Ordinary Differential Equations

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Modeling a spring-mass system

This lab describes an activity with a spring-mass system, designed to explore concepts related to modeling a real world system with wide applicability.

The goals of this project are:

- Deepen your understanding of linear second order homogeneous differential equations.
- Study a prototype model application for the harmonic oscillator, via a spring-mass system.
- Practice with data collection, analysis and interpretation.

For this lab you will need:

- A "spring". This can be a real spring, or, as a substitute, you can use a rubber band, or several smaller bands tied together.
- A "mass". The mass could be a roll of coins, a coffee mug (don't use your favorite one), a stapler, a can of food, or any attachable small dense object.
- Measuring devices: a measure tape for measuring distances, and a kitchen scale to measure masses. A timing device, such as a smart phone. A regular watch may be hard to use. You will have to decide which units to use, and adjust all constants accordingly.
- (Optional) A video-recording device, such as a video camera, cell phone or tablet.

Learning outcomes:
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Introduction

The spring-mass system is good model problem, since there are many related application problems for which the same mathematical equations apply.

This activity will instruct you on how to measure and model your own system, connecting the physical behavior of the model system with the mathematical features first introduced in Chapter 5 of Trench's text.

Figure **??** below demonstrates the situation when the mass is hung on a spring of length , stretching it by a distance Any movement of the mass can be measured from this equilibrium position. We make the arbitrary choice to call the upward direction positive, as noted in the figure.

To set up the system, the spring should hang from above by some fixture such as a hook. Place the spring near a wall, and attach the measuring tape behind the spring. If using a recording device, set it up some distance away, in such a way that both the spring-mass system and the measuring tape are visible. The image below illustrates a possible setup.

Model for the spring-mass system

The model for the spring-mass system has the following parameters:

- The mass, m.
- The relaxed length of the spring, L.
- The spring constant, k.
- The acceleration of gravity g.
- The coefficient of friction c.

The dynamic variable in the model is the displacement from the equilibrium position, which we denote by y. We assume that y is positive in the upward direction.

In this first exploration, we assume that the friction is negligible. That is, as a first approximation, we assume that $c = 0$.

Recall that we model the intensity of the force exerted by the spring by Hooke's law:

$$|F_{\text{spring}}| = k \cdot (\text{spring deformation})$$

where the spring deformation is represented by ΔL in Figure **??**.

Using this information, answer the following questions:

Problem 1 *What is the force on the spring due to gravity?*

$$F_G = \boxed{mg}$$

Let's now denote by y_R the y-coordinate corresponding to the relaxed spring.

Problem 2 *Using the fact that the total force at the equilibrium position is zero, determine an expression for y_R:*

$$y_R = \boxed{-mg/k}$$

Problem 3 *What is the force exerted by the spring if the mass is at a generic position y?*

$$F_S = \boxed{-k(y + mg/k)}$$

Problem 4 *Assuming that the coefficient of friction c is zero, what is the total force, $F_T = F_R + F_G$ acting on the spring? Simplify your answer as much as possible.*

$$F_T = \boxed{-ky}$$

Newton's Second Law states that:

$$(\text{mass}) \times (\text{acceleration}) = (\text{total force})$$

Problem 5 *Use Newton's Second Law to find a differential equation for the acceleration of the mass:*

$$y'' = \boxed{-\frac{k}{m}y}$$

Problem 6 *Find the general solution of this differential equation. Use c_1 and c_2 for the constants of integration.*

$$y(t) = \boxed{c_1 \cos\left(\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\right)}$$

Problem 7 *In the experiments we will perform, we will stretch the spring to a prescribed position y_0 and then let it go. This corresponds to the initial conditions $y(0) = y_0$ and $y'(0) = 0$. Find the solution of the differential equation that corresponds to this situation.*

$$y(t) = \boxed{y_0} \cos\left(\sqrt{\frac{k}{m}}\right)$$

Running the experiment with the system

Now set up the experiment. As an initial goal, let's try to establish the period of the oscillation.

Spring in Equilibrium Position Stretched by Mass

This was added by Felipe.

One more line by Paul here.