Templates

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Contents

1 An overview of representation theory

$$\begin{tikzcd} & \mathcal{C} = \textbf{Set} : \text{Set-theoretic representation} \\ & \mathcal{C} = \textbf{Vsp}_{\mathbb{K}} : \text{Linear representation} \\ \text{Functor } F : \textbf{Gp} \to \mathcal{C} \arrow[ruu] \arrow[ru] \arrow[r] \arrow[rd] \arrow[rdd] & \mathcal{C} = \textbf{AbGp} \\ & \mathcal{C} = \textbf{Pro} : \text{Projective representation} \\ & \mathcal{C} = \textbf{Aff} : \text{Affine representation} \\ & \mathcal{C} = \textbf{Top} \end{tikzcd}$$

Another perspective of representation functor: It is just group actions on different sets, so representations could be viewed as a continuation of group actions:

2 Basic definition of linear representations

2.1 Categorical stuff

Definition 2.1: Category of linear representations of finite group over a field

Let G be a finite group and $\mathbb{F}$ be a field. The cateogry of linear representation of G over $\mathbb{F}$, is denoted as $\textbf{LRep}_{\mathbb{F}}(G^{<\infty})$ with:
• Objects: Pairs (ρ, V). For each (ρ, V), V is a vector space over $\mathbb{F}$ and ρ is a linear representation of G on V.
• Morphisms: Equivariant maps.
• Compositions: The composition of equivariant maps.
• Identities: Identity equivariant maps.

2.1.1 Equivalent class of representation

Classify representation of G up to isomorphism: *Isomorphism class* is a collection of linear representations of G such that any two representations in that collection are isomorphic (in the sense of representation).

Lemma 2.1 () *Any element of an isomorphism class can be represented by* $(\tau, \mathbb{C}^n)$

2.2 New representations from old ones

2.2.1 Overview of new representations

direct sum $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$

tensor product $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$

Given $(\rho_1, V_1), (\rho_2, V_2)$ induced

Hom $(\sigma, \mathrm{Hom}(V_1, V_2))$

2.2.2 Direct sum of representations

2.2.3 Tensor product of representations

Definition 2.2: Tensor product of representations

Let (ρ_i, V_i) be representations of a group G. The **tensor product representation**, denoted $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$, has an underlying vector space $V_1 \otimes V_2$, given by the homomorphism $\rho_1 \otimes \rho_2 : G \to \mathrm{GL}(V_1 \otimes V_2)$ given by

$$\rho_1 \otimes \rho_2(g) := \rho_1(g) \otimes \rho_2(g)$$

where $\rho_1(g) \otimes \rho_2(g)$ is the tensor product of linear maps.
More explicitly,

$$\rho_1 \otimes \rho_2(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$$

or

$$\rho_1 \otimes \rho_2(g) \left(\sum_{i=1}^{n} v_i \otimes w_i \right) = \sum_{i=1}^{n} \rho_1(g)(v_i) \otimes \rho_2(g)(w_i)$$

2.3 Characters of representation

Definition 2.3: Character of representation

Given a linear representation $\rho : G \to \mathrm{GL}(V)$, its character χ_ρ is defined as

$$\chi_\rho : G \to \mathbb{C} \quad \chi_\rho(g) = \mathrm{Tr}(\rho(g)) \text{ or } g \mapsto \mathrm{Tr}(\rho(g))$$

Proposition 2.1: Properties of characters

Let (ρ, V) be a linear representation of G and χ_ρ is the chatacter of ρ. Then:
(1) χ_ρ only depends on the isomorphism class of ρ.
(2) χ_ρ is constant on each conjugacy class.
(3) $\chi_\rho(1_G) = \dim V$

Proof: (3) $\chi_\rho(1_G) = \mathrm{Tr}(\rho(1)) = \mathrm{Tr}(\mathrm{Id}_V) = \dim V$ □

Theorem 2.1: Identification of character group

$\exists$ a natural isomorphism

$$G^* \cong (G^{ab})^*$$

where $G^{ab} = G/G'$ and G' is the commutator subgroup of G.

Proof: Using the Universal property of abelianization, $\forall$ morphism $f : G \to \mathbb{C}^\times$, $\exists!$ morphism $g : G^{ab} \to \mathbb{C}^\times$ such that $g \circ \pi = f$. This proves the surjectivity of $(G^{ab})^* \to G^*$
injectivity □
So abelian group is isomorphic to its character group.

2.4 Special representations for specific groups

2.5 Decomposition of representations (as direct sum)

Definition 2.4: Subrepresentation

Let (ρ, V) be a representation of G and $U \leq V$ such that

$$\forall g \in G,\ \forall u \in U,\ \rho(g)(u) \in U$$

Then $(\rho|_U, U)$ is a subrepresentation of (ρ, V), where

$$\rho|_U : G \to \mathrm{GL}(U), \quad \text{by } g \mapsto \rho(g)|_U$$

Namely, $\rho|_U(g) := \rho(g)|_U$

2.5.1 An important subrepresentation: Projection

$$\begin{tikzcd}
(\rho, V) \arrow[r, leftrightarrow, "\pi\ [2]"] & (\rho|_{V^G}, V^G) \arrow[l, bend left, "\geq [1]"] \arrow[r, dashed, "{[3]}"] & \dim V^G
\end{tikzcd}$$

2.6 Maschke's theorem

2.7 A special decomposition

2.7.1 Dual representation

2.7.2 Structure of Hom representations

$$\begin{tikzcd}
(\rho, V) \arrow[dr, "{[1]}"'] & (\tau, W) \arrow[d, "{[1]}"] & \\
& (\sigma, \mathrm{Hom}(V, W)) \arrow[r, equal, "\sim\dagger\ [3]"] \arrow[d, "{[2]}"', "{(\tau,W)=(\mathbb{1},\mathbb{C})}"] & (\rho^\vee \otimes \tau, V^\vee \otimes W) \\
& (\rho^\vee, V^\vee) &
\end{tikzcd}$$

What is the relationship between $\mathrm{Hom}_G(V, W)$ and $\mathrm{Hom}(V, W)$? Illustrated by the following theorem, it is simply

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$$

Theorem 2.2: Identification of Hom

Let (ρ, V) and (τ, W) be two linear representations. $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$

Proof:

$$\begin{aligned}
\mathrm{Hom}(V, W)^G &= \{T : V \to W \mid \forall g \in G,\ \sigma(g)(T) = T\} \\
&= \left\{T : V \to W \mid \forall g \in G,\ \tau(g) \circ T \circ \rho(g)^{-1} = T\right\} \\
&= \{T : V \to W \mid \forall g \in G,\ \tau(g) \circ T = T \circ \rho(g)\} \\
&= \mathrm{Hom}_G(V, W)
\end{aligned}$$

□

Proposition 2.2: Character of $(\sigma, \mathrm{Hom}(V, W))$

Let (ρ, V) and (τ, W) be two linear representations. The character of $(\sigma, \mathrm{Hom}(V, W))$ is given by

$$\chi_\sigma = \chi_\tau \cdot \overline{\chi}_\rho$$

Lemma 2.1: Schur's lemma

Let (ρ, V) and (τ, W) be two irreducible linear representations of G. Then

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathbb{C} & (\rho, V) \cong (\tau, W) \\ 0 & \text{otherwise} \end{cases}$$

2.8 Class functions

struture of $\mathrm{Class}(G)$

$$\begin{tikzcd}
\text{Space of class functions } \mathrm{Class}(G) \arrow[r, "{[1]}"] \arrow[dr, "{[3]}"'] & \text{a vector space} \arrow[r, "{[2]}"] & \text{basis} \\
& \text{a Hermitian space} &
\end{tikzcd}$$

Definition 2.5: Class function

A function $f : G \to \mathbb{C}$ is a class function if

$$\forall g, h \in G,\ f(hgh^{-1}) = f(g)$$

i.e. f is constant on each conjugacy class.
The **space of class function**, denoted $\mathrm{Class}(G)$, is $\mathrm{Class}(G) := \{f : G \to \mathbb{C} | f \text{ is a class function}\}$.

3 Induced representation

3.1 Induced representations (first encounter)

3.2 Induced representations (second encounter)

As in the first encounter, let G be a finite group with a subgroup H. Let (ρ, W) be a linear representation of H. The representation of G induced by W is denoted $(\mathrm{ind}_H^G \rho, \mathrm{ind}_H^G W)$.

3.2.1 Characters of induced representations

3.2.2 Restriction to subgroups

Let G be a finite group with subgroups $H, K \leq G$ and (ρ, W) be a linear representation of H. Instead of inducing a representation from H to G and restricting it back to H, $\mathrm{ind}_H^G W$ is restricted to another subgroup K.
Here we need double cosets: [Dun23], [Ser77b]

Definition 3.1: Mackey's decomposition

Let G, H, K, S, W have the same meaning as in the prescribed context.
The representation $(\mathrm{res}_K^G \mathrm{ind}_H^G \rho, \mathrm{res}_K^G \mathrm{ind}_H^G W)$ can be decomposed into direct sum of $(\mathrm{ind}_{H_s}^G \rho, \mathrm{ind}_{H_s}^G W)$,

$$\mathrm{res}_K^G \mathrm{ind}_H^G W \cong \bigoplus_{s \in S \cong KG/H} \mathrm{ind}_{H_s}^K W_s$$

4 Character table of finite groups

4.1 The cyclic group

4.1.1 C_n

4.1.2 C_∞

4.2 The Dihedral group

4.2.1 D_4

4.2.2 D_n, n even, ≥ 2

4.2.3 D_n, n odd

Consider the group presentation $D_n = \langle x, y | y^2 = x^n = xyxy = 1 \rangle$

4.2.4 D_{nh}

4.2.5 D_∞

4.2.6 $D_{\infty h}$

4.3 Alternating group

4.3.1 A_4

4.4 Symmetric group

4.4.1 S_4

5 p-group representation

5.1 p-group notions revisited

- Solvable groups
- Supersolvable groups
- Nilpotent groups

The next lemma comes from [Ser77a]

Lemma 5.1: Property of fixed set of a p-group

Let G be a p-group and G acting on a finite set X. X^G is the set of elements of X fixed by G. Then,

$$|X| \equiv |X^G| \mod p$$

Theorem 5.1: Existence of fixed elements in p-group representation

Let V be a vector space over a field of characteristic $p > 0$ and G be a p-group. Let $\rho : G \to \mathrm{GL(V)}$ be a linear representation of G in V. V^G is the fixed set of V by G, where the action $G \curvearrowright V$ is induced by ρ. Then, $V^G \neq \{0\}$, i.e. $\exists$ non-zero element of V fixed by all $\rho(s), s \in G$

Proof: Pick an arbitrary $v \in V \backslash \{0\}$. Define the set X to be

$$X := \langle \rho(s)(v) | s \in G \rangle = \mathrm{span\{\rho(s)(v)|s \in G\} \subseteq V}$$

- X is an n-dimensional vector space for some n by definition of X. Hence, $X \cong \mathbb{F}^n_{p^k}$. and $|X| = p^m$ for some m.
- Applying lemma 5.1, we have $|X^G| \equiv |X| = p^m \equiv 0 \mod p$, but since $0 \in X^G$, the minimal possibility is $|V^G| \geq |X^G| \geq p$. Therefore, $X^G \neq \{0\}$. $\square$

Theorem 5.2: Irreducible representations of p-group

For any p-group G and any linear representation (ρ, V) of G, where V is a vector space over a field of characteristic $p > 0$. The only irreducible representation of (ρ, V) is the trivial representation.

5.2 Theory of Burnside and Blichfeldt

Theorem 5.3: Burnside's theorem

Let p, q be distinct primes and a, b be non-negative integers. Any group G of order $|G| = p^a q^b$ is solvable.

The next theorem asserts that for a supersolvable groups, in particular, for p-groups, every irreducible representation is induced from a 1-dimensional representation. (c.f. [Gor23])

Theorem 5.4: Blichfeldt's theorem

Let G be a supersolvable group and (ρ, V) be an irreducible representation of G, then $\exists$ a subgroup $J \leq G$ and an 1-dimensional representation ψ of J that

$$(\rho, V) \cong (\mathrm{ind}_J^G \psi, \mathrm{ind}_J^G \mathbb{C})$$

Proof: □

5.3 Brauer's theorem

References

[Ser77a] Jean-Pierre Serre. “Examples of induced representations”. In: *Linear Representations of Finite Groups*. New York, NY: Springer New York, 1977, pp. 61–67. ISBN: 978-1-4684-9458-7. DOI: 10.1007/978-1-4684-9458-7_8. URL: https://doi.org/10.1007/978-1-4684-9458-7_8.

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