Homological Algebras

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Contents

1 Additive and abelian category

Definition 1.1: Preadditive category

A category $\mathscr{C}$ is a **preadditive category** $\Leftrightarrow$
(1) $\forall A, B \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Mor}_{\mathscr{C}}(A, B)$ has an abelian group structure. The group law here is addition $+$.
(2) $\forall X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, the composition map

$$\mathrm{Mor}_{\mathscr{C}}(Y, Z) \times \mathrm{Mor}_{\mathscr{C}}(X, Y) \to \mathrm{Mor}_{\mathscr{C}}(X, Z) \quad (g, f) \mapsto g \circ f$$

is bilinear. If we write the map in the group law explicitly, then $\forall f_1, f_2, f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $g_1, g_2, g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$,

$$(g_1 + g_2) \circ f = g_1 \circ f +_2 \circ f, \quad g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$$

Lemma 1.1: Characterization of zero element in additive categories

Let $\mathscr{C}$ be a preadditive category. $Z \in \mathrm{Ob}(\mathscr{C})$ is an zero object $\Leftrightarrow$ $\mathrm{Mor}_C(Z, Z)$ has unique element, the trivial map.

Proof: $\Rightarrow$ Z is zero $\Rightarrow$ Z is initial $\Leftrightarrow$ $\forall A \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Mor}_{\mathscr{C}}(Z, A)$ contains unique element. Let $A := Z$.
$\Leftarrow$ Since $\mathscr{C}$ is an preadditive category, the zero map $0_{ZZ} \in \mathrm{Mor}_{\mathscr{C}}(Z, Z)$, $0_{ZZ} = \mathrm{id}_Z$. We will show Z is initial and the fact that Z is terminal/final is similar: $\forall A \in \mathrm{Ob}(\mathscr{C})$, $\forall f \in \mathrm{Mor}_{\mathscr{C}}(A, Z)$, $f = \mathrm{id}_Z \circ f = 0_{ZZ} \circ f = 0_{ZZ}$. So, the morphism from A to Z is unique. Then, Z is initial. □

Definition 1.2: Additive category

A category $\mathscr{C}$ is an **additive category** $\Leftrightarrow$ $\mathscr{C}$ is a preadditive category admitting all finitary biproducts. ???

Definition 1.3: Abelian category

2 Chain complexes

2.0.1 Categorical complexes

Definition 2.1: Complexes and differentials

Let $\mathscr{A}$ be an abelian category. A **complex** in $\mathscr{A}$, denoted $(A_\bullet, d_\bullet)$, is a sequence of objects and morphisms in $\mathscr{A}$, where $A_\bullet := \{A_i\}_i$ and $d_\bullet := \{d_i^A\}_i$:

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}^A} A_n \xrightarrow{d_n^A} A_{n-1} \xrightarrow{d_{n-1}^A} \cdots$$

such that $\forall n$, $d_{n+1}^A \circ d_n^A = 0$. The maps d_n^A are called **differentials**.

Definition 2.2: Morphism pf complexes

Let $(A_\bullet, d^A_\bullet)$ and $(B_\bullet, d^B_\bullet)$ be two complexes in an abelian category $\mathscr{A}$. A morphism between two complexes

$$f_\bullet : (A_\bullet, d^A_\bullet) \to (B_\bullet, d^B_\bullet)$$

is a collection of morphisms $f_n : A_n \to B_n$ such that all the squares in the following diagram commute, i.e. $\forall n, d^B_{n+1} \circ f_{n+1} = f_n \circ d^A_{n+1}$:

$$\begin{tikzcd}
\cdots \arrow[r, "d^A_{n+2}"] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "d^A_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^B_{n+2}"] & B_{n+1} \arrow[r, "d^B_{n+1}"] & B_n \arrow[r, "d^B_n"] & B_{n-1} \arrow[r, "d^B_{n-1}"] & \cdots
\end{tikzcd}$$

As the definitions suggest, there is a category whose objects are complexes in the Abelian category $\mathscr{A}$, $(A_\bullet, d^A_\bullet)$ and morphisms $\forall (A_\bullet, d^A_\bullet), (B_\bullet, d^B_\bullet) \in \mathrm{Ob}(R - \textbf{Comp})$,

$$\mathrm{Mor}((A_\bullet, d^A_\bullet), (B_\bullet, d^B_\bullet)) = \{f_\bullet : A_\bullet \to B_\bullet\}$$

Elementwise, $(f_\bullet)_n := f_n$. If there are two morphisms $f_\bullet : (A_\bullet, d^A_\bullet) \to (B_\bullet, d^B_\bullet)$ and $g_\bullet : (B_\bullet, d^B_\bullet) \to (C_\bullet, d^C_\bullet)$. The composition $g_\bullet \circ f_\bullet$ works elementwise as $(g_\bullet \circ f_\bullet)_n := (g_\bullet)_n \circ (f_\bullet)_n := g_n \circ f_n$. This definition satisfies those regulations on composition maps.
So, this category can be denoted as $\textbf{Comp}(\mathscr{A})$, $\textbf{Ch}(\mathscr{A})$. If $\mathscr{A}$ is understood, it is written in the notation **Comp**. When $\mathscr{A} =_{\mathbf{R}} \textbf{Mod}$, this category is usually written as $\textbf{Comp}({}_R\textbf{Mod})$ or ${}_R\textbf{Comp}$.

In fact, there is more information on $\textbf{Comp}(\mathscr{A})$, it is not only a category, but also an abelian category.

Proposition 2.1: Comp($\mathscr{A}$) is an abelian category

2.0.2 Exact sequence of complexes

There are two questions concerning the introducing the following concepts:
- Why study exact sequences?
- Why study the short exact sequences?

They are equally important. But, we will see the second one is independent of the first one. Let's explain the first question why study exact sequences

Definition 2.3: Exact sequence

Definition 2.4: Short exact sequence

One reason is that the following observation allows us to break up every exact complex into a larger number of short exact sequences.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & & & \mathrm{im}d_{i+1} = \mathrm{ker}d_i \arrow[ur] \arrow[dr, "\hookrightarrow"] & & & \\
M_{i+2} \arrow[rr, "d_{i+2}"] \arrow[dr, "d_{i+2}"] & & M_{i+1} \arrow[ur, "d_{i+1}"] \arrow[rr, "d_{i+1}"] & & M_i \arrow[rr, "d_i"] \arrow[dr, "d_i"] & & M_{i-1} \\
 & \mathrm{im}d_{i+2} = \mathrm{ker}d_{i+1} \arrow[ur, "\hookrightarrow"] \arrow[dr] & & & & \mathrm{im}d_i = \mathrm{ker}d_{i-1} \arrow[ur, "\hookrightarrow"] \arrow[dr] & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

This technique is used in the definition 4.2

Lemma 2.1: Five lemma

Let $\mathscr{A}$ be an abelian category. All objects are taken from $\mathscr{A}$ or **Gp** in the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

2.0.3 Classifying exact sequences: First encounter

To classify all exact sequences, we need introduce the decomposition as follows.

Definition 2.5: Split exact sequence

A short exact sequence of R-modules

$$0 \longrightarrow M_1 \longrightarrow N \longrightarrow M_2 \longrightarrow 0$$

splits if it is isomorphic to another exact[a] sequence

$$0 \longrightarrow M_1' \longrightarrow M_1' \oplus M_2' \longrightarrow M_2' \longrightarrow 0$$

such that the following diagram commutes, where $\sim$ means isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\sim"] & N \arrow[r] \arrow[d, "\sim"] & M_2 \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & M_1' \arrow[r] & M_1' \oplus M_2' \arrow[r] & M_2' \arrow[r] & 0
\end{tikzcd}$$

[a] from $m_1' \mapsto (m_1', 0)$ and $(m_1', m_2') \mapsto m_2'$

In the upper definition, the exact sequence is classified by the isomorphism and reduced to a simpler form. The classification has not yet done until introducing the tool–Ext functor.

2.0.4 Exact functors

3 Homology of a chain complex

3.1 Homology as a functor

3.1.1 Homology

Definition 3.1: Homology

Let the following be a complex in an abelian category $\mathscr{A}$:

$$M_\bullet : \cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

The i-th homology of that complex $M_\bullet$ is an object in $\mathscr{A}$:

$$H_i(M_\bullet) = \frac{\text{ker } d_i}{\text{im } d_{i+1}}$$

And $H_\bullet(M_\bullet)$ is a collection of R-modules indexed by $\mathbb{Z}$.

Remark Here i indicates the module M_i and consider the two components lying inside M_i.
Notice that

$$H_i(M_\bullet) = 0 \Leftrightarrow \text{ker } d_i = \text{im } d_{i+1} \Leftrightarrow \text{the complex } M_\bullet \text{ is exact at } M_i$$

So, if the sequence is not exact at somewhere, say M_i, then $H_i(M_\bullet) \neq 0$. The homology measures the failure of a complex from being exact.
Sometimes $\ker d_i$ and $\text{im} d_{i+1}$ have 'geometric' names and notations. M_n is called *n-chains*; $Z_n(M_\bullet) := \ker d_n$ is called *n-cycles* and $B_n(M_\bullet)$ is called *n-boundaries*. In these notations,

$$H_n(M_\bullet) = \frac{Z_n(M_\bullet)}{B_n(M_\bullet)}$$

3.1.2 Functoriality of H_n

Proposition 3.1: H_n as a functor

Let $\mathscr{A}$ be an abelian category. $\forall n$, $H_n : \textbf{Comp}(\mathscr{A}) \to \mathscr{A}$ is an additive functor.

3.2 Making new exact sequences from old

Lemma 3.1: The snake lemma

Let

$$\begin{tikzcd}
 & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\alpha_2"] & B_2 \arrow[r, "\beta_2"] & C_2 &
\end{tikzcd}$$

be a commutative diagram of R-modules with exact rows. Then there is an exact sequence

$$\begin{tikzcd}
\text{ker } f \arrow[r, "\alpha_1^\circ"] & \text{ker } g \arrow[r, "\beta_1^\circ"] & \text{ker } h \arrow[dll, rounded corners, "\delta"] \\
\text{coker } f \arrow[r, "\alpha_2^\circ"] & \text{coker } g \arrow[r, "\beta_2^\circ"] & \text{coker } h
\end{tikzcd}$$

Remark If the morphism of two exact sequences is given as (a extended version):

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\alpha_2"] & B_2 \arrow[r, "\beta_2"] & C_2 \arrow[r] & 0
\end{tikzcd}$$

then there is a corresponding exact sequence:

$$0 \longrightarrow \ker\ f \xrightarrow{\alpha_1^\circ} \ker\ g \xrightarrow{\beta_1^\circ} \ker\ h \xrightarrow{\delta} \mathrm{coker}\ f \xrightarrow{\alpha_2^\circ} \mathrm{coker}\ g \xrightarrow{\beta_2^\circ} \mathrm{coker}\ h \longrightarrow 0$$

Proof: Well-definedness of each map:
We will define $\alpha_1^\circ, \beta_1^\circ, \alpha_2^\circ, \beta_2^\circ$ and δ and show they are well-defined.
• The definitions of α_1° and β_1° are as follows:

$$\alpha_1^\circ := \alpha_1\Big|_{\ker\ f}^{\ker\ g} : A_1 \supseteq \ker\ f \longrightarrow \ker\ g \subseteq B_1 \quad \beta_1^\circ := \beta_1\Big|_{\ker\ g}^{\ker\ h} : B_1 \supseteq \ker\ g \longrightarrow \ker\ h \subseteq C_1$$

Here it is enough to show α_1° is well-defined, and the case for β_1° is similar. $\forall x \in \ker f$, $g(\alpha_1(x)) = \alpha_2(f(x)) = \alpha_2(0) = 0$, since α_2 is a homomorphism.
• $\alpha_2^\circ, \beta_2^\circ$ are defined as follows:

$$\alpha_2^\circ : \mathrm{coker} f \to \mathrm{coker} g \quad a_2 + \mathrm{im} f \mapsto \alpha_2(a_2) + \mathrm{im} g$$

The definition for g is similar and is omitted here. This α_2° is well-defined because it is independent of choice of representatives: $\forall a_2, a_2' \in A_2$ such that $a_2 - a_2' \in \mathrm{im} f$, $\exists a \in A_1$ makes $a_2 - a_2' = f(a)$. So, $\alpha_2(a_2 - a_2') = \alpha_2(f(a)) = g(\alpha_1(a)) \in \mathrm{im} g$. There is no worry about the image of $A_2/\mathrm{im} f$ is not in $B_2/\mathrm{im} g$.
• Let's define δ right now. The proof is constructive and the construction of δ is a diagram chasing illustrated in the following diagram:

$$\begin{tikzcd}
 & \ker f \arrow[r, "\alpha_1^\circ"] \arrow[d, hook, "\iota"] & \ker g \arrow[r, "\beta_1^\circ"] \arrow[d, hook, "\iota"] & \ker h \arrow[d, hook, "\iota"] \arrow[ddll, rounded corners, "\delta"] & \\
 & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\alpha_2"] \arrow[d] & B_2 \arrow[r, "\beta_2"] \arrow[d] & C_2 \arrow[d] & \\
 & \mathrm{coker}\, f \arrow[r, "\alpha_2^\circ"] & \mathrm{coker}\, g \arrow[r, "\beta_2^\circ"] & \mathrm{coker}\, h &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & x \arrow[d, squiggly, "(1)"] \\
 & y \arrow[d, squiggly, "(2.1.2)"'] & x \arrow[l, squiggly, "(2.1.1)"] \arrow[d, squiggly, "(2.2)"] \\
z \arrow[d, squiggly, "(4)"] & g(y) \arrow[l, squiggly, "(3)"] \arrow[r, dashed] & h(x) \\
z + \mathrm{im} f & &
\end{tikzcd}$$

(1) Starting from an element in $\ker h$, $\ker h$ has an inclusion to C_1. So, $\iota(x) = x$ is an element in C_1. (2) The exactness of C_1 implies that β_1 is surjective. So, $\exists y \in B_1$ such that $\beta_1(y) = x$. Consider $g(y)$. $g(y) \in \ker \beta_2$, since

$$\beta_2 \circ g(y) \overset{\dagger}{=} h \circ \beta_1(y) = h(x) \overset{\ddagger}{=} 0$$

†: from the commutativity of the diagram
‡: $x \in \ker h$
(3) Then exactness of B_2 implies $\ker \beta_2 = \mathrm{im}\alpha_2$. So, $g(y) \in \mathrm{im}\alpha_2$. Since α_2 is injective, there is a unique element $z \in A_2$ such that $\alpha_2(z) = g(y)$. (4) The natural projection maps z to $z + \mathrm{im} f$.
Therefore,

$$\boxed{\delta(x) := z + \mathrm{im} f}$$

It remains to prove that δ is well-defined and a homomorphism. Also, δ makes $\ker h$ and $\mathrm{coker} f$ being exact.
The well-definedness starts from this $x \in \ker h$. The first time we choose a representative is in $\beta_1^{-1}(\{x\})$. Suppose that there are two $y_1, y_2 \in B_1$ such that $\beta_1(y_1) = \beta_1(y_2) = x$. Because $\beta_1(y_1 - y_2) = \beta_1(y_1) - \beta_1(y_2) = 0$, $y_1 - y_2 \in \ker \beta_1 = \mathrm{im}\alpha_1$. So, $\exists w \in A_1$, making $y_1 - y_2 = \alpha_1(w)$. In the definition of δ, for $g(y_1), g(y_2)$ (all of them $\in \ker \beta_2 = \mathrm{im}\alpha_2$), $\exists!$ elements $z_1, z_2 \in A_2$ so that $\alpha_2(z_1) = g(y_1), \alpha_2(z_2) = g(y_2)$. Notice that

$$\alpha_2(f(w)) \overset{\dagger}{=} g(y_1 - y_2) \overset{\ddagger}{=} \alpha_2(z_1 - z_2)$$

†: $\alpha_2(f(w)) = g(\alpha_1(w)) = g(y_1 - y_2)$
‡: $\alpha_2(z_1 - z_2) = \alpha_2(z_1) - \alpha_2(z_2) = g(y_1) - g(y_2) = g(y_1 - y_2)$
Since α_2 is injective, $z_1 - z_2 = f(w) \in \mathrm{im} f$, showing that δ is well-defined.
Exactness at each position:
• The exactness of $\ker g$: $\mathrm{im}\alpha_1^\circ \subseteq \ker \beta_1^\circ$ is because $\forall x \in \ker f$, $(\beta_1^\circ \circ \alpha_1^\circ)(x) = (\beta_1 \circ \alpha_1)(x) = 0$.
To see $\ker \beta_1^\circ \subseteq \mathrm{im}\alpha_1^\circ$, let's start from an arbitrary element $x \in \ker \beta_1^\circ = \ker g \cap \ker \beta_1$. Rewriting $\mathrm{im}\alpha_1^\circ$, $\mathrm{im}\alpha_1^\circ = \alpha_1(\ker f)$. For one thing, $x \in \ker \beta_1$. So, $\exists a \in A_1$, $x = \alpha_1(a)$. For another thing, $x \in \ker g$. Hence, $0 = g(x) =$

$g(\alpha_1(a)) = \alpha_2(f(a))$. α_2 is a monomorphism. Therefore, $f(a) = 0$ and $a \in \ker f$.

- Exactness of $\ker h$:
- Exactness of $\mathrm{coker} f$:
- Exactness of $\mathrm{coker} g$: □

Theorem 3.1: Sequence of complexes to sequence of homology

Let $0 \longrightarrow A_\bullet \xrightarrow{\alpha_\bullet} B_\bullet \xrightarrow{\beta_\bullet} C_\bullet \longrightarrow 0$ be a short exact sequence of R-complexes. This R-complex induces a long exact sequence of homology:

$$\cdots \xrightarrow{\beta_{n+1}} H_{n+1}(C_\bullet) \xrightarrow{\delta_{n+1}} \boxed{H_n(A_\bullet) \xrightarrow{\alpha_n} H_n(B_\bullet) \xrightarrow{\beta_n} H_n(C_\bullet)} \xrightarrow{\delta_n} \boxed{H_{n-1}(A_\bullet) \xrightarrow{\alpha_{n-1}} H_{n-1}(B_\bullet) \xrightarrow{\beta_{n-1}} H_{n-1}(C_\bullet)} \xrightarrow{\delta_{n-1}} \cdots$$

Proof: □

Lemma 3.2: Naturality of the triangle

4 Injective, Projective and Flat modules

4.1 Definitions and properties

4.2 Resolutions

Definition 4.1: Projective, Injective resolutions and Enough projective/Injective

Let $\mathscr{A}$ be an abelian category. Let $M \in \mathrm{Ob}(\mathscr{A})$.
An **injective resolution** of M is an *exact* sequence

$$\mathbf{I} : 0 \longrightarrow M \xrightarrow{\eta} I_0 \xrightarrow{d^0} I_1 \xrightarrow{d^1} I_2 \xrightarrow{d^2} \cdots$$

such that $\forall j \geq 0$, I_j is injective.
A **projective resolution** of M is an *exact* sequence

$$\mathbf{P} : \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

such that $\forall j \geq 0$, P_j is projective.
$\mathbf{A}$ is **enough injective** $\Leftrightarrow \forall M \in \mathrm{Ob}(\mathbf{A})$, M has an injective resolution.
$\mathbf{A}$ is **enough projective** $\Leftrightarrow \forall M \in \mathrm{Ob}(\mathbf{A})$, M has a projective resolution.

Remark Beyond injective and projective resolutions, the notion *deleted injective resolution* and *deleted projective resolution* are useful:
A **deleted injective resolution** of M is the *complex*

$$\mathbf{I}^M : 0 \longrightarrow I_0 \xrightarrow{d^0} I_1 \xrightarrow{d^1} I_2 \xrightarrow{d^2} \cdots$$

Similarly, a **deleted surjective resolution** of M is the *complex*

$$\mathbf{P}_M : \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

These deleted forms do not cause any loss of information comparing to the original complex. For the deleted injective resolution case, $M \cong \eta(M) = \mathrm{im}\eta = \ker d^0$. So, M is engraved in the kernel of d^0. For projective (or free, flat) case, deleting loses nothing, for $\mathrm{coker} d_1 \cong M$.
The reverse process, recovering a resolution from its deleted form is called *augmenting*.

After this definition, a very natural question is: Do projective and injective resolutions exist for every objects of every abelian category, or at least for a special kind of abelian category, ${}_{\mathbf{R}}\mathbf{Mod}$. The answer should be positive. Otherwise, the definition is not useful enough. The next theorem gives a definite answer to this question.

Proposition 4.1: Existence of resolutions in ${}_{\mathrm{R}}$Mod

Definition 4.2: Syzygy and cosyzygy

Let $\mathscr{A}$ be an abelian category. Let $M \in \mathrm{Ob}(\mathscr{A})$.

For an injective resolution of M, $\mathbf{I} : 0 \longrightarrow M \xrightarrow{\eta} I_0 \xrightarrow{d^0} I_1 \xrightarrow{d^1} I_2 \xrightarrow{d^2} \cdots$, define $V_0 := \mathrm{coker}\eta$ and $\forall n \geq 0$, $V_n := \mathrm{coker} d^{n-1}$. Then, V^n is the n-th **cosyzygy** of $\mathbf{I}$.

For a projective resolution of M $\mathbf{P} : \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$, define $K_0 := \ker \epsilon$ and $\forall n \geq 1$, $K_n := \ker d_n$. Then, K_n is the n-th **syzygy** of $\mathbf{P}$.

Another definition is 'flat resolution'. It arises analogous to projective and injective resolutions but does not appear very often.

Definition 4.3: Flat resolution

5 Derived functors

5.1 Homotopic morphisms between complexes

Definition 5.1: Homotopic morphisms

Let $f_\bullet$, $g_\bullet$ be two extensions of f as morphisms between two complexes $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ as follows:

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "h_2"] & P_1 \arrow[r, "d_1"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "h_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "h_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d_3'"] & Q_2 \arrow[r, "d_2'"] & Q_1 \arrow[r, "d_1'"] & Q_0 \arrow[r, "\epsilon'"] & B \arrow[r] & 0
\end{tikzcd}$$

$f_\bullet$ and $g_\bullet$ are called **homotopic**, denoted $f_\bullet \simeq g_\bullet \Leftrightarrow \exists$ *homomorphisms* $s_i : P_i \to Q_{i+1}$ for all i such that the following relation holds

$$\forall n,\ f_n - h_n = s_{n-1} \circ d_n + d'_{n+1} \circ s_n$$

Lemma 5.1: Homotopy induces same map on homology

If $f_\bullet \simeq g_\bullet$, then they induces the same map $H_\bullet(P_\bullet) \to H_\bullet(Q_\bullet)$ on homology.

Lemma 5.2: Comparison lemma

If there is a commutative diagram of two rows of complexes

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\epsilon"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d'_3"] & Q_2 \arrow[r, "d'_2"] & Q_1 \arrow[r, "d'_1"] & Q_0 \arrow[r, "\epsilon'"] & B \arrow[r] & 0
\end{tikzcd}$$

where $P_\bullet \to A \to 0$ is a projective resolution of A and $Q_\bullet \to B \to 0$ is exact, then $\exists$ a morphism of complex $f_\bullet$ which extends f.

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d'_3"] & Q_2 \arrow[r, "d'_2"] & Q_1 \arrow[r, "d'_1"] & Q_0 \arrow[r, "\epsilon'"] & B \arrow[r] & 0
\end{tikzcd}$$

Moreover, any two such morphisms extending f are homotopic to each other.

Proof: First, we will prove the existence by induction:
(1.1) Base case: since the lower row is exact, the map $Q_0 \twoheadrightarrow B$ is surjective. The projectivity of P_0 implies the existence of f_0 which makes the following diagram commute:

$$\begin{tikzcd}
 & P_0 \arrow[d, "\epsilon"] \arrow[ld, dashed, "\exists f_0"'] \arrow[dd, bend left, "f\circ\epsilon"] & \\
 & A \arrow[d, "f"] & \\
Q_0 \arrow[r, two heads, "\epsilon'"] & B \arrow[r] & 0
\end{tikzcd}$$

(1.2) Inductive hypothesis+inductive step: suppose that $\exists\ f_0, f_1, \ldots, f_n$ where each f_i makes the square formed by $P_{i-1}, Q_{i-1}, P_i, Q_i$ commute. So, we have the diagram

$$\begin{tikzcd}
 & P_{n+1} \arrow[d, "f_n \circ d_{n+1}"] & \\
Q_{n+1} \arrow[r, two heads, "d'_{n+1}"] & Q_n \arrow[r, "d'_n"] & Q_{n-1}
\end{tikzcd}$$

The problem is d'_{n+1} is not necessarily surjective and d'_n might not goes to zero. To use the projectivity of P_{n+1}, Q_n can be replaced by $\ker d'_n = \mathrm{im} d'_{n+1}$.
One thing is left to check: to make the map $f_n \circ d_{n+1} : P_{n+1} \to Q_n$ work, $\mathrm{im} f_n \circ d_{n+1} \subseteq \mathrm{im} d'_{n+1} = \ker d'_n$ remains to check. But this is true from the following: $\forall x \in P_{n+1}$,

$$d'_n \circ (f_n \circ d_{n+1})(x) = (d'_n \circ f_n) \circ d_{n+1}(x) \overset{\dagger}{=} (f_{n-1} \circ d_n) \circ d_{n+1}(x) = f_{n-1} \circ (d_n \circ d_{n+1})(x) \overset{\ddagger}{=} 0$$

†: from the commutativity of the square
‡: because the upper row is a complex
Hence, the projectivity of P_{n+1} implies the existence of f_{n+1} which makes the square consisting of P_{n+1}, P_n, Q_{n+1} and Q_n commute.

$$\begin{tikzcd}
 & P_{n+1} \arrow[d, "f_n \circ d_{n+1}"] \arrow[ld, dashed, "\exists f_{n+1}"'] & \\
Q_{n+1} \arrow[r, two heads, "d'_{n+1}"] & \mathrm{im} d'_{n+1} \arrow[r, "d'_n"] & 0
\end{tikzcd}$$

Second, suppose that there are two morphisms of these complexes which are homotopic, i.e. $\exists\ f_\bullet \simeq g_\bullet$. Equivalently, let $g_n := f_n - h_n$, then we want to construct $\{s_n\}_n$ making the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, "g_2"] \arrow[dl, "s_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, "g_1"] \arrow[dl, "s_1"'] & P_0 \arrow[r, "0"] \arrow[d, "g_0"] \arrow[dl, "s_0"'] & 0 \arrow[dl, "s_{-1}"'] \\
\cdots \arrow[r, "d'_3"] & Q_2 \arrow[r, "d'_2"] & Q_1 \arrow[r, "d'_1"] & Q_0 \arrow[r, "0"] & 0
\end{tikzcd}$$

where each square is a commutative diagram. The commutativity of each square comes from the commutativity concerning f_n and g_n.
• for s_{-1}, $s_{-1} = 0$
• for s_0: Notice that B is deleted, so the lower sequence might not be exact at Q_0 in this new sequence. Using the exactness of the old sequence, let's replace Q_0 by $\mathrm{im} d'_{n+1} = \ker \epsilon'$. As the argument above, the well-definedness of $g_0 : P_0 \to \mathrm{im} d'_{n+1}$ is from:

$$\epsilon' \circ f_0 = \epsilon' \circ g_0 = f \circ \epsilon \Rightarrow \epsilon' \circ g_0 = 0 \Leftrightarrow \mathrm{im} g_0 \subseteq \ker \epsilon' = \mathrm{im} d'_1$$

Then, the projectivity of P_1 implies the existence of s_0 such that $g_0 = d'_1 \circ s_0 + s_{-1} \circ d'_0$, where $d'_0 := 0$.

$$\begin{tikzcd}
 & P_0 \arrow[dl, dashed, "\exists s_0"'] \arrow[d, "g_0"] & \\
Q_1 \arrow[r, two heads, "d'_1"] & \mathrm{im} d'_1 \arrow[r, "0"] & 0
\end{tikzcd}$$

• Suppose this works for all $s_0, s_1, \ldots, s_{n-1}$ and the equation $g_i = d'_{i+1} \circ s_i + s_{i-1} \circ d_i$ holds for each s_i $(i \leq n-1)$. Then, still, we can replace each Q_n by $\mathrm{im} d'_{n+1}$. There is a subtlety: we need to check $\mathrm{im}(g_n - s_{n-1} \circ d_n) \subseteq \mathrm{im} d'_{n+1}$. The exactness at each $Q_n \Rightarrow \ker d'_n = \mathrm{im} d'_{n+1}$. So, it suffices to check $d'_n \circ (g_n - s_{n-1} \circ d_n) = 0$,

$$d'_n \circ (g_n - s_{n-1} \circ d_n) \overset{\dagger}{=} g_{n-1} \circ d_n - (d'_n \circ s_{n-1}) \circ d_n \overset{\ddagger}{=} g_{n-1} \circ d_n - (g_{n-1} - s_{n-2} \circ d_{n-1}) \circ d_n = -s_{n-2} \circ (d_{n-1} \circ d_n) \overset{\dagger\dagger}{=} 0$$

†: $d'_n \circ g_n = g_{n-1} \circ d_n$
‡: By assumption, $g_{n-1} = d'_n \circ s_{n-1} + s_{n-2} \circ d_{n-1}$.
††: The upper chain is a complex. Hence, $d_{n-1} \circ d_n = 0$.
Hence, the projectivity of P_{n+1} implies the existence of s_{n+1} such that the following diagram commutes:

$$\begin{tikzcd}
 & P_{n+1} \arrow[dl, dashed, "\exists s_{n+1}"'] \arrow[d, "g_n - s_{n-1} \circ d_n"] & \\
Q_{n+1} \arrow[r, two heads, "d'_{n+1}"] & \mathrm{im} d'_{n+1} \arrow[r, "d'_n"] & 0
\end{tikzcd}$$

The commutativity of this diagram is exactly the equation $g_n = s_{n-1} \circ d_n + d'_{n+1} \circ s_{n+1}$, showing that $f_\bullet \simeq g_\bullet$. □

5.2 Left derived functors

Left(right) derived functors are defined for right-exact functors(left-exact). One way to think of left-derived(right-derived) functors is to consider them as recovering the information being lost by right-exact functors(correspondingly, left-exact functors): Suppose F is a right exact functor between two abelian categories, $\mathscr{A}$ and $\mathscr{C}$, $F : \mathscr{A} \to \mathscr{C}$. So, applying F to an exact sequence $0 \to A \to B \to C \to 0$, there is a right exact sequence $FA \to FB \to FC \to 0$. This F guarantees the exactness at FB and FC. But, it loses the information of exactness of FA, in other words, the injectivity of $FA \to FB$. To measure the loss of injectivity, $\ker(FA \to FB)$ is a good candidate because ??
Let's try to give the definition of left-derived functor of another functor:
Let $F : \mathscr{A} \to \mathscr{C}$ be a right-exact covariant functor between two abelian categories, where $\mathscr{A}$ is enough projective. Let $A, B \in \mathrm{Ob}(\mathscr{A})$ be arbitrary objects and $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ be an arbitrary morphism. The *left derived functor* of F at n, denoted L_nF, is defined by the action on A, B and f as follows:
• For objects, take the projective resolutions of A and B. Then, we get two exact sequences which are projective resolutions of A and B

$$\mathbf{P} : \cdots \xrightarrow{\tilde{d_3}} P_2 \xrightarrow{\tilde{d_2}} P_1 \xrightarrow{\tilde{d_1}} P_0 \xrightarrow{\tilde{d_0}} A \longrightarrow 0$$

$$\mathbf{Q} : \cdots \xrightarrow{\tilde{d'_3}} Q_2 \xrightarrow{\tilde{d'_2}} Q_1 \xrightarrow{\tilde{d'_1}} Q_0 \xrightarrow{\tilde{d'_0}} B \longrightarrow 0$$

Let $\mathbf{P}_A$ and $\mathbf{Q}_B$ be the A-deleted sequence and B-deleted sequences, respectively. Since there is a morphism $f : A \to B$, by the Comparison lemma 5.2, a morphism between two sequences, $f_\bullet = \{f_n\}_n$, can be constructed between two deleted sequences $\mathbf{P}_A$ and $\mathbf{Q}_B$ Exactness of deleted sequences

$$\begin{tikzcd}
\mathbf{P}_A : \arrow[d, "f_\bullet"] & \cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & 0 \\
\mathbf{Q}_B : & \cdots \arrow[r, "d'_3"] & Q_2 \arrow[r, "d'_2"] & Q_1 \arrow[r, "d'_1"] & Q_0 \arrow[r, "d'_0"] & 0
\end{tikzcd}$$

where the maps $d_i = \tilde{d}_i$ for all $i \geq 1$ and $d_0 := 0$, the maps $d'_i := \tilde{d}'_i$ for all $i \geq 1$ and $d'_0 := 0$.
The right-exact functor F induces these chains and morphism in $\mathscr{C}$ as

$$\begin{tikzcd}
F\mathbf{P}_A : \arrow[d, "Ff_\bullet"] & \cdots \arrow[r, "Fd_3"] & FP_2 \arrow[r, "Fd_2"] \arrow[d, "Ff_2"] & FP_1 \arrow[r, "Fd_1"] \arrow[d, "Ff_1"] & FP_0 \arrow[r, "Fd_0"] \arrow[d, "Ff_0"] & 0 \\
F\mathbf{Q}_B : & \cdots \arrow[r, "Fd'_3"] & FQ_2 \arrow[r, "Fd'_2"] & FQ_1 \arrow[r, "Fd'_1"] & FQ_0 \arrow[r, "Fd'_0"] & 0
\end{tikzcd}$$

So, L_nF acting on A as $\boxed{L_nF(A) := H_n(F\mathbf{P}_A)}$[1]. In particular, $L_0F(A) = F(A)$, since $L_0F(A) := \ker Fd_0/\mathrm{im}Fd_1 \overset{\dagger}{\cong} F(A)$.
†: Truncating the long sequence $\mathbf{P}$ into $P_1 \to P_0 \twoheadrightarrow A \to 0$, apply the right-exactness of F to this truncated sequence. Then, $F(P_0)$ is exact. Hence, $\mathrm{im}F\tilde{d}_1 = \ker F\tilde{d}_0$. Exactness of $FA \Rightarrow FP_0 \twoheadrightarrow FA$ is a surjective homomorphism. With $FP_0 = \ker Fd_0$ and $\mathrm{im}Fd_0 = \mathrm{im}F\tilde{d}_0$, $FP_0/\ker F\tilde{d}_0 \cong F(A)$.
As a functor, we have

$$\boxed{L_0F \cong F}$$

$\cong$ in what sense

• For morphisms, let $f : A \to B$ be a morphism in the category $\mathscr{A}$. L_nF has a natural way sending f to $L_nF(f)$. L_nF just goes into the category $\mathbf{Comp}(\mathscr{C})$ and applies the functor H_n to objects there. Analogously, apply H_n to the morphism $Ff_\bullet$. The 'natural' definition is $\boxed{L_nF(f) := H_n(Tf_\bullet)}$. Explicitly:

$$L_nF(f) : H_n(F\mathbf{P}_A) =: L_nF(A) \longrightarrow L_nF(B) := H_n(F\mathbf{P}_B) \qquad z + \mathrm{im}Fd_{n+1} \mapsto Ff_n(z) + \mathrm{im}Fd'_{n+1}$$

The definition of L_nF is completed, the work flow of defining L_nF is illustrated in the following diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[d, squiggly] \arrow[ddd, bend left, "L_nF"] \\
\mathbf{Comp}(\mathscr{A}) \arrow[d, "F"] \\
\mathbf{Comp}(\mathscr{C}) \arrow[d, "H_n"] \\
\mathscr{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, squiggly] & B \arrow[d, squiggly] \\
(P^A_\bullet, d^A_\bullet) \arrow[r, "f_\bullet"] \arrow[d, "F"] & (P^B_\bullet, d^B_\bullet) \arrow[d, "F"] \\
(FP^A_\bullet, Fd^A_\bullet) \arrow[r, "Ff_\bullet"] \arrow[d, "H_n"] & (FP^B_\bullet, Fd^B_\bullet) \arrow[d, "H_n"] \\
H_n(FP^A_\bullet, Fd^A_\bullet) \arrow[r, "H_nFf_\bullet"] & H_n(FP^B_\bullet, Fd^B_\bullet)
\end{tikzcd}$$

There is some subtlety on the definition: $L_F(A)$ depends on the choice of some resolution of A. It is crucial to make sure that two different resolutions never lead to different $L_nF(A)$. Otherwise, it would be problematic. To prove the two facts: independence of resolution and being an additive functor. The following two lemmas are needed.

Lemma 5.1 (Homotopy commutes with additive functors) *Let $\mathscr{A}, C$ be abelian categories and $F : \mathscr{A} \to \mathscr{C}$ be an additive functor. Then, homotopic extended morphisms of complexes in $\mathscr{A}$ is still homotopic after applying F. More precisely, let $(X_\bullet, d^X_\bullet)$ and $(Y_\bullet, d^Y_\bullet)$ be two complexes of $\mathscr{A}$. If $f_\bullet, g_\bullet : (X_\bullet, d^X_\bullet) \to (Y_\bullet, d^Y_\bullet)$ are two morphisms such that $f_\bullet \simeq g_\bullet$, then $Ff_\bullet \simeq Fg_\bullet$.*

Proof: (.1) $\{F(f_n)\}_n$ and $\{F(g_n)\}_n$ are lifts/extensions of Ff and Fg respectively:
Let's denote $h_\bullet := \{F(f_n)\}_n$ and $k_\bullet := \{F(g_n)\}_n$. The assumption $f_\bullet \simeq g_\bullet$ implies the existence of $\{s_i\}_{i \geq -1}$, such that $\forall n, f_n - g_n = d^Y_{n+1} \circ s_n + s_{n-1} \circ d^X_n \quad (*)$. Apply the functor F to $(*)$. Using the additivity and functoriality of F, we have

$$\forall n, F(f_n) - F(g_n) = F(d^Y_{n+1}) \circ F(s_n) + F(s_{n-1}) \circ F(d^X_n)$$

[1] This definition is not meaningless. If there is an sequence $P_\bullet$ that is exact at each place, the sequence being applied by a right-exact functor $FP_\bullet$ is not necessarily exact at each position. We will see a classical counterexample, $(-) \otimes M$.

Since each $F(f_n) : F(X_n) \to F(Y_n)$ is a morphism that makes every square in the induced morphism $Ff_\bullet$ commutative, $\{F(f_n)\}_n$ serves as an extension of Ff. So is $F(g_n)$. Define $s'_i := F(s_i)$. The existence of $\{s'_i\}_i$ yields $h_\bullet \simeq k_\bullet$.

(.2) Consider arbitrary extensions, $(Ff)_\bullet$ and $(Fg)_\bullet$, of Ff and Fg, respectively. We have

$$(Ff)_\bullet \overset{\dagger}{\simeq} h_\bullet \overset{(2.1)}{\simeq} k_\bullet \overset{\dagger}{\simeq} (Fg)_\bullet$$

†: By comparison lemma 5.2, both $(Ff)_\bullet$ and $h_\bullet$ are extensions of Ff and both $(Fg)_\bullet$ are extensions of $k_\bullet$. □

Lemma 5.2 (Homotopy induces same homology) *In an abelian category $\mathscr{C}$, if two maps between two complexes $\phi_\bullet, \psi_\bullet : (X_\bullet, d^X_\bullet) \to (Y_\bullet, d^Y_\bullet)$, are two morphisms such that $\phi_\bullet \simeq \psi_\bullet$, then $\forall n$, $H_n(\phi_\bullet) = H_n(\psi_\bullet)$.*

Proof: $\phi_\bullet \simeq \psi_\bullet$ implies that $\exists$ a collection of morphisms, $\{s_n : X_n \to Y_{n+1}\}_n$, such that $\phi_n - \psi_n = d^Y_{n+1} \circ s_n + s_{n-1} \circ d^X_n$. Evaluating this map at an element $w \in \ker d^X_n$,

$$(\phi_n - \psi_n)(w) = (d^Y_{n+1} \circ s_n)(w) + (s_{n-1} \circ d^X_n)(w) \overset{d^X_n(w)=0}{=} d^Y_{n+1}(s_n(w))$$

Hence, $\forall w \in \ker d^X_n$, $\forall n$, $\phi_n(w) - \psi_n(w) \in \mathrm{im} d^Y_{n+1} \Rightarrow \forall w$, $\forall n$, $[\phi_n(w)] = [\psi_n(w)]$, implying that $H_n(\phi_\bullet) = H_n(\phi_\bullet)$, for $H_n(\phi_\bullet) : [w] \mapsto [\phi_n(w)]$ and $H_n(\psi_\bullet) : [w] \mapsto [\psi_n(w)]$. □

Theorem 5.1: Independency of choice of resolution

The definition of L_nF for an additive functor $F : \mathscr{A} \to \mathscr{C}$ between two abelian categories is independent of the choice of resolutions. More precisely, if $P_\bullet : \mathbf{P}_A \to A \to 0$ and $Q_\bullet : \mathbf{Q}_A \to A \to 0$ are two projective resolutions of A, then $H_n(F\mathbf{P}_A) \cong H_n(F\mathbf{Q}_A)$.

Remark This enlightens us how to calculate the $L_nF(A)$ for some $A \in \mathrm{Ob}(\mathscr{A})$, because the resolution of A can be chosen as simple as possible.

Proof: Let $P_\bullet : \mathbf{P}_A \to A \to 0$ and $Q_\bullet : \mathbf{Q}_A \to A \to 0$ be two projective resolutions of A. The comparison lemma 5.2 extends the identity map id_A between $P_\bullet$ and $Q_\bullet$ to $f_\bullet$, $Q_\bullet$ and $P_\bullet$ to $g_\bullet$ illustrated as follows:

$$\begin{tikzcd}
P_\bullet : \arrow[d, "f_\bullet"] & \cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
Q_\bullet : \arrow[d, "g_\bullet"] & \cdots \arrow[r, "d'_3"] & Q_2 \arrow[r, "d'_2"] \arrow[d, "g_2"] & Q_1 \arrow[r, "d'_1"] \arrow[d, "g_1"] & Q_0 \arrow[r, "d'_0"] \arrow[d, "g_0"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
P_\bullet : & \cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & A \arrow[r] & 0
\end{tikzcd}$$

Notice that $g_\bullet \circ f_\bullet \simeq (\mathrm{id}_A)_\bullet$ by the diagram. Applying F to this homotopy and applying the lemma 5.1, we have

$$F(g_\bullet \circ f_\bullet) \simeq F((\mathrm{id}_A)_\bullet)$$

By functoriality of F, the left hand side is $F(g_\bullet \circ f_\bullet) = F(g_\bullet) \circ F(f_\bullet)$ and the right hand side is $F((\mathrm{id}_A)_\bullet) = (\mathrm{id}_{F(A)})_\bullet$. The lemma 5.2 further implies that $H_n(F(g_\bullet \circ f_\bullet)) = H_n(F((\mathrm{id}_A)_\bullet))$. Hence,

$$L_nF(g_\bullet \circ f_\bullet) = L_nF((\mathrm{id}_A)_\bullet)$$

Functoriality of H_n implies that $L_nF(g_\bullet \circ f_\bullet) = L_nF(g_\bullet) \circ L_nF(f_\bullet)$ and $L_nF((\mathrm{id}_A)_\bullet) = \mathrm{id}_{L_nFA}$. So,

$$L_nF(g_\bullet) \circ L_nF(f_\bullet) = \mathrm{id}_{L_nFA}$$

showing that $L_nF(f_\bullet) : L_nF(P_\bullet) \to L_nF(Q_\bullet)$ is an isomorphism. □

It remains to check that L_nF is indeed a functor.

Theorem 5.2: Functoriality of 'Left-derived functors'

Let $\mathscr{A}, \mathscr{C}$ be two abelian categories and $\mathscr{A}$ is enough projective. If $F : \mathscr{A} \to \mathscr{C}$ is an additive covariant(countravariant) functor, then $\forall n$, $L_nF : \mathscr{A} \to \mathscr{C}$ is also an additive covariant(countravariant) functor.

Proof: *Claim 1*: $(f+g)_\bullet \simeq f_\bullet + g_\bullet$.
$\forall X, Y \in \text{Mor}_{\mathscr{A}}(X,Y)$, fix resolutions of X and Y. The left hand side is an arbitrary extension morphisms of $f+g$ between fixed resolutions. The right hand side is elementwise defined as $(f_\bullet + g_\bullet)_n := (f_\bullet)_n + (g_\bullet)_n = f_n + g_n$. The fact that each $f_n + g_n : X_n \to Y_n$ is a morphism($\mathscr{A}$ is additive) and the commutativity more of diagram shows that $f_\bullet + g_\bullet$ is also an extension of $f+g$. By the comparison lemma 5.2, they are homotopic to each other.
With lemmas 5.1, 5.2, it is straightforward to show: • L_nF is a covariant functor; • L_nF is additive.
L_nF is a functor: need
Additivity: From claim 1 and 2, $F(f+g)_\bullet \simeq F(f_\bullet + g_\bullet) \overset{\spadesuit}{=} Ff_\bullet + Fg_\bullet$, where $\spadesuit$ is by the additivity of F. Overall,

$$L_nF((f+g)_\bullet) = H(F(f+g)_\bullet) \overset{\text{Lemma 5.2}}{=} H(Ff_\bullet + Fg_\bullet) = L_nF(f) + L_nF(g)$$

□

5.3 Right derived functors

5.3.1 Covariant right derived functors

Still, let $F : \mathscr{A} \to \mathscr{C}$ be a left-exact covariant functor between two abelian categories, where $\mathscr{A}$ is enough injective. Let $A, B \in \text{Ob}(\mathscr{A})$ be arbitrary objects and $f \in \text{Mor}_{\mathscr{A}}(A,B)$ be an arbitrary morphism. The *right derived functor* of F at n, denoted R^nF, is defined by the action on A, B and f as follows:
• For objects, take the injective resolutions of A and B. Then, we get two exact sequences which are injective resolutions of A and B

$$\mathbf{I} : \quad 0 \longrightarrow A \xrightarrow{\eta} I_0 \xrightarrow{d^0} I_1 \xrightarrow{d^1} I_2 \xrightarrow{d^2} \cdots$$
$$\mathbf{J} : \quad 0 \longrightarrow B \xrightarrow{\eta'} J_0 \xrightarrow{d^0} J_1 \xrightarrow{d'^1} J_2 \xrightarrow{d^2} \cdots$$

Let $\mathbf{I}^A$ and $\mathbf{J}^B$ be the A-deleted sequence and B-deleted sequences, respectively. Since there is a morphism $f : A \to B$, by the Comparison lemma 5.2, a morphism between two sequences, $f_\bullet = \{f_n\}_n$, can be constructed between two deleted sequences $\mathbf{I}^A$ and $\mathbf{J}^B$ Exactness of deleted sequences

$$\begin{tikzcd}
\mathbf{I}^A : \arrow[d, "f_\bullet"] & 0 \arrow[r] & I_0 \arrow[r, "d^0"] \arrow[d, "f_0"] & I_1 \arrow[r, "d'^1"] \arrow[d, "f_1"] & I_2 \arrow[r, "d^2"] \arrow[d, "f_2"] & \cdots \\
\mathbf{J}^B : & 0 \arrow[r] & J_0 \arrow[r, "d'^0"] & J_1 \arrow[r, "d'^{1\prime}"] & J_2 \arrow[r, "d'^2"] & \cdots
\end{tikzcd}$$

where the maps $d_i = \tilde{d}_i$ for all $i \geq 1$ and $d_0 := 0$, the maps $d'_i := \tilde{d}'_i$ for all $i \geq 1$ and $d'_0 := 0$.
The right-exact functor F induces these chains and morphism in $\mathscr{C}$ as

$$\begin{tikzcd}
F\mathbf{P}_A : \arrow[d, "Ff_\bullet"] & \cdots \arrow[r, "Fd_3"] & FP_2 \arrow[r, "Fd_2"] \arrow[d, "Ff_2"] & FP_1 \arrow[r, "Fd_1"] \arrow[d, "Ff_1"] & FP_0 \arrow[r, "Fd_0"] \arrow[d, "Ff_0"] & 0 \\
F\mathbf{Q}_B : & \cdots \arrow[r, "Fd'_3"] & FQ_2 \arrow[r, "Fd'_2"] & FQ_1 \arrow[r, "Fd'_1"] & FQ_0 \arrow[r, "Fd'_0"] & 0
\end{tikzcd}$$

So, L_nF acting on A as $\boxed{L_nF(A) := H_n(F\mathbf{P}_A)}$[2]. In particular, $L_0F(A) = F(A)$, since $L_0F(A) := \ker Fd_0/\text{im}Fd_1 \overset{\dagger}{\cong} F(A)$.
†: Truncating the long sequence $\mathbf{P}$ into $P_1 \to P_0 \twoheadrightarrow A \to 0$, apply the right-exactness of F to this truncated sequence. Then, $F(P_0)$ is exact. Hence, $\text{im}F\tilde{d}_1 = \ker F\tilde{d}_0$. Exactness of $FA \Rightarrow FP_0 \twoheadrightarrow FA$ is a surjective homomorphism. With $FP_0 = \ker Fd_0$ and $\text{im}Fd_0 = \text{im}F\tilde{d}_0$, $FP_0/\ker F\tilde{d}_0 \cong F(A)$.
As a functor, we have

$$\boxed{L_0F \cong F}$$

≅ in what sense
• For morphisms, let $f : A \to B$ be a morphism in the category $\mathscr{A}$. L_nF has a natural way sending f to $L_nF(f)$.

[2] This definition is not meaningless. If there is an sequence $P_\bullet$ that is exact at each place, the sequence being applied by a right-exact functor $FP_\bullet$ is not necessarily exact at each position. We will see a classical counterexample, $(-) \otimes M$.

L_nF just goes into the category $\mathbf{Comp}(\mathscr{C})$ and applies the functor H_n to objects there. Analogously, apply H_n to the morphism $Ff_\bullet$. The 'natural' definition is $\boxed{L_nF(f) := H_n(Ff_\bullet)}$. Explicitly:

$$L_nF(f) : H_n(F\mathbf{P}_A) =: L_nF(A) \longrightarrow L_nF(B) := H_n(F\mathbf{P}_B) \qquad z + \mathrm{im}Fd_{n+1} \mapsto Ff_n(z) + \mathrm{im}Fd'_{n+1}$$

The definition of L_nF is completed, the work flow of defining L_nF is illustrated in the following diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[d, squiggly] \arrow[ddd, bend left=60, "L_nF"] & A \arrow[rr, "f"] \arrow[d, squiggly] & & B \arrow[d, squiggly] \\
\mathbf{Comp}(\mathscr{A}) \arrow[d, "F"] & (P^A_\bullet, d^A_\bullet) \arrow[rr, "f_\bullet"] \arrow[d, "F"] & & (P^B_\bullet, d^B_\bullet) \arrow[d, "F"] \\
\mathbf{Comp}(\mathscr{C}) \arrow[d, "H_n"] & (FP^A_\bullet, Fd^A_\bullet) \arrow[rr, "Ff_\bullet"] \arrow[d, "H_n"] & & (FP^B_\bullet, Fd^B_\bullet) \arrow[d, "H_n"] \\
\mathscr{C} & H_n(FP^A_\bullet, Fd^A_\bullet) \arrow[rr, "H_nFf_\bullet"] & & H_n(FP^B_\bullet, Fd^B_\bullet)
\end{tikzcd}$$

5.3.2 Contravariant right-derived functors

5.4 Exact sequences and derived functors

5.5 A little summary

Let's recall the process how we define each two functors and compare the difference at the object level: let $\mathscr{A}$ be an abelian category. For $A, B \in \mathrm{Ob}(\mathscr{A})$,

$$\begin{tikzcd}
A \arrow[d, "\text{proj reso}"] & & B \arrow[dl, "\text{inj reso}"'] \arrow[dr, "\text{proj reso}"] & \\
\mathbf{P} \arrow[d, "\text{delete } A"] & \mathbf{I} \arrow[d, "\text{delete } B"] & & \mathbf{P} \arrow[d, "\text{delete } B"] \\
\mathbf{P}_A \arrow[d, "F \text{ covaiant}"] & \mathbf{I}_B \arrow[d, "F \text{ covaiant}"] & & \mathbf{P}_B \arrow[d, "G \text{ contravaiant}"] \\
F\mathbf{P}_A \arrow[d, squiggly] & F\mathbf{I}_B \arrow[d, squiggly] & & G\mathbf{P}_B \arrow[d, squiggly] \\
L_nF(A) := H_n(F\mathbf{P}_A) & R^nF(B) := H_n(F\mathbf{I}_B) & & R^nG(B) := H_n(G\mathbf{P}_B)
\end{tikzcd}$$

In each definition, the direction of the sequence applied by F is in the same direction. For L_nF, there is nothing to mention. For R^nF and R_nG, $F\mathbf{I}_B$ and $G\mathbf{P}_B$ are in the same direction. Left and right derived functor depends on a functor F, so

Exactness and derived-functors

Functors F	covariant		contravariant	
Exactness of F	right-exact	left-exact	right-exact	left-exact
Resolution applied	projective	injective	injective	projective
Derived functors applied	left-derived	right-derived	left-derived	right-derived
Examples of F	(-)$\otimes_R N$	$\mathrm{Hom}_R(M, -), \Gamma(X, -), (-)^G$	uncommon	$\mathrm{Hom}_R(-, N)$

From this table, one important observation is that the exactness of F and the direction of the derived functors are strongly related: the left-exact functor is binded with the right-derived functor, and the right-exact functor is binded with the left-derived functor. Each 'pack' of this property has no relation to any other properties.

6 Examples for left-derived functors

6.1 Tor functor

As we mentioned earlier, $(-) \otimes_R M$ and $\mathrm{Hom}(-, M)$ are two very important examples for right-exact and left-exact functors respectively. In previous subsections, we deal with abstract categories and functors, the most general abelian categories and additive functors. It is time to be down-to-earth and to embrace the categories and functors we are familiar with, the categories of R-modules ${}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$, and the functors $(-) \otimes_R M$ together with $\mathrm{Hom}(-, M)$.

6.1.1 Tor **functor**

Here we give the definition for Tor and functors and then show they are equivalent. For $A \in \mathrm{Ob}\mathbf{Mod}_R$ and $B \in \mathrm{Ob}_R\mathbf{Mod}$, we have two covariant right-exact functors: $(-) \otimes_R B$ and $A \otimes_R (-)$.

Definition 6.1: Tor **and functors**

For $A \in \mathrm{Ob}\mathbf{Mod}_R$ and $B \in \mathrm{Ob}_R\mathbf{Mod}$, Tor and tor are defined to be the left-derived functors

$$\mathrm{Tor}_n(-, B) := L_n((-) \otimes_R B) \quad \mathrm{tor}_n(A, -) := L_n(A \otimes_R (-))$$

Remark Explicitly,

$$\mathrm{Tor}_n(A, B) = L_n((-) \otimes_R B)(A) = H_n(((-) \otimes_R B)\mathbf{P}_A) = H_n(\mathbf{P}_A \otimes_R B)$$

and

$$\mathrm{tor}_n(A, B) = H_n(A \otimes_R \mathbf{P}_B)$$

One nice result about these two functors is $\forall n, \forall A, \forall B$,

$$\boxed{\mathrm{Tor}_n(A, B) \cong \mathrm{tor}_n(A, B)}$$

This is the theorem ss. To prove this, we need the following lemmas:

Corollary 6.1: Subscript reducing isomorphism

Let $\mathscr{A}$ be an abelian category with enough projectives and $A \in \mathrm{Ob}(\mathscr{A})$. Let $P_\bullet$ be a projective resolution of A:

$$P_\bullet : \quad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

Let $K_0 := \ker \epsilon$, $\forall i \geq 1$, $K_i := \ker d_i$. Then,

$$(L_{n+1}F)A \cong (L_nF)K_0 \cong (L_{n-1}F)K_1 \cong \cdots \cong (L_1F)K_{n-1}$$

Remark In particular, when $\mathscr{A} =_R \mathbf{Mod}$, $B \in \mathrm{Ob}(\mathscr{A})$ and $F = (-) \otimes_R B$, take the projective resolution of A and K_i as above. Then,

$$\mathrm{Tor}_{n+1}(A, B) \cong \mathrm{Tor}_n(K_0, B) \cong \mathrm{Tor}_{n-1}(K_1, B) \cong \cdots \cong \mathrm{Tor}_1(K_{n-1}, B)$$

Proof: The basic idea is to shortcut the sequence so that the truncated sequence is a projective resolution of some element in the original sequence.
For the sake of consistency, $\epsilon =: d_{-1}$ and $A =: K_{-1}$. Exactness of $P_\bullet \Rightarrow K_0 := \ker d_{-1} = \mathrm{im} d_0$. So, the new sequence

$$\begin{tikzcd}
 & \cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, equal] & P_1 \arrow[r, "d_1"] \arrow[d, equal] & K_0 \arrow[r] \arrow[d, equal] & 0 \\
P'_\bullet & \cdots \arrow[r, "\delta_2"] & P'_1 \arrow[r, "\delta_1"] & P'_0 \arrow[r, "\delta_0"] & K_0 \arrow[r] & 0
\end{tikzcd}$$

is exact, where $\forall n$, $P'_n = P_{n+1}$ and $\delta_n := d_{n+1}$. The lower sequence is hence a projective resolution of K_0. Let $(P'_\bullet)^{K_0}$ be the deleted projective resolution of K_0 and $(P_\bullet)_A$ be the deleted projective resolution of A. We have $\forall n$,

$$\begin{align} L_nF(K_0) &\overset{\dagger}{\cong} H_n(FP'_\bullet) = \frac{\ker \delta_n}{\mathrm{im}\delta_{n+1}} && \text{by definition of } K_0 \tag{1} \\ &= \frac{\ker d_{n+1}}{\mathrm{im}d_{n+2}} && \text{by definition of } \delta_n \tag{2} \\ &= H_{n+1}(FP_\bullet^A) \overset{\dagger}{\cong} L_{n+1}F(A) && \text{by definition of } A \tag{3} \end{align}$$

† from the theorem 5.1.
Iterating this process by truncating the sequence to $K_1, \ldots, K_n$, those isomorphisms follows. □

Theorem 6.1: Equivalence of Tor and tor

Let $A \in \operatorname{Ob}(\mathbf{Mod}_R)$ and $B \in \operatorname{Ob}({}_R\mathbf{Mod})$. Then, $\forall n$, $\operatorname{Tor}_n(A, B) \cong \operatorname{tor}_n(A, B)$.

Proof: (A.Zaks) This inducts on n.
Base case: when $n = 0$,
Inductive hypothesis: Suppose that for n, $\operatorname{Tor}_n(A, B) \cong \operatorname{tor}_n(A, B)$.
Inductive step: Consider a projective resolution of A and B respectively and factor it into SESs:

$$P_\bullet : \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0 \quad Q_\bullet : \cdots \xrightarrow{d_2'} Q_1 \xrightarrow{d_1'} Q_0 \xrightarrow{\epsilon'} B \to 0$$

The factorization of $P_\bullet$ by syzygies into short exact sequence is:

$$\begin{tikzcd}
 & X & 0 & W \\
Y & & &
\end{tikzcd}$$

□

Remark There is another direct proof. That version would not be introduced until the spectral sequence is studied.
In this proof, the projective resolution is not fully exploited. Each module in this projective resolution is projective hence flat. The proof only utilize the flatness of each term in the resolution. So, instead, the Tor and tor functor can be defined by taking flat resolutions(see [Rot09]).

6.1.2 Axiomatic characterization of Tor_n functors

6.1.3 Applications of Tor functor

From now on, R is an integral domain. Denote $Q := \operatorname{Frac}(R)$, $S := R - \{0\}$ and $K := Q/R$ (the quotient). These components form an exact sequence:

$$0 \longrightarrow R \longrightarrow Q \longrightarrow K \longrightarrow 0$$

By theorem oo, this SES and the functor Tor_n induces a LES:

$$\cdots \longrightarrow \operatorname{Tor}_2(K, A) \longrightarrow \operatorname{Tor}_1(R, A) \longrightarrow \operatorname{Tor}_1(Q, A) \longrightarrow \operatorname{Tor}_1(K, A) \longrightarrow \operatorname{Tor}_0(R, A) \longrightarrow \operatorname{Tor}_0(Q, A) \longrightarrow \operatorname{Tor}_0(K, A) \longrightarrow 0$$

Now, we want to demystify each spot in this long exact sequence:
(1) $\forall n \geq 1$, $\operatorname{Tor}_n(R, A) = 0$, since R is a free R-module $\Rightarrow$ is projective.
(2) $\forall n \geq 1$, $\operatorname{Tor}_n(Q, A) = 0$ since Q is flat. This is because the equivalency of $(-) \otimes_R Q :_R \mathbf{Mod} \to_{S^{-1}R} \mathbf{Mod}$ and $[S^{-1}] :_R \mathbf{Mod} \to_{S^{-1}R} \mathbf{Mod}$ as functors:

$$\begin{tikzcd}
\text{Localization} & \text{Tensor} \\
{[S^{-1}]} \arrow[r, leftrightarrow, "\sim"] & (-) \otimes_R S^{-1}R \\
\text{exact} \arrow[r, rightsquigarrow] & \text{exact} \arrow[d, Leftrightarrow] \\
 & S^{-1}R \text{ is flat}
\end{tikzcd}$$

(3) $\forall n \geq 2$, $\operatorname{Tor}_n(K, A) = 0$. This comes from the exactness at $\operatorname{Tor}_n(K, A)$.
The name Tor comes from the following reasons. Consider a 'functor' $(-)_t :_R \mathbf{Mod} \to_R \mathbf{Mod}$ that maps an R-module M to its torsion submodule M_t, and an R-module homomorphism $f : M \to N$ to $f_t : M_t \to N_t$ with $f_t = f|_{M_t}$.

Theorem 6.2: Origination of Tor

Under the settings above. $\operatorname{Tor}_1(K, -)$ and $(-)_t$ are naturally equivalent.

Proof: □

Tor_n for PID:

From now on, suppose that R is a PID and M, N are finitely generated R-modules. From the structure theorem for finitely generated modules over PIDs, M and N have the following decomposition:

$$M \cong R^s \oplus \bigoplus_{i=1}^{d} R/I_i \quad N \cong R^t \oplus \bigoplus_{j=1}^{e} R/J_j$$

By virtue of the fact that $L_n F$ is ???, each $\mathrm{Tor}_n(M,N)$ can be decomposed into

$$\mathrm{Tor}_n(M,N) = \mathrm{Tor}_n(R^s,R^t) \oplus \bigoplus_{i=1}^{d} \mathrm{Tor}_n(R/I_i,R^t) \oplus \bigoplus_{j=1}^{e} \mathrm{Tor}_n(R^s,R/J_j) \oplus \bigoplus_{\substack{1\le i\le d\\ 1\le j\le e}} \mathrm{Tor}_n(R/I_i,R/J_j)$$

Each $\mathrm{Tor}_n(*,*)$ with R^s or R^t appearing at $*$ vanishes because free implies projective. So, it reduces the calculation of $\mathrm{Tor}_n(M,N)$ into calculation of $\mathrm{Tor}_n(R/I,R/J)$ for some $I, J \trianglelefteq R$. The outcome is condensed in the following theorem:

Theorem 6.3: Tor **for PID**

Let R be a PID and $I, J \trianglelefteq R$. Then,

$$\mathrm{Tor}_n(R/I,R/J) = \begin{cases} R/(I+J) & n = 0 \\ (I\cap J)/IJ & n = 1 \\ 0 & n \ge 2 \end{cases}$$

7 Examples for right-derived functors

7.1 Ext functor

7.1.1 Applications of Ext 1: Classifying extensions of R-modules

Given two R-modules, how to nest them into a bigger one? This entails the knowledge of extensions. Let A, C be two R-modules (R is commutative). Then,

Definition 7.1: Extension of R-modules

Use the notations above. An R-module E is called an extension of A by C $\Leftrightarrow$ there is an exact sequence of R-modules ξ

$$\xi : \quad 0 \longrightarrow A \xrightarrow{f} E \xrightarrow{g} C \longrightarrow 0$$

Remark Such an extension ξ is called **split** $\Leftrightarrow$ a homomorphism h such that $g \circ h = \mathrm{id}_C$, called **section** of g.

Lemma 7.1 (Equivalence of split extensions) *Let $\xi : 0 \to A \xrightarrow{f} E \xrightarrow{g} C \to 0$ be an extension of R-modules A by C, ξ is split $\Leftrightarrow$ ξ is equivalent to the short exact sequence $0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0$* [3]*, where $\iota : a \mapsto (a,0)$ and $\pi : (0,c) \mapsto c$.*

Proof: $\Rightarrow$ If ξ is split, the goal is to show $E \cong f(A) \oplus g(C)$. This is equivalent to see $f(A) \cap g(C) = \{0\}$ and $f(A)g(C) = E$. The second one needs to pick a $y \in E$ and use the element $y - h(g(y))$.
$\Leftarrow$ Suppose $E \cong A' \oplus C'$ where $A \cong A'$ and $C \cong C'$. The equivalence gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"] & A' \oplus C' \arrow[r, "\pi"] & C \arrow[r] & 0
\end{tikzcd}$$

[3] Here $\oplus$ is the exterior direct sum.

Explicitly, suppose the isomorphisms are given by $\alpha : A \to A'$ and $\gamma : C \to C'$. Then, $\pi : (a, c) \mapsto \gamma(c)$. Define $\tilde{h} : C \to A' \oplus C'$ by $c \mapsto (0, \gamma^{-1}(c))$. It is quick to see that $\pi \circ \tilde{h} = \mathrm{id}_C$. Define $h := \varphi^{-1} \circ \tilde{h}$. This h is the desired one because

$$g \circ h = g \circ (\varphi^{-1} \circ \tilde{h}) = \pi \circ \varphi \circ \varphi^{-1} \circ \tilde{h} = \pi \circ \tilde{h} = \mathrm{id}_C$$

Hence, ξ is a split extension. □

Next, the notion of equivalence extension is needed. Because every extension keeps A and C and merely alters the middle E, equivalence should be given as

Definition 7.2: Equivalent extensions of R-modules

Let ξ, ξ' be two R-module extensions of A by C. ξ and ξ' are **equivalent**, denoted $\xi \sim \xi' \Leftrightarrow \exists$ R-module homomorphism $\varphi : E \to E'$ making the following diagram commutes.

$$\begin{tikzcd}
\xi : & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal, "\mathrm{id}_A"'] & E \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, equal, "\mathrm{id}_C"] & 0 \\
\xi' & 0 \arrow[r] & A \arrow[r, "f'"] & E' \arrow[r, "g'"] & C \arrow[r] & 0
\end{tikzcd}$$

Remark The equivalence between two extensions is not barely equivalent to the homomorphism between E and E'. Because this seemingly classifies E. But it in fact classifies the triple (E, f, g). The object E and the morphisms related to E should be considered as integrity.
φ is automatically an isomorphism. It suffices to see that φ is a bijection. • φ is *injective*. First notice that $\forall x \in \ker \varphi$, $g(x) = g'(\varphi(x)) = 0$. So, $\ker \varphi \subseteq \ker g = \mathrm{im} f$. For this x, $\exists y \in A$ such that $x = f(y)$. Using the first block, $f'(y) = \varphi(f(y)) = \varphi(x) = 0$. Then, $y \in \ker f'$. Since f' is injective, $y = 0$. This implies that $x = 0$. So, $\ker \varphi = \{0\}$. • φ is *surjective*. $\forall w \in E'$, we want to find an element $z \in E$ such that $w = \varphi(z)$. But first, consider $g'(w) \in C$. Since g is surjective, $\exists t \in E$ such that $g(t) = g'(w)$. In the meanwhile, $g(t) = g'(\varphi(t))$. So, $w - \varphi(t) \in \ker g' = \mathrm{im} f'$. So, w- $\varphi(t) = f'(a)$ for some $a \in A$. Notice that $f'(a) = \varphi(f(a))$. So, $w = \varphi(t) + \varphi(f(a)) = \varphi(t + f(a))$. Let $z := t + f(a) \in E$. This z is desired, showing φ is surjective. Hence, equivalence of two extensions $\Rightarrow$ $\varphi : E \to E'$ is an isomorphism. But, the converse is not true as suggested in the following example.
The example gives an isomorphism $E \to E'$ but the corresponding extensions are not equivalent:
Consider two short exact sequence ξ_1 and ξ_2:

$$\xi_1 : 0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{f_1} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \to 0 \quad \xi_2 : 0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \to 0$$

where $f_1 : 1 + p\mathbb{Z} \mapsto p + p^2\mathbb{Z}$ and $f_2 : 1 + p\mathbb{Z} \mapsto 2p + p^2\mathbb{Z}$. Suppose there is an isomorphism $\varphi : \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ making the diagram commutes. This φ determines a mapping $\overline{\varphi} : p^2\mathbb{Z} \to p^2\mathbb{Z}$ such that $\varphi(a + p^2\mathbb{Z}) = \overline{\varphi}(a) + p^2\mathbb{Z}$.

The first block $\varphi \circ f_1 = f_2 \circ \mathrm{id}_{\mathbb{Z}/p\mathbb{Z}}$ acting on $1 + p\mathbb{Z}$ yields $p \cdot \overline{\varphi}(1) \equiv 2p \bmod p^2$.

The second block $\mathrm{id}_{\mathbb{Z}/p\mathbb{Z}} \circ g = g \circ \varphi$ acting on $1 + p^{\mathbb{Z}}$ gives that $g(\overline{\varphi}(1) + p^2\mathbb{Z}) = g(1 + p^2\mathbb{Z})$, which implies that $\overline{\varphi}(1) - 1 \in \ker g \cong \mathbb{Z}/p\mathbb{Z}$. Hence, $\overline{\varphi}(1) - 1 \in p\mathbb{Z}$. This yields $\overline{\varphi}(1) = 1 + kp$ for some k.

But, $p \cdot \overline{\varphi}(1) = p + kp^2 \equiv p \not\equiv 2p \bmod p^2$, contradiction. So, these two extensions are not equivalent.

Let $[\xi] := \{\xi' : 0 \to A \to E \to C \to 0 | \xi' \text{ is exact}, \xi' \sim \xi\}$ be an equivalence class of exact sequences which are equivalent to ξ.
The notation $e(C, A)$ denotes the collection of all equivalence classes of exact sequences that are extensions of A by C. $e(C, A)$ is characterized by $\mathrm{Ext}^1(C, A)$. Why this The map between $e(C, A)$ and $\mathrm{Ext}^1(A, C)$ is given by the following process:
• First choose a representative, i.e. a SES $\xi :\ 0 \to A \to E \to C \to 0$ that is an extension of A by C, of some equivalence class $[\xi]$
• Then, pick a projective resolution of C, $\cdots \to 0 \to P_1 \to P_0 \to C \to 0$
• prolonging the left hand side of ξ by 0s, the identity map on C, id_C, could be extended into many maps by comparison lemma 5.2:

$$\begin{tikzcd}
P_\bullet & \cdots \arrow[r] & 0 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\alpha_0"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\
\xi & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & E \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

- Then, the map $\Psi : e(C,A) \to \mathrm{Ext}^1(C,A)$ is defined to be $[\xi] \mapsto \alpha_1 + \mathrm{im} F d_1$, where $F = \mathrm{Hom}_R(-, A)$.

Theorem 7.1: Ψ

The map $\Psi : e(C,A) \to \mathrm{Ext}^1(C,A) \quad [\xi] \mapsto \alpha_1 + \mathrm{im} F d_1$ is well-defined.

Proof: The definition of Ψ depends on two choices: one is for α_1 (the extension of id_C), and another is for the representative ξ.
Suppose that we have another extension of id_C at $P_1 \to A$, say α_1'. By the comparison lemma 5.2, $\exists$ morphisms $s_0 : P_0 \to A$ and $s_1 : P_1 \to 0$ such that $\alpha_1 - \alpha_1' = 0 \circ s_1 + s_0 \circ d_1 = s_0 \circ d_1$. Notice that $\mathrm{im} F d_1 = \{f \circ d_1 : f \in \mathrm{Hom}_R(P_0, A)\}$. So, $\alpha_1{}'_1 \in \mathrm{im} F d_1$.
Suppose that there is another representative of ξ, ξ'. ξ' is another extension of A by C that is isomorphic to ξ. Then, we have a composition of morphisms of SESs:

$$\begin{tikzcd}
P_\bullet & \cdots \arrow[r] & 0 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\alpha_0"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\
\xi & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & E \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\
\xi' & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

By definition of Ψ, it sends $[\xi']$ to the class of morphism from P_1 to A, which is $\mathrm{id}_A \circ \alpha_1 + \mathrm{im} F d_1 = \alpha_1 + \mathrm{im} F d_1$, showing that $\Psi([\xi]) = \Psi([\xi'])$. □

The map is used to characterize $e(C,A)$, so it should have nice property, being an isomorphism. To show this, we should construct an inverse of Ψ. Namely, for every α_1, find an extension of A by C.

7.1.2 Application of Ext 2: Baer sum

7.2 Sheaf cohomology

7.3 Group cohomology

Take the functor to be the invariant group functor $(-)^G :_{\mathbb{Z}[G]} \textbf{Mod} \to_{\mathbb{Z}} \textbf{Mod}$. This functor sends every $\mathbb{Z}[G]$-module A to its invariant submodule under the action of G, A^G. Since A^G is the part that is independent of the action of G, then it is viewed as a $\mathbb{Z}$-module. $(-)^G$ also takes every morphism f to its restriction. Consider an exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

The functor induces an left-exact sequence

$$0 \longrightarrow A^G \xrightarrow{f|_{A^G}} B^G \xrightarrow{g|_{B^G}} C^G$$

Because $\ker g|_{B^G} = \ker g \cap B^G = \mathrm{im} f \cap B^G = \mathrm{im} f|_{A^G}$. The last equality comes from the fact that f is G-equivariant and injective. For the sequence not necessarily being exact at C, here is an example: filling.
A central theme in homological algebra is to remedy the failure of exactness caused by applying a functor, typically by extending the resulting sequence into a long exact sequence. The treatment of the functor $(-)^G$ is no exception to this principle. However, before we can construct such an extension of the sequence $0 \to A^G \to B^G \to C^G$, a shift in perspective is necessary.
There are two perspectives for the invariants:
(1) The first perspective is 'internal'. A^G is viewed as something defined inside A. It is defined by checking whether individual elements $a \in A$ satisfy a certain property: remaining invariant under the action of every element of $g \in G$. This is an 'element-wise' or 'local' viewpoint.
(2) However, the core philosophy of homological algebra and categories are quite different. The center of study shifts away from the individual elements inside every single object. Instead, an object is understood by studying the morphisms between it and any other objects. This idea draws heavily from the Grothendieck school. Its essence, the so-called *point de vue relatif*(relative point of view), is that an object's identity is characterized by its morphisms with all the other objects. This philosophy inspires us to recast the 'internal property' A^G in terms of an 'external relationship', expressed as a collection of morphisms. We call this 'external perspective'.

How to make this shift of perspective operational? To talk about morphisms, we need another object X in addition to A, allowing us to study $\mathrm{Hom}(X, A)$. This object must serve as a universal probe to detect the specific property we are interested in, G-invariance across all the objects as $\mathbb{Z}[G]$-modules.

So, what object best embodies the concept of G-invariance? The answer is precisely $\mathbb{Z}$. This is because G-invariance signifies the triviality of G-action (G-action is ignored), leaving only the $\mathbb{Z}$-module structure.
Hence, it is natural to think the relationship between A^G and $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$.

Lemma 7.2 *Let A be a $\mathbb{Z}[G]$-module and $\mathbb{Z}$ have the trivial G-action on it. Then,*

$$\boxed{A^G \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)}$$

Proof: Define the map

$$\Psi : A^G \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \qquad a \mapsto \Big(f_a : 1 \mapsto a\Big)$$

Ψ is a group homomorphism(hence a $\mathbb{Z}$-module homomorphism) because $\forall x, y$, $\Psi(x+y) = f_{x+y} = f_x + f_y = \Psi_x + \Psi_y$. This map is injective because f_a is uniquely determined by the value assigned to 1. Ψ is also surjective because $\forall f \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$, $\forall g \in G$, ${}^g f(1) = f({}^g 1) = f(1)$. The first equality comes from G-equivariance of f and the second equality comes from triviality of the action $G \curvearrowright \mathbb{Z}$. So, $f(1) \in A^G$ and $f = \Psi(f(1))$. □

Remark Similarly, there is a general version:

$$\boxed{A \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A)}$$

Something deeper underlies here: as a left-exact covariant functor $(-)^G$, $(-)^G$ can be identified with the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$. We just abuse the notation H^n and define

$$H^n(G, A) := \big(R^n(-)^G\big)(A) \cong \big(R^n\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)\big)(A) =: \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

Especially, $H^0(G, A) = \mathrm{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}, A) \cong A^G$. From filling, the short exact sequence $0 \to A \to B \to C \to 0$ induces a long exact sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \longrightarrow H^2(G, A) \longrightarrow \cdots$$

Group cohomology was first conceived as a technical mechanism to measure and remedy the non-exactness of the functor $(-)^G$ only. But, it was quickly realized that these groups were not merely corrective tools. These groups themselves are subtle invariants, revealing profound structural insights and applications that extended far beyond the purely 'corrective' motivations. Consequently, they become a central object in research.

7.3.1 Calculation of cohomology groups 1: Choosing resolutions

Cohomology groups are concretely defined to be something derived from some resolution(either injective or projective). So, the standard way to calculate $H^n(G, A)$ for some group G and a $\mathbb{Z}[G]$-module A is to choose a $\mathbb{Z}[G]$-modules resolution of $\mathbb{Z}$. In fact, projective resolutions are simpler filling.
To assign each object and morphism concrete stuff on the following sequence

$$\mathbf{P}_{\mathbb{Z}} : \cdots \xrightarrow{d_3} Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

and make it a projective resolution. For some certain groups, the projective resolutions are easier to construct. Here are two examples for specific groups. The general construction of $\mathbf{P}_{\mathbb{Z}}$ follows.
(1) Let $G = \langle t \rangle$ of infinite order. Then,

$$0 \longrightarrow \mathbb{Z}[G] \xrightarrow{\cdot(t-1)} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a free (hence projective) resolution of $\mathbb{Z}$.
Let φ be the map $\cdot(t-1)$. Applying $\epsilon \circ \varphi$ for any element in $\mathbb{Z}[G]$, $\sum_{n=0}^{m} b_n t^n$, where m is the highest index of non-zero b_n,

$$(\epsilon \circ \varphi)\left(\sum_{n=0}^{m} b_n t^n\right) = \epsilon\Big(\sum_{n=0}^{m} b_n t^{n+1}\Big) - \epsilon\Big(\sum_{n=0}^{m} b_n t^n\Big) = 0$$

Fix an element $\sum_n^M a_n t^n$ in $\ker \epsilon$, where M is the highest integer making a_n non-zero. Hence, $\sum_{n=0}^M a_n = 0$. Suppose there is a $\sum_{n=0}^N b_n t^n$ such that

$$\sum_{n=0}^{M} a_n t^n = \sum_{n=0}^{N} b_n t^n (t-1) = -b_0 + \sum_{n=1}^{N} (b_{n-1} - b_n) t^n + b_N t^{N+1}$$

To equate both sides, first $M = N+1$. This equation also gives the coefficients of each b_n. $\forall 0 < N$, $b_n = -\sum_{i=0}^n a_i$. But, for $n = N$, there are two equations for b_N,

$b_N = -\sum_{i=0}^N a_i$

$b_N = a_{N+1}$

But, $\sum_{i=0}^{N+1} a_i = 0$ guarantees the existence of solutions of two equations by asserting they are equal. Therefore, we find an element $\sum_n \left(-\sum_{i=0}^n a_i\right) t^n$, showing that $\ker \epsilon \subseteq \mathrm{im}\varphi$.

(2) Let $G = \langle t \rangle$ of finite order n. Then,

$$\cdots \xrightarrow{\alpha} \mathbb{Z}[G] \xrightarrow{\beta} \mathbb{Z}[G] \xrightarrow{\alpha} \mathbb{Z}[G] \xrightarrow{\beta} \mathbb{Z}[G] \xrightarrow{\alpha} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a free resolution with $\alpha(x) = (t-1) \cdot x$ and $\beta(x) = \left(\sum_{i=0}^{t-1} t^i\right) \cdot x$.

Just as we did above, $\ker \epsilon = \mathrm{im}\alpha$. (Even though the group G has finite order now, just slightly modify the argument.)

It remains to see that $\ker \alpha = \mathrm{im}\beta$ and $\ker \beta = \mathrm{im}\alpha$.

For any $x \in \mathbb{Z}[G]$,

$$(\alpha \circ \beta)(x) = (t-1) \cdot \left(\sum_{i=0}^{t-1} t^i\right) \cdot x = \left(\sum_{i=0}^{n-1} t^i - \sum_{i=0}^{n-1} t^i\right) \cdot x = 0, \quad (\beta \circ \alpha)(x) = 0$$

For any $x = \sum_{i=0}^{n-1} a_i t^i \in \ker \alpha$, this implies that $\forall i$, a_i are all equal. Then, $x = a_0 \cdot \left(\sum_i t^i\right) \in \mathrm{im}\beta$.

(3) Now, let's consider the most general case. We do the following steps: Assume first that G is a finite group.

• For objects: $\mathbb{Z}$ can be thought of as a $\mathbb{Z}[G]$-module with null basis. Following this idea, the objects in the resolution can be taken to be $\mathbb{Z}[G]$-modules with G^0, G, G^2, etc, as basis. So, $\forall n \geq 0$, let Q_n be the free $\mathbb{Z}[G]$-module over the basis G^n. Due to historical reasons and for the sake of visual clarity, we adopt the notation

$$[x_1 | \dots | x_n] := (x_1, \dots, x_n)$$

Under this notation, Q_n is actually the free $\mathbb{Z}[G]$-module over the basis $\{[x_1|\dots|x_n] : x_1, \dots, x_n \in G\}$. Especially, Q_0 has $\{[\,]\}$ as its basis, meaning Q_0 is identical to $\mathbb{Z}[G]$ or $\mathrm{rank}_{\mathbb{Z}[G]} Q_0 = 1$. Similarly, $\mathrm{rank}_{\mathbb{Z}[G]} Q_n = |G|^n$.

• For morphisms: First consider ϵ, define it to be

$$\epsilon : Q_0 \to \mathbb{Z} \qquad \left(\sum_{g \in G} n_g g \cdot [\,]\right) \mapsto \sum_{g \in G} n_g$$

This is indeed a $\mathbb{Z}[G]$-module homomorphism.

Then, for $n \geq 1$, it suffices to define d_n and operate it on the basis of Q_n and then extend the map $\mathbb{Z}[G]$-linearly to whole Q_n. $d_n : Q_n \to Q_{n-1}$ works elementwise as follows:

$$d_n([x_1|\dots|x_n]) := x_1[x_2|\cdots|x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1|\cdots|x_{i-1}|x_i x_{i+1}|x_{i+2}|\dots|x_n] + (-1)^n [x_1|\dots|x_{n-1}]$$

There is no need to check this d_n is a $\mathbb{Z}[G]$-module homomorphism ($\mathbb{Z}[G]$-linear) because we specifies the rules on basis and *force* it to be $\mathbb{Z}[G]$-linear on Q_n.

For lower rank, this map looks milder:

$d_1([x_1]) = x_1[\,] - [\,]$

$d_2([x_1|x_2]) = x_1[x_2] - [x_1 x_2] + [x_1]$

$d_3([x_1|x_2|x_3]) = x_1[x_2|x_3] - [x_1 x_2|x_3] + [x_1|x_2 x_3] - [x_1|x_2]$

Now the definition work is done and it remains to check that this sequence is indeed a $\mathbb{Z}[G]$-module resolution.

Lemma 7.1: Bar resolution

Let Q_n and d_n ($\epsilon := d_0$) be the objects and morphisms defined above. Then,

$$\cdots \xrightarrow{d_3} Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a $\mathbb{Z}[G]$-module projective resolution of $\mathbb{Z}$.

Proof: □

To facilitate the discussion below, here is another identification. Objects in the complex(we just showed it is) resulting from the $Hom_{\mathbb{Z}[G]}(-, A)$ functor

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(Q_1, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(Q_2, A) \to \cdots$$

are cumbersome. A natural way to streamline $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, A)$ is to reduce functions to the basis G^n of Q_n:

$$\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, A) \cong \{\text{set-theoretical functions } f : G^n \to A\}$$

Lemma 7.3 *$A^{G^n} = \{$set-theoretical functions $f : G^n \to A\}$. A^{G^n} can naturally be equipped with a $\mathbb{Z}[G]$-module structure. Then, $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, A) \cong_{\mathbb{Z}[G]-\mathbf{Mod}} A^{G_n}$.*

Proof: For any $\mathbb{Z}[G]$-linear map $g : Q_n \to A$, define $\Phi : h \mapsto h|_{G^n}$. Then, $h|_{G^n}$ is a function $G^n \to A$. From right to left, send each f to its unique $\mathbb{Z}[G]$-linear extension to get the $\tilde{f} \in \mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, A)$, i.e. the map defined is $\Omega : \tilde{f} \hookleftarrow f$. $\Omega \circ \Phi(h) = h$ because h is uniquely determined by its acting on the basis. Hence, Φ is a bijection. □

Remark The key advantage of this identification is simplification, in two ways:
First, by focusing on A^{G^n}, a subset of $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, A)$, we are dealing with a more constrained (and likely simpler) set. Second, elements of A^{G^n} are simpler. This allows us to treat them as simpler objects, set-theoretical functions and forget the $\mathbb{Z}[G]$-linearity.

7.3.2 Calculations of group cohomology 2: formal calculations

Now we are trying to compute some group cohomology groups to feel how it works. The examples at our disposal are cyclic groups with infinite order and finite order, respectively.
Recall that the cohomology groups are independent of choice of resolutions. There are resolutions found previously. So, we shall use the resolution just found to do computations.
(1) First, consider $G = \langle t \rangle$ with infinite order. One resolution we found is

$$0 \longrightarrow \mathbb{Z}[G] \overset{\cdot(t-1)}{\longrightarrow} \mathbb{Z}[G] \overset{\epsilon}{\longrightarrow} \mathbb{Z} \longrightarrow 0$$

The corresponding exact sequence resulting from acting $\mathrm{Hom}_{\mathbb{Z}[G]}(-, A)$ is

$$0 \longleftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \overset{\cdot(t-1)^*}{\longleftarrow} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \longleftarrow 0$$

But this looks cumbersome to compute. One way to simplify this sequence is to exploit the isomorphism in the remark of lemma 7.1, $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \cong A$:

$$\begin{tikzcd}
0 & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \arrow[l, "d^{2*}"'] \arrow[d, "\Phi"', "\sim"] & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \arrow[l, "d^{1*}"'] \arrow[d, "\Phi"', "\sim"] & 0 \arrow[l, "d^{0*}"'] \\
0 & A \arrow[l, "\tilde{d}^2"] & A \arrow[l, "\tilde{d}^1"] & 0 \arrow[l, "\tilde{d}^0"]
\end{tikzcd}$$

Notice that $\forall n \geq 2$, $H^n(G, A) = 0$.
For $H^1(G, A) \cong \ker \tilde{d}^2 / \mathrm{im} \tilde{d}^1$, we can use the sequence below. $\ker \tilde{d}^2 = A$. For $\mathrm{im}\, \tilde{d}^1 = \{n \in A : \exists m \in A, \tilde{d}^1(m) = n\}$, by virtue of the commutativity of the diagram, $\tilde{d}^1 = \Phi \circ d^{1*} \circ \Phi^{-1}$. Let's compute $\tilde{d}^1(m)$:

$$\begin{aligned}
\tilde{d}^1(m) &= \Phi \circ d^{1*} \circ \Phi^{-1}(m) && \quad (4)\\
&= (\Phi \circ d^{1*})(f_m) && \text{construction of } \Phi : \Phi^{-1} : m \mapsto (f_m : 1 \mapsto m) \quad (5)\\
&= \Phi(f_m \circ d^1) && \text{definition of } d^{1*}\text{, where } d^1 = \cdot(t-1) \quad (6)\\
&= (f_m \circ d^1)(1) && \text{construction of } \Phi : \Phi : \theta \mapsto \theta(1) \quad (7)\\
&= f_m(t-1) && \quad (8)\\
&= (t-1) \cdot m && f_m : 1 \mapsto m \quad (9)
\end{aligned}$$

This shows that $\mathrm{im}\, \tilde{d}^1 = \{(t-1) \cdot m : m \in A\}$. Let $\delta := (t-1)$. Then, $H^1(G, A) \cong A/\delta A = A_G$. A_G is the co-invariant of A, i.e. the largest quotient fixed by G. [4]

[4] Here is the definition of A_G. Let $I_G A := \langle\{(g-1)m : g \in G, m \in A\}$. Then, $A_G := A/I_G A$.
A very important property of A_G is: It is the largest quotient fixed by $G(G \curvearrowright A/I_G A)$, more precisely $(A/I_G A)^G = A/I_G A$. 'Largest' means that there is no such submodule N of A with $N \subsetneqq I_G A$ such that $(A/N)^G = A/N$. Because any such submodule must contain all $(g-1)m$, $g \in G$ and $m \in A$, meaning $I_G A \subseteq N$.
In the context of cyclic group G, $I_G A$ gets simplified a lot. Because every g is of the form t^k, so $(t^k - 1)m = (t-1)(t^{k-1} + \cdots + 1)m$. Let $m' = (t^{k-1} + \cdots + 1)m$. $(g-1)m$ can still be written in the form $(t-1)m'$. So, in the cyclic group case, $I_G A = \{(t-1)m : m \in A\}$.

For $H^0(G,A)$, it is quick to see $H^0(G,A) \cong \ker \tilde{d}^1 = \{m \in A : (t-1)\cdot m = \tilde{d}^1(m) = 0\} = \{m \in A : tm = m\} = A^G$. In short,

$$H^0(G,A) \cong A^G, \quad H^1(G,A) \cong A/\delta A = A_G, \quad H^n(G,A) = 0 \text{ for } n \geq 2$$

(2) Second, consider $G = \langle t \rangle$ with order n. The resolution we found is

$$\cdots \xrightarrow{\alpha} \mathbb{Z}[G] \xrightarrow{\beta} \mathbb{Z}[G] \xrightarrow{\alpha} \mathbb{Z}[G] \xrightarrow{\beta} \mathbb{Z}[G] \xrightarrow{\alpha} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Let $\delta := (t-1)$ and $\eta := \sum_{i=0}^{n-1}$. Again, applying the $\mathrm{Hom}_{\mathbb{Z}[G]}(-,A)$ functor, we have the sequence:

$$\cdots \xleftarrow{d^3_\alpha} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],A) \xleftarrow{d^2_\beta} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],A) \xleftarrow{d^1_\alpha} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],A) \xleftarrow{d^0} 0$$

Once more, exploit the isomorphism $\mathrm{Hom}_{Z[G]}(\mathbb{Z}[G],A) \cong A$,

$$\begin{tikzcd}
\cdots & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],A) \arrow[l, "d^3_\alpha"'] \arrow[d, "\Phi"', "\sim"] & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],A) \arrow[l, "d^2_\beta"'] \arrow[d, "\Phi"', "\sim"] & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],A) \arrow[l, "d^1_\alpha"'] \arrow[d, "\Phi"', "\sim"] & 0 \arrow[l, "d^0"'] \\
\cdots & A \arrow[l, "d'^3_\alpha"] & A \arrow[l, "d'^2_\beta"] & A \arrow[l, "d'^1_\alpha"] & 0 \arrow[l, "d'^0"]
\end{tikzcd}$$

The calculations are quite similar to the infinite order case. $H^0(G,A) = A^G$.
For $n \geq 1$, $H^{2n}(G,A) = \ker d'^{2n+1}_\alpha / \mathrm{im} d'^{2n}_\beta$ and $H^{2n-1}(G,A) = \ker d'^{2n}_\beta / \mathrm{im} d'^{2n-1}_\alpha$.
Notice that $d'^{2n}_\beta(x) = (\Phi \circ d^{2n}_\beta \circ \Phi^{-1})(x) = \eta \cdot x$ and $d'^{2n-1}_\alpha(x) = \delta \cdot x$. So,

$$\ker d'^{2n+1} = \{x \in A : \delta \cdot x = 0\} = \{x \in A : \forall t \in G, (t-1)\cdot x = 0\} = \{x \in A : \forall t \in G, t \cdot x = x\} = A^G$$

For $\ker d'^{2n}$, calculations are same: $\ker d'^{2n} = \{x \in A : \eta \cdot x = 0\} = \{x \in A : \forall t \in G, \sum_{i=0}^{n-1} t^i \cdot x = 0\} =: A[\eta]$.
Similarly, $\mathrm{im} d'^{2n}_\beta = \{\eta \cdot x : x \in A\} = \eta A$ and $\mathrm{im} d'^{2n-1}_\alpha = \{\delta \cdot x : x \in A\} = \delta A$.
So, in conclusion,

$$H^0(G,A) \cong A^G, \quad H^{2n-1} \cong A[\eta]/\delta A, \quad H^{2n}(G,A) \cong A^G/\eta A, \text{ for } n \geq 1$$

(3) Some general(independent of groups) results on $H^n(G,A)$ hold. Now take G to be an arbitrary group. Still, A is a $\mathbb{Z}[G]$-module. Then,

$$H^0(G,A) \cong A^G$$

To see this, take an arbitrary resolution of $\mathbb{Z}$: $\cdots \to Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$. Applying $\mathrm{Hom}_{\mathbb{Z}[G]}(-,A)$ functor(left-exact contravariant) to it, we obtain a sequence only exact at $\mathrm{Hom}_{\{}\mathbb{Z}[G](\mathbb{Z},A)$ and $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_0,A)$:

$$\cdots \xleftarrow{d^{2*}} \mathrm{Hom}_{\mathbb{Z}[G]}(Q_1,A) \xleftarrow{d^{1*}} \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0,A) \xleftarrow{\epsilon^*} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A) \longleftarrow 0$$

The exactness at $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_0,A)$ and $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_0,A)$ yields injectivity of ϵ^* and $\ker d^{1*} = \mathrm{im}\epsilon^* \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A)$, respectively. While, the calculation of $H^n(G,A)$ uses the sequence deleting $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A)$ and keeps the remaining part the same:

$$\cdots \xleftarrow{d^{2*}} \mathrm{Hom}_{\mathbb{Z}[G]}(Q_1,A) \xleftarrow{d^{1*}} \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0,A) \longleftarrow 0$$

Lemma 7.1.$\Rightarrow$ the isomorphism $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A) \cong A^G$. Hence, $H^0(G,A) = \ker d^{1*} = \mathrm{im}\epsilon^* \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},A) \cong A^G$.

7.3.3 Application of group cohomology: Number theory

We now introduce applications of group cohomology to number theory and class field theory.
First, let's look at Hilbert's 90.

Theorem 7.2: Hilbert's 90

Let L/K be a finite Galois extension with Galois group $\mathrm{Gal}(L/K)$. Then, $H^1(\mathrm{Gal}(L/K), L^\times) = 0$.

Proof: $H^1(\mathrm{Gal}(L/K), L^\times) = \ker d^{2*}/\mathrm{im} d^{1*}$, where $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_2, L^\times) \xleftarrow{d^{2*}} \mathrm{Hom}_{\mathbb{Z}[G]}(Q_1, L^\times) \xleftarrow{d^{1*}} \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, L^\times)$. In this calculation, we still adopt the standard bar resolution of $\mathbb{Z}$, $\cdots \xleftarrow{d_2} Q_1 \xleftarrow{d_1} Q_0 \longleftarrow 0$.
For any $\varphi \in \ker d^{2*}$, $\varphi \in \ker d^{2*} \Leftrightarrow \varphi : Q_1 \to L^\times$ and $\varphi \circ d_2 = 1_{L^\times}$. To decode the condition $\varphi \circ d_2 = 1_{L^\times}$, apply it to basis element of Q_1, $\mathrm{Gal}(L/K)$. $\forall \tau, \sigma \in \mathrm{Gal}(L/K)$, we have

$$\begin{aligned}
1_{L^\times} = \varphi \circ d_2([\tau|\sigma]) &= \varphi(\tau[\sigma] - [\tau\sigma] + [\tau]) && (10)\\
&= \varphi(\tau[\sigma]) \cdot \varphi([\tau\sigma])^{-1} \cdot \varphi([\tau]) && \mathbb{Z}[\mathrm{Gal}(L/K)] - \text{linearity of } \varphi, \text{ operation of } L^\times \text{ is } \cdot \quad (11)\\
&= \varphi({}^{\tau}[\sigma]) \cdot \varphi([\tau\sigma])^{-1} \cdot \varphi([\tau]) && {}^{\tau}[\sigma] := \tau[\sigma] \quad (12)\\
&=^{\tau} \big(\varphi([\sigma])\big) \cdot \varphi([\tau\sigma])^{-1} \cdot \varphi([\tau]) && \mathbb{Z}[\mathrm{Gal}(L/K)] - \text{linearity or } \mathrm{Gal}(L/K) \text{ equivariance of } \varphi \quad (13)\\
&=^{\tau} \big(\varphi(\sigma)\big) \cdot \varphi(\tau\sigma)^{-1} \cdot \varphi(\tau) && [\sigma] = \sigma, [\tau] = \tau, [\tau\sigma] = \tau\sigma \text{ are basis of } Q_1 \quad (14)
\end{aligned}$$

(14) implicitly uses the above-mentioned identification. Transforming it, we finally have

$$\varphi(\tau\sigma) =^{\tau} \big(\varphi(\sigma)\big) \cdot \varphi(\tau) \tag{15}$$

A map $\psi \in \mathrm{im} d^{1*}$ if and only if $\exists$ a $\mathbb{Z}[G]$-module homomorphism $\alpha : G^0 = \mathbb{Z}[G] \cdot [\,] \to L^\times$ such that $\psi = \alpha \circ d_1$. Notice that α is uniquely determined by the value on the basis $[\,]$. Let $c := \alpha([\,])$. So, the existence of such an α is equivalent to the existence of such a c, under this: $\forall [\tau] \in Q_1$,

$$\psi([\tau]) = \alpha \circ d_1([\tau]) = \alpha(\tau[\,] - [\,]) = \tau(\alpha([\,])) \cdot \alpha([\,])^{-1} = \tau(c) \cdot c^{-1}$$

Now the task is transfered to find an element $c \in L^\times$ such that $\forall \tau \in \mathrm{Gal}(L/K)$, $\varphi(\tau) = \tau(c) \cdot c^{-1}$.
Let $\tilde{\sigma} := \sigma|_{L^\times}$, where $\sigma \in \mathrm{Gal}(L/K)$. Such a $\tilde{\sigma} : L^\times \to L^\times$ is a character. By Dedekind's lemma on the independence of characters, $\exists a \in L^\times$, such that

$$\sum_{\sigma \in \mathrm{Gal}(L/K)} \varphi(\sigma)\tilde{\sigma}(a) = \sum_{\sigma \in \mathrm{Gal}(L/K)} \varphi(\sigma)\sigma(a) \neq 0$$

Let $b := \sum_{\sigma \in \mathrm{Gal}(L/K)} \varphi(\sigma)\sigma(a)$. Applying τ,

$$\begin{aligned}
{}^{\tau}b &=^{\tau} \left(\sum_{\sigma \in \mathrm{Gal}(L/K)} \varphi(\sigma)\sigma(a) \right) && (16)\\
&= \sum_{\sigma \in \mathrm{Gal}(L/K)} {}^{\tau}\varphi(\sigma)\, {}^{\tau}\sigma(a) && (17)\\
&= \sum_{\sigma \in \mathrm{Gal}(L/K)} {}^{\tau}\varphi(\sigma)\tau\sigma(a) && (18)\\
&= \sum_{\sigma \in \mathrm{Gal}(L/K)} \varphi(\tau\sigma)\varphi(\tau)^{-1}\tau\sigma(a) && \text{by (15)} \quad (19)\\
&= \varphi(\tau)^{-1} \sum_{\eta \in \mathrm{Gal}(L/K)} {}^{\tau}\varphi(\eta)\eta(a) && (20)\\
&= \varphi(\tau)^{-1} \cdot b && (21)
\end{aligned}$$

Hence, $\varphi(\tau) = b \cdot ({}^{\tau}b)^{-1}$. Let $c := b^{-1}$. We have $\varphi(\tau) =^{\tau} c \cdot c^{-1}$. □
Remark As previous calculation suggested, elements of $\ker d^{2*}$ are twisted homomorphism. A function $f : G \to L^\times$ [5] is called twisted homomorphism $\Leftrightarrow$, $\forall \tau, \sigma \in G$,

$$f(\tau\sigma) =^{\tau} (f(\sigma)) \cdot f(\tau)$$

Comparing to a homomorphism $f(\tau\sigma) = f(\tau)f(\sigma)$, the term ${}^{\tau}(f(\sigma))$ is acted by τ, called 'twisted'. $H^1(G, A)$ is the quotient of the collection of twisted homomorphisms by the collection of 'trivial' homomorphisms. In other words, for any group G with an action on $L^\times$, $H^1(G, L^\times) = Z^1/B^1 = \ker d^{2*}/\mathrm{im} d^{1*}$ is a collection of classes of twisted homomorphism.
The Hilbert 90 says there is no 'non-trivial' twisted homomorphism $f : \mathrm{Gal}(L/K) \to L^\times$.[6] The word 'trivial' means

[5] Here G is not necessarily a Galois group, it can be any group has an action on $L^\times$.

[6] Here the $Z[G]$-module homomorphism $\mathbb{Z}[\mathrm{Gal}(L/K)] =: Q_1 :\to L^\times$ is identified with the set-theoretic function $\mathrm{Gal}(L/K) \to L^\times$.

a twisted homomorphism f is constructable via a simpler element $\mu \in L^\times$. More concretely, f is trivial if there is a $\mu \in L^\times$ that for every $\tau \in \mathrm{Gal}(L/K)$,

$$f(\tau) =^\tau \mu \cdot \mu^{-1}$$

This form comes from

$$\mathrm{im} d^{1*} = \{\varphi \circ d_1 : \varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, L^\times)\}$$

Hence, $\forall [\tau] \in \mathrm{Gal}(L/K)$ (basis of Q_1), $\varphi \circ d_1([\tau]) =^\tau \varphi([\,]) \cdot \varphi([\,])^{-1}$ ranging over all $\varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, L^\times)$. Notice that $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, L^\times) \cong L^\times$ by $\varphi \mapsto \varphi([\,])$. So, $\varphi \circ d^{1*}([\tau]) =^\tau \mu\mu^{-1}$ for every $\mu \in L^\times$. More explicitly,

$$\begin{aligned}
\{\varphi \circ d_1([\tau]) : \varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, L^\times)\} &= \{^\tau\varphi([\,])\varphi([\,])^{-1} : \varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, L^\times)\} \\
&\cong \{^\tau\varphi([\,])\varphi([\,])^{-1} : \varphi([\,]) \in L^\times\} \\
&= \{\tau(\mu)\mu^{-1} : \mu \in L^\times\}
\end{aligned}$$

Corollary 7.1: Normalized elements in a cyclic Galois extension

Let L/K be a finite cyclic Galois extension with Galois group $\mathrm{Gal}(L/K) = \langle g \rangle$ of order n. If $\lambda \in L^\times$ is an element whose norm to $K^\times$ equals to 1, i.e. $\prod_{i=0}^{n-1} g^i(\lambda) = 1$, then $\exists \mu \in L^\times$ such that $\lambda = g(\mu) \cdot \mu^{-1}$.

Remark There are two kinds of additions: one is the addition in $\mathbb{Z}[G]$; another is the 'addition' in a $\mathbb{Z}[G]$-module A. In the following example, the operation (or 'addition') is actually multiplication. So, more attention to the symbol is needed.

Proof: When $G := \mathrm{Gal}(L/K)$ is a finite cyclic group, we can choose the above-mentioned resolution of $\mathbb{Z}$. Then, we have $H^1(\mathrm{Gal}(L/K), L^\times) = A[\eta]/\delta A$, where $\eta = \sum_{i=0}^{n-1} g^i$ and $\delta = g - 1$. The Hilbert 90 theorem 7.2 asserts that $H^1(\mathrm{Gal}(L/K), L^\times) = 0$ implying that $L^\times[\eta] = \delta L^\times$.
Since

$$\begin{aligned}
{}^\eta\lambda &= ^{(1 +_{\mathbb{Z}[G]} g +_{\mathbb{Z}[G]} \cdots +_{\mathbb{Z}[G]} g^{n-1})} \lambda && +_{\mathbb{Z}[G]} \text{ is the addition in } \mathbb{Z}[G] && (22)\\
&= (^1\lambda) +_{L^\times} (^g\lambda) +_{L^\times} \cdots +_{L^\times} (^{g^{n-1}}\lambda) && +_{L^\times} \text{ is the addition in } L^\times && (23)\\
&= (^1\lambda) \cdot (^g\lambda) \cdot \cdots \cdot (^{g^{n-1}}\lambda) && +_{L^\times} \text{ is actually } \cdot && (24)\\
&= \prod_{i=0}^{n-1} g^i(\lambda) = 1 && ^{g^i}\lambda = g^i(\lambda), \text{ by assumption} && (25)
\end{aligned}$$

λ should be an element in the kernel of η, i.e.$\lambda \in L^\times[\eta] = \delta L^\times$. So, λ is in the image of δ. Hence, $\exists \mu \in L^\times$, such that $\lambda =^\delta \mu =^{g-1} \mu =^g \mu \cdot \mu^{-1} = g(\mu) \cdot \mu^{-1}$. □

Lemma 7.2

Still, let L/K be a finite Galois extension with Galois group $\mathrm{Gal}(L/K)$ and $\mu_n \subseteq K$ (K contains n distinct n-th roots of unity). Then, [a]

$$\frac{(L^\times)^n \cap K^\times}{(K^\times)^n} \cong \mathrm{Hom}(G, \mu_n)$$

[a] $(L^\times)^n = \{x^n : x \in L^\times\}$.

Proof: Notice that $0 \to \mu_n \xrightarrow{\iota} L^\times \xrightarrow{x \mapsto x^n} (L^\times)^n \to 0$ is an exact sequence. Let $G := \mathrm{Gal}(L/K)$. Then, apply the lemma 3.2 and use Hilbert's 90 7.2. We obtain a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(G,\mu_n) \arrow[r] \arrow[d, equal] & H^0(G,L^\times) \arrow[r] \arrow[d, equal] & H^0(G,L^\times)^n \arrow[r] \arrow[d, equal] & H^1(G,\mu_n) \arrow[r] \arrow[d, equal, "G\curvearrowright\mu_n \text{ trivially}"] & H^1(G,L^\times) \arrow[r] \arrow[d, equal, "\text{Hilbert' 90 7.2}"] & \cdots \\
0 \arrow[r] & {\mu_n}^G \arrow[r] \arrow[d, equal, "\mu_n\subseteq K"] & (L^\times)^G \arrow[r] \arrow[d, equal] & (L^{\times,n})^G \arrow[r] \arrow[d, equal] & \mathrm{Hom}(G,\mu_n) \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & \mu_n \arrow[r] & L^\times \cap K = K^\times \arrow[r] & (L^\times)^n \cap K^\times \arrow[r] & \mathrm{Hom}(G,\mu_n) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

The exactness at $(L^\times)^n \cap K^\times$ suggests the desired formula. □

7.3.4 Application of group cohomology: Group extension

The case of group extension is similar to the case of R-module extensions. To utilize the language of modules, if there are some group actions, it can be made into $\mathbb{Z}[G]$-modules. This bridge the gap between groups and R-modules. But, category of group is not an abelian category. Let A, C, E be groups. The notion of extensions of A by C is almost the same as in R-modules:

Definition 7.3: Extensions of groups

A group E is an extension of A by $C \Leftrightarrow$ there is an exact sequence of groups ξ

$$\xi : \quad 0 \longrightarrow A \xrightarrow{f} E \xrightarrow{g} C \longrightarrow 0$$

Now for this short exact sequence of groups $\xi : 0 \longrightarrow A \xrightarrow{f} E \xrightarrow{g} C \longrightarrow 0$, we try to see something. The first observation is $A \cong f(A) = \ker g$, so $f(A) \trianglelefteq E$. This fact implies that $\forall h \in C$, $hf(A)h^{-1} = f(A)$. (Also true for E. But, the goal is to see all structures of E jointed by A and C.)
For an arbitrary element $h \in C$, take a lift $\tilde{h} \in E$ of g, i.e. $g(\tilde{h}) = h$. So, every lift of h, $\tilde{h}$, induces an automorphism of $f(A)$:

$$\psi_{\tilde{h}} : f(A) \to f(A) \quad f(a) \mapsto \tilde{h} f(a) \tilde{h}^{-1}$$

Indeed, this equivalent to an automorphism of A by pulling the elements back. $A \to A \quad a \mapsto f^{-1}(\tilde{h} f(a) \tilde{h}^{-1})$.
It seems natural to define an action $C \curvearrowright A$. Notice that this is only for lifted elements of C in E. One more gap is between this and an action of C on A: If both $\tilde{h_1}$ and $\tilde{h_2}$ are lifts of h in E, then

$$\psi_{\tilde{h_1}} : f(a) \mapsto \tilde{h_1} f(a) \tilde{h_1}^{-1} \qquad \psi_{\tilde{h_2}} : f(a) \mapsto \tilde{h_2} f(a) \tilde{h_2}^{-1}$$

Since $g(\tilde{h_1}) = g(\tilde{h_2}) = g$, $\tilde{h_1}\tilde{h_2}^{-1} \in \ker g$. So, $\exists k \in f(A)$, $\tilde{h_2} = \tilde{h_1} k$.
In this case, $\psi_{\tilde{h_2}} : f(a) \mapsto \tilde{h_1}(kf(a)k^{-1})\tilde{h_1}^{-1}$. $\psi_{\tilde{h_2}} \circ \psi_{\tilde{h_1}}^{-1} \in \mathrm{Inn}(f(A))$. This implies that C barely induces a map

$$G \to \mathrm{Out}(f(A)) = \mathrm{Aut}(f(A))/\mathrm{Inn}(f(A))$$

Further, it also induces a map $C \to \mathrm{Out}(A)$. But $\mathrm{Out}(A)$ usually has complex structures beyond a module.
But, the case gets simplified if A is abelian. When A is commutative, $kf(a)k^{-1} = f(a)$ or $\mathrm{Inn}(f(A)) = \{1\}$. The $\mathrm{Out}(A)$ structure is beyond the scope of our study here. To make the question simplier to get started with, the condition A is abelian is put.
So, from now on, $\xi : 0 \to A \to E \to C \to 0$ is a short exact sequence with A being abelian and C being arbitrary.

The notion of equivalence is the same as in the module case.

Once an extension is given, there is a well-defined group action $C \curvearrowright A$ or $C \curvearrowright f(A)$: $\theta : G \to \mathrm{Aut}(f(A))$. The well-definedness is just from the argument above. Thus, θ equip A with a C-module structure, which can be extended linearly and uniquely to a $\mathbb{Z}[C]$-module structure on A. The $\mathbb{Z}[C]$ structure allows the discussion to fit into the frame of Ext functors and group cohomology. not quite clear .

Theorem 7.3: Classification of group extensions

Let A be an abelian group and C be an arbitrary group. Given an action $\theta : C \to \mathrm{Aut}(A)$, $H^2(C, A_\theta)$ classifies all equivalence classes of group extensions of A by C under the action θ. Then, the trivial element $0 \in H^2(C, A_\theta)$ corresponds to the trivial class of extension, the semi-direct product $A \rtimes_\theta C$.
If $H^2(C, A_\theta) = \{0\}$, then any extension of A by C is a semi-direct product and the complement of A in E is classified by $H^1(C, A_\theta)$.

Remark The reason why we write the cohomology group as $H^2(C, A_\theta)$ (sometimes $H^2_\theta(C, A)$) is from

$$H^2(C, A_\theta) = \mathrm{Ext}^2_{\mathbb{Z}[C]}(\mathbf{P}_\mathbb{Z}, A)$$

where the Ext functor depends on the $\mathbb{Z}[C]$-module structure of A. The structure of $\mathbb{Z}[C]$ is always the same for a fixed C, because it merely depends on C. While, different actions $C \curvearrowright A$ give different C-module structure of A. (Equivalently, different $\mathbb{Z}[C]$-module structure of A). So, each time we talk about $H^2(C, A_\theta)$, an action

$\theta : C \to \mathrm{Aut}(A)$ is already fixed beforehand.
After an action θ being fixed, this theorem suggests that there are possibly many extensions of A by C corresponds to this action. Each action has at least one extension corresponds to an action. This is shown in the following diagram:

$$\begin{tikzcd}
 & \text{Extension 1: trivial} \arrow[r, leftrightarrow] & 0 \in H^2(C, A_\theta) \\
\text{action } \theta \arrow[ur] \arrow[r] \arrow[dr] & \text{Extension 2} \arrow[r, leftrightarrow] & a \neq 0 \in H^2(C, A_\theta) \\
 & \text{Extension } \vdots \arrow[r, leftrightarrow] & \cdots \in H^2(C, A_\theta)
\end{tikzcd}$$

In the above-mentioned diagram, an 'extension' means an equivalent class of extension actually.
Proof: □

The complete picture of classification entails all actions θ and all extensions corresponds to a specific action. It is illustrated as follows:

$$\text{All extensions of } A \text{ by } C = \underbrace{\text{Actions } \theta : C \to \mathrm{Aut}(A)}_{\text{representation theory}} + \underbrace{\text{Extensions corresponding to } \theta}_{\text{Theorem 7.3, using } H^2(C, A_\theta)}$$

So, representation theory tells us how many actions are there up to isomorphism(equivalence). For each action get in previous step, calculation of $H^2(C, A_\theta)$ tells the number of extensions for this action. Combining these two steps together, we finally obtain all possible extensions of A by C up to equivalence.

Lemma 7.4 *If $|A| = n$, then $\forall i \geq 0$, $n \cdot H^i(C, A) = 0$. If $|G| = m$, then $\forall i \geq 1$, $m \cdot H^i(C, A) = 0$*

Corollary 7.2: Schur-Zassenhaus theorem

Let E be a finite group with an abelian subgroup $A \trianglelefteq E$(normal abelian) such that $(|A|, |E/A|) = 1$. Then, $H^2(E/A, A) = \{0\}$. Hence, $E = A \rtimes E/A$ and complements of A in E are conjugate.

Proof: By the previous lemma, $\forall n \geq 1$, $|A| \cdot H^n(E/A, A) = 0$ and $|E/A| \cdot H^n(E/A, A) = 0$ simultaneously. $(|A|, |E/A|) = 1$ implies that $\exists a, b \in \mathbb{Z}$, $a|A| + b|E/A| = 1$. Hence, $\forall n \geq 1$, $H^n(E/A, A) = \{0\}$. $n = 2$ is desired. Because $0 \to A \to E \to E/A \to 0$ is an extension of A by E/A. Theorem 7.3 asserts $E \cong A \rtimes E/A$.
The conjugacy classes are classified by $H^1(E/A, A) = \{0\}$. □

Example (1) Consider the alternating group A_4 and its Klein subgroup $K = \{(1), (12)(34), (13)(24), (14)(23)\}$. Notice $K \trianglelefteq A_4$ and K is abelian (normal abelian). $|A_4| = 12$, $|K| = 4$ and $|A_4/K| = 3$. So, the corollary 7.2 implies that $H^2(A_4/K, K) = 0$ and A_4 is the semi-direct product of K and its complement in A_4, i.e. of the form $K \rtimes H$ with $H \leq A_4$ and $|H| = 3$. (H can be taken as $\langle(123)\rangle$ for example.) The corollary also states that any two choices of H (two complements of K) are conjugate. Even though two '3-cycle' elements are not conjugate in A_4((123) and (132) are not conjugate in A_4), the subgroups generated by each of them are conjugate. So, this theorem provides an alignment on the subgroup level.
(2) Consider a group E of order $|E| = 11^2 \cdot 3 \cdot 5$.
By Sylow's theorem, the Sylow 11 subgroup G_{11} of E is normal in E. $|G_{11}| = 11^2$, so G_{11} is an abelian group. The quotient E/G_{11} has order 15. Thus, we can apply the corollary 7.2 to get $H^2(E/G_{11}, G_{11}) = 0$ and $E \cong G_{11} \rtimes E/G_{11}$. Since the order of E/G_{11} is of the type pq and $p \nmid (q-1)$, E/G_{11} is cyclic hence $\cong \mathbb{Z}/15\mathbb{Z}$. Hence, every group of order $11^2 \cdot 3 \cdot 5$ is an extension of G_{11} by $\mathbb{Z}/15\mathbb{Z}$:

$$0 \to G_{11} \to E \to \mathbb{Z}/15\mathbb{Z} \to 0$$

and $E \cong G_{11} \rtimes_\theta \mathbb{Z}/15\mathbb{Z}$
By fundamental theorem of abelian groups, the possibilities of G_{11} are either $\cong \mathbb{Z}/11^2\mathbb{Z}$ or $\cong (\mathbb{Z}/11\mathbb{Z})^2$:

Case 1: $G_{11} \cong \mathbb{Z}/121\mathbb{Z}$ (cyclic). $\mathrm{Aut}(\mathbb{Z}/121\mathbb{Z})$ is determined by where the generators go. So, $|\mathrm{Aut}(\mathbb{Z}/121\mathbb{Z})| = \varphi(121) = 110$. Let B be a subgroup of $\mathbb{Z}/15\mathbb{Z}$ of order 3. Restrict θ to $\theta|_B$. $\left|\mathrm{im}\theta|_B\right| = [B : \ker\theta|_B]$, so $\left|\mathrm{im}\theta|_B\right| \Big| 3$. Simultaneously, $\left|\mathrm{im}\theta|_B\right| \Big| 110$. So, the action $\theta|_B$ is trivial, implying $B \subseteq \ker\theta$. Hence, $B \trianglelefteq \mathbb{Z}/15\mathbb{Z}$.

This complement $\mathbb{Z}/15\mathbb{Z}$ of G_{11} can be intrinsically recovered from E: Let P_3 be an arbitrary Sylow-3 group of E, $P_3 \trianglelefteq E$. Take any Sylow-5 subgroup of E, P_5. Consider the internal direct product P_3P_5, which is a group because $P_3 \trianglelefteq E$. $|P_3P_5| = |P_3||P_5|/|P_3 \cap P_5| = 15$. It is a complement of A. Further calculation yields $G_{11} \cap (P_3P_5) = \{1\}$ and they consist a group of order $11^2 \cdot 3 \cdot 5$.

Case 2: $G_{11} \cong (\mathbb{Z}/11\mathbb{Z})^2 \cong \mathbb{F}_{11}^2$ (non-cyclic). $\mathrm{Aut}((\mathbb{Z}/11\mathbb{Z})^2) \cong \mathrm{GL}_2(\mathbb{Z}/11\mathbb{Z}) = \mathrm{GL}_2(\mathbb{F}_{11})$. $|\mathrm{Aut}(\mathbb{Z}/11\mathbb{Z})| = (11^2 - 1)(11^2 - 11)$. [7]

8 Introduction to spectral sequences

[7] This is from counting elements of $\mathrm{GL}_n(F)$ over a finite field F. As in GL_2, finding elements of $\mathrm{GL}_2(F)$ is equivalent to find all possible combinations of two non-zero linear independent vectors $(x, y)^\top$ with $x, y \in F$. For the first column, one can pick all vectors except $(0, 0)^\top$. For the second column, by fixing the first column one can pick all vectors besides the subspace generated by the first column. This subspace is of 11 elements including $(0, 0)^\top$. So, there are $11^2 - 11$ choices of elements for the second column.

References

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