Cantor Set

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1 Definition in $\mathbb{R}$

1.1 Construction of Cantor set

Definition 1.1: Cantor set

1.2 Representatives of elements of Cantor set

For each interval of C_n (each one is closed and of length $\frac{1}{3^n}$, and there are 2^n many of those intervals), let L_n be the collection of left endpoints of each interval of C_n. So, those end points have explicit forms:

Lemma 1.1: Left endpoints of Cantor set

Let L_n be the left endpoints of each interval in C_n. Then, each element in L_n has the form

$$\sum_{1\leq i\leq n} \frac{b_i}{3^i}, \quad \forall b_i, b_i \in \{0,2\}$$

So, each C_n has the form $C_n = L_n + [0, \frac{1}{3^n}]$.

Proof: Induct on n. When $n = 1$, the left endpoints of C_1 are 0 and $\frac{2}{3}$.
Suppose that when $n = k$, the left points are of the form $\sum_{i=1}^{k} \frac{b_i}{3^i}$. In this case, $C_n = \bigcup_{a\in L_n}[a, a + \frac{1}{3^n}]$ So, when $n = k + 1$, remove the middle third of each $[a, a + \frac{1}{3^n}]$. We have two sub-interval: $[a, \frac{1}{3^{n+1}}]$ and $[a + \frac{2}{3^{n+1}}, a + \frac{1}{3^n}]$. The left endpoints of C_{n+1} are L_n and $L_n + \frac{2}{3^{k+1}}$. So, every elements of $L_k \cup L_k + \frac{2}{3^{k+1}}$ is of the form $\sum_{1\leq i\leq k+1} \frac{b_i}{3^i}, \quad \forall b_i, b_i \in \{0,2\}$.
So, $\alpha \in C_n \Leftrightarrow \alpha$ lies in some interval of $C_n \Leftrightarrow \alpha = \sum_{i=1}^{n} \frac{b_i}{3^i} + y$ with $y \in [0, \frac{1}{3^n}]$ □

Theorem 1.1: Representation of Cantor set

Let $C = \bigcap_n C_n$ be the Cantor set and L_n be the left endpoints of C_n. Elements of C_n, C have the form respectively:

$$\begin{aligned}
\alpha \in C_n &\Leftrightarrow \alpha = x_n + y_n, \quad x_n \in L_n, y_n \in [0, \frac{1}{3^n}] \\
&\Leftrightarrow \alpha = \sum_{i\in\mathbb{N}_+} \frac{b_i}{3^i}, \quad \begin{cases} b_i \in \{0,2\} & i \leq n \\ b_i \in \{0,1,2\} & i > n \end{cases} \\
\alpha \in C &\Leftrightarrow \alpha = \sum_{i=1}^{\infty} \frac{b_i}{3^i} \quad \forall i, b_i \in \{0,2\}
\end{aligned}$$

Proof: The first $\Leftrightarrow$ comes from lemma 1.1. For the second $\Leftrightarrow$, fix an n. If $\alpha = \sum_{i=1}^{\infty} \frac{b_i}{3^i} = \sum_{i=1}^{n} \frac{b_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{3^i}$. Then,

$$0 \leq \sum_{i=n+1}^{\infty} \frac{b_i}{3^i} \leq \sum_{i=n+1}^{\infty} \frac{2}{3^i} = \frac{1}{3^n}$$

If $y_n \in [0, \frac{1}{3^n}]$, then
To show the form of every element of Cantor set $\mathcal{C}$. First, let's suppose that $\alpha \in \mathcal{C}$. So, $\forall n$, $\alpha \in C_n \Rightarrow \alpha = x_n + y_n$ with $x_n \in L_n$ and $y_n \in [0, \frac{1}{3^n}]$. Let's consider x_{n+1}.

- If α belongs to the first third interval of C_n, then $x_n = x_{n+1}$.
- If α belongs to the last third interval of C_n, then $x_{n+1} = x_n + \frac{2}{3^{n+1}}$.

No matter which case happens, $x_n \leq x_{n+1}$. So, (x_n) is a monotonically increasing sequence and bounded above by 1. By the monotone convergence theorem, $\lim_n x_n$ exists. Since $0 \leq y_n \leq \frac{1}{3^n}$, $0 \leq \lim_n y_n \leq \lim_n \frac{1}{3^n} = 0$. Then,

$$\alpha = \lim_n \alpha = \lim_n (x_n + y_n) = \lim_n x_n = \lim_n \sum_{i=1}^{n} \frac{b_i}{3^i} = \sum_{i=1}^{\infty} \frac{b_i}{3^i}$$

When $\alpha = \sum_{i=1}^{\infty} \frac{b_i}{3^i}$ with each $b_i \in \{0, 2\}$, for each n, this sum can be split into two parts: $\sum_{i=1}^{n} \frac{b_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{b_i}{3^i} \in L_n + [0, \frac{1}{3^n}] = C_n$. Hence, $\alpha \in \mathcal{C}$ □

1.3 Topological properties of Cantor sets

The Cantor set has many interesting properties as a topological space. Because of this, it is taken as a counterexample of many topological concepts. ([Pea20])

Theorem 1.2: Properties of Cantor set

Let $\mathcal{C}$ be the standard Cantor set.
Then, $\mathcal{C}$ is closed, compact, nowhere dense(does not contain any open interval), perfect, totally disconnected and uncountable.

Proof: Since each I_n is closed, the arbitrary intersection I_n is closed as well. In particular, $\overline{\mathcal{C}} = \mathcal{C}$
$\mathcal{C}$ is closed and bounded, so by Heine-Borel in $\mathbb{R}$, $\mathcal{C}$ is compact in $\mathbb{R}$.
To show $\mathcal{C}$ is nowhere dense, it suffices to show that $\operatorname{int} \mathcal{C} = \varnothing$, because $\operatorname{int} \overline{\mathcal{C}} = \operatorname{int} \mathcal{C}$. Suppose otherwise, $\operatorname{int} \mathcal{C} \neq \varnothing$, so then there must exist an open interval $(a, b) \subseteq \mathcal{C}$. Choose an n such that $\frac{1}{3^n} < b - a$. Then, $(a, b) \nsubseteq I_n \Rightarrow (a, b) \nsubseteq \mathcal{C}$, contradiction.
□
More interesting properties on Cantor set, such as the Lebesgue measure of $\mathcal{C}$ is 0, $\mathcal{C}$ is homeomorphic to 2-adic numbers $\mathbb{Z}_2$, remains to recover.

Theorem 1 (The Cantor function) *Define* $h : \mathcal{C} \to [0, 1]$ *by* $\sum_i \frac{b_i}{3^i} \mapsto \sum_i \frac{b_i}{2^{i+1}}$. *Then, this map is surjective.*

The map mentioned above can be used as an ingredient of a space-filling curve.

Theorem 2 (A space filling curve)

Theorem 3 (Sum of Cantor set) $\mathcal{C} + \mathcal{C} = [0, 2]$

1.4 Structure of Cantor set

Theorem 1.3: Structure of Cantor set

The map

$$\mathcal{C} \to \{0, 2\}^{\mathbb{N}} \quad x = \sum_i \frac{b_i}{3^i} \mapsto (b_i)_i$$

is a homeomorphism. In other words, $\{0, 2\}^{\mathbb{N}}$ can be changed into another notations, $\{0, 1\}^{\mathbb{N}}$ or $2^{\mathbb{N}}$. [a]

[a] $2 := \{0, 1\}$

This theorem gives us an option to generalize the Cantor set in $\mathbb{R}$ into a set in metric space. Because $2^{\mathbb{N}}$ captures the topological essence of $\mathcal{C}$ in $\mathbb{R}$ and this properties is not interfered by any other subordinate structure of $\mathbb{R}$.

2 Cantor set in metric space

Cantor space

Definition 2.1: Cantor set in metric spaces

Let X be a metric space and $C \subseteq X$. C is a **Cantor set** of $X \Leftrightarrow C$ is homeomorphic to $2^{\mathbb{N}}$.

Proposition 2.1: Topological properties for general Cantor set

Let X be a connected metric space. A Cantor set in X has empty interior and hence nowhere dense.

Proof: □

2.1 Constructing the Cantor set in $\mathbb{R}$: Second encounter

This time we construct the 'Cantor set' in $\mathbb{R}$ again. [Tse] The difference is that this time the Cantor set we construct satisfies the definition 2.1 not only 1.1. More intuitively speaking, this time the Cantor set has no constant length and absolute position: we do not force the deletion each time is one third of previous one, neither require the interval being removed is in the middle. This is the generalization we made. If shrinking each interval a point, then the relative position of each point/interval and the 'tree' are the same as we did in 1.1. From this perspective, the loss of information like length of interval does not affect the topology and this general definition captures the 'essence' of topology of definition 1.1.
In the following, $2^{<\mathbb{N}}$ denotes the collection of all finite binary sequence and $\frown$ denotes the concatenation of two sequences.

Definition 2.2: Cantor scheme

Let $(I_s)_{s\in 2^{<\mathbb{N}}}$ be a sequence of closed intervals in $\mathbb{R}$. $(I_s)_{s\in 2^{<\mathbb{N}}}$ is a **Cantor scheme** $\Leftrightarrow$ $\forall s \in 2^{<\mathbb{N}}$ the following conditions hold:
(1) $I_s \neq \varnothing$
(2) $I_{s\frown 0}, I_{s\frown 1} \subseteq I_s$
(3) $I_{s\frown 0} \cap I_{s\frown 1} = \varnothing$
(4) $\forall x \in 2^{\mathbb{N}}$, when $n \to \infty$ $\lambda(I_{x|n}) \to 0$

Consider a 'function' $f : 2^{\mathbb{N}} \to \mathbb{R}$ by $x \mapsto z \in \bigcap_n I_{x|n}$, say this function is the *function induced by Cantor scheme* $(I_s)_{s\in 2^{<\mathbb{N}}}$. This is indeed a function, because from (4) in definition 2.2, $\bigcap_n I_{x|n}$ is a singleton. So, this map is well-defined.
So, define $f(2^{\mathbb{N}})$ to be the *Cantor set*. Then,

$$f(2^{\mathbb{N}}) = \bigcup_{x\in 2^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} I_{x|_n} = \bigcap_{n\in\mathbb{N}} \bigcup_{x\in 2^{\mathbb{N}}} I_{x|_n} \overset{\dagger}{=} \bigcap_{n\in\mathbb{N}} \bigcup_{s\in 2^n} I_s$$

$\dagger$: because $\{x|_n : x \in 2^{\mathbb{N}}\} = 2^n$.

Theorem 2.1: Function induced by Cantor scheme is a topological embedding

Proof: • f is injective: $\forall x, y \in 2^{\mathbb{N}}$ with $x \neq y$. Let n be the first n that $x|_n \neq y|_n$. From (3) in definition 2.2, $I_{x|_n} \cap I_{y|_n} = \varnothing$. Hence, $f(x) \in I_{x|_n}$, $f(y) \in I_{y|_n} \Rightarrow f(x) \neq f(y)$ because $f(x) \subseteq \bigcap_n I_{x|_n} \subseteq I_{x|_n}$.
• f is surjective:
• f is continuous: □

Proposition 2.2: Interval containing Cantor set

Any interval $[a, b]$ contains a Cantor set. Such a Cantor set is called **obtained from** $[a, b]$.

Proof: We realize such a Cantor set by removing strictly open interval from $I_\varnothing := [a, b]$.
Fix $s \in 2^{<\mathbb{N}}$. Suppose that $I_s = [a_s, b_s]$. Set $U_s = (c_s, d_s)$, with $a_s < c_s < d_s < b_s$. Then, let $I_{s\frown 0} = [a_s, c_s]$ and $I_{s\frown 1} = [d_s, b_s]$. $I_{s\frown 0}, I_{s\frown 1}$ satisfy (2), (3) in definition 2.2.
More specifically, if U_s contains the middle point of I_s, then $I_{s\frown 0} \leq \frac{1}{2} I_s, I_{s\frown 1} \leq \frac{1}{2} I_s$. By induction, $I_s \leq \frac{1}{2^{-|s|}} I_\varnothing$. Here $|s|$ means the number of digits of s. So, as $|s| \to \infty$, $I_s \to 0$. (4) is satisfied. □

References

[Pea20] Jamie Pearce. *An introduction to the Cantor set*. Undergraduate Honors Thesis. Final project for Mathematics Honors Program, supervised by Dr. David R. Perez. May 2020. URL: https://web.ma.utexas.edu/users/drp/files/Spring2020Projects/drpfinalproject%20-%20Jamie%20Pearce.pdf.

[Tse] Anush Tserunyan. *Cantor Sets*. Lecture notes, McGill University Department of Mathematics. Part of Real Analysis or Topology course materials. URL: https://www.math.mcgill.ca/atserunyan/Teaching_notes/Cantor_sets.pdf.