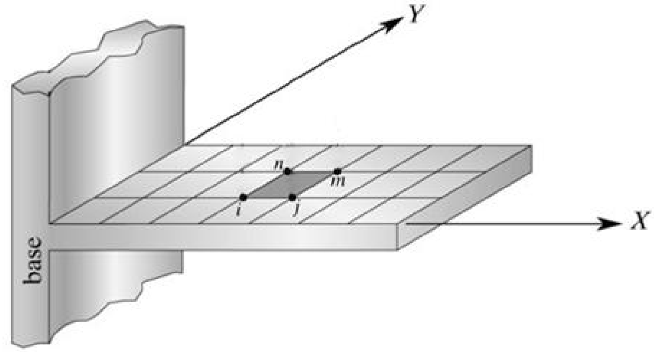
Two-Dimensional Elements

1 Introduction

<u>Where are we headed?</u>

1. Rectangular Elements
   - Linear
   - Quadratic (Quadrilateral)
2. Triangular Elements
   - Linear
   - Quadratic
3. Axisymmetric Elements
4. Isoparametric Elements
5. ANSYS Elements

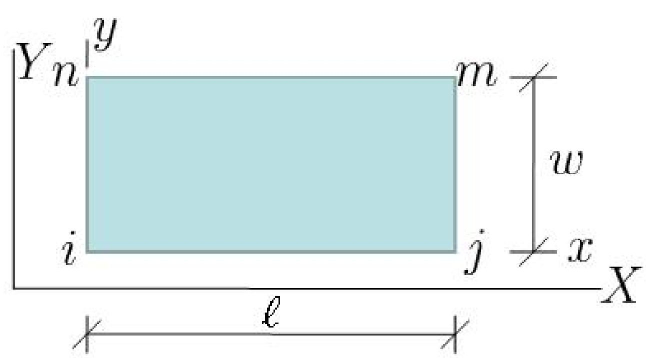
Ex: Heat Transfer through a fin surrounded by fluid



Temperature varies in both X and Y direction, so use 2-D elements

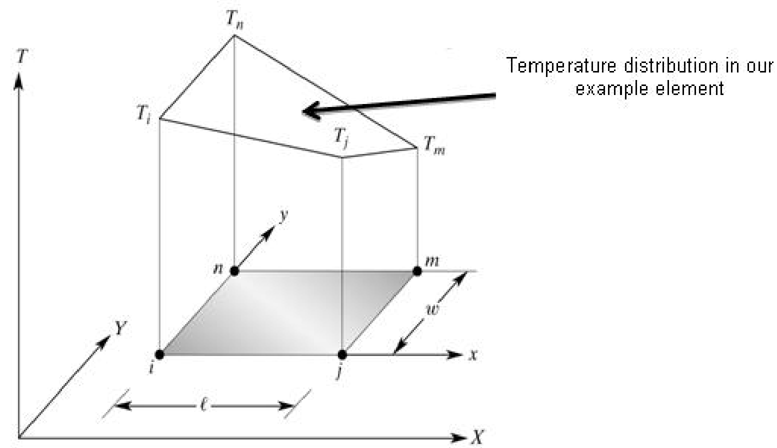
- 1-Dimensional solutions are approximated by *line segments*
- 2-Dimensional solutions are approximated by *plane segments*

2 2-D Linear Rectangular Element



2.1 Using Local Coordinates

Temperature Distribution



Approximate the temperature distribution in the rectangular element: $T^{(e)} = b_1 + b_2 x + b_3 y + b_4 xy$

2.1.1 Boundary Conditions

Four unknowns: b_1, b_2, b_3, and b_4. So we need four boundary conditions:

$T = T_i$ at $x = 0$ and $y = 0$
$T = T_j$ at $x = \ell$ and $y = 0$
$T = T_m$ at $x = \ell$ and $y = w$
$T = T_n$ at $x = 0$ and $y = w$

Substitute B.C.s into equation to create four equations with four unknowns. Solve equations simultaneously to obtain b_1, b_2, b_3, and b_4.

$$b_1 = T_i$$
$$b_2 = \frac{1}{\ell}\left(T_j - T_i\right)$$
$$b_3 = \frac{1}{w}\left(T_n - T_i\right)$$
$$b_4 = \frac{1}{\ell w}\left(T_i - T_j + T_m - T_n\right)$$

Substituting into equation ...

$$T^{(e)} = T_i + \frac{1}{\ell}\left(T_j - T_i\right)x + \frac{1}{w}\left(T_n - T_i\right)y + \frac{1}{\ell w}\left(T_i - T_j + T_m - T_n\right)xy$$

Algebraically rearranging to obtain shape functions

$$T^{(e)} = S_i T_i + S_j T_j + S_m T_m + S_n T_n$$

2.1.2 Shape Functions

Shape functions with local coordinates

$$S_i = \left(1 - \frac{x}{\ell}\right)\left(1 - \frac{y}{w}\right)$$
$$S_j = \frac{x}{\ell}\left(1 - \frac{y}{w}\right)$$
$$S_m = \frac{xy}{\ell w}$$
$$S_n = \frac{y}{w}\left(1 - \frac{x}{\ell}\right)$$

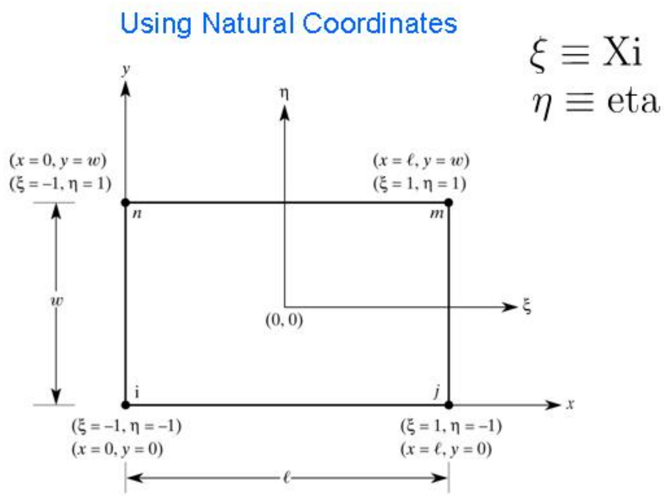
Using general ψ notation so equations will work for any variable, not just temperature.

$$\psi^{(e)} = S_i\psi_i + S_j\psi_j + S_m\psi_m + S_n\psi_n$$

Or

$$\psi^{(e)} = \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_m \\ \psi_n \end{Bmatrix}$$

2.2 Using Natural Coordinates



Relationship between Local coordinates (x and y) and the Natural coordinates (ξ and η).

$$\xi = \frac{2x}{\ell} - 1 \quad \eta = \frac{2y}{w} - 1$$

2-D linear shape functions in natural coordinates

$$S_i = \frac{1}{4}(1-\xi)(1-\eta) \quad S_j = \frac{1}{4}(1+\xi)(1-\eta)$$

$$S_m = \frac{1}{4}(1+\xi)(1+\eta) \quad S_n = \frac{1}{4}(1-\xi)(1+\eta)$$

2.2.1 Meeting Shape Function Properties

Notice shape function properties are retained for natural coordinates:

1. $S_i = 1$ at Node i (i.e. $\xi = -1,\ \eta = -1$)
   $S_j = 1$ at Node j (i.e. $\xi = 1,\ \eta = -1$)
   $S_m = 1$ at Node m (i.e. $\xi = 1,\ \eta = 1$)
   $S_n = 1$ at Node n (i.e. $\xi = -1,\ \eta = 1$)

2. $S_i = 0$ at all nodes other than i
   $S_j = 0$ at all nodes other than j
   $S_m = 0$ at all nodes other than m
   $S_n = 0$ at all nodes other than n

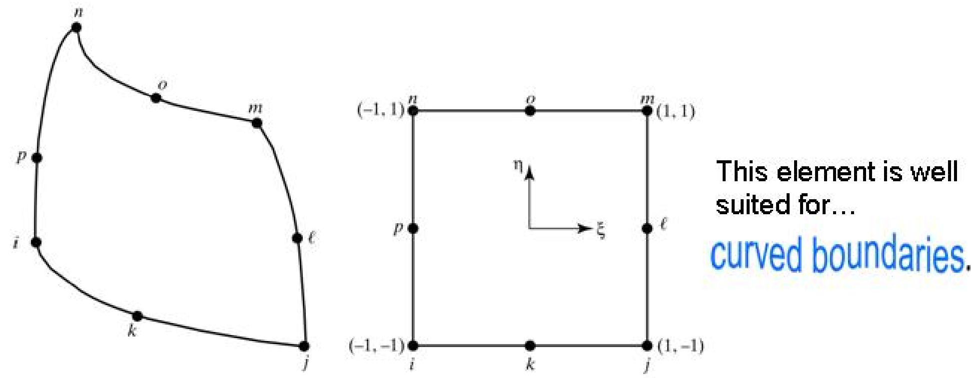
3. $S_i + S_j + S_m + S_n = 1$

A process similar to the Lagrange Functions can be used to generate these shape functions (see textbook).

3 Quadratic Rectangular (Quadrilateral) Elements

The 8-node Quadratic Quadrilateral Element is basically a higher order version of the 2-D 4-node quadrilateral element.

2-D Quadratic Quadrilateral Element (8-Node Element)



- For same number of elements, this element produces *better nodal results* than the linear 4-Node element.
- The approximate solution for this element in natural coordinates is:

$$\psi^{(e)} = b_1 + b_2\xi + b_3\eta + b_4\xi\eta + b_5\xi^2 + b_6\eta^2 + b_7\xi^2\eta + b_8\xi\eta^2$$

- Wow!
- Using B.C.'s, solving for unknowns $(b_1, b_2 \ldots b_8)$, and determining the shape functions is a *huge* task!!

Shape Functions
For corner nodes:

$$S_i = \frac{-1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \quad S_j = \frac{-1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta)$$

$$S_m = \frac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta) \quad S_n = \frac{-1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

For midpoint nodes:

$$S_k = \frac{1}{2}(1-\eta)(1-\xi^2) \quad S_\ell = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$S_o = \frac{1}{2}(1+\eta)(1-\xi^2) \quad S_p = \frac{1}{2}(1-\xi)(1-\eta^2)$$

General element solution with shape functions and ψ notation

$$\psi^{(e)} = S_i\psi_i + S_j\psi_j + S_m\psi_m + S_n\psi_n + S_k\psi_k + S_\ell\psi_\ell + S_o\psi_o + S_p\psi_p$$

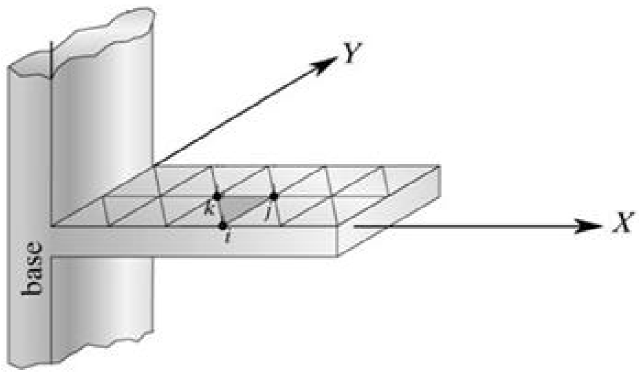
OR

$$\psi^{(e)} = \begin{bmatrix} S_i & S_j & S_m & S_n & S_k & S_\ell & S_o & S_p \end{bmatrix} \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_m \\ \psi_n \\ \psi_k \\ \psi_\ell \\ \psi_o \\ \psi_p \end{Bmatrix}$$

...and shape function properties still apply.

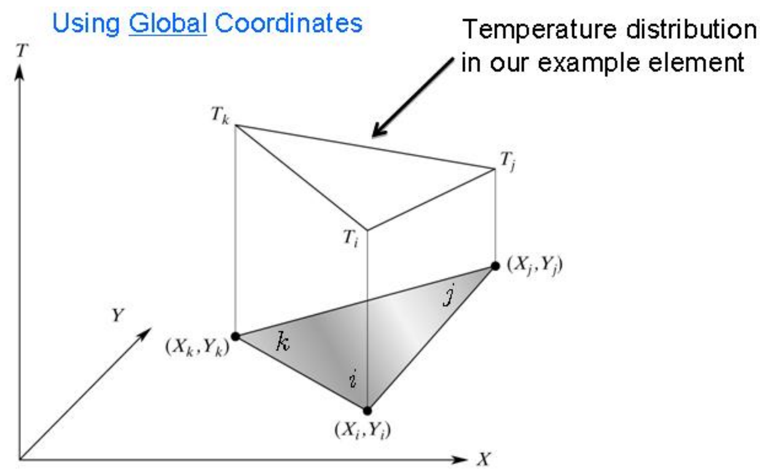
4 Linear Triangular Elements

Ex: Heat Transfer through a fin surrounded by fluid



Temperature varies in both X and Y direction, so use 2-D elements.

4.1 A Linear Triangular Element



- The linear triangular element is well suited for *curved boundaries*, more so than the linear quadrilateral element.
- The approximate temperature distribution in triangular elements is:
  $T^{(e)} = a_1 + a_2 X + a_3 Y$

Three unknowns, a_1, a_2, and a_3. So we need three boundary conditions:

$$T = T_i \text{ at } X = X_i \text{ and } Y = Y_i$$

$$T = T_j \text{ at } X = X_j \text{ and } Y = Y_j$$

$$T = T_k \text{ at } X = X_k \text{ and } Y = Y_k$$

Substitute B.C.s into equation to create three equations with three unknowns. Solve equations simultaneously to obtain a_1, a_2, and a_3.

$$a_1 = \frac{1}{2A}\left[\left(X_j Y_k - X_k Y_j\right) T_i + \left(X_k Y_i - X_i Y_k\right) T_j + \left(X_i Y_j - X_j Y_i\right) T_k\right]$$

$$a_2 = \frac{1}{2A}\left[\left(Y_j - Y_k\right) T_i + \left(Y_k - Y_i\right) T_j + \left(Y_i - Y_j\right) T_k\right]$$

$$a_3 = \frac{1}{2A}\left[\left(X_k - X_j\right) T_i + \left(X_i - X_k\right) T_j + \left(X_j - X_i\right) T_k\right]$$

where $A \equiv$ Area of triangle element, comes from ...

$$2A = X_i\left(Y_j - Y_k\right) + X_j\left(Y_k - Y_i\right) + X_k\left(Y_i - Y_j\right)$$

Algebraically rearranging to obtain shape functions

$$T^{(e)} = S_i T_i + S_j T_j + S_k T_k$$

4.2 Shape Functions
With global coordinates

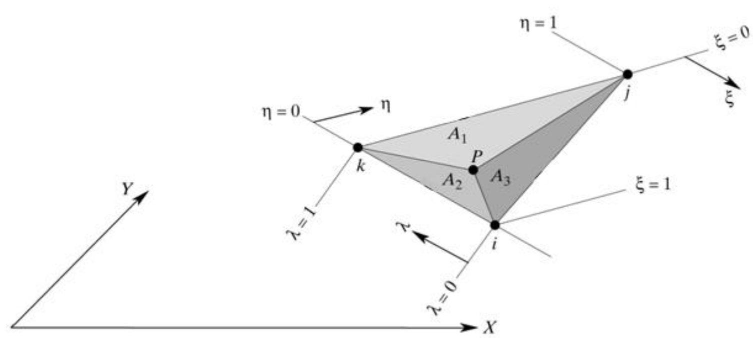
$$S_i = \frac{1}{2A}\left(\alpha_i + \beta_i X + \delta_i Y\right)$$
$$S_j = \frac{1}{2A}\left(\alpha_j + \beta_j X + \delta_j Y\right)$$
$$S_k = \frac{1}{2A}\left(\alpha_k + \beta_k X + \delta_k Y\right)$$

Where:

$$\alpha_i = X_j Y_k - X_k Y_j \quad \beta_i = Y_j - Y_k \quad \delta_i = X_k - X_j$$
$$\alpha_j = X_k Y_i - X_i Y_k \quad \beta_j = Y_k - Y_i \quad \delta_j = X_i - X_k$$
$$\alpha_k = X_i Y_j - X_j Y_i \quad \beta_k = Y_i - Y_j \quad \delta_k = X_j - X_i$$

The shape function properties are still enforced.

4.3 Linear Triangle Element using Natural Coordinates



where $\lambda \equiv$ Lambda. The coordinates of P define what areas A_1, A_2, and A_3 will be.

For triangle elements we use areas to define the natural (area) coordinates ξ, η,andλ:

$$\xi = \frac{A_1}{A}, \quad \eta = \frac{A_2}{A}, \quad \lambda = \frac{A_3}{A}$$

By observation we see that: $\xi + \eta + \lambda = 1$. Hey! This looks familiar! Through further investigation we find that ξ, η,andλ <u>are</u> the shape functions.
Therefore:

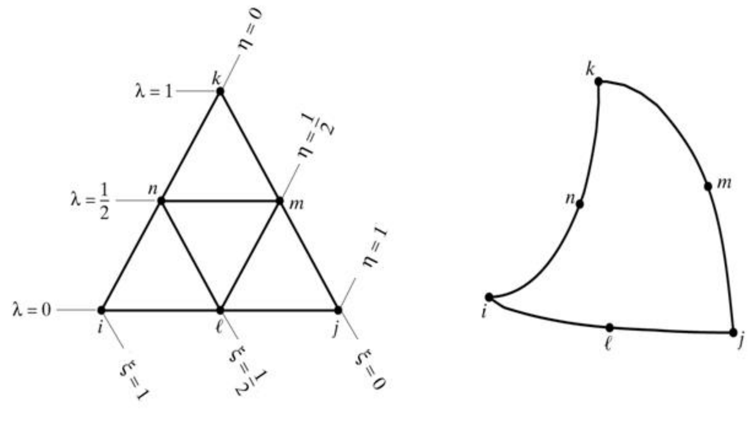
$$\xi = S_i \qquad \eta = S_j \qquad \lambda = S_k$$

where S_i, S_j, S_k are defined previously (substitute for nodal coordinates for A, $\alpha_i, \beta_i, \delta_i$)

5 Quadratic Triangular Elements

The 6-Node Quadratic Triangular Element is basically a higher order version of the 2-D 3-Node Triangular Element. For the same number of elements, this element produces *better nodal results* than the linear 3-Node element.

A Quadratic Triangular Element (6-Node Element)



Approximate solution of unknown variable:

$$\psi^{(e)} = a_1 + a_2 X + a_3 Y + a_4 X^2 + a_5 XY + a_6 Y^2$$

There are 6 unknown variables $(a_1, a_2, a_3, a_4, a_5, a_6)$ so we need 6 B.C.'s (Ex: T at each of 6 nodes)

Developing the shape functions in natural coordinates through the same process we obtain:

$$\begin{aligned}
&S_i = \xi\left(2\xi - 1\right)\\
&S_j = \eta\left(2\eta - 1\right)\\
&S_k = \lambda\left(2\lambda - 1\right) = 1 - 3\left(2 + \eta\right) + 2\left(\xi + \eta\right)^2\\
&S_\ell = 4\xi\eta\\
&S_m = 4\eta\lambda = 4\eta\left(1 - \xi - \eta\right)\\
&S_n = 4\xi\lambda = 4\xi\left(1 - \xi - \eta\right)
\end{aligned}$$

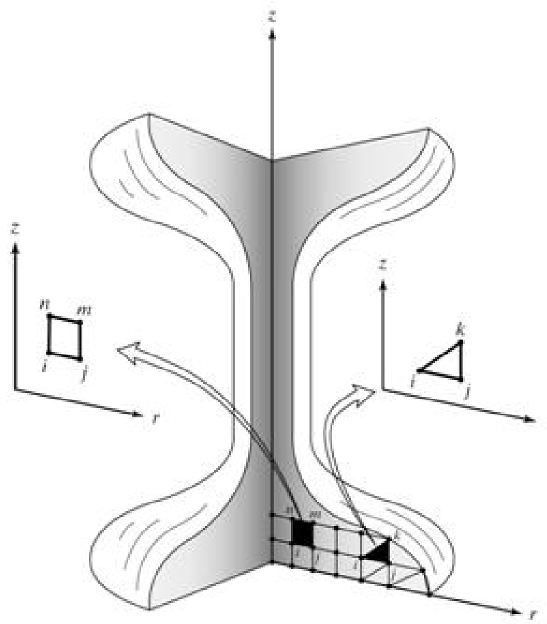
Again, the general solution for the 6-node Quadratic Triangular Element is:

$$\psi^{(e)} = \begin{bmatrix} S_i & S_j & S_k & S_\ell & S_m & S_n \end{bmatrix} \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_k \\ \psi_\ell \\ \psi_m \\ \psi_n \end{Bmatrix}$$

The spatial variable, ψ, can be temperature, velocity, displacement, etc. *This is a very commonly used element.*

6 Axisymmetric Elements

Axisymmetric elements are used in a special class of 3-D problems where geometry and loading (i.e. temperature, velocity, etc.) are **symmetrical about an axis**. Typically, the axis of symmetry is the z-axis and all the elements are “rotated” about it.



Notice that because of symmetry, we can use 2-D elements even though it is a 3-D problem.

Let’s examine two types of 2-D axisymmetric elements:

1. Triangular
2. Rectangular

6.1 Axisymmetric Triangular Elements

Recall the general solution for a Linear Triangular Element and the associated shape functions:

$$\psi^{(e)} = \begin{bmatrix} S_i & S_j & S_k \end{bmatrix} \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_k \end{Bmatrix}$$

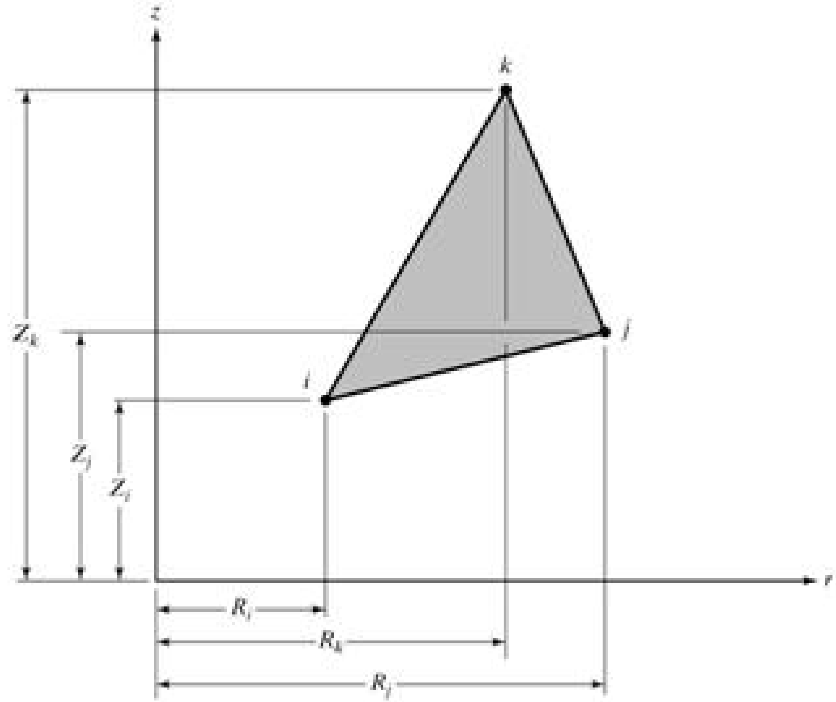
where:

$$S_i = \frac{1}{2A}\left(\alpha_i + \beta_i X + \delta_i Y\right)$$

$$S_j = \frac{1}{2A}\left(\alpha_j + \beta_j X + \delta_j Y\right)$$

$$S_k = \frac{1}{2A}\left(\alpha_k + \beta_k X + \delta_k Y\right)$$

Typically, axisymmetric triangular elements use “R” (i.e., radial) and “Z” coordinates. So let’s convert our shape functions from X-Y coordinates to R-Z coordinates



Therefore shape functions now are:

$$S_i = \frac{1}{2A}\left(\alpha_i + \beta_i r + \delta_i z\right)$$

$$S_j = \frac{1}{2A}\left(\alpha_j + \beta_j r + \delta_j z\right)$$

$$S_k = \frac{1}{2A}\left(\alpha_k + \beta_k r + \delta_k z\right)$$

where:

$$\alpha_i = R_j Z_k - R_k Z_j \quad \beta_i = Z_j - Z_k \quad \delta_i = R_k - R_j$$

$$\alpha_j = R_k Z_i - R_i Z_k \quad \beta_j = Z_k - Z_i \quad \delta_j = R_i - R_k$$

$$\alpha_k = R_i Z_j - R_j Z_i \quad \beta_k = Z_i - Z_j \quad \delta_k = R_j - R_i$$

6.2 Axisymmetric Rectangular Elements

Recall general solution for a Linear Rectangular Element and the associated shape functions:

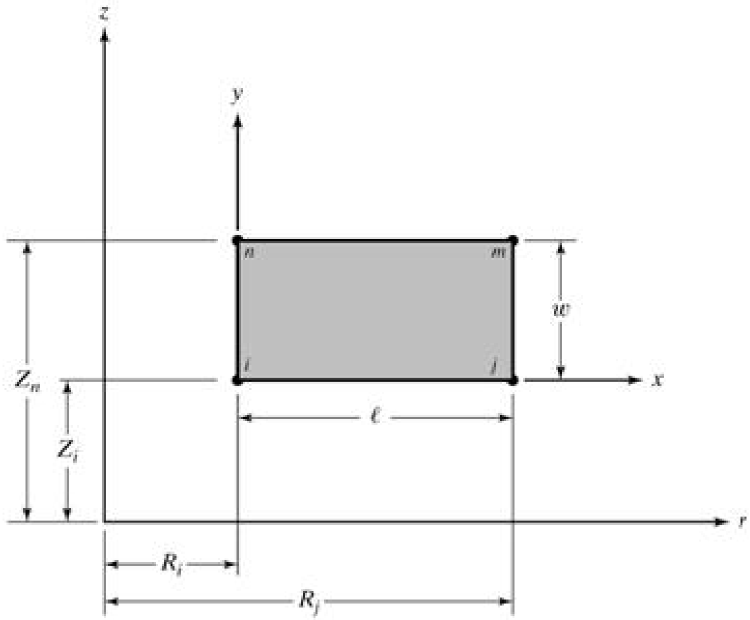
$$\psi^{(e)} = \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_m \\ \psi_n \end{Bmatrix}$$

where:

$$S_i = \left(1 - \frac{x}{\ell}\right)\left(1 - \frac{y}{w}\right) \quad S_m = \frac{xy}{\ell w}$$

$$S_j = \frac{x}{\ell}\left(1 - \frac{y}{w}\right) \quad S_n = \frac{y}{w}\left(1 - \frac{x}{\ell}\right)$$

Let's convert our shape functions from x-y coordinates to R-Z coordinates



The relationships between the x-y coordinates and the R-Z coordinates are:

$$r = R_i + x \quad \rightarrow \quad x = r - R_i$$

$$z = Z_i + y \quad \rightarrow \quad y = z - Z_i$$

Also:

$$\ell = R_j - R_i \;\text{ and }\; w = Z_n - Z_i$$

Substitute these relationships into the 2-D linear rectangular shape functions to obtain the axisymmetric rectangular shape functions.

The shape functions for an axisymmetric rectangular element are:

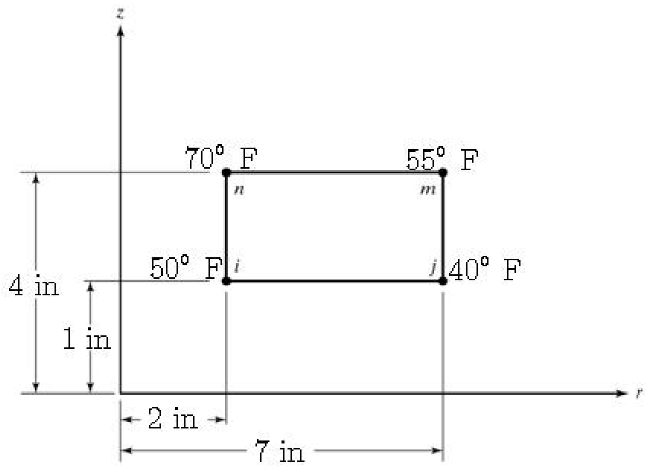
$$S_i = \left(\frac{R_j - r}{\ell}\right)\left(\frac{Z_n - z}{w}\right)$$

$$S_j = \left(\frac{r - R_i}{\ell}\right)\left(\frac{Z_n - z}{w}\right)$$

$$S_m = \left(\frac{r - R_i}{\ell}\right)\left(\frac{z - Z_i}{w}\right)$$

$$S_n = \left(\frac{R_j - r}{\ell}\right)\left(\frac{z - Z_i}{w}\right)$$

6.3 Example



Find: Temperature at r = 3.5 in and z = 3 in

Solution:

$$T^{(e)} = \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

$$S_i = \left(\frac{R_j - r}{\ell}\right)\left(\frac{Z_n - z}{w}\right) = \left(\frac{7 - 3.5}{5}\right)\left(\frac{4 - 3}{3}\right) = 0.233$$

$$S_j = \left(\frac{r - R_i}{\ell}\right)\left(\frac{Z_n - z}{w}\right) = \left(\frac{3.5 - 2}{5}\right)\left(\frac{4 - 3}{3}\right) = 0.100$$

$$S_m = \left(\frac{r - R_i}{\ell}\right)\left(\frac{z - Z_i}{w}\right) = \left(\frac{3.5 - 2}{5}\right)\left(\frac{3 - 1}{3}\right) = 0.200$$

$$S_n = \left(\frac{R_j - r}{\ell}\right)\left(\frac{z - Z_i}{w}\right) = \left(\frac{7 - 3.5}{5}\right)\left(\frac{3 - 1}{3}\right) = 0.467$$

$$T^{(e)} = \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

$$T^{(e)} = \begin{bmatrix} 0.233 & 0.100 & 0.200 & 0.467 \end{bmatrix} \begin{Bmatrix} 50^\circ \text{ F} \\ 40^\circ \text{ F} \\ 55^\circ \text{ F} \\ 70^\circ \text{ F} \end{Bmatrix}$$

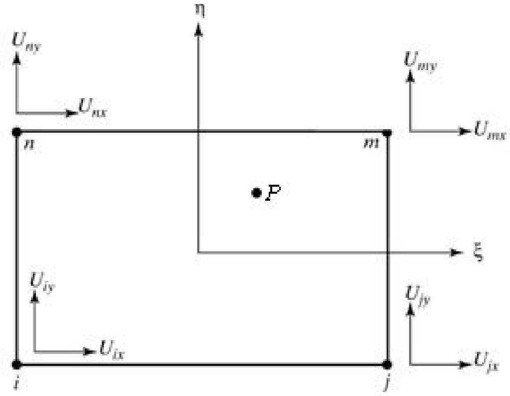
$$T^{(e)} = (0.233)\,(50^\circ \text{ F}) + (0.100)\,(40^\circ \text{ F}) + (0.200)\,(55^\circ \text{ F}) + (0.200)\,(55^\circ \text{ F}) + (0.467)\,(70^\circ \text{ F})$$

$$T^{(e)} = 59.3^\circ \text{ F}$$

7 Isoparametric Elements

Recall from a previous lecture, we use a single set of parameters (i.e. shape functions) to define the unknown variables displacements, velocity, temperature, etc., and use the same parameters (i.e. shape functions) to express the geometry, then we are using an "Isoparametric" Formulation. These types of elements are called Isoparametric Elements. (See Table 7.1 from Moaveni).

7.1 Quadrilateral Plane-Stress Element (Linear, Natural Coordinates)



U describes the nodal displacements in the Global Coordinates and $u - v$ describe the displacements within the element, x-y directions respectively.

$$u^{(e)} = S_i U_{ix} + S_j U_{jx} + S_m U_{mx} + S_n U_{nx}$$

$$v^{(e)} = S_i U_{iy} + S_j U_{jy} + S_m U_{my} + S_n U_{ny}$$

7.2 Elements

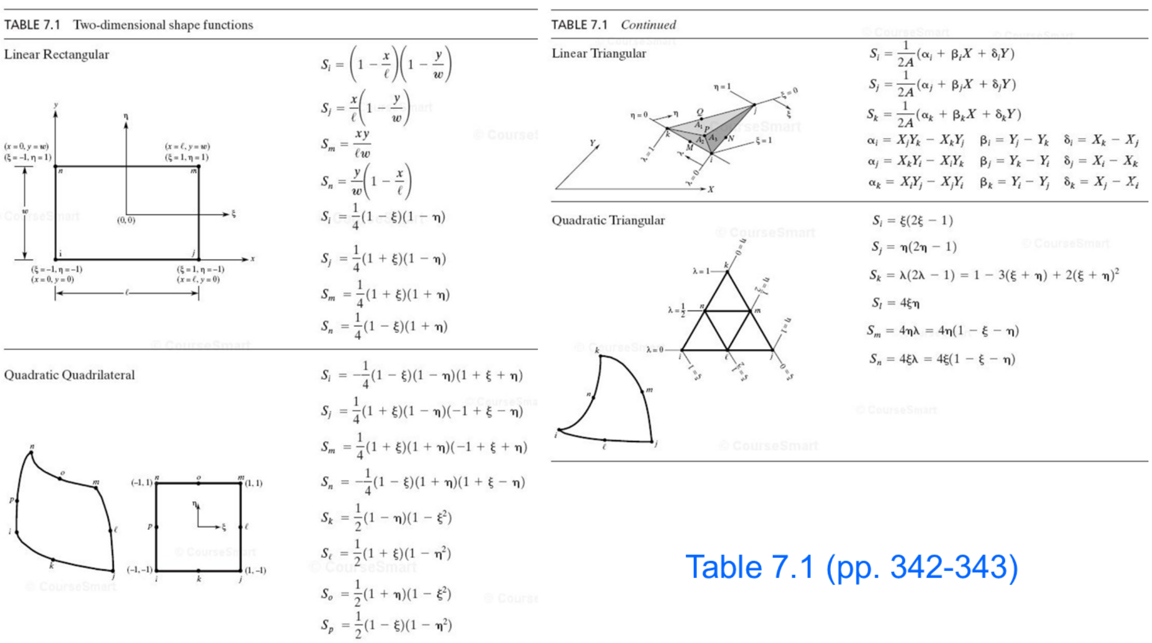


TABLE 7.1 Two-dimensional shape functions

Element	Shape functions
Linear Rectangular	$S_i = \left(1 - \frac{x}{\ell}\right)\left(1 - \frac{y}{w}\right)$ $S_j = \frac{x}{\ell}\left(1 - \frac{y}{w}\right)$ $S_m = \frac{xy}{\ell w}$ $S_n = \frac{y}{w}\left(1 - \frac{x}{\ell}\right)$ $S_i = \frac{1}{4}(1-\xi)(1-\eta)$ $S_j = \frac{1}{4}(1+\xi)(1-\eta)$ $S_m = \frac{1}{4}(1+\xi)(1+\eta)$ $S_n = \frac{1}{4}(1-\xi)(1+\eta)$
Quadratic Quadrilateral	$S_i = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$ $S_j = \frac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta)$ $S_m = \frac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta)$ $S_n = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$ $S_k = \frac{1}{2}(1-\eta)(1-\xi^2)$ $S_\ell = \frac{1}{2}(1+\xi)(1-\eta^2)$ $S_o = \frac{1}{2}(1+\eta)(1-\xi^2)$ $S_p = \frac{1}{2}(1-\xi)(1-\eta^2)$

TABLE 7.1 *Continued*

Element	Shape functions
Linear Triangular	$S_i = \frac{1}{2A}(\alpha_i + \beta_i X + \delta_i Y)$ $S_j = \frac{1}{2A}(\alpha_j + \beta_j X + \delta_j Y)$ $S_k = \frac{1}{2A}(\alpha_k + \beta_k X + \delta_k Y)$ $\alpha_i = X_j Y_k - X_k Y_j \quad \beta_i = Y_j - Y_k \quad \delta_i = X_k - X_j$ $\alpha_j = X_k Y_i - X_i Y_k \quad \beta_j = Y_k - Y_i \quad \delta_j = X_i - X_k$ $\alpha_k = X_i Y_j - X_j Y_i \quad \beta_k = Y_i - Y_j \quad \delta_k = X_j - X_i$
Quadratic Triangular	$S_i = \xi(2\xi - 1)$ $S_j = \eta(2\eta - 1)$ $S_k = \lambda(2\lambda - 1) = 1 - 3(\xi+\eta) + 2(\xi+\eta)^2$ $S_l = 4\xi\eta$ $S_m = 4\eta\lambda = 4\eta(1-\xi-\eta)$ $S_n = 4\xi\lambda = 4\xi(1-\xi-\eta)$

Table 7.1 (pp. 342-343)

In matrix form:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} S_i & 0 & S_j & 0 & S_m & 0 & S_n & 0 \\ 0 & S_i & 0 & S_j & 0 & S_m & 0 & S_n \end{bmatrix} \begin{Bmatrix} U_{ix} \\ U_{iy} \\ U_{jx} \\ U_{jy} \\ U_{mx} \\ U_{my} \\ U_{nx} \\ U_{ny} \end{Bmatrix}$$

Use the Natural Coordinate shape functions developed previously. These same shape functions can also be used to describe the position of any point within the element.

$$x = S_i x_i + S_j x_j + S_m x_m + S_n x_n$$
$$y = S_i y_i + S_j y_j + S_m y_m + S_n y_n$$