MACROECONOMICS II NOTES

Econ 713

Harpreet Singh
PhD student
McGill University
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1 Neo-classical growth model

1.1 Planner's Problem

We start with the standard workhorse modern macroeconomic model. The Planner's problem, in recursive formulation is given by:

$$V(k) = \max_{k^{'} \in [0, f(k)+(1-\delta)k]} \{u(f(k) + (1-\delta)k - k^{'}) + \beta V(k^{'})\} \tag{1}$$

Where, the environment of the model is given by:

1. Large number of identical households
   - capital endowment: each household starts with k_0 unit of capital
   - capital depreciates at a rate of $\delta \in [0, 1]$
   - Labor endowment: each household chooses labour $n_t \in [0, 1]$
   - labor supplied inelastically (population size of 1)
2. Consumption good: there is a production of a single consumption good using only two inputs capital and labor.
   - The production function $F(.)$ is standard neo-classical [1] $F(k, 1) = f(k)$
3. Preference of the representative agent are represented by a utility function $u(.)$[2] with discounting factor $\beta \in (0, 1)$
4. State space [3] is $X = [0, max\{k_0, \bar{k}\}]$, $\bar{k}$ is maximum possible capital stock, which can be seen if household decides to consume nothing. Therefore, $\bar{k} = f(\bar{k}) + (1-\delta)\bar{k}$

The solution of the above problem is given by a policy function g that maps from the state space to state space, that is $k^{'} = g(k)$, and the value function $V(k)$.
There are mainly three types of numerical computational method of solving the dynamic problem, all focusing on the value function.

[1] $F(k, n)$ is continuous differentiable, strictly increasing and concave in both arguments, CRS, follows inada conditions, and inputs are essential for production.

[2] Utility function is strictly increasing, strictly concave, with marginal utility at 0 consumption as $+\infty$

[3] A state variable (vector) at any period t is smallest set of variables that determines, (1) the feasible set of the controls, (2) the current return at t given x', and (3) the value tomorrow given

1. Value function iteration : This method proceeds by constructing a sequence of value functions and associated policy functions. The sequence is created by iterating on the recursive equation, starting from $V_0 = 0$, and continuing until V_j has converged.

2. Guess and verify : This method involves guessing and verifying a solution V to equation (1). This method relies on the uniqueness of the solution to the equation, but because it relies on luck in making a good guess, it is not generally available.

3. Howard's improvement algorithm : This method is also known as policy function iteration. In this pick a feasible policy, $g_0(x)$, and compute the value associated with operating forever with that policy. This method skips the optimization once in a while, iterate on the value function for a given g_j and then iterate over j to converge.

The solution through the first method of value function iteration can be obtained by transforming the continuous state space (of problem in (1)) to discrete state-space, and then iterating the value function on it. This can be achieved by dividing the continuous state space in intervals and replacing the state space X with $K = \{k_1, k_2, ..., k_n\}$ with k_i as the extreme points of the successive disjoint intervals. $(k_1 < k_2 < ... < k_n)$. The transformed problem can be written as:

$$V(k) = \max_{k^{'} \in [0, f(k)+(1-\delta)k],\, k^{'} \in K} \{u(f(k) + (1-\delta)k - k^{'}) + \beta V(k^{'})\} \qquad \forall k \in K$$

Now the problem reduces to fiding the appropriate n and K. Considering $k_0 < k^*$, where, $k* = g(k^*) =$ we may choose $k_1 = k_0, k_n = k^*$ and seting n such that the increasing n further will keep the value in ϵ NBD.

Now, how do we know that the value function will converge for sure? Noting that the bellman equation is a contraction mapping and using the contraction mapping theorem in the Banach space, we know there will be a unique fixed point. Therefore, we can implement recursive method of successive approximation based upon:

$$V_{j+1}(k) = \max_{k^{'} \in [0, f(k)+(1-\delta)k],\, k^{'} \in K} \{u(f(k) + (1-\delta)k - k^{'}) + \beta V_j(k^{'})\}$$

for some arbitrary initial guess V_0.

$$
\hat{V}_0 = \begin{bmatrix} V_0(k_1) \\ v_0(k_2) \\ \vdots \\ v_0(k_n) \end{bmatrix} = \begin{bmatrix} v_0^1 \\ v_0^2 \\ \vdots \\ v_0^n \end{bmatrix}
$$

Similarly, the initial guess for the policy function is given by:

$$
\hat{g}_j = \begin{bmatrix} g_j(k_1) \\ g_j(k_2) \\ \vdots \\ g_j(k_n) \end{bmatrix} = \begin{bmatrix} g_j^1 \\ g_j^2 \\ \vdots \\ g_j^n \end{bmatrix}
$$

Now, Step- j decision rule is

$$
\underset{(n\times 1)}{\hat{\mathbf{g}}_j} \equiv \begin{bmatrix} g_j(k_1) \\ g_j(k_2) \\ \vdots \\ g_j(k_n) \end{bmatrix} = \begin{bmatrix} g_j^1 \\ g_j^2 \\ \vdots \\ g_j^n \end{bmatrix}
$$

Considering the full depreciation for simplicity here, when $\delta = 1$ (for all j) current pay-off function is:

$$
\begin{aligned}
\underset{(n\times n)}{\hat{\mathbf{F}}} &\equiv \begin{bmatrix} u(f(k_1)-k_1) & u(f(k_1)-k_2) & \cdots & u(f(k_1)-k_n) \\ u(f(k_2)-k_1) & u(f(k_2)-k_2) & \cdots & u(f(k_2)-k_n) \\ \vdots & \vdots & \ddots & \vdots \\ u(f(k_n)-k_1) & u(f(k_n)-k_2) & \cdots & u(f(k_n)-k_n) \end{bmatrix} \\
&= \begin{bmatrix} F^{11} & F^{12} & \cdots & F^{1n} \\ F^{21} & F^{22} & \cdots & F^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F^{n1} & F^{n2} & \cdots & F^{nn} \end{bmatrix}
\end{aligned}
$$

Now note that at any given iteration the state is determined, and therefore the transpose of the j^{th} value iteration is multiplied by the vector of $\mathbf{1}$. Hence, Transformed model can be written as

$$
\hat{\mathbf{v}}_{j+1} = \max\left\{\hat{\mathbf{F}} + \beta\left(\mathbf{1}\hat{\mathbf{v}}_j^\top\right)\right\}
$$

where $\mathbf{1}_{(n\times 1)}$ is a vector of 1 's, and the max operator applies line-by-line. Now we do not know how efficient this is, therefore for the efficiency consideration we will compute the F only once and using the properties of the model such as monotone convergence of k to steady state, which implies that the policy function g is also monotone.

Howard's improvement algorithm can also be used for the efficiency consideration.

1.2 Competitive Equilibrium

In the Arrow-Debreu model, there are three kinds of agents in the economy: the households, the producers, and the market. The households born with a positive amount of endowment and sell all their endowment to the market in the beginning and buys whatever they want to consume. The households holds proportional ownership of the producers. The profit made by the j^{th} firm is distributed among the households. Ownership is exogenous. The households posses preferences over the bundles of commodities and the preferences are standard.

Market is only capable of choosing a market price vector. Others are price takers (no bargaining behavior), the main objective of the market to choose the prices such as the marker clears for all commodities in the market. Therefore, the market is playing the role of "Walrasian auctioneer".

There are various kind of possible decentralization, we will be discussing the Arrow-Debreu (time-0) market and sequential market. In the AD economy with time-0 markets of decentralization, market chooses sequence of time-0 price of the commodities at time t. Taking this problem in the form of recursive formulation, the state vector is now individual's capital stock (k) and aggregate capital stock (K). The price functions are $R(K)$, and $w(K)$, note that the price functions are the function of aggregate stock but individual's, implies that the individual's and firms are not price setters.

Given the above setup, the consumer's problem is the following (in recursive formulation):

$$V(k, K) = \max_{c,k'} \{u(c) + \beta V(k', K')\}$$

s.t.

$$\begin{aligned} c + k' &= [R(K) + 1 - \delta]k + w(K) \\ c, k' &\geq 0 \end{aligned}$$

with

$$K' = G(K).$$

Here $G(K)$ is policy function of the aggregate capital stock. In this case, the individual policy function is $g(k, K)$. This becomes significantly more complex in principle as there is a possibility of off-equilibrium consumer behavior.

Definition 1. *A Recursive Competitive Equilibrium (RCE) is a value function $V(k,K)$, a decision rule $g(k,K)$, an aggregate law of motion $G(K)$ and price functions $R(K), w(k)$ such that:*

1. *The representative firm optimizes given $R(K)$ and $w(K)$, that is*

$$R(K) = F_K(k^d, n^d)$$

$$w(K) = F_N(k^d, n^d)$$

2. *The representative household optimizes given $R(K)$, $w(K)$, and $G(K)$, that is $V(k,K)$ solves the household's maximization problem and $g(k,K)$ is the associated decision rule.*

3. *Market clears:*

$$k^d = K$$

$$n^d = 1$$

4. *Consistency, that is $G(k) = g(k,k)$*

The last point (4) states that the consumer has rational expectations. The consumer belief about the aggregate law of motion coincides with the actual law of motion in equilibrium. The rational expectations are formulated as a rational expectations equilibrium in terms of a fixed point of an operator that maps beliefs into optimal beliefs.
The solution of the model is to find $g(k,K), R(K), andw(K)$ s.t.

$$u'(c(k,K)) = \beta[R(g(k,K)) + 1 - \delta]u'(c(g(k,K), g(k,K))) \quad (2)$$

$$c(k,K) + g(k,K) = w(K) + [R(K) + 1 - \delta]k \quad (3)$$

$$R(K) = F_K(K,1) \quad (4)$$

$$w(K) = F_N(K,1) \quad (5)$$

$$k = K \quad (6)$$

The equality (2) above is Euler equation from derived from the FOC of the household problem, since, the utility function is standard we have it to be sufficient as well. Interpretation of the equation is that while making decision or choices the household evaluates the marginal cost of saving one unit to be equal to the discounted marginal utility of consuming the return of capital.

The equation (3) is the law of motion, left hand of this equation (c and g) is a function of (k, K) and w, R are the function of K, households are not price setters.
Equation (4) and (5) are from the firm's optimization problem, firm equates marginal product of labor to the wage and marginal product of capital to the return on the capital.
We can infer from the equation (6) that in equilibrium $k = K$ since we are solving the problem of representative agent problem. Though this does not mean that the individual is price setter, in all the previous equations the individual's capital is different from the aggregate capital. This last equations helps us to restrict to look for the decision rule along the equilibrium path: which is when $G(k) = g(k, k)$, and the pricing function $R(k), w(k)$.
To see this problem in the discrete state space, we need to find G as the fixed point, the steady state. Similarly as in the value function iteration, we put an outer loop on G.

RCE with variable labor: The RCE with the variable labor wold be similar with a little change in the consumer's problem and firm's optimality condition. Also, one more market is present here (labor market), therefore, the labor market also clears in the equilibrium.
Consumer's problem:

$$V(k, K) = \max_{c,k'} \{u(c, 1-n) + \beta V(k', K')\}$$

s.t.

$$\begin{aligned} c + k' &= [R(K) + 1 - \delta]k + w(K)n \\ c, k' &\geq 0 \\ n &\in [0, 1] \end{aligned}$$

with

$$K' = G^K(K)N = G^N(K)$$

and in this case the Firm's optimality will imply

$$\begin{aligned} R(K) &= F_K(K, G^N(K)) \\ w(K) &= F_N(K, G^N(K)) \end{aligned}$$

2 Inequality and Aggregation

What we have seen in the previous model(s) that the agents are homogeneous, now consider that the agents are heterogeneous. We can introduce heterogeneity in two ways in Macroeconomics.

1. Representative agent : aggregation
   In this aggregate mater for the inequality but converse in not the true. This model is more tractable and restrictive (off course).

2. No representative agent: In these kind of models there is a two way interaction, that inequality matters for aggregate and aggregate matters for inequality. Because of this two-way interaction, the problem becomes challenging, but interesting (potentially).

Consumer heterogeneity: Consumer heterogeneity can arise from various sourcs:

1. Wealth Endowments: Each agent in the economy differs with respect to the wealth endowment, agent born with unequal wealth. Now the inequality might or might not change over time.

2. Labor productivity: The productivity of each agent is different and constant over time. The productivity might be state dependent or may evolve over time, in both of the scenarios the agent's productivity may have dynamic nature (this can be the starting point on how do you invest in the human capital formation, or to increase in the productivity).

3. Preferences: The agent might have different preferences over leisure and consumption.

If marginal propensity of consumption is constant for all the agents then wealth distribution does not affect aggregates. Such kind of preferences are known as Gorman preferences[4]

[4]The standard consumer theory is based on one agent, but how can we use the consumer theory to say something about how our entire society is behaving. One way to think is to consider the entire society as one agent with endowment as the sum of all the agent's endowment, but under what circumstances we can represent the entire population as a single agent. Formally, $X(p, m^1, m^2, ..., m^n) = \sum_i x_i(p, m^i) = X(p, \sum_i m^i)$. This is true under quasi linear preferences and homothetic preferences, these are also known as Gorman preferences.

2.1 *Wealth Heterogeneity: A Simple variant*:

Consider there are N individuals in the economy, where every individual i is of type $\mu^i \in [0,1]$ such that $\sum_i \mu^i = 1$, preference of the agents are

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

with

$$u(c) = \begin{matrix} ln(\bar{c}+c) & \bar{c}+c \geq 0 \\ (\bar{c}+c)^{\pi} & \bar{c}+c \geq 0 \\ -\bar{c}\exp\left(-\eta c\right) & \end{matrix}$$

and the heterogeneity is with respect to wealth, which is a_0^i.

Here we will be focusing on the *log* case, note that $\bar{c}$ is subsistence consumption level with $\bar{c} \leq 0$. Therefore in this case of the utility function, the *inada* condition is same except the limit tends tends to $\bar{c}$.

In this model, firm owns the capital and therefore makes the investment decisions for each time period. The representative firm with $y_t = f(k_t)$. We will be considering the decentralization where firms owns the capital, consumers own firms, and Arrow-Debreu markets.

Given the structure of the decentralization, let p_t be the time zero price of y_t for all t with normalizing $p_0 = 1$. The firm solves the following problem:

$$V_0 = \max_{\{d_t, k_{t+1}\}} \sum_{t=0}^{\infty} p_t d_t$$

with k_0 given, and

$$d_t = f(k_t) - [k_{t+1} - (1-\delta)k_t]$$

Solving for the problem will give us the FOC as,

$$p_t = p_{t+1}\left(f'(k_{t+1}) + (1-\delta)\right)$$

Interpretation: The term in the bracket of the right hand side of the above equation is the total return from investment done at time t. $f'(k_{t+1})$ can also be seen as rental rate R_{t+1} , and therefore the whole term in bracket as real interest rate $1 + r_{t+1}$. Which gives $\frac{p_t}{p_{t+1}} = (1 + r_{t+1})$, i.e., 1 unit of consumption today will cost $(1 + r_{t+1})$

tomorrow. Note that here we have this for all t, which indicates stationarity (how you value tomorrow as compared to today is same as how you value 100^{th} and 101^{th} today).

Consumer i solves the following problem:

$$\max_{\{c_t^i\}} \sum_{i=0}^{\infty} \beta^t u(c_t^i)$$

subject to the budget constraint:

$$\sum_{i=0}^{\infty} p_t c_t^i = a_0^i$$

Given the change in the wealth endowment, we have each agents has a share in the aggregate value of the firm outcome, let this share be s_0^i, therefore $a_0^i = s_0^i V_0$. Also the share of wealth is $a_0 = \sum_i \mu^i s_0^i V_0 = V_0$, and (type of the agent i) μ^i , $s_0^i \in [0,1]$ which comes from assuming tha t $\sum_i \mu^i s_0^i = 1$.
Goal: To understand the dynamics or evolution of $\{s_0^i\}_{i=1}^N$ over time. Consumer problem gives the following FOC:

$$\beta^t u'(c_t^i) = \lambda^i p_t$$

given the utility function as log, we have

$$c_t^i = \frac{\beta^t}{\lambda^i p_t} - \bar{c}$$

where, λ^i is the multiplier on the budget constraint for agent i. Substituting the one in other gives us:

$$\sum_{t=0}^{\infty} p_t \left(\frac{\beta^t}{\lambda^i p_t} - \bar{c} \right) = a_0^i$$

$$\frac{1}{\lambda^i (1-\beta)} - \sum_t p_t \bar{c} = a_0^i$$

$$\frac{1}{\lambda^i} = \left(a_0^i + \sum_t p_t \bar{c} \right) (1-\beta)$$

$$\text{now, } c_t^i = \frac{\beta^t (1-\beta)(a_0^i + \sum_t p_t \bar{c})}{p_t} - \bar{c}$$

$$c_0^i = \frac{1}{\lambda^i} - \bar{c} = \Theta(\{p_t\}_{t=0}^{\infty}, \bar{c}) + (1-\beta) a_0^i$$

The first term above (Θ) would vanish if the utility function would have been $log(c)$, also the second term there is a constant multiplier $(1-\beta)$ which is marginal propensity to consume.
Digression: MPS: The marginal propensity to consume is the derivative of consumption function with respect to a_0^i, here we have it as β and marginal propensity to consume is $(1-\beta)$.
Whereas, the Average Propensity to Consume (APC) is the ratio of consumption and wealth, and average propensity to save (APS) is 1−APC. Therefore, in this case, we have APC $= \frac{\Theta}{a_0^i} + (1-\beta)$. Note that though the MPC is constant, the APC is variable and dependent on the individual endowment.

Steady State: In the steady state $k' = k$. While inequality does not matter for aggregates but aggregates matters for inequality, To see this let us start with the steady state:
$1 = \beta[f'(k) + 1 - \delta] \rightarrow f'(k) = \beta^{-1} - (1-\delta)$.

From the Euler equation we have, the ratio of price at t to price at $(t+1)$ is β^{-1}, that implies, $p_t = \beta^t$. This implies price declines over time and is proportional to how individuals value future periods as compared to current period. Also, the value of the overall asset is

$$a = \sum_{t=0}^{\infty} p_t d = \sum_{t=0}^{\infty} \beta^t [f(k) - \delta k] = \frac{f(k) - \delta k}{1-\beta}$$

The aggregate consumption and asset are given as,

$$c = f(k) - \delta k$$

$$\sum_{i=1}^{N} \mu^i a^i = a$$

Substituting in the individual level consumption, we have

$$c^i = \bar{c}[(1-\beta)\sum_{t=0}^{\infty} p_t - 1] + (1-\beta)a^i = (1-\beta)a^i$$

the above equation indicates that the consumption of the i^{th} individual is $(1-\beta)$ proportion of the a^i in the steady state, which is $s^i V$ and also in the steady state the $MPS = APS$
All of the above equations indicates that there are multiple distributions consistent with the steady-state. However, given the initial dis-

tribution, the economy converges to a single steady-state distribution.

Now the interesting question is how the distribution evolve over time and where does it converge. For that let us obtain individual-level wealth dynamics as function of prices and saving rate from consumer's budget constraint.

$$
\begin{aligned}
\sum_{\tau=t}^{\infty} p_\tau c_\tau^i &= p_t a_t^i \\
p_t c_t^i + \sum_{\tau=t+1}^{\infty} p_\tau c_\tau^i &= p_t a_t^i \\
p_t c_t^i + p_{t+1} a_{t+1}^i &= p_t a_t^i \\
\frac{c_t^i}{a_t^i} + \frac{p_{t+1} a_{t+1}^i}{p_t a_t^i} &= 1 \\
\frac{a_{t+1}^i}{a_t^i} &= \frac{p_t}{p_{t+1}} \left(1 - \frac{c_t^i}{a_t^i}\right)
\end{aligned}
$$

Note that in the last equation above, left hand side is the wealth dynamics, which is equal to the product of proportion of the price ratio and APS, from before, we know that the MPC is given by

$$
\frac{c_t^i}{a_t^i} = \frac{\Theta(\{p_t\}_{t=0}^{\infty}, \bar{c})}{a_t^i} + 1 - \beta
$$

Knowing that $a_t^i = s_t^i a_t$, we have

$$
\frac{a_{t+1}^i / a_t^i}{a_{t+1} / a_t} = \frac{1 - c_t^i / a_t^i}{1 - c_t / a_t} = \frac{s_{t+1}^i}{s_t^i}
$$

The above term will be greater than 1

$$
\iff \quad \frac{c_t^i}{a_t^i} < \frac{c_t}{a_t} \iff \Theta(\{p_t\}_{t=0}^{\infty}, \bar{c})(a_t^i - a_t) > 0 \iff \Theta(\{p_t\}_{t=0}^{\infty}, \bar{c}) a_t (s_t^i - 1) > 0
$$

This implies the wealth dynamics of consumer i depends on the sign of Θ and sign of $(s_t - 1)$

Based on the assumptions, we have that the sequence of $f'(k_t)$ is decreasing and $\frac{p_t}{P_{t+1}}$ converges to $1/\beta$ from above. This implies that $p_t \geq \beta^t$. All these conditions together with that $\bar{c} < 0$, we have:

$$
\Theta(\{p_t\}_{t=0}^{\infty}, \bar{c}) = \bar{c} \left[(1 - \beta) \sum_{t=0}^{\infty} p_t - 1 \right] \geq \bar{c} \left[(1 - \beta) \sum_{t=0}^{\infty} \beta^t - 1 \right] = 0
$$

The above inequality implies that as $\bar{c} < 0$ we have $\Theta > 0$

In this case, there will be ever widening wealth distribution along transition, without mobility, the rank is preserved. This is because, $\bar{c}$ generates saving rates which is increasing in wealth, $(APC \neq MPC)$. Aggregate economy is growing and induces everyone in the economy to save, though there will be decline in the interest rate. All individual including poor need to reach at the minimal consumption level $\bar{c}$, since marginal utility is very high. The one who born with higher wealth will save more and get more return, whereas the poor who born with less endowment will save very less as to reach the consumption level.

Note that the above conclusion is based on the single interest rate for all the consumers (poor or rich). To mitigate the inequality or ever widening wealth inequality with the saving interest rate different for poor and rich. This difference in the interest rate can be achieved through the credit constraints.

2.2 *Competitive Equilibrium without Aggregation*

In this section we will explore how the competitive equilibrium can be achieved without aggregation, relying on the Welfare theorems.
First welfare theorem Every competitive equilibrium is a Pareto optimal allocation. Whereas the *Second Welfare theorem* states that under certain assumptions Pareto optimal allocation, social welfare maximization, can be achieved as a competitive equilibrium with redistribution of the endowments.

For the benchmark model, we will solve the social planner's problem with given weights and then compute competitive equilibrium prices and ensure that those are affordable.
Consider the economy as before, endowment economy with heterogeneous agents, (two equally sized consumers). Without loss of generality, we assume that the aggregate endowment is 0 or $\sum_i a_0^i = 0$. Now the planner's problem can be formulated as follows:

$$\max_{(c_t^1, c_t^2)_{t=0}^{\infty}} \sum_t \beta^t [\alpha_1 u(c_t^1) + \alpha_2 u(c_t^2)]$$

:

Benchmark. Consider Planer solving the following problem

$$\max_{\left\{c_1^t, c_2^*\right\}_{t=-\infty}^{\infty}} \sum_{t=0}^{\infty} \left[\alpha_1 u^t \left(c_1^t\right) + \alpha_2 u^0 \left(c_2^t\right)\right]$$

subject to:

$$\sum_i c_t^i = \sum_i y_t = 2y_t \, \forall t$$

Here α_i is the weight planner gives to agent $i's$ utility. The weights are not need to be add up to 1. To solve the problem, we set up the Lagrangian,

$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\alpha_1 u_1^* \left(c_1^t\right) + \alpha_2 u_2 \left(c_2^t\right)\right) + \sum_{t=0}^{\infty} \lambda_t \left(\sum_i y_i^t - \sum_i c_i^t\right)$$

Solving for the first order conditions, we have

$$
\begin{aligned}
&\alpha_1 u_1'\left(c_1^t\right)+\partial_t(-1)=0 \quad \forall t \\
&\alpha_2 u_2'\left(c_2^t\right)+\alpha_t(-1)=0 \quad \forall t \\
&\Rightarrow \quad \alpha_i u_i'\left(c_i^t\right)=\alpha_j u_j'\left(c_j^t\right) \quad \forall t \\
&\Rightarrow \quad \frac{\alpha_i}{\alpha_j}=\frac{u_j'\left(c_j^t\right)}{u_i'\left(c_i^t\right)} \\
&u_i'\left(c_i^t\right)=\frac{\alpha_j}{\alpha_i} u_j'\left(c_j^t\right) \\
&c_i^t=\left(u_i'\right)^{-1}\left[\frac{\alpha_j}{\alpha_i} u_j'\left(c_j^t\right)\right] \\
&\sum_i y_i^t=\sum_i c_i^t=\sum_i\left(u_i'\right)^{-1}\left[\frac{\alpha_j}{\alpha_i} u_j'\left(c_j^t\right)\right] \\
&\Rightarrow \sum_i\left(u_i'\right)^{-1}\left[\frac{\alpha_j}{\alpha_i} u_j'\left(c_j^t\right)\right]=Y^t .
\end{aligned}
$$

This can be seen from the above equation that c_j is a function of aggregate endowment Y^t. Consider the case when the utility is of the form logarithmic, $u = \log$.

$$\frac{\alpha_1}{\alpha_2}=\frac{c_1^t}{c_2^t}$$

Now, consider the case when the agent is solving the problem, agent is representative agent, the agent is will be solving for the given time-o price at t^{th} time-period as P_t , Agent, problem is as follows:-

$$
\begin{aligned}
&\max_{\left\{C_t^i\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(C_i^t\right) \\
&\text{set. } \sum_{t=0}^{\infty} P_t C_t^i=\sum_{t=0}^{\infty} P_t y_t^i+P_0 a_0^i
\end{aligned}
$$

Setting up the Lagrangian, there will be only one constraint as there is only one time-0 constraint.

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^{t}u\left(C_{i}^{t}\right)+\lambda_{i}\left[\sum_{t=0}^{\infty}P_{t}y_{t}^{i}-\sum_{t=0}^{\infty}P_{t}C_{t}^{i}+P_{0}a_{0}^{i}\right]$$

Solving for the FOCs

$$\beta^{t}u'\left(c_{i}^{t}\right)+\lambda_{i}\left(-P_{t}\right)=0\quad\forall t\quad\forall i$$

Therefore, for both of the agents,,

$$\beta^{t}u'\left(c_{1}^{t}\right)=\lambda_{1}P_{t}\quad\forall t$$
$$\beta^{t}u'\left(C_{2}^{t}\right)=\lambda_{2}P_{t}\quad\forall t$$

The ratio of the marginal utilities

$$\frac{u'\left(c_{1}^{t}\right)}{u'\left(c_{2}^{t}\right)}=\frac{\lambda_{1}}{\lambda_{2}}\quad\forall t$$

Let us consider the case for simplicity that the utility function is logarithm function, $u=\log$:

$$\frac{\beta^{t}}{c_{1}^{t}}=\lambda_{1}P_{t}$$

$$c_{1}^{t}=\frac{\beta^{t}}{\lambda_{1}}P_{t}\quad\forall t$$

$$\sum_{t=0}^{\infty}P_{t}\frac{\beta^{t}}{\lambda_{i}P_{t}}=\sum_{t=0}^{\infty}P_{t}y_{t}^{i}+P_{0}a_{0}^{i}$$

$$\frac{1}{\lambda_{i}}\sum_{t=0}^{\infty}\beta^{t}=\sum_{t=0}^{\infty}p_{t}y_{t}^{i}+P_{0}\partial_{0}^{i}\quad\forall i$$

$$1/\lambda_{i}=(1-\beta)\left[\sum_{t=0}^{\infty}P_{t}y_{t}^{i}+P_{0}\partial_{0}^{i}\right]\quad\forall i$$

$$\Rightarrow\quad c_{i}^{t}=\beta^{t}\times\frac{1}{(1-\beta)\left[\sum_{t}P_{t}y_{t}^{i}+P_{0}a_{0}^{i}\right]}P_{t}$$

$\forall i$

Mapping with planner's problem $[\alpha_{1}=\alpha,\alpha_{2}=1-\alpha]$

planner's problem $[\alpha_{1}=\alpha,\alpha_{2}=1-\alpha]$

$$\frac{\sum_{t}P_{t}y_{t}+P_{0}a_{0}^{2}}{\sum_{t}P_{t}y_{t}+P_{0}a_{0}^{1}}=\frac{1-\alpha}{\alpha}\Rightarrow\alpha=\frac{\sum_{1}P_{t}y_{t}+P_{0}a_{0}^{i}}{2\sum_{t}P_{t}y_{t}+P_{0}\left(a_{0}^{i}+\alpha_{0}^{2}\right)}$$

$$\Rightarrow\alpha=\frac{\sum p_{t}y_{t}+{\alpha_{0}}^{i}}{2\Sigma_{t}p_{t}y_{t}}$$

Prices: $\frac{P_{t+1}}{P_t} = \frac{\theta_{t+1}}{\theta_t}$
(from Plane's problem)

$$\Rightarrow \frac{P_1}{P_0} = \frac{\theta_1}{\theta_0}$$
$$\text{let} \quad P_0 = \theta_0 = 1$$
$$\Rightarrow P_t = \theta_t$$

$\forall t$.

- Sequential markets Consumer (RA) problem

$$\max_{\left\{c_t^c \partial_{t+t}^i\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right)$$

subject to: $q_t a_{t+1}^i + c_t^i = y + \partial_t^i \quad \forall t$

$$\partial_0^i \geqslant 0$$

and $\lim_{T\to\infty} \frac{q_T a_{T+1}^i}{\prod_{s=1}^{T}(1+r_s)} \geqslant 0$
here. $r_{t+1} = \frac{1}{q_t} - 1$
in Recursive form:-

$$V\left(\partial_i, A_i\right) = \max_{\partial'_{i'}, c_i} \left\{u\left(c_i\right) + \beta v\left(\partial'_i, A'_i\right)\right\}$$

Subject to

$$q\left(A_i\right) a'_i + c'_i = y + a_i$$
$$A'_i = G\left(A_i\right)$$

FOL $\frac{\partial v(\partial_i, A_1)}{\partial \partial'_i} = 0 \Rightarrow 0 = u'\left(y - \partial_i - q\left(A_1\right)\partial'_i\right)\left[-q\left(A_i\right)\right] + \frac{\partial v\left(\partial'_i, A'_i\right)}{\partial \partial'_i}$
from envelope theorem, we know

$$\frac{\partial v\left(a'_i, A'_i\right)}{\partial \partial'_i} = u'\left(y - \partial'_i - q\left(A'_1\right)\partial''_i\right)$$

Substituting, we get

$$u'\left(y - \partial_i - q\left(A_1\right)\partial'_i\right) = \beta\left(1 + r\left(A_1\right)\right) u'\left(y - a'_i - q\left(A'_1\right)\partial''_i\right)$$

this is the Euler equation. $\left(1 + r\left(A_1\right) = \frac{1}{g(A_1)}\right)$
CEq^m with Heterogeneity: (Recursive)

A RCE is a value function $v(\partial_i, A_1)$, a decision rule $g(a_i, A_1)$ and a price function $q(A_1)$, such that:

1. g solves the consumer's problem given G and q. v is the associated value function.
2. Consistency:

$$G(a_1) = g(a_1, a_1)$$

3. Market clearing:

$$g(a_1, a_1) + g(-a_1, a_1) = 0$$

3 Uncertainty

3.1 *Introduction:*

In the real world scenarios, we do not observe the state of the world in future period with certainty. Therefore, models under certainty has a limited role in explaining the real problem in the economy. This uncertainty is primarily captured through the concept of the state space, which represents all possible conditions or states of the economy at a given time. These states are described by variables known as state variables. When these state variables are subject to randomness or variability, we use probability theory to model how consumers and the economy behave.

The first topic we'll delve into is the exploration of common stochastic processes in macroeconomics. Stochastic processes are mathematical models that describe the evolution of variables over time in a probabilistic manner. These processes help us understand how economic variables change and interact under uncertainty.

Secondly, we'll examine how individuals make decisions to maximize their utility when faced with uncertainty. Since outcomes are not certain, individuals must make choices based on probabilities and expected outcomes to maximize their satisfaction or utility.

Lastly, we'll define and discuss competitive equilibrium in the context of uncertainty. Competitive equilibrium refers to a state where markets are in balance, with supply equaling demand across all goods and services. We'll explore how this equilibrium is affected by uncertainty and its implications for the functioning of the economy. Before proceeding to these topic we will first define some basic concepts in stochastic process, these concepts are in the Appendix9.

3.2 *Markov Chains:*

3.3 *Neo-classical Growth Model with Uncertainty*

4 Market Structures

4.1 *Introduction:*

In macroeconomics, market structure refers to the organizations and characteristics of the markets where various agents engage in trading, borrowing, or lending different types of services and financial assets. Typically, there are two types of market structures: complete and incomplete. In both market structures, households or consumers face similar problems, albeit with adjustments in the available assets and changes in budget constraints. As discussed in the section on uncertainty, the economy can experience different types of shocks, which can be either idiosyncratic (specific to individuals) or aggregate (affecting all individuals). When individuals encounter uncertainty regarding potential negative shocks, their goal is to safeguard their consumption over their lifetime, particularly during times of adversity. To achieve this, they must make decisions regarding saving, investing, borrowing, or selling assets, considering how these actions may mitigate the impact of shocks on their consumption. This process involves anticipating adverse states, or negative shocks, and preparing for them by saving during favorable times, when income is higher. This strategy, known as consumption smoothing, allows individuals to maintain a more consistent level of consumption over time, thereby providing a degree of financial security during periods of uncertainty.

In this section we will mostly discuss the Complete market structure. We will approach this topic by introducing a simple problem of two period, representative agent with n states of the world in period 1.

4.2 *Sequential markets*

4.3 *Risk Sharing*

5 Asset pricing model

5.1 *Lucas Asset Pricing Model*

5.2 *Equity Premium Puzzle*

In this complete market asset pricing model, consider CRRA preferences, z (aggregate state) takes good (g) or bad (b). The consumer has access to three assets: two arrow securities (with price Q_z and quantity a_z), and a tree that lasts only for one period and the consumer receives one unit of a new tree every period as an endowment (tree price for quantity x at p_z). The problem of the representative consumer is to solve

$$V(a, x, z) = \max_{a'_g, a'_b, x'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \sum_{z'=g,b} \phi_{zz'} V(a', x', z')$$

s.t.

$$c + Q_g a'_g + Q_b a'_b + p_z(x' - 1) = a + xY_z$$

6 Incomplete Markets

Incomplete markets refer to situations where Arrow securities, which represent complete sets of securities for all possible states or shocks, are not available. In our previous studies, we focused on models with representative consumers, where the behavior of macroeconomic outcomes didn't rely on individual differences. We examined cases where shocks affected aggregate outcomes. Now, we'll delve into modeling scenarios where individuals experience idiosyncratic shocks. These shocks are not insurable due to the absence of complete markets.

Studying incomplete markets provides insights into the origins of wealth inequality, wealth mobility, and the consequences of financial market development. In the planner's problem, we observed that individual consumption solely relies on the aggregate endowment. When the aggregate endowment remains constant across different states, consumption becomes fully insured over time and across states for each individual. Mace (1991) tested this model, investigating whether, under perfect risk-sharing and Constant Relative Risk Aversion (CRRA) preferences, individual consumption should react to aggregate income shocks but not to changes in individual income. That is in the model, $\Delta \log c_t^i = \alpha_1 \Delta \log C_t + \alpha_2 \Delta \log y_t^i + \epsilon_t^i$ testing under the complete market is $H_0 : \alpha_1 = 1$ and $\alpha_2 = 0$ and testing under Autarky is $H_0 : \alpha_1 = 0$ and $\alpha_2 = 1$. Both of the hypothesis were rejected.

In the Lucas asset pricing model, we derived the asset prices and model the general equilibrium framework, whereas, now we will explore the partial equilibrium with prices as given. Therefore, now we refer to situations where certain risks or shocks cannot be fully mitigated through the purchase of insurance. In such markets, individuals cannot easily transfer the risk associated with adverse events to others, as there may not be available markets for insurance against these specific shocks.

Agents in incomplete markets must resort to "self-insurance" strategies to manage the impact of adverse shocks on their consumption. This involves managing a stock of a single asset, such as savings or investments, to cushion the effects of income fluctuations or other adverse events on their consumption levels, also known as consumption smoothing.

The effectiveness of self-insurance strategies depends on several factors:

1. Utility Function: The shape and properties of an individual's utility function play a crucial role in determining their willingness to sacrifice current consumption to build up savings or investments for future consumption smoothing. Different utility functions may lead to varying degrees of risk aversion and preferences for consumption smoothing.

2. Income Process: The nature of the income process, including its volatility, persistence, and correlation with other economic variables, influences the degree to which individuals need to self-insure. Higher income volatility or persistence of shocks may require individuals to hold larger buffers of assets to smooth consumption over time.

3. Existence of Borrowing Constraints: Borrowing constraints restrict an individual's ability to borrow against future income or assets, limiting their capacity to smooth consumption in the face of adverse shocks. Incomplete markets often coincide with the presence of borrowing constraints, which further complicate self-insurance strategies.

6.1 *Income Fluctuation Problem*

We will start modeling the income fluctuation problem in the basic incomplete market setting. Consider an endowment economy, where endowment $y_t \in \mathcal{Y} = \{y_1, y_2, ..., y_S\}$ follows a Markov chain with $\pi(y'|y) = Pr(y_{t+1} = y'|y_t = y)$ and $\Pi(y) = Pr(y_t = y)$ be the conditional and unconditional probability distribution of the state. We will consider the case when only one asset is available, risk-free bond and idiosyncratic element in endowment shocks. Given this, agents would like to self-insure, falling short of full insurance. There is a continuum of individuals of measure 1, and therefore, the the proportion of individuals being in a particular state is same as the probability distribution over [0,1]. There is no aggregate uncertainty, and therefore,

$$\bar{y} = \sum_{y} y\Pi(y)$$

Individual's type at time t is identified by state space.

Let us write the consumer problem, first, state variables are total wealth, income with the individual ω and also the endowment at time t. Since, the problem is stationary here, t would not be a state variable. In the recursive formulation, consumer problem is as follows:

$$V(\omega, y) = \max_{a' \geq -\phi} \{u(w - a') + \beta \sum_{y'} \pi(y'|y) V(w', y')\}$$

sub. to

$$c + a' \leq \omega$$
$$\omega' \equiv y' + (1 + r)a'$$

Here, ω is cash-on-hand. If shocks are persistent[5], there is no difference in taking a as a state variable in replace of the ω. Note that here the constraint on a' in the maximisation is credit constraints.

Consider the case of no uncertainty, and $\{y_t\}$ follows a deterministic path, or is a deterministic sequence of endowment. This will make 2 changes in the recursive formulation mentioned above, 1) time comes in as a state variable, and 2) no probability. One more condition or constraint is nPg (non-ponzi game), which requires agent should not die with debt, this requires that the agent at any time t can not borrow more than the life time income. Mathematically,

$$a_{t+1} \geq \sum_{\tau=0}^{\infty} \frac{c_{t+\tau}}{(1+r)^{\tau}} - \sum_{\tau=0}^{\infty} \frac{y_{t+\tau}}{(1+r)^{\tau}}$$

For the upper bound on the borrowing, imposing $c_t = 0 \, \forall \, t$ we get,

$$a_{t+1} \geq - \sum_{\tau=0}^{\infty} \frac{y_{t+\tau}}{(1+r)^{\tau}} \equiv= -\hat{b}_t$$

Present value budget constraint will determine the maximal feasible for an agent to pay in every period forever.

$$PV = \sum_{t=0}^{\infty} \frac{x}{(1+r)^t}$$
$$PV = \frac{x}{r}$$
$$x = PV.r$$

Here, PV is $\hat{b}_t$ at time t. therefore, annuity value of $\{y_t\}$ from t onwards is $r\hat{b}_t$. The natural borrowing constraint is the one which is least stringent, that satisfies the nPg condition. It is denoted by $\hat{b} = \sup_t \hat{b}_t$.

[5]A shock is persistent if its effect take longer to dissipate than the length of the time step. Mathematicaly, z is persistent if $p(z'|z) \neq p(z')$

We can definitely make this more tight, by increasing it by some small number. Note that it is same as imposing the nPg condition, it does not bind in the finite time as strictly positive consumption is optimal. By setting the borrowing constraint as 0 implies autarky, and there is some interval in between 0 and $\hat{b}$, these borrowing constraint (b) are known as ad-hoc and is potentially binding that is binding sometimes and does not bind sometime. We consider the borrowing limit as a function

$$\phi = \min\{b, \hat{b}\}$$

Coming back to the uncertainty, the worst case scenario for any individual is to receive minimum endowment throughout the life time. Consider $\inf y_t = y_1$, this gives us the natural borrowing limit as

$$a_{t+1} \geq -\sum_{\tau=0}^{\infty} \frac{y_1}{(1+r)^\tau} \equiv= -\frac{y_1}{r}$$

and this gives us the $\phi = \min\{b, \frac{y_1}{r}\}$, here b is the ad-hoc borrowing constraint.

Recall from the complete markets that consumption does not track individual's income, as that becomes the function of aggregate income. Here, in the incomplete market, we will see the consumption tracks the income. Solving the consumer problem under uncertainty, if the constraint $a' > -\phi$, we get the Euler equation as:

$$u'(c) \geq \beta(1+r) \sum_{y'} \pi(y'|y) u'(c')$$

The inequality within the Euler equation suggests that, due to borrowing constraints and incomplete markets, individuals are not at their optimum, and they strive to reach it. It's important to note that the borrowing limit is sometimes restrictive, depending on the income state or y. In contrast to complete markets where consumption is solely determined by aggregate income and individuals are fully insured, in incomplete markets, consumption varies with individual income. Furthermore, because of the occasionally binding borrowing constraints, consumption closely follows individual income, exceeding what can be attributed solely to the absence of certain markets.

Looking into the empirical counterpart, we wil go through calibrating the model based on the estimates from other studies. Model calibration refers to the process of adjusting the parameters of a theoretical economic model to match specific empirical data or targets. This involves fine-tuning the model's parameters such as coefficients, elasticities, or other structural features to ensure that the model generates outcomes that closely resemble observed economic phenomena or desired macroeconomic indicators. The goal of calibration is to enhance the model's predictive power and its ability to provide meaningful insights into real-world economic behavior and policy implications.

In our model, the empirical counterpart of $\phi(b)$ for the model calibration is to target empirical counterpart of the ratio of the log deviations in consumption to income i.e, $m(\phi) = \frac{std(log\, c_i)}{std(log\, y_i)}$. Logarithm just for the ease to interpret in terms of relative change and levels.

The range of $m(\phi)$ explains various models,

1. When $\phi = 0$, this implies no borrowing and thus everyone consumes their own endowment (forced to consume), and is a Autarky condition. This is also known as hand-to-mouth.

2. For the value of $\phi = y_1/r$ that is the natural borrowing limit, this gives us the lower bound on the ϕ, and $m(y_1/r) \in (0, 1)$ this implies that markets are incomplete.

3. If $\phi \in (0, y_1/r)$, we get $m(\phi) \in (m(y_1/r), 1)$, still incomplete, but this cannot be captured by the model, as it goes below the infimum of ϕ.

4. For the case when $m = 0$, we get the condition of complete markets.

Clearly, from the empirical evidence we are far from the complete markets, and therefore the model or our approach works here if $m(y_1/r)$ is lower than empirical counterpart.

6.2 ***Huggett Model***

We've covered the partial equilibrium aspect, and now we're transitioning to examine the General Equilibrium analysis of income fluctuations using the Huggett model as outlined in the Journal of Economic Dynamics and Control in 1993.
Consider the state space consist of 1) the individual level (ω, y) and 2)

the aggregate level ψ. ψ captures the overall cross-sectional distribution of (ω, y), which is an ∞ dimensional object, or simple the motion how individuals move over time. This becomes the aggregate variable as it captures the wealth and income of individual at every given point of time.

Given this, the consumer problem can be written as following,

$$V(\omega, y, \psi) = \max_{a' \geq -\phi} \left\{ u(\omega - a') + \beta \sum_{y'} \pi(y'|y) V(\omega', y', \psi') \right\}$$

s.t.

$$\begin{aligned} c + a' &\leq w \\ w' &\equiv y' + (1 + r(\psi'))a' \end{aligned}$$

with

$$\psi' = H(\psi)$$

Now, we will study the long run equilibrium, but before that let us define some definitions. Here, we will be looking into the subset of the individual state that is economically feasible and interesting to study, that is $(\omega, y) \in X = [\underline{\omega}, +\infty) \times \mathcal{Y}$ with an upper bound on ω. We denote this relevant subset (compact) as S. Let, $\mathcal{B}_S$ be the Borel σ algebra on S. Formally, we say that ψ is a probability measure defined over the space $(S, \mathcal{B}_S)$[6]

The transition of function is induced by the $a' = g(\omega, y)$ and π. This we can think of as probability of getting to the state (w', y') given the state today as (w, y). Formally, $P : S \times \mathcal{B}_S \to [0, 1]$, with the typical element of it as

$$P((\omega, y), ([\omega_l, \omega_h), \{y_l, ..., y_h\} = \sum_{y' \in \{y_l, ..., y_h\}} \pi(y'|y) \mathbf{1}(\omega' \in [\omega_l, \omega_h]).$$

$\mathbf{1}$ is an indicator function.
Now how the transition happens, the law of motion of ψ for a given

[6] A Borel Sigma algebra includes open sets, closed sets, countable unions, and intersections of these sets, making it a versatile tool for defining measurable sets and constructing functions and measures on the space. In probability theory, the Borel sigma algebra is crucial for defining probability measures on the real line or more general topological spaces. It forms the basis for constructing probability spaces and random variables, allowing us to define events and calculate probabilities in a rigorous manner. Here in our case it is simple as we have the cartesian product of a compact set of R and finite set.

$B \in \mathcal{B}_S$ is

$$\psi'(B) = \int_S P(x, B) d\psi(x)$$

The intuition behind the above motion equation is integrating (summing) over all the individuals who will land at the state B tomorrow given that they are at the state x today. There can be many individuals who lands there, since we restricted our population to continuum of 1, here the probability integration is just the proportion of the population.

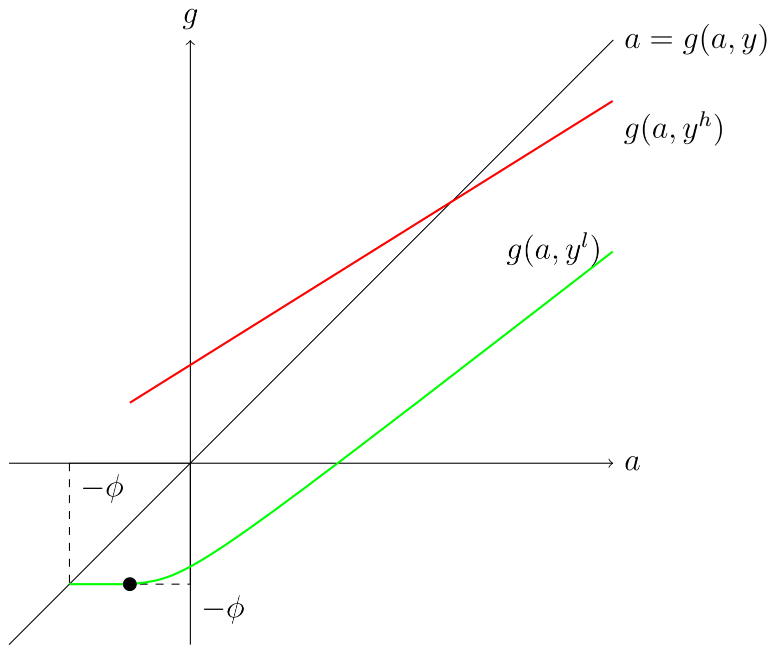
Definition 2. *A Stationary Recursive Competitive Equilibrium (SRCE) is a value function $V(k, K)$, a decision rule, or a policy function $a' = g(\omega, y)$, an interest rate r, an aggregate law of motion ψ over individual states such that:*

1. *Given r, consumer optimizes solving the consumer problem mentioned above representative.*
2. *Market clearing:*
$$\int_S g(\omega, y) d\psi = 0$$
3. *Stationary (Consistency), that is*
$$\psi(B) = \int_S P(x, B) d\psi(x)$$

Where P is induced by π and $g(\omega, y)$

A few things to note till now are, we are studying the workhorse model of the heterogeneous agent model. Mainly focusing on the ex-ante heterogeneity. We treat each agent similar ex-ante, which is described by the state (ω, y). We are focusing on the case where outcome, interest rate, and distribution ψ are constant in the steady-state. This does not rule out the micro-level action y, a, c. If this would have been a planner's problem the only uncertainty for the planner would be proportion of the individual changing their type, but not how many, as their will be convergence in the distribution and we know the proportion of individual transitioning from one state to another. We will also assume the Ergodic property, which implies that in the long run, the probability that an agent has been all type at least once is positive. In the other way, different individual will go through same experiences in the long-run. Though the short-term dynamics starting from different states can be different. Which becomes the key interest of this model.

Recalling the complete market scenario (for benchmark) the consumption of individual will be $c(y) = \bar{y}$ with the interest rate $r = \beta^{-1} - 1 = \rho$. The resource constraint equation will give us the arrow security prices. From the Monotone mixing condition and the theorem from the SLP, we know there is stationary probability distribution of the distribution.

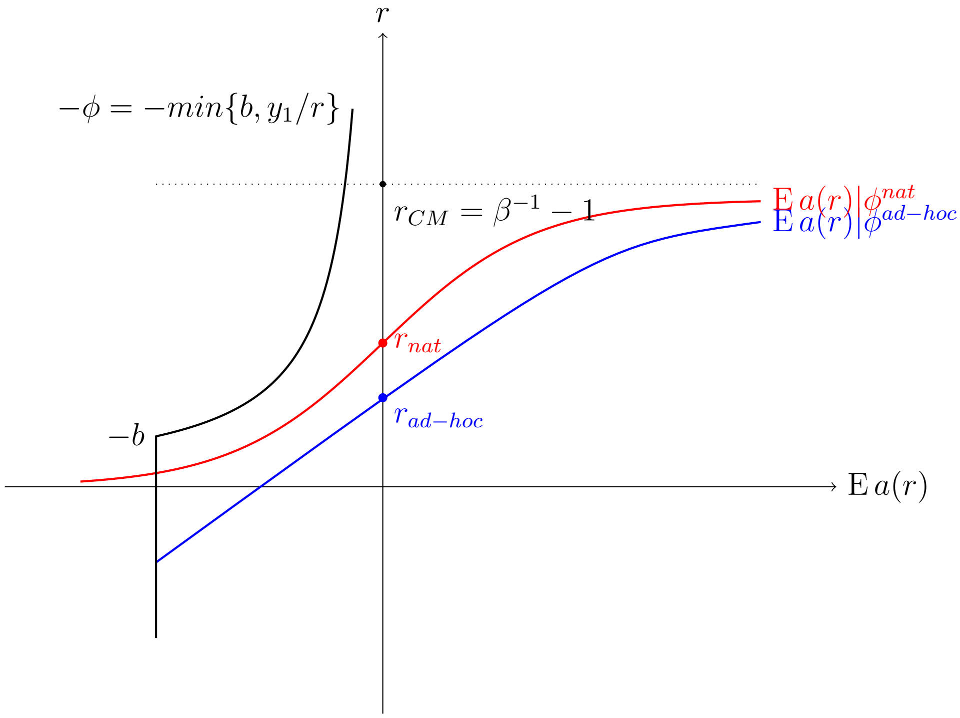


The graph above mentions how the policy function will be for low and high states of the nature. That is for the people who have binding borrowing condition and for whom borrowing conditions does not bind at some state of the world. It is to note that at the bottom of the green graph, the marginal propensity to save is less (next to the dot) and with the increase in the asset the saving increases (i.e., saving increases for those who are far away from them). The marginal propensity to save is the derivative of the curve here. Eventually it becomes parallel to the red line. For whom the marginal propensity to save is constant. It is under the low state of the world, who save less and consume and therefore, the green graph is convex near the black point in the figure. Now we will characterize the equilibrium under the Huggett framework. For the equilibrium, the market should clear and therefore the excess demand of the asset should be 0 as the interest rate reaches the equilibrium interest rate. The expression for the excess demand is

$$\mathrm{E}\, a(r) = \int_S g(\omega, y; r) d\psi_r$$

This implies that as $r \to \rho$ in the incomplete market $\lim_{r\to\rho} \mathrm{E}\, a(r) = +\infty$

In the incomplete market the agents are not able to have full insurance and therefore they do some precautionary savings or self-insurance, incomplete markets give them motive to save for consumption smoothing. This gives us that at the complete market interest rate, agents would demand very high amount of the asset to insure themselves. If compensation for time impatience is just right, then the demand for the asset is arbitrarily large, that is if β is low the interest rate is very large. Whereas, if the interest rate $r \to -1$, the excess demand $\mathrm{E}\, a(r)$ will be negative. People will save for nothing, but to borrow. This characterizes the equilibrium interest rate in incomplete market that equilibrium r will be less than ρ, given continuity.



This is evident from the figure above that the interest rate for the natural borrowing limit is above as compared to the ad-hoc borrowing limit. Tighter the credit constraint lower is the equilibrium interest rate.

Numerical solution: The numerical algorithm to find the equilibrium of the Huggett model involves two loops. One is an inner loop, which constitute of individual optimization, that is taking r as given. Second is an outer loop where we need to iterate over, such that the market clears.

- **Inner loop**: We will first discretize the continuous asset space into $\mathcal{A}$, where the lower bound is $a_1 = -\phi$ and the upper bound as a_{N_a} sufficiently high. We will put smaller intervals at the bottom to capture the curvature.
  Then for each $a_i \in \mathcal{A}$ perform value function iteration for the consumer problem (Bellman equation). The consumer problem is

$$v(a_i, y) = \max_{a' \geq -\phi} \{u(y + (1+r)a_i - a') + \beta \sum_{y'} \pi(y'|y)v(a', y')\}$$

  Restricting the $a' \in \mathcal{A}$, but more efficient and accurate would be to interpolate $v(a', y')$ (i.e., piece wise linearly) whenever we find $a' \notin \mathcal{A}$. It is better to disregard the credit constraint in the optimization and after the solution is observed, check whether the credit constraint is satisfied. If violated, impose $a' = -\phi$. For efficiency, using the monotoncity property would help, looping over top of $\mathcal{A}$ and then find where, $g(a_j, y) = -\phi$, this gives us that $\forall\, i < j \; g(a_j, y) = -\phi$ holds as well.

- **Outer loop**: The main purpose of the outer loop is to find the interest rate r such that the bond market clears. There are two procedures to compute $\mathrm{E}\, a(r)$, the expected excess demand of the asset.

  1. A very long simulation of a single agent, iterative jointly on the markov chain and decision rule. Performing piece wise linear interpolation to evaluate $g(a, y)$ at $a \notin \mathcal{A}$.
     Discarding the first few observations and approximate $\mathrm{E}\, a(r)$ with the following

$$\frac{1}{T - \hat{T}} \sum_{t=\hat{T}+1}^{T} a_t$$

     The cumulative histogram will yield CDF. For a smoother policy function to have less kinks, solve the bellman equation on finer asset grid. Then simulate using the new decision rule and alongwith the piece wise linear interpolation whenever $a' \notin \hat{\mathcal{A}}$

2. Another process involves computing the transition function. If $a' \in \mathcal{A}$, then iterate on the discete pdf ψ, then based on the iteration compute the vector $(1 \times N_a N_y)$ which has a typical element as $\psi(a, y) = Pr((a, y))$.
   Now, we can compute the transition function which can be arranged in the matrix P of $(N_a N_y \times N_a N_y)$ dimension, with the typical element as

   $$P((a, y), (a', y')) = Pr((a', y')|(a, y)) = \pi(y'|y)\mathbf{1}(g(a, y) = a')$$

   For the stationary distribution, this boils down to the problem of fixed point. That is solving the system of linear equations:

   $$\psi = \psi\mathbf{P}$$
   $$\psi(\mathbf{I} - \mathbf{P}) = 0$$

   here, ψ is the eigenvector of $\mathbf{P}$ associated with the unit eigenvalue.
   Now, if $a' \notin \mathcal{A}$ then discretizing the density with respect to $\mathcal{A}$, and using the piece wise linear interpolation to split the probabilities. For example, when $a_i < g(a, y) < a_{i+1}$, then,

   $$P((a, y), (a_i, y')) = \pi(y'|y)\frac{a_{i+1} - g(a, y)}{a_{i+1} - a_i}$$

   $$P((a, y), (a_{i+1}, y')) = \pi(y'|y)\frac{g(a, y) - a_i}{a_{i+1} - a_i}$$

   and market clearing condition $\sum_{(a,y)} g(a, y)\psi(a, y) \sim 0$

Transitional Dynamics For the case when the distribution ψ_0 is not the steady-state one, then distribution ψ_t will evolve deterministically. The problem is dependent on time and therefore t becomes a state variable. This implies that the value function for the consumer will be the following for any given $t \geq 0$

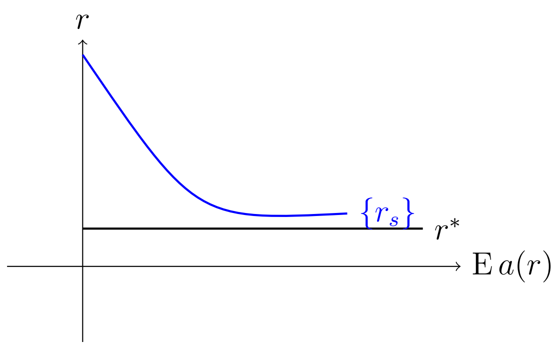
$$v_t(a, y) = \max_{a' \geq -\phi}\{u(y + (1 + r_t)a_i - a') + \beta\sum_{y'}\pi(y'|y)v_{t+1}(a', y')\}$$

given the perceived law of motion $\{r_s\}_{s=t}^{\infty}$. This gives the policy function as a function of t and therefore, $a_{t+1} = g_t(a, y)$. Computations for the transitional dynamics, algorithm steps are

1. Guess T, time to complete the transition.

2. Guess a sequence of interest rate, $\{r_s\}_{s=0}^{T}$ with the initial condition r_0 given. For example, as shown in the figure below, the sequence is converging, the sequence is a perceived law of motion and we guessed and fixed it.

3. Inner loop: Solve the consumer problem by backward induction, that is from $v_{T+1} = v$, where v solves the steady state problem and get the policy function, i,e., $g_t\{a, y\}_{t=0}^{T}$ and $v_t\{a, y\}_{t=0}^{T}$



4. outer loop: iterate forward on the economy transition function starting from ψ_0. This can be done through two ways, one is simulation (i) draw a large number (N) of initial conditions from ψ_0 and then (ii) starting from the $(a_0, y_0)^i$, now simulate for T periods using $\{g_t(a, y)\}_{t=0}^{T}$ and the Markov chain. second is Iterating on distribution. That is to iterate forward on $\psi_{t+1} = \psi_t P_t$ for T times starting from ψ_0, using the methods described previously in the second step of the outer loop in the numerical solution of the Huggett model.

5. Now check for the market clearing condition. $N^{-1} \sum_{i=1}^{N} a_t^i \sim 0$, if not update the sequence of interest rate accordingly.
   If not simulating, check whether $r_T \sim r$ or alternatively $\psi_T \sim \psi$. Otherwise increase T.

MIT shocks: Till now, we have seen the transitional dynamics and equilibrium properties. What happens when a surprise shock hits the economy, also known as MIT shock. Example for credit shock as the surprise shock here, that is when ϕ drops. Since, we are looking from the steady state state point of view, the distribution changes, now the initial distribution is $\psi_0 = \psi^{initial}$ and transition dynamics will also respond and $\psi_t \to \psi^{final}$. There are two interesting question,

1. What happens in the economy/comparative statics, that is how the aggregate distribution changes?
   This can be obtained through comparing the steady states before and after the shock. This is easier to compute but hard to interpret welfare effects. Average welfare loss, in terms of the **consumption equivalent**. For the steady state comparison we try to compute the value of λ_{ss} such that,

$$\int_S \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u((1-\lambda_{ss}) c_t^{initial}(a_0, y_0)) d\psi^{initial} = \int_S v^{initial}(a_0, y_0) d\psi^{final}$$

   where, if utility is isoelastic,

$$\lambda_{ss} = \frac{\int_S v^{final}(a_0, y_0) d\psi^{final}}{\int_S v^{initial}(a_0, y_0) d\psi^{initial}} - 1$$

   It is very hard to interpret λ_{ss} as it ignores the transitional adjustments, also the distribution has changed due to the shock not only the policy function.

2. What are the welfare effects that captures aggregate and distributional as well?
   This is period-0 comparison, a much harder to compute, but meaningful welfare. In period 0, we compute the value of λ_0 such that

$$\int_S \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u((1-\lambda_0) c_t^{initial}(a_0, y_0)) d\psi^{initial} = \int_S v^{initial}(a_0, y_0) d\psi^{final}$$

More on the MIT shock will be in the General equilibrium framework of incomplete markets, in Aiyagari and Krusell-Smith.

6.3 *Aiyagari Model*

Aiyagri model is an extension of the Huggett model. In this model it has production over and above the Huggett model. With production, consider a representative firm with neo-classical production function ($y = zF(K, N)$). There is no aggregate uncertainty, hence no covariate shocks. Consumers own's capital and labor. The consumer experience purely idiosyncratic shocks to the labor endowments: $s \in \{0,1\}$. These shocks are, whether the consumer is employed ($s = 1$) or not employed ($s = 0$). This process is governed by the Markov chain with transition probabilities as $\pi(s'|s)$, with the limiting probability as $\Pi(s)$.

We will be assuming that leisure is not valued, that is it does not enter the consumer's utility function and therefore there is inelastic labor supply, and is also time invariant. We consider the labor supply as $N = \Pi(1)$.

There is no insurance available against the unemployment (that during the negative shock), and there is only one risk free bond. This implies that the markets are incomplete.

The consumer solves the following problem:

$$v(\omega, s) = \max_{k' \geq -\phi} \{u(w - k') + \beta \sum_{s' \in \{0,1\}} \pi(s'|s) v(\omega', s')\}$$

where $\omega' = ws' + (1+r)k'$, we are assuming that total individual saving is k'. The constraint above mentions that the income will be addition of the return from the capital savings and the wage if $s = 1$.
Here the borrowing limit is $\phi = \min\{b, \frac{ws_{min}}{r}\}$. Since, s_{min} is 0, the borrowing constraint is equal to 0, whenever $b \geq 0$.

Once again we will be restricting the state space as a compact subset of $\mathcal{R} \times \{0,1\}$. Let, κ be a compact subset of $\mathcal{R}_+$ and S be the Cartesian product, i.e, $\kappa \times \{0,1\}$.

Now we will define the recursive competitve equilibrium in this context.

Definition 3. *A Recursive competitive equilibrium (RCE) is a value function $v(\omega, s)$, a decision rule $k' = g(\omega, s)$, prices r, w, and a probability measure ψ over the individual state space S, such that:*

1. *Consumers optimize given r and w.*
2. *Representative firm optimizes given r and w, i,e,*

$$r = F_K(K, N) - \delta$$

$$w = F_N(K, N)$$

3. *Capital market clears: $\int_S g(\omega, s) d\psi = K$*
4. *Labor market clears: $\Pi(1) = N$*

5. ψ is stationary, and consistent with the individual behavior:

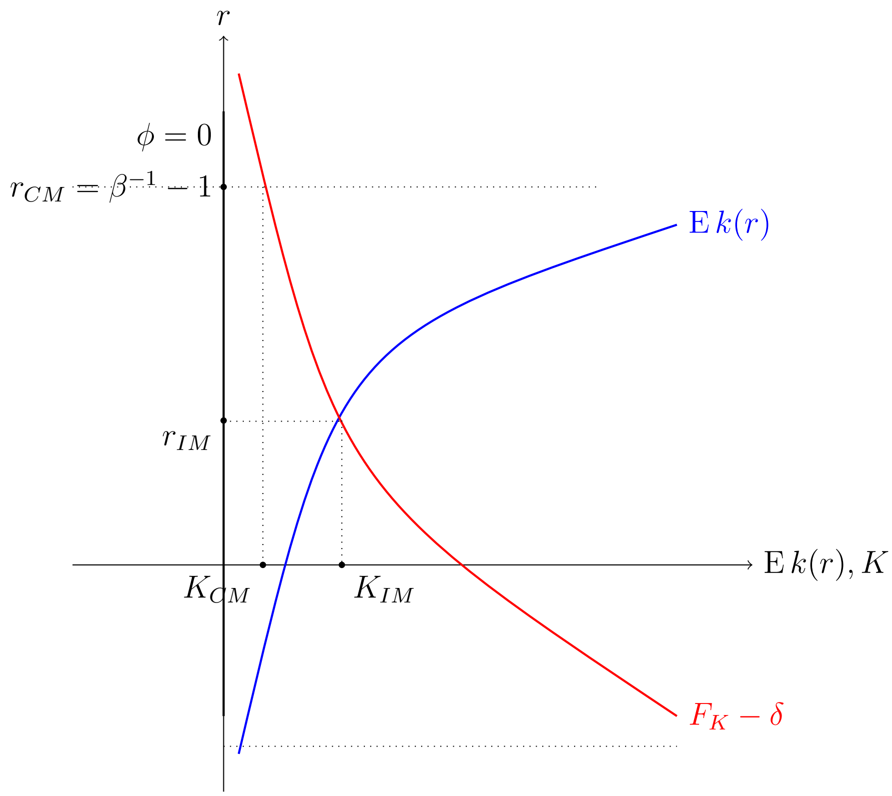
$$\psi(B) = \int_S P(x, B) d\psi(x) \quad \forall\, B \in \mathcal{B}S$$

Where P is the transition function induced by π and $g(\omega, s)$

As seen in the Huggett model, we will be plotting the aggregate capital, and impose the condition of capital market clearing along with the borrowing constraint of $\psi = 0$. Intuitively, as seen in the Huggett model, the equilibrium interest rate will be much lower than the complete market interest rate. The interest rate from the firm optimization traces the aggregate capital for which $r = F_K(K, N) - \delta$ and therefore is downward slopping, also the aggregate capital demanded by the firm approaches ∞ as the interest rate approaches δ.

Having $\phi = 0$, similar to the Huggett, here at the complete market interest rate the individual's capital supply goes to ∞. Moreover, as the interest rate approaches the level seen in a complete market, the inter-temporal effect diminishes. However, precautionary savings persist because of market incompleteness, leading individuals to aim for smoother consumption and risk mitigation in the face of uncertainty. Consequently, they supply higher levels of capital to achieve these objectives.

In the Aiyagari model, the equilibrium interest rate is lower compared to a hypothetical scenario where markets are complete, indicating constraints on borrowing. As a result, the capital market clears at a lower interest rate, ensuring that the total amount of capital supplied equals the total amount demanded. However, the equilibrium level of aggregate capital in the economy is higher. This discrepancy between the aggregate savings under incomplete and complete market equilibria ($K^{IM} - K^{CM}$) represents what we call aggregate precautionary savings. Essentially, individuals are supplying more capital and demanding more of the bond. Therefore, in equilibrium individuals save more in anticipation of future income shocks because they cannot fully insure against them due to market incompleteness.



This model highlights the significance of idiosyncratic risk in influencing macroeconomic aggregates. With higher aggregate capital in equilibrium, the overall output of the economy is larger compared to a scenario with complete markets. However, this seemingly positive outcome comes with a caveat. The over-accumulation of capital, driven by precautionary savings to mitigate idiosyncratic risk, leads to inefficiencies. Despite the increase in output, it doesn't necessarily translate to improved welfare for individuals.

Now we will be looking into the what happens when there is aggregate risk involved. Consider productivity shock z in the production function that follows Markov chain. We will be considering business cycle with heterogeneous agents. If s process depends on z that is known as counter cyclical employment risk. We will define the joint transition matrix with typical element $\pi(s', z'|s, z)$.

State variables: Here we still have the state ψ, along with s, z and ω. Now, the consumer problem is as follows:

$$v(\omega, s; \psi, z) = \max_{k' \geq -\phi} \{u(w - k') + \beta \sum_{s', z'} \pi(s', z'|s, z) v(\omega', s'; \psi', z')\}$$

Where, now the wage depends on the next period aggregate capital, next period productivity, and next period Labor force. Which gives

$$\omega' = w(K', N', z')s' + (1 + r(K', N', z'))k'$$

with,

$$\psi' = H(\psi, z, z')$$

$$N = \Pi(s = 1|z)$$

Note that in the law of motion of population in either of the state is given by H and is also a function of z' which is to find the ψ'.

Now the firm's problem, firm would solve the problem for given r, and w:

$$r = F_K(K, N) - \delta$$

$$w = F_N(K, N)$$

and

$$\psi(B) = \int_S P(x, B) d\psi(x) \quad \forall B \in \mathcal{B}S$$

The definition of the recursive competitive equilibrium remains the same as mentioned above in definition 3, except the change that now we are not in a steady-state with time-invariant ψ. Miao (JET, 2006), Cao (JET, 2020), the theoretical results on the recursive formulation of aggregate state, existence and uniqueness of RCE is non-trivial

6.4 *Krusell-Smith Approach*

The problem outlined above in the Aiyagrai model with the aggregate uncertainty poses significant challenges both in numerical and theoretical terms. Krusell and Smith (JPE 1998) proposed an approach to tackle this complexity. They suggested that rather than directly solving for the constraint on r, it suffices to focus on the mean of ψ, as it is a function of ψ. Their investigation delved into the bounded rationality of agents, suggesting that agents can only manage to track a limited number of moments of ψ. They initiated this exploration by considering the first moment, which entails agents tracking only the mean of ψ.

6.5 *Revisiting MIT shocks*

7 Industry Dynamics

7.1 *Hopenhayn model*

8 Labor Frictions

8.1 *McCall's Model*

8.2 *Mortensen-Pissarides-Diamond Model*

9 Appendix

In this appendix we will introduce and discuss some basic concepts from Probability and optimization.

9.1 Basic concepts in Stochastic Process

Definition 4. *A probability space is a mathematical object consisting of three elements:*

1. *A set Ω of all possible outcomes ω*
2. *A collection $\mathcal{F}$ of subsets of Ω that constitute the events to which probability is assigned (a $\sigma-$algebra*
3. *A set valued mapping $\mathbf{P}$ that assigns probability values to those events (elements of $\mathcal{F}$).*
   A Probability space is denoted by

$$(\Omega, \mathcal{F}, \mathbf{P})$$

Definition 5. *A $\sigma-$algebra $\mathcal{F}$ is a special kind of family of subsets of a space Ω that satisfy three properties:*

1. $\Omega \in \mathcal{F}$,
2. *$\mathcal{F}$ is closed under complementation: $E \in \mathcal{F} \rightarrow E^c \in \mathcal{F}$,*
3. *$\mathcal{F}$ is closed under countable union: if $\{E_i\}_{i=0}^{\infty}$ is a sequence of sets such that $E_i \in \mathcal{F} \, \forall \, i$, then $(\cup_{i=0}^{\infty} E_i) \in \mathcal{F}$,*

Definition 6. *A random variable is a function whose domain is the set of events Ω and whose image is the real numbers (or a subset of real numbers):*

$$x : \Omega \rightarrow \mathbf{R}$$

For any real number α define the set (E_α) of all elements of the sample space such that the function x takes the value less than α:

$$E_\alpha = \{\omega : x(\omega) < \alpha\}$$

Let us now define when does a function said to be measurable. The concept of measurability is needed so that we can assign probabilities to the events like $x < \alpha$ for all real number α

Definition 7. *A function x is said to be measurable with respect to the σ-algebra $\mathcal{F}$ (or sometimes denoted as $\mathcal{F}$-measurable) if the following holds:*

$$\forall \alpha \in \mathbf{R} : E_\alpha \in \mathcal{F}$$

10 *A Few Problems with proposed solutions:*

Q1) Setup and solve for the planner's problem in 2 agent economy with CRRA preferences under the complete market with Arrow-Debreu time-0 markets. Comment on the deterministic case as well.

Consider a consumer with a CRRA utility function and maximize the expected life-time utility over the finite state space.

$$\mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_{it}(s^t)^{1-v} - 1}{1-v}\right] = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \frac{c_{it}(s^t)^{1-v} - 1}{1-v} \pi(s^t)$$

Here, s^t is the state (including the history of states up to time t), $c_{it}(s^t)$ is the consumption of agent i at time t, $\pi(s^t)$ is the probability of the realization of state s^t from the viewpoint of time 0. We now assume that the markets are complete, that is at time 0, the market for contingency claims or Arrow securities for all states s^t opens. let, $p_t(s^t)$ be the time-0 price of an Arrow security that promises to pay one unit of consumption good at time t if the state s^t realizes. The budget constraint of the agent is,

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) c_i t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) w_{it}(s^t) = W_{i0}(\mathbf{p})$$

Here, $W_{i0}(\mathbf{p})$ is the wealth of the agent i at time 0, $w_{it}(s^t)$ is the state dependent endowment of agent i at time t if the state is s^t.[7] The Euler equation is:

$$\frac{1}{p_t(s^t)} \pi(s^t) c_{it}(s^t)^{-v} = \beta \frac{1}{p_{t+1}(s^{t+1})} \pi(s^{t+1}) c_{i,t+1}(s^{t+1})^{-v}$$

It is to notice that c_{it} can be rewritten as a function of c_{i0}. Then c_{i0} can be solved using the budget constraint. Consumption of agent i at each period can be expressed as $c_{it}(s^t) = g(s^t, \mathbf{p}) W_{i0}(\mathbf{p})$. Thus the consumption for consumers i and j satisfies the equation given the function g is not dependent on i . Therefore, taking the ratio of the

[7] What happens in deterministic case is the following:
The budget constraint becomes, $a_{i,t+1} + c_{it} \leq y_{it} + (1+r)a_{it}$, and given a_0 and $\lim_{T\to\infty} a_T/(1+r)^T = 0$ and can be written as, $\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} c_{it} \leq \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} y_{it}$

consumption for consumers i and j , we get

$$\frac{c_{it}(s^t)}{c_{jt}(s^t)} = \frac{W_{i0}(\mathbf{p})}{W_{j0}(\mathbf{p})}$$

Plugging this into the aggregate constraint, we get,

$$C_t(s^t) = \sum_i c_{it}(s^t) = g(s^t, \mathbf{p}) \sum_i W_{i0}(\mathbf{p}) = p_t(s^t) g(s^t, \mathbf{p}) \sum_i w_{it}(s^t)$$

The above implies that the distribution does not matter in the aggregation or for the macroeconomic outcome. Note, using the market cleaing condition for the equilibrium, the prices will be determined by equating $p_t(s^t)g(s^t, \mathbf{p}) = 1$ for each state.

The ratio of the consumption of individual i and j tells us that the policy change affects the rich and the poor in the same direction, and unless the policy is different for rich and poor. This is one of the motivation to study the incomplete markets to pin down the wealth distribution to a stationary wealth distribution and is unique.[8]. In the complete market scenario, we can not model or talk about the insurance policy on idiosyncratic shocks/risks. For example, the unemployment insurance, government-sponsored health insurance, etc. In real scenarios agents are not fully insured, and can access only to self-insurance, through some type of asset, bonds.

[8]The existence of the unique stationary distribution is in Appendix 9