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# 1

## Principle of Least Action

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You've all suffered through a course on Newtonian mechanics, and you all know how to calculate the way things move: you draw a pretty picture; you draw arrows representing forces; you add them all up; and then you use  $F = ma$  to figure out where things are heading next. All of this is rather impressive—it really is the way the world works and we can use it to understand things about Nature. For example, showing how the inverse square law for gravity explains Kepler's observations of planetary motion is one of the great achievements in the history of science.

However, there's a better way to do it. This better way was found about 150 years after Newton, when classical mechanics was reformulated by some of the giants of mathematical physics—people like Lagrange, Euler and Hamilton. This new way of doing things is better for a number of reasons:

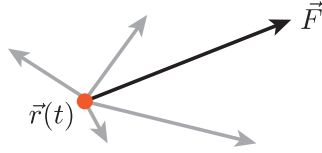
- Firstly, it's elegant. In fact, it's not just elegant: it's completely gorgeous. In a way that theoretical physics should be, and usually is, and in a way that the old Newtonian mechanics really isn't.
- Secondly, it's more powerful. It gives new methods to solve hard problems in a fairly straightforward manner. Moreover, it is the best way of exploiting the power of symmetries (see Lecture 6). And since these days all of physics is based on fundamental symmetry principles, it is really the way to go.
- Finally, and most importantly, it is universal. It provides a framework that can be extended to all other laws of physics, and reveals a deep relationship between classical mechanics and quantum mechanics. This is the real reason why it's so important.

In this lecture, I'll show you the key idea that leads to this new way of thinking. It's one of the most profound results in physics. But, like many profound results, it has a rubbish name. It's called the **principle of least action**.

## 1.1 A New Way of Looking at Things

### 1.1.1 Newtonian Mechanics

Let's start simple. Consider a single particle at position  $\vec{r}(t)$ , acted upon by a force  $\vec{F}$ . You might have gotten that force by adding up a bunch of different forces



Sir Isaac tells us that

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}. \quad (1.1.1)$$

The goal of classical mechanics is to solve this differential equation for different forces: gravity, electromagnetism, friction, etc. For conservative forces (gravity, electrostatics, but not friction), the force can be expressed in terms of a potential,

$$\vec{F} = -\vec{\nabla}V. \quad (1.1.2)$$

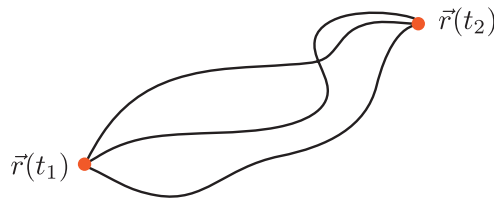
The potential  $V(\vec{r})$  depends on  $\vec{r}$ , but not on  $\dot{\vec{r}}$ . Newton's equation then reads

$$m\ddot{\vec{r}} = -\vec{\nabla}V. \quad (1.1.3)$$

This is a second-order differential equation, whose general solution has two integration constants. Physically this means that we need to specify the initial position  $\vec{r}(t_1)$  and velocity  $\dot{\vec{r}}(t_1)$  of the particle to figure out where it is going to end up.

### 1.1.2 A Better Way

Instead of specifying the initial position and velocity, let's instead choose to specify the initial and *final* positions,  $\vec{r}(t_1)$  and  $\vec{r}(t_2)$ , and consider the possible paths that connect them:



What path does the particle actually take?

Let's do something strange: to each path  $\vec{r}(t)$ , we assign a number which we call the *action*,

$$S[\vec{r}(t)] = \int_{t_1}^{t_2} dt \left( \frac{1}{2} m \dot{\vec{r}}^2 - V(\vec{r}) \right). \quad (1.1.4)$$

Notice the all important *minus* sign in the integrand. The action is the *difference* between the kinetic energy (K.E.) and the potential energy (P.E.) (while the total energy is the *sum*), integrated over the path. Here is an astounding claim:

*The true path taken by the particle is an extremum of the action  $S$ .*

Let us prove it:

You know how to find the extremum of a function—you differentiate and set it equal to zero. But the action not a function, it is a *functional*—a function of a function. That makes it a slightly different problem. You will learn how to solve problems of this type in next year’s “methods” course, when you learn about “calculus of variations”.

It is really not as fancy as it sounds, so let’s just do it for our problem: consider a given path  $\vec{r}(t)$ . We ask how the action changes when we change the path slightly

$$\vec{r}(t) \rightarrow \vec{r}(t) + \delta\vec{r}(t) , \quad (1.1.5)$$

but in such a way that we keep the end points of the path fixed

$$\delta\vec{r}(t_1) = \delta\vec{r}(t_2) = 0 . \quad (1.1.6)$$

The action for the perturbed path  $\vec{r} + \delta\vec{r}$  is

$$S[\vec{r} + \delta\vec{r}] = \int_{t_1}^{t_2} dt \left[ \frac{1}{2}m \left( \dot{\vec{r}}^2 + 2\dot{\vec{r}} \cdot \delta\dot{\vec{r}} + \delta\dot{\vec{r}}^2 \right) - V(\vec{r} + \delta\vec{r}) \right] . \quad (1.1.7)$$

We can Taylor expand the potential

$$V(\vec{r} + \delta\vec{r}) = V(\vec{r}) + \vec{\nabla}V \cdot \delta\vec{r} + \mathcal{O}(\delta\vec{r}^2) . \quad (1.1.8)$$

The difference between the action for the perturbed path and the unperturbed path is

$$\delta S \equiv S[\vec{r} + \delta\vec{r}] - S[\vec{r}] = \int_{t_1}^{t_2} dt \left[ m\dot{\vec{r}} \cdot \delta\dot{\vec{r}} - \vec{\nabla}V \cdot \delta\vec{r} \right] + \dots \quad (1.1.9)$$

$$= \int_{t_1}^{t_2} dt \left[ -m\ddot{\vec{r}} - \vec{\nabla}V \right] \cdot \delta\vec{r} + \left[ m\dot{\vec{r}} \cdot \delta\vec{r} \right]_{t_1}^{t_2} . \quad (1.1.10)$$

To go from the first to the second line we integrated by parts. This picks up a term that is evaluated at the boundaries  $t_1$  and  $t_2$ . However, this term vanishes since the end points are fixed, i.e.  $\delta\vec{r}(t_1) = \delta\vec{r}(t_2) = 0$ . Hence, we get

$$\delta S = \int_{t_1}^{t_2} dt \left[ -m\ddot{\vec{r}} - \vec{\nabla}V \right] \cdot \delta\vec{r} . \quad (1.1.11)$$

The condition that the path we started with is an extremum of the action is

$$\delta S = 0 . \quad (1.1.12)$$

This should hold for *all* changes  $\delta\vec{r}(t)$  that we could make to the path. The only way this can be true is if the expression in  $[\dots]$  is zero. This means

$$m\ddot{\vec{r}} = -\vec{\nabla}V . \quad (1.1.13)$$

But this is just Newton’s equation (1.1.3)! Requiring that the action is an extremum is equivalent to requiring that the path obeys Newton’s equation. It’s magical.

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Some bookkeeping: the integrand of the action is called the *Lagrangian*  $L(\vec{r}, \dot{\vec{r}}, t)$ , i.e.  $S = \int_{t_1}^{t_2} dt L$ . In our simple example, we have  $L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r})$ . From the Lagrangian we can define a generalized momentum and a generalized force,

$$\vec{P} \equiv \frac{\partial L}{\partial \dot{\vec{r}}} \quad \text{and} \quad \vec{F} \equiv \frac{\partial L}{\partial \vec{r}} . \quad (1.1.14)$$

In our simple example, these definitions reduce to the usual ones:  $\vec{P} = m\dot{\vec{r}}$  and  $\vec{F} = -\vec{\nabla}V$ . Newton's equation can then be written as

$$\frac{d\vec{P}}{dt} = \vec{F} \quad \Leftrightarrow \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\vec{r}}} \right) = \frac{\partial L}{\partial \vec{r}} . \quad (1.1.15)$$

This is called the *Euler-Lagrange equation*.

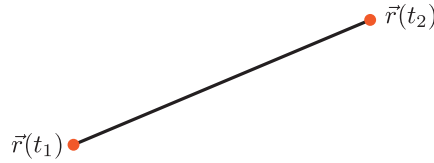
### 1.1.3 Examples

To get a bit more intuition for this strange way of doing mechanics, let us discuss two simple examples:

- Consider the Lagrangian of a free particle

$$L = \frac{1}{2}m\dot{\vec{r}}^2 . \quad (1.1.16)$$

In this case the least action principle implies that we want to minimize the kinetic energy over a fixed time. So the particle must take the most direct route, which is a straight line:

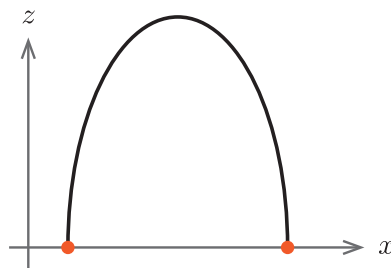


But, do we slow down to begin with, and then speed up? Or, do we accelerate like mad at the beginning and slow down near the end? Or, do we go at a uniform speed? The average speed is, of course, fixed to be the total distance over the total time. So, if you do anything but go at a uniform speed, then sometimes you are going too fast and sometimes you are going too slow. But, the mean of the square of something that deviates around an average, is always greater than the square of the mean. (If this isn't obvious, please think about it for a little bit.) Hence, to minimize the integrated kinetic energy—i.e. the action—the particle should go at a uniform speed. In the absence of a potential, this is of course what we should get.

- As a slightly more sophisticated example, consider a particle in a uniform gravitational field. The Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{z}^2 - mgz . \quad (1.1.17)$$

Imagine that the particle starts and ends at the same height  $z_0 = z(t_1) = z(t_2)$ , but moves horizontally from  $x_1 = x(t_1)$  to  $x_2 = x(t_2)$ . This time we don't want to go in a straight line. Instead, we can minimize the difference between K.E. and P.E. if we go up, where the P.E. is bigger. But we don't want to go too high either, since that requires a lot of K.E. to begin with. To strike the right balance, the particle takes a parabola:



At this point, you could complain: what is the big deal? I could easily do all of this with  $F = ma$ . While this is true for the simple examples that I gave so far, once the examples become more complicated,  $F = ma$  requires some ingenuity, while the least action approach stays completely fool-proof. No matter how complicated the setup, you just count all kinetic energies, subtract all potential energies and integrate over time. No messing around with vectors. You will see many examples of the power of the least action approach in future years. I promise you that you will fall in love with this way of doing physics!

## 1.2 Unification of Physics\*

Today, we use the Lagrangian method to describe all of physics, not just mechanics. All fundamental laws of physics can be expressed in terms of a least action principle. This is true for electromagnetism, special and general relativity, particle physics, and even more speculative pursuits that go beyond known laws of physics such as string theory.

To really explain this requires many concepts that you don't know yet. Once you learn these things, you will be able to write all of physics on a single line. For example, (nearly) every experiment ever performed can be explained by the Lagrangian of the Standard Model

$$\mathcal{L} \sim \underbrace{R}_{\text{Einstein}} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Maxwell Yang-Mills}} + \underbrace{i\bar{\psi}\gamma^\mu D_\mu\psi}_{\text{Dirac}} + \underbrace{|D_\mu h|^2 - V(|h|)}_{\text{Higgs}} + \underbrace{h\bar{\psi}\psi}_{\text{Yukawa}} . \quad (1.2.18)$$

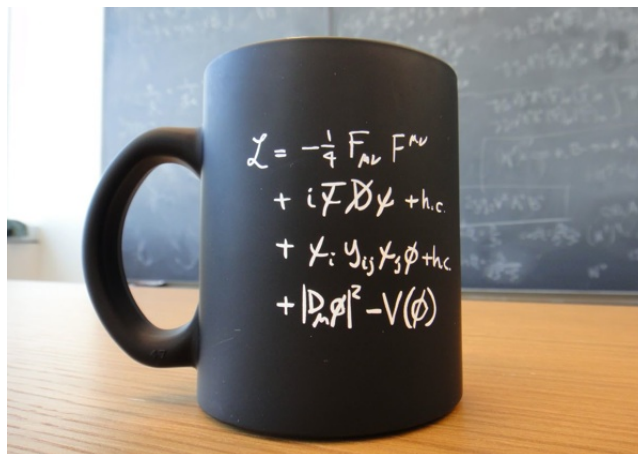
Don't worry if you don't understand many of the symbols. You are not supposed to. View this equation like art.

Let me at least tell you what each of the terms stands for:

The first two terms characterize all fundamental forces in Nature: The term 'Einstein' describes gravity. Black holes and the expansion of the universe follow from it (see Lectures 7 and 8). The next term, 'Maxwell', describes electric and magnetic forces (which, as we will see, are just different manifestations of a single (electromagnetic) force). A generalization of this, 'Yang-Mills', encodes the strong and the weak nuclear forces (we will see what these are in Lecture 5). The next term, 'Dirac', describes all matter particles (collectively called fermions)—things like electrons, quarks and neutrinos. Without the final two terms, 'Higgs' and 'Yukawa', these matter particles would be massless.

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The Lagrangian (1.2.18) is a compact way to describe all of fundamental physics. You can print it on a T-shirt or write it on a coffee mug:



## 1.3 From Classical to Quantum (and back)

### 1.3.1 Sniffing Out Paths

As we have seen, the principle of least action gives a very different way of looking at things:

In the Newtonian approach, the intuition for how particles move goes something like this: at each moment in time, the particle thinks “where do I go now?”. It looks around, sees the potential, differentiates it and says “ah-ha, I go this way.” Then, an infinitesimal moment later, it does it all again.

The Lagrangian approach suggests a rather different viewpoint: Now the particle is taking the path which minimizes the action. How does it know this is the minimal path? It is sniffing around, checking out all paths, before it decides: “I think I’ll go this way”.

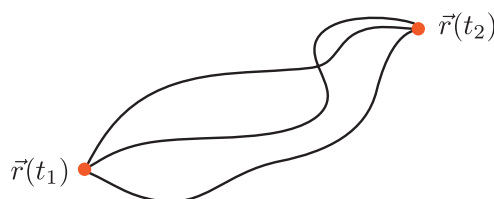
On some level, this philosophical pondering is meaningless. After all, we proved that the two ways of doing things are completely equivalent. This changes when we go beyond classical mechanics and discuss **quantum mechanics**. There we find that the particle really does sniff out every possible path!

### 1.3.2 Feynman’s Path Integral

*“Thirty-one years ago, Dick Feynman told me about his “sum over histories” version of quantum mechanics. “The electron does anything it likes,” he said. “It just goes in any direction at any speed, . . . however it likes, and then you add up the amplitudes and it gives you the wavefunction.” I said to him, “You’re crazy.” But he wasn’t.”*

Freeman Dyson

I have been lying to you. It is not true that a free particle only takes one path (the straight line) from  $\vec{r}(t_1)$  to  $\vec{r}(t_2)$ . Instead, according to quantum mechanics it takes *all possible paths*:



It can even do crazy things, like go backwards in time or first go to the Moon. According to quantum mechanics all these things can happen. But they happen *probabilistically*. At the deepest level, Nature is probabilistic and things happen by random chance. This is the key insight of quantum mechanics (see Lecture 2).

The probability that a particle starting at  $\vec{r}(t_1)$  will end up at  $\vec{r}(t_2)$  is expressed in terms of an *amplitude*  $\mathcal{A}$ , which is a complex number that can be thought of as the square root of the probability

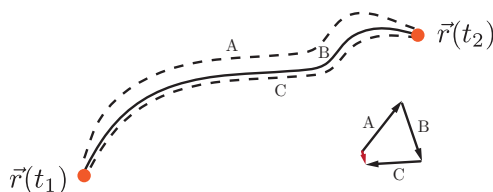
$$\text{Prob} = |\mathcal{A}|^2 . \quad (1.3.19)$$

To compute the amplitude, you must sum over all path that the particle can take, weighted by a *phase*

$$\mathcal{A} = \sum_{\text{paths}} e^{iS/\hbar} . \quad (1.3.20)$$

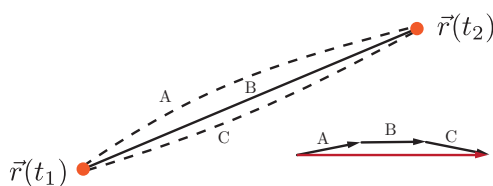
Here, the phase for each path is just the value of the action for the path in units of Planck's constant  $\hbar = 1.05 \times 10^{-34} \text{J}\cdot\text{s}$ . Recall that complex numbers can be represented as little arrows in a two-dimensional  $xy$ -plane. The length of the arrow represents the magnitude of the complex number, the phase represents its angle relative to say the  $x$ -axis. Hence, in eq. (1.3.20) each path can be represented by a little arrow with phase (angle) given by  $S/\hbar$ . Summing all arrows and then squaring gives the probability. This is Feynman's *path integral approach to quantum mechanics*. Sometimes it is called *sum over histories*.

The way to think about this is as follows: when a particle moves, it really does take all possible paths. However, away from the classical path (the straight line), the action varies wildly between neighbouring paths, and the sum of different paths therefore averages to zero:



i.e. far from the classical path, the little arrows representing the complex amplitudes of each path point in random directions and cancel each other out.

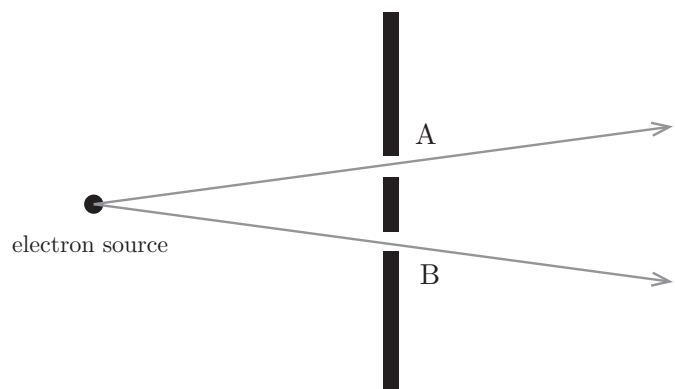
Only near the classical path do the phases of neighbouring paths reinforce each other:



i.e. near the classical path, the little arrows representing the complex amplitudes of each path point in the same direction and therefore add. This is how classical physics emerges from quantum physics. It is totally bizarre, but it is really the way the world works.

### 1.3.3 Seeing is Believing

You don't believe me that quantum particles really take all possible paths? Let me prove it to you. Consider an experiment that fires electrons at a screen with two slits, "one at a time":

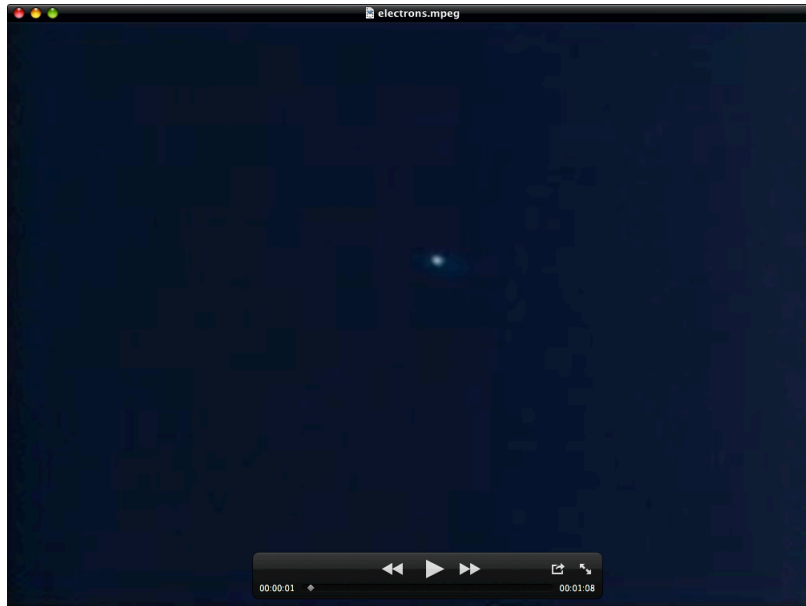


In Newtonian physics, each electron either takes path A OR B. We then expect the image behind the two slits just to be the sum of electrons going through slit A or B, i.e. we expect the electrons to build up a pair of stripes—one corresponding to electrons going through A, and one corresponding to electrons going through B.

Now watch this video of the actual experiment:

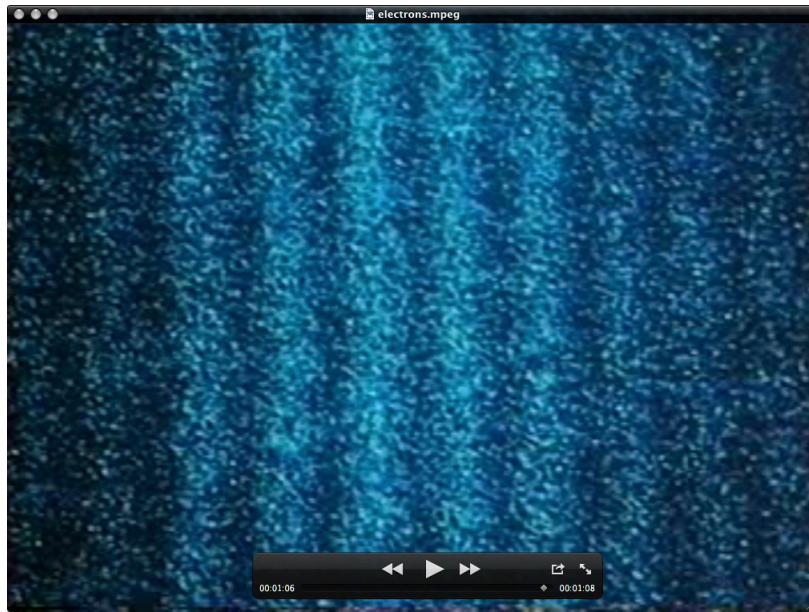
[www.damtp.cam.ac.uk/user/db275/electrons.mpeg](http://www.damtp.cam.ac.uk/user/db275/electrons.mpeg)

We see that the electrons are recorded one by one:





Slowly a pattern develops, until finally we see this:



We don't just see two strips, but a series of strips. In quantum physics, each electron seems to take the paths A AND B simultaneously, and interferes with itself! (Since the electrons are fired at the slits one by one, there is nobody else around to interfere with.) It may seem crazy, but it is really the way Nature ticks. You will learn more about this in various courses on **quantum mechanics** over the next few years. I will tell you a little bit more about the strange quantum world in Lecture 2.