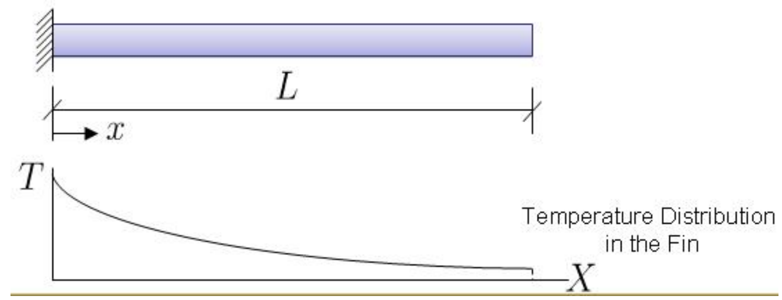
1-D Elements

1 Introduction

Where are we headed? We will be examining straight-line elements (i.e., 1-D) that contain an unknown variable.

Ex: Heat Transfer from a Fin



We will represent the spatial variation of the unknown variable (e.g., temperature) using the following functions:

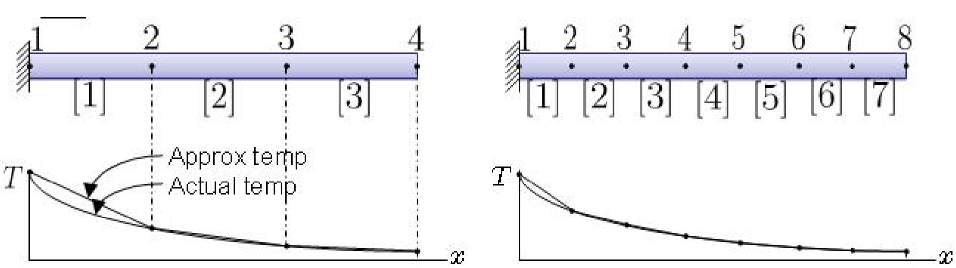
- Linear Functions (e.g., $T = Ax + B$)
- Quadratic Functions (e.g., $T = Ax^2 + Bx + C$)
- Cubic Functions (e.g., $T = Ax^3 + Bx^2 + Cx + D$)

We will also examine:

1. Coordinate systems
2. Special Elements called Isoparametric
3. Numerical Integration

2 Linear Elements

Returning to the heat transfer fin, Let's approximate the temperature distribution with a combination of linear functions.



Note:

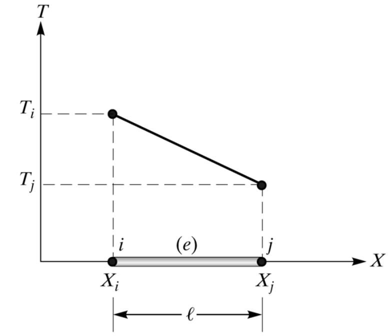
- **More elements...more accuracy** (and more time)
- Elements do not need to be the same length so make elements shorter on the left side (large gradient)

2.1 The Element

Approximate temperature within an element as a linear function

$$T^{(e)} = c_1 + c_2 X$$

B.C.'s:
$T = T_i$ at $X = X_i$
$T = T_j$ at $X = X_j$



Substitute B.C.'s into equation and solve simultaneously to obtain:

$$c_1 = \frac{T_i X_j - T_j X_i}{X_j - X_i} \qquad c_2 = \frac{T_j - T_i}{X_j - X_i}$$

Substituting

$$T^{(e)} = \frac{T_i X_j - T_j X_i}{X_j - X_i} + \frac{T_j - T_i}{X_j - X_i} X$$

As with Axial Elements, $T^{(e)}$ is the temperature in the element at any location X.
Rearranging:

$$T^{(e)} = \left(\frac{X_j - X}{X_j - X_i}\right) T_i + \left(\frac{X - X_i}{X_j - X_i}\right) T_j$$

If shape functions are:

$$S_i = \frac{X_j - X}{X_j - X_i} = \frac{X_j - X}{\ell} \quad \text{and} \quad S_j = \frac{X - X_i}{X_j - X_i} = \frac{X - X_i}{\ell}$$

Then: $T^{(e)} = S_i T_i + S_j T_j$

In matrix form,

$$T^{(e)} = \begin{bmatrix} S_i & S_j \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

Notice how this whole derivation is analogous to the axial elements derivation.

2.2 General Linear Elements

The preceding derivation using temperature can be used for any unknown variable where an approximate linear function is desired, such as temperature, deflection, or velocity.

Assuming an unknown variable, Ψ (Psi):

$$\Psi^{(e)} = c_1 + c_2 X$$

$$S_i = \frac{X_j - X}{\ell} \quad S_j = \frac{X - X_i}{\ell}$$

$$\Psi^{(e)} = \begin{bmatrix} S_i & S_j \end{bmatrix} \begin{Bmatrix} \Psi_i \\ \Psi_j \end{Bmatrix}$$

3 Properties of Shape Functions

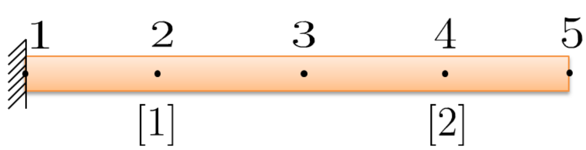
These properties seem trivial now, but will be *very useful later*:

1. Each shape functions has a value of 1 at its corresponding node. (i.e., $S_i = 1$ at $X = X_i$ and $S_j = 1$ at $X = X_j$)
2. All shape functions have a value of 0 at is adjacent node. (i.e., $S_i = 0$ at $X = X_j$ and $S_j = 0$ at $X = X_i$)
3. All shape functions, when added, equal to 1. (i.e., $S_i + S_j = 1$)
4. For *linear* shape functions the sum of the derivatives with respect to X is zero. (i.e., $\frac{dS_i}{dX} + \frac{dS_j}{dX} = 0$)

4 Quadratic Elements

- The *accuracy* of our finite element model can be *increased* via two different methods:
  - Increasing the number of linear elements
  - Using a higher order approximating function.
- Returning to the heat transfer fin, let's approximate the temperature distribution with a combination of quadratic elements.
- 1-D quadratic elements ...
  - Require three nodes to define the element
  - Need 3 B.C.'s for the 3 unknown coefficients

Ex:



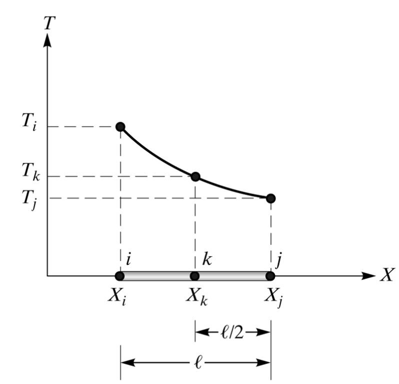
4.1 The Element

Node k is located in the middle of the element

Approximate temperature as a quadratic function

$$T^{(e)} = c_1 + c_2 X + c_3 X^2$$

B.C.'s:
$T = T_i$ at $X = X_i$
$T = T_j$ at $X = X_j$
$T = T_k$ at $X = X_k$



Substituting B.C.'s creates 3 algebraic Equations.

$$T_i = c_1 + c_2 X_i + c_3 X_i^2$$

$$T_j = c_1 + c_2 X_j + c_3 X_j^2$$

$$T_k = c_1 + c_2 X_k + c_3 X_k^2$$

Solving for c_1, c_2, and c_3 simultaneously and then substituting back in, we get:

$$T^{(e)} = \left[\frac{2}{\ell^2}(X - X_j)(X - X_k)\right] T_i + \left[\frac{2}{\ell^2}(X - X_i)(X - X_k)\right] T_j + \left[\frac{-4}{\ell^2}(X - X_i)(X - X_j)\right] T_k$$

If shape functions are:

$$S_i = \frac{2}{\ell^2}(X - X_j)(X - X_k)$$

$$S_j = \frac{2}{\ell^2}(X - X_i)(X - X_k)$$

$$S_k = \frac{-4}{\ell^2}(X - X_i)(X - X_j)$$

Then: $T^{(e)} = S_i T_i + S_j T_j + S_k T_k$

In matrix form:

$$T^{(e)} = \begin{bmatrix} S_i & S_j & S_k \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

4.2 General Quadratic Elements

This derivation works for any unknown variable. Assume unknown variable, ψ (psi)

$$\Psi^{(e)} = c_1 + c_2 X + c_3 X^2$$

The shape functions:

$$S_i = \frac{2}{\ell^2}(X - X_j)(X - X_k)$$

$$S_j = \frac{2}{\ell^2}(X - X_i)(X - X_k)$$

$$S_k = \frac{-4}{\ell^2}(X - X_i)(X - X_j)$$

In matrix form:

$$\Psi^{(e)} = \begin{bmatrix} S_i & S_j & S_k \end{bmatrix} \begin{Bmatrix} \Psi_i \\ \Psi_j \\ \Psi_k \end{Bmatrix}$$

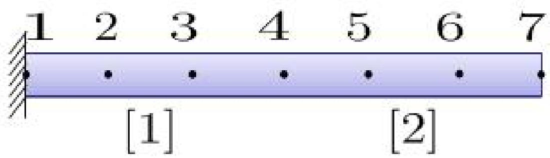
Notice the results of the shape functions at each node:

1. $S_i = 1$ at $X = X_i,\ S_j = 1$ at $X = X_j,\ S_k = 1$ at $X = X_k$
2. $S_i = 0$ at $X = X_j$ & $X_k,\ S_j = 0$ at $X = X_i$ & $X_k, S_k = 0$ at $X = X_i$ & X_j
3. $S_i + S_j + S_k = 1$

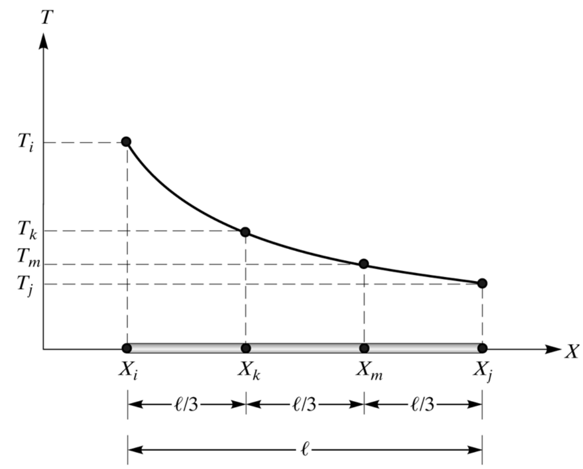
5 Cubic Elements

- We can achieve even more accuracy using a higher order approximation function.
- Let's approximate temperature distribution in the heat transfer fin using a combination of cubic elements.
- One-dimensional cubic elements require four nodes to define the element. (Need 4 B.C.'s for the 4 unknown coefficients.)

Ex:



5.1 The Element



The 4 nodes are equally spaced, $\ell/3$.
Approximate temperature as a cubic function

$$T^{(e)} = c_1 + c_2 X + c_3 X^2 + c_4 X^3$$

Using B.C.'s to solve for c_1, c_2, c_3 and c_4.
$T = T_i$ at $X = X_i$
$T = T_j$ at $X = X_j$
$T = T_k$ at $X = X_k$
$T = T_m$ at $X = X_m$

Resulting Equation

$$T^{(e)} = S_i T_i + S_j T_j + S_k T_k + S_m T_m$$

Where the shape functions are:

$$S_i = \frac{-9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m)$$
$$S_j = \frac{9}{2\ell^3}(X - X_i)(X - X_k)(X - X_m)$$
$$S_k = \frac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_m)$$
$$S_m = \frac{-27}{2\ell^3}(X - X_i)(X - X_j)(X - X_k)$$

Again, $T^{(e)}$ is the temperature in the element at any location X.

5.2 General Cubic Elements
This derivation works for any unknown variable.
Assume unknown variable, ψ (Psi)

$$\Psi^{(e)} = c_1 + c_2 X + c_3 X^2 + c_4 X^3$$

The shape functions are:

$$S_i = -\frac{9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m)$$

$$S_j = \frac{9}{2\ell^3}(X - X_i)(X - X_k)(X - X_m)$$
$$S_k = \frac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_m)$$
$$S_m = \frac{-27}{2\ell^3}(X - X_i)(X - X_j)(X - X_k)$$

In matrix form:

$$\Psi^{(e)} = \begin{bmatrix} S_i & S_j & S_k & S_m \end{bmatrix} \begin{Bmatrix} \Psi_i \\ \Psi_j \\ \Psi_k \\ \Psi_m \end{Bmatrix}$$

Notice the results of the shape functions at each node:

1. $S_i = 1$ at $X = X_i,\ S_j = 1$ at $X = X_j,$
   $S_k = 1$ at $X = X_k,\ S_m = 1$ at $X = X_m$

2. $S_i = 0$ at $X = X_j,\ X_k,\ \&\ X_m,\ S_j = 0$ at $X = X_i,\ X_k,\ \&\ X_m,$
   $S_k = 0$ at $X = X_i,\ X_j,\ \&\ X_m,\ S_m = 0$ at $X_i,\ X_j\ \&\ X_k$

3. $S_i + S_j + S_k + S_m = 1$

6 Lagrage Interpolation Functions

- Higher order approximating functions: much more cumbersome and difficult to obtain the shape functions

- Lagrange Method
  - Keeps us from solving for c_1, c_2, c_3, c_4 etc. by using simultaneous equations
  - Instead, we directly write shape functions by multiplying several linear functions.

General Lagrange Polynomial formula for developing shape functions:

$$S_K = \prod_{M=1}^{N} \frac{X - X_M \text{ omitting } (X - X_K)}{X_K - X_M \text{ omitting } (X_K - X_K)} = \frac{(X - X_1)(X - X_2)\cdots(X - X_N)}{(X_K - X_1)(X_K - X_2)\cdots(X_K - X_N)}$$

for an (N - 1)-order polynomial. Therefore, for a 3^{rd}-order polynomial, N = 4; for a 5^th-order polynomial, N = 6.

Example
Use the Lagrange Polynomial formula to find the shape functions for the cubic element.

$$N - 1 = 3 \ \therefore N = 4,\ K = 1,\ 2,\ 3,\ 4$$

$$S_K = \prod_{M=1}^{N} \frac{X - X_M \text{ omitting } (X - X_K)}{X_K - X_M \text{ omitting } (X_K - X_M)}$$

For node $i,\ K = 1$:

$$S_1 = \frac{(X - X_2)(X - X_3)(X - X_4)}{(X_1 - X_2)(X_1 - X_3)(X_1 - X_4)} = \frac{(X - X_k)(X - X_m)(X - X_j)}{(X_i - X_k)(X_i - X_m)(X_i - X_j)}$$

$$S_i = \frac{(X - X_j)(X - X_k)(X - X_m)}{(\frac{-\ell}{3})(\frac{-2\ell}{3})(-\ell)}$$

$$S_i = \frac{-9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m)$$

For node $k,\ K = 2$:

$$S_2 = \frac{(X - X_1)(X - X_3)(X - X_4)}{(X_2 - X_1)(X_2 - X_3)(X_2 - X_4)} = \frac{(X - X_i)(X - X_m)(X - X_j)}{(X_k - X_i)(X_k - X_m)(X_k - X_j)}$$

$$S_k = \frac{(X - X_i)(X - X_m)(X - X_j)}{(\frac{\ell}{3})(\frac{-\ell}{3})(\frac{-2\ell}{3})}$$

$$S_k = \frac{27}{2\ell^3}(X - X_i)(X - X_m)(X - X_j)$$

For node m, $K = 3$:

$$S_3 = \frac{(X - X_1)(X - X_2)(X - X_4)}{(X_3 - X_1)(X_3 - X_2)(X_3 - X_4)} = \frac{(X - X_i)(X - X_m)(X - X_j)}{(X_m - X_i)(X_m - X_k)(X_m - X_j)}$$

$$S_m = \frac{(X - X_i)(X - X_k)(X - X_j)}{(\frac{2\ell}{3})(\frac{\ell}{3})(\frac{-\ell}{3})}$$

$$S_m = \frac{-27}{2\ell^3}(X - X_i)(X - X_k)(X - X_j)$$

For node j, $K = 4$:

$$S_4 = \frac{(X - X_1)(X - X_2)(X - X_3)}{(X_4 - X_1)(X_4 - X_2)(X_4 - X_3)} = \frac{(X - X_i)(X - X_k)(X - X_m)}{(X_j - X_i)(X_j - X_k)(X_j - X_m)}$$

$$S_j = \frac{(X - X_i)(X - X_k)(X - X_m)}{(\ell)(\frac{2\ell}{3})(\frac{\ell}{3})}$$

$$S_j = \frac{9}{2\ell^3}(X - X_i)(X - X_k)(X - X_m)$$

Notice shape function properties are retained:

1. Value of 1 at its corresponding node
2. Value of 0 at its adjacent nodes
3. Sum of shape functions equal to 1

7 Introduction to Natural Coordinates and Isoparametric Elements

<u>Local Coordinates in 1-D</u>

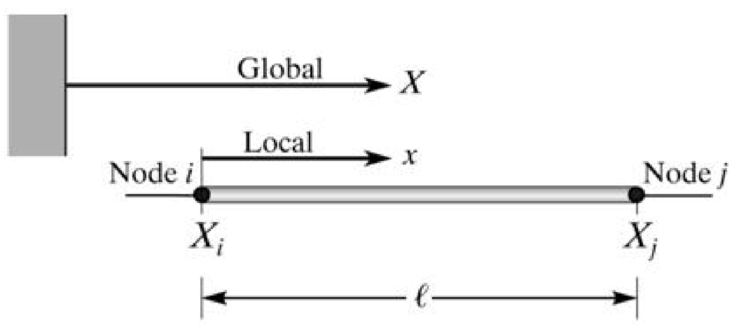


Figure 5-7
The relationship between a global coordinate X and a local coordinate x.

$$X = X_i + x$$

Linear shape functions in Local Coordinates

$$S_i = \frac{X_j - X}{\ell} \qquad S_j = \frac{X - X_i}{\ell}$$

$$S_i = \frac{X_j - (X_i + x)}{\ell} \qquad S_j = \frac{(X_i + x) - X_i}{\ell}$$

$$S_i = 1 - \frac{x}{\ell} \qquad S_j = \frac{x}{\ell}$$

Notice: $0 \le x \le \ell$

8 5.4 Natural Coordinates in 1-D

- Natural Coordinates are basically local coordinates in a dimensionless form.
- Natural Coordinates simplify numerical integration
- Notation is ξ, lower case Xi
- The objective is for ξ to be -1 at i and 1 at j.

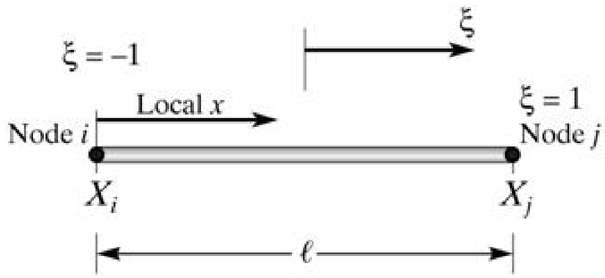


Figure 5-8
The relationship between the local coordinate x and the natural coordinate ξ.

$$\xi = \frac{2x}{\ell} - 1 \quad \text{or} \quad x = \frac{\ell}{2}(\xi + 1)$$

$$\text{at } x = 0 \;\rightarrow\; \xi = -1$$

$$\text{at } x = \ell \;\rightarrow\; \xi = 1$$

Let's see what Linear Shape Functions look like in natural coordinate form:

$$S_i = 1 - \frac{x}{\ell} \qquad\qquad S_j = \frac{x}{\ell}$$

$$S_i = 1 - \frac{1}{\ell}\left(\frac{\ell}{2}\right)(\xi + 1) \qquad\qquad S_j = \frac{1}{\ell}\left(\frac{\ell}{2}\right)(\xi + 1)$$

$$S_i = 1 - \frac{1}{2}(\xi + 1) \qquad\qquad S_j = \frac{1}{2}(1 + \xi)$$

$$S_i = 1 - \frac{1}{2}\xi - \frac{1}{2}$$

$$S_i = \frac{1}{2}(1 - \xi)$$

Notice shape function properties are retained for natural coordinates:

1. $S_i = 1$ at $X = X_i$ $(\xi = -1)$
   $S_j = 1$ at $X = X_j$ $(\xi = 1)$
2. $S_i = 0$ at $X = X_j$ $(\xi = 1)$
   $S_j = 0$ at $X = X_i$ $(\xi = -1)$
3. $S_i + S_j = 1$

9 5.5 Isoparametric Elements

We can use the same parameters $(S_i,\ S_j)$ to describe several different variables such as displacement, temperature, and velocity.
<u>Ex:</u>

$$u^{(e)} = S_i u_i + S_j u_j \qquad\qquad T^{(e)} = S_i T_i + S_j T_j$$

$$u^{(e)} = \frac{1}{2}(1 - \xi)u_i + \frac{1}{2}(1 + \xi)u_j \qquad\qquad T^{(e)} = \frac{1}{2}(1 - \xi)T_i + \frac{1}{2}(1 + \xi)T_j$$

- Since all these variables all use the same finite element formulation it is common to call it an Isoparametric formulation or Isoparametric element.

- Iso means "same." Therefore Isoparametric means "same parameters."
- This same formulation with local and natural coordinates can be performed on quadratic and cubic elements.
- These are summarized in Table 5.1

TABLE 5.1 One-dimensional shape functions

Interpolation function	In terms of global coordinate X $X_i \le X \le X_j$	In terms of local coordinate x $0 \le x \le \ell$	In terms of natural coordinate ξ $-1 \le \xi \le 1$
Linear	$S_i = \dfrac{X_j - X}{\ell}$	$S_i = 1 - \dfrac{x}{\ell}$	$S_i = \dfrac{1}{2}(1 - \xi)$
	$S_j = \dfrac{X - X_i}{\ell}$	$S_j = \dfrac{x}{\ell}$	$S_j = \dfrac{1}{2}(1 + \xi)$
Quadratic	$S_i = \dfrac{2}{\ell^2}(X - X_j)(X - X_k)$	$S_i = \left(\dfrac{x}{\ell} - 1\right)\left(2\left(\dfrac{x}{\ell}\right) - 1\right)$	$S_i = -\dfrac{1}{2}\xi(1 - \xi)$
	$S_j = \dfrac{2}{\ell^2}(X - X_i)(X - X_k)$	$S_j = \left(\dfrac{x}{\ell}\right)\left(2\left(\dfrac{x}{\ell}\right) - 1\right)$	$S_j = \dfrac{1}{2}\xi(1 + \xi)$
	$S_k = \dfrac{-4}{\ell^2}(X - X_i)(X - X_j)$	$S_k = 4\left(\dfrac{x}{\ell}\right)\left(1 - \left(\dfrac{x}{\ell}\right)\right)$	$S_k = (1 - \xi)(1 + \xi)$
Cubic	$S_i = -\dfrac{9}{2\ell^3}(X - X_j)(X - X_k)(X - X_m)$	$S_i = \dfrac{1}{2}\left(1 - \dfrac{x}{\ell}\right)\left(2 - 3\left(\dfrac{x}{\ell}\right)\right)\left(1 - 3\left(\dfrac{x}{\ell}\right)\right)$	$S_i = \dfrac{1}{16}(1 - \xi)(3\xi + 1)(3\xi - 1)$
	$S_j = \dfrac{9}{2\ell^3}(X - X_i)(X - X_k)(X - X_m)$	$S_j = \dfrac{1}{2}\left(\dfrac{x}{\ell}\right)\left(2 - 3\left(\dfrac{x}{\ell}\right)\right)\left(1 - 3\left(\dfrac{x}{\ell}\right)\right)$	$S_j = \dfrac{1}{16}(1 + \xi)(3\xi + 1)(3\xi - 1)$
	$S_k = \dfrac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_m)$	$S_k = \dfrac{9}{2}\left(\dfrac{x}{\ell}\right)\left(2 - 3\left(\dfrac{x}{\ell}\right)\right)\left(1 - \left(\dfrac{x}{\ell}\right)\right)$	$S_k = \dfrac{9}{16}(1 + \xi)(\xi - 1)(3\xi - 1)$
	$S_m = -\dfrac{27}{2\ell^3}(X - X_i)(X - X_j)(X - X_k)$	$S_m = \dfrac{9}{2}\left(\dfrac{x}{\ell}\right)\left(3\left(\dfrac{x}{\ell}\right) - 1\right)\left(1 - \left(\dfrac{x}{\ell}\right)\right)$	$S_m = \dfrac{9}{16}(1 + \xi)(1 - \xi)(3\xi + 1)$