# Solutions for Algebraic Geometry Hartshone

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## 2.1 Sheaves

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**Question 2.1**

Let $A$ be a ring and $X = \text{Spec}A$, let $f \in A$ and $D(f) := X \setminus V(\langle f \rangle)$. Show that

$$(D(f), \mathcal{O}_X|_{D(f)}) \cong (\text{Spec}A_f, \mathcal{O}_{\text{Spec}A_f})$$

**Question 2.2**

Let $(X, \mathcal{O}_X)$ be a scheme and $U \subseteq X$ be an open subset. Show that the *induced scheme* on $U$, $(U, \mathcal{O}_X|_U)$ is a scheme.

**Question 2.3**

(1) Show that $(X, \mathcal{O}_X)$ is reduced $\Leftrightarrow \forall P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent element.
(2) Let $(X, \mathcal{O}_X)$ be a scheme and $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$, where for any ring $A$, $A_{\text{red}} := A/I$ with $I$ the ideal of nilpotent elements of $A$. Show that the *reduced scheme* $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme and there is a morphism of schemes $(X, (\mathcal{O}_X)_{\text{red}}) \to (X, \mathcal{O}_X)$, which is a homeomorphism on the underlying topological space.
(3)

**Question 2.4**

Given a morphism $f : X \to \text{Spec}A$, and a associated map on sheaves $f^\# : \mathcal{O}_{\text{Spec}A} \to f_*\mathcal{O}_X$, we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$ by taking global sections.
Thus, there is a natural map

$$\alpha : \text{Mor}_{\mathbf{Sch}}((X, \mathcal{O}_X), (\text{Spec } A, \mathcal{O}_{\text{Spec } A})) \to \text{Mor}_{\mathbf{CRing}}(A, \Gamma(X, \mathcal{O}_X)) \quad [a]$$

Show that $\alpha$ is a bijection map.

[a] Hom is synonymous to Mor.

**Proof:** □

**Question 2.5: Final objects in category of schemes**

Describe Spec $\mathbb{Z}$. Show that Spec $\mathbb{Z}$ is a final object for the category of schemes.

**Proof:** Let $(X, \mathcal{O}_X)$ be an arbitrary scheme. By definition of schemes, $\exists$ an open cover of $X$, $\{U_i\}_{i \in I}$, such that $\forall i$, $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. i.e. $\forall i$, $\exists R_i$ a ring, such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (\text{Spec}R_i, \mathcal{O}_{\text{Spec } R_i})$.
Since $\mathbb{Z}$ is initial in the category **CRing**, then $\forall i$, $\exists!$ ring homomorphism $\mathbb{Z} \to R_i$ which induces unique morphism of affine schemes $(\text{Spec}R_i, \mathcal{O}_{\text{Spec } R_i}) \to (\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec}\mathbb{Z}})$. Hence, there is a unique morphism $(U_i, \mathcal{O}_X|_{U_i}) \to (\text{Spec}\mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}})$ by composing the above-mentioned isomorphism.
This result lifts to the unique morphism of schemes $(X, \mathcal{O}_X) \to (\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}})$ because the Hom of scheme, $\mathscr{H} := \text{Hom}_{\mathbf{Sch}}((X, \mathcal{O}_X), (\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}}))$ is a sheaf.

$$\mathscr{H}(U) := \text{Hom}_{\mathbf{Sch}}((U_i, \mathcal{O}_X|_{U_i}), (\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}}))$$

(So, the locality and gluing axioms extend morphisms on $(U_i, \mathcal{O}_X|_{U_i})$ globally.) □

#### Question 2.6: Initial objects in category of schemes

Describe Spec $\{0\}$. Show that Spec $\{0\}$ is an initial object for the category of schemes.

#### Question 2.7: Characterisation of tangent spaces

Let $(X, \mathcal{O}_X)$ be a scheme over a field $k$ and $k[\epsilon]/\langle\epsilon^2\rangle$ be the ring of dual numbers over $k$.
Show that giving a $k$-morphism $\varphi : \operatorname{Spec} k[\epsilon]/\langle\epsilon^2\rangle \to X \Leftrightarrow$ giving $x \in X$, rational over $k$ and an element of Zariski tangent space $T_x$.

**Remark** Some analysis on $k[\epsilon]/\langle\epsilon^2\rangle$:
(1) When $k = \mathbb{R}$, $\forall\ f(x) = \sum_i a_i x^i \in \left(\mathbb{R}[\epsilon]/\langle\epsilon^2\rangle\right)[x]$, $f(\overline{a+b\epsilon}) = f(\overline{a}) + \overline{b}f'(\overline{a})\overline{\epsilon}$.
(2) For general fields $k$, units are elements of the form $a + \langle\epsilon^2\rangle$ or $a + b\epsilon + \langle\epsilon^2\rangle$ with $a, b \neq 0$. So, non-units of $k[\epsilon]/\langle\epsilon^2\rangle$ are of the form $b\epsilon + \langle\epsilon^2\rangle$ with $b \neq 0$, which are in $\langle\epsilon\rangle/\langle\epsilon^2\rangle$.
Since every proper ideal cannot have any units, so it is a subset of the set of non-units. Hence a subset of $\langle\epsilon\rangle/\langle\epsilon^2\rangle$. Therefore, $\langle\epsilon\rangle/\langle\epsilon^2\rangle$ is the unique maximal ideal of $k[\epsilon]/\langle\epsilon^2\rangle$. Thus, $k[\epsilon]/\langle\epsilon^2\rangle$ is local.

**Proof:** $\Rightarrow$ Consider we have a morphism $\varphi : \operatorname{Spec} k[\epsilon]\Big/\langle\epsilon^2\rangle \to X$. Then, $x$ can be given by the point that the unique maximal ideal of $k[\epsilon]/\langle\epsilon^2\rangle$ goes, i.e. $x := \varphi(\langle\epsilon\rangle/\langle\epsilon^2\rangle)$ $\square$

#### Question 2.8: Anti-equivalence of Fld and Sch

The category of field **Fld** is antiequivalent to the category of scheme **Sch**.

#### Question 2.9: Generic points

#### Question 2.10: Spec $\mathbb{R}[x]$

#### Question 2.11: Spec $\mathbb{F}_p[x]$

#### Question 2.12: Glueing lemma

#### Question 2.13: Quasi-compactness of a topological space

(1) Show that a topological space is Noetherian $\Leftrightarrow$ every *open subset* is quasi-compact.
(2) Let $(X, \mathcal{O}_X)$ be an affine scheme. Show that $X$ is quasi-compact but not necessarily Noetherian.
(3) If $A$ is a Noetherian ring, then Spec$(A)$ is a Noetherian topological space.
(4) The converse of (3) is not always true. Give an example ($A$ is not Noetherian ring but Spec$(A)$ is a Noetherian topological space).

**Question 2.14: Projective scheme**

(1) Let $S$ be a graded ring. Show that $\operatorname{Proj} S = \varnothing$ iff $\forall x \in S_+$, $x$ is nilpotent.
(2) Let $\varphi : S \to T$ be a graded homomorphism of graded rings. $U := \{[\mathfrak{p}] \in \operatorname{Proj} T | \varphi(S_+) \not\subseteq \mathfrak{p}\}$. Show that $U$ is an open subset of $\operatorname{Proj} T$ and $\varphi$ determines a natural morphism $f : U \to \operatorname{Proj} S$.
(3) The morphism $f$ can be an isomorphism even when $\varphi$ is not an isomorphism. Show the following example:
Let $d_0$ be an integer. $\forall d \geq d_0$, $\varphi_d : S_d \to T_d$ is an isomorphism, then $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T =: U \to \operatorname{Proj} S$ is isomorphic.
(4) Let $V$ be a projective variety with homogeneous coordinate ring $S$. Show that $t(V) \cong \operatorname{Proj} S$.

**Proof:** (1) $\Rightarrow$ Suppose $\exists y \in S_+$ that is not a nilpotent. Then, we will show that there is a homogeneous prime ideal that satisfies $S_+ \not\subseteq$ this ideal.
$y = \sum_{i_1 \leq d \leq i_m} y_d$ with each $y_d \in S_d$. Then, $y$ is not a nilpotent implies that at least one $y_t$ in the sum is not a nilpotent. Let $T := \{1, y_t, y_t^2, \dots\}$. Consider the localization of $T^{-1}S =: S_{y_t}$. Since $y_t$ is not a nilpotent, $0 \notin T$ and $S_{y_t}$ is not a zero ring, which implies that the degree-0 localization $S_{(y_t)}$ is not a zero ring. [1] Furthermore, $\operatorname{Spec} S_{(y_t)}$ is non-empty because every non-zero ring has at least one maximal ideal. The following isomorphism

$$D_+(y_t) \cong \operatorname{Spec} S_{(y_t)}$$

implies that $D_+(y_t) \neq \varnothing \Rightarrow \operatorname{Proj} S \neq \varnothing$. ($D_+(y_t) \subseteq \operatorname{Proj} S$)
$\Leftarrow$ Suppose that $\forall x \in S_+$, $x$ is nilpotent. We will show that there is no homogeneous prime ideal $\mathfrak{p}$ that $S_+ \not\subseteq \mathfrak{p}$. Let $\mathfrak{p}$ be a homogeneous prime ideal. $\forall x \in S_+$, since $\exists n$, $x^n = 0 \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ because $\mathfrak{p}$ is prime, implying $S_+ \subseteq \mathfrak{p}$ for all homogeneous prime ideal $\mathfrak{p}$.
(2) □

**Question 2.15**

(1) Let $V$ be a variety over the algebraically closed field $k$. Show that $P \in t(V)$ is a closed point $\Leftrightarrow$ the residue field of $P$ is $k$.

**Question 2.16**

Let $(X, \mathcal{O}_X)$ be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ be a global section. $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. $X_f := \{x \in X : f_x \notin \mathfrak{m}_x\}$ where $f_x \in \mathcal{O}_{X,x}$ or a germ of the function $f$.
(1) Suppose $(U, \mathcal{O}_X|_U)$ is an *open affine scheme* of $(X, \mathcal{O}_X)$ with $U = \operatorname{Spec} B$. $f|_U \in \Gamma(U, \mathcal{O}_X|_U) \cong B$. Show that $X_f \cap U = D(f|_U)$. Conclude that $X_f$ is an open subset of $X$.
(2) Let $(X, \mathcal{O}_X)$ be quasi-affine and $a \in \Gamma(X, \mathcal{O}_X)$ such that $a|_{X_f} = 0$. Show that $\exists n > 0$, $f^n a = 0$.

**Proof:** (1) $U \cap X_f = \{x \in U | f_x \notin \mathfrak{m}_x\}$
(2) Let □

**Question 2.17: A criterion for affineness**

**Question 2.18**

Let $A$ be a ring commutative?. Show TFAE:
(1) $\operatorname{Spec} A$ is disconnected.
(2) $\exists$ non-zero orthogonal idempotents $e_1, e_2 \in A$, i.e. $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = 0$ and $e_1 + e_2 = 1$.
(3) $A \cong A_1 \times A_2$, where $A_1, A_2$ are two non-zero rings.

[1] This is true for all $S_y$ such that $y \in S_d$ with $d \geq 1$.

### 2.3 First properties of schemes

# 3 Cohomology

# 4 Curves

# 5 Surfaces