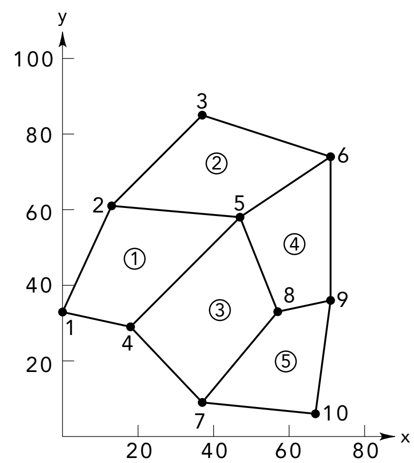
```
    vu=vx.^n;                  % nodal solution by collocation
    ve0=(n*(n-1))/8*vh.^2.*(vx(2:nelt+1).^(n-2)); % error
    ve0h=abs(ve0.*vh.^0.5);
%------- convergence test
    emax=max(ve0h); emin=min(ve0h);
    if((emax-emin)/emax<0.05) break; end
%-------mesh adaptation
    v1=(1./ve0h);
    vh=vh.*(v1.^0.4);
    vx=[0,v1]*L;
    for j=1:nelt
        vx(j+1)=vx(j)+vx(j+1); % new coordinate
    end
    vh=vx(2:nelt+1)-vx(1:nelt); % new lengths
end
```

Figure 1.3. Matlab© program for mesh adaptation

1.8 Example of application: rainfall problem

The finite element method of approximation is most often used for the discretization of partial differential equations. It may, however, be used to approximate a function known only at certain points of measurement.

In this section, we calculate the total rainfall for a region A using the measurements obtained by rain gages situated at certain points:



The total rainfall Q is defined from the rainfall $u(x, y)$ at all points by:

$$Q = \int_A u(x, y)\, \mathrm{d}A. \tag{1.73}$$

The rainfall u_i is known for 10 stations of coordinates $x_i\, y_i$. The data used in this example are taken from an article by Akin [AKI 71]. We will use a finite element approximation technique to evaluate (1.73).

a) Choice of nodes and elements

We select stations 1, 2, …, 10 as geometrical nodes and interpolation nodes. Their coordinates make up the CORG table:

Node	x_i (km)	y_i (km)
1	0.0	33.3
2	13.2	62.3
3	39.3	84.5
4	22.2	30.1
5	49.9	57.6
6	78.8	78.2
7	39.3	10.0
8	59.7	34.3
9	73.9	36.2
10	69.8	5.1

Region A is divided into five quadrilateral elements defined by the CONEC table:

Element	Nodes			
	i	j	k	l
1	1	4	5	2
2	2	5	6	3
3	4	7	8	5
4	5	8	9	6
5	7	10	9	8

The rainfall vector at the nodes (nodal values of u) is known in this problem (units: depth of rainfall in cm):

$$\{u_n\}^T = \langle 4.62 \;\; 3.81 \;\; 4.76 \;\; 5.45 \;\; 4.90 \;\; 10.35 \;\; 4.96 \;\; 4.26 \;\; 18.36 \;\; 15.69 \rangle$$

b) Approximation of $u(x, y)$ over each element

Choose a bilinear quadrilateral element as described in Example 1.16. For each element e, the approximate function $u(\xi, \eta)$ is:

$$u(\xi, \eta) = \langle P \rangle [P_n]^{-1} \{u_n\} \tag{1.74}$$

The geometrical transformation is:

$$x(\xi, \eta) = \langle P \rangle [P_n]^{-1} \{x_n\}$$
$$y(\xi, \eta) = \langle P \rangle [P_n]^{-1} \{y_n\}$$

where

$$\{u_n\}^T = \langle u_i \quad u_j \quad u_k \quad u_l \rangle$$
$$\{x_n\}^T = \langle x_i \quad x_j \quad x_k \quad x_l \rangle$$
$$\{y_n\}^T = \langle y_i \quad y_j \quad y_k \quad y_l \rangle$$

i, j, k and l being the numbers of the four nodes of the element, given by the CONEC table.

The determinant of the Jacobian matrix $\det(J)$ has already been evaluated in Example 1.18 in the form:

$$\det(J) = A_0 + A_1\,\xi + A_2\,\eta. \tag{1.75}$$

c) Evaluation of Q

The total rainfall Q is the sum of the rainfall Q^e over each element:

$$Q = \sum_{e=1}^{5} Q^e \tag{1.76}$$
$$Q^e = \int_{A^e} u(x, y)\, \mathrm{d}A$$
$$= \int_{-1}^{1} \int_{-1}^{1} u(\xi, \eta) \det(J)\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

Hence, after replacing u by approximation (1.74):

$$Q^e = \int_{-1}^{1}\int_{-1}^{1} \langle P \rangle [P_n]^{-1} \{u_n\} \det(J)\, d\xi\, d\eta$$

$$Q^e = \int_{-1}^{1}\int_{-1}^{1} (A_0 + A_1\,\xi + A_2\,\eta) \langle 1 \quad \xi \quad \eta \quad \xi\eta \rangle\, d\xi\, d\eta \cdot [P_n]^{-1} \{u_n\}. \tag{1.77}$$

After explicit integration, we organize Q^e in the form:

$$Q^e = \langle A_0 \quad \frac{A_1}{3} \quad \frac{A_2}{3} \rangle \begin{Bmatrix} u_i + u_j + u_k + u_l \\ -u_i + u_j + u_k - u_l \\ -u_i - u_j + u_k + u_l \end{Bmatrix} \tag{1.78}$$

where coefficients A_0, A_1 and A_2 are functions of the coordinates of the nodes, given in Example 1.18, and u_i, u_j, u_k, u_l are the values of rainfall at the four nodes of the element, extracted from $\{U_n\}$.

Finally, the table below shows the numerical calculation of (1.76) that uses (1.78), Example 1.18 and the CORG, CONEC and $\{U_n\}$ tables.

Element	A_0 (km^2)	A_1	A_2	Q^e (cm km^2)
1	228.18	1.64	55.04	4,261.41
2	241.65	−5.70	12.99	5,771.07
3	217.56	−25.18	−14.01	4,272.97
4	182.79	−65.72	29.70	6,954.45
5	159.37	15.94	−66.85	6,983.87

The total rainfall is $Q = \sum Q^e = 28{,}243.78$ cm km^2.

The approximate total area is $A = 4\sum A_0 = 4{,}118.21$ km^2.

The average height of rainfall is $u_m = \dfrac{Q}{A} = 6.86$ cm.

Important results

Nodal approximation of a function:

$$u(x) = \langle N \rangle \{u_n\}. \tag{1.9}$$

Transformation of the reference element into a real element:

$$\tau\colon \xi \to \mathbf{x}(\xi) = [\overline{N}(\xi)]\{\mathbf{x}_n\}. \tag{1.12}$$

Approximation of u on the reference element:

$$u(\xi) = \langle N(\xi) \rangle \{u_n\}. \tag{1.14}$$

Properties of the interpolation functions:

$$N_j(\xi_i) = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \tag{1.16}$$

$$\sum_{i=1}^{n_d} N_i(\xi)\ p(\xi_\iota) = p(\xi). \tag{1.31}$$

Construction of interpolation functions:

$$u(\xi) = \langle P(\xi) \rangle \{a\} \tag{1.18}$$

$$\{u_n\} = [P_n]\{a\} \tag{1.20}$$

$$\langle N(\xi) \rangle = \langle P(\xi) \rangle [P_n]^{-1}. \tag{1.24}$$

Transformation of first derivatives:

$$\{\partial_x\} = [j]\{\partial_\xi\} \tag{1.37b}$$

$$[j] = [J]^{-1} \tag{1.38}$$

$$[J] = \begin{bmatrix} \langle \overline{N}_{,\xi} \rangle \\ \langle \overline{N}_{,\eta} \rangle \\ \langle \overline{N}_{,\zeta} \rangle \end{bmatrix} \begin{bmatrix} \{x_n\} & \{y_n\} & \{z_n\} \end{bmatrix}. \tag{1.43}$$

Transformation of an integral:

$$\int_{V^e} f(\mathbf{x})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_{V^r} f(\mathbf{x}(\xi)) \det(J)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta\,. \tag{1.44}$$

Approximation errors:

$$\left| e \right|_s \le c\, l^{n-s} \left| u_{ex}(\mathbf{x}) \right|_n. \tag{1.57a}$$

Notations

$\langle a \rangle = \langle a_1 \quad a_2 \quad \dots \quad a_n \rangle$	general parameters of an approximation (or generalized variables)
$\langle a_x \rangle, \langle a_y \rangle, \langle a_z \rangle$	generalized coordinates of the element
$e(x) = u(x) - u_{ex}(x)$	approximation error
$[J], [j], \det(J)$	Jacobian matrix, its inverse and its determinant
n	number of interpolation nodes
n_d	number of degrees of freedom of an element
n^e	number of interpolation nodes of an element

n_{el}	number of elements
$\overline{n}$	number of geometrical nodes
$\overline{n}^e$	number of geometrical nodes of an element
$\langle N(x) \rangle = \langle N_1(x) \quad N_2(x) \quad \dots \rangle$	nodal interpolation functions over the real element
$\langle N(\xi) \rangle = \langle N_1(\xi) \quad N_2(\xi) \quad \dots \rangle$	interpolation functions over the reference element
$\langle \overline{N}(\xi) \rangle = \langle \overline{N}_1(\xi) \quad \overline{N}_2(\xi) \quad \dots \rangle$	geometrical transformation functions
$[P_n], [\overline{P}_n]$	nodal interpolation and geometrical transformation matrices
$\langle P(x) \rangle = \langle P_1(x) \quad P_2(x) \quad \dots \rangle$	polynomial basis of the approximation over the real element
$\langle P(\xi) \rangle = \langle P_1(\xi) \quad P_2(\xi) \quad \dots \rangle$	polynomial basis of the approximation over the reference element
$\langle \overline{P}(\xi) \rangle$	polynomial basis of the geometrical transformation
$[T_1], [T_2], [C_1], [C_2]$	transformation matrices for second derivatives
$u(x)$	approximate functions
$u_{ex}(x)$	exact functions
$u^e(x)$ or sometimes $u(x)$	approximate function over an element
$\langle u_n \rangle = \langle u_1 \quad u_2 \quad \dots \rangle$	nodal parameters or nodal variables
V	domain under study
V^e	domain corresponding to the element e
V^r	reference element domain
$\mathbf{x} = \langle x \quad y \quad z \rangle$	Cartesian coordinates of a point
$\mathbf{x}_i = \langle x_i \quad y_i \quad z_i \rangle$	coordinates of node i (geometrical or interpolation)
$\langle \mathbf{x}_n \rangle$	coordinates of the nodes of a real element
$\langle \overline{\mathbf{x}}_n \rangle$	coordinates of the geometrical nodes
$\langle \partial_x \rangle, \langle \partial_\xi \rangle$	differential operators: $\langle \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \rangle$ and $\langle \frac{\partial}{\partial \xi} \quad \frac{\partial}{\partial \eta} \quad \frac{\partial}{\partial \zeta} \rangle$
$\langle \partial_x^2 \rangle, \langle \partial_\xi^2 \rangle$	second-order differential operators defined by (1.45a) and (1.45b)
$\xi = \langle \xi \quad \eta \quad \zeta \rangle$	coordinates of a point of the reference element

$\langle \xi_n \rangle$	coordinates of the nodes of a reference element
$\langle \bar{\xi}_n \rangle$	coordinates of the geometrical nodes of a reference element
τ^e or τ	geometrical transformation for element e
$\lvert\ \rvert_s$ $\lVert\ \rVert_s^2$	maximum and root mean square norm for derivatives of order s of a function
CONEC	connectivity table
CORG	nodal coordinates table

Remarks

A vector a can be represented in three different ways:

a

$\langle a \rangle$ row vector

$\{a\}$ column vector

A matrix T, its transpose and its inverse are represented: $[T]$, $[T]^T$ and $[T]^{-1}$.

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