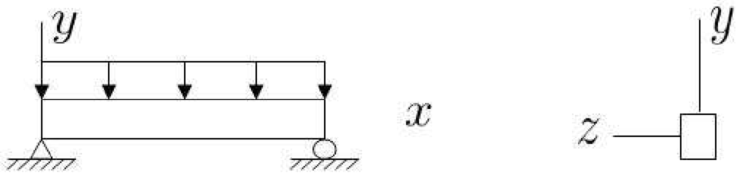
Beams: FE Formulation

1 Beam Refresher

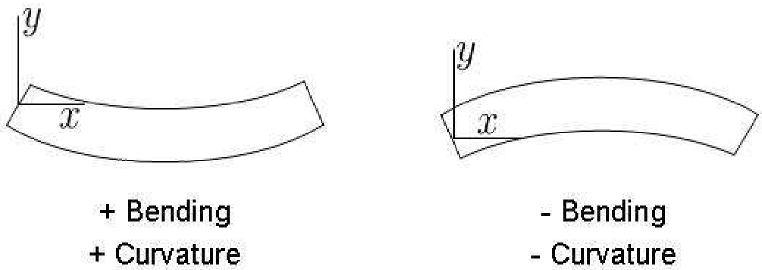
Definition: A beam is defined as a structural member whose *cross-sectional dimensions are relatively small compared to its length.*

- Primarily carry *lateral loading* that creates bending.
- Shear is typically ignored.

1.1 Sign Convention and Notation:



1.2 Members in Axial Loading



Governing Stress Equation (Flexure Formula)

$$\sigma = \frac{-Mc}{I} \quad \text{or} \quad \sigma = \frac{-My}{I}$$

(Negative sign means that positive values of y and c are in compression for positive bending.)

1.3 Governing Load & Shape Equations

$$\theta = \frac{dy}{dx}$$

$$M = EI\frac{d\theta}{dx} \quad \text{or} \quad M = EI\frac{d^2y}{dx^2}$$

$$V = \frac{dM}{dx} \quad \text{or} \quad V = EI\frac{d^3y}{dx^3}$$

$$\omega = \frac{dV}{dx} \quad \text{or} \quad \omega = EI\frac{d^4y}{dx^4}$$

1.4 Beam Tables

Common Loading and Shapes can be solved using Beam Tables such as this one (Table 4.1) from Moaveni's textbook.

TABLE 4.1 Deflections and slopes of beams under some typical loads and supports

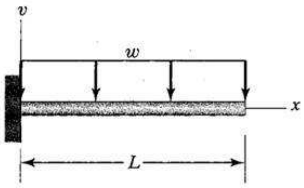
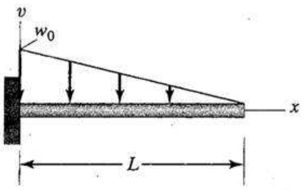
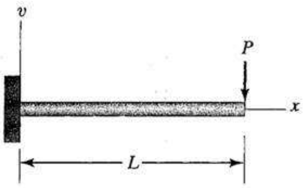
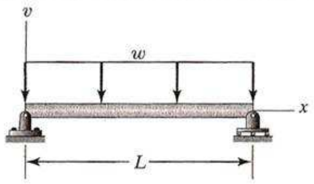
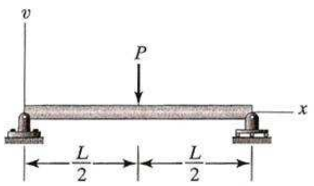
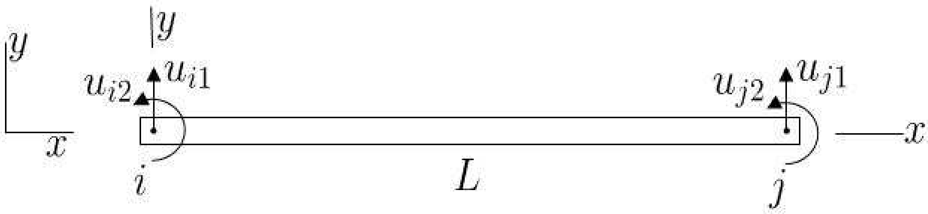
Beam Support and Load	Equation of Elastic Curve	Maximum Deflection	Slope
	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$	$v_{max} = \frac{-wL^4}{8EI}$	$\theta_{max} = \frac{-wL^3}{6EI}$
	$v = \frac{-w_0 x^2}{120LEI}(-x^3 + 5Lx^2 - 10L^2x + 10L^3)$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$\theta_{max} = \frac{-w_0 L^3}{24EI}$
	$v = \frac{-Px^2}{6EI}(3L - x)$	$v_{max} = \frac{-PL^3}{3EI}$	$\theta_{max} = \frac{-PL^2}{2EI}$

TABLE 4.1 (*continued*) Deflections and slopes of beams under some typical loads and supports

Beam Support and Load	Equation of Elastic Curve	Maximum Deflection	Slope
	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$	$v_{max} = \frac{-5wL^4}{384EI}$	$\theta_{max} = \frac{-wL^3}{24EI}$
	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ for $\left(x \le \frac{L}{2}\right)$	$v_{max} = \frac{-PL^3}{48EI}$	$\theta_{max} = \frac{-PL^2}{16EI}$

2 Finite Element Formulation of Beams

The beam element (we'll only use local coordinates since beams are horizontal.)



Only two degrees of freedom (DOF) at each node (Note: No Axial Displacement):

1. Lateral Displacement (unit: length)
2. Rotation/Slope (unit: radians)

Assume an equation for the deflected beam shape:

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3$$

3^{rd}-order polynomial with four unknown coefficients since we have four DOF.

2.1 Boundary Conditions:

$$y = u_{i1} \quad \text{at} \quad x = 0$$
$$\frac{dy}{dx} = u_{i2} \quad \text{at} \quad x = 0$$
$$y = u_{j1} \quad \text{at} \quad x = L$$
$$\frac{dy}{dx} = u_{j2} \quad \text{at} \quad x = L$$

Solve for $C_1,\ C_2,\ C_3,$ and C_4: 4 equations, 4 unknowns

2.2 Shape Functions

Substitute in $C_1,\ C_2,\ C_3,$ and C_4 and rearrange

$$y = S_{i1}u_{i1} + S_{i2}u_{i2} + S_{j1}u_{j1} + S_{j2}u_{j2}$$

Where:

$$S_{i1} = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \qquad S_{i2} = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$
$$S_{j1} = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \qquad S_{j2} = \frac{-x^2}{L} + \frac{x^3}{L^2}$$

$S_{i1},\ S_{i2},\ S_{j1},$ and S_{j2} are shape functions. y is the lateral position of the beam at any point x along the beam.

2.3 Strain Energy

Now create the *Stiffness-Displacement* relationship using the *Minimum Total Potential Energy* Equation . . . just like we did with axially loaded members.

$$\Lambda^{(e)} = \int_V \frac{\sigma\varepsilon}{2} dV$$
$$\downarrow$$
$$\Lambda^{(e)} = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx$$
$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(S_{i1}u_{i1} + S_{i2}u_{i2} + S_{j1}u_{j1} + S_{j2}u_{j2})$$

In Matrix form:

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\begin{bmatrix} S_{i1} & S_{i2} & S_{j1} & S_{j2}\end{bmatrix}\begin{Bmatrix} u_{i1} \\ u_{i2} \\ u_{j1} \\ u_{j2}\end{Bmatrix}$$

Solving for $\frac{\partial \Lambda^{(e)}}{\partial u_k}$, where u_k is $u_{i1},\ u_{i2},\ u_{j1},\ u_{j2}$

$$\frac{\partial \Lambda^{(e)}}{\partial u_k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{Bmatrix} u_{i1} \\ u_{i2} \\ u_{j1} \\ u_{j2}\end{Bmatrix}$$

Therefore the *Element Stiffness Matrix for a Beam Element* is:

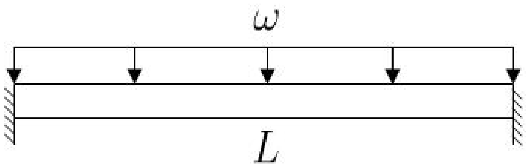
$$[\mathbf{K}]^{(e)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}$$

2.4 Load Matrix

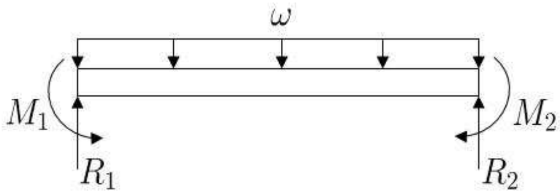
Two Methods (note: loads are not exclusively at nodes!!):

1. Minimize work done by the external loads: $\partial F u_k / \partial u_k$
   (We will examine this later with plane stress problems)
2. Solve for beam reaction forces at nodes (we will use this method for now)

2.5 Example: Beam Reaction Forces



2.5.1 FBD:



Statically indeterminate, BUT we can solve with governing equation for beam and B.C.'s.

2.5.2 Governing Equation:

$$\omega = EI\frac{d^4y}{dx^4}$$

Substituting ω (downward force) and integrating

$$\int -\omega dx = \int EI\frac{d^4y}{dx^4}dx$$

$$-\omega x + c_1 = EI\frac{d^3y}{dx^3} \quad (= V)$$

B.C. #1: $V = R_1$ at $x = 0$

$\therefore c_1 = R_1$. Substituting c_1 and integrating:

$$\int (-\omega x + R_1)dx = \int EI\frac{d^3y}{dx^3}dx$$

$$\frac{-1}{2}\omega x^2 + R_1 x + c_2 = EI\frac{d^2y}{dx^2} \quad (= M)$$

B.C. #2: $M = -M_1$ at $x = 0$

$\therefore c_2 = -M_1$. Substituting c_2 and integrating:

$$\int (\frac{-1}{2}\omega x^2 + R_1 x - M_1)dx = \int EI\frac{d^2y}{dx^2}dx$$

$$\frac{-1}{6}\omega x^3 + \frac{1}{2}R_1 x^2 - M_1 x + c_3 = EI\frac{dy}{dx} \quad (= \theta EI)$$

B.C. #3: $\theta = 0$ at $x = 0$

$\therefore c_3 = 0$. Substituting c_3 and integrating:

$$\int (\frac{-1}{6}\omega x^3 + \frac{1}{2}R_1 x^2 - M_1 x)dx = \int EI\frac{dy}{dx}dx$$

$$\frac{-1}{24}\omega x^4 + \frac{1}{6}R_1 x^3 - \frac{1}{2}M_1 x^2 + c_4 = EIy \ \ (= yEI)$$

B.C. #4: $y = 0$ at $x = 0$

$\therefore c_4 = 0$. Therefore, two equations with four unknowns:

1. $EI\theta = \frac{-1}{6}\omega x^3 + \frac{1}{2}R_1 x^2 - M_1 x$
2. $IEy = \frac{-1}{24}\omega x^4 + \frac{1}{6}R_1 x^3 - \frac{1}{2}M_1 x^2$

To obtain R_1 and M_1, we need to more B.C.'s:

$$\theta = \frac{dy}{dx} = 0 \ \text{ at } \ x = L$$

$$y = 0 \ \text{ at } \ x = L$$

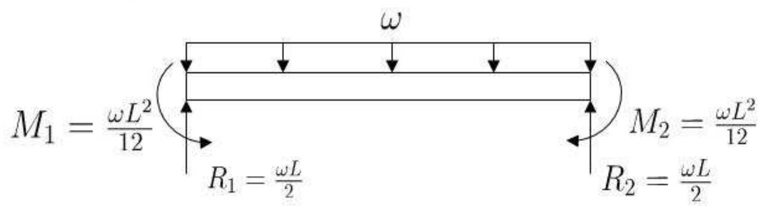
Substitute B.C.'s into equations 1 and 2 and solve for R_1 and M_1

$$R_1 = \frac{\omega L}{2} \quad M_1 = \frac{\omega L^2}{12}$$

Return to FBD and substitute R_1 and M_1 into equilibrium equations solving for R_2 and M_2.

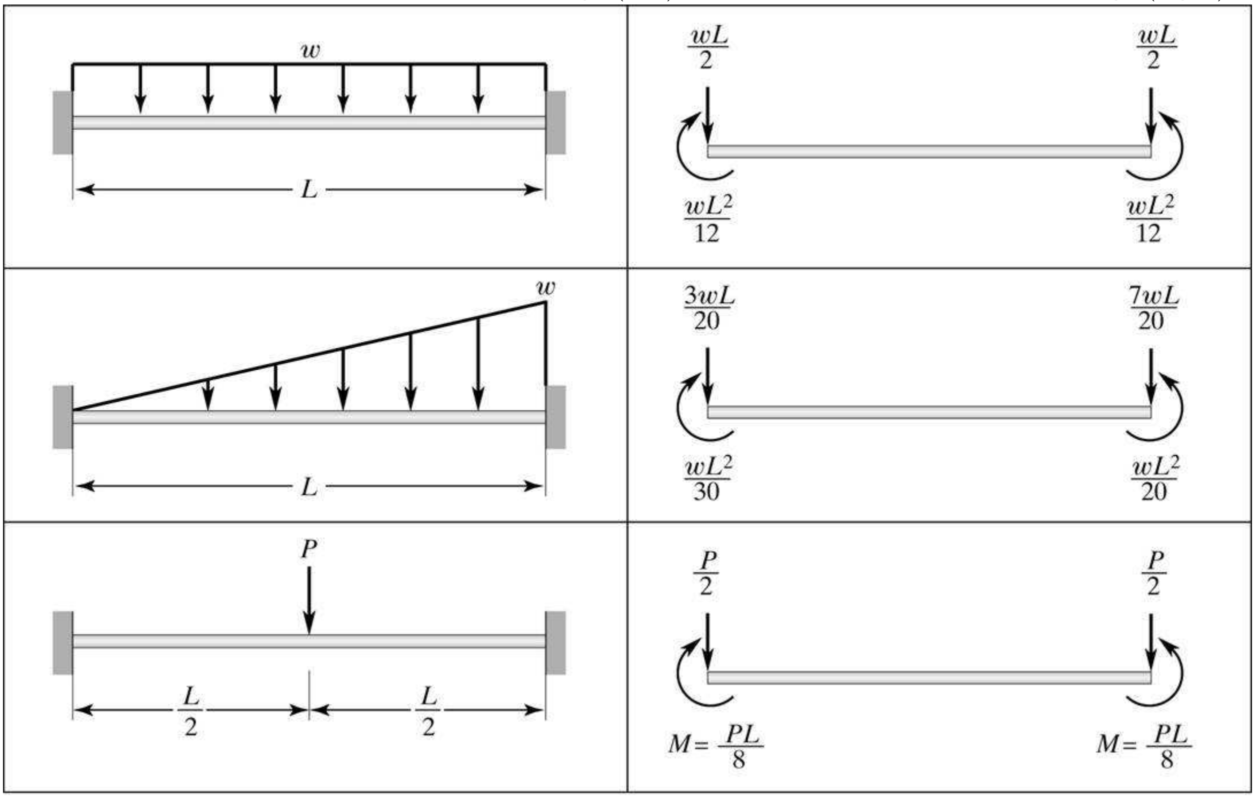
$$R_2 = \frac{\omega L}{2} \quad M_2 = \frac{\omega L^2}{12}$$

2.6 Reactions for Uniformly Distributed Load:

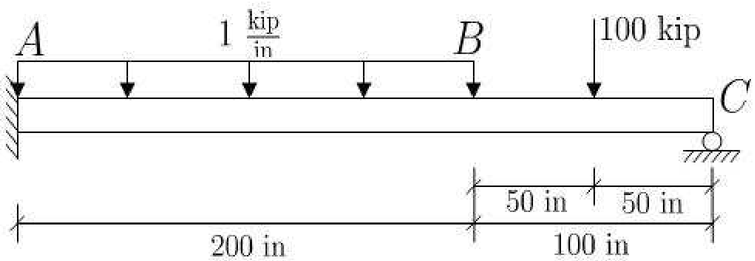


Reversing the signs, these reactions become the external forces or "Equivalent Nodal Loading" for the beam elements.

Table 4.2—Distributed or non-nodal loadings (left) and their equivalent nodal loadings (right)



3 Example



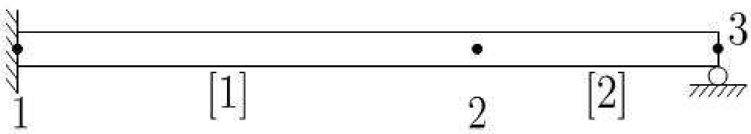
$$E = 10 \times 10^6 \text{ psi} \qquad\qquad I = 10,000 \text{ in}^4$$

3.1 Find:

Deflections at point B and C, and Reactions

3.2 Solution:

Use *two* elements: put a node at Point B to simplify load Matrix and finding displacements at B.



(Break into elements such that they match loading conditions available in Table 4.2)

3.2.1 Form Beam Element Stiffness Matrices

$$[\mathbf{K}]^{(e)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Element 1

$$[\mathbf{K}]^{(1)} = \frac{(10 \times 10^6 \text{ psi})(10,000 \text{ in}^4)}{(200 \text{ in})^3} \begin{bmatrix} 12 & 6(200) & -12 & 6(200) \\ 6(200) & 4(200)^2 & -6(200) & 2(200)^2 \\ -12 & -6(200) & 12 & -6(200) \\ 6(200) & 2(200)^2 & -6(200) & 4(200)^2 \end{bmatrix}$$

$$[\mathbf{K}]^{(1)} = \begin{array}{c} \begin{bmatrix} 0.150 & 15 & -0.150 & 15 \\ 15 & 2000 & -15 & 1000 \\ -.0150 & -15 & 0.150 & -15 \\ 15 & 1000 & -15 & 2000 \end{bmatrix} \end{array} \begin{array}{c} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{array} \times 10^6$$

Element 2

$$[\mathbf{K}]^{(2)} = \frac{(10 \times 10^6 \text{ psi})(10,000 \text{ in}^4)}{(100 \text{ in})^3} \begin{bmatrix} 12 & 6(100) & -12 & 6(100) \\ 6(100) & 4(100)^2 & -6(100) & 2(100)^2 \\ -12 & -6(100) & 12 & -6(100) \\ 6(100) & 2(100)^2 & -6(100) & 4(100)^2 \end{bmatrix}$$

$$[\mathbf{K}]^{(2)} = \begin{bmatrix} 1.2 & 60 & -1.2 & 60 \\ 60 & 4000 & -60 & 2000 \\ -1.2 & -60 & 1.2 & -60 \\ 60 & 2000 & -60 & 4000 \end{bmatrix} \begin{array}{c} u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{array} \times 10^6$$

3.2.2 Assemble Total Stiffness Matrix

Using the appropriate transpose matrix to change our 4x4 local form, elemental stiffness matrices into their 6x6 global forms, we can add the global forms of the elemental stiffness matrices to get the total stiffness matrix for the beam.

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} .15 & 15 & -0.15 & 15 & 0 & 0 \\ 15 & 2000 & -15 & 1000 & 0 & 0 \\ -0.15 & -15 & 0.15 + 1.2 & -15 + 60 & -1.2 & 60 \\ 15 & 1000 & -15 + 60 & 2000 + 4000 & -60 & 2000 \\ 0 & 0 & -1.2 & -60 & 1.2 & -60 \\ 0 & 0 & 60 & 2000 & -60 & 4000 \end{bmatrix} \begin{array}{c} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{array} \times 10^6$$

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} .15 & 15 & -0.15 & 15 & 0 & 0 \\ 15 & 2000 & -15 & 1000 & 0 & 0 \\ -0.15 & -15 & 1.35 & 45 & -1.2 & 60 \\ 15 & 1000 & 45 & 6000 & -60 & 2000 \\ 0 & 0 & -1.2 & -60 & 1.2 & -60 \\ 0 & 0 & 60 & 2000 & -60 & 4000 \end{bmatrix} \times 10^6$$

3.2.3 Form Element Load Matrices

Now we develop the element load matrices from Table 4.2.

$$\{\mathbf{F}\}^{(1)} = \begin{Bmatrix} -\omega L/2 \\ -\omega L^2/12 \\ -\omega L/2 \\ \omega L^2/12 \end{Bmatrix} \begin{array}{c} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{array} \quad \{\mathbf{F}\}^{(1)} = \begin{Bmatrix} \frac{-(1)(200)}{2} \\ \frac{-(1)(200)^2}{12} \\ \frac{-(1)(200)}{2} \\ \frac{(1)(200)}{12} \end{Bmatrix} \times 10^3 \quad \{\mathbf{F}\}^{(1)} = \begin{Bmatrix} -100 \\ -3333 \\ -100 \\ 3333 \end{Bmatrix} \times 10^3$$

$$\{\mathbf{F}\}^{(2)} = \begin{Bmatrix} -P/2 \\ -PL/8 \\ -P/2 \\ PL/8 \end{Bmatrix} \begin{array}{c} u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{array} \quad \{\mathbf{F}\}^{(2)} = \begin{Bmatrix} \frac{-100}{2} \\ \frac{-(100)(100)}{8} \\ \frac{-100}{2} \\ \frac{(100)(100)}{2} \end{Bmatrix} \times 10^3 \quad \{\mathbf{F}\}^{(2)} = \begin{Bmatrix} -50 \\ -1250 \\ -50 \\ 1250 \end{Bmatrix} \times 10^3$$

3.2.4 Assemble Global Load Matrix

Again, we can apply the same transpose matrices that we used for the elemental stiffness matrices to get the global form of the load matrix.

$$\{\mathbf{F}\}^{(G)} = \begin{Bmatrix} -100 \\ -3333 \\ -100-50 \\ 3333-1250 \\ -50 \\ 1250 \end{Bmatrix} \begin{matrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{matrix} \times 10^3$$

$$\{\mathbf{F}\}^{(G)} = \begin{Bmatrix} -100,000 \\ -3,333,333 \\ -150,000 \\ 2,083,333 \\ -50,0000 \\ 1,250,0000 \end{Bmatrix}$$

3.2.5 Global Matrix Equation

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$$

$$\begin{bmatrix} 0.15 & 15 & -0.15 & 15 & 0 & 0 \\ 15 & 2000 & -15 & 1000 & 0 & 0 \\ -0.15 & -15 & 1.35 & 45 & -1.2 & 60 \\ 15 & 1000 & 45 & 6000 & -60 & 2000 \\ 0 & 0 & -1.2 & -60 & 1.2 & -60 \\ 0 & 0 & 60 & 2000 & -60 & 4000 \end{bmatrix} \times 10^6 \begin{Bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{Bmatrix} = \begin{Bmatrix} -100,000 \\ -3,333,333 \\ -150,000 \\ 2,083,333 \\ -50,0000 \\ 1,250,0000 \end{Bmatrix}$$

Apply boundary conditions:

$$u_{11} = 0, \quad u_{12} = 0, \quad u_{31} = 0$$

The corresponding B.C. matrix will look like:

$$[\mathbf{BC}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.2.6 Reduced Global Matrix Equation

$$\begin{bmatrix} 1.35 & 45 & 60 \\ 45 & 6000 & 2000 \\ 60 & 2000 & 4000 \end{bmatrix} \times 10^6 \begin{Bmatrix} u_{21} \\ u_{22} \\ u_{32} \end{Bmatrix} = \begin{Bmatrix} -150,000 \\ 2,083,333 \\ 1,250,0000 \end{Bmatrix}$$

3.2.7 Solve with Mathcad/MATLAB

$$\begin{Bmatrix} u_{21} \\ u_{22} \\ u_{32} \end{Bmatrix} = \begin{Bmatrix} -0.4275 \text{ in} \\ 0.001574 \text{ rad} \\ 0.005938 \text{ rad} \end{Bmatrix}$$

3.2.8 Find Support and Nodal Reactions

$\{\mathbf{R}\} = [\mathbf{K}]\{\mathbf{u}\} - \{\mathbf{F}\}$

$$\begin{Bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \\ R_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 0.15 & 15 & -0.15 & 15 & 0 & 0 \\ 15 & 2000 & -15 & 1000 & 0 & 0 \\ -0.15 & -15 & 1.35 & 45 & -1.2 & 60 \\ 15 & 1000 & 45 & 6000 & -60 & 2000 \\ 0 & 0 & -1.2 & -60 & 1.2 & -60 \\ 0 & 0 & 60 & 2000 & -60 & 4000 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ 0 \\ -0.4275 \\ 0.0016 \\ 0 \\ 0.0059 \end{Bmatrix} - \begin{Bmatrix} -100,000 \\ -3,333,333 \\ -150,000 \\ 2,083,333 \\ -50,0000 \\ 1,250,0000 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \\ R_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 188 \text{ kips} \\ 113,200 \text{ kip-in} \\ 0 \\ 0 \\ 112 \text{ kips} \\ 0 \end{Bmatrix}$$