Analysis of Two-Dimensional Elements Heat Transfer Problems: Stiffness Matrices, ANSYS

1 Governing Equation

Three Modes of Heat Transfer

1. Conduction
2. Convection
3. Radiation

1.1 Fourier's Law for Conduction

Two-Dimensional Heat Transfer Rate is Defined by:

$$q_X = -kA\frac{\partial T}{\partial X}$$

$$q_Y = -kA\frac{\partial T}{\partial Y}$$

where:

$$q_X,\ q_Y \equiv \text{conduction heat transfer rate in X-Y Directions}$$

$$k \equiv \text{thermal conductivity of medium}$$

$$A \equiv \text{cross-sectional area of medium}$$

$$\frac{\partial T}{\partial X} \equiv \text{temperature gradients}$$

1.2 Conservation of Energy

For Two-Dimensional conduction with convective and radiation boundary conditions: (Steady state)

$$k_X\frac{\partial^2 T}{\partial X^2} + k_Y\frac{\partial^2 T}{\partial Y^2} + \dot{q} = 0$$

where:

$$\dot{q} \equiv \text{heat generation (i.e. input) per unit volume}$$

1.3 Various Boundary Conditions

1. Constant surface temperature due to fluid phase change such as condensation or evaporation

$$T\left(0,\ Y\right) = T_0$$

2. Heat loss or gain at surface may be neglected

$$\frac{\partial T}{\partial X}|_{(X=0,\ Y)} = 0$$

3. Constant heat flux (i.e., heat transfer rate per area) on surface

$$-k\frac{\partial T}{\partial X}|_{X=0} = q_0$$

4. Heat loss or gain at surface by convection

$$-k\frac{\partial T}{\partial X}|_{X=0,\ Y} = h\left[T\left(0,\ Y\right) - T_f\right]$$

5. Heat loss or gain at surface by radiation

6. Both conditions 4 and 5 simultaneously

2 Formulation: Linear Rectangular Elements

Equation for nodal temperatures within a linear 2-D Rectangular element:

$$T^{(e)} = \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

where the shape functions are:

$$S_i = \left(1 - \frac{X}{\ell}\right)\left(1 - \frac{Y}{w}\right)$$
$$S_j = \frac{X}{\ell}\left(1 - \frac{Y}{w}\right)$$
$$S_m = \frac{XY}{\ell w}$$
$$S_n = \frac{Y}{w}\left(1 - \frac{X}{\ell}\right)$$

2.1 Galerkin Approach

With the conduction heat transfer equation: remember the idea is to reduce residual (i.e. error) to zero. Galerkin uses an average with a weighting function of the same form as solution equation. Therefore our weight functions are the shape functions.

$$R_i^{(e)} = \int_A S_i \left(k_X \frac{\partial^2 T}{\partial X^2} + k_Y \frac{\partial^2 T}{\partial Y^2} + \dot{q}\right) dA = 0$$
$$R_j^{(e)} = \int_A S_j \left(k_X \frac{\partial^2 T}{\partial X^2} + k_Y \frac{\partial^2 T}{\partial Y^2} + \dot{q}\right) dA = 0$$
$$R_m^{(e)} = \int_A S_m \left(k_X \frac{\partial^2 T}{\partial X^2} + k_Y \frac{\partial^2 T}{\partial Y^2} + \dot{q}\right) dA = 0$$
$$R_n^{(e)} = \int_A S_n \left(k_X \frac{\partial^2 T}{\partial X^2} + k_Y \frac{\partial^2 T}{\partial Y^2} + \dot{q}\right) dA = 0$$

In matrix form

$$\int_A [\mathbf{S}]^T \left(k_X \frac{\partial^2 T}{\partial X^2} + k_Y \frac{\partial^2 T}{\partial Y^2} + \dot{q}\right) dA = 0$$

where: $[\mathbf{S}] = \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix}$. Separating into three integrals

$$\int_A [\mathbf{S}]^T \left(k_x \frac{\partial^2 T}{\partial X^2}\right) dA + \int_A [\mathbf{S}]^T \left(k_Y \frac{\partial^2 T}{\partial Y^2}\right) dA + \int_A [\mathbf{S}]^T \dot{q} dA = 0$$

Now, set:

$$C_1 = k_X$$
$$C_2 = k_Y \quad \psi = T$$
$$C_3 = \dot{q}$$

Therefore:

$$\int_A [\mathbf{S}]^T \left(C_1 \frac{\partial^2 \psi}{\partial X^2}\right) dA + \int_A [\mathbf{S}]^T \left(C_2 \frac{\partial^2 \psi}{\partial Y^2}\right) dA + \int_A [\mathbf{S}]^T C_3 dA = 0$$

Now the solution will apply to other problems that have similar governing differential equations.

Since our integrals are over an area, we will need to perform a very complicated double integral.

$$\int_0^\ell \int_0^w dY dX$$

The resulting "Stiffness" Matrix is:

$$[\mathbf{K}]^{(e)} = \frac{C_1 w}{6\ell}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{C_2 \ell}{6w}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$

where: $\ell,\ w \equiv$ Length of rectangular element sides.

2.2 Stiffness Matrices: Conduction

Substituting for C_1 and C_2, the conductance matrix is:

$$[\mathbf{K}]^{(e)} = \frac{k_X w}{6\ell}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{k_Y \ell}{6w}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$

2.3 Stiffness Matrices: Convection

Any convective boundary conditions along different <u>edges</u> will contribute to the conductance matrix with the following matrices:

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{jm}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{mn}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{ni}}{6}\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

2.4 Load Matices: Convection

The Load Matrices along the <u>edges</u> due to convection are:

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f\ell_{ij}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f\ell_{jm}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f\ell_{mn}}{2}\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f\ell_{ni}}{2}\begin{Bmatrix}1\\0\\0\\1\end{Bmatrix}$$

2.5 Load Matrices: Heat Generation

Any heat generation within an element will contribute to the thermal-load matrix for the element with:

$$\{\mathbf{F}\}^{(e)} = \frac{\dot{q}A}{4}\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}$$

2.6 Solve

$[\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{F}\}$

3 Formulation: Linear Triangular Elements

Equation for Nodal Temperatures within a linear 2-D Triangular element:

$$T^{(e)} = \begin{bmatrix}S_i & S_j & S_k\end{bmatrix}\begin{Bmatrix}T_i\\T_j\\T_k\end{Bmatrix}$$

where shape functions:

$$S_i = \frac{1}{2A}\left(\alpha_i + \beta_i X + \delta_i Y\right)$$
$$S_j = \frac{1}{2A}\left(\alpha_j + \beta_j X + \delta_j Y\right)$$
$$S_k = \frac{1}{2A}\left(\alpha_k + \beta_k X + \delta_k Y\right)$$

And:

$$2A = X_i\left(Y_j - Y_k\right) + X_j\left(Y_k - Y_i\right) + X_k\left(Y_i - Y_j\right)$$
$$\alpha_i = X_jY_k - X_kY_j \quad \beta_i = Y_j - Y_k \quad \delta_i = X_k - X_j$$
$$\alpha_j = X_kY_i - X_iY_k \quad \beta_j = Y_k - Y_i \quad \delta_j = X_i - X_k$$
$$\alpha_k = X_iY_j - X_jY_i \quad \beta_k = Y_i - Y_j \quad \delta_k = X_j - X_i$$

3.1 Galerkin Approach

Again, reduce Residual (i.e. error) to zero.

$$R = \int_A [\mathbf{S}]^T\left(k_X\frac{\partial^2 T}{\partial X^2} + k_Y\frac{\partial^2 T}{\partial Y^2} + \dot{q}\right)dA = 0$$

3.2 Stiffness Matrices: Conduction

$$[\mathbf{K}]^{(e)} = \frac{k_X}{4A}\begin{bmatrix}\beta_i^2 & \beta_i\beta_j & \beta_i\beta_k\\ \beta_i\beta_j & \beta_j^2 & \beta_j\beta_k\\ \beta_i\beta_k & \beta_j\beta_k & \beta_k^2\end{bmatrix} + \frac{k_Y}{4A}\begin{bmatrix}\delta_i^2 & \delta_i\delta_j & \delta_i\delta_k\\ \delta_i\delta_j & \delta_j^2 & \delta_j\delta_k\\ \delta_i\delta_k & \delta_j\delta_k & \delta_k^2\end{bmatrix}$$

where: $A \equiv$ Face area of element

3.3 Stiffness Matrices: Convection

For convective boundary conditions along different edges

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{jk}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{ki}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

3.4 Load Matrices: Convection

Thermal Load Matrices along the *edges* due to convection:

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f\ell_{ij}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f\ell_{jk}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f\ell_{ki}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

3.5 Load Matrices: Heat Generation

Heat Generation within an element will contribute to the thermal-load matrix:

$$\{\mathbf{F}\}^{(e)} = \frac{\dot{q}A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

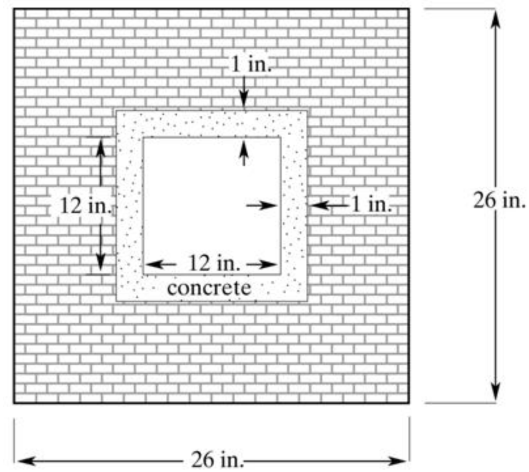
Solution: $[\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{F}\}$

4 ANSYS Example: Steady-State Heat Conduction

Small chimney constructed from two different materials

- Inner layer: concrete (k = 0.07 Btu/hr * in * °F)
- Outer layer: bricks (k = 0.04 Btu/hr * in * °F)

- Inside surface: exposed to hot gas
  - T = 140°F
  - h = 0.037 Btu/hr * in^2 * °F
- Outside surface
  - T = 10°F
  - h = 0.012 Btu/hr * in^2 * °F
- Dimensions of the chimney are shown in figure



Determine: state conditions; plot the heat fluxes through each layer