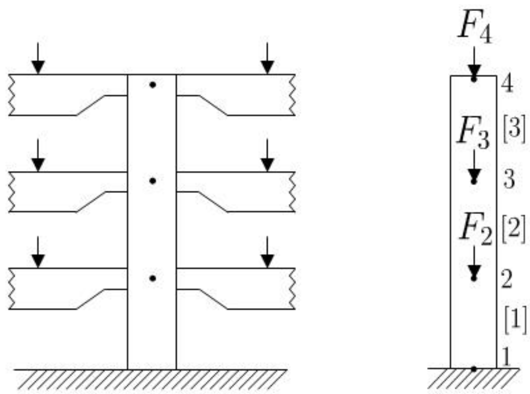
Axial Members and Intro. to Shape Functions

Introduction
In this section we will examine *general loading* of
- Structural members
- Machine components (push-pull and bending)

Torsion will be examined later.

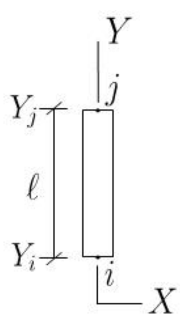
1 Members in Axial Loading

Linear Elements As an example, consider a column with multiple loadings.



Column carrying 3 floor loadings

Assume central axial loading (i.e. no bending)
Let's develop the *individual axial elements*



GCS
$Y_i,\ Y_j \equiv$ Global Location Of Nodes i and j
$U_i,\ U_j \equiv$ Nodal Deflections or Displacements i and j
$\ell = Y_j - Y_i$

Let's solve for unknown deflection, $u^{(e)}$, at any location, Y, within the element. Assuming a linear deflection distribution within the element:

$$u^{(e)} = C_1 + C_2 Y$$

Notice the form; equation for a straight line. (i.e. $Y = mx + b$). Again, $u^{(e)}$ is deflection at any point Y *within* the element. With two unknowns in equation, C_1, and C_2, we need two equations:

$$u = u_i \text{ @ } Y = Y_i$$
$$(1) \quad u_i = C_1 + C_2 Y_i$$

$$u = u_j \text{ @ } Y = Y_j$$
$$(2) \quad u_j = C_1 + C_2 Y_j$$

Solving (1) and (2) simultaneously, we get:

$$C_1 = \frac{u_i Y_j - u_j Y_i}{Y_j - Y_i} \qquad C_2 = \frac{u_j - u_i}{Y_j - Y_i}$$

C_2 is the slope and C_1 is the "Y-Intercept" of the equation ($u^{(e)} = C_1 + C_2 Y$)

$$u^{(e)} = \frac{u_i Y_j - u_j Y_i}{Y_j - Y_i} + \frac{u_j - u_i}{Y_j - Y_i} Y$$

Let's rearrange the equation and solve in terms of u_i and u_j

$$u^{(e)} = u_i \left(\frac{Y_j}{Y_j - Y_i} \right) - u_j \left(\frac{Y_i}{Y_j - Y_i} \right) + u_j \left(\frac{Y}{Y_j - Y_i} \right) - u_i \left(\frac{Y}{Y_j - Y_i} \right)$$

$$u^{(e)} = u_i \left(\frac{Y_j - Y}{Y_j - Y_i} \right) + u_j \left(\frac{Y - Y_i}{Y_j - Y_i} \right)$$

Let's define the terms in parentheses as

$$S_i = \left(\frac{Y_j - Y}{Y_j - Y_i} \right) \qquad S_j = \left(\frac{Y - Y_i}{Y_j - Y_i} \right)$$

where S_i and S_j are called *shape functions* and literally describe the "shape" or deflection of each element.

2 Shape Functions

Knowing $\ell = Y_j - Y_i$ from before

$$S_i = \frac{Y_j - Y}{\ell} \qquad S_j = \frac{Y - Y_i}{\ell}$$

Therefore

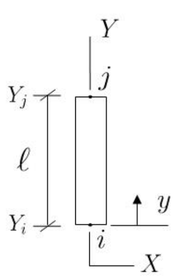
$$u^{(e)} = S_i u_i + S_j u_j$$

Again, $u^{(e)}$ is the deflection at any point Y within the element. In Matrix form:

$$u^{(e)} = [S_i S_j] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

This same approach can be used to approximate variation of any unknown variable such as temperature and velocity.

Let's look at the local coordinate system.



Notice that y starts at Node i. Thus, $Y = Y_i + y$.

Knowing that any location in an element can be described by $Y = Y_i + y$, we can substitute into our shape functions.

$$S_i = \frac{Y_j - Y}{\ell}$$
$$S_i = \frac{Y_j - (Y_i + y)}{\ell}$$
$$S_i = \frac{(Y_j - Y_i) - y}{\ell}$$
$$S_i = \frac{(\ell - y)}{\ell}$$
$$S_i = 1 - \frac{y}{\ell}$$

$$S_j = \frac{Y - Y_i}{\ell}$$
$$S_j = \frac{(Y_i + y) - Y_i}{\ell}$$
$$S_j = \frac{(Y_i - Y_i) + y}{\ell}$$
$$S_j = \frac{y}{\ell}$$

Notice that for $0 \leq y \leq \ell$,

$$S_i = 1 \text{ at } i,\ \ S_i = 0 \text{ at } j$$
$$S_j = 0 \text{ at } i,\ \ S_j = 1 \text{ at } j$$

3 Stiffness and Load Matrices for Axial Members

Let's develop stiffness and load matrices for axially loaded members using the minimum total potential energy formulation. First, we can find the Strain Energy in the element via

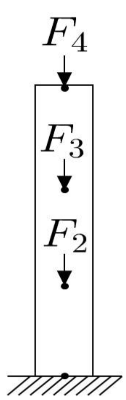
$$\Lambda^{(e)} = \int_V \frac{(\sigma\varepsilon)}{2} dV \qquad \text{where: } V \equiv Volume$$

From Hooke's Law:

$$\Lambda^{(e)} = \int_V \frac{(E\varepsilon^2)}{2} dV$$

The total potential energy for a body in a closed system:

$$\Pi = \Lambda_{\text{total}} - \text{External Work}$$



For a body with n elements and m nodes

$$\Pi = \sum_{e=1}^{n} \Lambda^{(e)} - \sum_{i=1}^{m} F_i u_i \ \ i \equiv \text{Different Node Values}$$

The system will reach equilibrium where the potential energy is at its minimum. Therefore set $\frac{\partial \Pi}{\partial u_i}$ equal to zero

$$\frac{\partial \Pi}{\partial u_i} = \frac{\partial}{\partial u_i} \sum_{e=1}^{n} \Lambda^{(e)} - \frac{\partial}{\partial u_i} \sum_{i=1}^{m} F_i u_i = 0 \quad \text{for } 1,\ 2, \ldots m$$

Knowing:

$$u^{(e)} = S_i u_i + S_j u_j$$
$$S_i = 1 - \frac{y}{\ell}$$
$$S_j = \frac{y}{\ell}$$

...and from Mechanics of Materials

$$\varepsilon = \frac{d\delta}{dx} \quad \text{or specifically } \varepsilon = \frac{du}{dy}$$

Then

$$\varepsilon = \frac{du}{dy}$$

$$\varepsilon = \frac{d}{dy}[S_i u_i + S_j u_j]$$

$$\varepsilon = \frac{d}{dy}[(1 - \frac{y}{\ell})u_i + \frac{y}{\ell}u_j]$$

$$\varepsilon = (\frac{-1}{\ell})u_i + (\frac{1}{\ell})u_j$$

$$\varepsilon = \frac{-u_i + u_j}{\ell}$$

Substitute strain, ε, into Strain Energy Equation, Λ:

$$\Lambda^{(e)} = \int_V \frac{E\varepsilon^2}{2} dV$$

$$\Lambda^{(e)} = \frac{E\varepsilon^2}{2} \int_V dV$$

$$\Lambda^{(e)} = \frac{E\varepsilon^2}{2}(A\ell)$$

$$\Lambda^{(e)} = \frac{AE\ell}{2}(\frac{-u_i + u_j}{\ell})^2$$

$$\Lambda^{(e)} = \frac{AE}{2\ell}({u_j}^2 - 2u_j u_i + {u_i}^2)$$

3.1 Minimizing Strain Energy

@ node i:

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{\partial}{\partial u_i}\left[\frac{AE}{2\ell}({u_j}^2 - 2u_j u_i + {u_i}^2)\right]$$

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{AE}{2\ell}(-2u_j + 2u_i)$$

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{AE}{\ell}(u_i - u_j)$$

@ node j:

$$\frac{\partial \Lambda^{(e)}}{\partial u_j} = \frac{\partial}{\partial u_j}[\frac{AE}{2\ell}({u_j}^2 - 2u_j u_i + {u_i}^2)]$$

$$\frac{\partial \Lambda^{(e)}}{\partial u_j} = \frac{AE}{2\ell}(2u_j - 2u_i)$$

$$\frac{\partial \Lambda^{(e)}}{\partial u_j} = \frac{AE}{\ell}(u_j - u_i)$$

3.2 External Work, $F_i u_i$

Minimizing:

$$\frac{\partial (F_i u_i)}{\partial u_i} = F_i$$

In Matrix form:

$$\begin{Bmatrix} \frac{\partial \Lambda^{(e)}}{\partial u_i} \\ \frac{\partial \Lambda^{(e)}}{\partial u_j} \end{Bmatrix} = \begin{bmatrix} \frac{AE}{\ell} & \frac{-AE}{\ell} \\ \frac{-AE}{\ell} & \frac{AE}{\ell} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial (F_i u_i)}{\partial u_i} \\ \frac{\partial (F_j u_j)}{\partial u_j} \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

Minimum total potential energy equation in Matrix form:

$$\frac{\partial \Pi^{(e)}}{\partial u} = \begin{Bmatrix} \frac{\partial \Lambda^{(e)}}{\partial u_i} \\ \frac{\partial \Lambda^{(e)}}{\partial u_j} \end{Bmatrix} - \begin{Bmatrix} \frac{\partial (F_i u_i)}{\partial u_i} \\ \frac{\partial (F_j u_j)}{\partial u_j} \end{Bmatrix} = 0$$

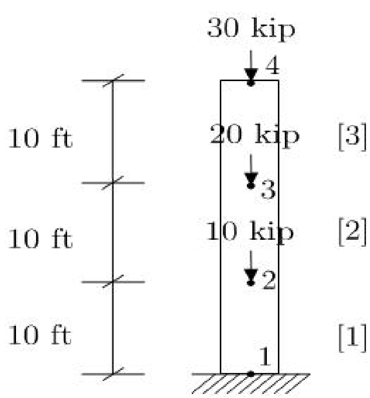
$$\begin{bmatrix} \frac{AE}{\ell} & \frac{-AE}{\ell} \\ \frac{-AE}{\ell} & \frac{AE}{\ell} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} - \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = 0$$

If $k = \frac{AE}{\ell}$

$$k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

So we have applied the minimum total potential energy function to Axial Members using the linear shape function. We now see that Axial Members act in the same manner as Trusses!

4 Example, Part 1



$$E = 30 \times 10^6 \ \frac{\text{lb}}{\text{in}^2}$$

$$A = 20 \text{ in}^2$$

4.1 Find:

1. Vertical Displacements of each floor level

2. Stress in each portion of column

4.2 Solution

Element Stiffness Matrix

$$[k]^{(e)} = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \text{where} \quad \frac{AE}{\ell} = \frac{(20\ \text{in}^2)(30 \times 10^6\ \frac{\text{lb}}{\text{in}^2})}{(10\ \text{ft})(12\ \frac{\text{in}}{\text{ft}})} = 5.00 \times 10^6\ \frac{\text{lb}}{\text{in}}$$

$$[k]^{(e)} = 5 \times 10^6\ \frac{\text{lb}}{\text{in}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k]^{(1)} = [k]^{(2)} = [k]^{(3)} = 5 \times 10^6\ \frac{\text{lb}}{\text{in}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Loading the element stiffness matrices in the the Global Stiffness Matrix, we get

$$[K]^{(G)} = 5 \times 10^6\ \frac{\text{lb}}{\text{in}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The Global Force Matrix is

$$\{F\}^{(G)} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10,000\ \text{lb} \\ -20,000\ \text{lb} \\ -30,000\ \text{lb} \end{Bmatrix}$$

Since the displacement at node is $U_1 = 0$, the Matrix Equation ($[K]^{(G)}\{U\} = \{F\}^{(G)}$) becomes

$$5 \times 10^6\ \frac{\text{lb}}{\text{in}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10,000\ \text{lb} \\ -20,000\ \text{lb} \\ -30,000\ \text{lb} \end{Bmatrix}$$

Solving with Matlab:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0140\ \text{in} \\ -0.0240\ \text{in} \\ -0.0300\ \text{in} \end{Bmatrix}$$

4.3 Post-processing

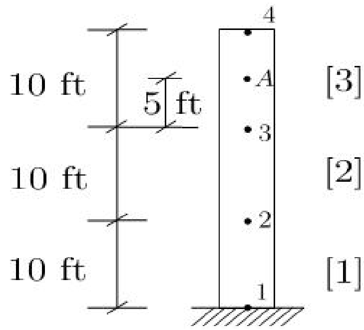
Axial Stresses ($\sigma = E\varepsilon$)

$$\sigma^{(1)} = \frac{E(u_j - u_i)}{\ell} = \frac{30 \times 10^6\ \frac{\text{lb}}{\text{in}^2}(-0.0140\ \text{in} - 0)}{120\ \text{in}} = -3500\ \text{psi C}$$

$$\sigma^{(2)} = \frac{E(u_j - u_i)}{\ell} = \frac{30 \times 10^6\ \frac{\text{lb}}{\text{in}^2}(-0.0240\ \text{in} - -0.0140\ \text{in})}{120\ \text{in}} = -2500\ \text{psi C}$$

$$\sigma^{(3)} = \frac{E(u_j - u_i)}{\ell} = \frac{30 \times 10^6\ \frac{\text{lb}}{\text{in}^2}(-0.0300\ \text{in} - -0.0240\ \text{in})}{120\ \text{in}} = -1500\ \text{psi C}$$

5 Example, Part 2



- Same Column
- Same Loads
- Same Elements and Nodes

$$\{U\} = \begin{Bmatrix} 0 \\ -0.0140 \text{ in} \\ -0.0240 \text{ in} \\ -0.0300 \text{ in} \end{Bmatrix} \quad \text{From previous problem}$$

5.1 *Find:*

Deflection at A

5.2 *Solution:*

Deflection within element using shape functions:

$$u^{(e)} = S_i u_i + S_j u_j$$

$$u^{(e)} = \left(\frac{Y_j - Y}{\ell}\right) u_i + \left(\frac{Y - Y_i}{\ell}\right) u_j$$

$$u^{(3)} = \left(\frac{Y_4 - Y}{\ell}\right) u_3 + \left(\frac{Y - Y_3}{\ell}\right) u_4$$

$$u^{(3)} = \left(\frac{360 \text{ in} - Y}{120 \text{ in}}\right)(-0.0240 \text{ in}) + \left(\frac{Y - 240 \text{ in}}{120 \text{ in}}\right)(-0.0300 \text{ in})$$

$$\text{At A, } u^{(3)} = \left(\frac{360 \text{ in} - 300 \text{ in}}{120 \text{ in}}\right)(-0.0240 \text{ in}) + \left(\frac{300 \text{ in} - 240 \text{ in}}{120 \text{ in}}\right)(-0.0300 \text{ in})$$

$$\therefore u^{(3)} = -0.0270 \text{ in at A}$$