Observer-based controller for position regulation of stepping motor

J. De León-Morales, R. Castro-Linares and O. Huerta Guevara

Abstract: The design of a controller-observer scheme for the exponential stabilisation of a permanent magnet stepper motor is proposed. The technique is based on sliding-mode techniques and nonlinear observers. Representing the stepper motor model as a singularly perturbed nonlinear system, a position regulation controller is obtained. Since this controller depends on the mechanical variables, load torque and equilibrium point, under the assumption that the rotor position is available for measurement, an observer design is presented to estimate the angular speed and load torque. Furthermore, a stability analysis of the closed-loop system is also made to provide sufficient conditions for the exponential stability of the full-order closed-loop system when the angular speed and load torque are estimated by means of the observer. The proposed scheme is applied to the model of a permanent-magnet stepper motor.

1 Introduction

Dynamic models obtained from theoretical considerations are frequently so complex that they may be impractical for control design purposes. Thus, several methods have been suggested in the literature for deriving reduced-order dynamic models from high-order models. For example, the integral manifold approach is sometimes used systematically to create adequate models of synchronous machines. Another method used for model reduction of large-scale systems is the singular perturbation method (see for example [1–3] and the References therein). This technique has also been used to study robustness in the presence of parasitic or unmodelled dynamics. Some of the advantages of the singular perturbation method are its applicability to nonlinear systems as well as its simplicity and good performance in many practical control situations. When the singular perturbation method is applied, the original system is decomposed into two subsystems of lower dimension, both described in different time scales. From this decomposition, a state feedback may be designed for each lower order subsystem combining them in a so-called composite feedback that is applied to the original system. In [4] and [5], an excellent description of the singular perturbation method is given.

On the other hand, sliding-mode control techniques have been extensively used when a robust control scheme is required; this is when dealing with systems that have uncertainties due to modelling errors and disturbance signals [6–8]. Moreover, a sliding controller is characterised as a high-speed switching controller that provides a robust means of controlling nonlinear systems by forcing the trajectories to reach a sliding manifold in finite time and stay on the manifold for all time. Owing to the switching behaviour of the controller some theoretical and practical problems rise. Thus, the idea of combining the singular perturbation method and the sliding-mode control technique represents a good possibility of achieving classical control objectives for nonlinear systems having unmodelled or parasitic dynamics and parametric uncertainties.

Generally, the controller resulting from the use of singular perturbation methods and sliding-mode techniques needs, however, to have information about the state vector of the plant. This could be possible if adequate sensors are available, but in most cases such sensors, if they exist, are expensive. The reduction of the number of sensors is an important problem for industrial applications. The sensors contribute to an increase in the complexity of the machinery and the cost of the installation. Therefore, it is necessary to estimate the states of the system by using state observers. The design of observers for nonlinear systems is a quite interesting research issue but, in general, a difficult one. To date, several methods have been suggested for the design of the observers. In [9], an excellent survey is made of different approaches proposed for such designs.

Permanent-magnet (PM) stepper motors are electromagnetic incremental motion devices that are very useful in industrial and research laboratory applications. These devices were originally designed to provided precise positioning control since they are open-loop stable to any step position and no feedback is needed to control them if the load torque in the rotor is greater than the detent torque. However, they have a step response with overshoot and relatively long setting time. Besides, loss of synchrony appears when steps of high frequency are given. It is thus necessary to develop control schemes to improve the performance of stepper motors. Feedback control methods for this devices are difficult to implement because they are highly nonlinear systems and it is expensive to have accurate measurement of some of its variables.

This paper reports the design of an observer-based controller for exponential stabilisation of a stepper motor combining the advantages of the singular perturbation methods and sliding-mode techniques by means of

IEE Proceedings online no. 20045066

doi: 10.1049/ip-cta:20045066

Paper first received 30th May 2004 and in revised form 13th January 2005. Originally published online 8th June 2005

J. De León-Morales and O. Huerta Guevara are with the Universidad Autónoma de Nuevo León, Facultad de Ingeniería Eléctrica y Mecánica, Apdo. Postal 148-F, 66451 San Nicolás de Los Garza; NL Mexico

R. Castro-Linares is with the CINVESTAV-IPN, Depto. de Ingeniería Electrica, Apdo. Postal 14-700, 07000 Mexico DF, Mexico,

E-mail: drjleon@hotmail.com

a nonlinear observer. Also, a study of the stability properties of the resultant closed-loop system is presented when the slow state is replaced by its estimate. Furthermore, a comparative study is included in which the performance of the proposed methodology is shown. An extension of this methodology is presented considering a sophisticated model of stepper motor.

2 Model and problem description

Permanent-magnet stepper motors are incremental-motion devices that convert digital pulse inputs to analogue output motion. The PM stepper motor basically consists of a rotor and a stator. The rotor features two axially slotted cylinders displaced by half a slot or 'tooth'; one of the cylinders or 'gears' is a permanent north magnet. The stator, which is also slotted, has a different number of teeth than the rotor gears, so that the rotor will never be aligned with the stator teeth. Each slot in the stator is an electromagnet, which can be alternatively made north or south (for more details see [10, 11]).

The mathematical model for a PM stepper motor is given by the following equations (see [11] for a detailed explanation and derivation of the model; in the Appendix a more sophisticated model is considered):

$$
\begin{aligned}
\frac{di_a}{dt} &= \frac{1}{L}[v_a - Ri_a + K_m\omega\sin(N_r\theta)] \\
\frac{di_b}{dt} &= \frac{1}{L}[v_b - Ri_b - K_m\omega\cos(N_r\theta)] \\
\frac{d\omega}{dt} &= \frac{1}{J}[-K_m i_a\sin(N_r\theta) + K_m i_b\cos(N_r\theta) \\
&\quad - B\omega - K_d\sin(4N_r\theta) - \tau_l] \\
\frac{d\theta}{dt} &= \omega
\end{aligned}
\tag{1}
$$

where i_a, i_b, and v_a, v_b are the currents and voltages in phases a and b, respectively, ω is the rotor angular speed and θ is the rotor angular position. L and R are the self-inductance and resistance of each phase winding, K_m is the motor torque constant, N_r is the number of rotor teeth, J is the rotor inertia, B is the viscous friction constant and τ_l is the load torque. The term $K_d\sin(4N_r\theta)$ represents the detent torque due to the permanent rotor magnet interacting with the magnetic material of the stator poles. In the model (1), we neglect the slight coupling between the phases, the small change in L as a function of θ, the variation in L due to magnetic saturation and the detent torque (since, in general, $K_d = 0\,\text{Nm}$). However, the identification procedure carried out in previous work suggests that the model (1) is adequate for control design (see [10] and the References therein). For a given constant pair $v_a = v_a^*$ and $v_b = v_b^*$, the left-hand sides of the differential equations in (1) are identically zero at an equilibrium point $(i_a, i_a, \omega, \theta) = (i_a^*, i_a^*, \omega^*, \theta^*) = (v_a^*/R, v_b^*/R, 0, \theta^*)$, where $K_d = 0\,\text{Nm}$ has been chosen. From $0 = -\frac{K_m}{JR}[v_a^*\sin(N_r\theta^*) - v_b^*\cos(N_r\theta^*)] - \frac{\tau_l}{J}$, two possible values for θ^* are obtained, i.e.

$$
\theta_\pm^* = \frac{2}{N_r}\arctan\left(\frac{K_m v_a^* \pm \sqrt{K_m^2(v_a^*)^2 + K_m^2(v_b^*)^2 - \tau_l^2R^2}}{-K_m v_b^* - \tau_l R}\right)
$$

By setting $\varepsilon = L$, and considering the following change of co-ordinates of the form $x = \mathrm{col}((\omega - \omega^*), (\theta - \theta^*))$ and $z = \mathrm{col}((i_a - i_a^*), (i_b - i_b^*))$. In addition, N_r is a known parameter, and we make the assignment

$$
\begin{aligned}
v_a - v_a^* &= -\sin(N_r(x_2 + \theta^*))u, \\
u_b - v_b^* &= \cos(N_r(x_2 + \theta^*))u
\end{aligned}
$$

where u is the new scalar control input. The model (1) can be put in the standard singular form

$$
\begin{aligned}
\dot{x} &= f_1(x) + F_1(x)z, \\
\varepsilon\dot{z} &= f_2(x) + F_2(x)z + g_2(x)u
\end{aligned}
$$

where $x(t_0) = x_0,\ z(t_0) = z_0$ and

$$
f_1(x) = \begin{pmatrix} K_4(i_b^*\cos(\alpha) - i_a^*\sin(\alpha)) - K_5x_1 - K_7 \\ x_1 \end{pmatrix},
$$

$$
F_1(x) = K_4\begin{pmatrix} -\sin(\alpha) & \cos(\alpha) \\ 0 & 0 \end{pmatrix},\quad f_2(x) = \begin{pmatrix} K_2x_1\sin(\alpha) \\ -K_2x_1\cos(\alpha) \end{pmatrix},
$$

$$
F_2(x) = \begin{pmatrix} -K_1 & 0 \\ 0 & -K_1 \end{pmatrix},\quad g_2(x) = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}
$$

with $K_1 = R$, $K_2 = K_m$, $K_3 = N_r$, $K_4 = K_m/J$, $K_5 = B/J$, $K_6 = K_d/J = 0$, $K_7 = \tau_l/J$ and $\alpha = K_3(x_2 + \theta^*)$.

Therefore, considering the above representation of the stepping motor, the goal is as follows: find a controller based on singular perturbation methods and sliding techniques such that the closed-loop system consisting of the system and the controller is exponentially stable in a desirable equilibrium point.

In general, a controller requires the full measurement of all variables of the system. However, only current measurements and position measurements are usually available in practice. Furthermore, it would also be adequate to estimate the load torque τ_l. Thus, to estimate this torque and the speed, an observer must be designed. Other types of observers for this kind of electromechanical devices have been designed and tested (see, for example [12], and the References therein). Then, the control problem addressed in this paper is as follows: Assuming that the physical parameters of the stepping motor are known and the measurable variables are the currents i_a, i_b and rotor angular position θ, find a controller based on singular perturbation methods and sliding techniques and design an observer to estimate the no measurable variables such that the overall closed-loop system is locally exponentially stable at the equilibrium point [Note 1].

In the Appendix, a more sophisticated model of the PM stepper motor is considered and, under suitable conditions, a control design is obtained using the procedure proposed in this paper.

3 Sliding-mode control

We now develop the control strategy for the stepper motor. To begin the development, the stepper motor model can be represented by the so-called standard singularly perturbed form:

$$
\begin{aligned}
\dot{x} &= f_1(x) + F_1(x)z & x(t_0) &= x_0 \\
\varepsilon\dot{z} &= f_2(x) + F_2(x)z + g_2(x)u & z(t_0) &= z_0
\end{aligned}
\tag{2}
$$

where $t_0 \geq 0$, $x \in B_x \subset R^n$ is the slow state, $z \in B_z \subset R^m$ is the fast state, $u \in R^r$ is the control input and $\varepsilon \in [0, 1)$, is

Note 1: This control problem can be extended to the case of constant references are considered to regulate the rotor position.

the small perturbation parameter. f_1, f_2, the columns of the matrices F_1, F_2, and g_2 are assumed to be bounded with their components being smooth functions of x. B_x and B_z denote closed and bounded subsets centred at the origin. It is also supposed that $f_1(0) = f_2(0) = 0$ and, for $u = 0$, the origin $(x, z) = (0, 0)$ is an isolated equilibrium state, and that $F_2(x)$ is nonsingular for all $x \in B_x$.

The slow reduced system is found by making $\varepsilon = 0$ in (2), obtaining the nth order slow system:

$$\begin{aligned} \dot{x}_s &= f(x_s) + g(x_s)u_s(x_s) \qquad x_s(t_0) = x_0 \\ z_s &= h(x_s) := -F_2^{-1}(x_s)[f_2(x_s) + g_2(x_s)u_s] \end{aligned} \tag{3}$$

where x_s, z_s and u_s denote the slow components of the original variables x, z and u, respectively, and

$$\begin{aligned} f(x_s) &= f_1(x_s) - F_1(x_s)F_2^{-1}(x_s)f_2(x_s) \\ g(x_s) &= -F_1(x_s)F_2^{-1}(x_s)g_2(x_s) \end{aligned} \tag{4}$$

$u_s(x_s)$in the first equation of (3) denotes the slow state feedback, which only depends on x_s.

The fast dynamics (or, equivalently, boundary layer system) is obtained by transforming the (slow) time scale t to the (fast) time scale $\tau := (t - t_0)/\varepsilon$ and introducing the deviation $\eta := z - h_e(x, \varepsilon)$. The original system (2) then becomes

$$\begin{aligned} \frac{d\tilde{x}}{d\tau} &= \varepsilon\{f_1(\tilde{x}) + F_1(\tilde{x})[\eta + h_e(\tilde{x}, \varepsilon)]\} \\ \frac{d\eta}{d\tau} &= f_2(\tilde{x}) + F_2(\tilde{x})[\eta + h_e(\tilde{x}, \varepsilon)] + g_2(\tilde{x})u \\ &\quad - \frac{\partial h_e(\tilde{x}, \varepsilon)}{\partial \tilde{x}}\frac{d\tilde{x}}{d\tau} \end{aligned} \tag{5}$$

where $\eta(0) = z_0 - h_e(x_0)$, $\tilde{z}(\tau) := z(\varepsilon\tau + t_0)$, with $\tilde{z}(0) = z_0$, and $\tilde{x}(\tau) := x(\varepsilon\tau + t_0)$, with $\tilde{x}(0) = x_0$.

The so-called composite control for the original system (2) is defined by

$$u(x, \eta, \epsilon) = u_{es}(x, \epsilon) + u_{ef}(x, \eta, \epsilon) \tag{6}$$

where u_{es} and u_{ef} denote the slow and fast components of the control, respectively. If $u_{es}(\tilde{x}, \epsilon)$ and $\partial h_e(\tilde{x}, \epsilon)/\partial\tilde{x}$ are bounded and $\tilde{x}$ remains relatively constant with respect to τ, then the term $\epsilon\partial h_e(\tilde{x}, \epsilon)/\partial\tilde{x}$ can be neglected for ϵ sufficiently small. Since the second equation of (5) defines the fast reduced subsystem, an O(ϵ) approximation can be obtained for this subsystem using the first equation of (4) and setting $\varepsilon = 0$ in (5), this is

$$\frac{d\eta_{apx}}{d\tau} = F_2(\tilde{x})\eta_{apx} + g_2(\tilde{x})u_f \tag{7}$$

where η_{apx}, $h_e(\tilde{x}, 0) = h(\tilde{x})$ and u_f are O(ϵ) approximations for η, $h_e(\tilde{x}, \varepsilon)$ and u_{ef} during the initial boundary layer and $\eta_{apx}(0) = z_0 - h(x_0, 0)$.

3.1 Sliding-mode control design

The sliding-mode control for the system (2) is designed in two stages. First, the slow control is designed for the slow subsystem (3). To do this, let us consider a $(n - r)$-dimensional slow nonlinear switching surface defined by

$$\sigma_s(x_s) = col(\sigma_{s_1}(x_s), \ldots, \sigma_{s_r}(x_s)) = 0 \tag{8}$$

where each function $\sigma_{s_i} : B_x \to R$, $i = 1, \ldots, r$; is a C^1 function such that $\sigma_{s_i}(0) = 0$. The equivalent control method [2] is used to determine the slow reduced system motion restricted to the slow switching surface $\sigma_s(x_s) = 0$, obtaining the slow equivalent control

$$u_{se} = -\left[\frac{\partial\sigma_s}{\partial x_s}g(x_s)\right]^{-1}\left[\frac{\partial\sigma_s}{\partial x_s}f(x_s)\right] \tag{9}$$

where the matrix $(\partial\sigma_s/\partial x_s)g(x_s)$ is assumed to be nonsingular for all $x_s \in B_x$.

Remark 1: The assumption that the matrix $(\partial\sigma_s/\partial x_s)g(x_s)$ is nonsingular is not restrictive. The switching surface should be considered to satisfy this condition. For the case that this matrix is not square, numerical methods can be used to obtain the pseudo-inverse of this term. Moreover, this assumption is met for electromechanical systems such as the induction motor, synchronous generator and several electrical machines. All of them depend on the switching surface and the control objective.

Substitution of (9) into (3) yields the slow sliding-mode equation

$$\dot{x}_s = f_e(x_s)$$

where $f_e(x_s) = \left\{I_n - g(x_s)\left[\frac{\partial\sigma_s}{\partial x_s}g(x_s)\right]^{-1}\frac{\partial\sigma_s}{\partial x_s}\right\}f(x_s)$, with I_n denoting the $n \times n$ identity matrix.

To complete the slow control design one sets [6, 7]

$$u_s = u_{se} + u_{sN} \tag{10}$$

where u_{se} is the slow equivalent control (9), which acts when the slow reduced system is restricted to $\sigma_s(x_s) = 0$, while u_{sN} acts when $\sigma_s(x_s) \neq 0$. In this work the control u_{sN} is selected as

$$u_{sN} = -\left[\frac{\partial\sigma_s}{\partial x_s}g(x_s)\right]^{-1}L_s(x_s)\sigma_s(x_s)$$

where $L_s(x_s)$ is a positive-definite matrix of dimension $r \times r$, whose components are C^0 bounded nonlinear real functions of x_s, such that $\|L_s(x_s)\| \leq \rho_s$, for all $x_s \in B_x$ with a constant $\rho_s > 0$. The equation that describes the projection of the slow subsystem motion outside $\sigma_s(x_s) = 0$ is given by

$$\dot{\sigma}_s(x_s) = -L_s(x_s)\sigma_s(x_s) \tag{11}$$

The stability properties of $\sigma_s(x_s) = 0$ in (11) can be studied by means of the Lyapunov function candidate $V(x_s) = \frac{1}{2}\sigma_s^T(x_s)\sigma_s(x_s)$, whose time derivative along (11) satisfies

$$\dot{V}(x_s) = -\sigma_s^T(x_s)L_s(x_s)\sigma_s(x_s), \quad \text{for all } x_s \in B_x$$

From the C^1 properties of $\sigma_s(x_s)$ one also has that $\|\sigma_s(x_s) - \sigma_s(0)\| \leq l_{\sigma_s}\|x_s\|$, $\forall\, x_s \in B_x$, where l_{σ_s} is the Lipschitz constant of $\sigma_s(x_s)$ with respect to x_s. Then, we obtain $\dot{V}(x_s) \leq -\rho_s a_1\|x_s\|^2$, where $a_1 = l_{\sigma_s}^2$. Thus, the existence of a slow sliding mode can be concluded.

The system (3) with the control (10) yields the slow reduced closed-loop system, which is represented as follows:

$$\dot{x}_s = f_e(x_s) + p_s(x_s, u_{sN}) \tag{12}$$

where $p_s(x_s, u_{sN}) = g(x_s)u_{sN}$.

We now introduce the following assumption:

A1: The equilibrium $x_s = 0$ of $\dot{x}_s = f_e(x_s) + p_s(x_s, u_{sN})$ is locally exponentially stable.

By a converse theorem of Lyapunov (see [4]), Assumption A1 assures the existence of a Lyapunov function $V_s = V_s(x_s)$ that satisfies

$$\begin{aligned} &c_1\|x_s\|^2 \le V_s \le c_2\|x_s\|^2, \\ &\frac{\partial V_s}{\partial x} f_e(x_s) + p_s(x_s, u_{sN}) \le -c_3\|x_s\|^2, \\ &\left|\frac{\partial V_s}{\partial x}\right| \le c_4\|x_s\| \end{aligned} \tag{13}$$

for some positive constants c_1, c_2, c_3 and c_4. One may use $V_s(x_s)$ as a Lyapunov function candidate to investigate the stability of the origin $x_s = 0$ as an equilibrium point for the system (12). Using Assumption A1 and (13), the time derivative of V_s along the trajectories of (13) satisfies $\dot{V}(x_s) \le -c_3\|x_s\|^2$, and the reduced slow system (13) is exponentially stable.

The fast control design for the subsystem (8) can be obtained in a similar way to the one used for the slow control. That is, one considers an $(m-r)$-dimensional fast switching surface defined by $\sigma_f(\eta_{apx}) = col(\sigma_{f_1}(\eta_{apx}), \ldots, \sigma_{f_r}(\eta_{apx})) = 0$, where each function $\sigma_{f_i} : B_z \to R$, $i = 1, \ldots r$; is also a C^1 function such that $\sigma_{f_i}(0) = 0$. The complete fast control takes the form

$$u_f = u_{fe} + u_{fN} \tag{14}$$

where u_{fe} is the fast equivalent control given by

$$u_{fe}(\tilde{x}, \eta_{apx}) = -\left[\frac{\partial \sigma_f}{\partial \eta_{apx}} g_2(\tilde{x})\right]^{-1} \left[\frac{\partial \sigma_f}{\partial \eta_{apx}} F_2(\tilde{x})\eta_{apx}\right] \tag{15}$$

and

$$u_{fN}(\tilde{x}, \eta_{apx}) = -\left[\frac{\partial \sigma_f}{\partial \eta_{apx}} g_2(\tilde{x})\right]^{-1} L_f(\eta_{apx})\sigma_f(\eta_{apx}) \tag{16}$$

In (15) and (16), the matrix $(\partial\sigma_f/\partial\eta_{apx})g_2(\tilde{x})$ is assumed to be nonsingular, for all $(\tilde{x}, \eta_{apx}) \in B_x \times B_z$, and $L_f(\eta_{apx})$ is a positive-definite matrix of dimension $r \times r$, whose components are C^0 bounded nonlinear real functions of η_{apx}, such that $\|L_f(\eta_{apx})\| \le \rho_f$, for all $(\tilde{x}, \eta_{apx}) \in B_x \times B_z$, with a constant ρ_f.

The projection of the fast subsystem motion outside $\sigma_f(\eta_{apx}) = 0$ is described by

$$\frac{d\sigma_f}{d\tau} = \frac{\partial \sigma_f}{\partial \eta_{apx}} \frac{d\eta_{apx}}{d\tau} = -L_f(\eta_{apx})\sigma_f(\eta_{apx}) \tag{17}$$

and arguments similar to the ones used for the slow subsystem motion can be applied to the system (17) to conclude the existence of a fast sliding mode.

When the complete fast control (14) is substituted in (7), the fast reduced closed-loop system takes the form

$$\frac{d\eta_{apx}}{d\tau} = g_c(\tilde{x}, \eta_{apx}) \tag{18}$$

where

$$\begin{aligned} g_c(\tilde{x}, \eta_{apx}) = F_2(\tilde{x})\eta_{apx} - g_2(\tilde{x}) &\left[\frac{\partial \sigma_f}{\partial \eta_{apx}} g_2(\tilde{x})\right]^{-1} \\ \times &\left[\frac{\partial \sigma_f}{\partial \eta_{apx}} F_2(\tilde{x})\eta_{apx} + L_f(\eta_{apx})\sigma_f(\eta_{apx})\right]. \end{aligned}$$

The following assumption is now introduced:

A2: The equilibrium $\eta_{apx} = 0$ of $d\eta_{apx}/d\tau = g_c(\tilde{x}, \eta_{apx})$ is locally exponentially stable.

From Assumption A2, by a converse theorem of Lyapunov (see [4]), there exists a Lyapunov function $W_f = W_f(\eta_{apx})$ that satisfies

$$\begin{aligned} &\bar{c}_1\|\eta_{apx}\|^2 \le W_f \le \bar{c}_2\|\eta_{apx}\|^2, \\ &\frac{\partial W_f}{\partial \eta_{apx}} g_c(\tilde{x}, \eta_{apx}) \le -\bar{c}_3\|\eta_{apx}\|^2, \\ &\left\|\frac{\partial W_f}{\partial \eta_{apx}}\right\| \le \bar{c}_4\|\eta_{apx}\| \end{aligned} \tag{19}$$

for some positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3$ and $\bar{c}_4$.

One may also use $W_f(\eta_{apx})$ as a Lyapunov function candidate to investigate the stability of the origin $\eta_{apx} = 0$ as an equilibrium point for the system (18). Using assumptions A2 and (19), the time derivative of W_f along the trajectories of (18) then satisfies $dW_f(\eta_{apx})/d\tau \le -\bar{c}_3\|\eta_{apx}\|^2$, and the reduced fast system (18) is exponentially stable.

The original slow and fast state variables are now used to construct the composite control (6), i.e. $u(x, \eta) = u_s(x) + u_f(x, \eta)$, where (10, 14)

$$u_s = -\left[\frac{\partial \sigma_s}{\partial x} g(x)\right]^{-1} \left[\frac{\partial \sigma_s}{\partial x} f(x) + L_s(x_s)\sigma_s(x_s)\right]$$

$$u_f = -\left[\frac{\partial \sigma_f}{\partial \eta} g_2(x)\right]^{-1} \left[\frac{\partial \sigma_f}{\partial \eta} F_2(x)\eta + L_f(\eta)\sigma_f(\eta)\right]$$

When the composite control (6,10,14) is substituted in (2), one obtains the closed-loop nonlinear singularly perturbed system

$$\begin{aligned} \dot{x} &= f_c(x, \eta) \\ \varepsilon\dot{\eta} &= g_c(x, \eta) - \epsilon\frac{\partial h}{\partial x}[f_c(x, \eta)] \end{aligned} \tag{20}$$

where $\eta = z - h(x)$, $x(t_o) = x_o$, $z(t_o) = z_o$ and

$$\begin{aligned} f_c(x, \eta) = f(x) + F_1(x)\eta - g(x) \\ \times \left[\frac{\partial \sigma_s}{\partial x} g(x)\right]^{-1} \left[\frac{\partial \sigma_s}{\partial x} f(x) + L_s(x)\sigma_s(x)\right]. \end{aligned}$$

In the present work the Lyapunov function candidates V_s and W_f are instrumental to investigate the stability properties of the closed-loop system obtained when the composite control $u = u_s + u_f$ is used and an observer is introduced to estimate the state of the original system.

4 Nonlinear estimator

Since the above control design depends on measurable and non-measurable variables, it is necessary to estimate those non-measurable variables in order to implement this controller. Hence, an observer design is presented to estimate the non-measurable variables.

It is clear that there is no systematic method to design an observer for a given nonlinear control system. However, several designs are available according to the specific characteristics of the nonlinear system considered.

In this paper, let us now consider the class of nonlinear singularly perturbed systems described by (2) together with an output variable $y \in R$, such that $y = q(x)$, where q is a continuously differentiable function of B_x and depends on the slow state [Note 2]. In addition, it is assumed that

Note 2: This observer design can be extended to the multi-output case.

the fast state z is an input vector to the slow subsystem (2) and that it is completely measurable.

Consider the nonlinear system

$$\dot{x} = f_1(x) + F_1(x)z \quad y = q(x) \tag{21}$$

where z is an input for the system. If there exists a mapping $T: R^n \to R^n$, which is a diffeomorphism from B_x onto $T(B_x)$, such that (21) can be written in the new co-ordinates as

$$\begin{aligned} \dot{\zeta} &= A\zeta + G(\zeta, z) \\ y &= C\zeta \end{aligned} \tag{22}$$

where $z = col(z_1, z_2, \ldots, z_m)$, $G(\zeta, z)$ is a $n \times m$ matrix with $G(\zeta, z) = col(g_1(\zeta_1, z),\ g_2(\zeta_1, \zeta_2, z), \ldots, g_n(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n, z))$, and the pair (A, C) is in the canonical observable form, that is

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $C = (1 \quad 0 \quad \cdots \quad 0)$. Then, we say that system (21) is uniformly observable for any input (see [13] for more details).

One now assumes that the mappings $g_j: R^i \to R$, for $i = 1, \ldots, n$ are globally Lipschitz.

Next, we will design an observer for systems (21) by exploiting this triangular structure. This property of the nonlinearity is important because it ensures the uniform observability of the system.

We can establish the following.

Theorem 1: Suppose system (21) is uniformly observable. Let also K be a $n \times 1$ constant column vector such that $Re\lambda\{(A - KC)\} \in C^-$. Then, system

$$\frac{d\hat{\zeta}}{dt} = A\hat{\zeta} + G(\hat{\zeta}, z) - R^{-1}(\ell)K(y - \hat{y}) \tag{23}$$

where $R(\ell) = diag(1, 1/\ell, \ldots, 1/\ell^{n-1})$, with ℓ a positive constant such that $\ell > 0$, is an exponential observer for system (22) whose dynamics can be arbitrarily fast.

Proof: Consider the change of co-ordinates $\xi = R(\ell)\zeta$. In these new co-ordinates, and due to the form of the matrix A and the column vector C, one has that $R(\ell)AR^{-1}(\ell) = \ell A$, $CR^{-1}(\ell) = C$, and the dynamics of system (22) can be written as

$$\begin{aligned} \dot{\xi} &= \ell A\xi + R(\ell)G(R^{-1}(\ell)\xi, z) \\ y &= C\xi \end{aligned}$$

In a similar way, the observer system (23) can be expressed as

$$\begin{aligned} \frac{d\hat{\xi}}{dt} &= \ell A\hat{\xi} + R(\ell)G(R^{-1}(\ell)\hat{\xi}, z) - \ell KC(\xi - \hat{\xi}) \\ \hat{y} &= C\hat{\xi} \end{aligned}$$

If we now define the estimation error as $e_\xi = \xi - \hat{\xi}$, its dynamics are given by

$$\dot{e}_\xi = \ell(A - KC)e_\xi + R(\ell)\Gamma(\xi, \hat{\xi}, z) \tag{24}$$

where $\Gamma(\xi, \hat{\xi}, z) = G(R^{-1}(\ell)\xi, z) - G(R^{-1}(\theta)\hat{\xi}, z)$. On the other hand, since, by assumption, the autonomous system $\dot{e}_\xi = \ell(A - KC)e_\xi$ is asymptotically stable, there exists a Lyapunov function $V_{e_\xi} = e_\xi^T P_e e_\xi$ with P_e a symmetric positive definite matrix, which is the unique solution of the Lyapunov equation (see [4]) $(A - KC)^T P_e + P_e(A - KC) = -Q_e$, where Q_e is an arbitrary symmetric positive definite matrix such that

$$c_1^*\|e_\xi\|^2 \le V_{e_\xi} \le c_2^*\|e_\xi\|^2, \quad \frac{\partial V_{e_\xi}}{\partial e_\xi}(A - KC)e_\xi \le -c_3^*\|e_\xi\|^2,$$

$$\left\|\frac{\partial V_{e_\xi}}{\partial e_\xi}\right\| \le c_4^*\|e_\xi\|$$

where $c_1^* = \lambda_{\min}(P_e)$, $c_2^* = \lambda_{\max}(P_e)$, $c_3^* = \lambda_{\min}(Q_e)$ and $c_4^* = 2\lambda_{\max}(P_e)$. Taking the time derivative of V_{e_ξ} along the trajectories of the estimation error dynamics (24) one has

$$\begin{aligned} \dot{V}_{e_\xi} &= -\ell c_3^*\|e_\xi\|^2 + 2e_\xi^T P_e R(\ell)\Gamma(\xi, \hat{\xi}, z) \\ &\le -\ell c_3^*\|e_\xi\|^2 + c_4^*\|e_\xi\|\|R(\ell)\Gamma(\xi, \hat{\xi}, z)\| \end{aligned}$$

Since the mappings g_is are globally Lipschitz, one also has that $\|R(\ell)\Gamma(\xi, \hat{\xi}, z)\| \le \rho\|e_\xi\|$, where ρ is a Lipschitz constant. Thus $\dot{V}_{e_\xi} \le -\alpha\|e_\xi\|^2$, where $\alpha = (\ell c_3^* - c_4^*\rho) > 0$. If ρ is small enough and satisfying the bound $\rho \le \bar{\rho} < \ell c_3^*/c_4^*$, the observer system (23) is exponentially stable with a convergence rate that can be made arbitrarily fast, that is $\dot{V}_{e_\xi} \le -\frac{\alpha}{c_2^*}V_{e_\xi}$.

5 Closed-loop stability

Suppose that a composite control (6, 10, 14) has been designed such that the nonlinear singularly perturbed system (20) is uniformly bounded, and that an observer (23), with exponential rate of convergence, is also designed. The fundamental question of knowing if the stability of the closed-loop system is preserved, when the state is replaced by its estimate in the control law, is now addressed. The purpose of this Section is to give sufficient conditions that ensure the stability of the closed-loop system with observer.

Let us consider the augmented closed-loop nonlinear singularly perturbed system described by

$$\begin{aligned} &\dot{x} = f(x) + F_1(x)\eta + g(x)u_s(\hat{x}), \quad x(t_0) = x_0 \\ &\varepsilon\dot{\eta} = F_2(x)\eta + g_2(x)u_f(\hat{x}, \eta) + \varepsilon\left(\frac{\partial h}{\partial \hat{\xi}}\right)\dot{e}_\xi \\ &\qquad - \varepsilon\left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial \hat{\xi}}\frac{\partial \gamma}{\partial x}\right)\left[f(x) + F_1(x)\eta + g(x)u_s(\hat{x})\right] \\ &\dot{e}_\xi = \ell(A - KC)e_\xi + R(\ell)\Gamma(\xi, \hat{\xi}, \eta + h(x)), \quad e_\xi(t_0) = e_{\xi_0} \\ &y = C\xi \end{aligned} \tag{25}$$

where $\eta(t_0) = z_0 - h(x_0)$ and ℓ, K are selected as in Section 3. Note that the composite control now depends on the estimate $\hat{x}$, where $x = T^{-1}(R^{-1}(\ell)\xi) = \gamma^{-1}(\xi)$. The system (25) can be rewritten as

$$
\begin{aligned}
\dot{x} &= f_c(x,\eta) + g(x)\Delta u_s(x,\hat{x}), \quad x(t_0) = x_0 \\
\varepsilon\dot{\eta} &= g_c(x,\eta) + g_2(x)\Delta u_f(x,\eta,\hat{x}) + \varepsilon\left(\frac{\partial h}{\partial \hat{\xi}}\right)\dot{e}_\xi \\
&\quad - \varepsilon\left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial \hat{\xi}}\frac{\partial \gamma}{\partial x}\right)\left\{f_c(x,\eta) + g(x)\Delta u_s(x,\hat{x})\right\} \\
\dot{e}_\xi &= \ell(A - KC)e_\xi + R(\ell)\Gamma(\xi,\hat{\xi},\eta + h(x)), \quad e_\xi(t_0) = e_{\xi_0} \\
y &= C\xi
\end{aligned} \tag{26}
$$

where $\eta(t_0) = z_0 - h(x_0)$ and f_c, g_c are defined as in Section 2, and

$$
\begin{aligned}
\Delta u_s(x,\hat{x}) &= u_s(\hat{x}) - u_s(x), \\
\Delta u_f(x,\hat{x},\eta) &= u_f(\hat{x},\eta) - u_f(x,\eta)
\end{aligned} \tag{27}
$$

From the properties of the functions involved in u_s and u_f, one has that Δu_s and Δu_f satisfy the local Lipschitz conditions

$$
\|\Delta u_s(x,\hat{x})\| \le m_s\|e_\xi\|, \quad \|\Delta u_f(x,\hat{x},\eta)\| \le m_f\|e_\xi\|
$$

for all $(x,\hat{x},\eta) \in B_x \times B_x \times B_z$, where m_s and m_f are the Lipschitz constants of $u_s(x)$ and $u_f(x,\eta)$ with respect to x and (x,η), respectively. From the fact that the columns of $g(x)$ and $g_2(x)$ are bounded, one has:

$$
\begin{aligned}
\|g(x)\Delta u_s(x,\hat{x})\| &\le m_0 m_s\|e_\xi\|, \\
\|g_2(x)\Delta u_f(x,\hat{x},\eta)\| &\le m_2 m_f\|e_\xi\|
\end{aligned} \tag{28}
$$

for all $(x,\hat{x},\eta) \in B_x \times B_x \times B_z$, and $e_\xi \in B_x$. m_0 and m_2 are some positive constants.

In view of the properties of all the functions involved in $f_c(x,\eta)$, this satisfies the local Lipschitz condition

$$
\begin{aligned}
\|f_c(x,\eta) - f_c(x,0)\| = \|F_1(x)\eta\| &\le l_{f\eta}\|\eta\|, \\
&\forall (x,\eta) \in B_x \times B_z
\end{aligned} \tag{29}
$$

where $l_{f\eta}$ is the Lipschitz constant of $f_c(x,\eta)$, with respect to the fast variable η. Furthermore, $f_c(0,0) = 0$; thus

$$
\|f_c(x,0)\| \le l_{fx_1}\|x\| \quad \forall x \in B_x \tag{30}
$$

where l_{fx_1} denotes the Lipschitz constant of $f_c(x,0)$ with respect to x. Also, from the continuous differentiability of $h(x)$ it follows that

$$
\left\|\frac{dh}{dx}\right\| \le l_{h_x}, \quad \left\|\frac{dh}{d\hat{\xi}}\right\| \le l_{h_{\hat{\xi}}}, \quad \left\|\frac{d\gamma}{dx}\right\| \le l_{\gamma_x}, \quad \forall x \in B_x, \quad \hat{\xi} \in B_{\hat{\xi}} \tag{31}
$$

where l_{h_x}, $l_{h_{\hat{\xi}}}$ and l_{γ_x} are positive constants. Now, set $\alpha_1 = c_3, \alpha_2 = \left(\frac{\bar{c}_3}{e} - \bar{c}_4 l_{f\eta}(l_{h_x} + l_{h_{\hat{\xi}}} l_{\gamma_x})\right)$, $\beta_1 = c_4 l_{f\eta} + \bar{c}_4(l_{h_x} + l_{h_{\hat{\xi}}} l_{\gamma_x})$ $l_{fx_1}, \beta_2 = c_4 m_0 m_s$, $\beta_3 = \bar{c}_4(l_{h_x} + l_{h_{\hat{\xi}}} l_{\gamma_x}) m_0 m_s + \frac{1}{e}\bar{c}_4 m_2 m_f + \bar{c}_4 l_{h_{\hat{\xi}}}$ $(\ell\alpha_M + \rho)$.

To study the stability properties of the closed-loop system, consider the following Lyapunov function

$$
L(x,\eta,e_\xi) = V_s(x) + W_f(\eta) + V_{e_\xi}(e_\xi).
$$

where $V_s(x), W_f(\eta)$, and $V_{e_\xi}(e_\xi)$ are the Lyapunov functions of the slow subsystem, the fast subsystem and the estimation error dynamics, respectively.

The following result gives sufficient conditions to assure the local exponential stability of the overall closed-loop nonlinear singularly perturbed system (26):

Theorem 2: Consider a nonlinear singularly perturbed system (2) in closed loop with the control (6, 10, 14) using an observer (23) for estimating the unmeasurable states, whose estimation error dynamics converges exponentially to zero. Thus, if there exist some numbers $0 < \delta_i < 1$, $i = 1,2,3,4$, such that $\mu_{co} = \min\{a', b', c'\} > 0$, where $a' = \alpha_1 - (\beta_1\delta_1/2) - (\beta_2\delta_2/2), b' = \alpha_2 - \beta_1/2\delta_1 - \beta_3\delta_3/2, c' = \alpha - \beta_2/2\delta_2 - \beta_3/2\delta_3$, for sufficiently small ϵ, then the augmented closed-loop nonlinear singularly perturbed system (26) is locally exponentially stable.

Sketch of proof: We proceed to compute the time derivative of L along the trajectories of each subsystem. Taking the norm of all terms and substituting the corresponding inequalities, after some computations it follows finally that

$$
\begin{aligned}
\dot{L}(x(t),\eta(t),e_\xi(t)) &\le -a'\|x\|^2 - b'\|\eta\|^2 - c'\|e\|^2 \\
&\le -\mu_{co} L(x(t),\eta(t),e_\xi(t)).
\end{aligned}
$$

This last inequality implies that

$$
L(x(t),\eta(t),e_\xi(t)) \le L(x_0,\eta_0,e_{\xi_0})e^{-\mu_{co}(t-t_0)}
$$

Then the states x, η, and e_ξ are locally exponentially stable for all $t \ge t_0$.

6 Application to PM stepper motor

In this Section, we present the procedure for designing the observer-based controller using the aforementioned method.

6.1 Control law design

When $\varepsilon = 0$, one obtains a unique root $z_s = h(x_s)$ (3) given by

$$
\begin{aligned}
z_1(s) = h_1(x_s) &= \frac{K_2 x_{s_1}\sin(\alpha_s) - \sin(\alpha_s)u_s(x_s)}{K_1} \\
z_2(s) = h_2(x_s) &= \frac{-K_2 x_{s_1}\cos(\alpha_s) + \cos(\alpha_s)u_s(x_s)}{K_1}
\end{aligned}
$$

and the slow reduced subsystem is given by (4) with

$$
f(x_s) = \begin{pmatrix} -\Omega x_{s_1} + \frac{K_4}{K_1}[v_b^*\cos(\alpha_s) - v_a^*\sin(\alpha_s)] - K_7 \\ x_{s_1} \end{pmatrix},
$$

$$
g(x_s) = \begin{pmatrix} \frac{K_4}{K_1} \\ 0 \end{pmatrix}
$$

where $\Omega = (K_4K_2/K_1) + K_5$ and $\alpha_s = K_3(x_{s_2} + \theta^*)$. Since it is desired that the rotor position tracks the reference signal, which in this case is the equilibrium point [Note 3], a slow nonlinear switching function is chosen as:

$$
\sigma_s(x_s) = s_1 x_{s_1} + s_2 x_{s_2}
$$

where s_1 and s_2 are constant real coefficients [Note 4]. This choice leads, in accordance with Section 2, to the slow control

Note 3: In this analysis, we do not consider time-varying reference signals. However, it is possible to include this kind of signal as long as some additional assumptions are imposed to guarantee the local exponential stability.

Note 4: Evidently, other switching surfaces can be considered. The choice depends on the control objective.

$$u_s(x_s) = u_{se}(x_s) + u_{s_N}(x_s) \tag{32}$$

with

$$\begin{aligned} u_{se} &= \frac{K_1}{K_4}\left[\left(\frac{K_4K_2}{K_1} + K_5 - \frac{s_2}{s_1}\right)x_{s_1} + K_7\right] \\ &\quad - [v_b^* \cos(\alpha_s) - v_a^* \sin(\alpha_s)] \\ u_{s_N} &= -\frac{K_1 L_s(x_s)}{s_1 K_4}(s_1 x_{s_1} + s_2 x_{s_2}) \end{aligned} \tag{33}$$

where $L_s(x_s) = l_s > 0$. On the other hand, (11) takes the form $\dot{\sigma}_s(x_s) = l_s \sigma_s(x_s)$, thus there exists a slow sliding mode. Substitution of the slow control (32, 33) into the slow reduced subsystem yields the slow reduced closed-loop system:

$$\begin{aligned} \dot{x}_s &= f_e(x_s) + p_s(x_s, u_{sN}) \\ &= A_s x_s \end{aligned}$$

where $A_s = \begin{pmatrix} -\left(\frac{s_2}{s_1} + l_s\right) & -\frac{s_2 l_s}{s_1} \\ 1 & 0 \end{pmatrix}$ and $p_s(x_s, u_{sN}) = 0$. By choosing $s_1 > 0$ and $s_2 > 0$, one guarantees the exponential stability of system $\dot{x}_s = A_s x_s$.

Since in the $\mathcal{O}(\epsilon)$ approximation of the exact fast subsystem, given by (18), the constant matrix F_2 has its two eigenvalues at $-K_1 = -R$, there is no need for fast control. That is, one sets $u_f = 0$, and the fast reduced closed-loop system is given by

$$\frac{d\eta_{apx}}{d\tau} = g_c(\tilde{x}, \eta_{apx}) = F_2 \eta_{apx}$$

Then, using the Lyapunov function candidate $W_f(\eta_{apx}) = \eta_{apx}^T P_f \eta_{apx}$, where P_f is a symmetric positive definite matrix, it is easy to prove that the equilibrium of this system is exponentially stable.

Finally, the composite control becomes

$$u(x) = u_s(x)$$

6.2 Speed and torque estimation

The controller designed above requires mechanical variables, load torque and equilibrium point. Then, for control implementation purposes, we must measure all of them. However, a reduction in the number of sensors reduces the cost of the overall control system. Assuming that the rotor position is usually available in practice using encoders and the load torque is unknown, an observer is designed to estimate the angular speed and load torque.

Let $x_3 = \tau_l / J$ so that now one has the assignment $x = \mathrm{col}((\omega - \omega^*), (\theta - \theta^*), \tau_l / J)$ for the slow state. This yields to the new augmented slow subsystem

$$\begin{aligned} \dot{x} &= \bar{f}_1(x) + \bar{F}_1(x) z \\ y &= q(x) \end{aligned} \tag{34}$$

where

$$\bar{f}_1(x) = \begin{pmatrix} K_4 i_b^* \cos(\alpha) - K_4 i_a^* \sin(\alpha) - K_5 x_1 - x_3 \\ x_1 \\ 0 \end{pmatrix},$$

$$\bar{F}_1(x) = \begin{pmatrix} -K_4 \sin(\alpha) & K_4 \cos(\alpha) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

with output $q(x) = x_2$, and z as the input of the system.

Using the observability condition given in [14] (Theorem 73, Section 7.3.2, p. 418), one can verify that the system (34) is locally observable; then an observer can be designed for this systems.

Remark 2: It is clear that other observers can be considered in this case. For instance, in [15] a position and velocity sensorless control for a brushless DC motor using an adaptive sliding mode observer is presented. Some conditions are required to determine the gain of the observer. However, when the angular speed is small and the motor load is large, these conditions are not satisfied. Then, the rotor position should be sensed. On the other hand, in [16] a sensorless observer for induction motors is proposed. This kind of observer can be considered in this work provided that one finds a change of coordinates to transform the stepper motor model to the suitable representation required.

Considering the following change of co-ordinates $\zeta_1 = x_2$, $\zeta_2 = x_1$, $\zeta_3 = K_4 i_b^* \cos(\alpha) - K_4 i_a^* \sin(\alpha) - K_5 x_1 - x_3$, the system (34) can be written in the form (22) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad G(\zeta, z) = \begin{pmatrix} 0 \\ \Psi_1 \\ \Psi_2 \end{pmatrix},$$

where $\Psi_1 = -K_4 \sin(\alpha) z_1 + K_4 \cos(\alpha) z_2$, $\Psi_2 = K_5 \Psi_1 + \zeta_2 K_3 K_4 [i_b^* \sin(\alpha) + i_a^* \cos(\alpha)] - K_5 \zeta_3$ and $\alpha = K_3(x_2 + \theta^*)$. Then, the observer for system (34) is given by (23) with the observer gain K selected in such a way that $(A - KC)$ is Hurwitz.

7 Simulation results

The stepper motor described by (1) was simulated together with the controller and observers designed above, using the following nominal values of motor parameters, which were chosen as in [17]: $R = 10\,\Omega$, $K_m = 0.113\,N - m/A$, $N_r = 50$, $B = 0.001$ Nm/rad/s, $K_d = 0$ Nm, $\tau_l = 0.05$ Nm, $J = 5.7 \times 10^{-6}$ Kgm^2 and $L = 0.0011$ H. Also, for a voltage pair $v_a^* = 2.1621$ V, $v_b^* = 5.4054$ V, the following equilibrium point was $i_a^* = 0.21621\,A$, $i_b^* = 0.54054\,A$, $\omega^* = 0$ rad/s, $\theta^* = 0.0065385$ rad. Furthermore, the control and observer parameters were chosen as follows $s_1 = 1$, $s_2 = 500$, $s_3 = 5000$, $l_s = 10000$, $\ell = 10$ and $K = \mathrm{col}(1, 1, 1)$.

In the following studies, the initial conditions of the motor variables and the estimates were fixed as $i_a(0) = 0.21621\,A$, $i_b(0) = 0.54054\,A$, $\omega(0) = 0$ rad/s, $\theta(0) = 0.031416$ rad is

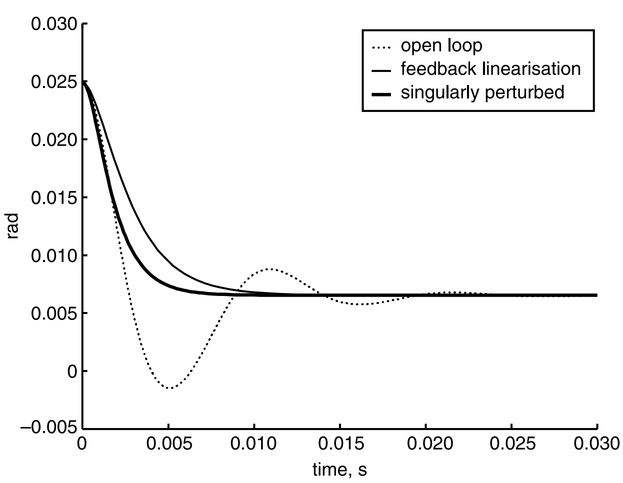

Fig. 1 *Rotor Position*

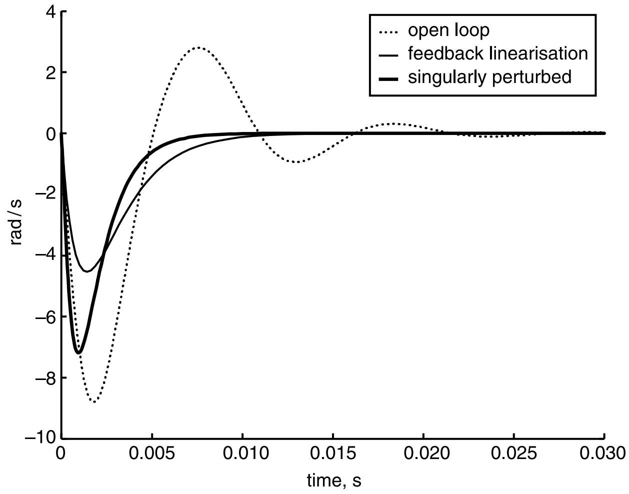

Fig. 2 *Angular speed*

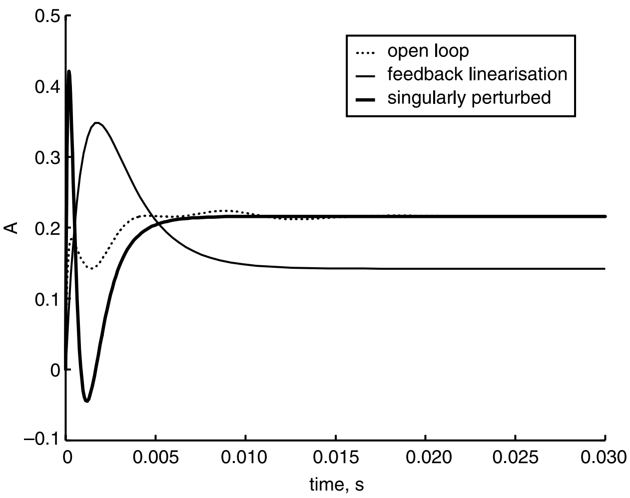

Fig. 3 *Phase current a*

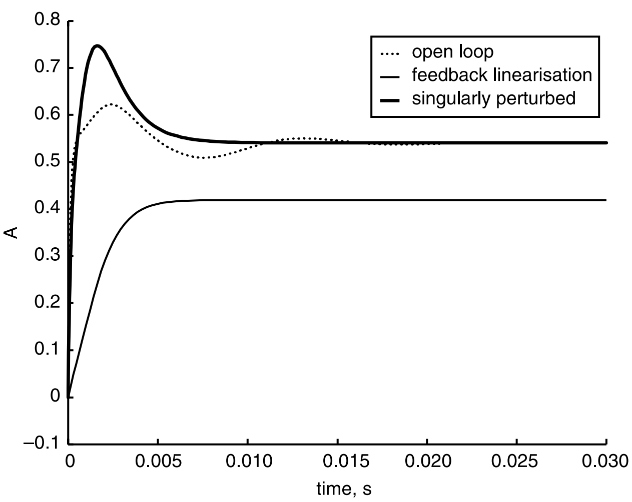

Fig. 4 *Phase current b*

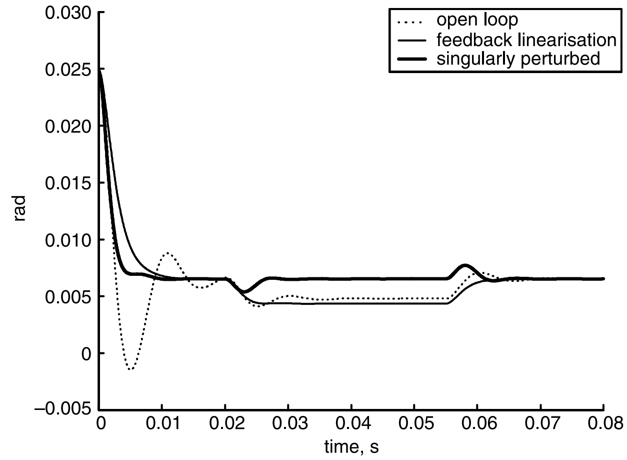

Fig. 5 *Rotor position (K_d variation)*

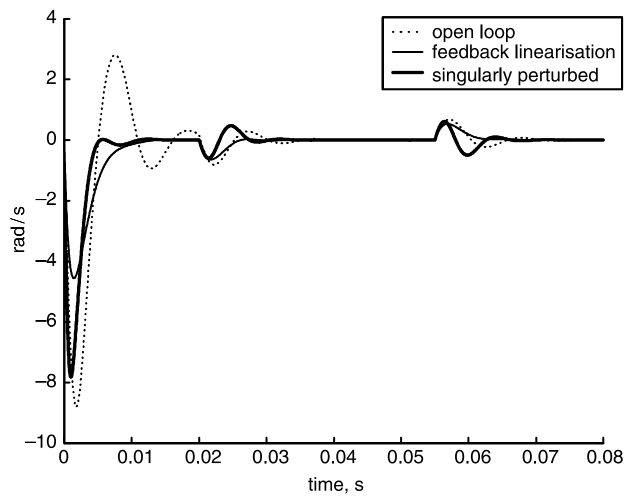

Fig. 6 *Angular speed (K_d variation)*

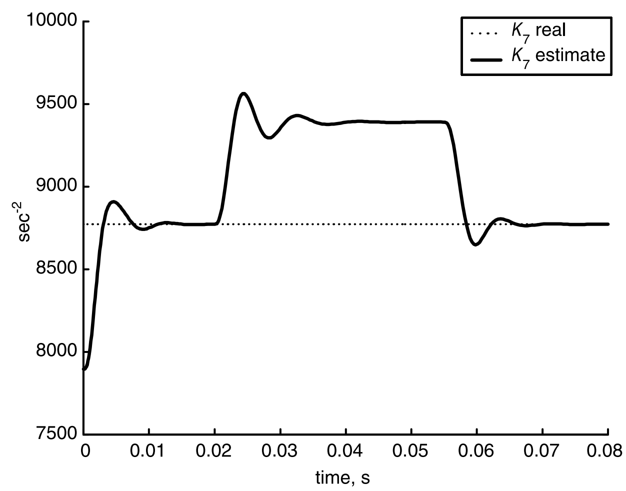

Fig. 7 *K_7 (K_d variation)*

the resolution limit of the device [17], $\hat{\omega}(0) = 0.001$ rad/s, $\hat{\theta} = 0$ rad and $\hat{\tau}_l(0) = 0.045$ Nm.

Furthermore, for comparison the exact linearisation controller proposed by [17] was also employed to show the performance with respect to the proposed control scheme. The time open-loop as well as the two closed-loop responses of the rotor angular speed and the rotor angular position are shown in Figs. 1 and 2, respectively. We can observe that the exact linearisation controller shows more oscillations than the proposed controller, which has a faster response with no oscillations.

In this work, the simulations were performed using the high-gain observer for implementing the proposed controller. In Figs. 3 and 4 the phase currents are shown.

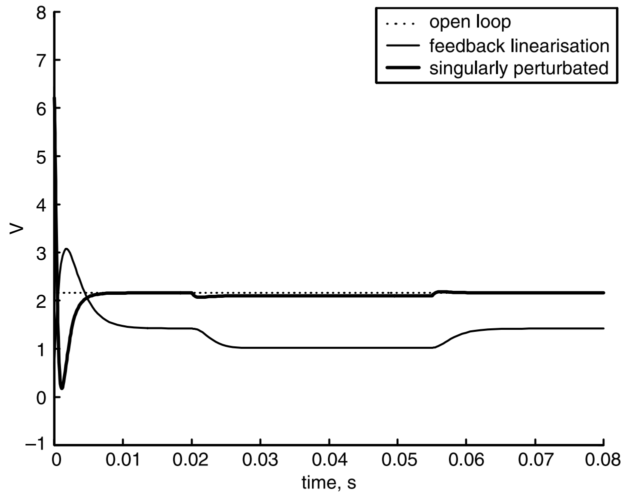

Fig. 8 *Control action v_a*

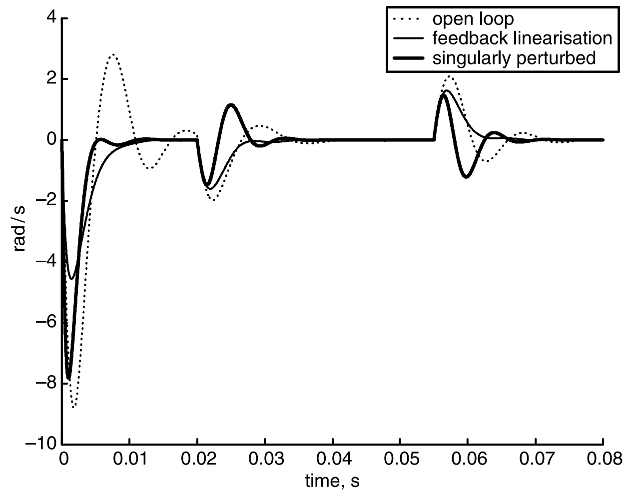

Fig. 11 *Angular speed (τ_l variation)*

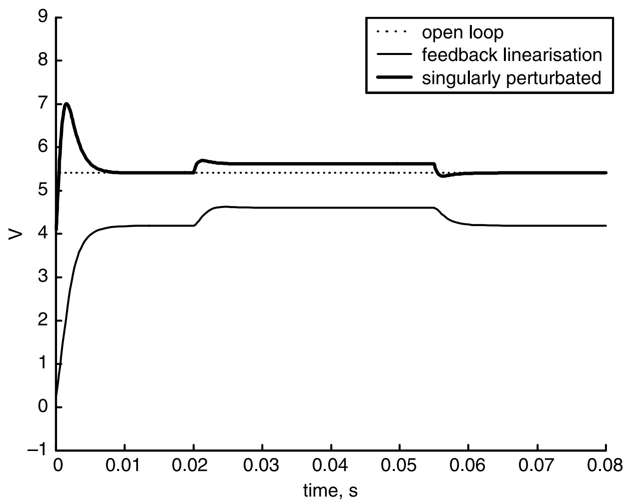

Fig. 9 *Control action v_b*

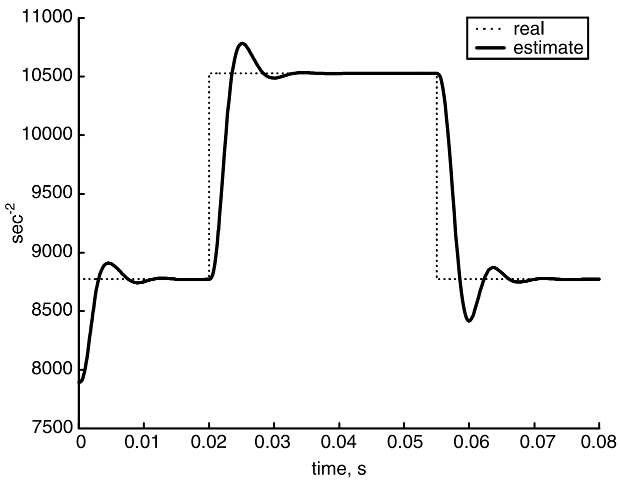

Fig. 12 *K_7 and its variation*

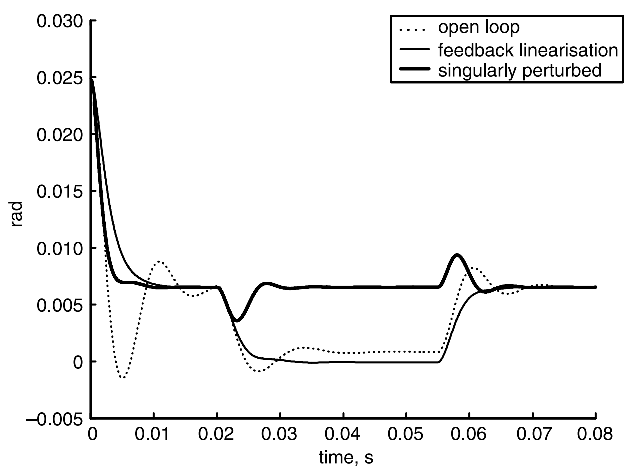

Fig. 10 *Rotor position (τ_l variation)*

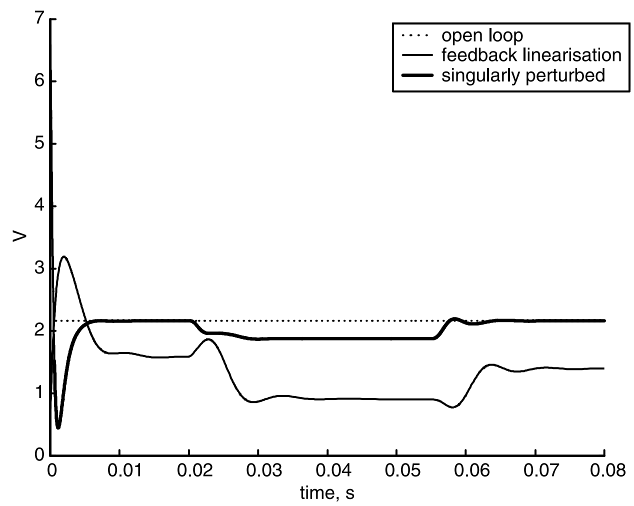

Fig. 13 *Control v_a action (τ_l variation)*

The value of the load torque τ_l and the parameter K_d were next changed to show the performance of the proposed methodology under parametric perturbations. In this case, τ_l was changed from 0.05 to 0.06 Nm and K_d from 0 to 0.0043 Nm, at $t = 0.02$ s and $t = 0.055$ s.

Furthermore, unlike in [17], an observer for estimating τ_l and ω was considered for implementing the exact linearisation controller. The simulations were performed using the same high-gain observer considered in Section 4.

The simulations results under K_d variation are shown in Figs. 5–9, and the plots under τ_l variation are shown in Figs. 10–14. These changes correspond to a variation in the torque of 20% and to a typical value of K_d, i.e. between 5 and 10% of the value of $K_m i_o$, where i_o is the rated current.

Finally, both parameters were changed simultaneously and theirs dynamic responses are given in Figs. 15–18.

The real and the estimate constant $K_7 = \tau_l/J$ under perturbations is given in Fig. 19. We can see that the observer performs well under parametric variations, without deterioration of the responses.

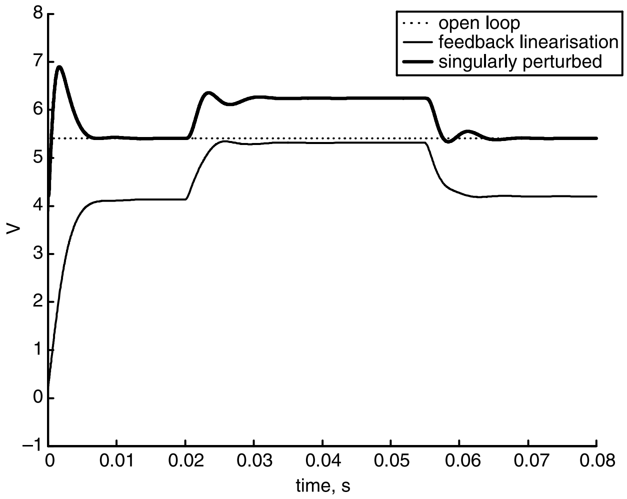

Fig. 14 *Control v_b action (τ_l variation)*

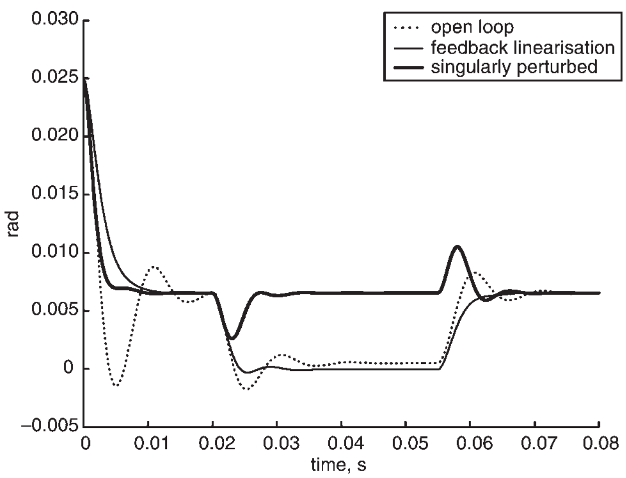

Fig. 15 *Rotor position (τ_l and K_d variations)*

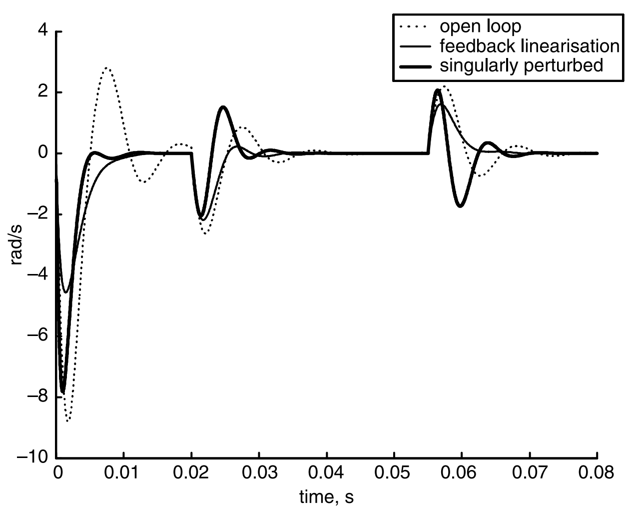

Fig. 16 *Angular speed (τ_l and K_d variations)*

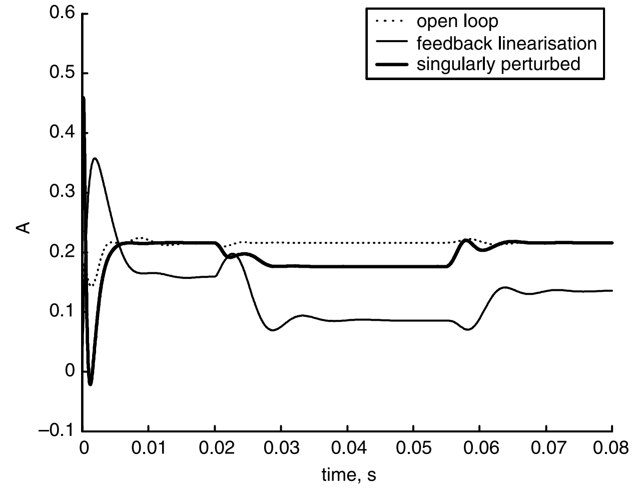

Fig. 17 *Phase current a*

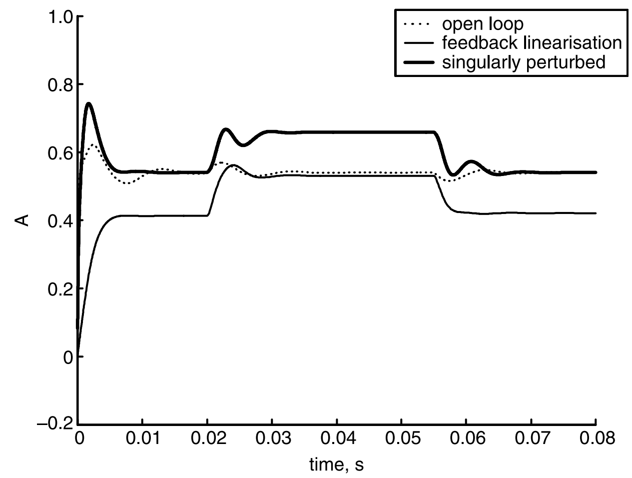

Fig. 18 *Phase current b*

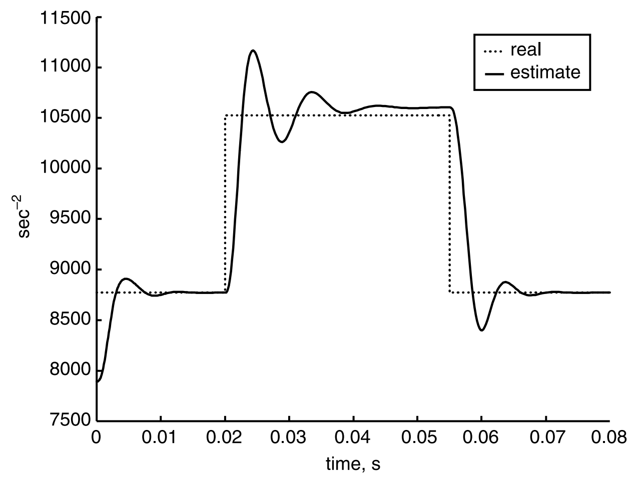

Fig. 19 *K_7 and its variation*

From the Figures we can say that the exact linearisation presents some degradation in the quality of the responses, i.e. a large overshoot, more oscillations and steady-state errors. On the other hand, the proposed observer-based control strategy exhibits the best transient behaviour and works well for different parameter and operating conditions.

8 Conclusions

In this paper, a controller-observer scheme has been presented and studied for a class of nonlinear singularly perturbed systems, where the dynamics are jointly linear in the fast variables and the control inputs, but nonlinear in the slow state variables. Assuming that the state of the system is available, a composite control is first designed using the sliding-mode technique in such a way that the closed-loop system is locally exponentially stable.

Considering that the system's output is a function of the slow state variable, the fast state variable is available and the slow system is observable, an observer has been designed to estimate the slow variable exponentially. In the paper, a set of sufficient conditions is given under which the exponential stability of the closed-loop system, together with the estimation error dynamics of the observer, can be guaranteed. These conditions are expressed in terms of those that assure the exponential stability of the closed-loop system without an observer and the exponential stability of the observer. Using the permanent-magnet stepper motor model with uncertainties in the load torque and detent torque, the controller–observer design has been illustrated and a comparative study with the exact linearisation control was carried out. Better results were obtained with the proposed scheme. An extension of the methodology proposed here would be to study a more general class of nonlinear singularly perturbed systems where the whole state is not available.

9 Acknowledgment

This work is supported by PAICYT, Mexico, under grant CA-866-04.

10 References

1 Saberi, A., and Khalil, H.K.: 'Quadratic-type Lyapunov functions for singular perturbed systems', *IEEE Trans. Autom. Control*, 1984, **29**, (6), pp. 542–550

2 Wilkelman, J.R., Chow, J.H., Allemong, J.J., and Kokotovic, P.V.: 'Multi-time-scale analysis of a power systems', *Automatica*, 1980, **16**, pp. 35–43

3 Xu, X., Mathur, R.M., Jiang, J., Rogers, G.J., and Kundur, P.: 'Modeling of generators and theirs controls in power systems simulations using singular perturbations', *IEEE Trans. Power Syst.*, 1998, **13**, (1), pp. 109–114

4 Khalil, H.K.: 'Nonlinear systems' (Mcmillan Publishing Company, 1996, 2nd edn.)

5 Kokotovic, P.V., Khalil, H.K., and O'Reilly, J.: 'Singular perturbation methods in control: analysis and design' (Academic Press, London, 1986)

6 Decarlo, R.A., Zak, S.H., and Matthews, G.P.: 'Variable structure control of nonlinear multivariable systems: a tutorial', *Proc. IEEE*, 1988, **76**, (3), pp. 212–233

7 Utkin, V.I.: 'Sliding modes in control and optimization' (Springer-Verlag, Berlin, Germany, 1992)

8 Sira-Ramirez, H., Spurgeron, S.K., and Zinober, A.S.I.: 'Robust observer-controller design for linear systems', *Lect. Notes Control Inf. Sci.*, 1994, **193**, pp. 161–180

9 Bornard, G., Celle-Couenne, F., and Gilles, G.: 'Observability and observers', in Fossard, A.J., and Normand-Cyrot, D. (Eds.): 'Systèmes non linéaires' (Masson, Paris, 1993), (1), pp. 177–221

10 Bodson, B., Chiason, N., Novotnak, R.T., and Rokowski, R.B.: 'High-performance non-linear feedback control of a permanent magnet stepper motor', *IEEE Trans. Control Syst. Tech.*, 1993, **1**, (1), pp. 5–14

11 Kenjo, T.: 'Stepping motors and their microprocessor controls' (Clarendon, Oxford, UK, 1984)

12 Chiasson, J.N., and Novotnak, R.T.: 'Nonlinear speed observer for the PM stepper motor', *IEEE Trans. Autom. Control.*, 1993, **38**, (10), pp. 1584–1588

13 Gauthier, J.P., Hammouri, H., and Othman, S.: 'A simple observer for nonlinear systems: application to bioreactors', *IEEE Trans. Autom. Control*, 1992, **37**, pp. 875–880

14 Vidyasagar, M.: 'Nonlinear systems analysis' (Prentice Hall, Englewood Cliffs, NJ, USA, 1993, 2nd edn.)

15 Furuhashi, T., Sangwongwanchi, S., and Okuma, S.: 'A position-and-velocity sensorless control for brushless DC motors using adaptive sliding mode observer', *IEEE Trans. Ind. Electron.*, 1992, **39**, (2), pp. 89–95

16 Ghanes, M., De Leon, J., and Glumineau, A.: 'Validation of an interconnected high gain observer for sensorless induction motor on low frequencies benchmark: application to an experimental set-up', *IEE Proc. Control Theory Appl.*, 1999, **152**, (4)

17 Zribi, M., and Chiasson, J.: 'Position control of a PM stepper motor by exact linearization', *IEEE Trans. Autom. Control*, 1991, **36**, (5), pp. 620–625

11 Appendix

We consider a more sophisticated model of the PM stepper motor for which we develop a control strategy based on singular perturbation methods and sliding mode techniques. The equations describing the dynamical behaviour of the PM stepper motor model are given by (see [17])

$$\begin{aligned}
v_a &= Ri_a + \frac{dL_A}{dt} + \frac{d\phi_A}{dt}\\
v_b &= Ri_b + \frac{dL_B}{dt} + \frac{d\phi_B}{dt}\\
J\frac{d\omega}{dt} &= i_a + \frac{d\phi_A}{d\theta} + i_b\frac{d\phi_B}{d\theta} - K_d\sin(4Nr\theta) - B\omega - \tau_L\\
\frac{d\theta}{dt} &= \omega
\end{aligned}$$

where $\phi_A = \phi_A(\theta)$ is the flux in phase a due to the permanent magnet rotor, $\phi_B = \phi_B(\theta)$ is the flux in phase b due to the permanent magnet rotor, $L_A = L_A(i_a, i_b, \theta)$ is the flux in phase a due to $i_a, i_b, L_B = L_B(i_a, i_b, \theta)$ is the flux in phase b due to i_a, i_b.

On the other hand, assuming that the matrix

$$N = \begin{pmatrix} \frac{\partial L_A}{\partial i_a} & \frac{\partial L_A}{\partial i_b} \\ \frac{\partial L_B}{\partial i_a} & \frac{\partial L_B}{\partial i_b} \end{pmatrix}$$

is nonsingular, then the above model can be represented as follows

$$\begin{pmatrix} \frac{d\omega}{dt} \\ \frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{K_d}{J}\sin(4N_r\theta) - \frac{B}{J}\omega - \frac{\tau_L}{J} \\ \omega \end{pmatrix} + \begin{pmatrix} \frac{1}{J}\frac{d\phi_A}{d\theta}\frac{1}{J}\frac{d\phi_B}{d\theta} 0 0 \end{pmatrix}\begin{pmatrix} i_a \\ i_b \end{pmatrix}$$

$$\begin{pmatrix} \frac{di_a}{dt} \\ \frac{di_b}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial L_A}{\partial i_a} & \frac{\partial L_A}{\partial i_b} \\ \frac{\partial L_B}{\partial i_a} & \frac{\partial L_B}{\partial i_b} \end{pmatrix}^{-1} \begin{pmatrix} v_a - Ri_a - \left(\frac{\partial L_A}{\partial\theta} + \frac{d\phi_A}{d\theta}\right)\omega \\ v_b - Ri_b - \left(\frac{\partial L_B}{\partial\theta} + \frac{d\phi_B}{d\theta}\right)\omega \end{pmatrix}$$

Furthermore, in order to represent this system in a singular perturbed form, we assume that the components of the matrix N are given by $\frac{\partial L_A}{\partial i_a} = \varepsilon\varphi_1, \frac{\partial L_A}{\partial i_b} = \varepsilon\varphi_2, \frac{\partial L_B}{\partial i_a} = \varepsilon\varphi_3, \frac{\partial L_B}{\partial i_b} = \varepsilon\varphi_4$, where ε is a small parameter, such that the second equation can be written as

$$\varepsilon\begin{pmatrix} \frac{di_a}{dt} \\ \frac{di_b}{dt} \end{pmatrix} = M^{-1}\begin{pmatrix} v_a - Ri_a - \left(\frac{\partial L_A}{\partial\theta} + \frac{d\phi_A}{d\theta}\right)\omega \\ v_b - Ri_b - \left(\frac{\partial L_B}{\partial\theta} + \frac{d\phi_B}{d\theta}\right)\omega \end{pmatrix}$$

where $M = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix}$. Defining $x = \text{col}(\omega - \omega^*, \theta - \theta^*)$, $z = \text{col}(i_a - i_a^*, i_b - i_b^*)$ and making the assignment

$v_a - v_a^* = -\sin(N_r(\theta+\theta^*))u, v_b - v_b^* = \cos(N_r(\theta+\theta^*))u$, where u is the new scalar control input, the system can be expressed as

$$\begin{aligned} \dot{x} &= f_1(x) + F_1(x)z \\ \varepsilon\dot{z} &= G(x,z,u) \end{aligned} \tag{35}$$

where

$$f_1(x) = \begin{pmatrix} -\frac{K_d}{J}\sin(4N_r(x_2+\theta^*)) - \frac{B}{J}x_1 - \frac{\tau_L}{J} + \frac{1}{J}\frac{d\phi_A}{d\theta}i_a^* + \frac{1}{J}\frac{d\phi_B}{d\theta}i_b^* \\ x_1 \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} \frac{1}{J}\frac{d\phi_A}{d\theta} & \frac{1}{J}\frac{d\phi_B}{d\theta} \\ 0 & 0 \end{pmatrix},$$

$$G(x,z,u) = M^{-1}\begin{pmatrix} -Rz_1 - \left(\frac{\partial L_A}{\partial\theta} + \frac{d\phi_A}{d\theta}\right)x_1 - \sin(N_r(x_2+\theta^*))u \\ -Rz_2 - \left(\frac{\partial L_B}{\partial\theta} + \frac{d\phi_B}{d\theta}\right)x_1 + \cos(N_r(x_2+\theta^*))u \end{pmatrix}$$

Remark 3: It is clear that this structure is nonlinear and different from those considered in this paper. However, the nonlinearities are found in the fast subsystem. In fact, as we will see, this difficulty can be overcome in the control design.

Now, to determine the slow subsystem, take $\varepsilon = 0$, then it follows that the roots of $G(x,z,u) = 0$ depend on the terms $\frac{\partial L_A}{\partial x_2} = \frac{\partial L_A(z_1,z_2,x_2)}{\partial x_2}$ and $\frac{\partial L_B}{\partial x_2} = \frac{\partial L_B(z_1,z_2,x_2)}{\partial x_2}$. Assuming that there exists a unique root and following the same ideas as in [5], the unique roots of $G(x,z,u) = 0$ are of the form $z = h(x,u)$, i.e.

$$\begin{aligned} z_1 &= h_1(x_1,x_2,u) = H_1(x_1,x_2) + H_2(x_1,x_2)u \\ z_2 &= h_2(x_1,x_2,u) = H_3(x_1,x_2) + H_4(x_1,x_2)u \end{aligned}$$

Replacing the roots of z in (35), it follows that

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{K_d}{J}\sin(4N_r(x_2+\theta^*)) - \frac{B}{J}x_1 - \frac{\tau_L}{J} + \frac{1}{J}\frac{d\phi_A(x_2)}{dx_2}i_a^* + \frac{1}{J}\frac{d\phi_B}{d\theta}i_b^* + \Psi_1 \\ x_1 \end{pmatrix}, + \begin{pmatrix} \Psi_2 \\ 0 \end{pmatrix}u$$

where $\Psi_1 = \frac{1}{J}\left(\frac{d\phi_A(x_2)}{dx_2}H_1(x_1,x_2) + \frac{d\phi_B(x_2)}{dx_2}H_3(x_1,x_2)\right)$ and $\Psi_2 = \frac{1}{J}\left(\frac{d\phi_A(x_2)}{dx_2}H_2(x_1,x_2) + \frac{d\phi_B(x_2)}{dx_2}H_4(x_1,x_2)\right)u$.

On the other hand, introducing the deviation $\eta := z - h(x,u)$ and transforming the time scale $\tau := (t-t_0)/\varepsilon$, since $h_1(x_1,x_2,u)$ and $h_2(x_1,x_2,u)$ are the roots of $G(x,z,u) = 0$, it follows that the fast subsystem is given by

$$\begin{pmatrix} \frac{d\eta_1}{d\tau} \\ \frac{d\eta_2}{d\tau} \end{pmatrix} = M^{-1}\begin{pmatrix} -\sin(N_r(x_2+\theta^*))u_f - R\eta_1 \\ \cos(N_r(x_2+\theta^*))u_f - R\eta_2 \end{pmatrix}$$

Taking $u_f = 0$, the resulting system has the form

$$\frac{d\eta}{d\tau} = -RM^{-1}\eta$$

To prove the stability of such a system, let $V(\eta) = \eta^T M\eta$ be a candidate Lyapunov function. The time derivative along the trajectories of the system is given by $\dot{V}(\eta) = -2\eta^T\eta \le -\mu V(\eta) < 0$. Hence, the fast subsystem is exponentially stable.

The slow control can be obtained using the following slow nonlinear switching function:

$$\sigma_s(x_s) = s_1 x_{s_1} + s_2 x_{s_2}$$

This choice leads, in accordance with Section 2, to the slow control

$$u_s(x_s) = u_{se}(x_s) + u_{s_N}(x_s)$$

with

$$u_{se} = \frac{-1}{\Psi_2}\left\{ -\frac{K_d}{J}\sin(4N_r(x_2+\theta^*)) - \frac{B}{J}x_1 - \frac{\tau_L}{J} + \frac{s_2}{s_1}x_1 + \frac{1}{J}\frac{d\phi_A(x_2)}{dx_2}i_a^* + \frac{1}{J}\frac{d\phi_B(x_2)}{dx_2}i_b^* + \Psi_1 \right\}$$

$$u_{s_N} = -\frac{L_s}{s_1\Psi_2}(s_1\omega + s_2(x_2+\theta^*))$$

To compare with the controller obtained with the simplified model, we consider that

$$\frac{d\phi_A}{d\theta} = -K_m\sin(N_r(x_2+\theta^*)),$$

$$\frac{d\phi_B}{d\theta} = K_m\cos(N_r(x_2+\theta^*)), \quad \frac{\partial L_A}{\partial i_a} = L, \quad \frac{\partial L_B}{\partial i_b} = L,$$

$$\frac{\partial L_A}{\partial i_b} = 0, \quad \frac{\partial L_B}{\partial i_a} = 0, \quad \frac{dL_A}{d\theta} = 0, \quad \frac{dL_B}{d\theta} = 0,$$

then, it follows that

$$\begin{aligned} H_1(x_1,x_2) &= \frac{K_m}{R}\sin(N_r(x_2+\theta^*))x_1, \\ H_2(x_1,x_2) &= -\frac{1}{R}\sin(N_r(x_2+\theta^*)), \\ H_3(x_1,x_2) &= -\frac{K_m}{R}\cos(N_r(x_2+\theta^*))x_1, \\ H_4(x_1,x_2) &= \frac{1}{R}\cos(N_r(x_2+\theta^*)). \end{aligned}$$

and

$$\Psi_1 = -\frac{1}{J}\frac{K_m^2}{R}x_1, \quad \Psi_2 = \frac{1}{J}\frac{K_m}{R}.$$

Finally, using these expressions, the controller for the simplified system is given by

$$\begin{aligned} u_{se} &= \frac{K_1}{K_4}\left[\left(\frac{K_4K_2}{K_1} + K_5 - \frac{s_2}{s_1}\right)x_{s_1} + K_7\right] \\ &\quad - [v_b^*\cos(\alpha) - v_a^*\sin(\alpha)] \\ u_{s_N} &= -\frac{K_1L_s}{s_1K_4}(s_1x_{s_1} + s_2x_{s_2}) \end{aligned}$$

where $K_1 = R$, $K_2 = K_m$, $K_3 = N_r$, $K_4 = K_m/J, K_5 = B/J$, $K_6 = K_d/J, K_7 = \tau_l/J$, $\alpha = K_3(x_2+\theta^*)$.