Algebraic Geometry

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March 2025

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1 Varieties and their morphisms

1.1 Affine space and affine variety

1.1.1 Basic notions

Definition 1.1: Affine algebraic sets

An **(affine) algebraic set** X is a vanishing set for some collection of polynomials, i.e. a subset $X \subseteq \mathbb{A}^n$ such that $X = V(S)$ for some $S \subseteq k[x_1, \ldots, x_n]$.

Proposition 1.1: Algebraic sets forming a topology

The union of two affine algebraic sets is still an affine algebraic sets. The intersection of any family of affine algebraic sets is still an affine algebraic set. $\varnothing$ and $\mathbb{A}^n(k)$ are affine algebraic sets.

Definition 1.2: Zariski topology for $\mathbb{A}^n$

Define the affine algebraic sets to be the closed set of $\mathbb{A}^n$. This topology is called the Zariski topology of $\mathbb{A}^n$.

From the proposition 1.1, this is indeed an topology. Another aspect is the collection of polynomials vanish on X, denoted $I(Y)$,

$$I(Y) := \{f \in k[x_1, \ldots, x_n] : \forall P \in Y, f(P) = 0\}$$

The remaining part is to study the relations between such an ideal and the vanishing sets. The properties of vanishing sets and ideals, with their interaction are given in the following theorem.

Proposition 1.2: Properties of ideals and vanishing sets

Let $S_1, S_2 \subseteq k[x_1, \ldots, x_n]$ and $X_1, X_2 \subseteq \mathbb{A}^n(k)$. Then,
(1) $S_1 \subseteq S_2 \Rightarrow V(S_2) \subseteq V(S_1)$.
(2) $X_1 \subseteq X_2 \Rightarrow I(X_2) \subseteq I(X_1)$.
(3) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
(4) $\forall S \subseteq k[x_1, \ldots, x_n]$, $S \subseteq I(V(S))$ and $V(S) = V(I(V(S)))$. In particular, if A is an affine algebraic set, then $A = V(I(A))$.
(5) $X \subseteq \mathbb{A}^n(k)$ as a subset, $X \subseteq V(I(X))$ and $I(X) = I(V(I(X)))$. In particular, if J is the ideal of some affine algebraic set, then $J = I(V(J))$.
(6) $\forall \mathfrak{a} \trianglelefteq k[x_1, \ldots, x_n]$, $\sqrt{\mathfrak{a}} = I(V(\mathfrak{a}))$. If J is the ideal of some affine algebraic set, then J is a radical ideal.
(7) $\forall X \subseteq \mathbb{A}^n(k)$, $\overline{X} = V(I(X))$.

Proof: (1) and (2) follows quickly from the definition.
For (4),
For (5), because $X \subseteq V(I(X))$ and because $V(I(X))$ is a closed set containing X, $\overline{X} \subseteq V(I(X))$. To show the other direction, pick any closed set W of $\mathbb{A}^n(k)$ containing X. We want $V(I(X)) \subseteq W$ for every such closed set W. As a closed set, $W = V(\mathfrak{a})$ for some $\mathfrak{a}$. $X \subseteq W = V(\mathfrak{a}) \Rightarrow \mathfrak{a} \subseteq I(V(\mathfrak{a})) \subseteq I(X)$. Applying V to this inclusion again, $V(I(X)) \subseteq V(\mathfrak{a}) = W$. Because $V(I(X)) \subseteq W$ for every closed W containing X.

$$V(I(X)) \subseteq \bigcap_{\substack{X \subseteq W \\ W \text{ is closed}}} W = \overline{X}$$

□

1.1.2 Irreducible components: affine varieties

An affine algebraic set might be the union of several smaller algebraic sets. Seeking the decomposition of an affine algebraic sets seems necessary. To decompose, the irreducible in the sense of topological space is at our disposal.

Two things are about to be mentioned: (1) How to tell an affine algebraic set is irreducible. (2) property of writing an affine algebraic set into irreducible ones.
Here is a characterisation of irreducible affine algebraic sets.

Proposition 1.3: Algebraic characterisation of irreducibility

An affine algebraic set $X \subseteq \mathbb{A}^n(k)$ is irreducible $\Leftrightarrow$ $I(X)$ is prime.

Proof: $\Rightarrow$ Suppose that $I(X)$ is not prime. Then, $\exists f, g,\ fg \in I(X)$ but $f, g \notin I(X)$. These imply that $X = V(I(X)) \subseteq V(fg)$ and $X \not\subseteq V(f), V(g)$. So, there is a decomposition $X = (X \cap V(f)) \cup (X \cap V(g))$ with $X \cap V(f), X \cap V(g) \subsetneqq X$.
$\Leftarrow$ If X is reducible with the decomposition $X = X_1 \cup X_2$. $X_i \subsetneqq X$ implies that $I(X) \subsetneqq I(X_i)$. For each i, let f_i be the element that $f_i \in I(X_i)$ but $f_i \notin I(X)$. But $f_1 f_2 \in I(X_1 \cup X_2) = I(X) \Rightarrow I(X)$ is not prime. $\square$

Next, the decomposition of an affine algebraic set only yields finitely many irreducible components. Suppose X is reducible, then $X = X_1 \cup X_2$. If any one of X_i is reducible, then X_i can be further decomposed. This process eventually terminates because: $k[x_1, \ldots, x_n]$ is a Noetherian ring for any field. In a Noetherian ring, any collection of ideals has the maximal element. The collection of affine algebraic sets $\{V_i\}$ are sets that appear during the decomposition. It has a counterpart $\{I(V_i)\}_i$ which is a collection of ideals of $k[x_1, \ldots, x_n]$. So, there is a maximal one $I(V_\alpha)$ for some α. Correspondingly, V_α is the minimal one, indicating the terminal of decomposition.

Uniqueness of decomposition

The building blocks of some object are the fundamental and simpler ones to study. For affine algebraic sets, their building blocks are those irreducible components. So, it is worth picking them out and assigning it a name:

Definition 1.3: Affine and quasi-affine variety

An **affine variety** $X \subseteq \mathbb{A}^n$ is an irreducible affine algebraic set.
A **quasi-affine** variety is an open subset of an affine variety.

$$\begin{tikzcd} \text{Affine algebraic sets} \arrow[d, "\text{irreducible components}"'] \\ \text{Affine varieties} \end{tikzcd} \qquad \begin{tikzcd} & X = \bigcup_j X_j \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ X_1 & \cdots & X_i & \cdots \end{tikzcd}$$

1.2 Projective spaces and projective varieties

The projective n-space is defined to be the quotient $\mathbb{A}^{n+1} - \{\mathbf{0}\}/_\sim$ where $\sim$ is the equivalence relation defined by $(a_0, a_1, \ldots, a_n) \sim (\lambda a_0, \lambda a_1, \ldots, \lambda a_n)$ for all $\lambda \in k - \{0\}$. Any representative of a point $P \in \mathbb{P}^n$ is called a set of homogeneous coordinate of P. Such a P is written explicitly as $P = [a_0 : a_1 : \cdots : a_n]$.
Then, for an $f \in k[x_1, x_2, \ldots, x_n]$, a function $f : \mathbb{P}^n \to k$ cannot be defined normally.
• Suppose $\deg f = d$, a natural way to define f is:

$$f([a_0 : a_1 : \cdots : a_n]) = f(a_0, a_1, \ldots, a_n)$$

But for another representative of $[a_0 : \cdots : a_n]$, $[\lambda a_0 : \cdots : \lambda a_n]$, the value of f is not invariant under different choice of representatives.
$(*)$ However, notice that being 0 is a property that is independent of choices of $[a_0 : \cdots : a_n]$ [1] if and only if f is a homogeneous polynomial: $\Leftarrow$ Suppose f is homogeneous. When $f([a_0 : \cdots : a_n]) = f(a_0, \ldots, a_n) = 0$, $\forall \lambda \in k - \{0\}$, $f([\lambda a_0 : \cdots : \lambda a_n]) = \lambda^d f(a_0, \ldots, a_n) = 0$.

$\Rightarrow$ Suppose that f is not homogeneous and $\deg f = d$, then f can be decomposed into sum of homogeneous polynomials: $f = \sum_{i=0}^{d} f_i$, with each f_i a homogeneous polynomial of degree i and at least two different terms are not zero. For some $P \in \mathbb{P}^n$, and one representative $P = [c_0 : \cdots : c_n]$, $f(c) = f(c_0, \ldots, c_n) = 0$. Here $c = (c_0, \ldots, c_n)$. If $f(P) = 0$ is well-defined, $f(\lambda c) = 0$ should work for all λ. We are going to see that $f(\lambda c_0, \ldots, \lambda c_n) \neq 0$ for infinitely many λ. Hence, $f(P) = 0$ is not well-defined. $f(\lambda c)$ can be viewed as a polynomial in λ, $G(\lambda)$:

$$G(\lambda) := f(\lambda c) = f_d(c)\lambda^d + f_{d-1}(\lambda c)\lambda^{d-1} + \cdots + f_1(c)\lambda + f_0(c)$$

[1] meaning $f(P) = 0$ is well-defined

As assumed, k is an algebraically closed field of char$k = 0$. k is an infinite field. But $G(\lambda)$ has at most d roots. So, there are infinitely many elements that make $f(\lambda c) \neq 0$.

• So, for f being homogeneous, a possible way to define function on $\mathbb{P}^n$ is

$$f([a_0 : \dots, a_n]) := \begin{cases} 0 & \text{if it is zero for some representative of } [a_0 : \cdots : a_n] \\ 1 & \text{else} \end{cases}$$

So far, we have got a function $f : \mathbb{P}^n \to \{0, 1\}$.

The vanishing sets and ideals are similar to those in affine case:

Definition 1.4: Projective algebraic sets

A **(projective) algebraic set** $X \subseteq \mathbb{P}^n$ is a vanishing set of some collection of homogeneous polynomials $S \subseteq k[x_1, \dots, x_n]$, i.e. $X = V(S)$.

Remark To make this definition play well, S could not be an arbitrary subset of $k[x_1, \dots, x_n]$. More explicitly, $V(S) = \{P \in \mathbb{P}^n : \forall f \in S, f(P) = 0\}$. The condition $f(P) = 0$ occurs. To make $f(P) = 0$ well-defined, the above-mentioned equivalence $(*)$ suggests that f should be homogeneous.

Proposition 1.4: Projective algebraic sets forming a topology

The union of two projective algebraic sets is still an projective algebraic sets. The intersection of any family of projective algebraic sets is still an projective algebraic set. $\varnothing$ and $\mathbb{P}^n(k)$ are projective algebraic sets.

Definition 1.5: Zariski topology for $\mathbb{P}^n$

Define the projective algebraic sets to be the closed set of $\mathbb{P}^n$. This topology is called the Zariski topology of $\mathbb{A}^n$.

From the proposition 1.1, this is indeed an topology.

1.2.1 Irreducible components: projective variety

Definition 1.6: Projective and quasi-projective variety

A **projective variety** $X \subseteq \mathbb{A}^n$ is an irreducible projective algebraic set.
A **quasi-projective** variety is an open subset of a projective variety.

Example (1) Grassmannian

1.2.2 Intertwining of affine and projective spaces

In this little section, the goal is to see that $\mathbb{P}^n$ has an open covering by $\mathbb{A}^n$. Whence, every projective variety has an open cover by affine varieties.

1.3 Correspondence: Hilbert Nullstellensatz

1.3.1 Affine Nullstellensatz

1.3.2 Projective Nullstellensatz

1.4 Dictionary of algebra and geometry

$$\begin{aligned} \text{irreducible closed sets} &\longleftrightarrow \text{prime ideals} \\ \text{points} &\longleftrightarrow \text{maximal ideals} \\ \text{closed subsets} &\longleftrightarrow \text{radical ideals} \end{aligned}$$

In this Correspondence, points are the 'minimal' irreducible closed sets. This coincides with the fact that maixmal ideals are prime.

1.5 Morphisms: Regular and Rational functions

1.5.1 Regular functions on affine quasi-affine varieties

Let X be a quasi-affine variety of $\mathbb{A}^n(k)$. Consider $\tilde{f} : X \to k$.

Definition 1.7: Regular functions on (quasi) affine varieties

A function $\tilde{f} : X \to k$ is **regular** at $P \in X \Leftrightarrow \exists$ open neighbourhood $U(\ni P) \subseteq X$, and $g, h \in k[x_1, \ldots, x_n]$, with $h \neq 0$ on U such that $\tilde{f} = \frac{g}{h}$ on U.
Such an $\tilde{f}$ is **regular** on $X \Leftrightarrow \tilde{f}$ is regular at every $P \in X$.

Lemma 1.1: Continuity of regular functions on (quasi) affine variety

Let $\tilde{f} : X \to \mathbb{A}^1(k)$ be a regular function on quasi-affine variety $X \subset \mathbb{A}^n(k)$. $\tilde{f}$ is continuous with respect to Zariski topology.

Proof: It suffices to show that $\forall$ closed subset W of $\mathbb{A}^1(k)$, $f^{-1}(W)$ is closed. But every closed subset of $\mathbb{A}^1(k)$ is a finite set. [2] So, it suffices to show that $\forall a \in k$, $f^{-1}(\{a\})$ is closed. This is further equivalent to the intersection of $f^{-1}(\{a\})$ and every open set in *an* open cover of X [3] is closed in X. In particular, $\forall P \in X$, let U_P be the neighbourhood of P where $f = \frac{g}{h}$ and $h \neq 0$ on U_P. Then,

$$f^{-1}(\{a\}) \cap U_P = \{P \in U_P : \frac{g(P)}{h(P)} = a\} = \{P \in U_P : (g - ah)(P) = 0\} = V(g - ah) \cap U_P$$

Hence, $f^{-1}(\{a\}) \cap U_P$ is closed in U_P.

□

1.5.2 Regular functions on affine quasi-affine varieties

For a quasi-projective variety $Y \subseteq \mathbb{P}^n$, there is a similar definition for regular maps.

Definition 1.8: Regular functions on (quasi) projective varieties

A function $\tilde{f} : Y \to k$ is **regular** at $P \in Y \Leftrightarrow \exists$ open neighbourhood $U(\ni P) \subseteq Y$, and $\exists$ *homogeneous* $g, h \in k[x_1, \ldots, x_n]$ *of the same degree*, with $h \neq 0$ on U such that $\tilde{f} = \frac{g}{h}$ on U.
Such an $\tilde{f}$ is **regular** on $Y \Leftrightarrow \tilde{f}$ is regular at every $P \in Y$.

Remark Even though $\tilde{g}, \tilde{h}$ are not 'normal' functions on Y, there quotient is when $h \neq 0$. Since every affine variety can be embedded into a projective variety. So is a quasi-affine variety. This inspire us that definition 1.8 should include 1.7 as its special cases. How does this definition-inclusion concretely work? By homogenization and dehomogenization!
Consider the inclusion $X \hookrightarrow \mathbb{A}^n(k) \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ where $\iota : (a_1, \ldots, a_n) \mapsto [1, a_1, \cdots, a_n]$. The f correpsonding to the function $\tilde{f} : X \to k$ can be homogenized to f_H

$$f_H(x_0, x_1, \ldots, x_n) = x_0^{\deg f} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

such that $\tilde{f_H}([1, a_1, \ldots, a_n]) = f_H(1, a_1, \ldots, a_n) = f(a_1, \ldots, a_n) = \tilde{f}((a_1, \ldots, a_n))$.

1.5.3 Rational maps and classification

As the fraction of a regular function over an affine variety or an affine set, rational functions consists of a field which encodes geometry of that affine variety into algebraic information. As a consequence, birational equivalence shows up to classify varieties.

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[2] Because a closed set is $V(S)$ for some $S \subseteq k[x]$. Combining the fact that $V(S) = V(\langle S \rangle)$ and $k[x]$ is a PID, $\langle S \rangle = \langle r \rangle$ for some $r \in k[x]$. r is a polynomial and it has $\deg r$ many (finitely many) zeros. Hence, $V(S) = V(r)$ is a finite set.
Another way to see is: Since every affine closed set is a finite union of irreducible closed sets. Irreducible closed sets of $\mathbb{A}^1(k)$ are of the form $V(x - a)$ and are singletons.

[3] Precisely, this equivalency is $Z \subseteq Y$ is closed $\Leftrightarrow \exists$ an open cover $\mathcal{U} = \{U_i\}_i$ of $Y, \forall i$, $U_i \cap Y$ is closed in U_i

Definition 1.9: Rational functions

A ration function on X is a element $\frac{f}{g}$, where $f, g \in k[X]$ are regular maps.

not well-defined for f/g

Rational maps for closed subset $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$: $\varphi : X \to Y \Leftrightarrow \varphi = (\frac{g_1}{h_1}, \dots, \frac{g_m}{h_m})$.
Rational maps for closed subset $X \subset \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$:

Definition 1.10: Rational maps

Let X, Y be two varieties. A rational map is an equivalence o

rational functions $f \in k(X)$ rational maps   rational map on varieties

1.6 Category of variety

1.7 Dimensions

1.8 New language: Spectrum of a ring

Using the 'dictionary', the vanishing set $V(f) = \{x \in \mathbb{A}^n(k) : f(x) = 0\}$ has a generalised version adapted to the Spec$k[x_1, \dots, x_n]$, $V(f) = \{[\mathfrak{p}] \in \text{Spec}k[x_1, \dots, x_n] : \langle f \rangle \subseteq \mathfrak{p}\}$ Are these generalised 'vanishing sets' still forming a topology by serving as closed sets? The answer is affirmative.
Open sets in Spec Distinguished open sets.

Proposition 1.5: Properties of distinguished open sets

Let R be a commutative ring and $f, g \in R$ being non-zero. Then the followings are equivalent:
(1) $D(f) \subseteq D(g)$ (2) $\exists n \geq 1,\ f^n \in \langle g \rangle$ (3) g is invertible in R_f[a]

[a] R_f is the localization of R at the multiplicative set $\{1, f, f^2, \dots\}$

Proof: (1) $\Rightarrow$ (2) $D(f) \subseteq D(g) \Leftrightarrow V(g) \subseteq V(f)$. Explicitly, $\forall$ prime ideal $\mathfrak{p}$ such that $\langle g \rangle \subseteq \mathfrak{p}$, $\langle f \rangle \subseteq \mathfrak{p}$. So, f is in $\bigcap_{\substack{[\mathfrak{p}] \in \text{Spec}R \\ g \in \mathfrak{p}}} \mathfrak{p} = \sqrt{\langle g \rangle}$, meaning that $\exists n \geq 1,\ f^n \in \langle g \rangle$.
(2) $\Rightarrow$ (1) If $f^n \in \langle g \rangle$ for some $n \geq 1$, then $\exists a \in R,\ f^n = ag$. $\forall$ prime ideal $\mathfrak{p}$ with $g \in \mathfrak{p}$, $f^n = ag \in \mathfrak{p}$. Primity of $\mathfrak{p}$ yields $f \in \mathfrak{p}$, showing $V(g) \subseteq V(f)$.
(2) $\Rightarrow$ (3) $f^n = ag$ for some $a \in R$, and $\frac{a}{f^n} \in R_f$.
(3) $\Rightarrow$ (2) If g is invertible, $\exists h \in R_f$ of the form $h = \frac{a}{f^m}$ such that $gh = \frac{ga}{f^m} = 1 = \frac{1}{1} \Leftrightarrow \exists r \in \{1, f, \dots\}$(so $R = f^k$ for some $k \geq 0$), $(ga - f^m) \cdot r = gaf^k - f^{m+k} = 0$. Let $n = m + k$, $f^n = (af^m)g \in \langle g \rangle$. □

2 Generalised language: scheme

3 Local properties

3.1 Singular and Non-singular points

3.2 Tangent spaces