Euler-Lagrange Dynamics Technical Conclusion

① Lagrange Equation : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$, $L = T - V$

Kinetic Potential

② Because of D'alembert : $m_i \ddot{r}_i \Leftrightarrow Q_j^*$

$-Q_j^* = q_j = Q_j + Q_K$, $j = 1, \cdots 3N\text{-}K$, $Q_j = M\ddot{r}_i$

势力 各广义坐标方向的外力

$\because -Q_j^* = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$ (1) ,

惯性力的广义力

when Q_j is conservative force

$Q_j = -\frac{\partial V}{\partial q_j} = \boxed{\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)} - \frac{\partial V}{\partial q_j}$ (2)

广义主动力 $\qquad \boxed{\;} = 0$

$\therefore$ ① stand

牛二在广义坐标下的表述，always true.

主动力 = { 势力 : $-\frac{\partial V}{\partial q_j}$ ⇐ 有势力做功改变势能 ; 外力 : Q_K }

if all Q_j are conservative,

$-Q_j^* = q_j \Leftrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$

if $\exists\, Q_K$ is non-conservative.

$-Q_j^* = q_j \Leftrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j$, $j = 1 \cdots k$

③ to prove (1) , $\frac{\partial T}{\partial \dot{r}_i} = M\dot{r}_i$

基本思想：

to get $M\ddot{r}_i$, derive $\frac{\partial T}{\partial \dot{r}_i}\delta r_i$

$\frac{\partial T}{\partial \dot{r}_i} = M\dot{r}_i$ & $\delta r_i = \sum_{j=1}^{3N-k}\frac{\partial r_i}{\partial q_j}\delta q_j$

$= \sum_{j=1}^{3N-k}\frac{\partial \dot{r}_i}{\partial \dot{q}_j}\delta q_j$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\delta r_i\right) = M\ddot{r}_i\delta r_i + M\dot{r}_i\frac{d}{dt}\delta r_i$$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\sum_{j=1}^{3N-k}\frac{\partial r_i}{\partial q_j}\delta q_j\right) = M\ddot{r}_i\delta r_i + \frac{\partial T}{\partial r_i}\delta \dot{r}_i$$

将牛二转化到广义坐标

$$\Leftrightarrow \sum_{j=1}^{3N-k}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\frac{\partial r_i}{\partial q_j}\right)\delta q_j - \frac{\partial T}{\partial r_i}\cdot\sum_{j=1}^{3N-k}\frac{\partial \dot{r}_i}{\partial q_j}\delta q_j = M\ddot{r}_i\delta r_i$$

④ D'alembert Proof

$$W = \sum_{i=1}^{N} \vec{F}_i \cdot \underset{\text{虚位移}}{\delta \vec{r}_i} \qquad N = 3N-K$$

$$= \sum_{i=1}^{N} \vec{F}_i \cdot \left(\sum_{j=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j\right)$$

$$= \sum_{j=1}^{3N-k} \underbrace{\left(\sum_{i=1}^{N} \frac{\partial \vec{r}_i}{\partial q_j} \vec{F}_i\right)}_{\text{合力在广义方向的分力(惯性力+主动力)}} \delta q_j$$

单个自由度在广义方向上的分量

外力、势力

$$= \sum_{j=1}^{3N-k} \underbrace{q_j}_{\text{广义力}} \underbrace{\delta q_j}_{\text{广义方向}} = 0 \qquad \Leftarrow \quad q_j = Q_j + Q_j^*$$

when Q_j is conservative force

$$Q_j = -\frac{\partial V}{\partial q_j} = \underbrace{\boxed{\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)}}_{=0} - \frac{\partial V}{\partial q_j}$$

⑤ Least Action: Lagrange say: when no non-consertive (outside) force the action is stationary (same) from x_0 to x_1, (non-relevant to path)

$\delta\left(\int_{x_0}^{x_1} L\, dx\right) = 0$, this lead to Lagrange's Equition

⑥ Calculate System Dynamics with Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_j} = Q_j, \quad j = 1, \cdots k,$$

generalized coordinate with non-conservative force Q_j

(1) define generalized coordinate, eliminate confined directions

(2) Calc $T = \sum \frac{1}{2}\dot{q}_j m \dot{q}_j$ & V. $L = T - V$.

(3) use $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_j} = Q_j$, get relation $\ddot{q}$ and q, u. etc