $\widetilde{v}_{e_H}$ can also be extended to another left invariant vector field $\widetilde{V} : H \to TH \in \mathfrak{X}^L(H)$. The reason why v_{e_G} and $\widetilde{v}_{e_H}$ could be extended to left invariant vector field is because $T_{e_G}G \cong \mathfrak{X}^L(G)$. And we are going to passing to f-related property through left invariant vector fields.
Since f is a Lie group homomorphism,

$$f \circ L_g = L_{f(g)} \circ f$$

where both sides are smooth maps, $f \circ L_g : G \to H$ and $L_{f(g)} \circ f : G \to H$.
From the fact that taking differential at a point is a functor, we obtain

$$\mathrm{d}(f \circ L_g)_{e_G} = \mathrm{d}(L_{f(g)} \circ f)_{e_G} \Rightarrow \mathrm{d}f_g \circ \mathrm{d}(L_g)_{e_G} = \mathrm{d}(L_{f(g)})_{e_H} \circ \mathrm{d}f_{e_G}$$

Applying it to v_{e_G}, we have:

$$\mathrm{d}f_g \circ \mathrm{d}(L_g)_{e_G}(v_{e_G}) = \mathrm{d}(L_{f(g)})_{e_H} \circ \mathrm{d}f_{e_G}(v_{e_G}) \tag{3}$$

$$\Rightarrow (\mathrm{d}f_g \circ \mathrm{d}(L_g)_{e_G})(v_{e_G}) = \mathrm{d}(L_{f(g)})_{e_H}(\widetilde{v}_{e_H}) \qquad \widetilde{v}_{e_H} := \mathrm{d}f_{e_G}(v_{e_G}) \tag{4}$$

$$\Rightarrow (\mathrm{d}f_g \circ \mathrm{d}(L_g)_{e_G})(V_{e_G}) = \mathrm{d}(L_{f(g)})_{e_H}(\widetilde{V}_{e_H}) \qquad \text{by definition of } V \& \widetilde{V} \tag{5}$$

$$\Rightarrow \mathrm{d}f_g(V_g) = \widetilde{V}_{f(g)} \qquad \text{by left invariance} \tag{6}$$

$$\tag{7}$$

So, V and $\widetilde{V}$ are f-related. It follows that the following vector fields $[V_v, V_w]$ and $[V_{\mathrm{d}f_{e_G}(v)}, V_{\mathrm{d}f_{e_G}(w)}]$ are also f-related. Hence,

$$\mathrm{d}f_{e_G}\left([v, w]_{T_{e_G}G}\right) := \mathrm{d}f_{e_G}\left([V_v, V_w]_{\mathfrak{X}^L(G)}(e_G)\right) \tag{8}$$

$$= \left[V_{\mathrm{d}f_{e_G}(v)}, V_{\mathrm{d}f_{e_G}(w)}\right]_{\mathfrak{X}^L(G)}(f(e_G)) \qquad \text{by } f - \text{relatedness} \tag{9}$$

$$= \left[V_{\mathrm{d}f_{e_G}(v)}, V_{\mathrm{d}f_{e_G}(w)}\right]_{\mathfrak{X}^L(G)}(e_H) \tag{10}$$

$$= [\mathrm{d}f_{e_G}(v), \mathrm{d}f_{e_G}(w)]_{T_{e_H}H} \tag{11}$$

Hence $\mathrm{d}f_{e_G}$ is a Lie algebra homomorphism. □

1.2.2 Matrix Lie algebras

Theorem 1.3: Matrix Lie algebras are (abstract) Lie algebras

Every matrix Lie algebras is an abstract Lie algebra.

It is immediate to think matrix Lie algebras are all abstract Lie algebras, but abstract ones do not belong to matrix ones. The next theorem claims every abstract Lie algebra can be embedded into a matrix one, revealing the importance of matrix Lie algebras. This is reminiscent of the Whitney's embedding theorem that embeds finite dimensional real smooth manifolds into some $\mathbb{R}^n$ and of the Weber's theorem that embeds a finite abelian extension into a cyclotomic one.

1.3 Interchanging between Lie groups and Lie algebras

1.3.1 From Lie groups to Lie algebras

To every Lie group, we can associate a Lie algebra whose underlying vector space is the tangent space of the Lie group at the identity element which captures the local structure of the Lie group at identity. *Informally, elements of the Lie algebra of a Lie group can be thought of as the elements of a Lie group that are 'infinitesimally close' to the identity.*

1.3.2 From Lie algebras to Lie groups: Exponential map

1.4 Lie group actions

In this subsection, we would like to study the Lie groups via group action on itself. There are two well-known group actions on itself: left/right multiplication and conjugation.

1.4.1 First action: Left multiplication

The reason we study the left multiplication is way beyond that it is one of the most important actions. The left multiplication has more sophisticated and deeper applications and reasons that appealing us:
(1) Comparing to smooth manifolds, the translation map $L_g : G \to G$ by $h \mapsto gh$ yields a linear isomorphism between two tangent spaces of the same Lie group G: $T_eG \xrightarrow{\sim} T_gG$. In general, for a smooth manifold M, it is impossible to have two different tangent space T_pM and T_qM to be isomorphic. This means that the tangent space for the whole manifold G (global) is a copy of the tangent space at e (local), connecting local and global information.
(2) The left multiplication map induces the definition of left invariant vector field. All left invariant vector fields has one-one correspondence to Lie algebras of G. So, the left multiplication map provides a way to generate Lie algebras.
(3) The left multiplication map is a diffeomorphism. This means many definitions can be defined only for the local of e and then generalize to the whole Lie group.
(4) The conjugation action $\Psi_g : G \to G$ by $h \mapsto ghg^{-1}$ can be viewed as a composition of some left and right multiplication: $\Psi_g = L_g \circ R_{g^{-1}}$.

Definition 1.2: Left translation

Remark We can define right translation map similarly. Notice that left/right translation maps are smooth, because by definition of Lie groups, the multiplication map is smooth. The left(right) multiplications are special multiplication maps.

1.4.2 Left invariant vector field of a Lie group

Definition 1.3: Left invariant vector field

Let G be a Lie group. Let $V : G \to TG \in \mathfrak{X}(G)$ be a smooth vector field. $L_g : G \to G, h \mapsto gh$ is the left translation map. V is **left invariant** $\Leftrightarrow \forall g \in G$, $\forall h \in G$, $\mathrm{d}(L_g)_h(V_h) = V_{gh}$.

$$\begin{tikzcd}
G \arrow[r, "V"] \arrow[d, "L_g"'] & TG \arrow[d, "\mathrm{d}(L_g)_h"] \\
G \arrow[r, "V"] & TG
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[r, mapsto, "V"] \arrow[d, mapsto, "L_g"'] & V_h \arrow[d, mapsto, "\mathrm{d}(L_g)_h"] \\
gh \arrow[r, mapsto, "V"] & V_{gh}
\end{tikzcd}$$

1.4.3 Second action: Conjugation

Let's consider another action $G \curvearrowright G$ by fixing an element $g \in G$ and the action is defined as: $\forall h \in G$, ${}^g h := ghg^{-1}$.

Definition 1.4: Adjoint map

Let G be a Lie group. $\forall g \in G$, the adjoint map $\Psi_g : G \to G$ is $h \mapsto ghg^{-1}$.

Remark $\Psi_g = L_g \circ R_{g^{-1}}$ is a smooth map because it is the composition of two smooth maps L_g and $R_{g^{-1}}$.

Since this map is smooth, it is possible to take the differential of Ψ_g at any point in G, especially at e. So, take the differential at e, $\mathrm{d}(\Psi_g)_e : T_eG \to T_eG$ is a *linear map* since taking differential is a functor from the category of smooth manifolds to the category of finite dimensional vector spaces. Moreover, this map is a *Lie algebra homomorphism* from theorem 1.2 (So, it is automatically a linear map). For simplicity, we give the map a new symbol $\mathrm{Ad}_g := \mathrm{d}(\Psi_g)_e$.
Altering the g, we get a new map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ with $\mathfrak{g} := T_eG$. Here are two perspectives to view this map: (1) as a linear representation, (2) as a Lie group homomorphism. Since $\mathfrak{g}$ is a finite dimensional vector space[1], and Ad is a group homomorphism:

$$\mathrm{Ad}(gh) = \mathrm{Ad}_{gh} = \mathrm{d}(\Psi_{gh})_e = \mathrm{d}(\Psi_g \circ \Psi_h)_e = \mathrm{d}(\Psi_g)_{\Psi_h(e)} \circ \mathrm{d}(\Psi_h)_e = \mathrm{d}(\Psi_g)_e \circ \mathrm{d}(\Psi_h)_e = \mathrm{Ad}_g \circ \mathrm{Ad}_h$$

We have not yet had the notion of 'representation' for an 'infinite' group. But, it can be co-opted by the representation of finite groups. As we checked, Ad is a group homomorphism. So, Ad is called *linear representation* of G, more precisely the *adjoint representation* of G. From another perspective, $\mathrm{GL}(\mathfrak{g})$ is a Lie group because $\mathfrak{g} \cong \mathbb{R}^n$ as a vector space and $\mathrm{GL}(\mathbb{R}^n)$ is a Lie group. In the meanwhile, Ad is a smooth map. Plus the group homomorphism checked above, it is a *Lie group homomorphism*.

Lemma 1.1 (Ad is C^∞) *For all finite dimensional $\mathbb{R}$-vector spaces, the map $\mathrm{Ad} : G \to \mathrm{GL}(V)$ is a smooth map.*

This reminds us to take the differential of Ad again to get $\mathrm{ad} := \mathrm{d}(\mathrm{Ad})_e$: $\mathrm{ad} : T_eG \to T_I\mathrm{GL}(\mathfrak{g})$. Denoting $\mathfrak{g} := T_eG$ and $\mathfrak{gl}(\mathfrak{g}) := T_I\mathrm{GL}(\mathfrak{g})$, the map ad is

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \quad X \mapsto \mathrm{ad}(X)$$

So, the next question is: how does the $\mathrm{ad}(X)$ work? To answer this, we should simplify $\mathfrak{gl}(\mathfrak{g})$ first.
As we assumed, $\mathfrak{g}$ is still a finite dimensional $\mathbb{R}$-vector space. Since $\mathrm{GL}(\mathfrak{g})$ is not only an open subset of $\mathrm{End}(\mathfrak{g})$, but an open submanifold of $\mathrm{End}(\mathfrak{g})$, so their tangent spaces are isomorphic as $\mathbb{R}$-vector spaces:

$$T_I\mathrm{GL}(\mathfrak{g}) \cong T_I\mathrm{End}(\mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$$

Because $\mathrm{End}(\mathfrak{g})$ is a vector space, the tangent space of it is isomorphic to itself. So, we have the second isomorphism. This chain of isomorphisms helps us demystify the heavy notation $T_I\mathrm{GL}(\mathfrak{g})$. So, the map ad can be written as

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

From now on, the effect of $\mathrm{ad}(X)$ is one-more-step clearer. But, it would be totally revealed by the following lemma:

Lemma 1.1: Effect of ad

Based on the setting above, $\mathrm{ad}(X) : Y \mapsto [X, Y]$, where the bracket comes from $\mathfrak{g}$.

Proof: □

1.4.4 Lie algebra representations: First encounter

The above-mentioned conclusion can not only be applied to $\mathfrak{g}$, but also for an arbitrary real vector space V. This substitution makes the concept more general. So, we are able to define the Lie algebra representations:

Definition 1.5: Lie algebra representations

[1] Because $\dim_{\mathbb{R}} G = \dim_{\mathbb{R}} \mathfrak{g}$, and by our assumption, we study finite dimensional $\mathbb{R}$-Lie group here.

1.5 One-parameter subgroups

A very crucial tool is the so-called 'one-parameter subgroup'.

Definition 1.6: One parameter subgroup

Let G be a real Lie group. Let $\gamma : \mathbb{R} \to G$ be a Lie group homomorphism from $(\mathbb{R}, +)$ to G. Then, the image $\gamma(G)$ is the **one-parameter subgroup**.

Remark Usually, in many contexts, the one-parameter subgroup refers to the curve/homomorphism γ. Accurately, this subgroup is the image. To avoid ambiguity, I will use the quote mark " ('one-parameter subgroup') when I refer the curve.

Lemma 1.2: Existence and uniqueness of 'one-parameter subgroup'

Let G be an $\mathbb{R}$-Lie group. $\forall v \in \mathfrak{g}$, $\exists!$ 'one-parameter subgroup' $\alpha_v : \mathbb{R} \to G$, such that $\alpha_v'(0) = v$.

Proof: **Existence**:
Uniqueness: Let α be a 'one-parameter subgroup' satisfying $\alpha'(0) = v$. Define a parametric curve $f : \mathbb{R} \to G$ by $f(\tau) := \alpha(t + \tau)$ for a fixed t. Since $\alpha(t + \tau) = \alpha(t)\alpha(\tau) = L_{\alpha(t)}(\alpha(\tau))$, $f(\tau) = L_{\alpha(t)}(\alpha(\tau))$. Take the differential of f, by the chain rule,

$$f'(\tau) = (L_{\alpha(t)} \circ \alpha)'(\tau) = \mathrm{d}(L_{\alpha(t)})_{\alpha(\tau)}(\alpha'(\tau))$$

Taking $\tau = 0$, we have $\alpha'(t) = f'(0) \overset{\dagger}{=} \mathrm{d}(L_{\alpha(t)})_e(v) \overset{\ddagger}{=} V_{\alpha(t)}$, where V is a left-invariant vector field of G.
$\dagger$: by assumption $\alpha(0) = e$ and $\alpha'(0) = v$.
$\ddagger$: by definition of left invariant vector field, and $v = V_e$, so $\mathrm{d}(L_{\alpha(t)})_e(v) = \mathrm{d}(L_{\alpha(t)})_e(V_e) = V_{\alpha(t)e}$.
The fact that $\alpha(0) = e$ and $\alpha'(t) = V_{\alpha(t)}$ implies that α is an integral curve to V on G at e α is defined on the whole $\mathbb{R}$. So, the uniqueness of integral curve on the maximal interval asserts the uniqueness of α. $\square$

1.5.1 Transporting 1-parameter subgroups to another

Lemma 1.2 (Transporting 1-parameter subgroups) *Let G, H be Lie groups with corresponding Lie algebras $\mathfrak{g}, \mathfrak{h}$. Let $f : G \to H$ be a Lie group homomorphism and $\mathrm{d}f : \mathfrak{g} \to \mathfrak{h}$ be the corresponding induced Lie algebra homomorphism such that $\mathrm{d}f(v) = w$. If α_v, α_w are the 'one-parameter subgroups' of G and H respectively, then*

$$f \circ \alpha_v = \alpha_w$$

Proof: We will show that $f \circ \alpha_v$ is a 'one-parameter subgroup' of H at e_H, and then by uniqueness of 'one-parameter subgroup' from lemma 1.2, $f \circ \alpha_v = \alpha_w$. Notice that

$$(f \circ \alpha_v)'(0) = \mathrm{d}f_{\alpha_v(0)}(\alpha_v'(0)) = \mathrm{d}f_{e_G}(v) = w$$

So, $f \circ \alpha_v$ is a 'one-parameter subgroup' of H at e_H $\square$

1.5.2 Applications of 1-parameter subgroups

This gives another language
(1) Another proof of theorem 1.2: In theorem 1.2, we actually use the isomorphism $T_eG \cong \mathfrak{X}^L(G)$. Now, we have a different way to view the Lie bracket on T_eG, and the new proof here mainly manipulates the bracket.

1.6 Exponential maps

Definition 1.7: (general) Exponential map

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. The exponential map is the map

$$\exp^{a} : \mathfrak{g} \to G \quad v \mapsto \alpha_v(1)$$

where $\alpha_v : \mathbb{R} \to G$ is the unique 'one-parameter subgroup' whose tangent vector at 0 is v.

[a] To stress the Lie group G, we use the notation $\exp_G$. When the context is clear, the subscript G can be omitted.

Theorem 1.4: Properties of exponential maps

Let $G, H, \mathfrak{g}, \mathfrak{h}$, $f : G \to H$ and $\mathrm{d}f : \mathfrak{g} \to \mathfrak{h}$ have the same setting as above. The exponential map satisfies the following properties:
(1) $\exp_G(0) = e_G$
(2) $\forall t \in \mathbb{R}$, $\forall v \in \mathfrak{g}$, $\exp_G(tv) = \alpha_{tv}(1) = \alpha_v(t)$, where α_v is the 'one-parameter subgroup' associated to v.
(3) The exponential map is funtorial in G. More precisely, the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{d}f"] \arrow[d, "\exp_G"'] & \mathfrak{h} \arrow[d, "\exp_H"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

Proof: (1)
(2) $\exp_g(tv) = \alpha_{tv}(1)$ is from definition. Consider $f(s) := \alpha_{tv}(s)$ and $g(s) := \alpha_v(st)$. Since $f'(0) = g'(0) = tv$, they are both 'one-parameter subgroups' of the same 'velocity' at 0. By the uniqueness from lemma 1.2, $f = g$. Take $s = 1$ to get the desired form.
(3) $\forall v \in \mathfrak{g}$,

$$(f \circ \exp_G)(v) \overset{(2)}{=} (f \circ \alpha_v)(1) \overset{\text{Lemma1.2}}{=} \alpha_{\mathrm{d}(f)(v)}(1) \overset{(2)}{=} \exp_H(\mathrm{d}(f)(v)) = (\exp_H \circ \mathrm{d}(f))(v)$$

$\square$

Corollary 1.1: Induced Lie group homomorphism

Let G, H be $\mathbb{R}$-Lie groups and $\mathfrak{g}, \mathfrak{h}$ be their corresponding Lie algebras. If G is connected, then the Lie group homomorphism $f : G \to H$ is uniquely determined by the Lie algebra homomorphism $\mathrm{d}f_e : \mathfrak{g} \to \mathfrak{h}$.

Proof: $\exp_G$ is a local diffeomorphism, meaning that $\exists$ open neighbourhood U of e_G, and $\exists$ open neighbourhood V of $0 \in \mathfrak{g}$, such that $\exp_g|_U$ is a diffeomorphism. From the functoriality in theorem 1.4, f can be locally determined by

$$f|_U = \exp_H \circ \mathrm{d}f \circ \exp_G|_U$$

Now we want to extend this map to the whole G. Let' s first consider the subgroup of G generated by U, $K := \langle U \rangle$. This K is open in G. Let $W = U \cup U^{-1}$. $W_1 = W$, $W_2 = WW$, and so on. So, we notice the following facts:
(1) U^{-1} is open. Because inverse map is a homeomorphism.
(2) W is open.
(3) $\forall i$, W_i is open. Base case is (2). Suppose this is true for W_k, then

$$W_{k+1} = W_k W = \bigcup_{g \in W_k} gW = \bigcup_{g \in W_k} L_g(W)$$

where L_g is a homeomorphism. Hence, W_{k+1} is a union of open sets.
Then, $H = \bigcup_{i \in \mathbb{N}} W_i$ implies that H is open. $\supseteq$ is quick. To see $H \subseteq \bigcup_i W_i$, this is because $\forall s \in H$ is of the form(reduced form) $s_1 s_2 \ldots s_n$, with each $s_i \in W$. Therefore, $s \in W_n$.
On the other hand, all the cosets of K in G except K, gK, are open in G, because of the same left translation argument: $gK = L_g(K)$ and L_g is a homeomorphism. So, K is a closed set. Since G is connected, $H = G$. $\square$ Let G be a Lie group. If G is abelian, then $[\cdot, \cdot] = 0$. There is a converse version of lemma for the connected case:

Corollary 1.2

1.6.1 Applications of exponential maps

Before going on, let's give a regulation of *abstract subgroup*. Suppose G is a Lie group. H is called an *abstract subgroup* of G if H is a smooth manifold and it is a subgroup of G(in the group-theoretical sense). Such an H is not necessarily a Lie subgroup of G, because there is no control of the compactability of the smooth structures of both G and H.

Theorem 1.5: Closed subgroup theorem(E. Cartan)

Let G be an $\mathbb{R}$-Lie group and H be an *abstract subgroup* of G. H is a submanifold of $G \Leftrightarrow H$ is closed in G.

Proof: $\square$

1.7 Connections between Lie groups and Lie algebras

1.7.1 General picture

What is the relation between a Lie group and a Lie algebra? The next theorem will show that given a Lie group, the tangent space at the identity of G is the unique Lie algebra corresponds to G. Conversely, given a Lie algebra, is there a unique Lie group G that corresponds to this Lie algebra? The answer is no. But, it could be yes if we just slightly restrict the condition.

$$\begin{tikzcd}
G \arrow[d, equal] & \mathfrak{g} \arrow[l, rightsquigarrow, "\exists! G, T_eG=\mathfrak{g}\ ?"'] \arrow[d, equal] \\
\text{Lie group} \arrow[d, equal] & \text{Lie algebra} \arrow[d, equal] \\
G \arrow[r, rightsquigarrow, "e"] & T_eG
\end{tikzcd}$$

The counter-examples for 'unique Lie group corresponds to the given Lie algebra' are:
(1) $\mathbb{R}$ and $\mathbb{S}^1$. Both of them have the same Lie algebra $\mathbb{R}$.
(2) SO_3 and SU_2.
(3) $\mathbb{R}_{>0}$ and $\mathbb{R}$.
(4) O_2 and SO_2.
Notice that the entire topology for both are different. One is simply-connected but another is not. So, it is reasonable to put the condition 'simply-connected' to Lie groups to make the inverse question to be true. So, we have the following correspondence.

Theorem 1.6: Lie algebra and Left invariant vector fields

The map $\mathfrak{X}^L(G) \to T_eG$ is a linear isomorphism.

We could join the content of abstract Lie groups, matrix Lie groups, abstract Lie algebras and matrix Lie algebra into one diagram:

$$\begin{tikzcd}
\text{matrix Lie groups} \arrow[r, "\text{Theorem}"] \arrow[d, "\text{Theorem1.1}"', "\subseteq"] & \text{Matrix Lie algebras} \arrow[d, "\subseteq"', "\text{Theorem1.3}"] \\
\text{(abstract) Lie groups} \arrow[r, "\text{Theorem1.6}"'] & \text{(abstract) Lie algebras}
\end{tikzcd}$$

1.7.2 Categorical Lie group-Lie algebra correspondence

The category of simply-connected and connected Lie groups is equivalent to the category of real finite dimensional Lie algebras.

Theorem 1.7

Let G be a connected abelian Lie group. Then:
(1) $\exp : \mathfrak{g} \to G$ is a Lie group homomorphism.
(2) $G \cong \mathfrak{g}/\ker\exp$
(3) $\exists$ basis $\gamma_1, \ldots, \gamma_n$ for $\mathfrak{g}$, $\exists k \in \{0, \ldots, n\}$, such that $\Gamma := \{\sum_i n_i\gamma_i|, n_i \in \mathbb{Z}\} =_{\mathbb{Z}} (\gamma_1, \ldots, \gamma_k)$. (4) $G \cong U(1)^k \times \mathbb{R}^{n-k}$
(5) $\exists$ a natural isomorphism $\ker(\exp) \cong \mathrm{Hom}(U(1), G)$ and $\pi_1(G) \cong \Gamma$. (6) If G is compact, then $G \cong U(1)^n$.

2 Compact Lie groups

Basic properties of compact Lie groups

Theorem 2.1

2.1 Representations on compact Lie groups

Let G be a $\mathbb{R}$-Lie group(might not be compact). If this group is compact, there is a volume-1 Haar measure on G. However, if this group G is not compact, then it is not able to define a Haar measure. How should we study the representation of G in this case? Recall the dictionary between connected Lie groups and Lie algebras. If we can find a compact Lie group G, such that $\mathfrak{g}'_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{C}}$ as an isomorphism between $\mathbb{C}$-Lie algebras, then the representation of G, $\rho : G \to \mathrm{GL}(V)$ can be induced from the representation of compact Lie group G', $\rho' : G' \to \mathrm{GL}(V)$, which we have more knowledge and control on the representation. This is illustrated by the following diagram:

$$\begin{tikzcd}
& \text{connected Lie group } G \arrow[dl, "G \text{ is compact}"'] \arrow[dr, "G \text{ is not compact}"] & & \\
\text{Haar measure} & & \text{Lie algebra } \mathfrak{g} \arrow[d] & \\
& & \text{complexification } \mathfrak{g}_{\mathbb{C}} & \text{a compact } G \text{ having } \mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g}'_{\mathbb{C}} \arrow[l, squiggly]
\end{tikzcd}$$

2.1.1 Integration on compact Lie groups