Math 518 Assignment 4

Guo Haoyang

November 2025

Contents

1 Introduction

1.(1) Suppose $\forall i$, f_i is a morphism. Then,
($\Leftarrow$.1) f is a continuous map.
Let W be any open subset of Y. $W = W \cap Y = \bigcup_i W \cap V_i$, where $W \cap V_i$ is open in V_i. Then,

$$f^{-1}(W) = \bigcup_i f^{-1}(W \cap V_i) = \bigcup_i f_i^{-1}(W \cap V_i)$$

Since each $f_i^{-1}(W \cap V_i)$ is open in X by the assumption that f_i is a morphism(hence continuous).
($\Leftarrow$.2) $\forall$ open set $V \subset Y$, and $\forall g \in k[V]$, $f_*(g) \in k[f^{-1}(V)]$.
Let W still be an arbitrary open subset of Y. Fix a $g \in k[W]$. We will see that $f_*(g) = g \circ f \in k[f^{-1}(W)]$. First, $f^{-1}(W) = \bigcup_i f^{-1}(W) \cap U_i$. $\{f^{-1}(W) \cap U_i\}_i$ is an open cover of $f^{-1}(W)$. So, it suffices to check that $(g \circ f)$ is a regular map on each open cover $f^{-1}(W) \cap U_i$. Then, by the gluing axiom, $g \circ f$ is automatically a regular map on $f^{-1}(W)$.
By the assumption, $(f_i)_*(g)$ is regular on each $f_i^{-1}(W \cap V_i)$, and

$$\begin{aligned} f|_{U_i}^{-1}(W \cap V_i) &= \{x \in X : f|_{U_i}(x) \in W \cap V_i\} && (1)\\ &= \{x \in U_i : f(x) \in W \cap V_i\} && (2)\\ &= U_i \cap \{x \in X : f(x) \in W \cap V_i\} && (3)\\ &= U_i \cap f^{-1}(W) \cap f^{-1}(V_i) && (4)\\ &\overset{\dagger}{=} f^{-1}(W) \cap U_i && (5) \end{aligned}$$

$\dagger$ is from the assumption $f(U_i) \subseteq V_i$. Hence, $U_i \subseteq f^{-1}(f(U_i)) \subseteq f^{-1}(V_i)$.
So, on each $f^{-1}(W) \cap U_i$, $(f_i)_*(g)|_{f^{-1}(W) \cap U_i} = (g \circ f|_{U_i})\big|_{f^{-1}(W) \cap U_i} = g \circ (f|_{f^{-1}(W) \cap U_i})$. By the calculation above, this is a regular map.
($\Rightarrow$.1)
For every $W_i \subset V_i$ open in V_i, $f_i^{-1}(W_i) = f^{-1}(W_i)$ is open in U because f is continuous.
($\Rightarrow$.2) Let $W' \subseteq V_i$ be any open set and let $g \in k[W']$ be a regular function on W'.Since W' is also open in Y, g is a regular function on W' (as an open subset of Y).Since f is a morphism, its pullback $f^*(g) = g \circ f$ must be a regular function on the open set $f^{-1}(W')$.The pullback under f_i is $f_i^*(g) = g \circ f_i$. This function is precisely the restriction of the regular function $f^*(g)$ to the open subset $f_i^{-1}(W') = f^{-1}(W') \cap U_i$.By the definition of regular functions (which form a sheaf), the restriction of a regular function to an open subset is still a regular function.Therefore, $f_i^*(g)$ is a regular function on $f_i^{-1}(W')$.
(2) Let $g \in k[V_i]$. By our clarification, g is a global regular function on V_i (i.e., $g \in \mathcal{O}_{V_i}(V_i) = k[V]$).By the definition of a morphism (Part 2 from above), the pullback $f_i^*(g) = g \circ f_i$ must be a regular function on the entire preimage $f_i^{-1}(V_i)$.Since the domain of f_i is U_i and the problem states $f(U_i) \subseteq V_i$, the preimage is $f_i^{-1}(V_i) = U_i$.Therefore, $f_i^*(g)$ is a global regular function on U_i.This means $f_i^*(g) \in \mathcal{O}_{U_i}(U_i)$, which is exactly the coordinate ring $k[U_i]$.Since this holds for all $g \in k[V_i]$, we have shown that $f_i^*(k[V_i]) \subseteq k[U_i]$.

2. X' and Y' are closed subvarieties $\Leftrightarrow$ $X' = V(I)$, $Y' = V(J)$ for some ideals I and J, with coordinate rings $k[X'] = k[x_1, \dots, x_n]/I$ and $k[Y'] = k[y_1, \dots, y_m]/J$.
Instead,we shall view $k[X'] = k[X]/I'$ and $k[Y'] = k[Y]/J'$ for some ideals I' and J'. Combining the notation $k[X] = k[x_1, \dots, x_n]/\tilde{I}$ and $k[Y] = k[y_1, \dots, y_n]/\tilde{J}$, and the third isomorphism theorems,

$$k[X'] = k[X]/I' \cong \frac{k[x_1, \dots, x_n]/\tilde{I}}{I'/\tilde{I}}$$

From 1 (2), it suffices to show that $f^* : k[Y'] \to k[X']$ is a k-algebra homomorphism. Given the map $\phi : k[Y] \to k[X]$, the condition $f(X') \subseteq Y' \Leftrightarrow \phi(J') \subseteq I'$ (*)
Define $\phi' : k[Y'] = k[X]/J' \to k[X']/I'$ by $g + J \mapsto \phi(g) + I$. $(*)$ guarantees that ϕ' is well-defined. By 1(b), this is equivalent to $f|_{X_i}$ is a morphism.

3. (1) $k[x, y] \subseteq \mathcal{O}_X(X)$: Any polynomial in $k[x, y]$ is a regular function on all of $\mathbb{A}^2$. Its restriction to the open subset X is therefore a regular function on X.
$\supseteq$: Let $g \in \mathcal{O}_X(X)$.X is covered by two affine open sets: $U_x = D(x)$ (where $x \neq 0$) and $U_y = D(y)$ (where $y \neq 0$).The ring of regular functions on U_x is $\mathcal{O}_X(U_x) = k[x, y]_x$ (localization by x). Thus, $g|_{U_x} = \frac{a}{x^n}$ for some $a \in k[x, y]$.

Similarly, $\mathcal{O}_X(U_y) = k[x,y]_y$. Thus, $g|_{U_y} = \frac{b}{y^m}$ for some $b \in k[x,y]$.
On the intersection $U_x \cap U_y = D(xy)$, these expressions must be equal:

$$\frac{a}{x^n} = \frac{b}{y^m} \implies a \cdot y^m = b \cdot x^n$$

which holds in the ring $k[x,y]_{xy}$. Since $k[x,y]$ is a UFD, this equality must hold in $k[x,y]$ itself.
In $k[x,y]$, since x and y are coprime, $x^n|a$.So, $a = a' \cdot x^n$ for some $a' \in k[x,y]$.
Substituting this back into the expression for g on U_x:

$$g|_{U_x} = \frac{a' \cdot x^n}{x^n} = a'$$

This shows that the function g is equal to the polynomial a' on the non-empty open set U_x.
By the axioms for varieties, g must be equal to a' everywhere on X.Therefore, g is a polynomial in $k[x,y]$.
(2) Assume X is isomorphic to some affine variety Y, via an isomorphism $f : X \to Y$. This isomorphism would induce a k-algebra isomorphism $f^* : k[Y] \to \mathcal{O}_X(X)$, where $k[Y]$ is the coordinate ring of Y.
From (1), we know $\mathcal{O}_X(X) = k[x,y]$. So, $k[Y] \cong k[x,y]$. Consider the ideal $I = \langle x, y \rangle$ in $\mathcal{O}_X(X)$. This is a proper ideal (it does not contain 1).Let $J \subseteq k[Y]$ be the corresponding proper ideal under the isomorphism.
On the Y (affine) side: By the Nullstellensatz, the vanishing set $V(J) \subseteq Y$ defined by the proper ideal J must be non-empty.
On the X (quasi-projective) side: The vanishing set $V_X(I) \subseteq X$ is:

$$V_X(I) = \{p \in X \mid x(p) = 0 \text{ and } y(p) = 0\}$$

$$V_X(I) = \{p \in \mathbb{A}^2 \setminus \{(0,0)\} \mid p = (0,0)\} = \emptyset$$

Contradiction: An isomorphism f must map $V_X(I)$ bijectively to $V(J)$. But $V_X(I)$ is empty, while $V(J)$ is non-empty.Therefore, X cannot be isomorphic to an affine variety.
(3) As seen in (a), $X = U_x \cup U_y$.$U_x = D(x)$ is a principal open set of the affine variety $\mathbb{A}^2$. Principal open sets of affine varieties are always affine.
Similarly, $U_y = D(y)$ is also an affine open set.X is the union of the two affine subvarieties U_x and U_y.From (2), we proved that X is not affine.This provides the required example.