Lyapunov Analysis

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① Given system dynamics $\dot{x} = f(x)$, f continuous

② Find $V(x) > 0$, where $V(x)$ is continuously-differentiable,
$V(0) = 0$,

$$\dot{V} = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$$

$\dot{V}(0) = 0$

③ Claim system is Lyapunov / Asymptotically / Exponentially stable
$(\dot{V} \leq 0)$ $(\dot{V} < 0)$ $(\dot{V} \leq -\alpha V)$

思考: ① HJB gives optimal costtogo function : $0 = \min_u \left[g(x,u) + \frac{\partial J^*}{\partial x} f(x,u) \right]$

$\Rightarrow \dot{J}^*(x) = -g(x, u^*)$ while $\dot{V}(x) \leq 0$

$\therefore$ Optimal cost togo has more strict math & computation requirement.

$J^*(x) \subset V(x)$

La Salle's Invariance Principle

① Given System dynamics $\dot{x} = f(x)$, f continuous.

② Find $V(x)$, $V(x) > 0$

$\dot{V}(x) \leq 0$

$x \to \infty \Rightarrow V(x) \to \infty$

③ Claim : x will converge to the largest invariant set G where $\dot{V}(x) = 0$

Def of invariant set : if $x(0)$ in invariant set, $\forall t$, $x(t)$ is in invariant set.

largest invariant set G is collection of fixed points.

思考: The system has many points where $\dot{V} = 0$, including fixed point.

But the system could only stay @ fixed point. not other transient point

How to find/check Lyapunov Function Satisfy

$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x)$. if $V(x)$ & $f(x)$ are complex, it is hard!

① Sample points x & check $V(x) > 0$, $\dot{V}(x) \leq 0$

② For Linear system $\dot{x} = Ax$, find $P > 0$ & $PA + A^T P < 0$

$$\therefore \begin{cases} V(x) = x^T P x , & P = P^T > 0 , \\ \dot{V}(x) = x^T P A x + x^T A^T P x < 0 \end{cases}$$

(matrix lyapunov function)

If A is stable, then $\exists P > 0, Q > 0$ st $PA + A^T P = -Q$.

in other words, for stable linear system, we can always find lyapunov func

③ Use SDP to solve (search) for lyapunov function

eg: Given linear system A,

$\underset{P}{\text{find}}$, st: $P \geq 0$, $PA + A^T P \leq 0$

④ Use SOS optimization. SOS means ≥ 0

$\underset{\alpha}{\text{find}}$, st $V_\alpha(x)$ is SOS

$-\dot{V}_\alpha(x) = -\frac{\partial V_\alpha(x)}{\partial x} \cdot f(x)$ is SOS

or S-procedure

$$V_\alpha(x) = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_k x_0 x_1 + \cdots + \alpha_L x_0^2 + \alpha_{L+1} x_1^2 + \cdots$$

$\underset{\alpha, \lambda}{\text{find}}$ st V is SOS , $-\dot{V} - \lambda(x) \cdot (\text{constraint})$ is SOS

constraint is equality or inequality that is ≥ 0

Sum of Square

Def: $p(x)$ is a polynomial.

we could find $\min_{x} p(x)$ easily if $p(x)$ is sum of square.

by $\max_{\lambda} \lambda$

st $p(x) - \lambda$ is sos

If $p(x)$ is sos, we could find λ, st $p(x) + \lambda \geq 0, \forall x$

($p(x)$ has min bound)

∴ ① we could find lyapunov $V_\alpha(x)$ by sos optimization

$$\text{find}_{\alpha}, \ \text{st} \quad V_\alpha(x) \text{ is SOS}$$
$$-\dot{V}_\alpha(x) = -\frac{\partial V_\alpha(x)}{\partial x} \cdot f(x) \text{ is SOS}$$

② we could verify lyapunov $V(x)$ by sos.

assume $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, Set $V(x) = C_0 + C_1 x_1 + C_2 x_2 + C_3 x_1^2 + C_4 x_1 x_2 + C_5 x_2^2$

$$\text{find}_{c} \quad \text{st} \quad V \text{ is SOS}$$
$$-\dot{V} \text{ is SOS}$$