Complex analysis

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Proposition 3.1: Laurent series coefficients calculation

Let f be holomorphic on $B_r(c)\ \{c\}$ (for some r) with a pole of order m at a. Then, the Laurent series coefficients of f near c, a_j ($j = -m, -m+1, \ldots, -1$), are

$$a_j = \frac{1}{(m+j)!}\left\{\left(\frac{\partial}{\partial z}\right)^{m+j}(z-c)^m f(z)\right\}\Bigg|_{z=c}$$

3.3 Residue theorem

Definition 3.1: Residue

Is there anything about essential sigularities?

Theorem 3.1: Residue theorem

Let $\{P_1, P_2, \ldots, P_n\}$ be poles of a function f defined on U and holomorphic on $U\backslash\{P_1, \ldots, P_n\}$. Let $\gamma : [0.1] \to U$ be a closed curve enclosing all $P_1, \ldots, P_n$. Then,

$$\oint_\gamma f = 2\pi i \sum_{i=1}^{n} \mathrm{Res}_f(P_i) \cdot \mathrm{Ind}_\gamma(P_i)$$

Remark The condition is for finitely many poles is not because this theorem only works for finitely many poles and fails for infinitely many poles. It also works for the infinitely many poles cases. The condition of finite poles vastly streamlines the theorem.
The intuition of this theorem is: the integration of f on γ that encloses poles $P_1, \ldots, P_n$ is equal to $2\pi i$ times of the integration on each small circle around P_i times the winding number of each circle.
When there are infinitely many poles, these poles must accumulate somewhere details .
Proof: Let s_i be the principal parts of Laurent series expansion of f at P_i (near P_i). More explicitly, if the Laurent series expansion of f at P_i is $\sum_{j=-n_i}^{\infty} a^i_{-j}(z-P_i)^j$, then $s_i = \sum_{j=1}^{n_i} a^i_{-j}(z-P_i)^j$.
Consider the function

$$f - \sum_i s_i$$ [1]

$f - \sum_i s_i$ is holomorphic on U. why? .
Hence, the integration of it on γ should vanish,

$$\oint_\gamma f - \sum_i s_i = 0 \Rightarrow \oint_\gamma f = \sum_i \oint_\gamma s_i$$

For each i,

$$\oint_\gamma s_i = \oint_\gamma \sum_j a^i_{-j}(z-P_i)^{-j} = \sum_{j=1}^{n_i} a^i_{-j} \oint_\gamma (z-P_i)^{-j}$$

[1] This process is removing all possible singularities of f on U.

From ??, for all $j > 1$, $\oint_\gamma (z-P_i)^{-j} = 0$ and for $j=1$, $\oint_\gamma (z-P_i)^{-1} = 2\pi i$. Notice that $\forall i$, $a^i_{-1} =: \mathrm{Res}_f(P_i)$. Hence,

$$\oint_\gamma f = \sum_i a^i_{-j} \cdot 2\pi i = 2\pi i \cdot \sum_i \mathrm{Res}_f(P_i)$$

□

To use residue theorem, we need a quick computation method for residue, as suggested in proposition 3.1, is

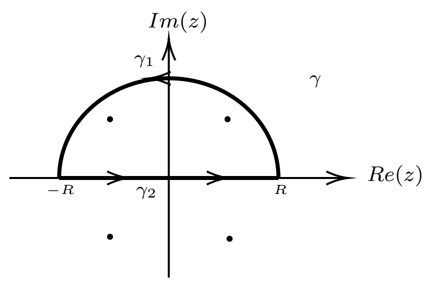
$$\mathrm{Res}_f(c) = \frac{1}{(m-1)!}\left\{\left(\frac{\partial}{\partial z}\right)^{m-1}(z-c)^m f(z)\right\}\Bigg|_{z=c}$$

3.3.1 Applications of residue theorem

The Residue theorem can be used to calculate some integrals in $\mathbb{R}$:
The question is to evaluate an integral $\int_I f$ with $I \subseteq \mathbb{R}$ and f a real-valued function. The basic idea is find a closed path $\gamma : I \to \mathbb{C}$ that some part of this path lies on the x-axis, which is related to I. At the same time, the function of real variable should be transferred to a complex-variable form such that its 'projection' onto the real axis is the

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx = \frac{\pi}{\sqrt{2}}$

Choose $\frac{z^2}{1+z^4}$. This $\frac{z^2}{1+z^4}$ has four poles, $e^{\frac{\pi}{4}i+\frac{\pi}{2}(n-1)i}$, $n = 1,2,3,4$. Choose the path to be $\gamma := \gamma_1 + \gamma_2$, where $\gamma_1 = Re^{it}$, with $R > 1$ (to enclose the singularities) $t \in [0,\pi]$ and γ_2 is the line on the real line axis from $-R$ to R. Namely, γ is the boundary of upper half disk of radius R centered at zero.



So, this path encloses the poles $e^{\frac{\pi}{4}}$ and $e^{\frac{3\pi}{4}i}$.
For the integral, one can first do the analysis of itself:

$$\oint_\gamma \frac{z^2}{1+z^4}dz = \int_{\gamma_1}\frac{z^2}{1+z^4}dz + \int_{\gamma_2}\frac{z^2}{1+z^4}dz$$

- the second term is $\int_{-R}^{R}\frac{x^2}{1+x^4}dx$.
- the first term is: $\int_{\gamma_1}\frac{z^2}{1+z^4}dz = \int_0^\pi \frac{R^2e^{2it}}{1+R^4e^{4it}}Rie^{it}dt$. Then, we do the estimation of the first curve:

$$\begin{aligned}\left|\int_0^\pi \frac{R^3ie^{3it}}{1+R^4e^{4it}}dt\right| &\le \int_0^\pi \left|\frac{Rie^{3it}}{1+R^4e^{4it}}\right|dt \\ &\le \int_0^\pi \frac{R^3}{R^4-1}dt && |1+R^4e^{4it}| \ge ||R^4e^{4it}|-1| \\ &= \frac{\pi R^3}{R^4-1}\end{aligned}$$

Take the limit of the radius of γ to ∞, i.e. $R \to \infty$. why? This does not change the value of $\oint_\gamma \frac{z^2}{1+z^4}dz$, i.e.

$$\lim_{R\to\infty}\oint_\gamma \frac{z^2}{1+z^4}dz = \oint_\gamma \frac{z^2}{1+z^4}dz$$

For the first term, $0 \le |\int_{\gamma_1}\frac{z^2}{1+z^4}dz| \le \frac{\pi R^3}{R^4-1}$ and $\lim_{R\to\infty}\frac{\pi R^3}{R^4-1} = 0$. So, $\lim_{R\to\infty}|\int_{\gamma_1}\frac{z^2}{1+z^4}dz| = 0$ implies that $\lim_{R\to\infty}\frac{z^2}{1+z^4}dz = 0$.

For the second term, $\lim_{R\to\infty}\int_{\gamma_2}\frac{x^2}{1+x^4}dx =: \int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx$.
Hence,

$$\oint_\gamma \frac{z^2}{1+z^4}dz = \int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx$$

From another perspective, the residue theorem 3.1 suggests winding number

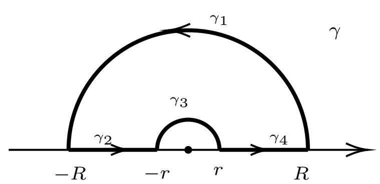
$$\oint_\gamma \frac{z^2}{1+z^4}dz = 2\pi i \cdot \left(\mathrm{Res}_{\frac{z^2}{1+z^4}}(e^{\frac{\pi}{4}i}) + \mathrm{Res}_{\frac{z^2}{1+z^4}}(e^{\frac{3\pi}{4}i})\right)$$

Let $b_i = e^{\frac{\pi}{4}i+\frac{\pi}{2}(n-1)i}$. Then, $\mathrm{Res}_{\frac{z^2}{1+z^4}}(b_1) = a_1^2(a_1-a_2)(a_1-a_3)(a_1-a_4) = \frac{1}{4}e^{-\frac{\pi}{4}i}$. Similarly, $\mathrm{Res}_{\frac{z^2}{1+z^4}}(a_2) = \frac{1}{4}e^{\frac{5\pi}{4}i}$.
So,

$$\oint_\gamma \frac{z^2}{1+z^4} = 2\pi i \cdot -\frac{\sqrt{2}}{4}i = \frac{\pi}{\sqrt{2}}$$

(2) $\displaystyle\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$

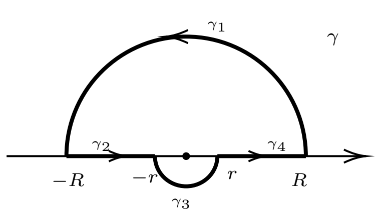
For this, we first choose the complexification of $\frac{\sin x}{x}$ to be $\frac{e^{iz}-e^{-iz}}{z}$ from $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. detect the poles This function has a pole $z=0$ of order 1. The contour should have some part on the x-axis. As before, we can choose the path γ to be boundary of the upper half disk centered at 0. But, this choice leads to a problem: To integrate $\oint_\gamma \frac{e^{iz}-e^{-iz}}{z}dz$, it splits into two parts: one part on γ_1 and another part on γ_2, where γ_1 and γ_2 are as (1). The integrand is holomorphic on the area enclosed by γ, so $\oint_\gamma \frac{e^{iz}-e^{-iz}}{z}dz = 0$. To get the desired integral, we need to push R to ∞. But, this leads to the integral of $\int_{\gamma_1}\frac{e^{iz}-e^{-iz}}{z}dz$ to ∞, which because e^{-iz} blows up on the upper half plane when the imaginary part of z goes to infinity.
Instead, e^{iz} does not make that happen. At the same time, $\frac{e^{iz}}{z}$ leads to a singularity(pole of order 1) at 0. But, this is the best case. Because the convergence is controlled and the poles can be dealt with by Residue theorem. So, for these reasons, $\frac{e^{iz}}{z}$ is chosen to be the complexification of $\frac{\sin x}{x}$.
Correspondingly, the path can avoid or enclose the pole at 0. It is seemingly easier to avoid 0. So, the path γ is chosen to be

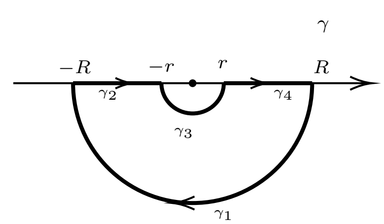


$$0 = \oint_\gamma \frac{e^{iz}}{z}dz = \int_{\gamma_1}\frac{e^{iz}}{z}dz + \int_r^R \frac{e^{-ix}}{x}dx + \int_{\gamma_3}\frac{e^{iz}}{z}dz + \int_r^R \frac{e^{ix}}{x}dx$$

To estimate $\int_{\gamma_1}\frac{e^{iz}}{z}dz$,

Remark One can also choose different contours such as follows:





(3) For $a>1$, $\displaystyle\int_0^\pi \frac{1}{a+\cos\theta}d\theta = \frac{\pi}{\sqrt{a^2-1}}$

Notice that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$. $e^{i\theta}$ with $\theta\in[0,\pi]$ is the curve being integrated. So, this is an inverse process of parametrizing an integration. Let $z=e^{i\theta}$. To get a closed curve, another half should be filled to get γ e^{it}, where $t\in[0,2\pi]$. curve?

$$\int_0^\pi \frac{1}{a+\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{a+\cos\theta}d\theta$$

For the integrand,

$$\frac{1}{a+\cos\theta} = \frac{1}{a+\frac{1}{2}\cdot z+\frac{1}{2z}} = \frac{2z}{z^2+2az+1}$$

and $dz = ie^{i\theta}d\theta = izd\theta$. Hence,

$$\int_0^\pi \frac{1}{a+\cos\theta}d\theta = -i\oint_\gamma \frac{1}{z^2+2az+1}dz$$

The function has two poles, $\alpha := -a+\sqrt{a^2-1}$ and $\beta := -a-\sqrt{a^2-1}$, each of order 1. Since $|\alpha|<1$ and $|\beta|>1$. Only α is enclosed by the curve γ. The residue of $\frac{1}{z^2+2az+1}$ at α is

$$\mathrm{Res}_{\frac{1}{z^2+2az+1}}(\alpha) = \frac{1}{\alpha-\beta} = \frac{1}{2\sqrt{a^2-1}}$$

and by residue theorem,

$$\int_0^\pi \frac{1}{a+\cos\theta}d\theta = -i\oint_\gamma \frac{1}{z^2+2az+1}dz = -i\cdot 2\pi i\cdot\frac{1}{2\sqrt{a^2-1}}\cdot 1 = \frac{\pi}{\sqrt{a^2-1}}$$

(4) $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}dx = 0$

(5) For $0<c<1$, $\displaystyle\int_0^\infty \frac{x^{-c}}{1+x}dx = \frac{\pi}{\sin\pi c}$

3.4 Counting zeros

4 Geometric Theory

4.1 Comformal mapping

4.2 Mobius transformation

5 More topics

5.1 Analytic continuation

5.2 Riemann surfaces