Maths: 14/15

S	O	L	V	E
3	3	2	3	3

Tot: 28/30

Davids Zakrevskis
Feedback exercises

Well done

FB1 ***Suppose that $f : A \mapsto B$ is a surjective function. Define the following relation on A:***

$$a_1 \sim a_2 \quad \textit{if and only if} \quad f(a_1) = f(a_2).$$

Show that this is an equivalence relation. Denote by $A/\sim$ the set of equivalence classes of $\sim$. Prove that

$$|A/\sim| = |B|$$

The relation on A:

$$a_1 \sim a_2 \;\text{ if and only if }\; f(a_1) = f(a_2)$$

is an equivalence relation because it is symmetric, transitive and reflexive.

Reflexive
Because $f(a) = f(a)$ hence $a \sim a$ $\quad \forall a \in A$

Transitive
Lets say $a_1 \sim a_2$ and $a_2 \sim a_3$ that means $f(a_1) = f(a_2) = f(a_3)$; therefore, $a_1 \sim a_3$

Symmetric
Because if $a_1 \sim a_2$ then $a_2 \sim a_1$ $\left[f(a_1) = f(a_2) \iff f(a_2) = f(a_1)\right]$

there is at least one $a \in A$ mapped to each $b \in B$

- Surjective means th~~at every b maps to at least one a~~ $\quad |B| \leqslant |A|$
- If every element of A would only relate to itself then $|A/\sim|$ would be equal to $|A|$.
- But if two elements of A both point to a same value of B, that means $f : A \mapsto B$ for a_i and a_j, $f(a_i) = f(a_j)$, $a_i, a_j \in A$.
- That also works for more than 2 elements of A, if multiple elements of A point to the same value of B the function $f : A \mapsto B$ takes the all of the specific a's as the parameters so $f(a_1) = f(a_2) = ... = f(a_n)$.
- If that is the case and no other values of A point to the same element of B the equivalence classes of a_i and a_j for example are the same, because both of them point to themselves and each other.
- So all elements of A that point to the same element of B have equal equivalence classes.
- Therefore; given that $f : A \mapsto B$ is a surjective function, it can be simply deduced that $|A/\sim| = |B|$

Idea is correct, but proper way to do it is to define $g : A/\sim \to B$, $g([a]) = f(a)$ and prove it is bijective.

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FB2 ***Suppose that G is a group with identity element e. Let $\alpha, \beta, \gamma \in G$ be arbitrary. Prove the following statements.***

By definition the group G satisfies closure, associativity, identity and inverse axioms.
In this case most importantly **inverse axiom** $(\alpha^{-1}\alpha = \alpha\alpha^{-1} = e)$
And **identity axiom** $(\alpha e = e\alpha = \alpha)$

(i) $\alpha\beta\gamma = e$ implies $\beta\gamma\alpha = e$. **Statement i**

?

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$\alpha\beta\gamma = e$

$\alpha^{-1}\alpha\beta\gamma = \alpha^{-1}e$ — I add operations on both sides of the equation to operation α^{-1}

$e\beta\gamma = \alpha^{-1}e$ — e is the identity operation and can be excluded ?

$\beta\gamma\alpha = \alpha^{-1}\alpha$ — I add operation α to both sides

$\beta\gamma\alpha = e$ — Therefore, $\alpha\beta\gamma = e$ implies $\beta\gamma\alpha = e$

(ii) $\alpha\beta\gamma = \alpha^{-1}$ implies $\beta\gamma\alpha = \alpha^{-1}$. **Statement ii**

$\alpha\beta\gamma = \alpha^{-1}$

$\alpha\beta\gamma\alpha = e$ — I add operation α to both sides

$\beta\gamma\alpha = \alpha^{-1}$ — I add operations on both sides of the equation to operation α^{-1}

Hence $\alpha\beta\gamma = \alpha^{-1}$ implies $\beta\gamma\alpha = \alpha^{-1}$

α, β, γ are elements of the group G, not operations. So you are multiplying your expressions on the left and on the right by elements of G.

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FB3 ***A parametric curve is described by the following equations***

$$\frac{dx}{dt} = x, \qquad y = \cos t, \qquad z = \sin t,$$

and passes through $\langle 1,1,0\rangle$ ***when*** $t = 0$***. By solving the ODE for*** $x(t)$***, or otherwise, find an expression for*** x ***in terms of*** t ***and use this to write the space curve as a vector function. Hence, find the unit tangent to the curve*** $\boldsymbol{T}(t)$ ***at the point*** $\langle 1,1,0\rangle$***.***

$$\frac{dx}{dt} = x \quad \Rightarrow \quad \frac{dx}{dt} = \frac{1}{x^{-1}} \quad \Rightarrow x^{-1}dx = dt$$

C, C_x, C_t are some constants

$$\int x^{-1}\,dx = \int dt \quad \Rightarrow \quad \ln x + C_x = t + C_t \quad \Rightarrow \quad \ln x = t + C$$

$$\ln x = t + C$$
$$x = e^{t+C}$$

It is given that the curve described by the parametric equations passes through $\langle 1,1,0\rangle$ when $t = 0$; therefore, when t=0 x=1.

$$x = e^{t+C} \quad \Rightarrow \quad 1 = e^{0+C} \quad \Rightarrow \quad e^0 = e^C \quad \Rightarrow \quad C = 0 \quad \Rightarrow \quad x = e^t$$

Full set of parametric equations

$$x = e^t, \qquad y = \cos t, \qquad z = \sin t,$$

Vector function

$$r(t) = e^t\mathbf{i} + \cos(t)\mathbf{j} + \sin(t)\mathbf{k} = \langle e^t, \cos t, \sin t\rangle$$

$$\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} \qquad r'(t) = \langle e^t, -\sin t, \cos t\rangle$$

$$\mathbf{T}(t) = \frac{\langle e^t, -\sin t, \cos t\rangle}{|\langle e^t, -\sin t, \cos t\rangle|} = \frac{\langle e^t, -\sin t, \cos t\rangle}{\sqrt{(e^t)^2 + (-\sin t)^2 + \cos^2 t}} = \frac{\langle e^t, -\sin t, \cos t\rangle}{\sqrt{e^{2t}+1}}$$

$$\mathbf{T}(0) = \frac{\langle e^0, -\sin 0, \cos 0\rangle}{\sqrt{e^0+1}} = \frac{\langle 1,0,1\rangle}{\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle = \left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right\rangle$$

$\mathbf{T}(t)$ at the point $\langle 1,1,0\rangle$ is $\left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right\rangle$

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