Algebraic Topology

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March 2025

Contents

1 Complexes

1.1 Simplicial complex

1.2 $\triangle$-complex

1.3 CW complex

CW complex is a generalization of both manifold and simplicial complex. How to see this? Instead of gluing simplexes, it glues topological balls in different dimensions.

2 Homotopy

Goals: (1)Distinguish concepts, tell their differences, (2) Be clear with examples under each concepts, (3)Take it as my own lar

2.1 Basic definition

$$\begin{tikzcd}
\text{Homotopy equivalence } X \simeq Y \arrow[r, "Y=\bullet"] \arrow[dr, "Y\subseteq X"] \arrow[ddr, "\text{special}"'] & \text{Contractible} \\
& \text{Retraction } r, r\circ\iota = \mathrm{id}_Y \\
& \text{Deformation retraction } r \simeq \mathrm{id}_Y \\
& \text{Strong deformation retraction}
\end{tikzcd}$$

Definition 2.1: Homotopy

Let X, Y be two topological spaces and $f_\nu : X \to Y$, $\nu = 0, 1$ be two continuous maps. f_0, f_1 are **homotopic**, denoted $f_0 \simeq f_1 \Leftrightarrow \exists$ a continuous map

$$F : X \times [0,1] \to Y$$

such that $\forall \nu$, $\forall x \in X$, $F(x, \nu) = f_\nu(x)$. In this case, F is called a **homotopy** from f_0 to f_1.

Homotopy is an equivalence relation. • Reflexivity: choose F such that $\forall \nu$, $F(x, \nu) = f(x)$.
• Symmetry: $f_0 \simeq f_1$ determines an F, choose $\tilde{F}$ to satisfy $\tilde{F}(x, \nu) = F(x, 1 - \nu)$, which is a homotopy making $f_1 \simeq f_0$.
• Transitivity: if $f \simeq g$ and $\tilde{f} \simeq \tilde{g}$ with $f, g : X \to Y$ and $\tilde{f}, \tilde{g} : Y \to Z$ being continuous maps, determining homotopy F and $\tilde{F}$ respectively. Then,

Definition 2.2: Homotopy equivalence

2.1.1 Variations of homotopy equivalence

Let X be a topological space and $Y \subset X$ be its subspace. Intuitively speaking [**Borsuk1931**], a retraction of X is a continuous mapping $X \to Y$ that **preserves the position of all points in** Y. A deformation retraction is a mapping that

Definition 2.3: Deformation retraction

Let X be a topological space and $Y \subseteq X$ be its subspace. Let $r : X \to Y$ be the retraction of Y in X.
Y is a deformation retraction of $X \Leftrightarrow r$ is homotopic to id_X

Remark $r \simeq \mathrm{id}_Y$

2.1.2 Homotopy groups

2.2 Homotopy equivalence and homeomorphism

Homeomorphism is a special type of homotopy equivalence. In homeomorphism, if $f : X \to Y$ is a homeomorphism and $g : Y \to X$ is its inverse, then $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ (here they are 'equal to',=, not only 'homotopy equivalent to',$\simeq$)

2.3 Homotopy invariant properties

Let X and Y be two topological spaces such that they are homotopic equivalent, i.e. $X \simeq Y$.
• Path connectivity
• Simply-connectedness
• (Singular) homology and cohomology group. The (singular) homology and cohomology groups of X and Y are isomorphic. $H_i(X) \cong H_i(Y)$ and $H^i(X) \cong H^i(Y)$
• Homotopy groups: the homotopy group of X and Y (both fundamental and higher) are isomorphic. i.e. if there is an homotopy equivalence $f : X \to Y$, then $\forall n$,

$$\pi_n(X, x_0) \cong \pi_n(Y, f(x_0))$$

2.3.1 Homotopy as a category

The topological spaces and being homotopic between them is a reminiscent of a new category: is there a category whose objects are topological spaces and whose morphisms are homotopy between two topological spaces? The answer is no, because if $f : X \to Y$ and $g : Y \to Z$ are two homotopy map, then there is no guarantee that $g \circ f$ is a homotopy.
However, if the morphisms are homotopic equivalence classes between two topological spaces $[f] := \{g : X \to Y : g \sim f\}$, then this makes a new category.

Definition 2.4: Homotopy category

The **homotopy category**, denoted **hTop**, is the category whose objects are topological spaces and whose morphisms are homotopy equivalence classes.

Remark This could be another example which reminds us that the morphisms in a category do not go to the element-level. It only describes the ability of combining two objects and does not manipulate the element directly. So, even though the equivalence classes do not look like a map, they still can be regarded as morphisms.
The well-definedness comes from the

3 Simplicial and Singular Homology

3.1 Simplicial homology

Definition 3.1: Simplex

Let $v_0, v_1, \ldots, v_p \in \mathbb{R}^n$ be $p+1$ affine independent [a] points in $\mathbb{R}^n$. A **geometric p-simplex** is a subset of $\mathbb{R}^n$, denoted $[v_0, v_1, \ldots, v_n]$,

$$[v_0, v_1, \ldots, v_n] := \left\{ \sum_{i=0}^{p} t_i v_i : \forall i, 0 \le t_i \le 1, \sum_{i=0}^{p} t_i = 1 \right\}$$

The **standard p-simplex**, denoted $\triangle_p$, is defined as

$$\triangle_p := [e_0, e_1, \ldots, e_p]$$

where e_i s are the standard basis vectors for $\mathbb{R}^p$.

[a] This can be thought as: $v_1 - v_0, \ldots, v_p - v_0$ are linearly independent.

3.2 Singular homology

To make triangle decomposition in a topological space X, it would be convenient to 'embeds' a geometric(more specifically, a standard) p-simplex into X. So,

Definition 3.2: Singular p-simplex

Let X be a topological space. A **singular p-simplex** is a continuous map $\sigma : \triangle_p \to X$.

Remark The word 'singular' here implies that σ might not be a nice embedding. The image might have 'singularites' that it does not look like a simplex at all [**hatcher**].

Definition 3.3: Singular chain group

The **singular chain group of X in degree** p, denoted by $S_p(X)$, is the *free abelian group* generated by the basis $\{$singular p-simplex in X, $\sigma\}$

Remark The operation in $S_p(X)$ is addition. The intuition of being free abelian groups is: those standard p-simplices(or singular p-simplices) are the fundamental building blocks of more complex objects. The 'addition' here corresponds to 'splicing', and we will see how to deal with the boundary nicely when splicing in the definition of boundary operators.

The following map is to embed a standard simplex into a simplex with one-dimension higher as a face of the higher dimensional simplex. $\forall i$, the *i-th face map in $\triangle_p$* is defined to be $F_{i,p} : \triangle_{p-1} \to \triangle_p$. So, each $F_{i,p}$ is a topological embedding with image $F_{i,p}(\triangle_{p-1}) = [e_0, \ldots, \hat{e_i}, \ldots, e_p]$, the face opposite e_i.

Definition 3.4: Boundary of singular p-simplex

The **boundary of singular p-simplex** $\sigma : \triangle_p \to X$, denoted $\partial_p \sigma$, is the map

$$\partial_p \sigma := \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}$$

3.2.1 Boundary operator

Since there is a notion on every singular p-simplices, and $S_p(X)$ is generated by all singular P–simplices σ. So, the 'boundary' can be extended to the whole singular chain group $S_p(X)$ as an operator:

Definition 3.5: Boundary operator

For X a topological space and its singular chain group in consecutive degrees p and $p-1$, the **boundary operator** between them, denoted ∂_p, is defined to be the map:

$$\partial_p : S_p(X) \to S_{p-1}(X) \quad \sigma \mapsto \partial_p \sigma := \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}$$

Boundary operator has a very important property:

Theorem 3.1: Nilpotency of boundary operator

Let $\sigma \in C_p(X)$. Then $\partial_{p-1}(\partial_p(\sigma)) = 0$.

Corollary 3.1: Exactness of singular chain groups

The collection of all singular chain groups of X, $\{S_p(X)\}_p$, together with the boundary operators forms an exact sequence. More explicitly, their is a long exact sequence

$$\cdots \xrightarrow{\partial_{p+2}} S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \xrightarrow{\partial_{p-1}} \cdots$$

Proof: The theorem 3.1 implies that $\mathrm{im}\partial_p \subseteq \ker \partial_{p-1}$. ☐

3.3 Singular homology groups

3.3.1 Properties of singular homology groups

From the category of topological spaces **Top** to the category of abelian groups .
Funtoriality of singular homology groups

Theorem 3.2: Funtoriality of singular homology groups

Let X, Y be topological spaces with a continuous map $f : X \to Y$. The induced map $f_p : H_p(X) \to H_p(Y)$ is a group homomorphism, $f_* : H_\bullet(X) \to H_\bullet(Y)$

Singular homology group is a topological invariant.
More specifically, this 'functor' could be refined, from the category of homotopy category **hTop** to the category of abelian groups **AbGp**. Recall that in the category **hTop**, objects are still topological spaces and the morphisms are equivalence classes of homotopic maps.
The next theorem implies how this 'functor' manipulates the morphisms:

Theorem 3.3

Let X, Y be topological spaces and $f, g : X \to Y$ are continuous and homotopic ($f \simeq g$). Then the two maps induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$

3.4 Singular homology group and fundamental group

Theorem 3.4: Hurwitz theorem