Classfication of Wallpaper Groups

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Contents

1 Basic abstraction

1.1 Isometry groups

Translations is an isometry and they form a group, translation group, denoted $\mathbb{T}$.

1.2 Wallpaper groups

The next question is how to translate wallpaper pattern into a mathematical language. First, a wallpaper pattern W consists some points of $\mathbb{R}^2$

2 Identification of wallpaper groups

In this section, the class of wallpaper groups are translated into another class of objects, the extension of groups. First, every wallpaper group can be split into two parts. The two parts has fixed patterns. From these two parts, any wallpaper group is an extension of those two groups. As an equivalence class, the collection of isomorphic class of wallpaper groups has a one-to-one correspondence to the collection of all equivalence classes of group extensions of those two parts.

$$\begin{tikzcd}
& \text{wallpaper groups} \arrow[dl, "\text{decompose}"'] \arrow[dr, "\text{decompose}"] & \\
\text{point group}\,\Gamma_0 \arrow[dr] & & \Gamma/T \arrow[dl] \\
& 0 \to T \to \Gamma \to \Gamma_0 \to 0 \arrow[uu] \arrow[d, dashed] & \\
& \text{structure of}\,\Gamma_0 - \text{module}\,T \arrow[d] & \\
& H^2(\Gamma_0, T) &
\end{tikzcd}$$

Homological algebra provides a way to classify all the group extensions of Γ_0 by T (up to equivalence), the second cohomology group $H^2(\Gamma_0, T)$. Therefore, after the identification work, the question of classifying wallpaper groups boils down to the question of calculating $H^2(\Gamma_0, T)$ for all possible Γ_0.

2.1 Structure of wallpaper groups

2.1.1 Decomposition of wallpaper groups

Proposition 2.1: Structure of Wallpaper groups

Let Γ be a wallpaper group with the translation lattice Λ[a], then the point group $\Gamma_0 \cong \Gamma/\Lambda$.

[a] the lattice $\mathbb{T} \cap \Gamma$

This isomorphism can be rewritten in a short exact sequence:

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Gamma_0 \longrightarrow 0$$

2.1.2 Compoenents of wallpaper groups

2.1.3 Crystallographic restrictions

2.2 Reassembly of wallpaper groups by extension

2.3 Dictionary between wallpaper groups and group extensions

3 Structure of Γ_0-module T

The reason why the Γ_0-module structure of T is important is for the following reasons: filling something
(1) inside the short exact sequence: $0 \to T \to \Gamma \to \Gamma_0 \to 0$.
(2) To recover the structure of T concretely,
As a Γ_0-module, there is an $\Gamma_0 \curvearrowright T$ action that is compatible with the abelian group structure of T. This action is equivalent to the group homomorphism $\rho : \Gamma_0 \to \mathrm{Aut}(T)$. Since $T \cong \mathbb{Z}^2$, $\mathrm{Aut}(T) \cong \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}(\mathbb{Z}^2)$. So, the structure of Γ_0-module T boils down to an integral representation $\rho : \Gamma_0 \to \mathrm{Aut}(T)$, meaning the space T is a $\mathbb{Z}$-module.
Also, a Γ_0-module can be uniquely extended to a $\mathbb{Z}[\Gamma_0]$-module linearly. Every $\mathbb{Z}[\Gamma_0]$-module is naturally a Γ_0-module. Up to the equivalence of category, T can be regarded as a $\mathbb{Z}[\Gamma_0]$-module which will facilitate the calculation of $H^2(\Gamma_0, T)$. But this distinction has no bearing on the subsequent study of representations. We will use these two terms interchangably.
Before diving into the representations, some crucial observations significantly streamline our task:
By looking at the kernel of a representation:

• First, Every representation boils down to the study of a faithful representation. Let $\rho : \Gamma_0 \to \mathrm{Aut}(T)$ be an arbitrary representation. Then, let $\rho' : \Gamma_0 \backslash \ker \rho \to \mathrm{Aut}(T)$ be the unique group homomorphism satisfying ρ', $\rho = \rho' \circ \pi$ with $\pi : \Gamma_0 \to \Gamma_0 / \ker \rho$. Such ρ' is a faithful representation.

• Second, all quotient groups of Γ_0 by a subgroup $K \leq \Gamma_0$, Γ_0/K, is isomorphic to the collection $\{C_n, D_n : n = 1, 2, 3, 4, 6\}$. In other words, this collection is closed under taking quotients. So, for any representation ρ, $\Gamma_0/\ker \rho$ must fall within the list of our interest, i.e. $\Gamma_0/\ker \rho \cong$ some group inside $\{C_n, D_n : n = 1, 2, 3, 4, 6\}$.

Thus, we need not worry about any omissions. By focusing solely on the faithful representations of 10 groups in the list, whether faithful or not, every possible Γ_0-module structure of T (arising from the representations) is exhaustively covered. It therefore suffices to confine our analysis to the faithful representations of each C_n and D_n.

For two faithful representations on Γ_0, ρ and τ, if they determine the same Γ_0-module structure T, the two representations are isomorphic. The notion of isomorphism is similar to the linear representation theory: $\rho \cong \tau \Leftrightarrow \forall g \in \Gamma_0, \exists \phi \in \mathrm{Aut}(T), \tau(g) = \phi \circ \rho(g) \circ \phi^{-1}$. It suffices to study the isomorphic classes of representations on each groups. This facilitates the question one more step.

Having established the simplification, our task now is to analyze these specific structures. Let's begin by fix a point group Γ_0 and a lattice T. Notice that this pairing does not necessarily determine a unique Γ_0-module structure. For example, we will see that there are two different D_1-module structures, called $D_{1,c}$ and $D_{1,p}$. This is the difference between integral representations and linear representations over some field. The theory of integral representation is the tool at our disposal to explore all the possible Γ_0-module structures that can arise.

$$\boxed{\textbf{Goal}: \text{Study all isomorphic classes of faithful integral representations } \Gamma_0 \to \mathrm{Aut}(T),\ \Gamma_0 = C_n, D_n}$$

3.1 Divergence between integral representations and linear representations

3.2 $\Gamma_0 = C_n$, $n = 1, 2, 3, 4, 6$

3.3 $\Gamma_0 = D_1$

Consider a faithful integral representation $\rho : D_1 \to \mathrm{Aut}(T)$, with presentation $D_1 = \langle s | s^2 = e \rangle$. Faithfulness implies that $\rho(s) \neq \mathrm{id}$. To discover all isomorphic classes of ρ satisfies $\rho(s) \neq \mathrm{id}$, first look at all possible components of T. As a $\mathbb{Z}[\Gamma_0]$-module of rank 2. T has only two possibilities: either it has non-trivial(rank 1) submodules, meaning it is decomposable, or it is indecomposable.
• Decomposable: For the case that T is decomposable, let A be some rank 1 $\mathbb{Z}[\Gamma_0]$-submodule of T.
• Indecomposable:

3.4 $\Gamma_0 = D_2$

3.5 $\Gamma_0 = D_3$

Consider $\rho : D_3 \to \mathrm{Aut}(T)$.

3.6 $\Gamma_0 = D_4$

3.7 $\Gamma_0 = D_6$

4 Calculation of $H^2(\Gamma_0, T)$

In last section, one of the components of Γ, Γ_0 has the following possibilities: $\{C_n, D_n : n = 1, 2, 3, 4, 6\}$. So, the goal is to calculate $H^2(\Gamma_0, T)$ for each $\Gamma_0 = C_n, D_n$. The computation for C_n is easier. For $H^2(D_n, T)$, many ways can be applied: Lyndon-Hochschild-Serre(LHS) spectral sequences. filling

4.1 Basic introduction of tools

4.2 Calculations