Probability Theory 101

x's PDF probability density function is denoted by $P(x)$

→ $P(x) \geqslant 0$

→ $P(x)\,dx \leqslant 1$

→ $\int P(x)\,dx = 1$

For a deterministic vector $x = x_0$, its PDF is a delta function

$$P(x) = \delta(x - x_0)$$

For two random vectors x_1 & x_2, their joint PDF is $\boxed{P(x_1, x_2)}$

☆ $\int P(x_1, x_2)\,dx_2 = P(x_1)$ (marginal distribution)

☆ Conditional probability $P(x_1|x_2) = P(x_1, x_2)/P(x_2)$

$$P(x_1|x_2) \cdot P(x_2) = P(x_1, x_2)$$

$$P(x_2|x_1) \cdot P(x_1) = P(x_1, x_2)$$

$$P(x_1|x_2)\,P(x_2) = P(x_2|x_1) \cdot P(x_1)$$

$$\boxed{P(x_1|x_2) = \frac{P(x_2|x_1) \cdot P(x_1)}{P(x_2)}}$$ Baye's formula

☆ Statistical independence $\boxed{P(x_1, x_2) = P(x_1)\,P(x_2)}$

→ For a sequence of vector $x_1, x_2, x_3 \dots x_n$ follows Chain Rule:

$$\boxed{P(x_1, x_2, x_3, \dots, x_n) = P(x_1)\,P(x_2|x_1)\,P(x_3|x_1, x_2)\,P(x_4|x_1, x_2, x_3) \dots P(x_n|x_1, x_2, x_{n-1})}$$

→ If this chain is know to be Markovian, this implies that each vector is statistically dependent only on its predecessor.

$$\boxed{P(x_1, x_2, x_3, \dots x_n) = P(x_1)\,P(x_2|x_1)\,P(x_3|x_2)\,P(x_4|x_3) \dots P(x_n|x_{n-1})}$$

Consider two independent random vectors $x_1 \in R^n$ and $x_2 \in R^n$

where, $x_1 \sim P(x_1)$ and $x_2 \sim Q(x_2)$

Let us define z as their sum, $z = x_1 + x_2$

The PDF of z is given by a convolution:

$$\boxed{G(z) = P(z) * Q(z)}$$

Proof: The probability for a value z should be obtained as sum of all the probabilities of the events (x_1, x_2) such that they sum to z

$$\boxed{G(z) = \int_{x_1} P(x_1)\, Q(z - x_1)\, dx_1 \equiv P(z) * Q(z)}$$

The expectation (expected value) of an arbitrary deterministic expression $f(x)$ over the random vector $x \sim P(x)$ is given by

$$\boxed{E_x(f(x)) = \int f(x)\, P(x)\, dx}$$

Few properties →

- Linearity: $E_x(f(x) + g(x)) = E_x(f(x)) + E_x(g(x))$
- Mean and covariance: $\boxed{\begin{aligned} m &= E_x(x) \\ \Sigma &= E_x((x-m)(x-m)^T) \end{aligned}}$

  mean

  mean

  covariance

- x_1 and x_2 are uncorrelated iff $\boxed{E_x(x_1^T x_2) = E_x(x_1)^T \cdot E_x(x_2)}$
- If x_1 & x_2 are independent, the necessarily $\boxed{E_x(x_1^T x_2) = E_x(x_1)^T \cdot E_x(x_2)}$

Conditional Expectation

For two random vector x and z with a conditional $P(x|z)$, the conditional expectation of x given z is:

$$\boxed{E_x(x|z) = \int x\, P(x|z)\, dx = \int x \frac{P(x, z)}{P(z)}\, dx}$$

$\Rightarrow$ The multivariate Gaussian PDF of a vector $x \in \mathbb{R}^n$ is given by

$\sigma \rightarrow$ covariance

$$P(x) = \underbrace{N}_{\text{Normal}}(x; \mu, \Sigma) = \sqrt{\frac{1}{(2\pi)^n |\sigma|}} \exp\left\{-\frac{1}{2}(x-\mu)^T \sigma^{-1} (x-\mu)\right\}$$

☆ Two uncorrelated gaussian random vectors are necessarily also independent.

☆ If $x \sim N(x; \mu, \sigma)$ and $z = Ax + b$ then $z \sim N(z; A\mu + b, A\sigma A^T)$

☆ If $x_1 \sim N(x_1; \mu_1, \sigma_1)$ and $x_2 \sim N(x_2; \mu_2, \sigma_2)$ are of same dimension and uncorrelated, then:

$$\boxed{x_1 + x_2 = z \sim N(z; \mu_1 + \mu_2, \sigma_1 + \sigma_2)}$$