Math 518 Assignment 2

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1 Solutions

1 (a) • $I = \ker\alpha$: $I \subseteq \ker\alpha$ because $\alpha(x^2 - y^3) = 0$ and $\alpha(y^2 - z^3) = 0$.
$\ker\alpha \subseteq I$. Suppose $f \in \ker\alpha$. Then, $f = q_1 \cdot (x^2 - y^3) + q_2 \cdot (y^2 - z^3) + r(x, y, z)$. By quotienting I, all terms containing x^2 and y^2 can be eliminated by $x^2 = y^3$ and $y^2 = z^3$. So, take a representative r of $\overline{r} \in k[V]/I$, r can be written as

$$r(x, y, z) = R_1(y, z) + xR_2(y, z)$$

But, $f \in \ker\alpha \Rightarrow \alpha(f) = \alpha(r(x, y, z)) = r(t^9, t^6, t^4) = R_1(t^6, t^4) + t^9 R_2(t^6, t^4) = 0$. If this polynomial over t is a zero polynomial, every term should have zero coefficient. But notice that all terms in $R_1(t^6, t^4)$ must be even degrees and all terms in $t^9 R_2(t^6, t^4)$ must be odd degrees. So, both R_1 and R_2 must be zero polynomials. Whence, $f \in I$.
• To show I is prime, it suffices to show that $k[x, y, z]/I$ is an integral domain. But by the isomorphism theorem, $k[x, y, z]/I = k[x, y, z]/\ker\alpha \cong \mathrm{im}\alpha = k[t^9, t^6, t^4]$ which is a subring of the integral domain $k[t]$. Hence, $k[x, y, z]/I$ is an integral domain.
(b) $\alpha : k[x, y, z] \to k[t]$ induces a map $\overline{\alpha} : k[V] := k[x, y, z]/I \to k[t]$ illustrated by the following diagram:

$$\begin{tikzcd}
k[x,y,z] \arrow[r, "\pi"] \arrow[dr, "\alpha"'] & k[x,y,z]/I \arrow[d, "\overline{\alpha}"] \\
& k[t]
\end{tikzcd}$$

Notice that $k[\mathbb{A}^1(k)] = k[t]/I(\mathbb{A}^1(k)) = k[t]/\{0\} = k[t]$. So, $\phi : \mathbb{A}^1(k) \to V(I)$ induces a map $\phi^* : k[V] \to k[t]$. Now we have to regulate ϕ to meet the requirement $\phi^* = \overline{\alpha}$. First, notice that $\forall f \in k[V]$, $\overline{\alpha}(f)(x, y, z) = f(t^9, t^6, t^4)$. Second, $\phi^*(f)(x, y, z) = f(\phi(t))$. Hence,

$$f(\phi(t)) = \phi^*(f)(x, y, z) = \overline{\alpha}(f)(x, y, z) = f(t^9, t^6, t^4)$$

which suggests a way to define ϕ by $t \mapsto (t^9, t^6, t^4)$.
ϕ is one-to-one, for $(t^9, t^6, t^4) = (0, 0, 0) \Rightarrow t = 0$.
ϕ is onto, because for any $(x, y, z) \in V(I)$ (i.e. $(x, y, z) \in \mathbb{A}^3(k)$, such that $x^2 = y^3$ and $y^2 = z^3$), if $z \neq 0$, let $t = \frac{x}{z^2}$; if $z = 0$ let $t = 0$.
Suppose that ϕ is an isomorphism. Then, $\phi^* = \overline{\alpha}$ should also be an isomorphism. But, $\mathrm{im}\phi^* = k[t^9, t^6, t^4] \subsetneqq k[t]$, contradiction.

2(a) On $V(x^2 + y^2 - 1)$, $\frac{1-\overline{y}}{\overline{x}}$ has an equivalent representation $\frac{\overline{x}}{1+\overline{y}}$. The first representation gives candidates for non-regular points: $(0, \pm 1)$. $(0, 1)$ is regular for the second representation, while $(0, -1)$ is not regular for the second one. So, the regular points of V are $V(x^2 + y^2 - 1)\backslash\{(0, -1)\}$.
(b) On $V(xw - yz)$, $\frac{\overline{x}}{\overline{y}} = \frac{\overline{z}}{\overline{w}}$, where $=$ means equivalent representations. For the first representative to be regular, $y \neq 0$. For the second one to be regular, $w \neq 0$. So, all the regular points of V are $\{(x, y, w, z) : (y, w) \neq (0, 0)\}$.

3 (a) Let $F = x^4 + y^4 - x^2y^2$, $G = x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$. Singular points of a curve are the points in the vanishing set of its all partial derivatives.
• To find singular points of F, compute $F_x := \frac{\partial F}{\partial x} = 2x(2x^2 - y^2)$ and $F_y := \frac{\partial F}{\partial y} = 2y(2y^2 - x^2)$. So all singular points are points in $V(F_x, F_y)$. $F_x = 0$ gives those points (x_0, y_0) such that $x_0 = 0$ and $y_0^2 = 2x_0^2$. Plugging those two kinds of solutions into $F_y = 0$ yields the only solution $(0, 0)$. Since this point is on the curve, then $(0, 0)$ is the only singular points of F.
• The process of finding singular points of G is similar. First compute $G_x = x^2 - 2x + y$ and $G_y = y^2 - 2y + x$. For some solution $(x_0, y_0) \in V(G_x, G_y)$. Plugging $G_x(x_0, y_0) = 0$ into $G_y(x_0, y_0) = 0$ to eliminate all y_0, we have $x_0(x_0 - 1)(x_0^2 - 3x_0 + 3) = 0$. Solutions are $x_0 = 0, 1, \frac{3 \pm \sqrt{3}i}{2}$. The only one $(x_0, y_0) \in V(G_x, G_y)$ and on G is $(1, 1)$. So, $(1, 1)$ is the only singular point of G.
(b) Now let's suppose that F is an irreducible plane curve. Suppose that the collection of singular points of $V(F, F_x, F_y)$ contains $V(F)$. Then, every points on F are singular points. So, $V(F_x), V(F_y) \subseteq V(F)$ implies that $F_x, F_y \in \langle F \rangle$. Hence, $F|F_x$ and $F|F_y$. But, the degrees of F_x and F_y are both $\deg F - 1$(on the field of charcacteristic 0). So, it is impossible that $F|F_x$ nor $F|F_y$. Here we get a contradiction. So, $V(F, F_x, F_y) \subsetneqq V(F)$.
When $F_x \neq 0$ and $F_y \neq 0$, F is irreducible $\Rightarrow$ F_x and F_y does not have a common factor, which means that $V(F, F_x, F_y)$ is a finite set.
WHen $F_x = 0$ or $F_y = 0$, wlog, suppose that $F_x = 0$, then F only depends on y. If F is irreducible, then $F(y) = y - c$ for some $c \in k$, which does not have any singularities.

4. If $x = 0$, then $xy = xyz = 0$. So the image of $(x, y, z) \mapsto (x, xy, xyz)$ is $S := \{(u, v, w) \in \mathbb{A}^3(k) : u \neq 0\} \cup \{(0,0,0)\}$. Notice that $U := \{(u, v, w) : u \neq 0\}$ is an (Zariski) open set in $\mathbb{A}^3(k)$ because its complement $\mathbb{A}^3(k) \backslash S = V(u)$ is determined by an Zariski closed set.
• S is not Zariski closed because if it were, $\overline{S} = S \subsetneqq \mathbb{A}^3(k)$. But any Zariski open set is dense, $U \subseteq S \Rightarrow \mathbb{A}^3(k) = \overline{U} = \overline{S}$, contradiction.
• S is not Zariski open. Consider $T := \mathbb{A}^3(k) \backslash S = \{(0, v, w) : (v, w) \neq (0,0)\}$. Notice that $T \subsetneqq V(u)$. Suppose S is open, then T is closed and T must be determined by some certain set $J \subseteq k[x, y, z]$ of polynomials. On one hand, $V(J)$ is closed and then $\overline{T} = \overline{V(J)} = V(J) = T$. On the other hand, T is dense in $V(u)$. This is because $\mathbb{A}^3(k) - \{(0,0,0)\}$ is open in $\mathbb{A}^3(k)$(because its complement is the vanishing set of $k[x, y, z]$) and $\overline{\mathbb{A}^3(k) - \{(0,0,0)\}} = \mathbb{A}^3(k)$. Therefore, $\overline{V(J)} = \overline{\mathbb{A}^3(k) - \{(0,0,0)\} \bigcap V(u)} = \overline{\mathbb{A}^3(k) \cap V(u)}$. Hence, $\overline{T} = V(u)$, contradiction.
• S is Zariski dense because of what is stated in 'not Zariski closed part', $\mathbb{A}^3(k) = \overline{U} = \overline{S}$.

5. All elements of $\mathcal{O}_q(W)$ are rational functions that are regular at q. Hence, each of them is of the form $\frac{g}{h}$ with $g, h \in k[W]$ and $h(q) \neq 0$.
Define the extension $\Phi^* : \mathcal{O}_q(W) \to \mathcal{O}_p(V)$ of ϕ^* to be

$$\Phi^* : \mathcal{O}_q(W) \to \mathcal{O}_p(V) \quad \frac{g}{h} \mapsto \frac{\phi^*(g)}{\phi^*(h)}$$

• This Φ^* is well-defined because $\phi^*(g), \phi^*(h) \in k[V]$ and $\phi^*(h)(p) = h(\phi(p)) = h(q) \neq 0$.
• Φ^* is a ring homomorphism from properties of ϕ^* and properties of fraction fields.
• The extension Φ^* is unique. Suppose there is another extension $\Psi^* : \mathcal{O}_q(W) \to \mathcal{O}_p(V)$, then for any $\frac{g}{h} \in \mathcal{O}_q(W)$,

$$\begin{aligned}
\Psi^*\left(\frac{g}{h}\right) &= \Psi^*(g)\Psi^*(h)^{-1} && \Psi^* \text{ is also a ring homomorphism} && (1)\\
&= \phi^*(g)\phi^*(h)^{-1} && \Psi^* \text{ is an extension of } \phi^* && (2)\\
&= \Phi^*(g)\Phi^*(h)^{-1} && \Phi^* \text{ is an extension of } \phi^* && (3)\\
&= \Phi^*\left(\frac{g}{h}\right) && && (4)
\end{aligned}$$

Hence, Ψ^* and Φ^* agrees on every elements of $\mathcal{O}_q(W)$ implying that $\Psi^* = \Phi^*$.
Recall $\mathfrak{m}_q(W) = \{f \in \mathcal{O}_q(W) : f(q) = 0\}$. $\forall f \in \mathfrak{m}_q(W)$, choose an representative of $f = \frac{g}{h}$. So, $f(q) = 0 \Leftrightarrow g(q) = 0$. Then,

$$\Phi^*(f)(p) = \frac{\phi^*(g)(p)}{\phi^*(h)(p)} = \frac{g(q)}{h(q)} = 0 \quad \Rightarrow \quad \Phi^*(f) \in \mathfrak{m}_p(V)$$

which implies that $\Phi^*(\mathfrak{m}_q(W)) \subseteq \mathfrak{m}_p(V)$.