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Feedback exercises

$\frac{26}{30}$

S	D	U	U	E	Math
3	1	3	3	3	$\frac{13}{15}$

FB1 ***Construct the particular solution to the homogenoeus linear second order ODE***

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 50x = 0,$$

subject to initial conditions

$$x(0) = -\frac{1}{3}, \qquad \frac{dx}{dt}(0) = 0.$$

Find the value of x ***at*** $t = 3\pi/5$**.**

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 50x = 0 \iff x'' + 10x' + 50x = 0$$

$$r^2 + 10r + 50 = 0 \qquad \text{auxillary equation}$$

$$r_{1,2} = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot 50}}{2 \cdot 1} = \frac{-10 \pm \sqrt{-100}}{2} = \frac{-10 \pm 10i}{2} = -5 \pm 5i$$

A second order linear, homogeneous ODE with complex conjugate roots with non-zero real part has the form

$$x(t) = e^{\alpha x}(A\cos\beta x + B\sin\beta x) \quad \text{where} \quad r_{1,2} = \alpha \pm \beta i$$

$$x(t) = e^{-5x}(A\cos 5x + B\sin 5x)$$

$$\frac{dx}{dt} = e^{-5x} \cdot -5(A\cos 5x + B\sin 5x) + e^{-5x}(-A\sin 5x \cdot 5 + B\cos 5x \cdot 5) =$$

$$= \frac{-5}{e^{5x}}(A\cos 5x + B\sin 5x) + \frac{-5}{x^{5x}}(A\sin 5x - B\cos 5x) = \frac{-5}{e^{5x}}(A\cos 5x + B\sin 5x + A\sin 5x - B\cos 5x) =$$

$$= \frac{-5}{e^{5x}}(\cos 5x(A - B) + \sin 5x(A + B))$$

$$\frac{dx}{dt}(0) = 0 \implies \frac{-5}{e^0}(\cos 0(A - B) + \sin 0(A + B)) = 0 \implies -5(A - B) = 0 \implies \mathbf{A{=}B}$$

$$x(0) = -\frac{1}{3} \implies e^0(A\cos 0 + B\sin 0) = -\frac{1}{3} \implies A = -\frac{1}{3} \implies B = -\frac{1}{3}$$

$$x(t) = -\frac{1}{3}e^{-5x}(\cos 5x + \sin 5x) = \frac{-1}{3e^{5x}}(\cos 5x + \sin 5x)$$

$$x(3\pi/5) = \frac{-1}{3e^{5\cdot\frac{3\pi}{5}}}(\cos 5 \cdot \frac{3\pi}{5} + \sin 5 \cdot \frac{3\pi}{5}) = \frac{-1}{3e^{3\pi}}(\cos 3\pi + \sin 3\pi) = \frac{-1}{3e^{3\pi}}(-1 + 0) = \frac{1}{3e^{3\pi}}$$

$$\text{Value of x at, } t = 3\pi/5 \qquad x(3\pi/5) = \frac{1}{3e^{3\pi}}$$

$\frac{5}{5}$

FB2 ***Construct the group table for the additive group $\mathbb{Z}_3$.***

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$			
$\overline{1}$			
$\overline{2}$			

Show that $\mathbb{Z}_3$ is a subgroup of $\mathbb{Z}_9$.

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0+0}$	$\overline{1+0}$	$\overline{2+0}$
$\overline{1}$	$\overline{0+1}$	$\overline{1+1}$	$\overline{2+1}$
$\overline{2}$	$\overline{0+2}$	$\overline{2+1}$	$\overline{2+2}$

The complete collection of cyclic groups under addition is ($\mathbb{Z}$, e, $\mathbb{Z}(mod\ n)$), n $\in \mathbb{N}$, $\mathbb{Z}_3 \Longrightarrow$ integers (mod 3)
The identity is $\overline{0}$ and the cyclic generator is $\overline{1}$

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

$\mathbb{Z}_3$ can't be disqualified to be a subgroup of $\mathbb{Z}_9$ by Legrange's Theorem since $|\mathbb{Z}_3|$ divides $|\mathbb{Z}_9|$. (0)

It is clear that the operation is addition because if it would be multiplicative, then $\mathbb{Z}_9$ and $\mathbb{Z}_3$ wouldn't be a groups, their elements wouldn't have inverses in $\mathbb{Z}_9$ and $\mathbb{Z}_3$ respectively.

All elements of $\mathbb{Z}_3$ are in $\mathbb{Z}_9$, hence $\mathbb{Z}_3$ is a subset of $\mathbb{Z}_9$

$\mathbb{Z}_3$ is a subset of $\mathbb{Z}_9$ and by proposition 26.1 it also is a group because
1) The identity is in $\mathbb{Z}_3$, $0 \in \mathbb{Z}_3$
2) All combinations of $\mathbb{Z}_3$ elements are in $\mathbb{Z}_3$, $\quad x * y \in \mathbb{Z}_3$ for any $x, y \in \mathbb{Z}_3$
3) Every elements inverse is in $\mathbb{Z}_3$

$\frac{3}{5}$

Because it is a subset and a group $\mathbb{Z}_3$ is a subgroup of $\mathbb{Z}_9$.

This proves group
but not subgroup

FB3 ***Determine the left and right cosets in $\mathbb{Z}_6$ of the subgroup $\langle\overline{3}\rangle$ generated by $\overline{3}$***

At first it might seem that the operation is unknown and the subgroups can't be derived beyond the abstract operation $*$, but it is clear that the operation must be addition. If the cyclic groups operation would be multiplicative the resulting set wouldn't be a group because it wouldn't have inverses in $\mathbb{Z}_6$

3

$\langle\overline{3}\rangle = \{\overline{0}, \overline{3}\}$ A cyclic subgroup of order 2

All cyclic groups are Abelian, $a * b = b * a$, hence cyclic group left and right cosets are equal.

left and right coset of $\langle\overline{3}\rangle$ containing x, where $x \in \mathbb{Z}_6$ $\quad \langle\overline{3}\rangle * x = x * \langle\overline{3}\rangle = \{x * n | n \in \langle\overline{3}\rangle\} = \{x * 0, x * 3\}$

And because the operation is addition $\{\ x * 0, x * 3\ \} = \{\ x + 0, x + 3\ \} = \{\ x, x + 3\ \}$

left and right coset of $\langle\overline{3}\rangle$ containing 0 $\quad \langle\overline{3}\rangle * 0 = 0 * \langle\overline{3}\rangle = \{0 * n | n \in \langle\overline{3}\rangle\} = \{0 * 0, 0 * 3\} = \{0, 3\}$
left and right coset of $\langle\overline{3}\rangle$ containing 1 $\quad \langle\overline{3}\rangle * 1 = 1 * \langle\overline{3}\rangle = \{1 * n | n \in \langle\overline{3}\rangle\} = \{1 * 0, 1 * 3\} = \{1, 4\}$
left and right coset of $\langle\overline{3}\rangle$ containing 2 $\quad \langle\overline{3}\rangle * 2 = 2 * \langle\overline{3}\rangle = \{2 * n | n \in \langle\overline{3}\rangle\} = \{2 * 0, 2 * 3\} = \{2, 5\}$
left and right coset of $\langle\overline{3}\rangle$ containing 3 $\quad \langle\overline{3}\rangle * 3 = 3 * \langle\overline{3}\rangle = \{3 * n | n \in \langle\overline{3}\rangle\} = \{3 * 0, 3 * 3\} = \{3, 0\}$
left and right coset of $\langle\overline{3}\rangle$ containing 4 $\quad \langle\overline{3}\rangle * 4 = 4 * \langle\overline{3}\rangle = \{4 * n | n \in \langle\overline{3}\rangle\} = \{4 * 0, 4 * 3\} = \{4, 1\}$
left and right coset of $\langle\overline{3}\rangle$ containing 5 $\quad \langle\overline{3}\rangle * 5 = 5 * \langle\overline{3}\rangle = \{5 * n | n \in \langle\overline{3}\rangle\} = \{5 * 0, 5 * 3\} = \{5, 2\}$

left and right coset of $\langle\overline{3}\rangle$ containing 0 = left and right coset of $\langle\overline{3}\rangle$ containing 3 = {3,0}
left and right coset of $\langle\overline{3}\rangle$ containing 1 = left and right coset of $\langle\overline{3}\rangle$ containing 4 = {4,1}
left and right coset of $\langle\overline{3}\rangle$ containing 2 = left and right coset of $\langle\overline{3}\rangle$ containing 5 = {5,2}

$\frac{5}{5}$