Algebraic Geometry

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Contents

1 Varieties and their morphisms

1.1 Affine space and affine variety

1.1.1 Basic notions

Definition 1.1: Affine algebraic sets

An **(affine) algebraic set** X is a vanishing set for some collection of polynomials, i.e. a subset $X \subseteq \mathbb{A}^n$ such that $X = V(S)$ for some $S \subseteq k[x_1, \ldots, x_n]$.

Example $V(k[x_1, \ldots, x_n]) = \varnothing$, $V(\langle x_1, \ldots, x_n \rangle) = \{0\}$ and $V(\varnothing) = V(\{0\}) = \mathbb{A}^n(k)$
The smallest ideal containing $\varnothing$ is $\{0\}$, because an ideal at least contain 0.

Proposition 1.1: Algebraic sets forming a topology

The union of two affine algebraic sets is still an affine algebraic sets. The intersection of any family of affine algebraic sets is still an affine algebraic set. $\varnothing$ and $\mathbb{A}^n(k)$ are affine algebraic sets.

Definition 1.2: Zariski topology for $\mathbb{A}^n$

Define the affine algebraic sets to be the closed set of $\mathbb{A}^n$. This topology is called the Zariski topology of $\mathbb{A}^n$.

From the proposition 1.1, this is indeed an topology. Another aspect is the collection of polynomials vanish on X, denoted $I(Y)$,

$$I(Y) := \{f \in k[x_1, \ldots, x_n] : \forall P \in Y, f(P) = 0\}$$

The remaining part is to study the relations between such an ideal and the vanishing sets. The properties of vanishing sets and ideals, with their interaction are given in the following theorem.

Proposition 1.2: Properties of ideals and vanishing sets

Let $S_1, S_2 \subseteq k[x_1, \ldots, x_n]$ and $X_1, X_2 \subseteq \mathbb{A}^n(k)$. Then,
(1) $S_1 \subseteq S_2 \Rightarrow V(S_2) \subseteq V(S_1)$.
(2) $X_1 \subseteq X_2 \Rightarrow I(X_2) \subseteq I(X_1)$.
(3) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
(4) $\forall S \subseteq k[x_1, \ldots, x_n]$, $S \subseteq I(V(S))$ and $V(S) = V(I(V(S)))$. In particular, if A is an affine algebraic set, then $A = V(I(A))$.
(5) $X \subseteq \mathbb{A}^n(k)$ as a subset, $X \subseteq V(I(X))$ and $I(X) = I(V(I(X)))$. In particular, if J is the ideal of some affine algebraic set, then $J = I(V(J))$.
(6) $\forall \mathfrak{a} \trianglelefteq k[x_1, \ldots, x_n]$, $\sqrt{\mathfrak{a}} = I(V(\mathfrak{a}))$. If J is the ideal of some affine algebraic set, then J is a radical ideal.
(7) $\forall X \subseteq \mathbb{A}^n(k)$, $\overline{X} = V(I(X))$.

Proof: (1) and (2) follows quickly from the definition.
For (4),
For (5), because $X \subseteq V(I(X))$ and because $V(I(X))$ is a closed set containing X, $\overline{X} \subseteq V(I(X))$. To show the other direction, pick any closed set W of $\mathbb{A}^n(k)$ containing X. We want $V(I(X)) \subseteq W$ for every such closed set W. As a closed set, $W = V(\mathfrak{a})$ for some $\mathfrak{a}$. $X \subseteq W = V(\mathfrak{a}) \Rightarrow \mathfrak{a} \subseteq I(V(\mathfrak{a})) \subseteq I(X)$. Applying V to this inclusion again, $V(I(X)) \subseteq V(\mathfrak{a}) = W$. Because $V(I(X)) \subseteq W$ for every closed W containing X.

$$V(I(X)) \subseteq \bigcap_{\substack{X \subseteq W \\ W \text{ is closed}}} W = \overline{X}$$

$\square$

1.1.2 Irreducible components: affine varieties

An affine algebraic set might be the union of several smaller algebraic sets. Seeking the decomposition of an affine algebraic sets seems necessary. To decompose, the irreducibility in the sense of topological space is at our disposal. Two things are about to be mentioned: (1) How to tell an affine algebraic set is irreducible. (2) property of writing an affine algebraic set into irreducible ones.
Here is a characterisation of irreducible affine algebraic sets.

Proposition 1.3: Algebraic characterisation of irreducibility

An affine algebraic set $X \subseteq \mathbb{A}^n(k)$ is irreducible $\Leftrightarrow$ $I(X)$ is prime.

Proof: $\Rightarrow$ Suppose that $I(X)$ is not prime. Then, $\exists f, g,\ fg \in I(X)$ but $f, g \notin I(X)$. These imply that $X = V(I(X)) \subseteq V(fg)$ and $X \nsubseteq V(f), V(g)$. So, there is a decomposition $X = (X \cap V(f)) \cup (X \cap V(g))$ with $X \cap V(f), X \cap V(g) \subsetneqq X$.
$\Leftarrow$ If X is reducible with the decomposition $X = X_1 \cup X_2$. $X_i \subsetneqq X$ implies that $I(X) \subsetneqq I(X_i)$. For each i, let f_i be the element that $f_i \in I(X_i)$ but $f_i \notin I(X)$. But $f_1 f_2 \in I(X_1 \cup X_2) = I(X) \Rightarrow I(X)$ is not prime. $\square$

Next, the decomposition of an affine algebraic set only yields finitely many irreducible components. Suppose X is reducible, then $X = X_1 \cup X_2$. If any one of X_i is reducible, then X_i can be further decomposed. This process eventually terminates because: $k[x_1, \dots, x_n]$ is a Noetherian ring for any field. In a Noetherian ring, any collection of ideals has the maximal element. The collection of affine algebraic sets $\{V_i\}$ are sets that appear during the decomposition. It has a counterpart $\{I(V_i)\}_i$ which is a collection of ideals of $k[x_1, \dots, x_n]$. So, there is a maximal one $I(V_\alpha)$ for some α. Correspondingly, V_α is the minimal one, indicating the terminal of decomposition.

Uniqueness of decomposition

The building blocks of some object are the fundamental and simpler ones to study. For affine algebraic sets, their building blocks are those irreducible components. So, it is worth picking them out and assigning it a name:

Definition 1.3: Affine and quasi-affine variety

An **affine variety** $X \subseteq \mathbb{A}^n$ is an irreducible affine algebraic set.
A **quasi-affine** variety is a)n((Zariski) open subset of an affine variety.

$$\begin{tikzcd} \text{Affine algebraic sets} \arrow[d, "\text{irreducible components}"'] \\ \text{Affine varieties} \end{tikzcd} \qquad \begin{tikzcd} & X = \bigcup_j X_j \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ X_1 & \cdots & X_i & \cdots \end{tikzcd}$$

1.2 Projective spaces and projective varieties

The projective n-space is defined to be the quotient $\mathbb{A}^{n+1} - \{\mathbf{0}\}/_\sim$ where $\sim$ is the equivalence relation defined by $(a_0, a_1, \dots, a_n) \sim (\lambda a_0, \lambda a_1, \dots, \lambda a_n)$ for all $\lambda \in k - \{0\}$. Any representative of a point $P \in \mathbb{P}^n$ is called a set of homogeneous coordinate of P. Such a P is written explicitly as $P = [a_0 : a_1 : \cdots : a_n]$.
Then, for an $f \in k[x_1, x_2, \dots, x_n]$, a function $f : \mathbb{P}^n \to k$ cannot be defined normally.
• Suppose $\deg f = d$, a natural way to define f is:

$$f([a_0 : a_1 : \cdots : a_n]) = f(a_0, a_1, \dots, a_n)$$

But for another representative of $[a_0 : \cdots : a_n]$, $[\lambda a_0 : \cdots : \lambda a_n]$, the value of f is not invariant under different choice of representatives.
$(*)$ However, notice that being 0 is a property that is independent of choices of $[a_0 : \cdots : a_n]$ [1] if and only if f is a homogeneous polynomial: $\Leftarrow$ Suppose f is homogeneous. When $f([a_0 : \cdots : a_n]) = f(a_0, \dots, a_n) = 0$, $\forall \lambda \in k - \{0\}$, $f([\lambda a_0 : \cdots : \lambda a_n]) = \lambda^d f(a_0, \dots, a_n) = 0$.

$\Rightarrow$ Suppose that f is not homogeneous and $\deg f = d$, then f can be decomposed into sum of homogeneous polynomials: $f = \sum_{i=0}^d f_i$, with each f_i a homogeneous polynomial of degree i and at least two different terms are not

[1] meaning $f(P) = 0$ is well-defined

zero. For some $P \in \mathbb{P}^n$, and one representative $P = [c_0 : \cdots : c_n]$, $f(c) = f(c_0, \ldots, c_n) = 0$. Here $c = (c_0, \ldots, c_n)$. If $f(P) = 0$ is well-defined, $f(\lambda c) = 0$ should work for all λ. We are going to see that $f(\lambda c_0, \ldots, \lambda c_n) \neq 0$ for infinitely many λ. Hence, $f(P) = 0$ is not well-defined. $f(\lambda c)$ can be viewed as a polynomial in λ, $G(\lambda)$:

$$G(\lambda) := f(\lambda c) = f_d(c)\lambda^d + f_{d-1}(\lambda c)\lambda^{d-1} + \cdots + f_1(c)\lambda + f_0(c)$$

As assumed, k is an algebraically closed field of char$k = 0$. k is an infinite field. But $G(\lambda)$ has at most d roots. So, there are infinitely many elements that make $f(\lambda c) \neq 0$.

• So, for f being homogeneous, a possible way to define function on $\mathbb{P}^n$ is

$$f([a_0 : \ldots, a_n]) := \begin{cases} 0 & \text{if it is zero for some representative of } [a_0 : \cdots : a_n] \\ 1 & \text{else} \end{cases}$$

So far, we have got a function $f : \mathbb{P}^n \to \{0, 1\}$.

The vanishing sets and ideals are similar to those in affine case:

Definition 1.4: Projective algebraic sets

A **(projective) algebraic set** $X \subseteq \mathbb{P}^n$ is a vanishing set of some collection of homogeneous polynomials $S \subseteq k[x_1, \ldots, x_n]$, i.e. $X = V(S)$.

Remark To make this definition play well, S could not be an arbitrary subset of $k[x_1, \ldots, x_n]$. More explicitly, $V(S) = \{P \in \mathbb{P}^n : \forall f \in S, f(P) = 0\}$. The condition $f(P) = 0$ occurs. To make $f(P) = 0$ well-defined, the above-mentioned equivalence $(*)$ suggests that f should be homogeneous.

Proposition 1.4: Projective algebraic sets forming a topology

The union of two projective algebraic sets is still an projective algebraic sets. The intersection of any family of projective algebraic sets is still an projective algebraic set. $\varnothing$ and $\mathbb{P}^n(k)$ are projective algebraic sets.

Definition 1.5: Zariski topology for $\mathbb{P}^n$

Define the projective algebraic sets to be the closed set of $\mathbb{P}^n$. This topology is called the Zariski topology of $\mathbb{A}^n$.

From the proposition 1.1, this is indeed an topology.

1.2.1 Irreducible components: projective variety

Definition 1.6: Projective and quasi-projective variety

A **projective variety** $X \subseteq \mathbb{P}^n(k)$ is an irreducible projective algebraic set.
A **quasi-projective** variety is a)n((Zariski) open subset of a projective variety.

Example (1) Grassmannian

1.2.2 Intertwining of affine and projective spaces

In this little section, the goal is to see that $\mathbb{P}^n$ has an open covering by $\mathbb{A}^n$. Whence, every projective variety has an open cover by affine varieties.

1.3 Correspondence: Hilbert Nullstellensatz

Definition 1.7: Coordinate ring

1.3.1 Affine Nullstellensatz

1.3.2 Projective Nullstellensatz

1.4 Dictionary of algebra and geometry

$$\begin{aligned} \text{irreducible closed sets} &\longleftrightarrow \text{prime ideals} \\ \text{points} &\longleftrightarrow \text{maximal ideals} \\ \text{closed subsets} &\longleftrightarrow \text{radical ideals} \end{aligned}$$

In this Correspondence, points are the 'minimal' irreducible closed sets. This coincides with the fact that maixmal ideals are prime.

1.5 Morphisms: Regular and Rational functions

1.5.1 Regular functions/maps on affine quasi-affine varieties

The hierarchy of definition of morphisms is:

$$\begin{tikzcd} \text{Regular functions } f : \mathbb{A}^n(k) \supseteq X \to \mathbb{A}^1(k) \arrow[r, rightsquigarrow] \arrow[d] & \text{Locally: All regular functions on } U : \mathcal{O}_X(U) \arrow[d] \\ \text{Regular maps } \varphi : \mathbb{A}^n \supseteq X \to Y \subseteq \mathbb{A}^m(k) \arrow[r, rightsquigarrow, "\text{induces}"] & \text{morphism between them} : \varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U)) \end{tikzcd}$$

In the diagram above, $X \subseteq \mathbb{A}^n(k)$ and $Y \subseteq \mathbb{A}^m(k)$ are quasi-affine varieties. When $U = X$, the induced morphism becomes

Let X be a quasi-affine variety of $\mathbb{A}^n(k)$. Consider $\tilde{f} : X \to \mathbb{A}^1(k)$.

Definition 1.8: Regular functions/maps on (quasi) affine varieties

A function $\tilde{f} : X \to \mathbb{A}^1(k)$ is a **regular function** at $P \in X \Leftrightarrow \exists$ open neighbourhood $U(\ni P) \subseteq X$, and $g, h \in k[X] = k[x_1, \ldots, x_n]/I(X)$, with $h \neq 0$ on U such that $f(x) = \frac{g}{h}(x)$ on U.
The collection of all regular functions on U can be assign a symbol $\mathcal{O}_X(U)$.
A function $\tilde{f} : X \to Y$ is a **regular map** at $P \in X \Leftrightarrow \exists$ open neighbourhood $U(\ni P) \subseteq X$, and $\exists \tilde{f}_1, \ldots, \tilde{f}_m \in \mathcal{O}_X(U)$, such that $\forall x \in U$, $\tilde{f}(x) = (\tilde{f}_1(x), \ldots, \tilde{f}_m(x))$.
Such an $\tilde{f}$ is **regular** (as a function or a map) on $X \Leftrightarrow \tilde{f}$ is regular at every $P \in X$.

Remark A biregular function is a regular function which has an inverse function that is also regular. Biregular maps are isomorphisms for category of affine varieties.

Example • A regular function is a regular map. So, from now on, we will only say regular map.

Now we have two objects, the ring of regular functions on U,$\mathcal{O}_X(U)$; and the coordinate ring $k[X]$. The ring of regular functions is somehow geometric, while the coordinate ring $k[X]$ is algebraic. To distinguish their elements, $\tilde{f}$ is for elements of $\mathcal{O}_X(U)$ while f or $\overline{f}$ means an element in $k[X]$. The function/map $\tilde{f}$ can be regarded as the evaluation homomorphism of f.

In the following parts, we will identify them (here X is just temporarily an affine variety/ *closed*):

$$\mathcal{O}_X(X) \cong k[X]$$

For a quasi-affine variety X, this is not always true. Consider $X = \mathbb{A}^1(k)\backslash\{0\}$. $(f : t \mapsto \frac{1}{t}) \in \mathcal{O}_X(X)$. But, f cannot be written as a polynomial even locally. Notice that $\mathcal{O}_X(X) \cong k[t, \frac{1}{t}] = k[t]_t$, as a special case of the following:

We can even identify more, $\mathcal{O}_X(U)$ with some ring. Explicitly, here U should take the distinguished open sets $D(\tilde{f})$, then it can be identified with the localization of $k[X]$ at $\{1, f, f^2, \ldots\}$, i.e.

$$\mathcal{O}_X(D(\tilde{f})) \cong k[X]_f$$

Before that, let's get some sense of the **properties of regular functions**.

(1) The first thing is *continuity*:

Lemma 1.1: Continuity of regular functions on (quasi) affine variety

Let $\tilde{f} : X \to \mathbb{A}^1(k)$ be a regular function on quasi-affine variety $X \subset \mathbb{A}^n(k)$. $\tilde{f}$ is continuous with respect to Zariski topology.

Proof: It suffices to show that $\forall$ closed subset W of $\mathbb{A}^1(k)$, $f^{-1}(W)$ is closed. But every closed subset of $\mathbb{A}^1(k)$ is a finite set. [2] So, it suffices to show that $\forall a \in k$, $f^{-1}(\{a\})$ is closed. This is further equivalent to the intersection of $f^{-1}(\{a\})$ and every open set in *an* open cover of X [3] is closed in X. In particular, $\forall P \in X$, let U_P be the neighbourhood of P where $f = \frac{g}{h}$ and $h \neq 0$ on U_P. Then,

$$f^{-1}(\{a\}) \cap U_P = \{P \in U_P : \frac{g(P)}{h(P)} = a\} = \{P \in U_P : (g - ah)(P) = 0\} = V(g - ah) \cap U_P$$

Hence, $f^{-1}(\{a\}) \cap U_P$ is closed in U_P. $\square$

(2) The second thing is how to *reconcile local and global regular functions/maps*:
Even though a regular map can be locally written as a fraction of polynomials, but when it is regular globally, the h in the fraction $\frac{g}{h}$ must be a constant function. So, a global regular map is barely a polynomial over k:

Corollary 1.1: Globally regular= constant

Let $\tilde{f} : X \to Y$ be a regular map on the whole X. Then, the corresponding f is a polynomial.

Proof: $\square$

The morphism between two rings/algebras of regular functions
Now. Given the morphism/ regular maps $f : X \to Y$, and an open subvariety U of Y, let's define the map

$$f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$

by

$$\varphi \mapsto \varphi \circ f$$

This map is well-defined: $\varphi \circ f$ is a regular function on $f^{-1}(U)$. Since $f(f^{-1}(U)) \subseteq U$, it is okay to define $\varphi \circ f$ on $f^{-1}(U)$. $\varphi \circ f$ is regular because the composition of two regular functions is regular.
Now, we give a characterisation when f^* is *injective*:

Proposition 1.5: Characterisation of injectivity of the induced map

The morphism $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is injective $\Leftrightarrow$ the image $f(X)$ is dense in Y.

Proof: Let's give an equivalent condition for injectivity of f^*: Take an arbitrary $u \in \ker f^*$, which $\Leftrightarrow u \circ f = f^*(u) = 0 \Leftrightarrow f(X) \subseteq \ker u = V(u)$. Since $\ker u$ is closed, the condition $f(X) \subseteq \ker u \Leftrightarrow \overline{f(X)} \subseteq \ker u \Leftrightarrow u \in I(\overline{f(X)})$. So, $u \in \ker f^* \Leftrightarrow u \in I(\overline{f(X)})$ implies that $\ker f^* = I(\overline{f(X)})$.
In particular, $\ker f^* = \{0\} \Leftrightarrow I(\overline{f(X)}) = \{0\}$. Since $\overline{f(X)} \subseteq Y \Rightarrow I(Y) \subseteq I(\overline{f(X)})$, $\ker f^* = \{0\} \Leftrightarrow I(Y) = I(\overline{f(X)}) = \{0\} \Leftrightarrow Y = \overline{f(X)}$. Thus, injectivity of f^* is equivalent to $\overline{f(X)} = Y$. $\square$
Remark Certainly when $f(X) = Y$, f^* is injective. It is also possible that $f(X)$ is dense in Y but $f(X) \subsetneqq Y$: Let $X = V(xy - 1)$. The map $\tilde{f} : X \to \mathbb{A}^1(k)$ is defined by $(a, b) \mapsto a$.

This is a regular function because $\tilde{f}$ can be represented by a polynomial(monomial) $F(x, y) = 1 \cdot x + 0 \cdot y$, such that $\forall (a, b)$, $\tilde{f}(a, b) = F(a, b)$.

$\tilde{f}(X) = \mathbb{A}^1 \backslash \{0\} \subsetneqq \mathbb{A}^1(k)$. While, $\overline{f(X)} = \mathbb{A}^1(k)$ because closed sets of $\mathbb{A}^1(k)$ are either the whole space $\mathbb{A}^1(k)$ or a finite set. Since $f(X)$ has infinite many points, it is dense in $\mathbb{A}^1(k)$.

1.5.2 Regular functions on projective, quasi-projective varieties

For a quasi-projective variety $Y \subseteq \mathbb{P}^n$, there is a similar definition for regular maps.

[2] Because a closed set is $V(S)$ for some $S \subseteq k[x]$. Combining the fact that $V(S) = V(\langle S \rangle)$ and $k[x]$ is a PID, $\langle S \rangle = \langle r \rangle$ for some $r \in k[x]$. r is a polynomial and it has $\deg r$ many (finitely many) zeros. Hence, $V(S) = V(r)$ is a finite set.
Another way to see is: Since every affine closed set is a finite union of irreducible closed sets. Irreducible closed sets of $\mathbb{A}^1(k)$ are of the form $V(x - a)$ and are singletons.
[3] Precisely, this equivalency is $Z \subseteq Y$ is closed $\Leftrightarrow \exists$ an open cover $\mathcal{U} = \{U_i\}_i$ of Y,$\forall i$, $U_i \cap Y$ is closed in U_i

Definition 1.9: Regular functions on (quasi) projective varieties

A function $\tilde{f} : Y \to \mathbb{A}^1(k)$ is **regular** at $P \in Y \Leftrightarrow \exists$ open neighbourhood $U(\ni P) \subseteq Y$, and $\exists$ *homogeneous* $g, h \in k[x_1, \ldots, x_n]$ *of the same degree*, with $h \neq 0$ on U such that $\tilde{f} = \frac{g}{h}$ on U.
Such an $\tilde{f}$ is **regular** on $Y \Leftrightarrow \tilde{f}$ is regular at every $P \in Y$.

Remark Even though $\tilde{g}, \tilde{h}$ are not 'normal' functions on Y, there quotient is when $h \neq 0$. Since every affine variety can be embedded into a projective variety. So is a quasi-affine variety. This inspire us that definition 1.9 should include 1.8 as its special cases. How does this definition-inclusion concretely work? By homogenization and dehomogenization!
Consider the inclusion $X \hookrightarrow \mathbb{A}^n(k) \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ where $\iota : (a_1, \ldots, a_n) \mapsto [1, a_1, \cdots, a_n]$. The f correpsonding to the function $\tilde{f} : X \to k$ can be homogenized to f_H

$$f_H(x_0, x_1, \ldots, x_n) = x_0^{\deg f} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

such that $\tilde{f_H}([1, a_1, \ldots, a_n]) = f_H(1, a_1, \ldots, a_n) = f(a_1, \ldots, a_n) = \tilde{f}((a_1, \ldots, a_n))$.

1.5.3 Rational maps and classification

As the fraction of a regular function over an affine variety or an affine set, rational functions consists of a field which encodes geometry of that affine variety into algebraic information. As a consequence, birational equivalence shows up to classify varieties.

Definition 1.10: Rational functions

A ration function on X is a element $\frac{f}{g}$, where $f, g \in \mathcal{O}_X(X)$ are regular maps.

not well-defined for f/g

Rational maps for closed subset $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$: $\varphi : X \to Y \Leftrightarrow \varphi = (\frac{g_1}{h_1}, \ldots, \frac{g_m}{h_m})$.
Rational maps for closed subset $X \subset \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$:

Definition 1.11: Rational maps

Let X, Y be two varieties. A rational map is an equivalence o

rational functions $f \in k(X)$ rational maps rational map on varieties

Definition 1.12: Stalk of rational functions

Definition 1.13: Birational maps

Let $\varphi : X \dashrightarrow Y$ be a rational map. φ is **birational** $\Leftrightarrow \exists$ inverse map ψ of φ, $\psi : Y \dashrightarrow X$ is also rational.

1.6 Quasi-Projective variety

Definition 1.14: Sub quasi-projective varieties

Let X be a quasi-projective variety. A sub quasi-projective variety Z of X is a subset of X and Z itself is also a quasi-projective variety.
Moreover, when Z is an (Zariski) open subset of V, Z is called an open sub (quasi-projective) variety. Similar for closed case.

Remark When Z is an open sub (quasi-projective) variety, Z itself is a quasi-projective variety. Since Z is open X and X is open in some projective variety Y. Z is open in Y implies that it is quasi-projective.
When Z is a closed sub (quasi-projective) variety, Z itself is Two pathways:
(1) Fulton's way: embedding V into an 'ugly' space $\mathbb{P}^{n_1}(k) \times \cdots \times \mathbb{A}^m(k)$.
(2) The standard way: embedding V into a projective space $\mathbb{P}^N$.
The way how 'Fulton' deals with a quasi-projective saves a lot of efforts to study a product of varieties. Consider a product $V \times W$ with $V \subseteq \mathbb{P}^r$ and $W \subseteq \mathbb{P}^s$. To see $V \times W$ is a quasi-projective variety, the standard definition requires an embedding of a product into a projective variety in the space $\mathbb{P}^N$. But, it is usually hard to get such an embedding. However, the Segre embedding reconciles them.
While, in Fulton's book, he wants to avoid such work at the cost of embedding the product into an 'ugly' space. In this 'ugly' space, checking whether a product is a quasi-projective variety is easy: as long as each component is a projective variety as in definition 1.6. Here we show the equivalence under the "ugly' space frame'. So, from now on one can use it without any hesitation.

Corollary 1.2: Equivalence of quasi-projective varieties

Let V be a non-empty irreducible algebraic set $\subseteq \mathbb{P}^{n_1}(k) \times \cdots \times \mathbb{A}^m(k)$. A set $X \subseteq \mathbb{P}^{n_1}(k) \times \cdots \times \mathbb{A}^m(k)$ is a **quasi-projective variety** $\Leftrightarrow$ X is an open subset of V.

Proof: The product space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \times \mathbb{P}^m$ (and by extension, products involving affine spaces $\mathbb{A}^m$ which are open subsets of $\mathbb{P}^m$) can be embedded as a closed subvariety into a higher-dimensional projective space $\mathbb{P}^N$ via the Segre embedding

$$\sigma : \mathbb{P}^{n_1}(k) \times \cdots \times \mathbb{P}^m(k) \hookrightarrow \mathbb{P}^N$$

V can be embedded into $\mathbb{P}^N(k)$ and the image of V, $\sigma(V)$, is also an irreducible algebraic set in this $\mathbb{P}^N$, where irreducibility comes from: σ is a topological embedding and closedness of $\sigma(V)$ is because σ is a closed immersion. Thus, $\sigma(V)$ is a projective variety. Also, σ induces an isomorphism of varieties $V \cong \sigma(V)$. Thus, we regard V as a projective variety.
Hence, X is a quasi-projective variety (openness is preserved by σ) by definition 1.6, showing these two definitions are equivalent. $\square$

1.6.1 Morphisms of quasi-projective varieties

Let X be a quasi-projective variety.

$$\Gamma(U, \mathcal{O}_X) := \{\text{rational functions } \varphi \text{ on } X\text{: } \varphi \text{ has definition on each } p \in U\}$$

This means that a $\varphi \in \Gamma(U, \mathcal{O}_X)$ might not have definition on the whole X. If $X = \mathbb{C}$, $\Gamma(U, \mathcal{O}_X)$ is all meromorphic functions that are defined on U. not quite sure.
Remark $\Gamma(U, \mathcal{O}_X)$ is a ring and $\mathcal{O}_X$ is indeed a sheaf. When X is an affine variety, $\Gamma(X, \mathcal{O}_X)$ is the coordinate ring of X. Let k^U be the set of all functions from U to k. Then, there is a mapping

$$\Gamma(U, \mathcal{O}_X) \to k^U \quad \varphi \mapsto \big(p \mapsto \varphi(p)\big)$$

which is a ring homomorphism. This mapping allows us to regard $\Gamma(U, \mathcal{O}_X)$ as real functions (real=genuine). To see this is indeed an embedding/injective, we have the following proposition

Proposition 1.6: Injectivity of embedding as a function

Proof: $\square$
Remark This shows not only the injectivity of the mapping. But, it also conveys that
(1) A rational function is zero locally implies it is zero globally. It has a counterpart in real/complex analysis, the identity theorem or analytic continuation:
If f, g are analytic functions on a domain D (open and connected subset of $\mathbb{R}$ or $\mathbb{C}$), and $f = g$ on SD with $p \in S$ an accumulation point in D, then $f = g$ on D.
This property is not shared for a C^∞ function. A smooth function are 'soft', i.e. it could be locally zero but not globally zero.
While, a holomorphic/meromorphic function, or a polynomial/rational function, is rigid, meaning some strong

restrictions put on it. This proposition utilizes the fact that open sets in Zariski topology are 'huge' and dense, which is from the property(rigidity) of polynomials. Each polynomial has finite zeros, so its complements, those open sets are almost the whole space. For holomorphic functions, even though the topology is Euclidean, the rigidity of holomorphic functions compensates the problem of topology being too fine. So, local information reveals the global information. (2) The second thing is a morphism between two sheaves can be extracted: define $\mathscr{F}_X$ to be $\mathscr{F}_X(U) := k^U$. The map mentioned above yields a morphism

$$\mathcal{O}_X \to \mathscr{F}_X$$

1.7 Product of varieties

1.8 Abstract variety

We just introduced affine, projective varieties and embedded them into a projective space as quasi-projective spaces. A subtle assumption is the existence of the 'background', a huge projective space $\mathbb{P}^n$.
But this idea has some drawbacks:
(1) It might be hard to find an embedding: Consider a torus $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$. This is not a variety at the first glance until we find a way to embed it into some projective space $\mathbb{P}^N(k)$. Indeed, such an embedding does exist: the Segre embedding

$$\mathbb{P}^1(k) \times \mathbb{P}^1(k) \to \mathbb{P}^1(k) \qquad ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$$ [4]

(2) It leads some problem when studying some intrinsic properties: The classic definitions of varieties, affine/ projective varieties or quasi-projective varieties, rely an embedding into an ambient space, and this embedding was used to define the topology on the variety and the regular functions on the variety.
But there are many ways to embed such a space into an ambient space $\mathbb{P}^N$. For each embedding, there is a certain kind of coordinates to represent this varieties. For example, for some varieties V that can be embedded into $\iota : V \to \mathbb{P}^1(k)$, it can also be embedded into $\mathbb{P}^2(k)$ by composing the Veronese embedding $\nu : \mathbb{P}^1(k) \hookrightarrow \mathbb{P}^2(k)$. The first embedding ι uses coordinates of the form $[x : y]$ while $\nu \circ \iota$ uses the coordinates $[u : v : w]$.
Some intrinsic properties of a variety does not rely on the coordinates. But, under the extrinsic definition (embedding it into an ambient space), their are many embeddings. So, to check that a property does not rely on a certain embedding, we have to check it works for any two such embeddings.

These problems requires an intrinsic definition of a variety to avoid unnecessary difficulties.

$$\text{affine, projective varieties} \subseteq \text{quasi-projective varieties} \subseteq \text{abstract varieties}$$

1.9 Category of variety

1.10 Dimensions

1.11 New language: Spectrum of a ring

Using the 'dictionary', the vanishing set $V(f) = \{x \in \mathbb{A}^n(k) : f(x) = 0\}$ has a generalised version adapted to the Spec$k[x_1, \dots, x_n]$, $V(f) = \{[\mathfrak{p}] \in \text{Spec}k[x_1, \dots, x_n] : \langle f \rangle \subseteq \mathfrak{p}\}$ Are these generalised 'vanishing sets' still forming a topology by serving as closed sets? The answer is affirmative.
Open sets in Spec Distinguished open sets.

Proposition 1.7: Properties of distinguished open sets

Let R be a commutative ring and $f, g \in R$ being non-zero. Then the followings are equivalent:
(1) $D(f) \subseteq D(g)$ (2) $\exists n \geq 1,\ f^n \in \langle g \rangle$ (3) g is invertible in R_f [a]

[a] R_f is the localization of R at the multiplicative set $\{1, f, f^2, \dots\}$

Proof: (1) $\Rightarrow$ (2) $D(f) \subseteq D(g) \Leftrightarrow V(g) \subseteq V(f)$. Explicitly, $\forall$ prime ideal $\mathfrak{p}$ such that $\langle g \rangle \subseteq \mathfrak{p}$, $\langle f \rangle \subseteq \mathfrak{p}$. So, f is in $\bigcap_{\substack{[\mathfrak{p}] \in \text{Spec}R \\ g \in \mathfrak{p}}} \mathfrak{p} = \sqrt{\langle g \rangle}$, meaning that $\exists n \geq 1,\ f^n \in \langle g \rangle$.
(2) $\Rightarrow$ (1) If $f^n \in \langle g \rangle$ for some $n \geq 1$, then $\exists a \in R,\ f^n = ag$. $\forall$ prime ideal $\mathfrak{p}$ with $g \in \mathfrak{p}$, $f^n = ag \in \mathfrak{p}$. Primity of $\mathfrak{p}$

[4] In general, the Segre embedding is

$$\mathbb{P}^m(k) \times \mathbb{P}^n(k) \to \mathbb{P}^{(m+1)(n+1)-1}(k)$$

via $([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \mapsto [x_0y_0 : \cdots : x_0y_n : \cdots : x_my_n]$

yields $f \in \mathfrak{p}$, showing $V(g) \subseteq V(f)$.
(2) $\Rightarrow$ (3) $f^n = ag$ for some $a \in R$, and $\frac{a}{f^n} \in R_f$.
(3) $\Rightarrow$ (2) If g is invertible, $\exists h \in R_f$ of the form $h = \frac{a}{f^m}$ such that $gh = \frac{ga}{f^m} = 1 = \frac{1}{1} \Leftrightarrow \exists r \in \{1, f, \dots\}$(so $R = f^k$ for some $k \geq 0$), $(ga - f^m) \cdot r = gaf^k - f^{m+k} = 0$. Let $n = m + k$, $f^n = (af^m)g \in \langle g \rangle$. □

2 Local properties

2.1 Singularities and regularities

2.2 Tangent spaces

3 An introduction to birational geometry

3.1 Resolution of singularities

Let X, Y be two varieties with Y being non-singular. Then, a *surjective* and *birational* map $Y \to X$ is called a **resolving of singularities**.

3.2 Blow-ups

The blow-up is important in basically two senses:
(1) As a collection of examples of birational maps.
(2) As a canonical way of resolving sigularities.

3.2.1 First encounter: intuition

Now, we first do the blow-up in affine varieties to get some intuition. After this, blowing-ups have many levels of generalization: blowing-up of an arbitrary variety, blowing-up a scheme.

Definition 3.1: Blowing-up of an affine variety

In an affine space $\mathbb{A}^n(k)$, let $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n(k))$ be the closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ determined by the following: $\forall i, j$,

$$x_i y_j = x_j y_i$$

with $(x_1, \dots, x_n) \in \mathbb{A}^n$ and $[y_1 : \cdots : y_n] \in \mathbb{P}^{n-1}$.
Then, $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n(k))$ is called the **blowing-up** of the base space $\mathbb{A}^n$ at $\{0\}$. $\{0\}$ is called **the center of blow-up** and $\mathbb{A}^n$ is the base space.

Remark The space $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ is always fixed.
Explicitly, $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n) = V(S)$ with $S = \{x_i y_j - x_j y_i | 1 \leq i, j \leq n\}$. Each $x_i y_j - x_j y_i$ is homogeneous for all y_k of degree 1. So, $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ is **closed**.
Let's look at $\mathbb{A}^2$ case to get make more sense of this definition. The process of blowing-up is kind of lifting the plane $\mathbb{A}^2$ to a 'helicoid' in a larger space. Intuitively, this process is just picking every points on a line with a fixed slope in the plane, and then lift this line with a certain slope to the position above the line in $\mathbb{A}^2$ with height being the slope in the space. More than that, if one trace any point in $\mathbb{A}^2$ except the center of blow-up vertically, it should meet the blowing-up space $\mathrm{Bl}_{\{0\}}(\mathbb{A}^2)$ only once. The next theorem rigorously reflects such intuition.
For some curves in $\mathbb{A}^2$ with sigularities, $\mathrm{Bl}_{\{0\}}(\mathbb{A}^2)$ provides a stage that those curves can live in without any singularities. So, intuitively, any curve should be non-singular after being embedded in the blowing-up space.

Theorem 3.1: Properties of blowing-ups

The morphism $\varphi : \mathrm{Bl}_{\{0\}}(\mathbb{A}^n(k)) \to \mathbb{A}^n$ has the following properties:
(1) $\varphi^{-1}(0) \cong \mathbb{P}^{n-1}$. Here $\varphi^{-1}(0)$ is called the exceptional fibre.
(2) $\varphi|_{\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)\setminus\varphi^{-1}(0)} : \mathrm{Bl}_{\{0\}}(\mathbb{A}^n)\setminus\varphi^{-1}(0) \to \mathbb{A}^n(k)\setminus\{0\}$ is an isomorphism.
(3) There is an bijection

$$\{\text{lines of } \mathbb{A}^n \text{ through } 0\ \} \longleftrightarrow \varphi^{-1}(0) \quad l \mapsto p_l$$

Moreover, $\overline{\varphi^{-1}(l-\{0\})} \cap l = p_l$.
(4) $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n(k))$ is irreducible.

Proof: (1) Since $\varphi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}(k)$. The isomorphism comes naturally.
(2) Let $X := \mathrm{Bl}_{\{0\}}(\mathbb{A}^n(k))$. Define a function

$$g : \mathbb{A}^n(k)\setminus\{0\} \to X\setminus\varphi^{-1} \quad x \mapsto (x, [x])$$

where $x = (x_1, \cdots, x_n)$ and $[x] = [x_1 : \cdots : x_n]$. It is quick to see that $\varphi|_{X\setminus\varphi^{-1}(0)} \circ g : x \mapsto (x, [x]) \mapsto x$ is an identity. To see that $g \circ \varphi|_{X\setminus\varphi^{-1}(0)} = \mathrm{id}_{X\setminus\varphi^{-1}(0)}$, it suffices to see that: $\varphi|_{X\setminus\varphi^{-1}(0)}$ is injective since for any $(x, [y]) \in X\setminus\varphi^{-1}(0)$, $\exists x_i \neq 0$. Hence, $\forall j$, $x_i y_j = x_j y_i$ yields

$$y_j = x_i^{-1} x_j y_i$$

The projective coordinate of $[y]$ also restricts that for this i, $y_i \neq 0$. Hence, $[x] = [y] \Rightarrow g \circ \varphi|_{X\setminus\varphi^{-1}(0)} = \mathrm{id}_{X\setminus\varphi^{-1}(0)}$. So, $\varphi|_{X\setminus\varphi^{-1}(0)}$ is a bijection.
Also, $\varphi|_{X\setminus\varphi^{-1}(0)}$ is well-defined and a morphism.
(3) Write a line l in the form

$$l = \{t(x_1, x_2, \cdots, x_n) | t \in k\}$$

The map from the collection of lines to $\varphi^{-1}(0)$ is

$$l \leftrightarrow [x_1 : x_2 : \cdots : x_n]$$

Then, $\varphi^{-1}(l - \{0\}) = \{(tx, [x]) : t \in k, t \neq 0\}$ for some $x \in \mathbb{A}^n(k)$. Take any $f \in I(\varphi^{-1}(l - \{0\}))$. Notice that $\underline{f(0, [x]) = 0}$ $(t = 0)$. Hence, $(0, [x]) \in V(I(\varphi^{-1}(l - \{0\})))$. From the propsition 1.2 (7), $V(I(\varphi^{-1}(l - \{0\}))) = \overline{\varphi^{-1}(l - \{0\})}$. Also, $(0, [x]) \in \varphi^{-1}(0)$.

□

Remark explanation of each entry