Math 518 Assignment 3

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Contents

1 Solutions

1. $\forall i$, $H_i = \mathbb{P}^n(k) \backslash U_i$. Hence,

$$\bigcap_{i=1}^{m} H_i = \mathbb{P}^n(k) \Big\backslash \bigcup_{i=1}^{m} U_i$$

Notice that $\bigcup_{i=1}^{m} U_i = \{[x_1, x_2, \ldots, x_{n+1}] : x_1 \neq 0 \text{ or } \ldots \text{ or } x_m \neq 0\}$. Hence,

$$\mathbb{P}^n(k) \Big\backslash \bigcup_{i=1}^{m} U_i = \{[0, 0, \ldots, 0, x_{m+1}, \ldots, x_{n+1}] : x_{m+1} \neq 0 \text{ or } \ldots \text{ or } x_{n+1} \neq 0\}$$

Now, for every $m + 1 \leq j \leq n + 1$, let $V_j := \{[0, 0, \ldots, 0, x_{m+1}, \ldots, x_{n+1}] : x_j \neq 0\}$. Then,

$$\mathbb{P}^n(k) \Big\backslash \bigcup_{i=1}^{m} U_i = \bigcup_{j=m+1}^{n+1} V_j$$

The condition $m \leq n$ implies that at least one such V_j exists. Each $V_j \neq \varnothing$. Hence, $\mathbb{P}^n(k) \Big\backslash \bigcup_{i=1}^{m} U_i \neq \varnothing$ implying that $\bigcap_{i=1}^{m} H_i \neq \varnothing$.

2. By definition, $I(V)^* = \{f^* | f \in I(V)\} = \{f^* | f \in \langle f_1, f_2 \rangle\}$. Since $z - xy = z - x^3 - x \cdot (y - x^2) = f_2 - x f_1 \in I(V)$, its homogenization $zw - xy = (z - xy)^* \in I(V)^*$. On the other hand, $\langle f_1^*, f_2^* \rangle = \langle yw - x^2, zw^2 - x^3 \rangle$. $zw - xy$ is not in $\langle f_1, f_2 \rangle$ because the least degree of polynomials in this ideal is 2, and $yw - x^2$ is not associate with $zw - xy$.

3. If $W = V$, done. Now, suppose that $V \subsetneqq W$ and suppose $V \supsetneqq V_1 \subsetneqq \cdots \supsetneqq V_n$ is the chain whose length reaches the $\dim V(f)$. Then, $W \supsetneqq V \supsetneqq V_1 \subsetneqq \cdots \supsetneqq V_n$ is another chain whose length is the supremum over all such chains of W. Then, $\dim W = \dim V + 1$. Since f is an irreducible polynomial, $V = 1$. Hence, $\dim W = \dim \mathbb{P}^n(k)$. $W \subseteq \mathbb{P}^n(k)$ implies that $W = \mathbb{P}^n(k)$ in this case.
For H_∞, notice that it could be determined by an irreducible homogeneous polynomial $f := x_{n+1}$. In this sense, $H_\infty = V(f) = V(x_{n+1})$. By assumption, W contains H_∞. The first fact tells that either $W = H_\infty$ or $W = \mathbb{P}^n(k)$.
• When $W = \mathbb{P}^n(k)$, $W_* = W \cap U_{n+1} = U_{n+1} \cong \mathbb{A}^n(k)$. For $(W_*)^*$, $(W_*)^* = (\mathbb{A}^n(k))^* = \mathbb{P}^n(k)$.
• When $W = H_\infty$, $W_* = H_\infty \cap U_{n+1}$. H_∞ contains all elements $[x_1 : \cdots : x_n : 0]$, while all elements in U_{n+1} must have $x_{n+1} \neq 0$. So, $W_* = \varnothing$. For $(W_*)^*$, $(W_*)^* = \varnothing^* = V(I(\varnothing)^*)$. Notice that $I(\varnothing) = k[x_1, \ldots, x_n]$. It is homogenization is all the polynomials $k[x_1, \ldots, x_{n+1}]$. So, its vanishing set is empty.

4. This can be classified by the height of homogeneous prime ideals $\operatorname{ht}(P)$. Then, $\operatorname{ht}(P) = \operatorname{codim}(V(P))$
• For $\mathbb{P}^1(k)$, when $\operatorname{ht}(P) = 0$, $P = 0$. $V(P) = \mathbb{P}^1(k)$. When the height equals 1, the codimension of $V(P)$ is 1. $V(P) = V(f)$ for some irreducible polynomial. Since k is algebraically closed, $V(f)$ only has finite solutions. When the height of some homogeneous ideal P corresponds to a variety equals 2, the codimension of this variety is 0. The variety is $\varnothing$ in this case. So, all possibilities are $\mathbb{P}^1(k)$, finite sets and $\varnothing$.
• For $\mathbb{P}^2(k)$, the cases are similar. All varieties are $\mathbb{P}^2(k)$, hyperplanes correponding to an irreducible homogeneous polynomial, single points and $\varnothing$.