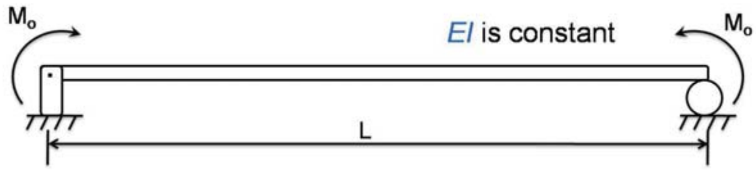
Weighted Residual Formulations and Verification of Results

Review: Formulations

- Direct Formulation
- Minimum Total Potential Energy Formulation
- Weighted Residual Formulation
  - Collocation Method
  - Subdomain Method
  - Galerkin's Method

Now which to use? It depends on the application. Remember these are being used on much more complicated models that may have 1000's or 10000's (or more!) of nodes. The best approach is the easiest, the fastest, and most accurate (i.e., forces to zero the error at one point of interest, average error, etc.) but usually it is a trade-off between all these criteria. This determination goes beyond this course, but it is a valuable consideration to keep in mind.

Example



where Y is the deflection distance and the boundary conditions are $Y = 0$ at $x = 0$ and $dY/dx = 0$ at $x = L/2$.

1 Exact Solution

Take the governing differential equation...

$$EI\frac{d^2Y}{dx^2} = M(x)$$

where $M(x) = M_o$ for all values of x in this problem.

Integrating with respect to x, we get:

$$\frac{dY}{dx} = \frac{M_o}{EI}x + C_1$$

Applying the boundary condition of $dY/dx = 0$ at $x = L/2$, we find

$$C_1 = -\frac{M_oL}{2EI}$$

Substituting for C_1 and integrating a second time with respect to x,

$$Y = \frac{M_o}{EI}\frac{x^2}{2} - \frac{M_oL}{2EI}x + C_2$$

Applying the boundary condition of $Y = 0$ at $x = 0$, we find that $C_2 = 0$. Therefore, our exact solution for deflection in a beam under pure bending is

$$Y = \frac{M_ox^2}{2EI} - \frac{M_oLx}{2EI} = \frac{M}{2EI}(x^2 - Lx)$$

2 Weighted Residual: Process

1. Start with the Governing Differential Equation
2. Assume a Solution Equation
3. This assumed Solution Equation must satisfy initial and boundary conditions
4. Because the assumed Solution is not exact, there will be Residuals (or Errors) remaining
5. Force the Residual (or Error) to zero at various points or intervals using the different Methods

This may be a bit confusing ... so check the other examples in the textbook!

Governing Differential Equation
As above, the bending moment equation,

$$EI\frac{d^2Y}{dx^2} = M(x)$$

where $M(x) = M_o$ for all values of x in this problem.

Assume a Solution Equation
We are trying to find Y. We'll assume an approximate solution of:

$$Y = A\sin\frac{\pi x}{L}$$

where A is an unknown coefficient, and we see that the boundary conditions listed above work in this solution.

Determine the Residual (Error) Function
Rearranging the differential equation:

$$EI\frac{d^2Y}{dx^2} - M_o = 0$$

For exact solutions of Y, the differential equation will equal zero. If Y is not exactly correct, there will be error or Residual, R.

$$R = EI\frac{d^2Y}{dx^2} - M_o$$

Where:

$$Y = A\sin\frac{\pi x}{L}$$

$$\frac{dY}{dx} = A\frac{\pi}{L}\cos\frac{\pi x}{L}$$

$$\frac{d^2Y}{dx^2} = -A\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

Combining, we get the error function:

$$R = EI\left(-A\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}\right) - M_o$$

2.1 Collocation Method

In this method, the Residual equation, R, is forced to be zero at as many points as there are unknown coefficients. Since there is only one point to have no error (i.e., one unknown coefficient, A) . . . how about trying in the middle at . . .

$$x = \frac{L}{2}$$

$$\therefore R \text{ at } \left(x = \frac{L}{2}\right) = EIA\frac{\pi^2}{L^2}\sin\frac{\pi}{L}\left(\frac{L}{2}\right) + M_o = 0$$

Solving for A:

$$A = -\frac{M_oL^2}{EI\pi^2}$$

Substituting A back into our assumed solution:

$$\therefore Y = -\frac{M_oL^2}{EI\pi^2}\sin\frac{\pi x}{L}$$

Note: "A" would have been different if we had chosen a point other than $x = L/2$.

2.2 Subdomain Method

In this method, the integral of the Residual equation, R, is forced to be zero over as many subintervals as there are unknown coefficients.

$$\int_0^L R\,dx = \int_0^L \left[EI\left(-A\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}\right) - M_o\right]\,dx = 0$$

$$\left(AEI\frac{\pi}{L}\cos\frac{\pi x}{L} - M_o x\right)\Big|_0^L = 0$$

$$\left(-AEI\frac{\pi}{L} - M_o L - AEI\frac{\pi}{L}\right) = 0$$

$$A = -\frac{M_o L^2}{2\pi EI}$$

Substituting A back into our assumed solution:

$$Y = -\frac{M_o L^2}{2\pi EI}\sin\frac{\pi x}{L}$$

Note: if we had 2 unknown coefficients ... say A and B, then we would have two intervals:

$$\int_0^{L/2} R\,dx = 0$$

$$\int_{L/2}^{L} R\,dx = 0$$

Solve for A and B. (If we had 3 unknown coefficients, the intervals would be $L/3$, etc.)

2.3 Galerkin Method

In this method, the integral of the Residual equation, R, times a weighting function, Φ, is forced to be zero. The weighting functions are chosen to be the same form as each part of the approximate solution.

$$\int_0^L \Phi_i R\,dx = 0 \qquad i = 1, 2, \ldots \text{ \# of unknown coefficients}$$

Example 1

In this example:

$$\Phi_i = \sin\frac{\pi x}{L}$$

Substitute in Φ_i and R; solve for A:

$$\int_0^L \sin\frac{\pi x}{L}\left[EI\left(-A\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}\right) - M_o\right]\,dx = 0 \quad\longrightarrow\quad A = -\frac{4M_o L^2}{\pi^3 EI}$$

Substituting A back into our assumed solution:

$$Y = -\frac{4M_o L^2}{\pi^3 EI}\sin\frac{\pi x}{L}$$

Example 2

If the approximate solution had been ...

$$Y = Ax + Bx^2 + Cx^3$$

then:

$$\Phi_1 = x, \quad \Phi_2 = x^2, \quad \Phi_3 = x^3$$

There would be three equations to solve for A, B, and C.

$$\int_0^L \Phi_1 R \, dx = 0$$

$$\int_0^L \Phi_2 R \, dx = 0$$

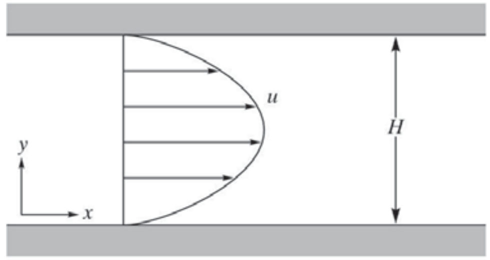
$$\int_0^L \Phi_3 R \, dx = 0$$

Galerkin Method is *one of the more common FE solutions*.

2.4 Comparison of Methods

	Exact	Collocation	Subdomain	Galerkin
$x =$	$\frac{M_o}{2EI}(x^2 - Lx)$	$-\frac{M_o L^2}{\pi^2 EI} \sin \frac{\pi x}{L}$	$-\frac{M_o L^2}{2\pi EI} \sin \frac{\pi x}{L}$	$-\frac{4M_o L^2}{\pi^3 EI} \sin \frac{\pi x}{L}$
0	0	0	0	0
$L/4$	$-0.094\frac{M_o L^2}{EI}$	$-0.072\frac{M_o L^2}{EI}$	$-0.113\frac{M_o L^2}{EI}$	$-0.091\frac{M_o L^2}{EI}$
$L/3$	$-0.111\frac{M_o L^2}{EI}$	$-0.088\frac{M_o L^2}{EI}$	$-0.138\frac{M_o L^2}{EI}$	$-0.112\frac{M_o L^2}{EI}$
$L/2$	$-0.125\frac{M_o L^2}{EI}$	$-0.101\frac{M_o L^2}{EI}$	$-0.159\frac{M_o L^2}{EI}$	$-0.129\frac{M_o L^2}{EI}$

3 Example #2



The leakage flow of hydraulic fluid through the gap between a piston-cylinder arrangement may be modeled as laminar flow of fluid between infinite parallel plates, as shown in the accompanying figure. This model offers reasonable results for relatively small gaps. The differential equation governing the flow is

$$\mu \frac{d^2u}{dy^2} = \frac{dP}{dx}$$

where μ is the dynamic viscosity of the hydraulic fluid, u is the fluid velocity, and dP/dx is the pressure drop and is constant. Derive the equation for the exact fluid velocities. Assume an approximate fluid velocity solution of the form

$$u(y) = A\left[\sin\left(\frac{\pi y}{H}\right)\right]$$

Use the following methods to evaluate A: (a) the collocation method, (b) the subdomain method, and (c) the Galerkin method. Compare the approximate results to the exact solution.