Algebraic Geometry: Scheme-based

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2.1 Basic definitions

Definition 2.1: Induced sheaf

Let $(X, \tau, \mathcal{O}_X)$ be a locally ringed space and $U \overset{\text{open}}{\subseteq} X$. The **induced sheaf** by U, denoted $\mathcal{O}_X|_U$, is defined elementwise:

$$\forall V \overset{\text{open}}{\subseteq} U, \quad \mathcal{O}_X|_U(V) := \mathcal{O}_X(V)$$

$$\begin{tikzcd}
\text{Sheaves are functors } \mathcal{F}, \mathcal{G}, \cdots \arrow[r] \arrow[d, "\text{special: from index category}"'] & \text{morphisms } \mathcal{F} \to \mathcal{G} \text{ natural transformation} & \\
\text{Structure sheaves } \mathcal{O}_X \text{ induces direct systems} \arrow[d, "\varinjlim \text{ for that direct system}"', "x \in X"] & & \in \mathrm{Ob}(\textbf{CRing}) \arrow[d, dash] \\
\text{Stalk of structure sheaf } \mathcal{O}_{X,x} := \varinjlim\limits_{U \ni x} \mathcal{O}_X(U) \arrow[r, shift left, "\text{concrete version}"] & \text{Germs of functions} \arrow[l, shift left, "\text{abstract version}"] & \bigsqcup\limits_{U(\ni x) \in \tau} \mathcal{O}_X(U) \Big/ \sim
\end{tikzcd}$$

$$\varinjlim_{U \ni x} \mathcal{O}_X(U) \cong \bigsqcup_{U(\ni x) \in \tau} \mathcal{O}_X(U) \Big/ \sim$$

Definition 2.2: Pushforward sheaf

Let $(X, \tau, \mathcal{O}_X)$ and $(Y, \sigma, \mathcal{O}_Y)$ be two ringed spaces. Let $f : X \to Y$ be a continuous map. The **pushforward** or **direct image** of $\mathcal{O}_Y$, denoted $f_*\mathcal{O}_X$, is defined elementwise as

$$\forall U \overset{\text{open}}{\subseteq} Y, \quad f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U))$$

$f_*\mathcal{O}_X$ is also called the **pushforward** of $\mathcal{O}_X$ by f.

Lemma 2.1 *This so-called pusforward sheaf $f_*\mathcal{O}_X$ is a sheaf.*

Proof: (1) $\forall U \overset{\text{open}}{\subseteq} Y$, since f is continuous, $f^{-1}(U) \overset{\text{open}}{\subseteq} X$. Then by definition,

$$f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$$

is a set(abelian group, commutative ring), because $\mathcal{O}_X$ is a sheaf: $\forall V \overset{\text{open}}{\subseteq} X$, $\mathcal{O}_X(V)$ is a set(abelian group, commutative ring).

(2) $\forall U \overset{\text{open}}{\subseteq} Y$, $\forall V \subseteq U$, with V open in X, $\exists$ a 'function' [1] $\mathrm{res}_{U,V} : f_*\mathcal{O}_X(U) \to f_*\mathcal{O}_X(V)$, a restriction morphism, has two additional properties: • $\forall U \overset{\text{open}}{\subseteq} X$, $\mathrm{res}_{U,U} = \mathrm{id}_{f_*\mathcal{O}_X(U)}$

• If $W \subseteq V \subseteq U$, with $W, V, U \overset{\text{open}}{\subseteq} X$, then $\mathrm{res}_{U,W} = \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V}$

It is not hard to verify that • the existence of $\mathrm{res}_{U,V}$ and • those two properties come from definition below:

$$\mathrm{res}_{U,V} := \mathrm{res}_{f^{-1}(U), f^{-1}(V)}$$

(3) (Locality) $f^{-1}(U) \overset{\text{open}}{\subseteq} X$ and let $\{U_i\}_{i \in I}$ be any open cover of U, with $\forall i, U_i \subseteq U$. Then $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of $f^{-1}(U)$. To be compatible with the notation before, $\forall s, t \in f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$:

$$\begin{aligned}
& \mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t) && \text{assumption} && (1)\\
\Rightarrow\ & \mathrm{res}'_{f^{-1}(U), f^{-1}(U_i)}(s) =: \mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t) := \mathrm{res}'_{f^{-1}(U), f^{-1}(U_i)}(t) && \text{definition} && (2)\\
\Rightarrow\ & \mathrm{res}'_{f^{-1}(U), f^{-1}(U_i)}(s) = \mathrm{res}'_{f^{-1}(U), f^{-1}(U_i)}(t) && && (3)\\
\Rightarrow\ & s = t && \text{locality of } \mathcal{O}_X && (4)
\end{aligned}$$

(4) (Gluing) $f^{-1}(U) \overset{\text{open}}{\subseteq} X$ Let $\{U_i\}_{i\in I}$ be any open cover of U, with $\forall i, U_i \subseteq U$. Then $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of $f^{-1}(U)$. $\{s_i \in \mathcal{O}_X(U_i)\}_{i\in I}$ is a family of sections.

$$\forall i,j \in I, s_i\big|_{U_i\cap U_j} = s_j\big|_{U_i\cap U_j} \Rightarrow \exists s \in \mathcal{O}_X(U) : \forall i \in I, s\big|_{U_i} = s_i$$

$$\begin{aligned}
& \mathrm{res}_{U,U_i\cap U_j}(s_i) = \mathrm{res}_{U,U_i\cap U_j}(s_j) && \text{assumption} \quad (5)\\
\Rightarrow\ & \mathrm{res}'_{f^{-1}(U),f^{-1}(U_i\cap U_j)}(s_i) =: \mathrm{res}_{U,U_i}(s_i) = \mathrm{res}_{U,U_i}(s_j) := \mathrm{res}'_{f^{-1}(U),f^{-1}(U_i\cap U_j)}(s_j) && \text{definition} \quad (6)\\
\Rightarrow\ & \mathrm{res}'_{f^{-1}(U),f^{-1}(U_i)\cap f^{-1}(U_j)}(s_i) = \mathrm{res}'_{f^{-1}(U),f^{-1}(U_i)\cap f^{-1}(U_j)}(s_j) && (7)\\
\Rightarrow\ & \exists s \in \mathcal{O}_X(f^{-1}(U)) : \forall i \in I, \mathrm{res}'_{f^{-1}(U),f^{-1}(U_i)} = s_i && \text{gluability in } \mathcal{O}_X \quad (8)\\
\Leftrightarrow\ & \exists s \in f_*\mathcal{O}_X(U) : \forall i \in I, \mathrm{res}_{U,U_i} = s_i && \text{definition} \quad (9)
\end{aligned}$$

□

2.2 Local information: Stalk and germs

Definition 2.3: Stalk of a sheaf

Let $(X, \tau, \mathcal{O}_X)$ be a ringed space. Given $x \in X$, the **stalk** of the sheaf $\mathcal{O}_X$ **at** x, denoted $\mathcal{O}_{X,x}$, is defined as

$$\mathcal{O}_{X,x} := \varinjlim_{U\ni x} \mathcal{O}_X(U)$$

Remark (1) To see that this definition is well-defined, we have Sheaf induces a direct system. Then it is possible to define the notion of direct limit.
(2) This is a direct limit in the category **CRing**.
(3) In general, $\mathcal{O}_{X,x}$ is merely a commutative ring. We do not know whether it is a local ring up to now.

Stalk generalises the notion of germs of functions.
Let $x \in X$ and U be an open neighbourhood of x. The germs of functions of $f \in \mathcal{O}_X(U)$ at x is defined as the followings:
For $f \in \mathcal{O}_X(V)$ and $g \in \mathcal{O}_X(W)$, with $V, W \subseteq U$, $(f, V) \sim (g, W) \Leftrightarrow V, W$ are open neighbourhood of x and $f|_{V\cap W} = g|_{V\cap W}$. Then, $[(f, U)]$ is defined to be the equivalence class of (f, U). Sometimes, we use another notation $f_x := [(f, U)]$. Let $\mathrm{Germ}_x(X)$ be the collection of all germs of functions at x. Explicitly,

$$\mathrm{Germ}_x(X) = \{f_x | x \in X\} = \{[(f, U)] | f \in \mathcal{O}_X(U), U \text{ is an open neighbourhood of } x\}$$

• The germs of functions at x is a stalk because it is a direct limit of the direct system $\{\mathcal{O}_X(U)\}_U$:

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[rr, "\mathrm{res}_{U,V}"] \arrow[dr, "\varphi_U"'] \arrow[ddr, bend right, "\psi_U"'] & & \mathcal{O}_X(V) \arrow[dl, "\varphi_V"] \arrow[ddl, bend left, "\psi_V"] \\
& \mathrm{Germ}_x(X) \arrow[d, dashed, "\exists!\Phi"] & \\
& Y &
\end{tikzcd}$$

Define $\varphi_U : f \mapsto [(f, U)]$ for all open neighbourhood of x, U. Clearly, $\varphi_U = \varphi_V \circ \mathrm{res}_{U,V}$. Also, for any commutative ring Y, suppose that ψ_U is another ring homomorphism such that $\psi_U = \psi_V \circ \mathrm{res}_{U,V}$. Then, define $\Phi : [(f, U)] \mapsto \psi_U(f)$. This is well-defined because for another representative $[(g, V)] = [(f, U)]$ such that $V \subseteq U$, $\psi_U(f) = (\psi_V \circ \mathrm{res}_{U,V})(f) = \psi_V(g)$ by $f|_V = g|_V$. This Φ is also a ring homomorphism because ψ_U is.
The definition of Φ automatically makes the diagram being commutative. Moreover, Φ is unique. ??

Importance of stalk and relations between sheaf and stalk:
• Stalk $\Rightarrow$ Sheaf: In many situations, knowing the stalks of a sheaf is enough to control the sheaf itself.
• Sheaf $\Rightarrow$ Stalk: the global information present in a sheaf typically carry less information.

Definition 2.4: Locally-ringed space

A ringed space $(X, \tau, \mathcal{O}_X)$ is called a locally ringed space $\Leftrightarrow \forall x \in X$, $\mathcal{O}_{X,x}$ is a local ring.
In the language of stalk 2.3, a locally ringed space is a ringed space $(X, \tau, \mathcal{O}_X)$ such that all stalks of $\mathcal{O}_X$ are local rings.

Elements of $(X, \mathcal{O}_X)$ are denoted as $(U, f) \in (X, \mathcal{O}_X)$.

Example (1)
(2) A Riemann surface $\widehat{\mathbb{C}}$ with its sheaf $\mathcal{O}_{\widehat{\mathbb{C}}}$ of analytic functions, i.e. $(\widehat{\mathbb{C}}, \mathcal{O}_{\widehat{\mathbb{C}}})$

Given a morphism of ringed spaces, $(f, f^{\#}) : \mathcal{O}_Y \to f_*\mathcal{O}_X$.
We have a collection of morphisms: $f_U^{\#} : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$. Consider the direct limit $O_{Y,f(x)}$ and $\mathcal{O}_{X,x}$, we want $f^{\#}$ somehow connects the two local rings, in other words, ring homomorphism between them.
Let $I := \{U \subseteq X | x \in U \text{ open sets}\}$ and $J := \{V \subseteq Y | f(x) \in V \text{ open sets}\}$.
Since $f^{\#}$ gives a morphism between two collections of rings, it naturally induces a map

$$\mathcal{O}_{Y,f(x)} := \varinjlim_{V \in J} \mathcal{O}_Y(V) \longrightarrow \varinjlim_{V \in J} f_*\mathcal{O}_X(V) = \varinjlim_{V \in J} \mathcal{O}_X(f^{-1}(V)) \xrightarrow{\dagger} \varinjlim_{U \in I} \mathcal{O}_X(U) =: \mathcal{O}_{X,x}$$

†: This is because $\{f^{-1}(V)\}_{V \in J} \subseteq \{U\}_{U \in I} = I$, since f is continuous
$\subseteq$: $\forall V \in J$, $f^{-1}(V)$ is open in X and $f(x) \in V \Rightarrow x \in f^{-1}(V)$ so $f^{-1}(V) \in I$

Definition 2.5: Morphism between locally ringed spaces

Let $(X, \tau, \mathcal{O}_X)$ and $(Y, \sigma, \mathcal{O}_Y)$ be two locally ringed spaces. A morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$, denoted

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

given by the following data:
- A continuous map $f : X \to Y$.
- A morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$, where $f_*\mathcal{O}_X$ is the pushforward of $\mathcal{O}_Y$ by f.
- The morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a local homomorphism, i.e.

$$(f^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$$

where $\mathfrak{m}_x \overset{\max}{\trianglelefteq} \mathcal{O}_{X,x}$ and $\mathfrak{m}_{f(x)} \overset{\max}{\trianglelefteq} \mathcal{O}_{Y,f(x)}$.

Remark Intuition Change: the (3) into the form of $f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. should be $\mathcal{O}_{X,x}$ instead of $(f^*\mathcal{O}_Y)_{f(x)}$.
The condition $(f^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ can be reformulated as $f^{\#}(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$:
Because $(X, \tau, \mathcal{O}_X)$ is a locally ringed space, $\mathcal{O}_{X,x}$ is a local ring. Let its unique maximal ideal be $\mathfrak{m}_x$.
$\mathfrak{m}_x$ is prime. So, $(f^{\#})^{-1}(\mathfrak{m}_x)$ is a prime ideal of $\mathcal{O}_{Y,f(x)}$.
$(Y, \sigma, \mathcal{O}_Y)$ is a locally ringed space as well. So, $\mathcal{O}_{Y,f(x)}$ is a local ring,unique maximal ideal $\mathfrak{m}_{f(x)}$. By uniqueness and maximality of $\mathfrak{m}_x$, we know that

$$(f^{\#})^{-1}(\mathfrak{m}_x) \subseteq \mathfrak{m}_{f(x)}$$

So, to say $(f^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$, it suffices to recast it as $\mathfrak{m}_{f(x)} \subseteq (f^{\#})^{-1}(\mathfrak{m}_x) \overset{\dagger}{\Leftrightarrow} f^{\#}(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$.
†: $\Rightarrow$ Assume $\mathfrak{m}_{f(x)} \subseteq (f^{\#})^{-1}(\mathfrak{m}_x)$, then $f^{\#}(\mathfrak{m}_{f(x)}) \subseteq f^{\#} \circ (f^{\#})^{-1}(\mathfrak{m}_x) \subseteq \mathfrak{m}_x$.
$\Leftarrow$ Assume $f^{\#}(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$, then $\mathfrak{m}_{f(x)} \subseteq (f^{\#})^{-1} \circ f^{\#}(\mathfrak{m}_{f(x)}) \subseteq (f^{\#})^{-1}(\mathfrak{m}_x)$.

Example must have

Here is a diagram to upshot the hierarchy of different concepts.

Category **Objects** **Morphisms**

$$
\begin{tikzcd}
\text{set} \arrow[d] & X \arrow[d, "\tau-\text{Zariski topology}"'] \\
\text{topological space} \arrow[d] & (X,\tau) \arrow[d, "\mathcal{O}_X-\text{sheaf}"'] \\
\text{ringed space} \arrow[d] & (X,\tau,\mathcal{O}_X) \arrow[d, "\forall x\in X, \mathcal{O}_{X,x}\text{local}"'] \\
\text{locally ringed space} & (X,\tau,\mathcal{O}_X)
\end{tikzcd}
$$

Proposition 2.1: Global-Local

Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of rings on (X,τ). If $\exists$ a morphism of sheaves $\eta : \mathcal{F} \to \mathcal{G}$, then η induces a ring homomorphism η_x between rings $\mathcal{F}_x$ and $\mathcal{G}_x$:

$$\eta_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

where $\mathcal{F}_x := \varinjlim_{U(\ni x)\in\tau} \mathcal{F}(U)$ and $\mathcal{G}_x := \varinjlim_{U(\ni x)\in\tau} \mathcal{G}(U)$

Proof: [1] This is directly from morphism between direct system induces a morphism of direct limits. □

Proposition 2.2: Local-Global

Given a morphism between two ringed spaces $(f, f^{\#}) : (X,\tau,\mathcal{O}_X) \to (Y,\sigma,\mathcal{O}_Y)$.
If $\forall x \in X$, the induced stalk homomorphism $f^{\#}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism, then $f^{\#}$ is an isomorphism of the corresponding structure sheaves.

2.2.1 Sheafification

Definition 2.6: Sheafification

Let $\mathcal{F}$ be a presheaf. The **sheafification** of $\mathcal{F}$ is a sheaf $\mathcal{F}^{\text{sh}}$ such that
$\exists$ morphism of presheaf $\alpha : \mathcal{F} \to \mathcal{F}^{\text{sh}}$, $\forall$ sheaf $\mathcal{G}$ with a morphism of presheaf $\beta : \mathcal{F} \to \mathcal{G}$, $\exists!$ morphism of sheaf $\Phi : \mathcal{F}^{\text{sh}} \to \mathcal{G}$ such that the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\alpha-\text{pre}"] \arrow[dr, "\beta-\text{pre}"'] & \mathcal{F}^{\text{sh}} \arrow[d, dashed, "\exists!\Phi-\text{sheaf}"] \\
& \mathcal{G}
\end{tikzcd}
$$

[1] Note that we don't know if $(X,\tau,\mathcal{F})$ ($\mathcal{G}$ and so on) is a locally ringed space, so we don't know in priori if $\mathcal{F}_x$ and $\mathcal{G}_x$ are local rings.

2.3 Local properties of sheaves

3 Schemes

3.1 Preliminaries

Proposition 3.1: $(\mathbf{Spec}(R), \mathcal{O}_{\mathbf{Spec}(R)})$ a ringed space

Let R be a commutative ring and let $\mathrm{Spec}(R)$ be its spectrum. Define $\mathcal{O}_{\mathrm{Spec}(R)}$ as the following: $\forall U \overset{\text{open}}{\subseteq} \mathrm{Spec}(R)$, let

$$\mathcal{O}_{\mathrm{Spec}(R)}(U) := \{f | f \text{ is a function on } U \text{ such that } \forall [\mathfrak{p}] \in U, \exists V \overset{\text{open}}{\subseteq} U, f([\mathfrak{p}]) \in R_{\mathfrak{p}}\}$$

Then, $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is a ringed space.

Proposition 3.2: Stalk structure of $\mathcal{O}_{\mathbf{Spec}(R)}$

$$\mathcal{O}_{\mathrm{Spec}(R),[\mathfrak{p}]} \cong_{\mathbf{CRing}} R_{\mathfrak{p}}$$

3.2 Affine schemes and schemes

3.2.1 Affine scheme

Definition 3.1: Affine scheme

Let $(X, \tau, \mathcal{O}_X)$ be a locally ringed space. $(X, \tau, \mathcal{O}_X)$ is an **affine scheme** $\Leftrightarrow \exists R \in$ **CRing**,

$$(X, \tau, \mathcal{O}_X) \cong (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$$

where $\cong$ is an isomorphism between **locally** ringed spaces.

Example (1) $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ itself is an affine scheme.

3.2.2 Scheme

Definition 3.2: Scheme

A scheme $(X, \mathcal{O}_X)$ is a locally ringed space that admits an open cover $\{U_i\}_i$ of X, such that $\forall i$, $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

3.2.3 Zariski Tangent space

This is an example to see how the locally ringed space 2.4 is applied. Let's consider the simplest case: The stalk of $\mathcal{O}_X$ at $x \in X$ is explicitly considered as the collections of germs of functions $\mathrm{Germ}_x(X)$. So, for every open neighbourhood U of x, we have a function

$$\psi_U : \mathcal{O}_x(U) \to \mathcal{O}_{X,x} := \mathrm{Germ}_x(X) \quad f \mapsto f_x$$

as suggested above. At the same time, $\mathcal{O}_X, x$ is a local ring with the unique maximal ideal $\mathfrak{m}_x$. Let f_x be an element of $\mathcal{O}_{X,x}$. From a proposition in local rings, $f_x \notin \mathcal{O}_{X,x}^{\times}$ [2] $\Leftrightarrow f_x \in \mathfrak{m}_x$ ($\Leftrightarrow f(x) := \overline{f_x} = 0$ in $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$) Let $\lambda := f_x - g_x$ be the constant, then $f_x = \lambda + g_x$. We know how to differentiate a constant λ. Thus, the differentiation of any f at x, $D : \mathcal{O}_{X,x} \to \kappa(x)$ can be reduced to its maximal ideals $g_x \in \mathfrak{m}_x$. So, D can be streamlined to $D : \mathfrak{m}_x \to \kappa(x)$. Furthermore, for two functions $g_x, h_x \in \mathfrak{m}_x$, the differentiation of their product should be $D(f_x g_x) = D(f_x)g_x + f_x D(g_x) = \mathfrak{m}_x = \overline{0}$ via the Leibniz rule. So, product of $\mathfrak{m}_x$, $\mathfrak{m}_x^2$, lies in the kernel of D. The new map $D : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \kappa(x)$ should be well-defined.

[2] $\mathcal{O}_{X,x}^{\times}$: units of $\mathcal{O}_{X,x}$.

3.2.4 Tangent space

This is a local construction. To define the tangent space at a point $x \in X$, we focus on the local behavior of functions. The natural domain for differentiation is the stalk $\mathcal{O}_{X,x}$, which captures the germs of functions near x.

A tangent vector at x is defined as a k-linear derivation $D : \mathcal{O}_{X,x} \to \kappa(x)$, satisfying the Leibniz rule:

$$D(f_x g_x) = f(x)D(g_x) + g(x)D(f_x)$$

where $f(x), g(x) \in \kappa(x)$ are the values of the germs at x.

We can simplify this structure by analyzing the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$:

1. **Decomposition:** Any germ $f_x \in \mathcal{O}_{X,x}$ can be uniquely written as $f_x = \lambda + g_x$, where $\lambda = f(x)$ is a constant (identifying $k \subset \mathcal{O}_{X,x}$) and $g_x \in \mathfrak{m}_x$ is a germ vanishing at x.

2. **Vanishing on Constants:** Since D is k-linear and satisfies the Leibniz rule, $D(\lambda) = 0$ for any constant $\lambda \in k$. Thus, $D(f_x) = D(g_x)$. The derivation is completely determined by its action on $\mathfrak{m}_x$.

3. **Vanishing on $\mathfrak{m}_x^2$:** Consider any two elements $g_x, h_x \in \mathfrak{m}_x$. Their product $g_x h_x$ lies in $\mathfrak{m}_x^2$. Applying the Leibniz rule:
$$D(g_x h_x) = g(x)D(h_x) + h(x)D(g_x) = 0 \cdot D(h_x) + 0 \cdot D(g_x) = 0$$
Here we used the fact that $g(x) = 0$ and $h(x) = 0$.

Conclusion: Any derivation D vanishes on $\mathfrak{m}_x^2$. Consequently, D induces a well-defined linear map on the quotient space:
$$\bar{D} : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \kappa(x)$$
This creates a natural isomorphism between the tangent space $T_x X$ (the space of derivations) and the dual of the cotangent space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

3.2.5 Projective scheme

Let S be a graded ring. So, naturally $S = \bigoplus_{d \geq 0} S_d$. The irrelevant ideal $S_+ = \bigoplus_{d>0} S_d$, which is a homogeneous ideal. A homogeneous ideal of S is written as $\mathfrak{p}_h$ to distinguish it from an arbitrary ideal $\mathfrak{p}$.
Proj S is defined to be the collection of all homogeneous prime ideals of S that do not contain all of S_+:

$$\text{Proj } S := \{[\mathfrak{p}_h] : \mathfrak{p}_h \trianglelefteq S \text{ is prime}, S_+ \not\subseteq \mathfrak{p}_h\}$$

Closed sets of Proj S are defined to be $V(\mathfrak{p}_h) := \{[\mathfrak{q}] \in \text{Proj } S : \mathfrak{p}_h \subseteq \mathfrak{q}\}$.
The following lemma checks the collection of $V(\mathfrak{p}_h)$ indeed play the role of closed sets.

Lemma 3.1: Closed sets of Proj S

Before we define the concept of sheaf, let's define the ring $S_{(\mathfrak{p}_h)}$. $S_{(\mathfrak{p}_h)}$ is defined to be the homogeneous elements of degree 0 of the localization $T^{-1}S$ with T the multiplicative set $\bigcup_d S_d \Big\backslash \mathfrak{p}_h$.
Explicitly,

$$S_{(\mathfrak{p}_h)} = \left\{ \frac{f}{g} : \exists d > 0 \; f, g \in S_d, g \notin \mathfrak{p}_h \right\} = \left\{ \frac{f}{g} : \deg(f) = \deg(g) > 0, f, g \text{ are homogeneous}, g \notin \mathfrak{p}_h \right\}$$

So, $S_{(\mathfrak{p}_h)}$ is a subring of the localized ring $S_{\mathfrak{p}_h}$ consisting of the elements of degree 0.
Now, we try to define the structure sheaf of Proj S, denoted $\mathcal{O}_{\text{Proj } S}$, as follows: For an open subset U of Proj S, define

$$\mathcal{O}_{\text{Proj } S}(U) = \left\{ s : U \to \coprod_{\mathfrak{p}_h} S_{(\mathfrak{p}_h)} : s \text{ is locally quotient} \right\}$$

More explicitly,

$$\mathcal{O}_{\text{Proj } S}(U) = \left\{ s : U \to \coprod_{\mathfrak{p}_h} S_{(\mathfrak{p}_h)} : \forall \mathfrak{p}_h \in U, \exists V(\ni \mathfrak{p}_h) \subseteq U \text{ open }, \forall \mathfrak{q}_h \in V, s(\mathfrak{q}_h) = \frac{a}{f} \in S_{(\mathfrak{p}_h)} \right\}$$

$\mathcal{O}_{\text{Proj } S}$ is a sheaf: (1) (2)
(3) Let $\{U_i\}_i$ be an open cover of U. $\forall i$, $s|_{U_i} = 0$. $\forall \mathfrak{p}_h \in U$, $\exists i$, $\mathfrak{p}_h \in U_i$, $s(\mathfrak{p}_h) = s|_{U_i}(\mathfrak{p}_h) = 0$.
(4) Still, take the same $\{U_i\}_i$. Define $s : U \to \coprod_{\mathfrak{p}_h} S_{(\mathfrak{p}_h)}$ by $\mathfrak{p}_h \mapsto s_i(\mathfrak{p}_h)$ where i is the index of U_i that $\mathfrak{p}_h \in U_i$. This definition is well-defined because for i, j such that $\mathfrak{p}_h \in U_i \cap U_j$, $s_i(\mathfrak{p}_h) = s_i|_{U_i \cap U_j}(\mathfrak{p}_h) = s_j|_{U_i \cap U_j}(\mathfrak{p}_h) = s_j(\mathfrak{p}_h)$.
Let's define a basis of open sets. $\forall f \in S_+$ homogeneous, i.e. $\forall f \in \bigcup_{d>0} S_d$,

$$D_+(f) := \{[\mathfrak{p}_h] \in \text{Proj } S : f \notin \mathfrak{p}_h\}$$

$D_+(f)$ is open because $D_+(f) = \text{Proj } S \backslash \{[\mathfrak{p}_h] \in \text{Proj } S : f \in \mathfrak{p}_h\} = \text{Proj } S \backslash V(\langle f \rangle)$. $\langle f \rangle$ is generated by a homogeneous element, so it is a homogeneous ideal that does not contain S_+ and $V(\langle f \rangle)$ is a closed set in S.

Proposition 3.3: Relative structures of $(\text{Proj } S, \mathcal{O}_{\text{Proj } S})$

Let S be a graded ring.
(1) Stalk of $\mathcal{O}_{\text{Proj } S}$: the structure of the stalk of $\mathcal{O}_{\text{Proj } S}$ at $[\mathfrak{p}_h] \in \text{Proj } S$, $\mathcal{O}_{\text{Proj } S, \mathfrak{p}}$ is a local ring $S_{\mathfrak{p}}$, i.e.

$$\mathcal{O}_{\text{Proj } S, \mathfrak{p}_h} \cong S_{\mathfrak{p}_h}$$

(2) Open sets of $\mathcal{O}_{\text{Proj } S}$: $\{D_+(f)\}_{f \in \bigcup_{d>0} S_d}$ is an open cover of Proj$S$. In addition, there is an isomorphism of locally ringed space:

$$(D_+(f), \mathcal{O}_{\text{Proj } S}|_{D_+(f)}) \cong (\text{Spec } S_{(\langle f \rangle)}, \mathcal{O}_{\text{Spec } S_{\langle f \rangle}})$$

$S_{(\langle f \rangle)}$ is the subring of S_f of elements of degree 0.
(3) Scheme: $(\text{Proj } S, \mathcal{O}_{\text{Proj } S})$ is a scheme.

Remark (1) just says that $(\text{Proj } S, \mathcal{O}_{\text{Proj } S})$ is a locally ringed space. (2) reveals that this locally ringed space is covered by open affine schemes. Hence, it is a scheme. So, (3) is a consequence of (2).
Proof: □
Remark Let R be a ring. The projective n-space over R is a scheme $\mathbb{P}^n(R) \cong \text{Proj } R[x_1, \ldots, x_n]$ Up to now, we have more correspondence between the language from classic AG and the schemes.

	affine cases	projective cases
dd	Spec $R[x_1, \cdots, x_n] \cong \mathbb{A}^n(R)$	Proj $R[x_1, \ldots, x_n] \cong \mathbb{P}^n(R)$

3.2.6 Product of schemes

Definition 3.3: Schemes over a fixed scheme

Let $(S, \mathcal{O}_S)$ be a fixed scheme. A scheme $(Y, \mathcal{O}_Y)$ is **over** $(S, \mathcal{O}_S)$ $\Leftrightarrow$ there is a morphism of schemes $(\varphi, \varphi^\#) : (Y, \mathcal{O}_Y) \to (S, \mathcal{O}_S)$.
A morphism between two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ over $(S, \mathcal{O}_S)$ is a morphism of schemes $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ that commutes with the morphism to $(S, \mathcal{O}_S)$, i.e. a morphism $(f, f^\#)$ that makes the following diagram commute:

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[rr, "{(f, f^\#)}"] \arrow[dr, "{(\psi, \psi^\#)}"'] & & (Y, \mathcal{O}_Y) \arrow[dl, "{(\varphi, \varphi^\#)}"] \\
& (S, \mathcal{O}_S) &
\end{tikzcd}$$

3.3 Correspondence between variety language and scheme language

3.4 Properties of schemes

3.4.1 Local properties: Reducedness, Normality, Regularity and singularity, Cohen-Macauley

Definition 3.4: Reducedness

A scheme $(X, \mathcal{O}_X)$ is reduced $\Leftrightarrow$ $\forall$ open set of X, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

3.4.2 Topological properties: Connectedness, Irreducibility, Quasi-compactness

A scheme $(X, \mathcal{O}_X)$ is connected $\Leftrightarrow$ its topological space X is connected.

3.4.3 Mixed: Integral

3.4.4 Gluing properties: Separatedness, properness

4 Sheaf of modules

4.1 $\mathcal{O}_X$-modules, ideal sheaves

Definition 4.1: $\mathcal{O}_X$-modules

Let $(X, \tau, \mathcal{O}_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$-modules** or **$\mathcal{O}_X$-module** over $(X, \tau, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ such that
(1) $\forall U \overset{\text{open}}{\subseteq} X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.
(2) For any sheaf satisfies (1), $\mathcal{F}$, it is compatible with the restriction map. i.e.$\forall V \subseteq U$ and $V, U \in \tau$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] \arrow[d, "\mathrm{res}'_{U,V} \times \mathrm{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] & \mathcal{F}(V)
\end{tikzcd}$$

Remark To further decode this, using element $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$, through one path we finally have $r|_V \cdot f|_V$ and through another path we have $r \cdot f\Big|_V$. Compatibility $\Leftrightarrow$ $r|_V \cdot f|_V = r \cdot f\Big|_V$.

4.2 Quasi-coherent sheaves: 'Finite dimensional' modules

4.3 Coherent sheaves: The 'Noetherian' case

5 Projective schemes and projective morphisms: Example of a quasi-coherent sheaves

5.1 Quasi-coherent sheaf on $\mathrm{Proj}\, S$

5.2 Serre's twisting sheaves $\mathcal{O}(n)$

5.3 Blow-ups

6 Sheaf cohomology: machinary

measure of failure of gluing local information into global

6.1 Some motivations

6.2 Čech cohomology

6.3 Cohomology of $\mathcal{O}_X$-modules

7 Sheaf cohomology: exmaples of calculations

7.1 An important example: cohomology of a projective scheme $H^i(\mathbb{P}^n(k), \mathcal{O}(m))$

7.2 Cohomology of quasi-coherent modules

8 Bundles and divisors: Toolkits for geometors

8.1 Line bundles: Locally free sheaves

8.1.1 Vector bundles

Definition 8.1: Real vector bundle

A real vector bundle is a tuple $(X, E, \pi, \{\varphi_U\}_U)$ such that:
(1) X is a topological space, called the base space.
(2) E is a locally trivial topological space, called the total space.
(3) $\pi : E \to X$ is a continuous surjective map, called the bundle projection.
(4) $\forall x \in X$, $\pi^{-1}(\{x\})$ is a finite dimensional vector space.
The local triviality of E means that $\forall$ open set U of X, $\pi^{-1}(U)$ is homeomorphic to $U \times \mathbb{R}^k$, i.e. $\exists$ a homeomorphism $\varphi_U : U \times \mathbb{R}^k \to \pi^{-1}(U)$, called **local trivialization** that makes the following diagram commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[ll, "\varphi_U"'] \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

where p is given by: $\forall x \in X$, $\forall v \in \mathbb{R}^k$, $p(x, v) = x$
The vector space structure is explicitly given by: Fix an $x \in X$. $\mathbb{R}^k \to \pi^{-1}(\{x\})$ with $v \mapsto \varphi_U(x, v)$ is an linear isomorphism.

Remark In this definition, as a topological space, $\pi^{-1}(\{x\})$ is homeomorphic to the space $\{x\} \times \mathbb{R}^k$. This comes from the homeomorphism $\varphi_U : U \times \mathbb{R}^k \to \pi^{-1}(U)$. In the meanwhile, $\pi^{-1}(\{x\})$ also has an algebraic structure, which is given explicitly by the linear isomorphism to $\mathbb{R}^k$, or $\{x\} \times \mathbb{R}^k$. The homeomorphism is not enough to determine the linear isomorphism. Consider an example: $f : \mathbb{R} \to \mathbb{R}$ by $v \mapsto v^3$, which is a homeomorphism but not a linear map.
In one word, structures of $\pi^{-1}(\{x\})$ align perfectly with structures of $\{x\} \times \mathbb{R}^k$ in two levels.
When $k = 1$, the space E or the tuple $(X, E, \pi, \{U_i\}_i)$ is called a **line bundle**.
When $E = X \times \mathbb{R}^k$, it is called a **trivial bundle**.

Sections of a vector bundle:
Transition functions:
There is one subtlety needs to deal with. Since the local trivializations work on each open subset of X in the definition. For the intersection of two open sets, they have to match to each other.
Supose there are two local trivializations on U_α and U_β, $\varphi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha)$ and $\varphi_\beta : U_\beta \times \mathbb{R}^k \to \pi^{-1}(U_\beta)$. Consider there composition on $U_\alpha \cap U_\beta$, $\varphi_\beta^{-1} \circ \varphi_\alpha : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$, which maps $(x, v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to some $(x, u) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$. Since $\varphi_\alpha(x, v) \in \pi^{-1}(\{x\})$ and $\varphi_\beta^{-1}|_{\pi^{-1}(\{x\})} : \pi^{-1}(\{x\}) \to \{x\} \times \mathbb{R}^k$ is a homeomorphism.
Furthermore, $u = T_x(v)$ for some $T_x \in \mathrm{GL}(\mathbb{R}^k)$. Because $\varphi_\alpha|_{\pi^{-1}}$ and $\varphi_\beta|_{\pi^{-1}}$ are linear isomorphisms. Fix an x.

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "\sim"] \arrow[rr, bend right] & \{x\} \times \mathbb{R}^k \arrow[r, "\sim"] & \pi^{-1}(\{x\}) \arrow[r, "\sim"] \arrow[rr, bend right] & \{x\} \times \mathbb{R}^k \arrow[r, "\sim"] & \mathbb{R}^k \\
v \arrow[r, mapsto] \arrow[rr, mapsto, bend right] & (x,v) \arrow[r, mapsto] & \varphi_\alpha(x,v) \arrow[r, mapsto] \arrow[rr, mapsto, bend right] & (x,u) \arrow[r, mapsto] & u
\end{tikzcd}$$

is a linear isomorphism. Notice that this linear isomorphism depends on x. So, $x \mapsto T_x$ is actually a map, called $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$.
So, more explicitly, for an $x \in U_\alpha \cap U_\beta$, $\varphi_\beta^{-1} \circ \varphi_\alpha(x,v) = (x, \big(g_{\beta\alpha}(x)\big)(v))$.

Operations of vector bundles:

8.1.2 Invertible sheaves

terms	Geometry perspective	Algebraic perspective
	Vector bundle $\pi : E \to X$	Locally free sheaves $\mathcal{E}$
	Fiber E_x	Sheaf $\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$
	Section $s : X \to E$	Elements $s \in \Gamma(X, \mathcal{E})$

8.1.3 Correspondence between line bundles and invertible sheaves

8.1.4 Embedding schemes into a projective space

8.1.5 Classification of vector bundles over the projective line

8.2 Divisors

8.3 Correspondence between line bundles and divisors

9 Curve Theory

9.1 Riemann-Roch

9.2 Classification of curves in $\mathbb{P}^3$

10 Surface Theory

11 Calculus

11.1 Differentials

11.2 Sheaf of Kähler differentials