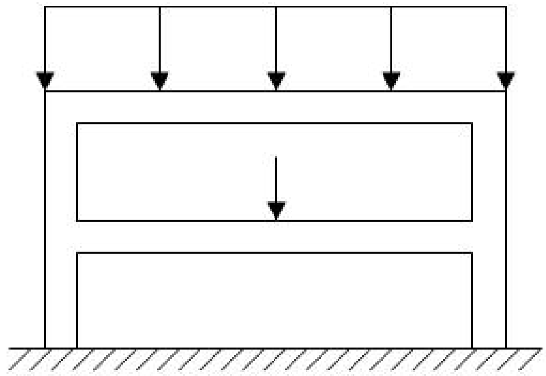
Frames: FE Formulation

1 Finite Element Formulation of Frames

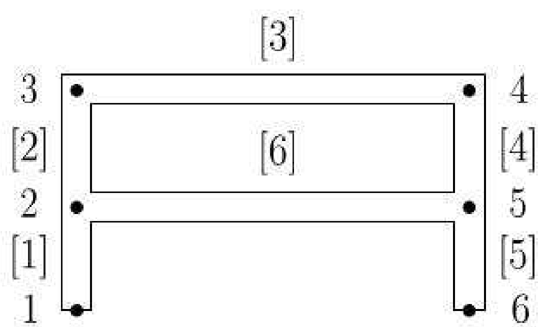
Frames are structural members that have rigidly connected joints (i.e. welded or bolted)

Ex:



1.1 Reduce Frame to several elements

Ex:

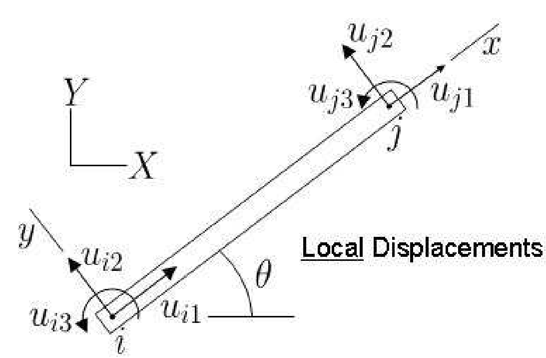


- Notice that individual elements will experience rotation, lateral displacement, and axial deformation
- Frame elements are a combination of axial members and beams

1.2 The Frame Element

Each node has 3 degrees of freedom:
$u_{i1},\ u_{j1} \equiv$ Axial Displacement
$u_{i2},\ u_{j2} \equiv$ Lateral Displacement
$u_{i3},\ u_{j3} \equiv$ Rotation

Notice 2 frames of reference:
$X,\ Y \equiv$ Global Coordinate System
$x,\ y \equiv$ Local Coordinate System



We need a transformation matrix to relate global coordinates to local coordinates (i.e., $\{\mathbf{U}\} = [\mathbf{T}]\{\mathbf{u}\}$)

$$[\mathbf{T}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that some books use a transformation matrix that is the inverse (or transpose) of the truss transformation matrix. The reason is because they are using the following relationship: $\{\mathbf{u}\} = [\mathbf{T}]\{\mathbf{U}\}$

The **Local** stiffness matrix for **Bending** is:

$$[\mathbf{k}]^{(e)}_{\text{bending}} = \frac{EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{array}{c} u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{j1} \\ u_{j2} \\ u_{j3} \end{array}$$

The **Local** stiffness matrix for **Axial** Loading is:

$$[\mathbf{k}]^{(e)}_{\text{axial}} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{j1} \\ u_{j2} \\ u_{j3} \end{array}$$

The **Local** stiffness matrix for a **Frame Element** is the addition of the bending and axial loading stiffness matrices.

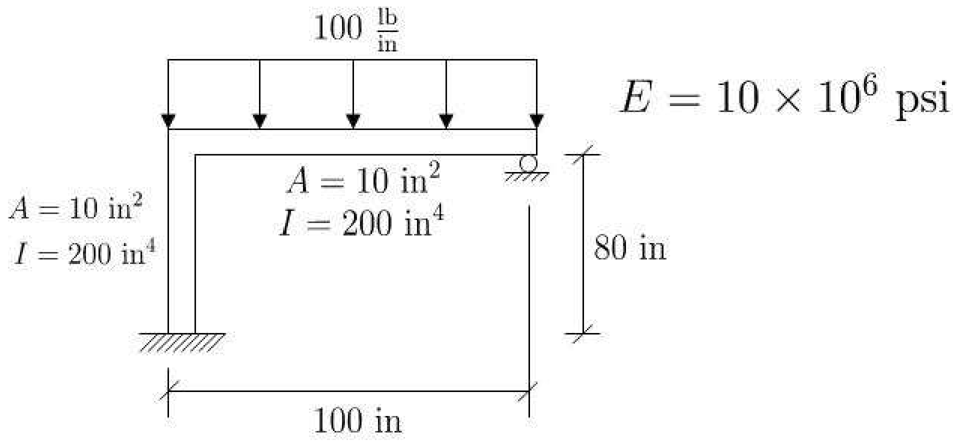
$$[\mathbf{k}]^{(e)}_{xy} = [\mathbf{k}]^{(e)}_{\text{bending}} + [\mathbf{k}]^{(e)}_{\text{axial}}$$

$$[\mathbf{k}]^{(e)}_{xy} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{array}{c} u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{j1} \\ u_{j2} \\ u_{j3} \end{array}$$

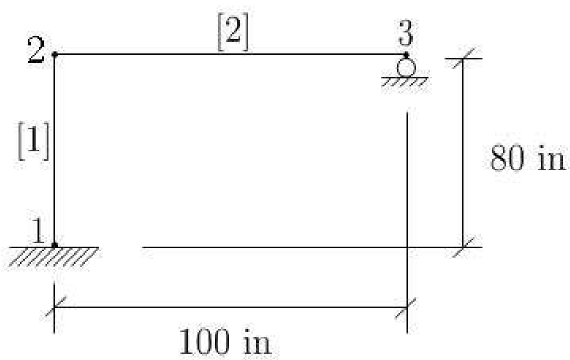
The **Global** stiffness matrix for a **Frame Element** is developed using the transformation matrix and its transpose.

$$[\mathbf{K}]^{(e)} = [\mathbf{T}][\mathbf{k}]^{(e)}_{xy}[\mathbf{T}]^{-1}$$

2 Example



Find: Displacements
Solution
Using two elements:



2.0.1 Form frame element stiffness matrices in Local CS

$$[\mathbf{k}]_{xy}^{(e)} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Element 1

$$[\mathbf{k}]_{xy}^{(1)} = \begin{bmatrix} 1,250,000 & 0 & 0 & -1,250,000 & 0 & 0 \\ 0 & 46,875 & 1,875,000 & 0 & -46,875 & 1,875,000 \\ 0 & 1,875,000 & 100,000,000 & 0 & -1,875,000 & 50,000,000 \\ -1,250,000 & 0 & 0 & 1,250,000 & 0 & 0 \\ 0 & -46,875 & -1,875,000 & 0 & 46,875 & -1,875,000 \\ 0 & 1,875,000 & 50,000,000 & 0 & -1,875,000 & 100,000,000 \end{bmatrix}$$

Element 2

$$[\mathbf{k}]_{xy}^{(2)} = \begin{bmatrix} 1,000,000 & 0 & 0 & -1,000,000 & 0 & 0 \\ 0 & 24,000 & 1,200,000 & 0 & -24,000 & 1,200,000 \\ 0 & 1,200,000 & 80,000,000 & 0 & -1,200,000 & 40,000,000 \\ -1,000,000 & 0 & 0 & 1,000,000 & 0 & 0 \\ 0 & -24,000 & -1,200,000 & 0 & 24,000 & -1,200,000 \\ 0 & 1,200,000 & 40,000,000 & 0 & -1,200,000 & 80,000,000 \end{bmatrix}$$

2.0.2 Develop Transformation Matrices

Element 1 $(\theta = 90°)$

$$[\mathbf{T}]^{(1)} = \begin{bmatrix} \cos 90° & -\sin 90° & 0 & 0 & 0 & 0 \\ \sin 90° & \cos 90° & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 90° & -\sin 90° & 0 \\ 0 & 0 & 0 & \sin 90° & \cos 90° & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Element 2 $(\theta = 0°)$

$$[\mathbf{T}]^{(2)} = \begin{bmatrix} \cos 0° & -\sin 0° & 0 & 0 & 0 & 0 \\ \sin 0° & \cos 0° & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 0° & -\sin 0° & 0 \\ 0 & 0 & 0 & \sin 0° & \cos 0° & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$[\mathbf{T}]^{(2)}$ is the identity matrix because global CS and Local CS are the same for Element 2.

2.0.3 Determine Global Element Stiffness Matrices

Element 1:

$$[\mathbf{K}]^{(1)} = [\mathbf{T}]^{(1)}[\mathbf{k}]^{(1)}_{xy}[\mathbf{T}]^{(1)-1}$$

$$[\mathbf{K}]^{(1)} = \begin{bmatrix} 46,875 & 0 & -1,875,000 & -46,875 & 0 & -1,875,000 \\ 0 & 1,250,000 & 0 & 0 & -1,250,000 & 0 \\ -1,875,000 & 0 & 100,000,000 & 1,875,000 & 0 & 50,000,000 \\ -46,875 & 0 & 1,875,000 & 46,875 & 0 & 1,875,000 \\ 0 & -1,250,000 & 0 & 0 & 1,250,000 & 0 \\ -1,875,000 & 0 & 50,000,000 & 1,875,000 & 0 & 100,000,000 \end{bmatrix}$$

Element 2:

$$[\mathbf{K}]^{(2)} = [\mathbf{k}]^{(2)}_{xy}$$

2.0.4 Assemble Total Stiffness Matrix

In order to get the Global Element Stiffness Matrices (6x6) into the Total Stiffness Matrix (9x9), we need to apply the transpose matrix (9x6) for each element.

Element 1:

$$[\mathit{Trans1}] = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} U_{11} \\ U_{12} \\ U_{13} \\ U_{21} \\ U_{22} \\ U_{23} \\ U_{31} \\ U_{32} \\ U_{33} \end{array}$$

$$[\mathbf{K}]^{(1G)} = [\mathit{Trans1}][\mathbf{K}]^{(1)}[\mathit{Trans1}]^T$$

Element 2:

$$[\mathit{Trans2}] = \begin{array}{c} \begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{bmatrix} \end{array} \begin{array}{c} U_{11}\\U_{12}\\U_{13}\\U_{21}\\U_{22}\\U_{23}\\U_{31}\\U_{32}\\U_{33} \end{array}$$

$$[\mathbf{K}]^{(2G)} = [\mathit{Trans2}][\mathbf{K}]^{(2)}[\mathit{Trans2}]^T$$

Combining our Global Element Stiffness Matrices, which are now in Global form, we get:

$$[\mathbf{K}]^{(G)} = [\mathbf{K}]^{(1G)} + [\mathbf{K}]^{(2G)}$$

$$[\mathbf{K}]^{(G)} = 10^6 \begin{bmatrix} .046875 & 0 & -1.875 & -.046875 & 0 & -1.875 & 0 & 0 & 0\\ 0 & 1.25 & 0 & 0 & -1.25 & 0 & 0 & 0 & 0\\ -1.875 & 0 & 100 & 1.875 & 0 & 50 & 0 & 0 & 0\\ -.046875 & 0 & 1.875 & .046875+1 & 0+0 & 1.875+0 & -1 & 0 & 0\\ 0 & -1.25 & 0 & 0+0 & 1.25+.024 & 0+1.2 & 0 & -.024 & 1.2\\ -1.875 & 0 & 50 & 1.875+0 & 0+1.2 & 100+80 & 0 & -1.2 & 40\\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -.024 & -1.2 & 0 & .024 & -1.2\\ 0 & 0 & 0 & 0 & 1.2 & 40 & 0 & -1.2 & 80 \end{bmatrix} \begin{array}{c} U_{11}\\U_{12}\\U_{13}\\U_{21}\\U_{22}\\U_{23}\\U_{31}\\U_{32}\\U_{33} \end{array}$$

2.0.5 Load Matrix

$$\{\mathbf{F}\} = \left\{ \begin{array}{c} 0\\0\\0\\0\\ \frac{-\omega L}{2}\\ \frac{-\omega L^2}{12}\\ 0\\ \frac{-\omega L}{2}\\ \frac{\omega L^2}{12} \end{array} \right\} = \left\{ \begin{array}{c} 0\\0\\0\\0\\ \frac{-(100\ \frac{\text{lb}}{\text{in}})(100\ \text{in})}{2}\\ \frac{-(100\ \frac{\text{lb}}{\text{in}})(100\ \text{in})^2}{12}\\ 0\\ \frac{-(100\ \frac{\text{lb}}{\text{in}})(100\ \text{in})}{2}\\ \frac{(100\ \frac{\text{lb}}{\text{in}})(100\ \text{in})^2}{12} \end{array} \right\}$$

$$\{\mathbf{F}\} = \left\{ \begin{array}{c} 0\\0\\0\\0\\ -5,000\ \text{lb}\\ -83,333\ \text{lb-in}\\ 0\\ -5,000\ \text{lb}\\ 83,333\ \text{lb-in} \end{array} \right\} = \begin{array}{c} U_{11}\\U_{12}\\U_{13}\\U_{21}\\U_{22}\\U_{23}\\U_{31}\\U_{32}\\U_{33} \end{array}$$

2.0.6 Enforce Boundary Conditions

Since we have four boundary conditions we want to enforce,

$$U_{11} = U_{12} = U_{13} = 0, \quad U_{32} = 0$$

...our stiffness matrix (9x9) and load matrix (9x1) need to be cut down to a (5x5) and (5x1), respectively. We can do this by creating a boundary condition transpose matrix (9x5).

$$[\mathbf{BC}] = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_{11} \\ U_{12} \\ U_{13} \\ U_{21} \\ U_{22} \\ U_{23} \\ U_{31} \\ U_{32} \\ U_{33} \end{array}$$

Applying the boundary condition transpose matrix, we get:

$$[\mathbf{K}]^{(G,reduced)} = [\mathbf{BC}]^T[\mathbf{K}]^{(G)}[\mathbf{BC}]$$

$$[\mathbf{F}]^{(reduced)} = [\mathbf{BC}]^T[\mathbf{F}]$$

And combining them,

$$[\mathbf{K}]^{(G,reduced)}\mathbf{U} = [\mathbf{F}]^{(reduced)}$$

$$\begin{bmatrix} 1,046,875 & 0 & -1,875,000 & -1,000,000 & 0 \\ 0 & 1,274,000 & 1,200,000 & 0 & 1,200,000 \\ 1,875,000 & 1,200,000 & 180,000,000 & 0 & 40,000,000 \\ -1,000,000 & 0 & 0 & 1,000,000 & 0 \\ 0 & 1,200,000 & 40,000,000 & 0 & 80,000,000 \end{bmatrix} \begin{Bmatrix} U_{21} \\ U_{22} \\ U_{23} \\ U_{31} \\ U_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5000 \\ -83,333 \\ 0 \\ 83,333 \end{Bmatrix}$$

2.0.7 Solve with Mathcad or MATLAB

$$\begin{Bmatrix} U_{21} \\ U_{22} \\ U_{23} \\ U_{31} \\ U_{33} \end{Bmatrix} = \begin{Bmatrix} 0.0576 \text{ in} \\ -0.0043 \text{ in} \\ -0.0014 \text{ rad} \\ 0.0576 \text{ in} \\ 0.0018 \text{ rad} \end{Bmatrix}$$

3 Using ANSYS with 2-D and 3-D Frames

BEAM188: the ANSYS 3-D beam element with tension, compression, bending about different axes, and twisting (torsion). Each node has 6 DOF (stiffness matrix is 12x12):

- X-translation
- Y-translation
- Z-translation
- X-rotation
- Y-rotation
- Z-rotation

3.1 Input Data Required

1. Node location
2. Cross-sectional area: for $\sigma_P = \frac{P}{A}$
3. Area moment of inertia: for $\sigma_M = -\frac{Mc}{I}$
4. Cross-sectional height (h): for $\sigma_M = -\frac{Mc}{I}$, where $c = \frac{h}{2}$
5. Modulus of elasticity: for converting deflection to stress

Finding Stresses

- Must first copy results to "Element Table"
- Items obtained using Item Label and Sequence Numbers
- Refer to Element Reference Manual in "Help"

4 Verification of Results

ALWAYS VERIFY RESULTS!!!

For example:

1. Pass a cutting through member and find internal load or stress using statics.
2. Find reaction using statics.

5 Examples

Example 4.5

Consider the overhang frame shown in Figure 4-13. The frame is made of steel, with $E = 30 \times 10^6$ lb/in^2. The cross-sectional areas and the second moment of areas for the two W12x26 members are shown in Figure 4-13. The frame is fixed as shown in the figure, and we are interested in determining the deformation of the frame under the given distributed load.

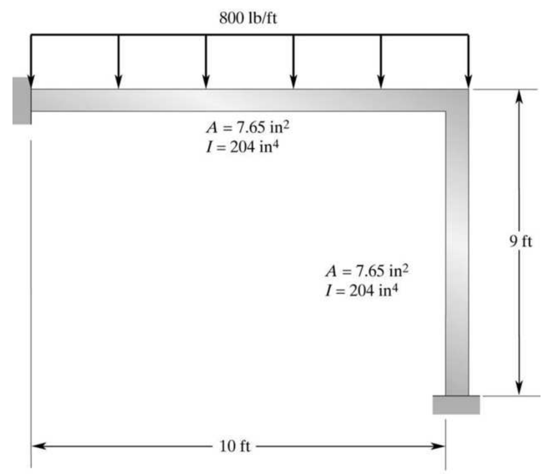


Figure 4-13
An overhang frame supporting a distributed load.