MACROECONOMICS II NOTES

Econ 713

Harpreet Singh
PhD student
McGill University
Jan. 2024

Contents

1 Neo-classical growth model **3**

1.1 Lecture 1 3

1.2 Lecture 2 6

1 Neo-classical growth model

1.1 Lecture 1

Review of Neo-classical Growth Models:

Deterministic case

We start with the standard workhorse modern macroeconomic model. The Planner's problem, in recursive formulation is given by:

$$V(k) = \max_{k^{'} \in [0, f(k)+(1-\delta)k]} \{u(f(k) + (1-\delta)k - k^{'}) + \beta V(k^{'})\} \tag{1}$$

Where, the environment of the model is given by:

1. Large number of identical households
    - capital endowment: each household starts with k_0 unit of capital
    - capital depreciates at a rate of $\delta \in [0, 1]$
    - Labor endowment: each household chooses labour $n_t \in [0, 1]$
    - labor supplied inelastically (population size of 1)
2. Consumption good: there is a production of a single consumption good using only two inputs capital and labor.
    - The production function $F(.)$ is standard neo-classical [1] $F(k, 1) = f(k)$
3. Preference of the representative agent are represented by a utility function $u(.)$[2] with discounting factor $\beta \in (0, 1)$
4. State space [3] is $X = [0, max\{k_0, \bar{k}\}]$, $\bar{k}$ is maximum possible capital stock, which can be seen if household decides to consume nothing. Therefore, $\bar{k} = f(\bar{k}) + (1-\delta)\bar{k}$

The solution of the above problem is given by a policy function g that maps from the state space to state space, that is $k^{'} = g(k)$, and the value function $V(k)$.
There are mainly three types of numerical computational method of solving the dynamic problem, all focusing on the value function.

1. Value function iteration : This method proceeds by constructing a sequence of value functions and associated policy functions. The sequence is created by iterating on the recursive equation, starting from $V_0 = 0$, and continuing until V_j has converged.
2. Guess and verify : This method involves guessing and verifying a solution V to equation (1). This method relies on the uniqueness of the solution to the equation, but because it relies on luck in making a good guess, it is not generally available.

[1] $F(k, n)$ is continuous differentiable, strictly increasing and concave in both arguments, CRS, follows inada conditions, and inputs are essential for production.

[2] Utility function is strictly increasing, strictly concave, with marginal utility at 0 consumption as $+\infty$

[3] A state variable (vector) at any period t is smallest set of variables that determines, (1) the feasible set of the controls, (2) the current return at t given x', and (3) the value tomorrow given

3. Howard's improvement algorithm : This method is also known as policy function iteration. In this pick a feasible policy, $g_0(x)$, and compute the value associated with operating forever with that policy. This method skips the optimization once in a while, iterate on the value function for a given g_j and then iterate over j to converge.

The solution through the first method of value function iteration can be obtained by transforming the continuous state space (of problem in (1)) to discrete state-space, and then iterating the value function on it. This can be achieved by dividing the continuous state space in intervals and replacing the state space X with $K = \{k_1, k_2, ..., k_n\}$ with k_i as the extreme points of the successive disjoint intervals. $(k_1 < k_2 < ... < k_n)$.
The transformed problem can be written as:

$$V(k) = \max_{k^{'} \in [0, f(k)+(1-\delta)k],\, k^{'} \in K} \{u(f(k) + (1-\delta)k - k^{'}) + \beta V(k^{'})\}$$
$$\forall k \in K$$

Now the problem reduces to fiding the appropriate n and K. Considering $k_0 < k^*$, where, $k* = g(k^*) =$ we may choose $k_1 = k_0, k_n = k^*$ and seting n such that the increasing n further will keep the value in ϵ NBD.
Now, how do we know that the value function will converge for sure? Noting that the bellman equation is a contraction mapping and using the contraction mapping theorem in the Banach space, we know there will be a unique fixed point. Therefore, we can implement recursive method of successive approximation based upon:

$$V_{j+1}(k) = \max_{k^{'} \in [0, f(k)+(1-\delta)k],\, k^{'} \in K} \{u(f(k) + (1-\delta)k - k^{'}) + \beta V_j(k^{'})\}$$

for some arbitrary initial guess V_0.

$$\hat{V_0} = \begin{bmatrix} V_0\,(k_1) \\ v_0\,(k_2) \\ \vdots \\ v_0\,(k_n) \end{bmatrix} = \begin{bmatrix} v_0^1 \\ v_0^2 \\ \vdots \\ v_0^n \end{bmatrix}$$

Similarly, the initial guess for the policy function is given by:

$$\hat{g_j} = \begin{bmatrix} g_j\,(k_1) \\ g_j\,(k_2) \\ \vdots \\ g_j\,(k_n) \end{bmatrix} = \begin{bmatrix} g_j^1 \\ g_j^2 \\ \vdots \\ g_j^n \end{bmatrix}$$

Now, Step- j decision rule is

$$\underset{(n \times 1)}{\hat{\mathbf{g}}_j} \equiv \begin{bmatrix} g_j\,(k_1) \\ g_j\,(k_2) \\ \vdots \\ g_j\,(k_n) \end{bmatrix} = \begin{bmatrix} g_j^1 \\ g_j^2 \\ \vdots \\ g_j^n \end{bmatrix}$$

Considering the full depriciation for simplicity here, when $\delta = 1$ (for all j) current pay-off function is:

$$\underset{(n\times n)}{\hat{\mathbf{F}}} \equiv \begin{bmatrix} u\left(f\left(k_1\right)-k_1\right) & u\left(f\left(k_1\right)-k_2\right) & \cdots & u\left(f\left(k_1\right)-k_n\right) \\ u\left(f\left(k_2\right)-k_1\right) & u\left(f\left(k_2\right)-k_2\right) & \cdots & u\left(f\left(k_2\right)-k_n\right) \\ \vdots & \vdots & \ddots & \vdots \\ u\left(f\left(k_n\right)-k_1\right) & u\left(f\left(k_n\right)-k_2\right) & \cdots & u\left(f\left(k_n\right)-k_n\right) \end{bmatrix}$$

$$= \begin{bmatrix} F^{11} & F^{12} & \cdots & F^{1n} \\ F^{21} & F^{22} & \cdots & F^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F^{n1} & F^{n2} & \cdots & F^{nn} \end{bmatrix}$$

Now note that at any given iteration the state is determined, and therefore the transpose of the j^{th} value iteration is multiplied by the vector of $\mathbf{1}$. Hence, Transformed model can be written as

$$\hat{\mathbf{v}}_{j+1} = \max\left\{\hat{\mathbf{F}} + \beta\left(\mathbf{1}\hat{\mathbf{v}}_j^{\top}\right)\right\}$$

where $\mathbf{1}_{(n\times 1)}$ is a vector of 1 's, and the max operator applies line-by-line. Now we do not know how efficient this is, therefore for the efficiency consideration we will compute the F only once and using the properties of the model such as monotone convergence of k to steady state, which implies that the policy function g is also monotone.

Howard's improvement algorithm can also be used for the efficiency consideration.

1.2 Lecture 2

Competitive Equilibrium

In the Arrow-Debreu model, there are three kinds of agents in the economy: the households, the producers, and the market. The households born with a positive amount of endowment and sell all their endowment to the market in the beginning and buys whatever they want to consume. The households holds proportional ownership of the producers. The profit made by the j^{th} firm is distributed among the households. Ownership is exogenous. The households posses preferences over the bundles of commodities and the preferences are standard.

Market is only capable of choosing a market price vector. Others are price takers (no bargaining behavior), the main objective of the market to choose the prices such as the marker clears for all commodities in the market. Therefore, the market is playing the role of "Walrasian auctioneer".

There are various kind of possible decentralization, we will be discussing the Arrow-Debreu (time-0) market and sequential market. In the AD economy with time-0 markets of decentralization, market chooses sequence of time-0 price of the commodities at time t. Taking this problem in the form of recursive formulation, the state vector is now individual's capital stock (k) and aggregate capital stock (K). The price functions are $R(K)$, and $w(K)$, note that the price functions are the function of aggregate stock but individual's, implies that the individual's and firms are not price setters.

Given the above setup, the consumer's problem is the following (in recursive formulation):

$$V(k, K) = \max_{c,k'} \{u(c) + \beta V(k', K')\}$$

s.t.

$$\begin{aligned} c + k' &= [R(K) + 1 - \delta]k + w(K) \\ c, k' &\geq 0 \end{aligned}$$

with

$$K' = G(K).$$

Here $G(K)$ is policy function of the aggregate capital stock. In this case, the individual policy function is $g(k, K)$. This becomes significantly more complex in principle as there is a possibility of off-equilibrium consumer behavior.

Definition 1. *A Recursive Competitive Equilibrium (RCE) is a value function $V(k, K)$, a decision rule $g(k, K)$, an aggregate law of motion $G(K)$ and price functions $R(K), w(k)$ such that:*

1. *The representative firm optimizes given $R(K)$ and $w(K)$, that is*

$$R(K) = F_K(k^d, n^d)$$

$$w(K) = F_N(k^d, n^d)$$

2. *The representative household optimizes given $R(K)$, $w(K)$, and $G(K)$, that is $V(k,K)$ solves the household's maximization problem and $g(k,K)$ is the associated decision rule.*

3. *Market clears:*

$$k^d = K$$

$$n^d = 1$$

4. *Consistency, that is $G(k) = g(k,k)$*

The last point (4) states that the consumer has rational expectations. The consumer belief about the aggregate law of motion coincides with the actual law of motion in equilibrium. The rational expectations are formulated as a rational expectations equilibrium in terms of a fixed point of an operator that maps beliefs into optimal beliefs.
The solution of the model is to find $g(k,K), R(K), andw(K)$ s.t.

$$u'(c(k,K)) = \beta[R(g(k,K)) + 1 - \delta]u'(C(g(k,K), g(k,K))) \tag{2}$$

$$c(k,K) + g(k,K) = w(K) + [R(K) + 1 - \delta]k \tag{3}$$

$$R(K) = F_K(K,1) \tag{4}$$

$$w(K) = F_N(K,1) \tag{5}$$

$$k = K \tag{6}$$

The equality (2) above is Euler equation from derived from the FOC of the household problem, since, the utility function is standard we have it to be sufficient as well. Interpretation of the equation is that while making decision or choices the household evaluates the marginal cost of saving one unit to be equal to the discounted marginal utility of consuming the return of capital.
The equation (3) is the law of motion, left hand of this equation (c and g) is a function of (k,K) and w, R are the function of K, households are not price setters.
Equation (4) and (5) are from the firm's optimization problem, firm equates marginal product of labor to the wage and marginal product of capital to the return on the capital.

Equation (6)?