Multilinear Algebra

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1 Basic notions

Consider a group action $S_n \curvearrowright V^n$ by $\forall \sigma \in S_n$, $\forall \mathbf{v} := (v_1, \dots, v_n) \in V^n$,

$$^{\sigma}\mathbf{v} := (v_{\sigma(1)}, \dots, v_{\sigma(n)})$$

This is a group action. For id, the action is trivial. For $\tau, \sigma \in S_n$, ${}^{\tau}({}^{\sigma}\mathbf{v}) ={}^{\tau} (v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(n)})$.
Now, let's consider another action S_n on the collection of all multilinear maps from V^n to W: $\{\psi : V^n \to W : \psi \text{ is a multilinear map}\}$, by taking arbitrary $\sigma \in S_n$ and ψ,

$$^{\sigma}\psi := \Big(\mathbf{v} \mapsto \psi({}^{\sigma}\mathbf{v})\Big)$$

2 Symmetric multilinear map

3 Alternating multilinear map

4 Exterior algebra

The motivation for exterior algebra originates from Grassmann. He first tried to use a form to unify all the 'rigid' objects in various dimensional, for instance, lines , plane and paralleloids in $\mathbb{R}^3$. His idea was not valued until many years later when Cartan studied the ??.
Given a vector space V over a field F, the intuition of an exterior algebra of power i (with $i \leq \dim_F V$), $\bigwedge^i(V)$, might be all the possible i-dimensional subspaces of V spanned by i elements in some basis of V. More concretely, for $\bigwedge^2(\mathbb{R}^3)$, it consists of all the elements that could span the xy, xz and yz planes.
To make the definition more comprehensive. Instead of a vector space, the context is put into free R-modules.
For the heuristic reason, let's use the vector space to discuss and get better motivations.

Definition 4.1: Exterior Algebra

Let R be a commutative ring and M be a free R-module. The exterior algebra, denoted $\bigwedge(M)$, and a unital R-algebra homomorphism $i : M \to \bigwedge(M)$ are the unique ones satisfies: $\forall$ unital R-algebra A and $\forall$ R-linear map $\varphi : M \to A$ with $\forall v \in V$, $\varphi(v)\varphi(v) = 0$, $\exists!$ R-linear homomorphism $\Phi : \bigwedge(M) \to A$ such that $\Phi \circ i = \varphi$, i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "\varphi"'] & \bigwedge(M) \arrow[d, "\exists!\Phi"] \\ & A \end{tikzcd}$$

Construction of an exterior algebra: To construct such a most general R-algebra, start with the most general associative R-algebra containing M, the tensor algebra $T(M)$. Then, force the alternating property by taking a quotient. Take I to be the two-sided ideal of $T(M)$ generated by $\{m \otimes m : m \in M\}$. Then, the quotient

$$T(M)/I$$

is a candidate for the exterior algebra of M. [1]

4.1 Wedge product

4.2 Gradation of exterior algebras

4.2.1 k-th exterior power

For each gradation component $\bigwedge^k(M)$ of $\bigwedge(M)$, it can be depicted by another universal property. But notice that when we study $\bigwedge(M)$, it is always be considered as an R-algebra. But its pieces, $\bigwedge^k(M)$ loses the multiplication.

[1] The quotient is still an algebra, because $T(M)/I$ is a ring and I can be viewed as an R-submodule of M and $T(M)$, hence $T(M)/I$ has all structures.

Because elements $\alpha, \beta \in \bigwedge^k(M)$ are not closed under multiplication $\wedge$: $\alpha \wedge \beta \in \bigwedge^{2k}(M)$. So, usually $\bigwedge^k(M)$ is only considered as an R-module. This is also reflected in the universal property below.

Definition 4.2: Universal property of k-th exterior power

Let R be a commutative ring and M, N be free R-modules. The k-th **exterior power** of M, $\bigwedge^k(M)$, is defined to be the unique object with unique alternating multilinear $i : M^k \to \bigwedge^k(M)$ satisfying the following universal property: $\forall$ alternating multilinear map $\varphi : M^k \to N$, $\exists!$ R-module homomorphism $\Phi : \bigwedge^k(M) \to N$ such that $\Phi \circ i = \varphi$, i.e. the following diagram commutes:

$$\begin{tikzcd} M^k \arrow[r, "i"] \arrow[dr, "\varphi"'] & \bigwedge^k(M) \arrow[d, "\exists!\Phi"] \\ & N \end{tikzcd}$$

Proposition 4.1: Different spans of the ideal

$m \otimes m$ can be written in another form $m_1 \otimes \cdots \otimes m_n$ with $m_i = m_j$ for some $i \neq j$.

4.2.2 Basis of $\bigwedge^k(M)$

From now on, the center is to study the basic properties $\bigwedge^k(M)$ as an R-module.

Corollary 4.1: Basis of gradation components $\bigwedge^k(M)$

Let M be a free R-module over a commutative ring R with basis $\{m_1, \ldots, m_n\}$. Then, one basis of $\bigwedge^k(M)$ is

$$\{m_{i_1} \wedge \cdots \wedge m_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

When $k > n$, $\bigwedge^k(M) = \{0\}$. In particular, $\text{rank}_R \bigwedge^k(M) = \binom{n}{k}$.

Remark Especially, when $k = n$, $\bigwedge^n(M) = R(m_1 \wedge m_2 \wedge \cdots \wedge m_n)$.
If M is not a free R-module, then it is probably false to have $\bigwedge^k(M) = 0$ for $k > n$. Here is an example. Let $R = \mathbb{Z}$ and consider $\mathbb{Z}[x, y]$. The ideal $\langle x, y \rangle$ is an R-submodule of $\mathbb{Z}[x, y]$, but it is not free because $x \cdot y - y \cdot x = 0$. We want to show that $I \bigwedge I \neq 0$.

First construct a map $\varphi : I \times I \to R/I \cong \mathbb{Z}$ by $(ax + by, cx + dy) \mapsto (ad - bc) + I$. This is a well-defined non-trivial alternating bilinear map. [2] By the universal property of k-component of $\bigwedge(M)$, $\exists!$ non-trivial R-linear map $\Phi : I \bigwedge I \to R/I$. But this implies that $I \bigwedge I \neq 0$.

Proof: $\square$

Example (1) Let V be a 1-dimensional vector space over F with basis element v. Then, every element of $\bigwedge^k(V)$ is $a_1 v \wedge a_2 v \wedge \cdots \wedge a_k v = (a_1 a_2 \cdots a_k) v \wedge \cdots \wedge v$. Since $v \wedge v = 0$, $\bigwedge^0(V) = F$, $\bigwedge^1(V) = Fv = V$, $\bigwedge^i(V) = 0$ for $i \geq 2$. As a graded algebra, $\bigwedge(V) = F \oplus V \oplus 0 \oplus \cdots$.
(2) Let V be a vector space with $\dim_F(V) = 2$, with basis element v and w. Then, every element is $(a_1 v + b_1 w) \wedge \cdots \wedge (a_k v + b_k w)$. It follows that $\bigwedge^i(V) = 0$ for $i \geq 3$ because either v or w appear twice in every term of the expansion. For $\bigwedge^2(V)$, $\bigwedge^2(V) = T^2(V)/I$ where I is the ideal generated by the set $\{x \otimes x : x \in V\}$. $T^2(V)$ has a basis consisting of $v \otimes v$, $v \otimes w$, $w \otimes v$ and $w \otimes w$. But every element in I is

$$(av + bw) \otimes (av + bw) = a^2(v \otimes v) + ab(v \otimes w + w \otimes v) + b^2 w \otimes w$$

Notice that $v \otimes v$, $v \otimes w + w \otimes v$, $v \otimes w$, $w \otimes w$ is another basis for $T^2(V)$. As a subspace, I contains three elements of the basis of $T^2(V)$ but $v \otimes w \notin I$. So, it is 3-dim and $\bigwedge^2(V)$ is 1-dim. Further, $v \otimes w \notin T^2(V) \Rightarrow v \wedge w \neq 0 \in T^2(V)$, which gives $\bigwedge^2(V) = F(v \wedge w)$. As a graded algebra, $\bigwedge^2(V) = F \oplus V \oplus F(v \wedge w) \oplus 0 \oplus \cdots$.

[2] Well-defined: Suppose another representation of $(ax + by, cx + dy)$ is $(a'x + b'y, c'x + d'y)$, then $a' = a + yf$ and $b' = b - xf$ for some $f \in \mathbb{Z}[x, y]$. This is because $(a - a')x = (b' - b)y$ and it must be $x | b' - b$ and $y | a - a'$. Take $f = \frac{b - b'}{x}$.
Non-trivial: $\varphi(x, y) = 1 + I$ is not zero.
Bilinear and alternating follows easily from the definition of φ.

4.2.3 Characterizing by a vector space

Consider the space of multilinear alternating R-linear maps $M^k \to R$, denoted $\mathrm{Alt}_R(M^k, R)$. By the universal property of k-the exterior algebras 4.2, there is a natural equivalence [3]

$$\mathrm{Alt}_R(M^k, R) \cong \left(\bigwedge^k(M)\right)^*$$

As long as M is a finitely generated R-module, its dual module $M^* = \mathrm{Hom}_R(M, R)$ is guaranteed to be a free-module and $M \cong M^*$. Under this condition,

$$\bigwedge^k(M^*) \cong \left(\bigwedge^k(M)\right)^*$$

4.2.4 Interior product

There is already an operation for lifting the degree of $\bigwedge^k(V)$. Another dual operation for degrading the operation is interior product. From now on, let $M^* := \mathrm{Hom}_R(M, R)$ be the dual module of M, where R is a commutative ring. The goal is to find a mapping $\bigwedge^k(M) \to \bigwedge^{k-1}(M)$ and how this map interacts with the wedge product. Let's just observe the basic cases, $k = 0, 1, 2$, and then generalize this rule to arbitrary k.
A natural way of degrading is to pick a $u \in M^*$, $v \in M$, and valuate u on v: $u(v) \in R$. This map should analogue such a process. Keep $u \in M^*$ and name the map $i_u^k : \bigwedge^k(M) \to \bigwedge^{k-1}(M)$.
For $k = 0$ and $v \in \bigwedge^0(M)$, set $i_u^0(v) := 0$ always. For $k = 1$ and $\bigwedge^1(M)$, set $i_u^1(v) := u(v) \in R = \bigwedge^0(M)$.
In fact, there is a unique extension of $\bigwedge^1(M) \to \bigwedge^0(M)$ to $\bigwedge^k(M) \to \bigwedge^{k-1}(M)$. To avoid a degree k, the map i_u^k can also be regarded as a product, called *interior product*, denoted $u \lrcorner v$. We have

$$u \lrcorner v = i_u^k(v)$$

Let's continue extending the meaning of this $\lrcorner$. For $k = 2$, take $x, y \in \bigwedge^1(M)$. Then, we have $x \wedge y \in \bigwedge^2(M)$. $\lrcorner$ should be the unique product that extending $\lrcorner$:M* $\times \bigwedge^1(M) \to \bigwedge^0(M)$. This is equivalent to seeking the unique product $\lrcorner : M^* \times \bigwedge^k(M) \to \bigwedge^{k-1}(M)$ such that: (1) $u \lrcorner v = u(v)$ for any $v \in \bigwedge^1(M)$ and (2) $\lrcorner(u, \bigwedge^k(M)) \subseteq \bigwedge^{k-1}(M)$.
For $u \lrcorner (x \wedge y)$, it is a vector in $\bigwedge^1(M)$ and dependent of x and y. To make i_u^2 linear, $\lrcorner(x \wedge y)$ should be a linear combination of x and y,

$$u \lrcorner (x \wedge y) = c_1 x + c_2 y$$

where c_1 and c_2 also depend on x and y. Applying i_u^1, $i_u^1 \circ i_u^2(x \wedge y) = c_1 u(x) + c_2 u(y) = 0$. One plausible choice of coefficients is $c_1 = -u(y)$ and $c_2 = u(x)$. So,

$$u \lrcorner (x \wedge y) = -u(y)x + u(x)y = u(x) \wedge y - u(y) \wedge x = (u \lrcorner x) \wedge y + (-1)x \wedge (u \lrcorner y)$$

- The first generalization admitted is the interaction between $u \in M^*$ and $x \wedge y$ where $x \in \bigwedge^r(M)$ and $y \in \bigwedge^s(M)$,

$$u \lrcorner (x \wedge y) = (u \lrcorner x) \wedge y + (-1)^r x \wedge (u \lrcorner y)$$

- The second generalization is for $u \in \bigwedge^l(M^*)$.

An easier and more acceptable way to establish the interior product is use the identification $\mathrm{Alt}_R(M^k, R) \cong \left(\bigwedge^k(M)\right)^*$.

[3] The map $I : \mathrm{Alt}_R(M^k, R) \cong \left(\bigwedge^k(M)\right)^*$ given by $f \mapsto \Phi_f$ is well-defined by the universal property applying to f, where $f : M^k \to R$ and Φ_f is the unique R-linear map corresponding to f. For two morphisms f and g, if $\Phi_f = \Phi_g$, they must agree on M^k: $f = \Phi_f \circ i = \Phi_g \circ i = g$. So, I is injective. Surjectivity comes from composing Φ with i. The linearity also comes from the R-linearity and uniqueness of Φ_f.

Theorem 4.1: Axiomatic characterization and properties of interior product

Let M be a finitely generated free R-module. The interior product on $\bigwedge^k(M)$ satisfies the following properties $\forall u \in M^*$ and $\forall v \in M$
(1) $\forall k$, $\iota_u : \bigwedge^k(M) \to \bigwedge^{k-1}(M)$
(2) $\iota_u(v) = u(v)$
(3) It satisfies the graded derivation of degree -1: $\forall x \in \bigwedge^r(M)$ and $\forall y \in \bigwedge^s(M)$,

$$\iota_u(x \wedge y) = \iota_u(x) \wedge y + (-1)^{\deg x} x \wedge \iota_u(y)$$

4.2.5 Application to algebraic geometry: Plüker embeddings

Given a vector space V over F, with $\dim_F(V) = n$. The projective space $\mathbb{P}(V)$ parametrises all the lines of V (all one-dimensional subspace of V). In more condensed language,

$$\mathbb{P}(V) \leftrightsquigarrow \{L_1 : L_1 \leq V, \dim_F L_1 = 1\}$$

The collection on the right hand side is exactly the Grassmannian or Grassmann variety $\mathrm{Gr}_1(V)$. Now the goal is to generalize this ideal to any $\mathrm{Gr}_r(V)$ for $r \leq n$. Are there any counterparts(projective spaces) corresponding to $\mathrm{Gr}_r(V)$?
Since all the r-dimensional subspaces are taken into consideration, the exterior product $\bigwedge^r(V)$ is a huge vector space that contains every possible r-dimensional subspaces. There should be a map $\mathrm{Gr}_r(V) \to \bigwedge^r(V)$. But as in the case of V and $\mathbb{P}(V)$, V itself has many elements that describe the same information(the same line) that are compressed into one in $\mathbb{P}(V)$. To remove the redundancy in $\bigwedge^r(V)$, its projective space is better than itself. The effective map should be

$$\mathrm{Gr}_r(V) \to \mathbb{P}\left(\bigwedge^r(V)\right)$$

To define this map concretely, start from an r-dimensional subspace L_r of V, pick a basis $\{e_1, \ldots, e_r\}$ for this L_r and send this basis to $e_1 \wedge \cdots \wedge e_r$. Finally, take the equivalent class $[e_1 \wedge \cdots \wedge e_r] \in \mathbb{P}(\bigwedge^r(V))$. So, this map is defined elementwise as

$$L_r = \mathrm{span}\{e_1, \ldots, e_r\} \mapsto P_{L_r} := [e_1 \wedge \cdots \wedge e_r]$$

• This map is well-defined or independent of the choice of basis. Take another basis of L_r, $\{f_1, \ldots, f_r\}$. This map gives another representation $[f_1 \wedge \cdots \wedge f_r]$. Both $e_1 \wedge \cdots \wedge e_r$ and $f_1 \wedge \cdots \wedge f_r$ are elements of $\bigwedge^r(L_r)$. Consider a base-change map $T : L_r \to L_r$ by $e_i \mapsto f_i$ for all i. By funtoriality of $\bigwedge^r$, this T induces a linear map $\bigwedge^r(T) : \bigwedge^r(L_r) \to \bigwedge^r(L_r)$ such that

$$\bigwedge^r(T)\left(e_1 \wedge \cdots \wedge e_r\right) := T(e_1) \wedge \cdots \wedge T(e_r) = f_1 \wedge \cdots \wedge f_r$$

In the meanwhile, $\bigwedge^r(T)(e_1 \wedge \cdots \wedge e_r) = \det(T) e_1 \wedge \cdots \wedge e_r$. Hence, $[f_1 \wedge \cdots \wedge f_r] = [e_1 \wedge \cdots \wedge e_r]$.

4.3 Funtoriality of exterior algebras

4.3.1 Morphisms between two exterior algebras

Let M and N be two free modules over a commutative ring R and $f : M \to N$ be a R-linear map. Apply the universal property in 4.1 to M and N.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "\varphi"'] & \bigwedge(M) \arrow[d, dashed, "\exists ! \Phi"] \\ & \bigwedge(N) \end{tikzcd}$$

The unique R-algebra homomorphism Φ can be assigned a new symbol $\bigwedge(f)$.