Solutions for algebraic curves by Fulton

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Contents

1 Affine algebraic sets

1.1 The Hilbert basis theorem

Question 1.1: 1.22

Let I be an ideal of a ring R and consider $\pi : R \to R/I$. Then,
(1) There is a one-to-one correspondence

$$\{\text{ideals of } R/I\} \longleftrightarrow \{\text{ideals of } R \text{ containing } I\}$$

More concretely, $\forall\ J' \trianglelefteq R/I$, $\pi^{-1}(J') \trianglelefteq R$ containing I. $\forall\ J \trianglelefteq R$ containing I, $\pi(J) \trianglelefteq R/I$.
(2) J' is radical $\Leftrightarrow$ J is radical. So, this one-to-one correpsondence can be refined to

$$\{\text{radical ideals of } R/I\} \longleftrightarrow \{\text{radical ideals of } R \text{ containing } I\}$$

(3) J is finitely generated $\Rightarrow$ J' is finitely generated, and R is Noetherian $\Rightarrow$ R/I is Noetherian.

1.2 Irreducible components for algebraic sets

Question 1.2: 1.25

Question 1.3: 1.26

Show that $F = Y^2 + X^2(X-1)^2 \in \mathbb{R}[X,Y]$ is irreducible but $V(F)$ is reducible.

Solutions: F only has factors $Y + iX(X-1)$ and $Y - iX(X-1)$ which are not in $\mathbb{R}[X,Y]$. So, F is irreducible over $\mathbb{R}$. $V(F) = \{(0,0),(1,0)\}$ is a union of two singletons hence is reducible.

Question 1.4: 1.27

Question 1.5: 1.28

Question 1.6: 1.29

When k is infinite, $\mathbb{A}^n(k)$ is irreducible.

Solutions: Suppose that $\mathbb{A}^n(k)$ is reducible, it has a unique decomposition into irreducible irredundant components, $\mathbb{A}^n(k) = \bigcup_i V_i$, with every $V_i \subsetneqq \mathbb{A}^n(k)$. But, $I(\mathbb{A}^n(k)) = 0$ is true if and only if k is a infinite field. While $I(\bigcup_i V_i) = \bigcap_i I(V_i)$. Each $I(V_i) \neq 0$ because $V_i \subsetneqq \mathbb{A}^n(k)$ implies that $I(\mathbb{A}^n(k)) \subsetneqq I(V_i)$. Contradiction.

1.3 Hilbert Nullstellensatz

Question 1.7: 1.38

Let k be an algebraically closed field. Let $I \trianglelefteq k[x_1,\dots,x_n]$ and $U = V(I)$. Show that there is a one-to-one correspondence:

$$\{\text{algebraic subsets of } U\} \longleftrightarrow \{\text{radical ideals in } k[x_1,\dots,x_n]/I\}$$

Solution:

2 Affine varieties

2.1 Coordinate rings

Question 2.1: 2.1

Question 2.2: 2.2

Question 2.3: 2.3