Analysis of 1-D Problems

1 Where are we headed?

We are going to analyze 1-D problems in...

1. Heat transfer problems
2. Fluid mechanics problems
3. Using ANSYS

2 Heat Transfer with 1-D Linear Elements

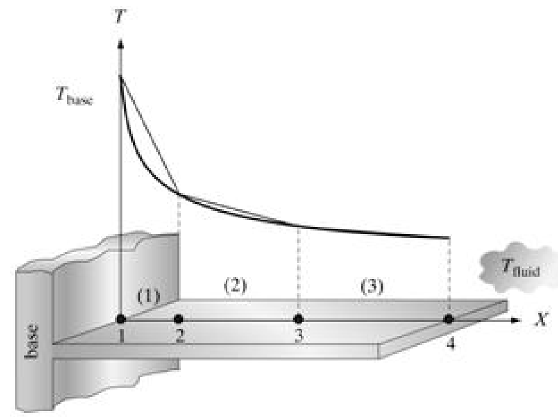
Ex: Heat transfer through a fin surrounded by a fluid

Heat source: base
Heat transfer mode:
1. Conduction in the x-direction through fin
2. Convection to the surrounding fluid through the fin surfaces

The Governing Differential Equation is:

$$kA\frac{d^2T}{dx^2} - hpT + hpT_f = 0$$

where:
$k \equiv$ Thermal Conductivity
$A \equiv$ Cross-sectional area in direction of conduction
$h \equiv$ Convective heat transfer coefficient
$p \equiv$ Fin perimeter
$T_f \equiv$ Temperature of surrounding fluid
$T_b \equiv$ Temperature of base(i.e. source)



We are using linear elements so we need two boundary conditions.

1. $T = T_b$ at $x = 0$
2. A defined tip condition (one of the follwing)
   - Long fin
     $$T = T_f \text{ at } x = L$$
   - Adiabatic tip
     $$-kA\frac{dT}{dx} = 0 \text{ at } x = L$$
   - Convection
     $$-kA\frac{dT}{dx} = hA(T_L - T_f) \text{ at } x = L$$

The governing differential equation represents an energy balance of the fin. We will approximate the solution with linear elements of the form:

$$T^{(e)} = S_iT_i + S_jT_j$$

where:

$$S_i = \frac{X_j - X}{\ell}, \quad S_j = \frac{X - X_i}{\ell}$$

In matrix form:

$$T^{(e)} = \begin{bmatrix} S_i & S_j \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

The general form of our differential equation is:

$$C_1 \frac{d^2T}{dx^2} + C_2 T + C_3 = 0$$

where:

$$C_1 = kA$$

$$C_2 = -hp$$

$$C_3 = hpT_f$$

Let's go one step further and make our differential equation very general and applicable to other problems ... set $T = \psi$. Now we need to solve this thing using ... The Galerkin Formulation!!!

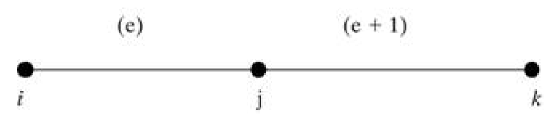
Remember the basic idea with weighted residuals is to assume an approximate solution, substitute into the governing D.E., and solve. The result is residual or error. We then work to set that error or residual equal to zero in different manners.

With Galerkin's formulation, we multiply the residual by a weighting function, average it, and set it equal to zero. The residual equation is:

$$R_i = \int_0^L \Phi_i (\text{D.E.}) dx = 0$$

The weighting function must be of the same form as the approximate solution. Since we are using $\psi^{(e)} = S_i\psi_i + S_j\psi_j$ as our approximate solution, we will use shape functions as the weighting function.

Consider three nodes in a row with elements between

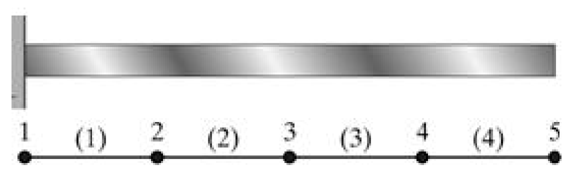


The residual equation for node j then becomes:

$$R_j = R_j^{(e)} + R_j^{(e+1)}$$

$$R_j = \int_{X_i}^{X_j} S_j{}^{(e)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(e)} dX + \int_{X_j}^{X_k} S_j^{(e+1)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(e+1)} dX = 0$$

We will have a residual equation for each node in the finite element model. For example, if:



Then:

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Where:

$$\begin{aligned} R_2 &= R_2^{(1)} + R_2^{(2)} \\ R_3 &= R_3^{(2)} + R_3^{(3)} \\ R_4 &= R_4^{(3)} + R_4^{(4)} \\ &\vdots \end{aligned}$$

Let's examine the Nodal Residual Equations for nodes 1, 2, 3, 4 ... **You can draw in the arrows!**

$$R_1 = \int_{X_1}^{X_2} S_1^{(1)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(1)} dX = 0$$

$R_2 = R_2^{(1)} + R_2^{(2)}$

$$R_2 = \int_{X_1}^{X_2} S_2^{(1)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(1)} dX + \int_{X_2}^{X_3} S_2^{(2)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C3 \right]^{(2)} dX = 0$$

$R_3 = R_3^{(2)} + R_3^{(3)}$

$$R_3 = \int_{X_2}^{X_3} S_3^{(2)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(2)} dX + \int_{X_3}^{X_4} S_3^{(3)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C3 \right]^{(3)} dX = 0$$

$R_4 = R_4^{(3)} + R_4^{(4)}$

$$R_4 = \int_{X_3}^{X_4} S_4^{(3)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(3)} dX + \int_{X_4}^{X_5} S_4^{(4)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C3 \right]^{(4)} dX = 0$$

$$\vdots$$

In general for element (e) with nodes i and j

$$R_i^{(e)} = \int_{X_i}^{X_j} S_i^{(e)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(e)} dX \quad (1)$$

$$R_j^{(e)} = \int_{X_j}^{X_k} S_j^{(e)} \left[C_1 \frac{d^2\psi}{dX^2} + C_2\psi + C_3 \right]^{(e)} dX \quad (2)$$

Now, lets manipulate the 2nd-order term from the governing equation into a 1st-order term. Take the derivative of $(S_i \frac{d\psi}{dX})$ using the chain rule.

$$\frac{d}{dX}\left(S_i \frac{d\psi}{dX} \right) = S_i \frac{d^2\psi}{dX^2} + \frac{dS_i}{dX}\frac{d\psi}{dX}$$

Rearrange algebraically

$$S_i \frac{d^2\psi}{dX^2} = \frac{d}{dX}\left(S_i \frac{d\psi}{dX} \right) - \frac{dS_i}{dX}\frac{d\psi}{dX} \quad (3)$$

Substitute Equation 3 into Equation 1 and solve ... knowing $S_i = (\frac{X_j - X}{\ell})$ and $\psi = (\frac{X_j - X}{\ell}\psi_i + \frac{X - X_i}{\ell}\psi_j)$:

$$R_i^{(e)} = \int_{X_i}^{X_j} C_1 \left[\frac{d}{dX}\left(S_i \frac{d\psi}{dX} \right) \right] dX + \int_{X_i}^{X_j} C_1 \left(\frac{-dS_i}{dX}\frac{d\psi}{dX} \right) dX + \int_{X_i}^{X_j} S_i (C_2\psi) dX + \int_{X_i}^{X_j} S_i C_3 dX$$

$$R_i^{(e)} = \overbrace{C_1 S_i \frac{d\psi}{dX} \mid_{X=X_j} - C_1 S_i \frac{d\psi}{dX} \mid_{X=X_i}} + \overbrace{\int_{X_i}^{X_j} C_1 \left[\frac{-d(\frac{X_j - X}{\ell})}{dX} \frac{d}{dX}\left(\frac{X_j - X}{\ell}\psi_i + \frac{X - X_i}{\ell}\psi_j \right) \right] dX}$$
$$+ \overbrace{\frac{C_2\ell}{3}\psi_i + \frac{C_2\ell}{6}\psi_j} + \overbrace{C_3\frac{\ell}{2}}$$

Knowing $S_i = 1$ at $X = X_i$ and $S_i = 0$ at $X = X_j$

$$R_i^{(e)} = -C_1 \frac{d\psi}{dX} \mid_{X=X_i} - \frac{C_1}{\ell}(\psi_i - \psi_j) + \frac{C_2\ell}{3}\psi_i + \frac{C_2\ell}{6}\psi_j + C_3\frac{\ell}{2}$$

Substituting Equation 3 into Equation 2 and solving gives a similar result:

$$R_j^{(e)} = C_1 \frac{d\psi}{dX} \mid_{X=X_j} - \frac{C_1}{\ell}(-\psi_i + \psi_j) + \frac{C_2 \ell}{6}\psi_i + \frac{C_2 \ell}{3}\psi_j + C_3 \frac{\ell}{2}$$

Wow!! Remember ψ is the temperature for our problem.

Assemble into Matrix form

$$\begin{Bmatrix} R_i \\ R_j \end{Bmatrix} = \begin{Bmatrix} -C_1 \frac{d\psi}{dX} \mid_{X=X_i} \\ C_1 \frac{d\psi}{dX} \mid_{X=X_j} \end{Bmatrix} - \frac{C_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \psi_i \\ \psi_j \end{Bmatrix} + \frac{C_2 \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \psi_i \\ \psi_j \end{Bmatrix} + \frac{C_3 \ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Setting the residual equations to zero gives:

$$\begin{Bmatrix} C_1 \frac{d\psi}{dX} \mid_{X=X_i} \\ -C_1 \frac{d\psi}{dX} \mid_{X=X_j} \end{Bmatrix} + \frac{C_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \psi_i \\ \psi_j \end{Bmatrix} + \frac{-C_2 \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \psi_i \\ \psi_j \end{Bmatrix} = \frac{C_3 \ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

2.1 All Elements

Notice the form!

$$\begin{Bmatrix} C_1 \frac{d\psi}{dX} \mid_{X=X_i} \\ -C_1 \frac{d\psi}{dX} \mid_{X=X_j} \end{Bmatrix} + \left\{ [\mathbf{K}]_{C_1}^{(e)} + [\mathbf{K}]_{C_2}^{(e)} \right\} \begin{Bmatrix} \psi_i \\ \psi_j \end{Bmatrix} = \{\mathbf{F}\}_{C_3}^{(e)}$$

Substituting for the constants, we get

$$\begin{Bmatrix} C_1 \frac{d\psi}{dX} \mid_{X=X_i} \\ -C_1 \frac{d\psi}{dX} \mid_{X=X_j} \end{Bmatrix} = \begin{Bmatrix} kA \frac{dT}{dX} \mid_{X=X_i} \\ -kA \frac{dT}{dX} \mid_{X=X_j} \end{Bmatrix}$$

$$[\mathbf{K}]_{C_1}^{(e)} = \frac{C_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{kA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \leftarrow$$

$$[\mathbf{K}]_{C_2}^{(e)} = \frac{-C_2 \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{hp\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \leftarrow$$

$$\{\mathbf{F}\}_{C_3}^{(e)} = \frac{C_3 \ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{hp\ell T_f}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

We can make an additional substitution to the first term when we assume the 3^{rd} B.C. from page 280,

$$-kA\frac{dT}{dX} = hA(T_L - T_f) \text{ at } X = L$$

Therefore:

$$\begin{Bmatrix} C_1 \frac{d\psi}{dX} \mid_{X=X_i} \\ -C_1 \frac{d\psi}{dX} \mid_{X=X_j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ hA(T_j - T_f) \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} - \begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix}$$

There are two parts to this result: an element "stiffness" contribution and a forcing function contribution.

$$[\mathbf{K}]_{\text{B.C.}}^{(e)} = \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}$$

$$\{\mathbf{F}\}_{\text{B.C.}}^{(e)} = \begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix}$$

Now combine the whole thing!!

$$[\mathbf{K}]^{(e)}\{\mathbf{T}\} = \{\mathbf{F}\}^{(e)}$$

2.2 *Tip Element ONLY*

For only the tip element, $[\mathbf{K}]^{(e)} = [\mathbf{K}]^{(e)}_{C_1} + [\mathbf{K}]^{(e)}_{C_2} + [\mathbf{K}]^{(e)}_{\text{B.C.}}$

$$[\mathbf{K}]^{(e)} = \left\{ \frac{kA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hp\ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix} \right\}$$

and $\{\mathbf{F}\}^{(e)} = \{\mathbf{F}\}^{(e)}_{C_3} + \{\mathbf{F}\}^{(e)}_{\text{B.C.}}$

$$\{\mathbf{F}\}^{(e)} = \frac{hp\ell T_f}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix}$$

Wow!! This should all look familiar! Now assemble global stiffness matrix and load matrix for the whole finite element model. Solve same as we did the mechanics solution: $[\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{F}\}$. See two examples in textbook.

2.3 *Calculating Heat Loss*

Heat Loss (Q, or $\dot{Q}$ from our heat transfer textbook) from fin due to convection on surfaces

$$Q_{\text{total}} = \sum Q^{(e)}$$

$$Q^{(e)} = \int_{X_i}^{X_j} hp\left[(S_i T_i + S_j T_j) - T_f\right] dX$$

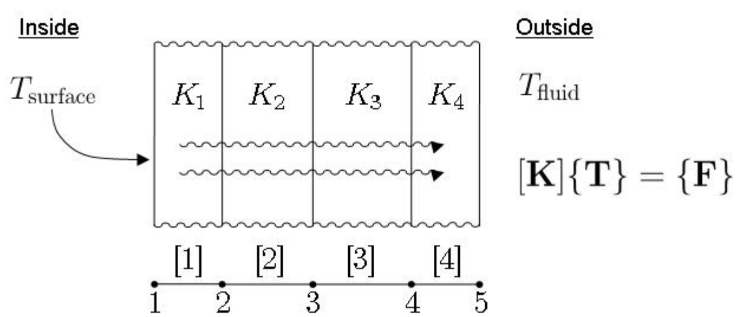
$$Q^{(e)} = \int_{X_i}^{X_j} hp\,(T - T_f)\, dX$$

$$Q^{(e)} = hp\ell\left[\left(\frac{T_i + T_j}{2}\right) - T_f\right]$$

where: $h \equiv$ Convective Heat Transfer Coefficient
$p \equiv$ Fin Perimeter
$\ell \equiv$ Element Length

3 Conductance Problems with Convection Only at End

<u>Ex:</u> Conduction through wall and convection to outside



In this example, convection only occurs at Node 5

Therefore, for all elements except (4):

$$[\mathbf{K}]^{(e)} = \left\{ \frac{kA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \cancelto{0}{\frac{hp\ell}{6}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \cancelto{0}{hA} \end{bmatrix} \right\}$$

$$[\mathbf{K}]^{(1)} = \frac{k_1 A}{\ell_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\mathbf{K}]^{(2)} = \frac{k_2 A}{\ell_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\mathbf{K}]^{(3)} = \frac{k_3 A}{\ell_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\mathbf{K}]^{(4)} = \frac{k_4 A}{\ell_4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}$$

Again, for all elements except (4):

$$\{\mathbf{F}\}^{(e)} = \cancelto{0}{\frac{hp\ell T_f}{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \cancelto{0}{hAT_f} \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(1)} = \{\mathbf{F}\}^{(2)} = \{\mathbf{F}\}^{(3)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(4)} = \begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix}$$

Assemble Global Stiffness Matrix and Global Force Matrix

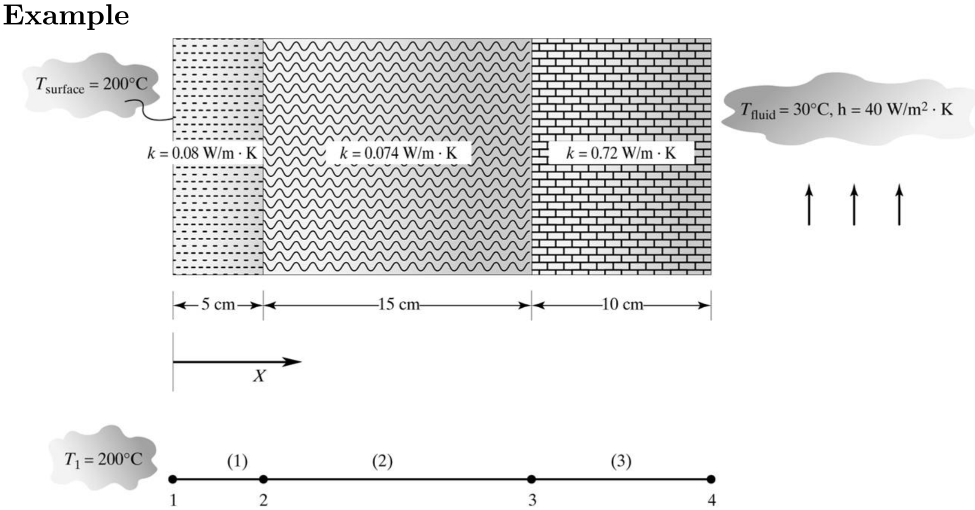
$$\begin{bmatrix} \\ \mathbf{K}^{(G)} \\ \\ \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} T_{\text{surface}} \\ 0 \\ 0 \\ 0 \\ hAT_f \end{Bmatrix}$$

where we have included the B.C.: $T_1 = T_{\text{surface}}$

4 Examples of One-Dimensional Elements in ANSYS

- **LINK31**: Models radiation heat transfer between two points in space.
- **LINK33**: Uniaxial heat conduction element. Models transfer of heat between its two nodes via conduction mode. DOF is temperature. Defined by cross-sectional area and thermal conductivity.
- **LINK34**: Uniaxial heat convection element. Models transfer of heat between its two nodes via convection mode. DOF is temperature. Defined by surface area and convection heat transfer (film) coefficient.

ANSYS Example

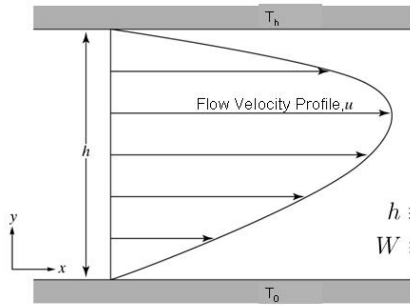


5 Fluid Mechanics with 1-D Linear Elements

Ex: Laminar Flow in a channel

The governing differential equation is:

$$\mu \frac{d^2u}{dy^2} - \frac{dp}{dx} = 0$$



$h \equiv$ height
$W \equiv$ width z-dir.

where:

$u \equiv$ fluid velocity
$\mu \equiv$ dynamic viscosity of the fluid
$\frac{dp}{dx} \equiv$ pressure drop in direction of flow (x-direct)
$T \equiv$ temperature on surfaces and in the fluid

Boundary Conditions (two for linear elements):

$$u = 0 \text{ at } y = 0 \qquad\qquad u = 0 \text{ at } y = h$$

Note that the governing equation is of the same general form as the equation we developed for Heat Transfer:

$$C_1 \frac{d^2\psi}{dx^2} + C_2\psi + C_3 = 0$$

where:

$$\begin{aligned} \psi &= u \\ x &= y \\ C_1 &= \mu \\ C_2 &= 0 \\ C_3 &= \tfrac{-dp}{dx} \end{aligned}$$

From derivation of $[K]^{(e)}$ in Heat Transfer

$$[\mathbf{K}]^{(e)} = [\mathbf{K}]^{(e)}_{C_1} + [\mathbf{K}]^{(e)}_{C_2}$$

where:

$$[\mathbf{K}]^{(e)}_{C_1} = \frac{C_1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{\mu}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\mathbf{K}]^{(e)}_{C_2} = \frac{-C_2\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0$$

Therefore:

$$[\mathbf{K}]^{(e)} = \frac{\mu}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \leftarrow \text{Flow Resistance Matrix}$$

From derivation of $\{\mathbf{F}\}^{(e)}$ in Heat Transfer

$$\{\mathbf{F}\}^{(e)} = \frac{C_3\ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

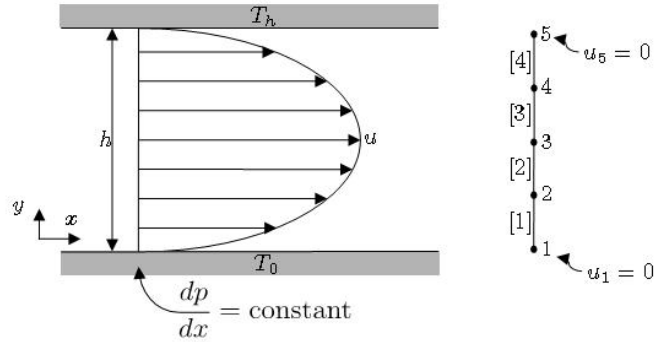
$$\{\mathbf{F}\}^{(e)} = \frac{-\frac{dp}{dx}\ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \leftarrow \text{Flow Forcing Matrix}$$

For fully-developed laminar flow, $\frac{dp}{dx}$ is a constant

As before, assemble global "stiffness" and "force" matrices. Then solve the Global Matrix Equation:

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$$

6 Typical 1-D Fluid Mechanics Finite Element Model



μ is fluid viscosity at each element

6.1 Mass Flow Rate in the Channel

$$\dot{m}_{\text{total}} = \sum \dot{m}^{(e)}$$

$$\dot{m}_{\text{total}} = \dot{m}^{(1)} + \dot{m}^{(2)} + \dot{m}^{(3)} + \ldots$$

$$\dot{m}^{(e)} = \int_{Y_i}^{Y_j} \rho W u dY$$

where:

$\rho \equiv$ Fluid density at each element
$W \equiv$ Channel width (in z-direction)
$u \equiv$ Element flow velocity

$$\dot{m}^{(e)} = \int_{Y_i}^{Y_j} \rho W (S_i u_i + S_j u_j) dY$$

$$\dot{m}^{(e)} = \rho W \ell \left(\frac{u_i + u_j}{2} \right)$$

7 Verify Results

Again, **answers must be verified** to some degree of certainty!! (See book for conservation of energy in each element.)