Review of Linear Algebra & Introduction to Mathcad

1 Basic Definitions

Matrix size is determined from the number of rows and columns: m Rows $\times$ n Columns

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$[\mathbf{A}]$ is a m x n matrix

Terms

- Matrix: capital letters in brackets, Ex: $[\mathbf{A}]$
- Elements within matrix: lower case letters, Ex: a_{12}
- Column Matrix: in { }, Ex:
$$\{\mathbf{A}\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix}$$
- Row Matrix: in [ ], Ex:
$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$
- Diagonal Matrix: elements only along the principal diagonal, all others zero ($a_{ij} = 0$ when $i \neq j$), Ex.:
$$[\mathbf{A}] = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$
- Identity or Unit Matrix: elements equal 1 on the principal diagonal, all others equal zero, Ex.:
$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2 Matrix Addition and Subtraction

Each matrix must have the same number of rows and columns. Ex.: $[\mathbf{A}]_{m\times n}$ and $[\mathbf{B}]_{m\times n}$
$[\mathbf{A}]$ and $[\mathbf{B}]$ must both have dimensions $m \times n$

$$[\mathbf{A}] + [\mathbf{B}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$[\mathbf{A}] + [\mathbf{B}] = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$$

$$[\mathbf{A}] - [\mathbf{B}] = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \end{bmatrix}$$

For: $[\mathbf{A}] + [\mathbf{B}] = [\mathbf{C}]$
$c_{ij} = a_{ij} + b_{ij}$ where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$

3 Matrix Multiplication

3.1 Scalar

To multiply a matrix by a scalar, each element is multiplied by the scalar.

$$b[\mathbf{A}] = b \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} ba_{11} & ba_{12} & ba_{13} \\ ba_{21} & ba_{22} & ba_{23} \\ ba_{31} & ba_{32} & ba_{33} \end{bmatrix}$$

3.2 Multiplying Two Matrices

If: $[\mathbf{A}]$ $[\mathbf{B}]$
Then: $[\mathbf{A}] \equiv$ Pre-multiplier Matrix, $[\mathbf{B}] \equiv$ Post-multiplier Matrix

The number of columns in the pre-multiplier *MUST equal* the number of rows in the post-multiplier matrix!

$$[\mathbf{A}]_{m\times n}[\mathbf{B}]_{n\times p} = [\mathbf{C}]_{m\times p}$$

If:

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix}$$

$$[\mathbf{B}] = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$$

$$[\mathbf{A}]_{5\times 3} \quad \text{and} \quad [\mathbf{B}]_{3\times 4}$$

Then: $[\mathbf{A}][\mathbf{B}] = [\mathbf{C}]_{5\times 4}$, where the solution for each element in [C] is:

$$c_{mp} = \sum_{k=1}^{n} a_{mk}b_{kp}$$

3.3 Matrix Multipication: Example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \\ c_{51} & c_{52} & c_{53} & c_{54} \end{bmatrix}$$

For example, c_{23} equals Row 2 of $[\mathbf{A}]$ times Column 3 of $[\mathbf{B}]$

$$c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$$

For example, c_{54} equals Row 5 of $[\mathbf{A}]$ times Column 4 of $[\mathbf{B}]$

$$c_{54} = a_{51}b_{14} + a_{52}b_{24} + a_{53}b_{34}$$

For example, c_{22} equals Row 2 of $[\mathbf{A}]$ times Column 2 of $[\mathbf{B}]$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$$

3.4 Multiplying More than One Matrix

Commutative: No!

$$[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}]$$

Associative: Yes!

$$[\mathbf{A}]([\mathbf{B}][\mathbf{C}]) = ([\mathbf{A}][\mathbf{B}])[\mathbf{C}]$$

Distributive: Yes!

$$([\mathbf{A}] + [\mathbf{B}])[\mathbf{C}] = [\mathbf{A}][\mathbf{C}] + [\mathbf{B}][\mathbf{C}]$$

$$[\mathbf{A}]([\mathbf{B}] + [\mathbf{C}]) = [\mathbf{A}][\mathbf{B}] + [\mathbf{A}][\mathbf{C}]$$

For a Square Matrix:

$$[\mathbf{A}]^n = \overbrace{[\mathbf{A}][\mathbf{A}]\ldots[\mathbf{A}]}^{n\ \text{times}}$$

$$[\mathbf{I}][\mathbf{A}] = [\mathbf{A}][\mathbf{I}] = [\mathbf{A}], \quad \text{where } [\mathbf{I}] = \text{Identity matrix}$$

4 Partitioning a Matrix

Partitioned matrices require less computer memory to perform operations:

$$[\mathbf{A}] = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where

$$[\mathbf{A}_{11}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad [\mathbf{A}_{12}] = \begin{bmatrix} a_{14} & a_{15} & a_{16} \\ a_{24} & a_{25} & a_{26} \end{bmatrix}$$

$$[\mathbf{A}_{21}] = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \qquad [\mathbf{A}_{22}] = \begin{bmatrix} a_{34} & a_{35} & a_{36} \\ a_{44} & a_{45} & a_{46} \\ a_{54} & a_{55} & a_{56} \end{bmatrix}$$

4.1 Add & Subtract

If: $[\mathbf{B}]$ is 5x6 and is partitioned the same as $[\mathbf{A}]$

Then:

$$[\mathbf{A}] + [\mathbf{B}] = \begin{bmatrix} \mathbf{A}_{11} + \mathbf{B}_{11} & \mathbf{A}_{12} + \mathbf{B}_{12} \\ \mathbf{A}_{21} + \mathbf{B}_{21} & \mathbf{A}_{22} + \mathbf{B}_{22} \end{bmatrix}$$

4.2 Multiplying

Matrix multiplication can be performed *only* when the *number of Columns* in the Pre-multiplier matrix is *equal* to the *number of Rows* in the Post-multiplier matrix.

$$[\mathbf{C}] = \left[\begin{array}{cc|c} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ \hline c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63} \end{array}\right] = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$$

$$[\mathbf{C}_{11}] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} \qquad \{\mathbf{C}_{12}\} = \begin{Bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{Bmatrix}$$

$$[\mathbf{C}_{21}] = \begin{bmatrix} c_{41} & c_{42} \\ c_{51} & c_{52} \\ c_{61} & c_{62} \end{bmatrix} \qquad \{\mathbf{C}_{22}\} = \begin{Bmatrix} c_{43} \\ c_{53} \\ c_{63} \end{Bmatrix}$$

Multiplying [**C**] by [**A**] from above:

$$[\mathbf{D}] = [\mathbf{A}][\mathbf{C}] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$$

$$[\mathbf{D}] = [\mathbf{A}][\mathbf{C}] = \begin{bmatrix} \mathbf{A}_{11}\mathbf{C}_{11} + \mathbf{A}_{12}\mathbf{C}_{21} & \mathbf{A}_{11}\mathbf{C}_{12} + \mathbf{A}_{12}\mathbf{C}_{22} \\ \mathbf{A}_{21}\mathbf{C}_{11} + \mathbf{A}_{22}\mathbf{C}_{21} & \mathbf{A}_{21}\mathbf{C}_{12} + \mathbf{A}_{22}\mathbf{C}_{22} \end{bmatrix}$$

$$[\mathbf{D}] = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix}$$

Therefore, we get...

$$[\mathbf{D}_{11}] = [\mathbf{A}_{11}][\mathbf{C}_{11}] + [\mathbf{A}_{12}][\mathbf{C}_{21}] \qquad [\mathbf{D}_{12}] = [\mathbf{A}_{11}]\{\mathbf{C}_{12}\} + [\mathbf{A}_{12}]\{\mathbf{C}_{22}\}$$

$$[\mathbf{D}_{21}] = [\mathbf{A}_{21}][\mathbf{C}_{11}] + [\mathbf{A}_{22}][\mathbf{C}_{21}] \qquad [\mathbf{D}_{22}] = [\mathbf{A}_{21}]\{\mathbf{C}_{12}\} + [\mathbf{A}_{22}]\{\mathbf{C}_{22}\}$$

5 Transposing Matrices

Finite Element formulations frequently can be solved more efficiently through transposing matrices. Transposing a matrix creates a new matrix with Columns created from Rows of the original matrix.
Ex.:

$$\underbrace{[\mathbf{K}]^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}}_{\text{Element Stiffness Matrix}}$$

$$\underbrace{[\mathbf{K}]^{(1G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Its position in the Global Stiffness Matrix}}$$

5.1 Transpose Example

If

$$[\mathbf{A}_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

And

$$[\mathbf{A}_1]^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$[\mathbf{K}]^{(1G)} = [\mathbf{A}_1]^T [\mathbf{K}]^{(1)} [\mathbf{A}_1]$$

$$[\mathbf{K}]^{(1G)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $[\mathbf{A}_1]^T$ is the Transpose of $[\mathbf{A}_1]$

5.2 Transpose Example 2

This can be repeated for each Element Matrix

$$[\mathbf{K}]^{(2G)} = [\mathbf{A}_2]^T [\mathbf{K}]^{(2)} [\mathbf{A}_2]$$

Where

$$[\mathbf{A}_2] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$[\mathbf{A}_2]^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}]^{(2G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5.3 Transpose Summary

In general, to obtain the Transpose of a matrix $[\mathbf{X}]$ of size $m \times n$, the first row of $[\mathbf{X}]$ becomes the first column of $[\mathbf{X}]^T$. The second row of $[\mathbf{X}]$ becomes the second column of $[\mathbf{X}]^T$. The third row of $[\mathbf{X}]$ becomes the third column of $[\mathbf{X}]^T$ and so on. $[\mathbf{X}]^T$ will be of size $n \times m$.

Transpose Operations:

$$([A] + [B] + \ldots + [X])^T = [A]^T + [B]^T + \ldots + [X]^T$$

$$([A][B] \ldots [X])^T = [X]^T \ldots [B]^T [A]^T$$

6 Symmetric Matrix

A symmetric matrix must be a Square Matrix where the Elements are $a_{mn} = a_{nm}$.
Therefore, $[\mathbf{A}] = [\mathbf{A}]^T$.

Ex.:

$$\begin{bmatrix} 2 & 3 & 6 & 10 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 6 & 4 & 9 & 3 & 6 \\ 10 & 2 & 3 & 10 & 2 \\ 3 & 1 & 6 & 2 & 5 \end{bmatrix}$$

7 Determinant of a Matrix

Determinants are *only* defined for a Square Matrices and will result in a *single* number. The Determinant of a matrix is used for:

1. Solving a set of Simultaneous Equations
2. Determining the Inverse of a Matrix
3. Forming the Characteristic equations for a dynamic problem (Eigenvalue Problem)

Notation:

$$\mathbf{Det}[\mathbf{A}], \ \mathbf{det}[\mathbf{A}], \ |\mathbf{A}|, \ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

7.1 Definition

For ...

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

... the determinant is:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

7.2 Solution Procedures: Minor Lower Elements Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Where:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

7.3 Solution Procedures: Direct Expansion Method

(Note this cannot be used for Higher-Order determinants!!)

2 x 2 case: *Add* Solid Arrows, *Subtract* Dashed Arrows

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}b_{21}$$

3 x 3 case:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

7.4 Example of Solution: Option 1

Method 1 is more versatile, Ex.:

$$\mathbf{Det}[\mathbf{C}] = \begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix}$$

$$\begin{matrix} \boxed{c_{11}} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{matrix} \qquad \begin{matrix} c_{11} & \boxed{c_{12}} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{matrix} \qquad \begin{matrix} c_{11} & c_{12} & \boxed{c_{13}} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{matrix} \qquad \begin{matrix} c_{11} & c_{12} & c_{13} & \boxed{c_{14}} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{matrix}$$

$$\mathbf{Det}[\mathbf{C}] = c_{11} \begin{vmatrix} c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{vmatrix} - c_{12} \begin{vmatrix} c_{21} & c_{23} & c_{24} \\ c_{31} & c_{33} & c_{34} \\ c_{41} & c_{43} & c_{44} \end{vmatrix} + c_{13} \begin{vmatrix} c_{21} & c_{22} & c_{24} \\ c_{31} & c_{32} & c_{34} \\ c_{41} & c_{42} & c_{44} \end{vmatrix} - c_{14} \begin{vmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{vmatrix}$$

Minors of Minors

$$\begin{vmatrix} c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{vmatrix}$$

$$\begin{matrix} \boxed{c_{22}} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{matrix} \qquad \begin{matrix} c_{22} & \boxed{c_{23}} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{matrix} \qquad \begin{matrix} c_{22} & c_{23} & \boxed{c_{24}} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{matrix}$$

$$\begin{vmatrix} c_{22} & c_{23} & c_{24} \\ c_{32} & c_{33} & c_{34} \\ c_{42} & c_{43} & c_{44} \end{vmatrix} = c_{22} \begin{vmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{vmatrix} - c_{23} \begin{vmatrix} c_{32} & c_{34} \\ c_{42} & c_{44} \end{vmatrix} + c_{24} \begin{vmatrix} c_{32} & c_{33} \\ c_{42} & c_{43} \end{vmatrix}$$

8 Solution of Simultaneous Equations: Cramer's Rule

General form of linear algebraic equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

In matrix form:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Final solution:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

OR we can write it as...

$$\begin{matrix} \{\text{B}\} \text{ in column 1} \\ \downarrow \\ x_1 = \frac{\textbf{Det}[\mathbf{A}]}{\textbf{Det}[\mathbf{A}]} \end{matrix} \qquad \begin{matrix} \{\text{B}\} \text{ in column 2} \\ \downarrow \\ x_2 = \frac{\textbf{Det}[\mathbf{A}]}{\textbf{Det}[\mathbf{A}]} \end{matrix} \qquad \begin{matrix} \{\text{B}\} \text{ in column 3} \\ \downarrow \\ x_3 = \frac{\textbf{Det}[\mathbf{A}]}{\textbf{Det}[\mathbf{A}]} \end{matrix}$$

9 Gauss Elimination Method

Ex.: Linear Algebraic Equations:

$$\begin{matrix} & 2x_2 & & +x_4 & = 0 \\ 2x_1 & +2x_2 & +3x_3 & +2x_4 & = -2 \\ 4x_1 & -3x_2 & & +x_4 & = -7 \\ 6x_1 & +x_2 & -6x_3 & -5x_4 & = 6 \end{matrix}$$

Matrix form:

$$[\mathbf{A}]\{\mathbf{X}\} = \{\mathbf{B}\}$$

$$\begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 4 & -3 & 0 & 1 \\ 6 & 1 & -6 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2 \\ -7 \\ 6 \end{Bmatrix}$$

Augmented Coefficient Matrix: ([A] & {B} Combined)

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

9.1 Swap Row 1 with the row containing the largest value in Column 1 (i.e. Row 4)

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

9.2 Make all Elements in first Column (except Row 1) = 0

Start by dividing Row 1 by Column 1 term (i.e. 6):

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Multiply Row 1 by Column 1 term in Row 2 (i.e. 2) and subtract it from Row 2:

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ (2-2) & (2-{}^{2}/_{6}) & (3--2) & (2-{}^{-10}/_{6}) & (-2-2) \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Multiply Row 1 by Column 1 term in Row 3 (i.e. 4) and subtract it from Row 3

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & {}^{5}/_{3} & 5 & {}^{11}/_{3} & -4 \\ (4-4) & (-3-{}^{4}/_{6}) & (0--4) & (1-{}^{-20}/_{6}) & (-7-4) \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Multiply Row 1 by Column 1 term in Row 4 (i.e. 0) and subtract it from Row 4

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & {}^{5}/_{3} & 5 & {}^{11}/_{3} & -4 \\ 0 & {}^{-22}/_{6} & 4 & {}^{13}/_{3} & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

9.3 Swap Row 2 with the row containing the largest value in Column 2, Except Row 1 (i.e. Row 3)

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & {}^{-22}/_{6} & 4 & {}^{13}/_{3} & -11 \\ 0 & {}^{5}/_{3} & 5 & {}^{11}/_{3} & -4 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

9.4 Make all elements in Column 2 (Except Rows 1 & 2) = 0

Start by dividing Row 2 by Column 2 term (i.e. -22/6)

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} & 3 \\ 0 & {}^{5}/_{3} & 5 & {}^{11}/_{3} & -4 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Multiply Row 2 by Column 2 term in Row 3 (i.e. 5/3) and subtract it from Row 3

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} & 3 \\ (0) & ({}^{5}/_{3} - {}^{5}/_{3}) & (5 - {}^{-60}/_{33}) & ({}^{11}/_{3} - {}^{-65}/_{33}) & (-4 - {}^{15}/_{3}) \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

Multiply Row 2 by Column 2 term in Row 4 (i.e. 2) and subtract it from Row 4

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} & 3 \\ 0 & 0 & {}^{225}/_{33} & {}^{186}/_{33} & -9 \\ 0 & 0 & {}^{24}/_{11} & {}^{37}/_{11} & -6 \end{bmatrix}$$

9.5 Make elements in Column 3 (except Rows 1, 2, & 3) = 0

Start by dividing Row 3 by Column 3 term (i.e. 225/33)

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} & 3 \\ 0 & 0 & 1 & {}^{186}/_{225} & {}^{-297}/_{225} \\ 0 & 0 & {}^{24}/_{11} & {}^{37}/_{11} & -6 \end{bmatrix}$$

Multiply Row 3 by Column 3 term in Row 4 (i.e. 24/11) and subtract it from Row 4

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} & 3 \\ 0 & 0 & 1 & {}^{186}/_{225} & {}^{-297}/_{225} \\ 0 & 0 & ({}^{24}/_{11}) - ({}^{24}/_{11}) & {}^{37}/_{11} - ({}^{186}/_{225})({}^{24}/_{11}) & -6 - ({}^{-297}/_{225})({}^{24}/_{11}) \end{bmatrix}$$

9.6 Resolution

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} & 3 \\ 0 & 0 & 1 & {}^{186}/_{225} & {}^{-297}/_{225} \\ 0 & 0 & 0 & {}^{3861}/_{2475} & {}^{-7722}/_{2475} \end{bmatrix}$$

Divide Row 4 by Column 4 term (i.e. 3861/2475)

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{-5}/_{6} & 1 \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} & 3 \\ 0 & 0 & 1 & {}^{186}/_{225} & {}^{-297}/_{225} \\ 0 & 0 & 0 & 1 & {}^{-7722}/_{3861} \end{bmatrix}$$

Returning to Original Matrix form

$$\begin{bmatrix} 1 & {}^{1}/_{6} & -1 & {}^{5}/_{6} \\ 0 & 1 & {}^{-12}/_{11} & {}^{-13}/_{11} \\ 0 & 0 & 1 & {}^{186}/_{225} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ {}^{-297}/_{225} \\ {}^{-7722}/_{3861} \end{Bmatrix}$$

Therefore:

$$x_4 = {}^{-7722}/_{3861} \text{ or } x_4 = -2.00$$

Back substituting one-by-one and solving:

$$x_3 = 0.333$$
$$x_2 = 1.00$$
$$x_1 = -0.500$$

10 Using MATLAB (pp. 102-105)

TABLE 2.1 MATLAB's basic scalar operators

Operation	Symbol	Example: $x = 5$ and $y = 3$	Result
Addition	+	$x + y$	8
Subtraction	−	$x - y$	2
Multiplication	*	$x * y$	15
Division	/	$(x + y)/2$	4
Raised to a power	^	x^2	25

Enter Matrices as follows:

$$[\mathbf{A}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

>> A = [1 2 3;4 5 6;7 8 9]

TABLE 2.2 Examples of MATLAB's matrix operations

Operation	Symbols or Commands	Example: *A* and *B* are matrices that you have defined
Addition	+	*A* + *B*
Subtraction	−	*A* − *B*
Multiplication	*	*A* * *B*
Transpose	*matrix name'*	*A'*
Inverse	inv(*matrix name*)	inv(*A*)
Determinant	det(*matrix name*)	det(*A*)
eigenvalues	eig(*matrix name*)	eig(*A*)
Matrix left division (uses Gauss elimination to solve a set of linear equations)	\	see Example 2.8

EXAMPLE 2.1 Revisited

Given the following matrices:

$$[A] = \begin{bmatrix} 0 & 5 & 0 \\ 8 & 3 & 7 \\ 9 & -2 & 9 \end{bmatrix}, [B] = \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 3 \\ 1 & 3 & -4 \end{bmatrix}, \text{ and } \{C\} = \begin{Bmatrix} -1 \\ 2 \\ 5 \end{Bmatrix}$$

using MATLAB, perform the following operations:

a. $[A] + [B] = ?$, **b.** $[A] - [B] = ?$, **c.** $3[A] = ?$, **d.** $[A][B] = ?$, **e.** $[A]\{C\} = ?$
f. $[A]^2 = ?$

Also compute $[A]^T$ and the determinant of $[A]$.

```
To get started, select "MATLAB Help" from the Help menu.
>> A = [0 5 0;8 3 7;9 -2 9]
A =

  0   5   0
  8   3   7
  9  -2   9

>> B = [4 6 -2;7 2 3;1 3 -4]
B =

  4   6  -2
  7   2   3
  1   3  -4

>> C = [-1;2;5]
C =

 -1
  2
  5
```

```
≫ A + B
ans =
   4   11   -2
  15    5   10
  10    1    5
≫ A-B
ans =
-4   -1   °2
1    1   °4
8    5   13
≫ 3*A
ans =
   0   15   °0
  24   °9   21
  27   °6   27
≫ A*B
ans =
  35   10    15
  60   75   -35
  31   77   -60
≫ A*C
ans =
  10
  33
  32
≫ A^2
ans =
  40   15   35
  87   35   84
  65   21   67
≫ A'
ans =
  0   8    9
  5   3   -2
  0   7    9
≫ det(A)
ans =
−45
≫
```

EXAMPLE 2.8

Solve the following set of equations using the Gauss elimination and by inverting the $[A]$ matrix and multiplying it by the $\{b\}$ matrix.

$$2x_1 + x_2 + x_3 = 13$$
$$3x_1 + 2x_2 + 4x_3 = 32$$
$$5x_1 - x_2 + 3x_3 = 17$$
$$[A] = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 4 \\ 5 & -1 & 3 \end{bmatrix} \text{ and } \{b\} = \begin{Bmatrix} 13 \\ 32 \\ 17 \end{Bmatrix}$$

We first use the MATLAB matrix left division operator \ to solve this problem. The \ operator solves the problem using Gauss elimination. We then solve the problem using the **inv** command.

```
        To get started, select "MATLAB Help" from the Help menu.
>>A=[2 1 1;3 2 4;5 -1 3]
A =
  2  1  1
  3  2  4
  5 -1  3
>>b=[13;32;17]
b =
  13
  32
  17
>>x=A\b
x =
  2.0000
  5.0000
  4.0000
>>x=inv(A)*b
x =
  2.0000
  5.0000
  4.0000
```