PhD Economics Math Camp 2024

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Course Description

The course is designed to equip students with the mathematical techniques and concepts essential for the first-year PhD sequence in Economics. Its primary aim is to both review and introduce core mathematical concepts in Real Analysis, Linear Algebra, and Calculus. Throughout the course, students will develop a robust understanding of mathematical rigor and formality. Emphasis will be placed on problem-solving skills and the practical application of these mathematical tools.

Although the course is largely self-contained, it assumes that students have previously completed coursework in elementary analysis and single-variable calculus. Additionally, a basic familiarity with linear algebra is expected. This foundation will enable students to engage effectively with the course material and develop a deeper comprehension of the mathematical principles crucial for advanced economic theory and research.

The initial lecture will include a brief assessment to gauge current understanding. This will help identify areas which requires revision or additional focus.

Assignments & Evaluation: This course is not a mandatory course, hence there is no evaluation and no assignment. However, if you are not familiar with the content, it is highly recommended that you review the materials.

Course Material: The following notes are compiled from various books, along with some of my own contributions. *Mathematics for Economists* by Simon and Blume provides an excellent introduction to these topics. If you're interested in delving deeper into specific areas covered in this course, I recommend *A First Course in Optimization Theory* by Rangarajan Sundaram for optimization, *Linear Algebra* by Hoffman and Kunze for linear algebra, and *A First Course in Probability Theory* by Sheldon Ross for Probability.

Contents

1 Preliminaries

1.1 Propositions

A proposition is a statement which is either true or false. Let P be a proposition. It can have two 'truth values': True, False.
$\sim P$ is the negation of P. $\sim P$ is a statement that is 'opposite' to P. $\sim P$ takes the truth value True when P is False and $\sim P$ takes the truth value False when P is True.

Truth Values for Two Propositions

For any 2 propositions P and Q, there are 4 possible assignments of truth values:

1. P is True and Q is True
2. P is True and Q is False
3. P is False and Q is True
4. P is False and Q is False

Combining Propositions

We can combine 2 propositions. $(P \wedge Q)$ is a new proposition. This combined proposition means both P and Q. It is true in case 1 and false in cases 2, 3, and 4. $(P \vee Q)$ is another proposition. This combined proposition means either P or Q or both. So it is true in cases 1, 2, 3 and false in case 4.

Propositions with Quantifiers

Most of the time we would deal with propositions given an underlying 'state'. Let A be the set of all relevant states. Instead of checking just P, we shall check $P(x)$ that is the validity of P at x.

For all

Suppose a proposition P is true at all states $x \in A$. Our new proposition with quantifier is, $P(x)$ is True $\forall x$.

There exists

If there is at least one $x \in A$ such that P is True at state x. Our new proposition with quantifier is, $\exists x$ such that $P(x)$ is True (there exists).

Negation of Proposition with Quantifiers

Negation of [$P(x)$ is True $\forall x$]: $\exists x$ such that $P(x)$ is False

Negation of [$\exists x$ such that $P(x)$ is True]: $P(x)$ is False $\forall x$

1.2 Logical Implication

P implies Q (denoted by $P \Rightarrow Q$): This itself is a Proposition. We check the validity of propositions $P(x)$ and $Q(x)$ for all $x \in A$. The Proposition $P \Rightarrow Q$ is True if for all $x \in A$ either of the following three hold:

1. $P(x)$ is True and $Q(x)$ is True
2. $P(x)$ is False and $Q(x)$ is True
3. $P(x)$ is False and $Q(x)$ is False

The Proposition $P \Rightarrow Q$ is False if there exists at least one $x \in A$ such that $P(x)$ is True and $Q(x)$ is False.

Note that the set $\{x \mid P(x) \text{ is True}\}$ is a subset of $\{x \mid Q(x) \text{ is True}\}$. We also say that Q is necessary for P. Or P is sufficient for Q.

$Q \Rightarrow P$ is called the converse of the statement $P \Rightarrow Q$. There is no logical connection between these two statements. However, if both are True, then we denote that by $P \Leftrightarrow Q$.

Useful Observation

$[P \Rightarrow Q]$ and $[\sim Q \Rightarrow \sim P]$ are the same statement.
Therefore, to prove $[P \Rightarrow Q]$ it is equivalent (and sometimes easier) to prove $[\sim Q \Rightarrow \sim P]$.

1.3 Set

A set is a collection of elements. Let X be a set. If x is an element in X then $x \in X$, otherwise $x \notin X$. A set with no element is called an empty set and denoted by Φ.
$A \subseteq B$ (A is a subset of B): All elements of A are member of B. Note that $\Phi \subseteq A$ for any set A.

$A \subset B$ (A is strict subset of B): $A \subseteq B$ and there is at least one x such that $x \in B$ but $x \notin A$.

$A = B$ (A is equal to B): If $A \subseteq B$ and $B \subseteq A$ then $A = B$. That is, all elements in A and B are the same.

A^c (Complement of A): Suppose all sets under consideration are subset of a fixed set U (which can be context specific). $A^c = \{x \in U | x \notin A\}$.

Operations on Sets

Basic set operations:
Union of A and B: $A \cup B = \{x | x \in A \text{ or } x \in B\}$
Intersection of A and B: $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
A set minus B: $A - B = \{x | x \in A \text{ and } x \notin B\}$.
Some Useful results

- Distributive laws:
  1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  2. $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$
- De Morgan's laws:
  1. $(A \cup B)^c = A^c \cap B^c$
  2. $(A \cap B)^c = A^c \cup B^c$

Proof of Distributive law

We want to show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. We prove this by showing $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

First, let's show that LHS $\subseteq$ RHS.
Take any $x \in A \cap (B \cup C)$. This means $x \in A$ and $x \in B \cup C$. Moreover, $x \in B \cup C$ means either (i) $x \in B$, or (ii) $x \in C$.

If (i) is the case, then $x \in B$ and $x \in A$ together mean that $x \in A \cap B$. Hence $x \in (A \cap B) \cup (A \cap C)$ because $(A \cap B) \cup (A \cap C)$ is a superset of $(A \cap B)$.

On the other hand, if (ii) is the case, then $x \in C$ and $x \in A$ mean that $x \in A \cap C$. Hence, once again, $x \in (A \cap B) \cup (A \cap C)$ because $(A \cap B) \cup (A \cap C)$ is a superset of $(A \cap C)$.

Thus no matter which of case (i) or (ii) is true, we get $x \in (A \cap B) \cup (A \cap C)$. Since we started from an arbitrary member x of $A \cap (B \cup C)$, we can claim that each member of $A \cap (B \cup C)$ is also a member of $(A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Next, let's show that RHS $\subseteq$ LHS. Take any $y \in (A \cap B) \cup (A \cap C)$. This means either (i) $y \in (A \cap B)$ or (ii) $y \in (A \cap C)$. We shall show that in both cases $y \in A \cap (B \cup C)$.

If (i) is true, then $y \in (A \cap B)$. It means $y \in A$ and $y \in B$. But $y \in B$ makes $y \in B \cup C$, because $B \cup C$ is a superset of B. Thus we have $y \in A$ and $y \in B \cup C$, together which means $y \in A \cap (B \cup C)$.

We can follow the same steps for case (ii). Here $y \in (A \cap C)$, means $y \in A$ and $y \in C$. But $y \in C$ makes $y \in B \cup C$, because $B \cup C$ is a superset of C. Once again $y \in A$ and $y \in B \cup C$ together mean $y \in A \cap (B \cup C)$.

Irrespective of case (i) or (ii), we obtain that y is a member of $A \cap (B \cup C)$. Thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

1.4 Cartesian Product

Take two sets A and B. The Cartesian product of A and B is,

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

A typical element of $A \times B$ is an ordered pair (x, y) where x is from A and y is from B.
Similarly, we can define the Cartesian product of n sets, $A_1 \times A_2 \times \cdots \times A_n$.

A special case: $A_1 = A_2 = \cdots = A_n$, is often denoted by A^n.
We shall focus on $\mathbb{R}^n$, where $\mathbb{R}$ is the set of all real numbers.

A Vector in $\mathbb{R}^n$

A vector in $\mathbb{R}^n$: $x = (x_1, x_2, \ldots, x_n)$ where $x_k \in \mathbb{R}$ for all k.

Some Special Vectors

$$0 = (0, \ldots, 0), \quad e_j = (0, 0, \ldots, \underbrace{1}_{j\text{th position}}, \ldots, 0)$$

Basic Operations on Vectors

Sum:

$$x + y = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

Negation:

$$-x = (-x_1, -x_2, \ldots, -x_n)$$

Difference:

$$x - y = (x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$$

Scalar Multiplication:

$$\text{Let } \alpha \in \mathbb{R}, \quad \alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

Useful Observations

Let $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$:

1. $x + y = y + x$
2. $(\alpha_1 + \alpha_2)x = \alpha_1 x + \alpha_2 x$
3. $\alpha(x + y) = \alpha x + \alpha y$

1.5 Inner Product of Two Vectors

Definition: This operation assigns a number to each pair of vectors. This is also connected to the angle between two vectors (we will skip this geometric interpretation for now).

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Useful Observations

Let $\alpha \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^n$:

1. $x \cdot y = y \cdot x$
2. $x \cdot (y + z) = x \cdot y + x \cdot z$
3. $x \cdot (\alpha y) = \alpha(x \cdot y)$

Norm

Norm is the length of a vector. It is the distance between 0 and x.

$$\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Useful Observations

Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$:

1. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|$

Two Important Inequalities

Let $x, y \in \mathbb{R}^n$:

1. **Cauchy-Schwarz Inequality**:
   - $|x \cdot y| \leq \|x\|\|y\|$
   - Equality holds if and only if $x = \alpha y$ for some $\alpha \in \mathbb{R}$
2. **Triangle Inequality**:
   - $\|x + y\| \leq \|x\| + \|y\|$
   - Equality holds if and only if $x = \alpha y$ for some $\alpha > 0$

Proof of Cauchy-Schwarz Inequality

(i)

Define $z = x + ty$. We know that $z \cdot z \geq 0$ and equality holds if $z = 0$.

$$z \cdot z = (x + ty) \cdot (x + ty) = x \cdot x + t^2(y \cdot y) + 2t(x \cdot y) = \|x\|^2 + t^2\|y\|^2 + 2t(x \cdot y) \geq 0$$

To eliminate the linear term $2t(x \cdot y)$ in the quadratic expression, we choose $t = -\frac{x \cdot y}{\|y\|^2}$. This simplifies the inequality:

$$\|x\|^2 + \left(-\frac{x \cdot y}{\|y\|^2}\right)^2 \|y\|^2 + 2\left(-\frac{x \cdot y}{\|y\|^2}\right)(x \cdot y) \geq 0$$

Simplifying further,

$$\|x\|^2 + \frac{(x \cdot y)^2}{\|y\|^2} - 2\frac{(x \cdot y)^2}{\|y\|^2} \geq 0$$

$$\|x\|^2 - \frac{(x \cdot y)^2}{\|y\|^2} \geq 0$$

Multiplying both sides by $\|y\|^2$,

$$\|x\|^2\|y\|^2 \geq (x \cdot y)^2$$

Thus,

$$|x \cdot y| \leq \|x\|\|y\|$$

Qn: Why do we use this particular t?

By choosing $t = -\frac{x \cdot y}{\|y\|^2}$, we eliminate the linear term in the quadratic expression $\|x\|^2 + t^2\|y\|^2 + 2t(x \cdot y)$, making it easier to derive the inequality. This specific choice of t sets the derivative of the quadratic expression with respect to t to zero, simplifying the analysis and leading directly to the Cauchy-Schwarz inequality.

(ii)

Equality holds $\Rightarrow x + ty = 0 \Rightarrow x = -ty$.

If $x = \alpha y$, check that the Cauchy-Schwarz inequality holds with equality.

Proof of Triangle Inequality

(i)

$$\|x + y\|^2 = (x + y) \cdot (x + y) = \|x\|^2 + \|y\|^2 + 2(x \cdot y) \leq \|x\|^2 + \|y\|^2 + 2(\|x\|\|y\|)$$

The last inequality follows from the Cauchy-Schwarz inequality.

Taking the square root, we obtain the Triangle inequality.

$$\|x + y\| \leq \|x\| + \|y\|$$

(ii)

Equality holds $\Rightarrow$ Cauchy-Schwarz holds with equality $\Rightarrow x = \alpha y$.

If $x = \alpha y$, check that the Triangle inequality holds with equality.

1.6 Relation

A relation is a way of showing a connection or relationship between elements of two sets. A relation can be defined as a subset of the cartesian product of two sets.

Let A and B be sets. A relation R from A to B is a subset of $A \times B$. If $(a, b) \in R$, we say that a is related to b by R and write aRb.

Notation

- aRb: a is related to b by the relation R.
- $R(a) = b$: a maps to b under the relation R.

Types of Relations

1. **Reflexive Relation**: A relation R on a set A is called reflexive if every element of A is related to itself. Formally, R is reflexive if $(a, a) \in R$ for all $a \in A$.
2. **Symmetric Relation**: A relation R on a set A is called symmetric if for all $a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$.
3. **Transitive Relation**: A relation R on a set A is called transitive if for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
4. **Equivalence Relation**: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive. An equivalence relation partitions the set A into disjoint equivalence classes.
5. **Partial Order Relation**: A relation R on a set A is called a partial order relation if it is reflexive, antisymmetric (i.e., if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$), and transitive. A set with a partial order relation is called a partially ordered set (poset).

1.7 Function and Related Concepts

Take two sets A and B. A function f is a rule that maps each element of A to exactly one element in B. That is, $f(a) \in B \; \forall a \in A$. We denote this by $f : A \to B$.

A is called the domain of f.

Notes

1. More than one element of A can be mapped to the same element in B.
2. Some members of B may not be reached by f.

Take any $A_0 \subseteq A$. If A_0 is the restricted domain, then the image/range of A_0 under f is the set $f(A_0) = \{b \in B \mid f(a) = b, a \in A_0\}$.

Example: Indifference curve

Inverse of a Function

Take a function $f : A \to B$. The inverse of f, denoted by f^{-1}, is the route back from B to A. For each $b \in f(A)$, $f^{-1}(b) = \{a \in A \mid f(a) = b\}$.

Note that f^{-1} is not necessarily a function. (Question: Why?)

Take any $B_0 \subseteq B$. The preimage of B_0 under f is $f^{-1}(B_0) = \{a \in A \mid f(a) = b, b \in B_0\}$.

Composition of Functions

We can compose two functions to get a new function in the following manner. Suppose $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is the composite function which maps each element of A to exactly one element in C, through a temporary stop at B.

$$(g \circ f)(a) = g(f(a)) \ \forall a \in A$$

1.8 Countability and Cardinality in Mathematics

Cardinality

Cardinality is a measure of the "size" of a set. It indicates the number of elements in a set. There are different types of cardinalities depending on whether the set is finite or infinite.

Finite Sets

For finite sets, the cardinality is simply the number of elements in the set. For example, if $A = \{1, 2, 3\}$, then the cardinality of A, denoted by $|A|$, is 3.

Infinite Sets

For infinite sets, cardinality can be more complex. Infinite sets can have different sizes of infinity. The most common types of infinite cardinalities are:

- $\aleph_0$ (aleph-null): This is the cardinality of the set of natural numbers $\mathbb{N}$. A set with cardinality $\aleph_0$ is called countably infinite.
- $\mathfrak{c}$ (the cardinality of the continuum): This is the cardinality of the set of real numbers $\mathbb{R}$. A set with cardinality $\mathfrak{c}$ is called uncountably infinite.

Countability

Countability is a property that describes whether the elements of a set can be put into one-to-one correspondence with the natural numbers.

Countable Sets

A set A is countable if there exists a bijection between A and $\mathbb{N}$, the set of natural numbers. This means that the elements of A can be listed in a sequence $a_1, a_2, a_3, \ldots$. A set can be countably infinite or finite.

Examples of Countable Sets:

- The set of natural numbers $\mathbb{N}$ is countable.
- The set of integers $\mathbb{Z}$ is countable.
- The set of rational numbers $\mathbb{Q}$ is countable.

Uncountable Sets

A set B is uncountable if it is not countable. This means there is no bijection between B and $\mathbb{N}$.

Examples of Uncountable Sets:

- The set of real numbers $\mathbb{R}$ is uncountable.
- The set of all subsets of natural numbers (the power set of $\mathbb{N}$) is uncountable.

Comparing Cardinalities

To compare the cardinalities of two sets A and B, we use the concept of bijections:

- If there exists a bijection between A and B, then A and B have the same cardinality.
- If there exists an injection from A to B but not a bijection, then the cardinality of A is less than or equal to the cardinality of B.
- If there exists a surjection from A to B but not a bijection, then the cardinality of B is less than or equal to the cardinality of A.

2 Real Analysis

Real Analysis is a branch of mathematics dealing with real numbers and real-valued functions. It provides a rigorous foundation for calculus and is crucial for understanding many economic models. We will be focusing on normed linear spaces. Unless specified otherwise, the space we will be in is $\mathbb{R}^k$ with the Euclidean norm $\|x\| = \left(\sum_{i=1}^k x_i^2\right)^{1/2}$. Although there are many other norms, let's start with the definition of the norm.

2.1 Norm

Definition 2.1 (Norm)**.** *A norm in a vector space V is defined as a function assigning to each vector x a non-negative real number $\|x\|$, such that:*

1. *For all x, $\|x\| \geq 0$ with equality if and only if $x = 0$,*
2. *If $c \in \mathbb{R}$, $\|cx\| = |c|\|x\|$,*
3. *$\|x + y\| \leq \|x\| + \|y\|$.*

The last requirement is known as the triangle inequality, which follows for the Euclidean norm from the Cauchy-Schwarz inequality (for proof, see Section 1.5). Taking any $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$, consider the following quadratic in z:

$$f(z) = (x_1 z + y_1)^2 + \cdots + (x_k z + y_k)^2 = \|x\|^2 z^2 + 2(x \cdot y)z + \|y\|^2$$

There are various other normed linear spaces, such as the space of bounded continuous functions defined on an interval of real numbers with the sup norm. However, in the remainder of this course, we will focus on finite-dimensional spaces. Some of the concepts below apply to both finite and infinite-dimensional spaces, so we will sometimes denote the underlying space as V. Mostly, it will help to think of V as simply $\mathbb{R}^k$ and to visualize concepts in $\mathbb{R}^2$.

We will measure the distance between vectors using $\|x-y\| = \left(\sum_{i=1}^k (x_i - y_i)^2\right)^{1/2}$. This is an intuitive notion of distance using the Pythagorean theorem. Let us now define the notion of distance or metric.

2.2 Metric Space

Definition 2.2 (Metric Space)**.** *A metric ρ on a set V is a function that assigns to each pair of elements a non-negative real number, such that:*

1. *For all $x, y \in V$, $\rho(x, y) \geq 0$ with equality if and only if $x = y$,*

2. $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in V$ *(symmetry),*

3. *For all* $x, y, z \in V$, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Note that the last inequality is the triangle inequality for metrics. If we have a vector space and have defined a norm on it (such as the Euclidean norm), then this norm induces the metric $\|x - y\|$.
Check for any norm this is indeed a metric

Example: Discrete Metric

Let X be any set, and define the function $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Then it is straightforward to check (do it!) that d is a metric on X, called the **discrete metric**. Here, the distance between any two distinct points is always 1.

2.3 Open and Closed Sets

Definition 2.3 (Open sets)**.** *Let* $\epsilon > 0$ *and* $x \in V$*. The open ball centered at* x *with radius* ϵ *is defined as*

$$B(x, \epsilon) = \{y \mid \|x - y\| < \epsilon\}$$

We will see if $V = \mathbb{R}$, $\mathcal{B}(x, \epsilon)$ is the open interval $(x - \epsilon, x + \epsilon)$. If $V = \mathbb{R}^2$, it is an open disk centered at x. The boundary of the disk is traced by the Pythagorean theorem.

Example: Supremum Metric

After the standard metric spaces $\mathbb{R}^n$, this example will perhaps be the most important.

First, recall that a function $f : X \to \mathbb{R}$ from a set X to $\mathbb{R}$ is **bounded** if there is some $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$. In other words, this says that the set $\{f(x) \mid x \in X\}$ of values of f is a bounded subset of $\mathbb{R}$. Note that because of this, the set of values of a bounded function has a supremum as a consequence of the completeness axiom for $\mathbb{R}$.

Let $C_b([a, b])$ denote the space of real-valued, bounded functions on the closed interval $[a, b]$:

$$C_b([a, b]) := \{f : [a, b] \to \mathbb{R} \mid f \text{ is bounded}\}.$$

The function d given by

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [a, b]\}$$

defines a metric on $C_b([a,b])$ called the **supremum** (or **sup** for short) metric. Again, you should try to verify on your own that this is indeed a metric.

A few remarks are in order. First, if f and g are bounded, then so is $f-g$, so the supremum used in the definition of d actually exists in $\mathbb{R}$. Thus the definition of d makes sense. Now, here is what this definition is saying: take two bounded functions f and g, and look at the distance between $f(x)$ and $g(x)$ over all possible x—the distance between f and g is defined to be the "largest" such distance. (I put the word "largest" in quotation marks since the supremum defining the sup metric is not necessarily in the set $\{|f(x)-g(x)| \mid x \in [a,b]\}$ itself, but you should intuitively think of it as the largest such value.) Also, note that the same definition works for bounded functions on all of $\mathbb{R}$. The notation

$$\sup_{x\in[a,b]} |f(x)-g(x)|$$

for the supremum used in the definition of $d(f,g)$ is also commonly used.

Eventually, it will help to try to "visualize" what this metric "looks like"—we will talk about this in class as we go on. As a start, consider the functions $f(x)=\cos x$ and $g(x)=-\cos x$ on $\mathbb{R}$. Draw their graphs and ask yourself: what is the largest possible distance between the values of f and g at points $x \in \mathbb{R}$? From the picture it should be easy to see that the answer is 2, say when $x=0$. Because of this, 2 is indeed the distance between f and g with respect to the sup metric; the sup metric will give us a way to tell how "close" or how "far apart" two functions are from each other in terms of their graphs.

For now onwards, we will use $\mathbb{R}^k$ with Euclidean norm and its induced metric.

Exercise 1

Show that $\|x-y\|$ defined by $\max\{|x_1-y_1|,\ldots,|x_k-y_k|\}$, for all $x,y \in \mathbb{R}^k$, is a metric (i.e., satisfies the three requirements of a metric). In the space $\mathbb{R}^2$, sketch $B(0,1)$, the open ball centered at 0, the origin, of radius 1, in this metric.

Thus, $\|x-y\| = \max\{|x_1-y_1|,\ldots,|x_k-y_k|\}$ is a metric.

In the space $\mathbb{R}^2$, the open ball $B(0,1)$, centered at the origin with radius 1 in this metric, is a square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$.

Interior and Open Sets

Let $S \subset V$. x is an interior point of S if $B(x,\epsilon) \subset S$, for some $\epsilon > 0$. S is an open set if all points of S are interior points. On the other hand, S is a closed set if and only if S^c is an open set.

Examples

Open in $\mathbb{R}$ vs. open in $\mathbb{R}^2$. The empty set and the entire space are open sets.

There is an alternative, equivalent, and convenient way to define closed sets. x is an adherent point of S, or adheres to S, if every $B(x, \epsilon)$ contains a point belonging to S. Note that this does not necessarily mean that x is in S. (However, if $x \in S$, then x adheres to S, of course).

Lemma 1

A set S is closed if and only if it contains all its adherent points.

Proof

Suppose S is closed, so S^c is open. Let x adhere to S. We want to show that $x \in S$. Suppose not. Then $x \in S^c$, and since S^c is open, x is an interior point of S^c. So there is some $\epsilon > 0$ such that $B(x, \epsilon) \subset S^c$; this does not have any points from S. So x cannot be an adherent point of S. Contradiction.

Conversely, suppose S contains all its adherent points. To show S is closed, we show S^c is open. We show that all the points in S^c are interior points. Let $x \in S^c$, Since x does not adhere to S. it must be the case that for some $\epsilon > 0$, $B(x, \epsilon) \subset S^c$

Remarks

1. Singleton and finite sets are closed.
2. Countable sets need not be closed.
3. Empty set and the entire space are closed sets.

Now we will relate the idea of closedness to the convergence of sequences.

Definition 2.4 (Sequence)**.** *A sequence in V is a function $x : N \to V$. We will denote the image as $\{x_1, x_2, ...\}$ or $\{x^1, x^2, ...\}$. In short, we will write it as (x_n) or x^n.*

Definition 2.5 (Convergence)**.** *A sequence $(x^n)_{n=1}^{\infty}$ of points in V converges to x if for every $\epsilon > 0$ there exists a positive integer N s.t. $n \geq N$ implies $||x^n - x|| < \epsilon$.*

Note that writing this definition in terms of open ball, for every open ball $B(x, \epsilon)$, we can find N s.t. for all points x^n following x^N, x^n lies in $B(x, \epsilon)$.

To clarify the convergence concept, we will look at some examples.

1. $x_n = 1/n$, $n = 1, 2, ...$ is a sequence of real numbers converging to zero. Indeed, take any $\epsilon > 0$. To show that there exists N such that $n > N$ implies $|x_n - 0| = 1/n < \epsilon$. We work with the equation $1/n = \epsilon$. So if we choose m, s.t. $1/m = \epsilon$, we get $m = 1/\epsilon$. So let N be the smallest natural number such that $N > 1/\epsilon$. Then for every $n \geq N$, we have $n > 1/\epsilon$ or $1/n < \epsilon$

2. $x_n = \frac{2n-3}{3n+4}$ converges to $2/3$ as $n \to \infty$.

3. $x_n = \frac{2n^3+3n}{n^3-4}$ converges to 2 as $n \to \infty$.

Here are some algebraic properties of sequence.

Exercise 2

If (a_n) and (b_n) converge to a and b respectively, then

- $(a_n + b_n)$ converges to $a + b$.
- $a_n b_n$ converges to ab.
- If $(a_n) \to a$ and a_n, a are all non-zero, then $1/a_n \to 1/a$.

Now let's get back to convergence of sequences of vectors in $\mathbb{R}^k$.

$$\mathbf{x}_n = \left(\frac{1}{n}, \frac{1}{n}\right), \quad n = 1, 2, \ldots$$

is a sequence of vectors in $\mathbb{R}^2$ converging to the origin. More generally, a sequence converges in $\mathbb{R}^k$ if and only if all the coordinate sequences converge, as can be visualized in the example here using hypotenuses and legs of triangles.

Theorem 2.1. A sequence $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$ in $\mathbb{R}^k$ if and only if for every $i \in \{1, \ldots, k\}$, the coordinate sequence $\{x_{n,i}\}$ converges to x_i.

Proof. Suppose $\mathbf{x}_n \to \mathbf{x}$ in $\mathbb{R}^k$. Let $\epsilon > 0$. Since

$$(x_{n,i} - x_i)^2 \leq \sum_{j=1}^{k} (x_{n,j} - x_j)^2,$$

taking square roots implies $|x_{n,i} - x_i| \leq \|\mathbf{x}_n - \mathbf{x}\|$, so for every $n \geq N$ such that

$$\|\mathbf{x}_n - \mathbf{x}\| < \epsilon, \quad |x_{n,i} - x_i| < \epsilon.$$

Conversely, if all the coordinate sequences converge to the coordinates of the point $\mathbf{x}$, then there exists a positive integer N such that $n \geq N$ implies

$$|x_{n,i} - x_i| < \frac{\epsilon}{\sqrt{k}},$$

for every coordinate i. Squaring, adding across all i, and taking square roots, we have

$$\|\mathbf{x}_n - \mathbf{x}\| < \epsilon.$$

Note that if $(\mathbf{x}_n) \to \mathbf{x}$, $(\mathbf{y}_n) \to \mathbf{y}$, then $(\mathbf{x}_n + \mathbf{y}_n) \to (\mathbf{x} + \mathbf{y})$. Indeed, for every $\epsilon > 0$, there exists N such that $n \geq N$ implies

$$\|\mathbf{x}_n - \mathbf{x}\| < \frac{\epsilon}{2} \quad \text{and} \quad \|\mathbf{y}_n - \mathbf{y}\| < \frac{\epsilon}{2}.$$

So

$$\|(\mathbf{x}_n+\mathbf{y}_n)-(\mathbf{x}+\mathbf{y})\| = \|(\mathbf{x}_n-\mathbf{x})+(\mathbf{y}_n-\mathbf{y})\| \leq \|\mathbf{x}_n-\mathbf{x}\|+\|\mathbf{y}_n-\mathbf{y}\| \quad \text{(by the triangle inequality)},$$

and this is less than $\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Closed sets can be characterized in terms of convergent sequences as follows.

lemma 2.1.1. A set S is closed if and only if for every sequence $(\mathbf{x}_n)$ lying in S, $\mathbf{x}_n \to \mathbf{x}$ implies $\mathbf{x} \in S$.

Proof. Suppose S is closed. Take any sequence $(\mathbf{x}_n)$ that converges to a point $\mathbf{x}$. Then for every $B(\mathbf{x}, \epsilon)$, we can find a member $\mathbf{x}_n$ of the sequence lying in this open ball. So, $\mathbf{x}$ adheres to S. Since S is closed, it must contain this adherent point $\mathbf{x}$.

Conversely, suppose the set S has the property that whenever $(\mathbf{x}_n) \subset S$ converges to $\mathbf{x}$, $\mathbf{x} \in S$. Take a point y that adheres to S. Take the successively smaller open balls $B(y, 1/n)$, $n = 1, 2, 3, \ldots$. We can find, in each such open ball, a point y_n from the set S (since y adheres to S). These points need not be all distinct, but since the open balls have radii converging to 0, $y_n \to y$. Thus by the convergence property of S, $y \in S$. So, any adherent point y of S actually belongs to S

Some important result

Convergence of Non-negative Sequences
1. If (a_n) is a sequence of real numbers all greater than or equal to 0, and $a_n \to a$, then $a \geq 0$. The reason is that for all k, $a_k \in [0, \infty)$ which is a closed set and hence must contain the limit a.

Supremum and Infimum

Let $S \subset \mathbb{R}$. u is an upper bound of S if $u \geq a$, for every $a \in S$. s is the supremum or least upper bound of S (called $\sup S$), if s is an upper bound of S, and $s \leq u$, for every upper bound u of S.

We say that a set S of real numbers is bounded above if there exists an upper bound, i.e., a real number M such that $a \leq M, \forall a \in S$. The most important property of a supremum, which we'll by and large take here as given, is the following:

Completeness Property of Real Numbers: Every set S of real numbers that is bounded above has a supremum.

Examples: $S = (0, 1)$, $D = [0, 1]$, $K =$ set of all numbers in the sequence $1 - \frac{1}{2^n}, n = 1, 2, 3, \ldots$. The supremum of all these sets is 1, and this does not belong to S or to K. When $\sup S$ belongs to S, it is called the maximum of S, for obvious reasons. Another important property of suprema is the following.

lemma 2.1.2. For every $\epsilon > 0$, there exists a number $a \in S$ such that $a > \sup S - \epsilon$. Note that this means that $\sup S$ is an adherent point of S.

Proof. Suppose that for some $\epsilon > 0$, there is no number $a \in S$ such that $a > \sup S - \epsilon$. So, every $a \in S$ must then satisfy $a \leq \sup S - \epsilon$. But then, $\sup S - \epsilon$ is an upper bound of S that is less than $\sup S$. This implies that $\sup S$ is not in fact the supremum of S. Contradiction. □

lemma 2.1.3. If a set S of real numbers is bounded above and closed, then it has a maximum.

Proof. Since it is bounded above, it has a supremum, $\sup S$. $\sup S$ is an adherent point of S (by the above lemma). S is closed so it contains all its adherent points, including $\sup S$. Hence $\sup S$ is the max of S. □

Corresponding to the notion of supremum or least upper bound of a set S of real numbers, is the notion of infimum or greatest lower bound of S. A number l is a lower bound of S if $l \leq a, \forall a \in S$. The infimum of S is a number s such that s is a lower bound of S, and $s \geq l$, for all lower bounds l of S. We call the infimum of S, $\inf S$. Let $-S$ be the set of numbers of the form $-a$, for all $a \in S$.

Fact: $\sup S = -\inf(-S)$.

So, sup and inf are intimately related. By the completeness property of real numbers, if $S \subset \mathbb{R}$ is bounded below, (i.e., there exists m such that $m \leq a, \forall a \in S$), it has an infimum. If S is closed and bounded below, it has a minimum.

A set $S \subset \mathbb{R}$ is said to be bounded if it is bounded above and bounded below. We can extend the lemma above along obvious lines as follows:

Theorem 2.2. If $S \subset \mathbb{R}$ is closed and bounded, then it has a Maximum and a Minimum.

For a more general normed linear space V, we define boundedness as follows. A set $S \subset V$ is bounded if there exists an open ball $B(0, M)$ such that $S \subset B(0, M)$.

2.4 Subsequences

Suppose (x_n) is a sequence in V. (Note the change in notation, from superscript to subscript.

Let $m(k)$ be an increasing function from the natural numbers to the natural numbers. So, $l > n$ implies $m(l) > m(n)$. A subsequence $(x_{m(k)})$ of (x_n) is an infinite sequence whose k-th member is the $m(k)$-th member of the original sequence.

Example: The idea is that to get a subsequence from (x_n), you strike out some members, keeping the remaining members' positions the same.

Fact: If a sequence (x_n) converges to x, then all its subsequences converge to x.

Proof. Take an arbitrary $\epsilon > 0$. So, there exists N such that $n \geq N$ implies $\|x_n - x\| < \epsilon$. This implies, for any subsequence $(x_{m(k)})$, that $k \geq N$ implies $\|x_{m(k)} - x\| < \epsilon$. □

However, if a sequence does not converge anywhere, it can still have (lots of) subsequences that converge. For example, let $(x_n) \equiv ((-1)^n)$, $n = 1, 2, \ldots$. Then, (x_n) does not converge; but the subsequences $(y_m) = -1, -1, -1, \ldots$ and $(z_m) = 1, 1, 1, \ldots$ both converge, to different limits. (Such points are called limit points of the mother sequence (x_n)).

2.5 Monotone Sequences and Bolzano-Weierstrass Theorem

Theorem 2.3. Every bounded and increasing sequence of real numbers converges (to its supremum).

Proof. Let $a \equiv \sup(x_n)$. Take any $\epsilon > 0$. By the above discussion, there exists some $x_N \in (a - \epsilon, a]$. And since (x_n) is an increasing sequence, we have that for all $k \geq N, x_k \in (a - \epsilon, a]$. So $(x_n) \to a$. □

A similar conclusion holds for decreasing bounded sequences. Note that:

Theorem 2.4. Every sequence of real numbers has a monotone subsequence.

Proof. For the sequence (x_k), let $A_n = \{x_k | k \geq n\}$, $n = 1, 2, \ldots$. If any one of these sets A_n does not have a maximum, we can pull out an increasing sequence. For instance, suppose A_1 does not have a max. Then let $x_{k_1} = x_1$. Let x_{k_2} be the first member of the sequence (x_k) that is greater than x_1, and so on.

On the other hand, if all A_n have maxes, then we can pull out a decreasing subsequence. Let $x_{k_1} = \max\{A_1\}$, $x_{k_2} = \max\{A_{k_1+1}\}$, $x_{k_3} = \max\{A_{k_2+1}\}$, and so on. □

It follows from the above two theorems, that we have:

Theorem 2.5 (Bolzano-Weierstrass Theorem)**.** Every bounded sequence of real numbers has a convergent subsequence.

2.6 Cantor's Nested Intervals Theorem

Theorem 2.6. If $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots$ is a nested sequence of closed intervals, then $\bigcap_{m=1}^{\infty} [a_m, b_m]$ is nonempty. Moreover, if $b_m - a_m \to 0$, then this intersection is a single point.

Proof. Because of the nesting, $a_1 \leq a_2 \leq \ldots \leq b_2 \leq b_1$. So, (a_k) is bounded and increasing and so has a supremum, say a; (b_k) is bounded and decreasing and has an infimum, say b; and $a \leq b$. So, $[a, b] \subseteq [a_m, b_m], m = 1, 2, \ldots$, and therefore lies in the intersection $\bigcap_{m=1}^{\infty} [a_m, b_m]$; which is therefore nonempty. Moreover, if $b_m - a_m \to 0$, then by sandwiching, $a = b$ and the intersection is a single point. □

2.7 Compact Sets

Definition 2.6. *Let V be a normed linear space. A set $S \subset V$ is compact or sequentially compact if every sequence (x_n) in S has a subsequence that converges to a point in S.*

Theorem 2.7. Suppose $S \subset \mathbb{R}^n$. Then S is compact if and only if it is closed and bounded.

Proof (Sketch). Suppose S is closed and bounded. We can show it's compact using a pigeonhole-like argument; let's sketch it here. Since S is bounded, we can cover it in a closed rectangle $R_0 = I_1 \times \ldots \times I_n$, where $I_i, i = 1, \ldots, n$ are closed intervals.

Take a sequence (x_n) in S. Divide the rectangle in two: $I_1^1 \times \ldots \times I_n$ and $I_1^2 \times \ldots \times I_n$, where $I_1^1 \cup I_1^2 = I_1$ is the union of 2 intervals. Then, there's an infinity of members of (x_n) in at least one of these smaller rectangles, call this R_1. Divide R_1 into 2 smaller rectangles, say by dividing I_2 into 2 smaller intervals; we'll find an infinity of members of (x_n) in at least one of these rectangles, call it R_2. This process goes on ad infinitum, and we find an infinity of members of (x_n) in the rectangles $R_0 \supset R_1 \supset R_2 \supset \ldots$. By the Cantor Intersection Theorem, $\bigcap_{i=0}^{\infty} R_i$ is a single point; call this point x.

Now we can choose points $y_i \in R_i$, $i = 1, 2, \ldots$ such that each y_i is some member of (x_n); because the R_i's collapse to x, it is easy to show that (y_m) is a subsequence that converges to x. Moreover, the y_i's lie in S, and S is closed; so $x \in S$.

Conversely, suppose S is compact.

(i) Then it is bounded. For suppose not. Then we can construct a sequence (x_n) in S such that for every $n = 1, 2, \ldots$, $\|x_n\| > n$. But then, no subsequence of (x_n) can converge to a point in S. Indeed, take any point $x \in S$ and any subsequence $(x_{m(n)})$ of (x_n). Then

$\|x_{m(n)}\| = \|x_{m(n)} - x + x\| \leq \|x_{m(n)} - x\| + \|x\|$ (The inequality above is due to the triangle inequality).

So,

$$\|x_{m(n)} - x\| \geq \|x_{m(n)}\| - \|x\| \geq n - \|x\|,$$

and the RHS becomes larger with n. So $(x_{m(n)})$ does not converge to x.

(ii) S is also closed. Take any sequence (x_n) in S that converges to x. Then, all subsequences of (x_n) converge to x, and since S is compact, (x_n) has a subsequence converging to a point in S. So, this point of limit is x, and $x \in S$. So, S is closed. $\square$

2.8 Continuity of Functions

Definition 2.7. *A function $F : \mathbb{R}^k \to \mathbb{R}^m$ is continuous at $x \in \mathbb{R}^k$, if for every sequence (x_n) that converges to x in $\mathbb{R}^k$, the image sequence $(f(x_n))$ converges to $f(x)$ in $\mathbb{R}^m$.*

Example of point discontinuity.

Example of continuous function on discrete space.

F is continuous on $S \subset \mathbb{R}^k$, if it is continuous at every point $x \in S$.

Examples: The real-valued function $F(x) = x$ is continuous using this definition, almost trivially, since (x_n) and x are identical to $(F(x_n))$ and $F(x)$ respectively.

$F(x) = x^2$ is continuous. We want to show that if (x_n) converges to x, then $(F(x_n)) = x_n^2$ converges to $F(x) = x^2$. This follows from the exercise above on limits: $x_n \to x$, $x_n \to x$ implies $x_n \cdot x_n \to x \cdot x = x^2$.

By extension, polynomials are continuous functions.

Exercise 2.1. Consider the real-valued function $f(x) = \sqrt{x}$? Show that it is continuous at 0. Extend the argument to show it is continuous at an arbitrary x.

Exercise 2.2. Continuity of compositions. Let $f : \mathbb{R}^k \to \mathbb{R}^m$ be continuous at some x_0 and $g : \mathbb{R}^m \to \mathbb{R}^p$ be continuous at $f(x_0)$. Then $h : \mathbb{R}^k \to \mathbb{R}^p$ defined by $h(x) = g(f(x))\forall x$ is continuous at x.

Because of the above results on continuity of polynomials, square root, and compositions, for example, the real-valued function $\sqrt{2x^2 + 3}$ is continuous.

We may talk a little about the coordinate functions of $F : \mathbb{R}^n \to \mathbb{R}^m$:

$$(F_1(x_1, \ldots, x_n), \ldots, F_m(x_1, \ldots, x_n)).$$

Example: $F(x_1, x_2) = (x_1 + x_2, x_1^2 + x_2^2)$. This is continuous because (i) F_1 and F_2 are continuous; e.g. let $x^k \to x$. Then the coordinates $x_1^k \to x_1$ and $x_2^k \to x_2$. So $F_1(x^k) = x_1^k + x_2^k \to x_1 + x_2 = F_1(x)$.

(ii) Since the coordinate sequences $F_1(x^k) \to F_1(x)$ and $F_2(x^k) \to F_2(x)$, $F(x^k) \equiv (F_1(x^k), F_2(x^k)) \to F(x) = (F_1(x), F_2(x))$.

There is an equivalent, (ϵ, δ) definition of continuity. We won't get a chance to explore this much, but this definition is the one that gets modified when we move to the notion of uniform continuity.

Definition 2.8. *A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x \in \mathbb{R}^n$, if for every $\epsilon > 0, \exists \delta > 0$ such that if for any $y \in \mathbb{R}^n$ we have $\|x - y\| < \delta$, then $\|F(x) - F(y)\| < \epsilon$.*

So if there is a hurdle of size ϵ around $F(x)$, then, if point y is close enough to x, $F(y)$ cannot overcome the hurdle.

Theorem 2.8. The two definitions above are equivalent.

Proof. Suppose there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exists a y with $\|x - y\| < \delta$ and $\|F(x) - F(y)\| \geq \epsilon$. Then for this particular ϵ, we can choose a sequence of $\delta_k = 1/k$ and x_k with $\|x - x_k\| < 1/k$, such that $\|F(x) - F(x_k)\| \geq \epsilon$. So, $(x_k) \to x$ but $(F(x_k))$ does not converge to $F(x)$, staying always outside the ϵ-band of $F(x)$.

Conversely, suppose there exists a sequence (x_k) that converges to x, but $(F(x_k))$ does not converge to $F(x)$. So, there exists $\epsilon > 0$ such that for every positive integer N, there exists $k \geq N$ for which $\|F(x_k) - F(x)\| \geq \epsilon$. Then, for this specific ϵ, there does not exist any $\delta > 0$ such that for all y with $\|x - y\| < \delta$ we have $\|F(x) - F(y)\| < \epsilon$; for we can find for any such δ, one of the x_k's such that $\|x_k - x\| < \delta$, so $\|F(x_k) - F(x)\| \geq \epsilon$. □

Here is an immediate upshot of the latter definition. Suppose $F : \mathbb{R} \to \mathbb{R}$ is continuous at x. If $F(x) > 0$, then there is an open interval $(x - \delta, x + \delta)$ such that

if y is in this interval, then $F(y) > 0$. The idea is that we can take an $\epsilon = F(x)/2$, say, and use the (ϵ, δ) definition. A similar statement will hold if $F(x) < 0$.

Exercise 2.3. Show that $f(x) = \sqrt{x}$ is continuous at a specific c in the domain using the (ϵ, δ) definition of continuity.

We use this fact now in the following result.

Theorem 2.9 (Intermediate Value Theorem)**.** Suppose $F : \mathbb{R} \to \mathbb{R}$ is continuous on an interval $[a, b]$ and $F(a)$ and $F(b)$ are of opposite signs. Then there exists $c \in (a, b)$ such that $F(c) = 0$.

Proof. Suppose WLOG that $F(a) > 0$, $F(b) < 0$ (i.e., for the other case just consider the function $-F$). Then the set

$$S = \{x \in [a, b] | F(x) \geq 0\}$$

is bounded above. Indeed, b is an upper bound of S. By the completeness property of real numbers therefore, S has a supremum, $\sup S = c$, say. This means that c is the smallest number for which, for all $x \in S$ such that $F(x) \geq 0$, we have $x \leq c$.

It can't be that $F(c) > 0$, for then by continuity, there is an $x \in S$, $x > c$, such that $F(x) > 0$ so c is not an upper bound of S. It can't be that $F(c) < 0$. For, by continuity, there is an interval $(c - \delta, c]$ such that every y in this interval satisfies $F(y) < 0$, and $x \leq y$, $\forall x \in S$. But then, c is not the least upper bound of S. So, it must be that $F(c) = 0$. □

As an application, you may want to prove the following corollary, a simple fixed point theorem.

Exercise 2.4. Suppose $f : [a, b] \to [a, b]$ is a continuous function. Then there exists $x^* \in [a, b]$ such that $x^* = f(x^*)$.

2.9 Existence of Optima

Before moving to the optimization we will look into the existence of such a This theorem of Weierstrass gives a sufficient condition for a maximum and minimum to exist, for an optimization problem.

Theorem 16 (Weierstrass): Let $S \subset \mathbb{R}^n$ be compact and let $f : S \to \mathbb{R}$ be continuous. Then f has a maximum and minimum on S; i.e., there exist $z_1, z_2 \in S$ such that

$$f(z_2) \leq f(x) \leq f(z_1), \quad \forall x \in S.$$

The idea is that the continuity of f preserves compactness; i.e. since S is compact and f is continuous, the image set $f(S)$ is compact. This holds irrespective of the

space $f(S)$ is in; but since f is real-valued, $f(S)$ is a compact set of real numbers, and therefore must have a max and a min, by a result in Chapter 1.

Proof:

Let (y_k) be a sequence in $f(S)$. So, for every k, there is an $x_k \in S$ such that $y_k = f(x_k)$. Since $(x_k), k = 1, 2, \ldots$ is a sequence in the compact set S, it has a subsequence $(x_{m(k)})$ that converges to a point x in S. Since f is continuous, the image sequence $(f(x_{m(k)}))$ converges to $f(x)$, which is obviously in $f(S)$.

So we've found a convergent subsequence $(y_{m(k)}) = (f(x_{m(k)}))$ of (y_k); hence $f(S)$ is compact. This means the set $f(S)$ of real numbers is closed and bounded; so, it has at least one maximum and at least one minimum.

Example: Let $p_1 = p_2 = 1$, $I = 10$. Maximize $U(x_1, x_2) = x_1 x_2$ subject to the budget constraint. Here, the budget set is compact, since the prices are positive. We can see that the image of the budget set S under the function U (or the range of U), is $U(S) = [0, 25]$. This is compact, and so U attains a max (25) and a min (0) on S.

The fact that $U(S)$ is in fact an interval has to do with another property of continuity of the objective: such functions preserve connectedness in addition to preserving compactness of the set S, and here, the budget set is a connected set.

In class, examples and applications of Weierstrass' theorem to utility maximization and cost minimization from Rangarajan Sundaram:

Examples:

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$. Domain is closed but not bounded, f is continuous.
2. $f : (0, 1) \to \mathbb{R}, f(x) = x$. $(0, 1)$ is bounded but not closed, f continuous.
3. $f : [-1, 1] \to \mathbb{R}, f(x) = x$ if $x \in (-1, 1)$, $f(x) = 0$ otherwise. f has two discontinuities, domain is compact.

On the other hand, example (4) has a non-compact set and an everywhere discontinuous function, yet a max and min exist. $f : \mathbb{R}_{++} \to \mathbb{R}, f(x) = 1$ if x is rational, $f(x) = 0$ otherwise.

Suppose x is irrational. Take a sequence (x_n) of rational numbers converging to x. Then $f(x_n)$ converges to 1 while $f(x) = 0$. So f is not continuous at any irrational number. By a similar argument, f is not continuous at any rational x either.

3 Optimization

In this section, we will explore the methods for determining the optimal values of an objective function, both in the presence and absence of equality constraints. Before delving into these optimization techniques, it is essential to revisit some foundational concepts, specifically the definitions of limits and derivatives. These fundamental mathematical tools are crucial for understanding how to approach optimization problems, as they provide the necessary framework for analyzing the behavior of functions and identifying points of maximum or minimum value.

The definition of the derivatives uses the notion of the limit of a function. As defined earlier, x is *an accumulation point* or *limit point* of a set S if for every $\epsilon > 0$, Consider a neighborhood $B(x;\epsilon)$ that includes a point from the set S other than x itself. Let's begin by examining a function $g : S \to \mathbb{R}$, where $S \subset \mathbb{R}$. Assume x is a limit point of S. We say that the limit of $g(y)$ as y approaches x is L, denoted as $\lim_{y\to x} g(y) = L$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|y - x| < \delta$ and $y \neq x$, it follows that $|g(y) - L| < \epsilon$.

This definition can also be interpreted as follows: for any sequence (x_n) converging to x, where $x_n \neq x$ for all n, the sequence $g(x_n)$ converges to L. In other words, the limit $\lim_{y\to x} g(y) = L$ means that for any sequence (x_n) approaching x, the corresponding values of the function $g(x_n)$ approach L, provided that none of the x_n are equal to x.

Definition 3.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is defined to be differentiable at a point x if there exists a real number $a \in \mathbb{R}$ such that*

$$\lim_{y\to x} \left(\frac{f(y) - f(x)}{y - x} - a \right) = 0$$

Here, when we say the limit is 0 as $y \to x$, we require this to hold for all sequences (y_n) that converge to x. The number a is unique and corresponds to the slope of the tangent line to the graph of f at the point x. This number is denoted by $f'(x)$. The expression can also be rewritten as:

$$\lim_{y\to x} \left(\frac{f(y) - f(x) - a(y - x)}{y - x} \right) = 0$$

It's important to observe that this indicates the numerator approaches zero more rapidly than the denominator.

Additionally, this implies that for values of y close to x, the function $f(y)$ can be

well-approximated by the expression $f(x)+a(y-x)$. Moreover, the point $(y, f(x)+a(y-x))$ lies on the tangent line that passes through $(x, f(x))$. This can be visualized through a graphical representation, as shown in class.

This method of defining differentiability can be extended to more general functions.

Definition 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. The function f is said to be differentiable at a point x if there exists an $m \times n$ matrix A such that*

$$\lim_{y \to x} \frac{\|f(y) - f(x) - A(y-x)\|}{\|y - x\|} = 0$$

In the case of a single variable, the existence of the scalar a corresponds to the existence of a tangent line. In the more general case, the existence of the matrix A implies the existence of tangent planes to the graphs of the m component functions $f = (f_1, \ldots, f_m)$, where each component function maps from $\mathbb{R}^n$ to $\mathbb{R}$. In other words, this definition is related to finding the 'best' linear approximation to the function f at the point x.

To better understand this, consider setting $h = y - x$ in the definition above, so that $y = x + h$. Then, in the case of a single variable, the numerator $f(x + h)$ is approximated by the affine function $f(x) + ah = f(x) + f'(x)h$. In the general case, $f(x + h)$ is approximated by the affine function $f(x) + Ah$.

It can be shown that, with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, the matrix A is equal to $Df(x)$, the $m \times n$ matrix of partial derivatives of f evaluated at x. The matrix $Df(x)$ is known as the derivative of f at x:

$$Df(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}$$

Here, the i-th row contains the partial derivatives of the component function f_i with respect to the variables $x_1, x_2, \ldots, x_n$. Since there are m component functions, there are m rows of partial derivatives.

According to this convention, if $f : \mathbb{R}^n \to \mathbb{R}$, then $Df(x)$ is a row matrix:

$$Df(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$$

On the other hand, if $f : \mathbb{R} \to \mathbb{R}^m$, this means that f has m component functions $f_1, \ldots, f_m$, each mapping from $\mathbb{R}$ to $\mathbb{R}$. In this case, the derivative matrix $Df(x)$

has m rows, one for each component function, but only one column:

$$Df(x) = \begin{pmatrix} f_1'(x) \\ \vdots \\ f_m'(x) \end{pmatrix}$$

where the superscript T denotes the transpose; we have written it this way to save space.

Next, let's recall the concept of a partial derivative. Let $f : \mathbb{R}^n \to \mathbb{R}$. The partial derivative $\frac{\partial f(x)}{\partial x_j}$ is defined as a number a_j that satisfies

$$\lim_{t \to 0} \frac{f(x_1, \ldots, x_j + t, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_n) - a_j t}{t} = 0$$

In other words:

$$\frac{\partial f(x)}{\partial x_j} = a_j = \lim_{t \to 0} \frac{f(x_1, \ldots, x_j + t, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_n)}{t}$$

We also want to discuss partial derivatives and directional derivatives using more concise notation.

Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the unit vector in $\mathbb{R}^n$ along the j-th axis. Then,

$$\frac{\partial f(x)}{\partial x_j} = \lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t}$$

This means that the partial derivative of f with respect to x_j, evaluated at the point x, is essentially the derivative of a single-variable function. We take the $(n-1)$-dimensional surface of the function f and slice it parallel to the j-th axis, such that the point x lies on this slice or plane. We then obtain a function defined on this plane, and its derivative is the relevant partial derivative.

To be more precise about this one-variable function defined on the slice or plane, note that the single variable $t \in \mathbb{R}$ is first mapped to a vector $x + te_j \in \mathbb{R}^n$, and then that vector is mapped to a real number $f(x + te_j)$. So, let $\phi : \mathbb{R} \to \mathbb{R}^n$ be defined by $\phi(t) = x + te_j$ for all $t \in \mathbb{R}$. Then the one-variable function we are interested in is $g : \mathbb{R} \to \mathbb{R}$, defined by $g(t) = f(\phi(t))$ for all $t \in \mathbb{R}$; this is the composition of f and ϕ.

In addition to slicing the surface of functions that map from $\mathbb{R}^n$ to $\mathbb{R}$ in the directions of the coordinate axes, we can also slice them in any direction and obtain a function defined on the slicing plane. This is the concept of a directional derivative.

Recall that if $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, then the set of all points that can be written as $x + th$, for some $t \in \mathbb{R}$, forms a line through x in the direction of h.

Definition 3.3. *The directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ in the direction of $h \in \mathbb{R}^n$, denoted by $Df(x; h)$, is defined as*

$$\lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t}$$

If $t \to 0^+$ is replaced by $t \to 0$, we obtain the two-sided directional derivative.

We now demonstrate that, with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, the matrix A in the definition of differentiability (Definition 7) is equal to $Df(x)$. To see this, consider the slightly less general case of a function $f : \mathbb{R}^n \to \mathbb{R}$. If f is differentiable at x, there exists a $1 \times n$ matrix $A = (a_1, \ldots, a_n)$ satisfying the definition above, i.e.,

$$\lim_{h \to \theta} \frac{\|f(x + h) - f(x) - Ah\|}{\|h\|} = 0$$

In particular, this must hold if we choose $h = (0, \ldots, 0, t, 0, \ldots, 0)$ with $h_j = t \to 0$. That is,

$$\lim_{t \to 0} \frac{\|f(x_1, \ldots, x_j + t, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_n) - a_j t\|}{t} = 0$$

But from the limit on the left-hand side, we know that a_j must equal the partial derivative $\frac{\partial f(x)}{\partial x_j}$.

For a function $F : \mathbb{R}^n \to \mathbb{R}^m$, we will use f_i to denote the i-th component function. Then,

$$\frac{\partial f_i(x)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(x_1, \ldots, x_j + t, \ldots, x_n) - f_i(x_1, \ldots, x_j, \ldots, x_n)}{t}$$

A function is differentiable on a set S if it is differentiable at every point in S. A function f is continuously differentiable if it is differentiable and all its partial derivatives are continuous.

3.1 Interior Optima

Definition 3.4. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. A point z is called a local maximum (or local minimum) of f on a set $S \subset \mathbb{R}^n$ if there exists some $\epsilon > 0$ such that for all $x \in B(z; \epsilon) \cap S$, we have $f(z) \geq f(x)$ (or $f(z) \leq f(x)$, respectively). Here, $B(z; \epsilon)$ represents a neighborhood around z.*

It's important to note that $B(z; \epsilon)$ is intersected with S because the neighborhood $B(z; \epsilon)$ might not lie entirely within the set S. However, if z is in the interior of S, this intersection can be disregarded. In this case, z is said to be an interior local

maximum or minimum of f on S if there exists some $\epsilon > 0$ such that $f(z) \geq f(x)$ (or $f(z) \leq f(x)$) for all $x \in B(z; \epsilon)$.

First-Order Necessary Condition

We now present a necessary condition for a point to be an interior local maximum or minimum; specifically, this condition states that the derivative at such a point must be zero. The reasoning behind this is that if the derivative were not zero, one could slightly move away from the point and either increase or decrease the value of the function.

Theorem 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subset \mathbb{R}^n$, and let x^* be a local maximum or minimum of f on S, where x^* lies in the interior of S. If f is differentiable at x^*, then the derivative $Df(x^*)$ must equal the zero vector θ.
Here, $\theta = (0, \ldots, 0)$ represents the origin, and $Df(x^*)$ is the gradient vector given by $Df(x^*) = \left(\frac{\partial f(x^*)}{\partial x_1}, \ldots, \frac{\partial f(x^*)}{\partial x_n}\right)$.

Proof:

Proof. Let x^* be an interior local maximum (a similar proof holds for a minimum).

Step 1: First, consider the case where $n = 1$. Take any sequence $\{y_k\}$ where $y_k < x^*$ and $y_k \to x^*$, and another sequence $\{z_k\}$ where $z_k > x^*$ and $z_k \to x^*$. Since x^* is a local maximum, for k large enough, we have:

$$\frac{f(z_k) - f(x^*)}{z_k - x^*} \leq 0 \leq \frac{f(y_k) - f(x^*)}{y_k - x^*}$$

Taking the limits of these inequalities, and considering that $(-1, 0]$ and $[0, 1)$ are closed sets, we obtain:

$$f'(x^*) \leq 0 \leq f'(x^*)$$

Thus, $f'(x^*) = 0$.

Step 2: Now consider the case where $n > 1$. For any direction along the j-th axis, define a function $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(x^* + te_j)$, where e_j is the unit vector along the j-th axis. Notice that $g(0) = f(x^*)$.

Since x^* is a local maximum of f, we have $f(x^*) \geq f(x^* + te_j)$ for t sufficiently small. This implies that $g(0)$ is a local interior maximum, meaning $g'(0) = 0$.

By the Chain Rule, we get:

$$g'(0) = Df(\phi(0)) \cdot D\phi(0) = Df(x^*) \cdot e_j = \frac{\partial f(x^*)}{\partial x_j}$$

This shows that the partial derivative with respect to x_j is zero.

Note: While this condition is necessary, it is not sufficient for a local maximum or minimum. For example, the function $f(x) = x^3$ has a zero derivative at $x = 0$, yet $x = 0$ is not a local extremum.

□

3.2 Finding Global Optima

To find global optima, we can use the results from the above theorem. A global maximum is also a local maximum, which may either be in the interior of the feasible set S or on its boundary. If we are certain that the global maximum lies in the interior, it must be among the solutions to the first-order necessary condition for interior optima. Therefore, by solving this condition, evaluating $f(x)$ at each solution, and comparing these values to those on the boundary, we can identify the global maximum at the point z (or points) where $f(z)$ is the highest.

Second-Order Conditions

Definition 3.5. *A point x is called a strict local maximum of f on S if there exists some $\epsilon > 0$ such that $f(x) > f(y)$ for all $y \in B(x; \epsilon) \cap S$ with $y \neq x$.*

We represent the Hessian matrix or the matrix of second derivatives of f by $D^2 f$.

Theorem 3.2. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function on $S \subset \mathbb{R}^n$, and x is an interior point of S.

1. (*Necessary Condition*) If f has a local maximum (or local minimum) at x, then $D^2 f(x)$ is negative semidefinite (or positive semidefinite).
2. (*Sufficient Condition*) If $Df(x) = \theta$ and $D^2 f(x)$ is negative definite (or positive definite) at x, then x is a strict local maximum (or minimum) of f on S.

These results follow from the second-order Taylor series approximation around the local maximum or minimum. For instance,

$$f(x) = f(x^*) + Df(x^*) \cdot (x - x^*) + \frac{1}{2}(x - x^*)^\top D^2 f(x^*)(x - x^*) + R_2(x - x^*)$$

where $R_2(\cdot)$ is a remainder term of order smaller than two. If x^* is an interior local maximum or minimum, then $Df(x^*) = 0$ (a vector of zeros), so the sign of the quadratic form in the second-order term will determine the sign of $f(x) - f(x^*)$.

Examples to Illustrate:

- (i) The second-order necessary condition (SONC) is not sufficient: $f(x) = x^3$.
- (ii) Semidefiniteness cannot be replaced by definiteness: $f(x) = x^4$.
- (iii) These conditions are for local, not global optima: $f(x) = 2x^3 - 3x^2$.
- (iv) Strategy for using the conditions to identify global optima: $f(x) = 4x^3 - 5x^2 + 2x$ on $S = [0, 1]$.

4 Optimization with Equality constraints

We explored the existence of both local and global optima for a function, as well as the first and second order necessary conditions. Moving forward, we will examine optimization problems with constraints, focusing on equality constraints in this chapter and addressing inequality constraints in the following chapter.

4.1 Introduction

Consider the problem, we are given an objective function $f : \mathbb{R}^n \to \mathbb{R}$ that we aim to either maximize or minimize, subject to k constraints. Specifically, there are k functions $g_1 : \mathbb{R}^n \to \mathbb{R}, g_2 : \mathbb{R}^n \to \mathbb{R}, \ldots, g_k : \mathbb{R}^n \to \mathbb{R}$, and the problem is to:

$$\text{Maximize } f(x) \text{ over all } x \in \mathbb{R}^n \text{ such that } g_1(x) = 0, \ldots, g_k(x) = 0.$$

We can express this problem more compactly by collecting the constraint functions (viewed as components) into a single function $g : \mathbb{R}^n \to \mathbb{R}^k$, where $g(x) = (g_1(x), \ldots, g_k(x))$. The optimization problem can then be restated as:

$$\text{Maximize } f(x) \text{ over all } x \in \mathbb{R}^n \text{ such that } g(x) = \theta_{1\times k}.$$

The Theorem of Lagrange provides necessary conditions for a local optimum x^*. By "local optimum," we mean that the value $f(x^*)$ is either a maximum or minimum when compared to the values of $f(x)$ for all x within some open set U containing x^*, where x satisfies the k constraints. Thus, the theorem deals with the necessary conditions for a maximum or minimum of $f(x)$ over all $x \in S$, where $S = U \cap \{x \in \mathbb{R}^n \mid g(x) = \theta\}$ for some open set U. We are restricting to the open set U as we are focusing on the interior optima.

To illustrate the principle of *no arbitrage* underlying a maximum, consider the following example. While a more general illustration with more than one constraint would require some additional tools from the theory of linear inequalities, the key idea is that the Lagrange multiplier captures how the constraint is distributed across the variables. We will now see the idea of Lagrange theorem through a simple example of utility maximization.

Example Suppose x^* solves Maximize $U(x_1, x_2)$ subject to $I - p_1x_1 - p_2x_2 = 0$, and suppose $x^* \gg \theta$.

In this scenario, reallocating a small amount of income from one good to another does not increase utility. Let an increment in income $dI > 0$ be shifted from good 1 to good 2. Then $dx_1 = \frac{dI}{p_1} > 0$ and $dx_2 = -\frac{dI}{p_1} < 0$. This reallocation satisfies the

budget constraint, as:

$$p_1(x_1 + dx_1) + p_2(x_2 + dx_2) = I.$$

The change in utility is:

$$dU = U_1 dx_1 + U_2 dx_2 = \left(\frac{U_1}{p_1} - \frac{U_2}{p_2}\right) dI \leq 0,$$

since the change in utility cannot be positive at a maximum. Therefore:

$$\frac{U_1}{p_1} - \frac{U_2}{p_2} \leq 0 \quad (1).$$

Similarly, if income $dI > 0$ is shifted from good 2 to good 1, it does not increase utility, leading to:

$$-\frac{U_1}{p_1} + \frac{U_2}{p_2} \geq 0 \quad (2).$$

From inequalities (1) and (2), we conclude:

$$\frac{U_1(x^*)}{p_1} = \frac{U_2(x^*)}{p_2} = \lambda^* \quad (3).$$

This implies that the marginal utility per unit of income is equalized across goods at the optimum, i.e., $\frac{U_1(x^*)}{p_1} = \frac{U_2(x^*)}{p_2}$. Consequently, we have $U_1(x^*) = \lambda^* p_1$ and $U_2(x^*) = \lambda^* p_2$. Along with the budget constraint $p_1 x_1^* + p_2 x_2^* = I$, these are the first-order necessary conditions (FONC) of the Lagrangean function:

$$\text{Maximize } L(x, \lambda) = U(x_1, x_2) + \lambda[I - p_1 x_1 - p_2 x_2].$$

Now consider the case of the more general problem, suppose $F : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}$, and suppose x^* solves Maximize $F(x)$ subject to $c - G(x) = 0$. Consider a change dx in x^* that respects the constraint $G(x) = c$. That is:

$$dG = G_1 dx_1 + G_2 dx_2 = 0.$$

Therefore:

$$G_1 dx_1 = -G_2 dx_2 = dc, \quad \text{so } dx_1 = \frac{dc}{G_1},\ dx_2 = -\frac{dc}{G_2}.$$

If $dc > 0$, then our change dx implies $dx_1 > 0$, $dx_2 < 0$. At the maximum x^*, the

function F does not increase. Hence:

$$dF = F_1 dx_1 + F_2 dx_2 \leq 0, \quad \text{or } \left(\frac{F_1}{G_1} - \frac{F_2}{G_2}\right) dc \leq 0.$$

Similarly, it can be shown that $\left(\frac{F_1}{G_1} - \frac{F_2}{G_2}\right) \geq 0$. Therefore:

$$\frac{F_1(x^*)}{G_1(x^*)} = \frac{F_2(x^*)}{G_2(x^*)} = \lambda^* \quad (4).$$

Note that we have assumed $G_1(x^*)$ and $G_2(x^*)$ are not both zero at x^*. This is known as the constraint qualification.

Again, note that equation (4) can be derived as the first-order necessary condition for the problem:

$$\text{Maximize } L(x, \lambda) = F(x) + \lambda[c - G(x)].$$

Now, let's explore the economic intuition behind the equations and conditions we derived for the optima by revisiting the same utility maximization example we discussed earlier. At the optimum (x^*, λ^*), suppose you increase income by ΔI.

Buying more of good 1 increases utility by approximately $\frac{U_1(x^*)}{p_1}\Delta I$. Similarly, buying more of good 2 increases utility by approximately $\frac{U_2(x^*)}{p_2}\Delta I$.

At the optimum, $\frac{U_1(x^*)}{p_1} = \frac{U_2(x^*)}{p_2} = \lambda^*$. Therefore, in either case, utility increases by $\lambda^* \Delta I$. Thus, λ^* represents the rate at which the objective (in this case, utility) increases at the optimum when the constraint is relaxed slightly.

This interpretation extends to the more general case: If $G(x) = c$ and c is increased by Δc, suppose only x_1 is increased. Then:

$$\Delta G = g_1 dx_1 = \Delta c, \quad \text{so } dx_1 = \frac{\Delta c}{G_1}.$$

At x^*, F increases by:

$$dF = F_1 dx_1 = \frac{F_1(x^*)}{G_1(x^*)}\Delta c = \lambda^* \Delta c.$$

If instead x_2 is changed, F increases by $dF = F_2 dx_2 = \frac{F_2(x^*)}{G_2(x^*)}\Delta c = \lambda^* \Delta c$.

4.2 The Theorem of Lagrange

The setup is as follows. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the objective function, and let $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, k$ where $k < n$ be the constraint functions.

Define $g : \mathbb{R}^n \to \mathbb{R}^k$ as the function given by $g(x) = (g_1(x), \ldots, g_k(x))$.

The gradient of f at x is denoted by:

$$Df(x) = \left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \right).$$

The gradient of g at x is represented by the $k \times n$ matrix:

$$Dg(x) = \begin{pmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k(x)}{\partial x_1} & \cdots & \frac{\partial g_k(x)}{\partial x_n} \end{pmatrix} = \begin{pmatrix} Dg_1(x) \\ \vdots \\ Dg_k(x) \end{pmatrix}.$$

The matrix $Dg(x)$ is a $k \times n$ matrix, representing the Jacobian of the constraint functions. The following theorem provides a necessary condition for x^* to be a local maximum or minimum under the given constraints. Specifically, x^* is a local maximum (or minimum) of the function f on the constraint set $\{x \in \mathbb{R}^n \mid g_i(x) = 0 \text{ for } i = 1, \ldots, k\}$ if $f(x^*) \geq f(x)$ (or $f(x^*) \leq f(x)$) for all x within some open set U containing x^*, where x satisfies the constraints $g_i(x) = 0$ for $i = 1, \ldots, k$. Therefore, x^* is a maximum (or minimum) on the set $S = U \cap \{x \in \mathbb{R}^n \mid g_i(x) = 0 \text{ for } i = 1, \ldots, k\}$.

Theorem 4.1 (Theorem of Lagrange)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, k$, with $k < n$, be continuously differentiable (C^1) functions. Suppose x^* is a local maximum or minimum of f on the set $S = U \cap \{x \in \mathbb{R}^n \mid g_i(x) = 0 \text{ for } i = 1, \ldots, k\}$, for some open set $U \subseteq \mathbb{R}^n$. Then, there exist real numbers μ^* and $\lambda_1^*, \ldots, \lambda_k^*$, not all zero, such that:

$$\mu^* Df(x^*) + \sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = \theta_{1 \times n}.$$

Moreover, if the rank of the Jacobian matrix $Dg(x^*)$ is k, then we can normalize by setting $\mu^* = 1$.

Notes:

1. The condition $k < n$ ensures that the constraint set is non-trivial. Consider how a linear constraint of the form $a \cdot x = 0$ (or $a \cdot x = c$) defines a set of points that form an $(n-1)$-dimensional subspace. The $1 \times n$ matrix a has rank 1, leading to a nullity of $(n-1)$. Similarly, each constraint $g_i(x) = 0$ for $i = 1, \ldots, n$ defines an $(n-1)$-dimensional manifold. The intersection of these manifolds forms the

constraint set, which is an $(n-k)$-dimensional space. To ensure that this set is non-empty and contains more than one point, we require $k < n$.

2. The condition $\text{rank}(Dg(x^*)) = k$ is referred to as the constraint qualification (CQ). The first part of the theorem states that at a local maximum or minimum, assuming the continuous differentiability of f and g_i for $i = 1, \ldots, k$, the vectors $Df(x^*)$, $Dg_1(x^*)$, $\ldots$, $Dg_k(x^*)$ are linearly dependent. The CQ essentially assumes that the vectors $Dg_1(x^*)$, $\ldots$, $Dg_k(x^*)$ are linearly independent. If this condition holds, the equation:

$$\mu Df(x^*) + \sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = \theta$$

implies that μ cannot be zero. If $\mu = 0$, then we would have $\sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = \theta$. Given the linear independence of the gradients $Dg_1(x^*)$, $\ldots$, $Dg_k(x^*)$, this would imply that $\lambda_i^* = 0$ for all $i = 1, \ldots, k$, which is not possible since not all λ_i^* can be zero. Thus, if the CQ holds, $\mu \neq 0$, and we can divide through by μ to normalize the equation.

3. In most practical applications, the CQ holds, and it is standard to check whether the CQ is satisfied before proceeding with the analysis. If the CQ is satisfied, the condition:

$$\mu^* Df(x^*) + \sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = \theta$$

subsumes the following n equations (with $\mu = 1$):

$$\frac{\partial f(x^*)}{\partial x_j} + \sum_{i=1}^{k} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0, \quad \text{for } j = 1, \ldots, n.$$

This leads to the standard method for finding maxima or minima under equality constraints by setting up a Lagrangean function:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x),$$

and solving the first-order necessary conditions (FONC):

$$\frac{\partial L(x, \lambda)}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0, \quad \text{for } j = 1, \ldots, n,$$

$$\frac{\partial L(x, \lambda)}{\partial \lambda_i} = g_i(x) = 0, \quad \text{for } i = 1, \ldots, k.$$

These form a system of $(n+k)$ equations in $(n+k)$ variables $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k$.

Limitations

The procedure described above, while powerful, may not always yield the correct results. There are a few situations where it may fail:

1. **Global Optima May Not Exist:** In some cases, a global optimum might not exist. For example, consider the problem:

$$\text{Maximize } f(x, y) = x^3 + y^3 \text{ subject to } g(x, y) = x - y = 0.$$

(You may verify this example yourself.)

2. **Violation of the Constraint Qualification (CQ) at the Optimum:** If the CQ is violated at the point of interest, the first-order necessary conditions (FONC) may not hold at the global optimum. For instance, consider the following problem:

$$\text{Maximize } f(x, y) = -y \text{ subject to } g(x, y) = y^3 - x^2 = 0.$$

(You may verify this example as well.)

In such cases, the standard Lagrangian approach might not apply, and alternative methods may need to be considered to correctly identify the global optimum.

Second-Order Conditions

In constrained optimization problems, second-order conditions are crucial in determining whether a point is a local maximum or minimum. These conditions are characterized by the definiteness or semi-definiteness of the Hessian matrix of the Lagrangean function, which is the appropriate function to analyze in this context.

Unlike in unconstrained optimization, where we check the curvature of the objective function over all possible directions, in constrained optimization, we only need to consider directions that satisfy the constraints. Specifically, second-order conditions inform us about the curvature of the objective function around a candidate optimal point, x^*, and how the function behaves when we move from x^* to a nearby point that also satisfies the constraints.

When moving from x^* to a nearby point x, the movement is restricted by the constraints. This means that any such move must ensure that all constraint functions $g_i(x) = 0$ are still satisfied. Mathematically, this requires that the differential $dg_i(x)$ satisfies:

$$dg_i(x) = Dg_i(x) \cdot dx = 0,$$

where dx is a vector representing the difference $x - x^*$, and $Dg_i(x)$ is the gradient of

the i-th constraint function. The condition $Dg_i(x) \cdot dx = 0$ implies that the vector dx must be orthogonal to the gradient of $g_i(x)$. Therefore, it is sufficient to evaluate the appropriate quadratic form (associated with the Hessian) only for those vectors dx that are orthogonal to all gradients of the constraint functions.

To see this more clearly, consider parameterizing the curve that describes a constraint by setting $g_i(x(t)) = 0$. Using the Chain Rule, we have:

$$Dg_i(x(t)) \cdot \frac{dx(t)}{dt} = 0.$$

Here, $\frac{dx(t)}{dt}$ is the tangent vector to the curve $x(t)$. The equation above implies that $Dg_i(x(t))$, the gradient of $g_i(x(t))$, is orthogonal to the tangent vector of the curve. This geometric insight has applications, such as in utility maximization for two goods, where the first-order condition $MRS_{xy} = \frac{p_x}{p_y}$ arises naturally from this framework.

In second-order conditions, we examine the definiteness or semi-definiteness of the Hessian matrix of the Lagrangean function, $D^2L(x^*, \lambda^*)$, with respect to all vectors dx that are orthogonal to the gradients of each constraint function. This allows us to approximate the behavior of the function near x^* while still satisfying the constraints.

Given that the Lagrangean function is defined as:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x),$$

the Hessian matrix $D^2L(x, \lambda)$ is given by:

$$D^2L(x, \lambda) = D^2f(x) + \sum_{i=1}^{k} \lambda_i D^2g_i(x),$$

where the Hessian matrices of $f(x)$ and $g_i(x)$ are denoted as:

$$D^2f(x) = \begin{pmatrix} f_{11}(x) & \dots & f_{1n}(x) \\ \vdots & \ddots & \vdots \\ f_{n1}(x) & \dots & f_{nn}(x) \end{pmatrix},$$

$$D^2g_i(x) = \begin{pmatrix} g_{i11}(x) & \dots & g_{i1n}(x) \\ \vdots & \ddots & \vdots \\ g_{in1}(x) & \dots & g_{inn}(x) \end{pmatrix}.$$

Thus, the Hessian matrix of the Lagrangean function is:

$$D^2L(x,\lambda) = \begin{pmatrix} f_{11}(x) + \sum_{i=1}^{k} \lambda_i g_{i11}(x) & \dots & f_{1n}(x) + \sum_{i=1}^{k} \lambda_i g_{i1n}(x) \\ \vdots & \ddots & \vdots \\ f_{n1}(x) + \sum_{i=1}^{k} \lambda_i g_{in1}(x) & \dots & f_{nn}(x) + \sum_{i=1}^{k} \lambda_i g_{inn}(x) \end{pmatrix}.$$

This matrix, $D^2L(x,\lambda)$, represents the second derivatives of the Lagrangean function with respect to the variables x. It is a symmetric matrix, which means that it is appropriate to work with its associated quadratic form.

At a given point $x^* \in \mathbb{R}^n$, the matrix $Dg(x^*)$ is given by:

$$Dg(x^*) = \begin{pmatrix} Dg_1(x^*) \\ \vdots \\ Dg_k(x^*) \end{pmatrix}.$$

The set of all vectors dx that are orthogonal to all the gradients of the constraint functions at x^* forms the Null Space of $Dg(x^*)$. This Null Space, denoted as $N(Dg(x^*))$, is defined as:

$$N(Dg(x^*)) = \{dx \in \mathbb{R}^n \mid Dg(x^*) \cdot dx = \theta_{k\times 1}\}.$$

Theorem 4.2. Suppose there exists (x^*, λ^*) such that $\text{Rank}(Dg(x^*)) = k$ and:

$$Df(x^*) + \sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = \theta.$$

Then:

1. **(Necessary Condition):** If f has a local maximum (or minimum) on S at the point x^*, then for all $dx \in N(Dg(x^*))$:

$$dx^T D^2L(x^*, \lambda^*)dx \leq 0 \quad (\text{or } \geq 0).$$

2. **(Sufficient Condition):** If for all $dx \in N(Dg(x^*))$, $dx \neq \theta$, we have:

$$dx^T D^2L(x^*, \lambda^*)dx < 0 \quad (\text{or } > 0),$$

then x^* is a strict local maximum (or strict local minimum) of f on S.

4.3 The Implicit Function Theorem

Theorem 4.3. Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is continuously differentiable (i.e., C^1), and suppose that there exists a point $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $F(x^*, y^*) = 0$. Additionally, assume that the Jacobian matrix $DF_y(x^*, y^*)$ (the derivative of F with respect to y) has full rank m. Then, there exist open sets U containing x^* and V containing y^*, and a continuously differentiable function $f : U \to V$ such that:

$$F(x, f(x)) = 0 \quad \text{for all } x \in U.$$

Moreover, the derivative of f at x^* is given by:

$$Df(x^*) = -[DF_y(x^*, y^*)]^{-1} DF_x(x^*, y^*),$$

where DF_x and DF_y represent the partial derivatives of F with respect to x and y, respectively.

It is important to note that we could consider the equation $F(x, y) = c$ for some constant $c \in \mathbb{R}^m$ without altering the theorem's conclusions. Although the proof of this theorem involves complex mathematical concepts and will not be covered in this course, a simpler proof for the case where $n = m = 1$ is provided at the end of this chapter.

To understand the result, consider differentiating the equation $F(x^*, f(x^*)) = 0$ using the Chain Rule:

$$DF_x(x^*, y^*) + DF_y(x^*, y^*) Df(x^*) = 0.$$

From this equation, we can derive the expression for $Df(x^*)$ as:

$$Df(x^*) = -[DF_y(x^*, y^*)]^{-1} DF_x(x^*, y^*).$$

This is a direct result of the Implicit Function Theorem.

To provide a more detailed interpretation, consider the composition of functions. Define $h(x) = (x, f(x))$, then the derivative $Dh(x)$ is given by:

$$Dh(x) = \begin{pmatrix} I \\ Df(x) \end{pmatrix},$$

where I is the identity matrix. On the other hand, the derivative of F can be written as:

$$DF(h(x)) = \left(DF_x(h(x)) \mid DF_y(h(x))\right),$$

where the vertical bar denotes the partitioning of the matrix. Matrix multiplication using these partitions leads directly to the earlier expression for $Df(x^*)$.

Interpretation with Indifference Curve:

To motivate the Implicit Function Theorem, consider an economic example involving an indifference curve. Suppose we have a utility function F defined over two goods, with the utility level fixed at some value c. Thus, the function $F(x, y) = c$ represents an indifference curve. Let (x^*, y^*) be a point on this curve, meaning that $F(x^*, y^*) = c$.

According to the Implicit Function Theorem, under its assumptions, there exists a function f such that, for any x close to x^*, there is a unique y such that $F(x, y) = c$. This implies that y can be expressed as a function of x, i.e., $y = f(x)$. This establishes a functional relationship between x and y on the indifference curve.

Geometric Interpretation:

You can visualize this by drawing an indifference curve corresponding to $F(x, y) = c$. As x varies slightly from x^*, y adjusts accordingly to maintain the same level of utility c, and this adjustment is described by the function $f(x)$.

Moreover, the theorem provides a formula for the derivative of the implicit function:

$$f'(x^*) = -\frac{F_x}{F_y},$$

where F_x and F_y are the partial derivatives of F with respect to x and y, respectively, evaluated at (x^*, y^*). This result reveals that the marginal rate of substitution (MRS) between the two goods, which is represented by the left-hand side, equals the ratio of the marginal utilities of the goods, which is represented by the right-hand side.

An important assumption in applying this theorem is that the partial derivative F_y evaluated at (x^*, y^*) is non-zero. This ensures that the function f exists and is well-defined.

To find the derivative of the implicit function, remember that starting from $F(x, y) = c$, we "totally differentiate" the equation to obtain:

$$F_x\, dx + F_y\, dy = 0.$$

Rearranging terms, we find:

$$\frac{dy}{dx} = -\frac{F_x}{F_y},$$

which gives us the derivative of the implicit function $f'(x^*)$.

5 Linear Algebra

5.1 Introduction

In mathematics, the concept of a vector space is foundational in many areas, including linear algebra, functional analysis, and many applied fields. To fully understand vector spaces, it is crucial to first grasp the notion of a field, as vector spaces are defined over fields.

The goal is to extend the familiar concepts we encounter in a two-dimensional plane, where we can add vectors and scale them using real numbers (scalars), in a way that these operations adhere to specific rules. In a plane, vector addition and scalar multiplication follow well-defined patterns that we aim to generalize to more abstract settings. While in many cases, our scalars will be real numbers, the concept can be broadened to allow the set of scalars to be any mathematical structure known as a field, which supports addition, subtraction, multiplication, and division (except by zero). This generalization will enable us to explore vector-like structures beyond the simple 2D or 3D spaces, applying the same principles in more complex or higher-dimensional contexts. Further discussion on the nature and properties of vectors will follow.

Definition 5.1. *A field F is a set equipped with two operations, addition and multiplication, which satisfy certain axioms. The critical property of a field is that it allows for the addition, subtraction, multiplication, and division of any two elements, with the exception that division by zero is not defined. Formally, for any $a, b \in F$:*

- $a + b \in F$
- $a - b \in F$
- $ab \in F$
- $\frac{a}{b} \in F$ *(provided $b \neq 0$)*

These operations must satisfy properties such as commutativity, associativity, distributivity, the existence of an additive identity (0), and a multiplicative identity (1).

5.2 Vector Space

Definition 5.2. *A vector space V over a field F is a set whose elements, called vectors, can be added together and multiplied by scalars from F. Formally, a vector space consists of:*

- *A set V (elements of V are called vectors).*

- *A field F (elements of F are called scalars).*
- *Two operations:*
    1. ***Vector addition**: An operation that takes any two vectors $v, w \in V$ and produces a third vector $v + w \in V$.*
    2. ***Scalar multiplication**: An operation that takes a scalar $c \in F$ and a vector $v \in V$, and produces a new vector $cv \in V$.*

These operations must satisfy the following axioms:

1. ***Associativity of vector addition**: For all $u, v, w \in V$, the equation $(u + v) + w = u + (v + w)$ holds.*
2. ***Existence of a zero vector**: There exists an element $0 \in V$ such that for every $u \in V$, $u + 0 = u$.*
3. ***Existence of additive inverses**: For every $u \in V$, there exists a vector $-u \in V$ such that $u + (-u) = 0$.*
4. ***Associativity of scalar multiplication**: For any $a, b \in F$ and any $u \in V$, $(ab)u = a(bu)$.*
5. ***Distributivity**: For all $a, b \in F$ and $u, v \in V$, $(a + b)u = au + bu$ and $a(u + v) = au + av$.*
6. ***Unitarity**: For every vector $u \in V$, $1u = u$, where 1 denotes the multiplicative identity in F.*

In words: A vector space (or linear space) over a field F is a set V equipped with two operations: vector addition and scalar multiplication. These operations must satisfy specific axioms (such as commutativity, associativity, and distributivity) for all vectors $\mathbf{v_1}, \mathbf{v_2}$ in V and scalars c in F.

Examples of Vector Spaces

Infinite Sequences of Real Numbers

Let V be the set of all infinite sequences of real numbers $(x_1, x_2, x_3, \dots)$, and let the field of scalars be $\mathbb{R}$. Define the operations of vector addition and scalar multiplication as follows:

$$(x_1, x_2, x_3, \dots) + (y_1, y_2, y_3, \dots) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots)$$

$$c(x_1, x_2, x_3, \dots) = (cx_1, cx_2, cx_3, \dots)$$

Verifying that these operations satisfy the axioms of a vector space is straightforward. For instance, vector addition is associative, and the zero vector is the sequence $(0, 0, 0, \dots)$. Every sequence also has an additive inverse, and the other axioms are similarly easy to verify.

Continuous Functions

Let V be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, and let the field of scalars be $\mathbb{R}$. Define vector addition and scalar multiplication as follows:

- For any $f, g \in V$, their sum $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in \mathbb{R}$.
- For any scalar $c \in \mathbb{R}$ and any $f \in V$, the scalar multiple cf is defined by $(cf)(x) = c \cdot f(x)$ for all $x \in \mathbb{R}$.

To ensure that these operations are valid, one must verify that the sum of two continuous functions is continuous, and that multiplying a continuous function by a real number yields another continuous function. These facts are well-known results from calculus.

Exercise 5.1. Show formally that the set of continuous functions defined as above forms a vector space over field $\mathbb{R}$

Over complex number field

The complex vector space $\mathbb{C}^n$ over $\mathbb{C}$ is another example. Here, c, x_i, and y_i are complex numbers, with vector addition and scalar multiplication as in the real case but involving complex arithmetic.

We have often seen the vectors as arrows from the origin, we will now understand vectors beyond arrows from origin. A frequently encountered example of a vector space is the set of column matrices of real numbers. For instance, consider a column matrix like:

$$\mathbf{v} = (4, 2)^T$$

This can be visualized as an arrow starting from the origin of the plane and ending at the point $(4, 2)$. This visualization gives rise to the idea that "vectors are arrows," which is a specific way to represent vectors in this particular vector space. However, for other vector spaces, such as those consisting of functions or infinite sequences, this arrow representation is not applicable.

Subspaces

Definition 5.3. *A subset S of a vector space V is called a subspace if it is itself a vector space under the same operations defined on V. Specifically, S must be closed under vector addition and scalar multiplication, meaning for any $\mathbf{v_1}, \mathbf{v_2} \in S$ and any scalar $c \in F$, both $\mathbf{v_1} + \mathbf{v_2} \in S$ and $c\mathbf{v_1} \in S$.*

Examples of Subspaces

Consider the vector space $\mathbb{R}^3$:

- Every line through the origin is a subspace of $\mathbb{R}^3$.
- Any plane passing through the origin is also a subspace of $\mathbb{R}^3$.

Exercise 5.2. Show that a quarter-plane (where $x, y \geq 0$) is not a subspace of $\mathbb{R}^2$

5.3 Basis of a vector space

Definition 5.4 (Linear combination)**.** *Given vectors $\mathbf{v_1}, \ldots, \mathbf{v_m}$ in a vector space V, a linear combination of these vectors is any vector of the form:*

$$c_1\mathbf{v_1} + \cdots + c_m\mathbf{v_m},$$

where $c_1, \ldots, c_m$ are scalars from F.

Definition 5.5 (Span)**.** *Given vectors $\mathbf{v_1}, \ldots, \mathbf{v_m}$ in a vector space V, the set of all possible linear combinations of $\mathbf{v_1}, \ldots, \mathbf{v_m}$ forms a subspace of V called the span of $\{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$, denoted as span$\{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$.*

Definition 5.6 (Linear Dependence)**.** *A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is said to be linearly dependent if there exist scalars $c_1, \ldots, c_n$, not all zero, such that:*

$$c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n} = \mathbf{0}.$$

If no such scalars exist (except for the trivial case where all are zero), the vectors are *linearly independent*.

Exercise 5.3. Check that for all subsets of the linear independent set of vectors are also linear independent.

Checking Linear Dependence in $\mathbb{R}^n$

To determine if a set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ in $\mathbb{R}^n$ is linearly dependent, express each vector in terms of its coordinates and check if the system of equations $A\mathbf{c} = \mathbf{0}$ has a non-trivial solution, where A is a matrix whose columns are the vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$.

Definition 5.7 (Basis)**.** *A basis for a vector space V is a set of linearly independent vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ that span V. The dimension of V, denoted dim(V), is the number of vectors in any basis of V.*

Theorem 5.1. If $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis for a vector space V, then every vector $\mathbf{v} \in V$ has a unique representation as:

$$\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n},$$

where $c_1, \ldots, c_n$ are scalars in F.

Proof. Suppose two sets of scalars $(c_1, \ldots, c_n)$ and $(c'_1, \ldots, c'_n)$ represent $\mathbf{v}$ relative to the basis. Then

$$\mathbf{v} = \sum_{i=1}^{n} c_i\mathbf{v}_i = \sum_{i=1}^{n} c'_i\mathbf{v}_i.$$

So,

$$\sum_{i=1}^{n} c_i\mathbf{v}_i - \sum_{i=1}^{n} c'_i\mathbf{v}_i = \sum_{i=1}^{n}(c_i - c'_i)\mathbf{v}_i = 0.$$

By the linear independence (LI) of the basis vectors, we have $c_i - c'_i = 0$, or $c_i = c'_i$ for all $i \in \{1, \ldots, n\}$. □

Theorem 5.2. If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are bases for a vector space V, then $m = n$.

Proof. Suppose without loss of generality (WLOG) that $m \geq n$. Since the $\{v_i\}$'s form a basis, we have

$$w_1 = \sum_{i=1}^{n} c_i v_i,$$

with at least one $c_i \neq 0$ (since $w_1 \neq 0$ by the linear independence (LI) of the w_i's).

WLOG, let $c_1 \neq 0$; then

$$v_1 = \frac{1}{c_1}w_1 - \sum_{i=2}^{n} \frac{c_i}{c_1} v_i.$$

So, the set $\{w_1, v_2, \ldots, v_n\}$ spans V. It is also LI, as $\{v_2, \ldots, v_n\}$ are LI and w_1 is not a linear combination (LC) of them (if it were, c_1 would be 0). Thus, $\{w_1, v_2, \ldots, v_n\}$ is a basis.

We can iterate this argument n times, replacing v_2 with w_2 and so on. After n replacements, we get that $\{w_1, \ldots, w_n\}$ is a basis. But then $\{w_1, \ldots, w_m\}$ cannot have any additional vectors: since this is a LI set, an additional vector, say w_{n+1}, cannot be a LC of $\{w_1, \ldots, w_n\}$, contradicting the fact that this latter set is a basis. Therefore, $m = n$. □

Definition 5.8 (Dimension)**.** *The number of elements in any basis of V is called the dimension of V, denoted* $\dim(V)$*. If this is finite, we say V is a finite-dimensional vector space.*

Note: Our definitions of *linearly dependent* (LD) and *linearly independent* (LI), as well as the subsequent results, are all for finite sets of vectors. The definitions are slightly modified if the topic of discussion is infinite-dimensional vector spaces. The function space $C[a, b]$ is one such space—functions in this space are also important in economics but are outside the scope of the present discussion.

Definition 5.9 (Linear Transformation)**.** *A linear transformation T from a vector space V to a vector space W is a function that satisfies the following properties for all $\mathbf{v_1}, \mathbf{v_2} \in V$ and scalars $c_1, c_2 \in F$:*

- $T(\mathbf{v_1} + \mathbf{v_2}) = T(\mathbf{v_1}) + T(\mathbf{v_2})$,
- $T(c\mathbf{v}) = cT(\mathbf{v})$.

Examples:

Rotation in $\mathbb{R}^2$
Consider the rotation of vectors in $\mathbb{R}^2$ by an angle θ. This transformation satisfies both linearity conditions, making it a linear transformation.

Matrix:
A matrix A_{mn} is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T(\mathbf{x}) = A\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

This function is linear because for all $c_1, c_2 \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have

$$A(c_1\mathbf{x} + c_2\mathbf{y}) = c_1 A\mathbf{x} + c_2 A\mathbf{y}.$$

In a later section, we will discuss the point that any linear transformation between finite-dimensional vector spaces can be "represented" as a matrix by fixing bases in the domain and co-domain.

Definition 5.10. *Let $T : V \to W$ be a linear transformation (LT). The range space of T, denoted $R(T)$, is the range of the function, i.e.,*

$$R(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\}.$$

In the context of linear transformations, the range is called the *range space* because it is a subspace. Indeed, let $w_1, w_2 \in R(T)$ and c_1, c_2 be scalars. So, $w_1 = T(v_1)$ and $w_2 = T(v_2)$ for some vectors $v_1, v_2 \in V$. Searching for a preimage for $c_1w_1 + c_2w_2$:

$$c_1w_1 + c_2w_2 = c_1T(v_1) + c_2T(v_2) = T(c_1v_1 + c_2v_2),$$

the last equality holds since T is linear. So, $c_1w_1 + c_2w_2$ has preimage $c_1v_1 + c_2v_2$, and is hence in $R(T)$. Thus, $R(T)$ is a subspace.

Let $A_{mn} = (a_1, a_2, \ldots, a_n)$ be written in terms of its n columns. Since $A\mathbf{x} = x_1a_1 + x_2a_2 + \cdots + x_na_n$, $R(A)$ or the set of all images like $A\mathbf{x}$ is equal to span$\{a_1, \ldots, a_n\}$, the subspace spanned by the columns of A.

Exercise 5.4. Try to draw the range space of the following matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

Since $R(T)$ is a subspace, it has a dimension. The dimension of $R(T)$ (denoted $\dim(R(T))$) is called the *Rank* of T. A matrix A_{mn} with k linearly independent (LI) columns has the property that its columns span a k-dimensional subspace. This dimension therefore corresponds to the notion of Rank(A) being equal to the number of LI columns of A.

Earlier, we examined the properties of the range of a linear transformation. Now, we'll shift our focus to understanding the properties of the domain of the linear transformation T.

Definition 5.11. *The Null Space or Kernel of an LT $T : V \to W$, denoted $N(T)$, is the subset of vectors in V with image 0 in W; i.e.,*

$$N(T) = \{v \in V \mid T(v) = 0\}.$$

The dimension of $N(T)$ is called as Nullity of T.

Exercise 5.5. Try to find the nullity of the linear transformation represented by the matrix from the *exercise 5.4*.

Remarks:
Notice in the last two exercises the LT represented by a matrix of 2×2, that their Rank and Nullity add up to 2, which is also the dimension of the domain $\mathbb{R}^2$.
More generally, suppose A is an $m \times n$ matrix, and suppose its null space has dimension k. That is, a k-dimensional subspace of vectors in $\mathbb{R}^n$ all map to the origin in $\mathbb{R}^m$. Then there remain $n-k$ dimensions of $\mathbb{R}^n$ free, and to map the remaining vectors, we need a space of dimension $n-k$ in $\mathbb{R}^m$.

We will use the following fact. Let $\{v_1, \ldots, v_k\}$ be an LI set of vectors in a finite-dimensional vector space V. We can extend this to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$. If $\text{span}(\{v_1, \ldots, v_k\}) = V$, we need go no further. Otherwise, there exists a vector, call it $v_{k+1} \notin \text{span}(\{v_1, \ldots, v_k\})$. So, $\{v_1, \ldots, v_k, v_{k+1}\}$ is LI. If it spans V, stop; otherwise iterate the argument. The expansion of this set must stop at some finite set $\{v_1, \ldots, v_n\}$, because if it doesn't, the vector space is not finite-dimensional. This set therefore will span V, and it is LI by construction. Hence, it is a basis for V. The following theorem considers an LT $T : V \to W$, where V and W are finite-dimensional vector spaces (FDVS).

Theorem 5.3 (Fundamental Theorem of Linear Algebra)**. The Fundamental Theorem of Linear Algebra:**
Theorem:
Let $T : V \to W$. Then $\dim(V) = \text{Rank}(T) + \text{Nullity}(T)$.

Proof. Suppose that $\{v_1, \ldots, v_k\}$ is a basis for $N(T)$. We extend this to $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$, a basis for V in the domain, so that $\dim(V) = n$, and $\text{Nullity}(T) = k \leq n$. Then it suffices to show that $\dim R(T) = n - k$ or, $\{w_1, \ldots, w_{n-k}\} \equiv \{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for $R(T)$.

Let's conjecture that $\{T(v_{k+1}), \ldots, T(v_n)\}$ forms a basis of the image T.

Part (i). Show span$(\{T(v_{k+1}), \ldots, T(v_n)\}) = R(T)$**.**
Let $w \in R(T)$, so $w = T(v)$ for some $v \in V$. In terms of the chosen basis, let $v = a_1 v_1 + \cdots + a_k v_k + a_{k+1} v_{k+1} + \cdots + a_n v_n$, for some scalars $a_1, \ldots, a_n$. So,

$$w = T(v) = T\left(\sum_{i=1}^{k} a_i v_i\right) + T\left(\sum_{i=k+1}^{n} a_i v_i\right) = \sum_{i=k+1}^{n} a_i T(v_i).$$

The second equality is due to the linearity of T; the third because $\left(\sum_{i=1}^{k} a_i v_i\right) \in N(T)$ and T is linear. Since w is arbitrary in $R(T)$ and is a linear combination of $\{T(v_{k+1}), \ldots, T(v_n)\}$, this set spans $R(T)$.

Part (ii). Show $\{T(v_{k+1}), \ldots, T(v_n)\}$ **is LI.**

Let $c_{k+1}T(v_{k+1}) + \cdots + c_nT(v_n) = 0$ (for scalars $c_{k+1}, \ldots, c_n$)

$$\iff T(c_{k+1}v_{k+1} + \cdots + c_nv_n) = 0 \quad \text{[by linearity]}$$

$$\implies c_{k+1}v_{k+1} + \cdots + c_nv_n \in N(T) \quad \text{[by definition of nullity]}.$$

But then, since $\{v_1, \ldots, v_k\}$ span $N(T)$, there exist some scalars $c_1, \ldots, c_k$ such that

$$c_{k+1}v_{k+1} + \cdots + c_nv_n = c_1v_1 + \cdots + c_kv_k.$$

So,

$$c_1v_1 + \cdots + c_kv_k - c_{k+1}v_{k+1} - \cdots - c_nv_n = 0.$$

But, since $\{v_1, \ldots, v_n\}$ form a basis of V, they should be LI, so all the c_i's equal 0, so that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is LI and span the range.

For an $m \times n$ matrix A, we have $n = \text{Rank}(A) + \text{Nullity}(A)$.

□

5.4 System of Linear Equations

Given $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}^m$, $\mathbf{x} = (x_1, \ldots, x_n)$ is a solution to the system of linear equations (Note that $m \leq n$)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

if $\mathbf{x}$ is a preimage of $\mathbf{b}$ under the function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}, \forall \mathbf{v} \in \mathbb{R}^n$, where A is the $m \times n$ matrix in the above system. More compactly, we can write the linear system of equations as the equation $A\mathbf{x} = \mathbf{b}$.

lemma 5.3.1. Let A be an $m \times n$ real matrix, $\mathbf{b}$ a vector in $\mathbb{R}^m$, and $\mathbf{x}_0 \in \mathbb{R}^n$ satisfy $A\mathbf{x}_0 = \mathbf{b}$. Then the set of all solutions to the equation $A\mathbf{x} = \mathbf{b}$ (i.e., $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$) is the set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{x}_0 + \mathbf{w}, \text{ for some } \mathbf{w} \in N(A)\}$.

Proof. Suppose $\mathbf{x}$ is a general solution of $A\mathbf{x} = \mathbf{b}$. Now $\mathbf{x} = \mathbf{x}_0 + \mathbf{w}$, where $\mathbf{w} = \mathbf{x} - \mathbf{x}_0$ such that

$$A\mathbf{w} = A(\mathbf{x} - \mathbf{x}_0) = A\mathbf{x} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = 0,$$

so $\mathbf{w} \in N(A)$.

Conversely, suppose $\mathbf{x} = \mathbf{x}_0 + \mathbf{w}$ where $\mathbf{x}_0$ is a solution and $\mathbf{w} \in N(A)$. Then

$$A\mathbf{x} = A(\mathbf{x}_0 + \mathbf{w}) = A\mathbf{x}_0 + A\mathbf{w} = \mathbf{b} + 0,$$

so $\mathbf{x}$ is a solution.

Geometrically, $N(A) = \{\mathbf{w} \in \mathbb{R}^n \mid A\mathbf{w} = 0\}$ is a subspace. The above lemma tells us that the set of solutions to $A\mathbf{x} = \mathbf{b}$ can be written as $\{\mathbf{x}_0\} + N(A)$, where $\mathbf{x}_0$ is one solution, a parallel translate of this subspace.

□

Note that if Nullity$(A) \geq 1$, then the system of equations has multiple solutions if there are any solutions at all. To understand the lemma, consider a system of equations with

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Exercise 5.6. Draw the solution set and the 1-dimensional subspace $N(A)$.

Theorem 5.4. A be an $n \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^n$. The system $A\mathbf{x} = \mathbf{b}$:

1. has a unique solution if Rank$(A) = n$.
2. may have no solution if Rank$(A) < n$. But if a solution exists, there are multiple solutions.

Proof. (i) Rank$(A) = n \implies R(A) = \mathbb{R}^n$, i.e., A is surjective, so a solution (preimage) exists for every $\mathbf{b} \in \mathbb{R}^n$. Moreover, Nullity$(A) = 0$, so $N(A) = \{0\}$, so the solution set (a translate of $N(A)$) has a single element—the solution is unique. (ii) Rank$(A) < n \implies R(A) \subset \mathbb{R}^n$, so there exists $\mathbf{b}$ in the codomain with no preimage/solution to the equation system. However, Nullity$(A) \geq 1$. So if $\mathbf{b} \in R(A)$, so that there is a solution (i.e., $\mathbf{b}$ has a preimage), we have multiple solutions—the translate of a 1- or higher-dimensional subspace. □

Theorem 5.5. (Fewer equations than variables) Let A be an $m \times n$ matrix, $m < n$, and let $\mathbf{b} \in \mathbb{R}^m$. The system $A\mathbf{x} = \mathbf{b}$:

1. has multiple solutions if Rank$(A) = m$.
2. If Rank$(A) < m$, a solution need not exist; if it does exist, then there are multiple solutions.

Proof. (i) Rank$(A) = m \implies R(A) = \mathbb{R}^m$, since $R(A)$ is then an m-dimensional subspace of $\mathbb{R}^m$. So A is surjective, and every $\mathbf{b} \in \mathbb{R}^m$ has a preimage. Moreover,

Nullity$(A) = n - \text{Rank}(A) = n - m > 0$, so the solution set, which is a translate of $N(A)$, holds multiple solutions.
(ii) Proved along similar lines as (ii) of the previous theorem.
So if A is square and of full rank, the system is "identified"; if A has fewer rows than columns and is of full rank, the system is "overidentified". In the other cases, the system is not identified.

□

Theorem 5.6. (More equations than variables) Let A be an $m \times n$ matrix, $m > n$, and $\mathbf{b} \in \mathbb{R}^m$. Then a solution to $A\mathbf{x} = \mathbf{b}$ need not exist. If a solution does exist, it is unique if $\text{Rank}(A) = n$ and not unique if $\text{Rank}(A) < n$.

Exercise 5.7. Proove the above theorem.

5.4.1 Linear Implicit Function Theorem

Theorem 5.7. Consider the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1k}x_k + a_{1k+1}x_{k+1} + \cdots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mk}x_k + a_{mk+1}x_{k+1} + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Fix $x_{k+1}, \ldots, x_n$ at values $x^*_{k+1}, \ldots, x^*_n$ respectively. For each such choice of "parameters", there exists a unique set of values $x^*_1, \ldots, x^*_k$ that solves the equation system if and only if:

1. $k = m$
2. $\text{Rank}\left(\begin{pmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mk} \end{pmatrix}\right) = k$

Proof. For each equation i, carry $a_{ik+1}x^*_{k+1} + \cdots + a_{in}x^*_n$ to the right-hand side. Then from the two theorems above (*Theorem 5.4 and Theorem 5.5*), the only way for a solution to always exist and be unique is if the underlying matrix is square and of full rank. □

6 Spectral Theory

Spectral theory, originating from functional analysis in mathematics, has found intriguing applications in economics, particularly in the analysis of dynamic systems, networks, and financial markets. It involves the study of eigenvalues and eigenvectors of matrices or operators, which can be used to understand the behavior of economic systems over time. For example, in network analysis, spectral theory helps to identify influential nodes or sectors within an economy by examining the spectrum of the adjacency matrix. In finance, it aids in portfolio optimization and risk management by analyzing the covariance matrix of asset returns. Additionally, spectral methods are employed in the study of economic growth models and in understanding the diffusion of shocks across interconnected markets. By offering a deeper understanding of the underlying structure of complex economic systems, spectral theory provides powerful tools for economists to analyze stability, resilience, and systemic risk.

Definition 6.1. *Let $T : V \to V$ be a linear transformation (LT). A scalar λ is called an eigenvalue of T if there exists a non-zero vector $v \in V$ such that $T(v) = \lambda v$. The vector v is then referred to as an eigenvector of T corresponding to the eigenvalue λ.*

Note:
(1) If v is an eigenvector corresponding to λ, then any scalar multiple of v, denoted cv for any scalar c, is also an eigenvector corresponding to λ. This is because

$$T(cv) = cT(v) = c\lambda v = \lambda(cv).$$

Furthermore, if v and v' are eigenvectors corresponding to the same eigenvalue λ, then any linear combination of v and v', say $c_1v + c_2v'$ for scalars c_1 and c_2, is also an eigenvector corresponding to λ. This follows from:

$$T(c_1v + c_2v') = c_1T(v) + c_2T(v') = c_1\lambda v + c_2\lambda v' = \lambda(c_1v + c_2v').$$

Thus, c_1v+c_2v' is also an eigenvector corresponding to λ. This means that the set of all eigenvectors corresponding to a particular eigenvalue, along with the zero vector, forms a subspace of V. This subspace is known as the *eigenspace* corresponding to the eigenvalue λ.

(2) A matrix for which all non-zero vectors are eigenvectors is the identity matrix I_n. For all non-zero $\mathbf{x} \in \mathbb{R}^n$, we have $I_n\mathbf{x} = 1\mathbf{x}$; all such $\mathbf{x}$ are eigenvectors corresponding to the eigenvalue 1.

Definition 6.2. *The set of all eigenvalues of a linear transformation T is referred to as the spectrum of T, and is denoted by $\sigma(T)$.*

6.1 Characteristic Equation

Given that $T(\mathbf{v}) = \lambda\mathbf{v}$, we can rewrite this as $(T - \lambda I)(\mathbf{v}) = 0$, meaning that the eigenvector $\mathbf{v}$ corresponding to the eigenvalue λ lies in the null space of the linear transformation $(T - \lambda I)$, along with the zero vector. This null space, denoted by $N(T - \lambda I)$, is known as the *eigenspace* associated with λ.

For there to be a non-zero solution $\mathbf{v}$ to $(T - \lambda I)(\mathbf{v}) = 0$, the nullity of $(T - \lambda I)$ must be greater than zero, i.e., $\text{Nullity}(T - \lambda I) > 0$, which implies that the rank of $(T - \lambda I)$ must be less than the dimension of V: $\text{Rank}(T - \lambda I) < \dim(V)$.

Now, let the matrix A_{nn} represent T. Thus, we have $\text{Rank}(A - \lambda I) < n$, which implies that the determinant $\det(A - \lambda I) = 0$. The determinant $\det(A - \lambda I)$ can be written as:

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix} = 0.$$

This determinant is an nth-degree polynomial in λ, and can be expressed as:

$$f(\lambda) = \det(A - \lambda I) = (-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + \cdots + b_1(-\lambda) + b_0.$$

Alternatively, in its factored form, this polynomial can be written as:

$$f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda).$$

The equation $\det(A - \lambda I) = 0$ has n solutions (roots), which may be repeated and may also be complex. If λ is a complex root of the equation, then its conjugate $\overline{\lambda}$ is also a root.

By comparing the expanded polynomial form of $f(\lambda)$ and its factored form, we obtain the following relationships:

$$b_{n-1} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$b_{n-2} = \sum_{j>i} \lambda_i\lambda_j = \lambda_1\lambda_2 + \cdots + \lambda_1\lambda_n + \lambda_2\lambda_3 + \cdots + \lambda_{n-1}\lambda_n,$$

$$b_r = \sum_{k>j>i} \lambda_i\lambda_j\lambda_k \quad [\text{each term is the product of } r \text{ of the } \lambda_i],$$

$$\vdots$$

$$b_0 = \lambda_1\lambda_2\lambda_3\ldots\lambda_n.$$

These coefficients $b_{n-1}, b_{n-2}, \ldots, b_0$ represent the sums and products of the eigenvalues of the matrix A.