Math 518 assignment 1

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Contents

1. (a) Let $\mathfrak{p}$ be a prime ideal. By definition of $\text{rad}(\mathfrak{p})$, $\mathfrak{p} \subseteq \text{rad}(\mathfrak{p})$. For the other direction, fix an arbitrary element $a \in \text{rad}(\mathfrak{p})$, so $\exists n$, $a^n \in \mathfrak{p}$. The primity indicates that either $a \in \mathfrak{p}$ or $a^{n-1} \in \mathfrak{p}$. If it is the first case, done. If the second case comes, continue this process and finally $a \in \mathfrak{p}$, showing $\text{rad}(\mathfrak{p}) \subseteq \mathfrak{p}$.
(b) Since for every ideal I, $I \subseteq \text{rad}(I)$. The inverse inclusion property of V gives $V(\text{rad}(I)) \subseteq V(I)$. Now fix an $a \in V(I)$. For this a, $\forall f \in I$, $f(a) = 0$. Now take an arbitrary element $g \in V(\text{rad}(I))$, $\exists n$, $g^n \in I$. So, $g^n(a) = g(a)^n = 0$. Since k has no zero divisors, the similar argument as in (a) shows that $g(a) = 0$, indicating that $a \in V(\text{rad}(I))$.
(c) It suffices to show $\text{rad}(I(V)) \subseteq I(V)$. Take any $f \in$rad(I(V)),$\forall n$, $f^n(P) = 0$ for all $P \in I(V)$. Since $k[x_1, \ldots, x_n]$ is an integral domain. $f(P) = 0$ finally holds, showing that $f \in I(V)$.

2. (a) $IJ \subseteq I \cap J$ holds for every ideal. Because IJ consists of finite sum of elements of I and J. Each term is inside I and J.
I and J are coprime implies $\exists a \in I$ and $b \in J$, $a + b = 1$. $\forall x \in I \cap J$. $x = xa + xb \in IJ$.
(b) First, let's decompose this V into irreducible components:

$$V(xy^2, (x-1)(y+1)^2) = (V(x) \cup V(y)) \cap (V(x-1) \cup V(y+1)) = V(xy) \cup V((x-1)(y+1)) = V(xy, (x-1)(y+1))$$

Then, $I(V) = I(V(xy, (x-1)(y+1)) = I(V(\langle xy, (x-1)(y+1)\rangle)) = \langle xy, (x-1)(y+1)\rangle)$. So, the generators are xy and $(x-1)(y+1)$.

3. First, we show $\subseteq$. This is because $W \subseteq V(I(W))$ and by definition of Zariski closed sets, $V(I(W))$ is closed. For $\supseteq$,

4. (a) $V(y - x^2)$ is irreducible. Because $I(V(y - x^2)) = \langle y - x^2 \rangle$, which is a prime ideal because $y - x^2$ itself is irreducible as a polynomial. Then, use the proposition 1 in chapter 1.5.
(b) Plugging $x^2 - z^2 = 1$ into $x^2 + y^2 = 1$, we have $y^2 + z^2 = 0$. So,

$$V(x^2 + y^2 - 1, x^2 - z^2 - 1) = V((y + iz)(y - iz), x^2 - z^2 - 1) = V(y + iz, x^2 - z^2 - 1) \cup V(y - iz, x^2 - z^2 - 1)$$

(c)

$$\begin{aligned} V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) &= V((y^2 - x)(y^2 + x), (y^2 + x)(y - x)(y + x)) && (1) \\ &= V(y^2 - x, y + x) \cup V(y^2 - x, y - x) \cup V(y^2 + x) && (2) \end{aligned}$$

This decomposition is possible since k is assumed to be algebraically closed.
(d) In case (a) and (c), the irreducible components do not change when the field is changed into $\mathbb{R}$(or any non-algebraically closed field), because the decomposition there does not depend on the algebraical-closedness. For (b), the case changes a little bit: when the field is chosen to be $\mathbb{R}$, $V(y^2 + z^2, x^2 - z^2 - 1)$ itself is already irreducible.

5. First, let find a set $S \subseteq k[x, y, z]$, such that $V = V(S)$: $V = V(y - x^2, z - xy) = V(\langle y - x^2, z - xy\rangle)$. $I(V) = \langle y - x^2, z - xy\rangle$. So, generators are $y - x^2$ and $z - xy$. The irreducibility comes from:

$$k[x, y, z]/I(V) = k[x, y, z]/\langle y - x^2, z - xy\rangle \cong k[t]$$

by $t \mapsto x + I(V)$. $k[t]$ is an integral domain. Hence, $I(V)$ is prime and then V is irreducible.

6. (a) False. Consider $V(y)$(the x-axis) and $V(y^2 - x^2 + 1)$. Both are irreducible. Their intersection is $V(y^2 - x^2 + 1, y) = V(y, x^2 - 1) = \{(1, 0), (0, 1)\}$. Assume that (k)$\neq 2$. Then, this set is reducible becase it can be written into the union of two singletons, which are irreducible.
(b) False. Consider $V(y - x^2) = \{(x, x^2) : x \in k\}$, which is closed in Zariski topology. But, it is not closed in $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$. Suppose $V(y - x^2)$ is closed in the product. Then, $\forall (a, b)$ with b$\neq a^2$, $\exists$ a basis element $U \times V$ containing (a, b) and $U \times V \subseteq V(y - x^2)^c$. So, $\forall c \in U$, $c^2 \in V^c$. And since non-empty Zariski closed sets are finite. Since U is infinite, $\{c^2, c \in U\}$ is also infinite, a contradiction.