① Ex. 13 p 200

Trapezoidal: $\int_{x_0}^{x_1} f(x)\,dx = \frac{x_1 - x_0}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)$

Simpson: $\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi)$

with $x_0 = a$, $x_2 = b$, $x_1 = a+h$ and $h = \frac{b-a}{2}$

$\hookrightarrow \int_0^2 f(x)\,dx$, $h = \frac{b-a}{2} = \frac{2-0}{2} = 1$

$\frac{2-0}{2}[f(0) + f(2)] = 4 \Leftrightarrow f(0) + f(2) = 4$

$\frac{2-0}{6}[f(0) + 4f(1) + f(2)] = 2 \Leftrightarrow f(0) + 4f(1) + f(2) = 6$

$\Rightarrow f(0) + 4f(1) + f(2) = 4f(1) + \underbrace{f(0) + f(2)}_{=4} = 6$

$\Leftrightarrow 4f(1) + 4 = 6 \Leftrightarrow f(1) = \frac{1}{2}$

(2) Ex. 28 p. 201

Theorem 4.2: $\int_a^b f(x)\,dx = \sum_{i=0}^{3} a_i f(x_i) + \frac{h^5 f^{(4)}(\xi)}{4!} \int_0^3 t(t-1)(t-2)(t-3)\,dt$

Error-Term:

1. $\int_0^3 t(t-1)(t-2)(t-3)\,dt$

$= \int_0^3 t^4 - 6t^3 + 11t^2 - 6t\,dt$

$= \left[\frac{t^5}{5} - \frac{6}{4}t^4 + \frac{11}{3}t^3 - 3t^2 \right]_0^3$

$= -\frac{9}{10}$

$\Rightarrow \int_a^b f(x)\,dx = \sum_{i=0}^{3} a_i f(x_i) - \frac{9h^5}{240} f^{(4)}(\xi) = \sum_{i=0}^{3} a_i f(x_i) - \frac{3h^5}{80} f^{(4)}(\xi)$

with $a_i = h \int_0^3 \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-j}{i-j}\,dx$

$\Rightarrow a_0 = \int_{x_0}^{x_3} \prod_{j=1}^{3} \frac{x-x_j}{x_0-x_j}\,dx = \int_{x_0}^{x_3} \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx$

With: $x_0 = a,\ x_1 = a + \frac{b-a}{3} = \frac{2a+b}{3}$

$$x_2 = a + \frac{2(b-a)}{3} = \frac{a+2b}{3},\quad x_3 = b$$

$$= \int_a^b \frac{(x - \frac{2a+b}{3})(x - \frac{a+2b}{3})(x-b)}{(a - \frac{2a+b}{3})(a - \frac{a+2b}{3})(a-b)}\, dx = \frac{b-a}{8}$$

$$= \frac{3}{8} \cdot h$$

$$a_1 = \int_a^b \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\, dx$$

$$= \int_a^b \frac{(x-a)(x - \frac{a+2b}{3})(x-b)}{(\frac{2a+b}{3} - a)(\frac{2a+b}{3} - \frac{a+2b}{3})(\frac{2a+b}{3} - b)}\, dx$$

$$= \int_a^b \frac{(x-a)(x - \frac{a+2b}{3})(x-b)}{(\frac{b-a}{3})(\frac{a-b}{3})(\frac{2a-2b}{3})}\, dx = -\frac{3}{8}(a-b) = \frac{3(b-a)}{8}$$

$$= \frac{9}{8} h$$

$$a_2 = \int_a^b \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\, dx$$

$$= \int_a^b \frac{(x-a)(x - \frac{2a+b}{3})(x-b)}{(\frac{a+2b}{3} - a)(\frac{a+2b}{3} - \frac{2a+b}{3})(\frac{a+2b}{3} - b)}\, dx$$

$$= \frac{3(b-a)}{8} = \frac{9}{8} h$$

$$a_3 = \int_a^b \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,dx$$

$$= \int_a^b \frac{(x-a)\left(x-\frac{2a+b}{3}\right)\left(x-\frac{a+2b}{3}\right)}{(b-a)\left(b-\frac{2a+b}{3}\right)\left(b-\frac{a+2b}{3}\right)}\,dx$$

$$= \frac{b-a}{8} = \frac{3}{8}h$$

$\rightarrow a_0 = \frac{3}{8}h,\ a_1 = \frac{9}{8}h,\ a_2 = \frac{9}{8}h,\ a_3 = \frac{3}{8}h$

$$\hookrightarrow \int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8}h\,f(x_0) + \frac{9}{8}h\,f(x_1) + \frac{9}{8}h\,f(x_2) + \frac{3}{8}h\,f(x_3) - \frac{3h^5}{80}f^{(4)}(\xi)$$

$$= \frac{3}{8}h\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right) - \frac{3h^5}{80}f^{(4)}(\xi)$$

③ Ex. 19 p. 201

Actual Result | Approximation

deg 0: $\int_{-1}^{1} 1 \, dx = [x]_{-1}^{1} = 0$ | $\int_{-1}^{1} 1 dx = 1 - 1 = 0$

deg 1: $\int_{-1}^{1} x \, dx = [\frac{x^2}{2}]_{-1}^{1} = 0$ | $\int_{-1}^{1} x \, dx = -\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = 0$

deg 2: $\int_{-1}^{1} x^2 dx = [\frac{x^3}{3}]_{-1}^{1} = 0$ | $\int_{-1}^{1} x^2 dx = \left(-\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{2}{3}$

the degree of precision is 1.

④ Ex. 26 p. 201

$$\int_{x_0}^{x_2} x \, dx = \left[\frac{x^2}{2} \right]_{x_0}^{x_2} = \frac{(x_0+2h)^2}{2} - \frac{x_0^2}{2} = \frac{x_0^2 + 4x_0h + 4h^2 - x_0^2}{2}$$

$$= 2x_0h + 2h^2$$

because Simpson's rule is exact here: $= a_0x_0 + a_1(x_0+h) + a_2(x_0+2h)$

$$\int_{x_0}^{x_2} x^2 dx = \left[\frac{x^3}{3} \right]_{x_0}^{x_2} = \frac{(x_0+2h)^3 - x_0^3}{3}$$

$$= 2x_0^2h + 4x_0h + \frac{8}{3}h^3$$

$$= a_0x_0^2 + a_1(x_0+h)^2 + a_2(x_0+2h)^2$$

$$= a_0x_0^2 + a_1x_0^2 + 2a_1x_0h + a_1h^2 + a_2x_0^2 + 4a_2x_0h + 4a_2h^2$$

$$\int_{x_0}^{x_2} x^3 dx = \left[\frac{x^4}{4} \right]_{x_0}^{x_2} = \frac{(x_0+2h)^4 - x_0^4}{4}$$

$$= 2x_0^3h + 6x_0^2h^2 + 8x_0h^3 + 4h^4$$

$$= a_0x_0^3 + a_1(x_0+h)^3 + a_3(x_0+2h)^3$$

We get an Equation System

$$\text{I} \quad 2x_0h + 2h^2 = a_0x_0 + a_1(x_0+h) + a_2(x_0+2h)$$

$$\text{II} \quad 2x_0^2h + 4x_0h + \frac{8}{3}h^3 = a_0x_0^2 + a_1(x_0+h)^2 + a_2(x_0+2h)^2$$

$$\text{III} \quad 2x_0^3h + 6x_0^2h^2 + 8x_0h^3 + 4h^4 = a_0x_0^3 + a_1(x_0+h)^3 + a_3(x_0+2h)^3$$

that is solved when using the parameters of the Simpson-Formula for a_0, a_1, and a_2:

$$a_0 = \frac{h}{3}, \quad a_1 = \frac{4h}{3}, \quad a_2 = \frac{h}{3}$$

Error: $f(x) = x^4 \quad f'(x) = 4x^3 \quad f''(x) = 12x^2 \quad f'''(x) = 24x \quad f^{(4)}(x) = 24$

$$\int_{x_0}^{x_2} x^4 = \left[\frac{x^5}{5}\right]_{x_0}^{x_2} = \frac{(x_0+2h)^5 - x_0^5}{5}$$

$$= \frac{h}{3}\left(x_0^4 + 4(x_0+h)^4 + (x_0+2h)^4\right) + 24 \cdot k$$

This equation for the error is solved when using the error parameter of the Simpson rule:

$$k = -\frac{h^5}{90}$$

5 Ex. 18 p. 234

b) $\int_0^1 x^2 e^{-x}\,dx = 2-\frac{5}{e}$ (exact result)

<u>Gaussian Quadrature</u>: $n=2$

Transform to integral on interval $[-1,1]$:

$$\int_0^1 f(x)\,dx = \int_{-1}^{1} f\left(\frac{(1-0)t+(1+0)}{2}\right)\frac{(1-0)}{2}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} f\left(\frac{t+1}{2}\right) dt$$

$$\Rightarrow \int_0^1 x^2 e^{-x}\,dx = \frac{1}{2}\int_{-1}^{1}\left(\frac{t+1}{2}\right)^2 e^{-\left(\frac{t+1}{2}\right)}\,dt$$

$$\approx \frac{1}{2}\left(f(-0.5773502692+1)+f(0.5773502692+1)\right)$$

$$\approx 0.159418\ldots$$

Absolute error: $\left|2-\frac{5}{e} - 0.159418\ldots\right| \approx 0.001192$

$$= 1.192\cdot 10^{-3}$$

⑥ P. 235 Ex. 12

We want the formula to be exact for polynomials $1, x, x^2, x^3$ and x^4.

We get the equations:

$$1: \quad \int_{-1}^{1} 1\,dx = [x]_{-1}^{1} = 2 = a + b + c$$

$$x: \quad \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0 = -a + c + d + e$$

$$x^2: \quad \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} = a + c - 2d + 2e$$

$$x^3: \quad \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0 = -a + c + 3d + 3e$$

$$x^4: \quad \int_{-1}^{1} x^4\,dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5} = a + c - 4d + 4e$$

We get the system of equations:

$$\begin{aligned}
&\text{I} && a + b + c = 2 \\
&\text{II} && -a + c + d + e = 0 \\
&\text{III} && a + c - 2d + 2e = \tfrac{2}{3} \\
&\text{IV} && -a + c + 3d + 3e = 0 \\
&\text{V} && a + c - 4d + 4e = \tfrac{2}{5}
\end{aligned}$$

which is solved for the values:

$$a = \frac{7}{15},\quad b = \frac{16}{15},\quad c = \frac{7}{15},\quad d = \frac{1}{15},\quad e = -\frac{1}{15}$$

$\Rightarrow$ so the Quadrature Formula with degree of precision 4 is

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{15}\left(7f(-1) + 16f(0) + 7f(1) + f'(-1) - f'(1)\right)$$

⑦ Ex. 9 p. 273

a) & b) See Jupyter

c) $|y(t_i) - w_i| \leq 0.1$

Eq. 5.10 $\Rightarrow |y(t_i) - w_i| \leq \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right)$

$$\frac{\delta f(t,y)}{\delta y} = \frac{2}{t}$$

$$|f(t,y) - f(t,y')| \leq L|y - y'|$$

$$t \in [1,2] \Rightarrow |f(t,y) - f(t,y')| \leq \frac{2}{1}|y - y'|$$

$$\Rightarrow L = 2$$

Exact Solution: $y(t) = t^2(e^t - e)$

$$y'(t) = 2t(e^t - e) + e^t t^2$$

$$y''(t) = 2(e^t - e) + 2te^t + 2te^t + e^t t^2$$

$$= 4te^t + e^t t^2 + 2(e^t - e)$$

increasing function $\Rightarrow$ $|y''(t)| \leq |y''(2)| \quad \forall\, t \in [1,2]$

$$= 8e^2 + 4e^2 + 2e^2 - e = 14e^2 - e$$

$$= M$$

$$\frac{h \cdot M}{2L}\left(e^{L \cdot (t_i - a)} - 1\right)$$

$$\Rightarrow \quad \frac{h(14e^2 - e)}{4}\left(e^{2(t_i - 1)} - 1\right) \leq 0.1$$

$$\Leftrightarrow \quad \frac{h(14e^2 - e)}{4} \leq \frac{0.1}{e^{2(t_i - 1)} - 1}$$

$$\Leftrightarrow \quad h \leq \frac{0.4}{\left(e^{2(t_i - 1)} - 1\right)(14e^2 - e)}$$

Depending on the the value of t_i at which y_{t_i} is computed, the value for h can be computed with this formula. For example: if we want to compute $y(1.1)$ with the given error bound, we get the following restriction for h:

$$h \leq \frac{0.4}{\left(e^{2(1.1 - 1)} - 1\right)(14e^2 - e)} \approx 0.00884$$

(8) Ex. 5c p. 280

$$y' = \frac{(y^2 + y)}{t}, \quad 1 \le t \le 3, \quad y(1) = -2, \quad h = 0.5$$

we need the derivative wrt t of $f(t, y(t)) = \frac{y(t)^2 + y(t)}{t}$

$$f'(t,y) = \frac{t(2y'y + y') - (y^2+y)}{t^2}$$

So with $y' = \frac{y^2+y}{t}$, $f'(t,y) = \frac{t \cdot (2y \cdot \frac{y^2+y}{t} + \frac{y^2+y}{t}) - y^2 - y}{t^2}$

$$= \frac{2y\,(y^2+y)}{t^2}$$

So, $T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$

$$= \frac{w_i^2 + w_i}{t_i} + \frac{0.5}{2}\left(\frac{2w_i\,(w_i^2 + w_i)}{t_i^2}\right)$$

$$= \frac{w_i^2 + w_i}{t_i}\left(1 + \frac{0.5}{2} \cdot \frac{2w_i}{t_i}\right)$$

We have $1 \leq t \leq 3$ and $h = 0.5$, so that gives us

$$\frac{b-a}{N} = h \iff \frac{3-1}{N} = 0.5 \iff N = 4$$

$$t_i = 1 + i \cdot 0.5$$

So the second order Method becomes:

$$w_0 = -2$$

$$w_{i+1} = w_i + 0.5 \cdot \left(\frac{w_i^2 + w_i}{1 + 0.5i} \left(1 + \frac{w_i}{2(1+0.5i)}\right)\right)$$

See the Jupyter Notebook for the N solutions.

(9) Ex. 29 p. 293

We have $y' = f(t,y) = -y + t + 1$, $0 \le t \le 1$, $y(0) = 1$ and $t_i = hi$

<u>Midpoint:</u> $w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$

$$\Rightarrow w_{i+1} = w_i + h\left(-\left(w_i + \frac{h}{2} f(t_i, w_i)\right) + t_i + \frac{h}{2} + 1\right)$$

$$= w_i + h\left(-\left(w_i + \frac{h}{2}(-w_i + t_i + 1)\right) + t_i + \frac{h}{2} + 1\right)$$

$$= w_i + h\left(-w_i - \frac{h}{2}(-w_i + t_i + 1) + t_i + \frac{h}{2} + 1\right)$$

$$= w_i + h\left(-w_i + w_i \frac{h}{2} - \frac{t_i h}{2} - \cancel{\frac{h}{2}} + t_i + \cancel{\frac{h}{2}} + 1\right)$$

$[t_i = h \cdot i]$
$$= w_i + h\left(\left(\frac{h}{2} - 1\right) w_i + \left(1 - \frac{h}{2}\right) h \cdot i + 1\right)$$

<u>Modified Euler:</u> $w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))\right)$

$$= w_i + \frac{h}{2}\left(-w_i + t_i + 1 + \left(-\left(w_i + h f(t_i, w_i)\right) + t_{i+1} + 1\right)\right)$$

$$= w_i + \frac{h}{2}\left(-w_i + t_i + 1 + \left(-\left(w_i + h(-w_i + t_i + 1)\right) + t_{i+1} + 1\right)\right)$$

$$= w_i + \frac{h}{2}\left(-w_i + t_i + 1 - w_i - h(-w_i + t_i + 1) + t_{i+1} + 1\right)$$

$$= w_i + \frac{h}{2}\left(-w_i + t_i + 1 - w_i + hw_i - ht_i - h + t_{i+1} + 1\right)$$

$$= w_i + \frac{h}{2}\left(-2w_i + 2 + t_i + t_{i+1} - ht_i + hw_i - h\right)$$

$$= w_i + \frac{h}{2}\left(-2w_i + 2 + h \cdot i + h \cdot (i+1) - h^2 \cdot i + h \cdot w_i - h\right)$$

$$= w_i + \frac{h}{2}\left(-2w_i + 2 + h(i + i \cancel{+1} - h \cdot i + w_i \cancel{-1})\right)$$

$$= w_i + \frac{h}{2}\left(-2w_i + 2 + h(i(2-h) + w_i)\right)$$

$$= w_i + h\left(-w_i + 1 + \frac{h}{2} i(2-h) + \frac{h}{2} w_i\right)$$

$$= w_i + h\left(\left(\frac{h}{2} - 1\right) w_i + \frac{h}{2} i(2-h) + 1\right)$$

$$= w_i + h\left(\left(\frac{h}{2} - 1\right) w_i + \left(1 - \frac{h}{2}\right) h_i + 1\right)$$

→ which can now be recognized as the same as the Midpoint Method (see last step of simplification

This is true because both the Midpoint Method and the modified Euler Method are derived from Runge-Kutta Methods, and in the case of this problem the higher order Taylor-

Approximation of the Modified Euler Method with 4 parameters reduces to a 3-parameter Approximation, which happens to be equivalent to the Midpoint Method.

(10) Ex. 12 for problem 1a, p. 356

<u>Backward Euler Method:</u>

$w_{i+1} = w_i + h f(t_{i+1}, w_{i+1})$ for $i = 0, \dots, N-1$, $h = 0.1 \Rightarrow N = 10$, $y(0) = e$

$y' = f(t, y) = -9y$

$\Rightarrow w_{i+1} = w_i - 0.9 w_{i+1}$

<u>Newton-Iteration</u> for w_{i+1}

take $w_{i+1}^{(0)} = w_i$

$$w_{i+1}^{(k)} = w_{i+1}^{(k-1)} - \frac{g(w_{i+1}^{(k-1)})}{g'(w_{i+1}^{(k-1)})}$$

with $g(w_{i+1}) = 1.9 w_{i+1} - w_i$

$\rightarrow$ So the function g always depends on the last w_i

$$\Rightarrow \quad w_{i+1}^{(k)} = w_{i+1}^{(k-1)} - \frac{1.9 w_{i+1}^{(k-1)} - w_i}{1.9}$$

Results for Backward Euler Method, Euler Method and Actual Solution: See Jupyter