Analysis of Multivariable Functions

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These are meant to be notes on real functions of several variables, if possible, there will be enough generality to make the results applicable in any complete metric space over a field. We assume all basic linear algebra knowledge (basis, linear independence, vector space axioms, dimension of vector spaces, etc.).

1 Functions of Several Variables

We start our exposition by defining the space of all linear transformations between two vector spaces.

Definition 1.1. (a) Let $L(X, Y)$ be the set of all linear transformations of the vector space X into the vector space Y. For Linear Operators, the domain and codomain are the same, so we shall write $L(X)$. If A_1, A_2 $\in L(X, Y)$ and if c_1, c_2 are scalars, define

$$(c_1A_1 + c_2A_2)\boldsymbol{x} = c_1A_1\boldsymbol{x} + c_2A_2\boldsymbol{x}$$

Where $\boldsymbol{x} \in X$. Then we clearly have that: $c_1A_1 + c_2A_2 \in L(X, Y)$. So we have closure under linear combinations.

(b) If X, Y, Z are vector spaces, and if $A \in L(X, Y)$ and $B \in L(Y, Z)$, we define the product of two linear operators to be:

$$BA\boldsymbol{x} = B(A\boldsymbol{x})$$

This shows us that $BA \in L(X, Z)$.

(c) For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, define the **norm** of A as follows:

$$\|A\| = \left\{ \sup_{\boldsymbol{x}\in\mathbb{R}^n} |A\boldsymbol{x}| : |\boldsymbol{x}| \leq 1 \right\}$$

The inequality:

$$|A\boldsymbol{x}| \leq \|A\| \, |\boldsymbol{x}|$$

holds for any $\boldsymbol{x} \in \mathbb{R}^n$. If λ is such that $|A\boldsymbol{x}| \leq \lambda \, |\boldsymbol{x}|$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $\|A\| \leq \lambda$

We may immediately see that the space of all linear transformations from X to Y: $L(X,Y)$, is indeed a **normed vector space**. This allows us to talk about linear maps, not as functions acting on elements of a vector space, but as elements themselves.

We present our first theorem here:

Theorem 1. *We will begin with some results about real-valued multivariate functions.*

(a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| \leq \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ onto $\mathbb{R}^m$.

(b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$, and c is a scalar, then:

$$\|A + B\| \leq \|A\| + \|B\|$$

$$\|cA\| = |c| \, \|A\|$$

The distance between A and B is defined as $\|A - B\|$, making $L(\mathbb{R}^n, \mathbb{R}^m)$ a metric space.

(c) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then:

$$\|BA\| \leq \|B\| \, \|A\|$$

Proof. (a) Let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbb{R}^n$ and let $\boldsymbol{x} = \sum c_i e_i$, and $|\boldsymbol{x}| \leq 1$ so that each $|c_i| \leq 1$. Then:

$$|A\boldsymbol{x}| = \left| A \sum c_i e_i \right| = \left| \sum c_i A e_i \right| \leq \sum |c_i| \, |Ae_i|$$

Where we use the triangle inequality in the last inequality. Continuing this:

$$\sum |c_i| \, |Ae_i| \leq \sum |Ae_i| \leq \sum \|A\| \, |e_i| = \|A\| \sum |e_i| = n \, \|A\|$$

The norm of A is bounded as we have that:

$$\|A\| \leq \sum |Ae_i| < \infty$$

The uniform continuity follows as we may simply use:

$$|A\boldsymbol{x} - A\boldsymbol{y}| \leq \|A\| \, |\boldsymbol{x} - \boldsymbol{y}|$$

So if we choose $|\boldsymbol{x} - \boldsymbol{y}| < \delta$, then we may choose $\varepsilon = \|A\| \, \delta$ so that:

$$|A\boldsymbol{x} - A\boldsymbol{y}| < \varepsilon$$

Thus, A is uniformly continuous on $\mathbb{R}^n$.

(b) This is just a computation.

$$|(A+B)\boldsymbol{x}| = |(A+B)\boldsymbol{x}| \leq |A\boldsymbol{x}| + |B\boldsymbol{x}| \leq (\|A\| + \|B\|)\,|\boldsymbol{x}|$$

We may also verify the triangle inequality:

$$\|A-C\| = \|(A-B)+(B-C)\| \leq \|A-B\| + \|B-C\|$$

in a similar fashion. Take:

$$|(A-C)\boldsymbol{x}| = |(A-B)\boldsymbol{x} + (B-C)\boldsymbol{x}| \leq |(A-B)\boldsymbol{x}| + |(B-C)\boldsymbol{x}| \leq (\|A-B\| + \|B-C\|)\,|\boldsymbol{x}|$$

But we know that:

$$|(A-C)\boldsymbol{x}| \leq \|A-C\|\,|\boldsymbol{x}|$$

Thus, we have that:

$$\|A-C\| \leq \|A-B\| + \|B-C\|$$

(c) We also verify this in a similar fashion as the previous part:

$$|BA\boldsymbol{x}| = |B(A\boldsymbol{x})| \leq \|B\|\,|A\boldsymbol{x}| \leq \|B\|\,\|A\|\,|\boldsymbol{x}|$$

However, we again have that:

$$|BA\boldsymbol{x}| \leq \|BA\|\,|x|$$

So the inequality:

$$\|BA\| \leq \|B\|\,\|A\|$$

follows.

The metric $\|\cdot\|$ on our space of linear transformations gives us all the notions of topology in our function space now. This will help us develop our understanding of spaces of functions rather than viewing functions as maps that act on elements. Our proofs here connect the view of functions as mappings of elements and as elements in their own right.

□

Theorem 2. *Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$.*

(a) If $A \in \Omega$, $B \in L(\mathbb{R}^n)$, and

$$\|B-A\| \cdot \left\|A^{-1}\right\| < 1$$

then $B \in \Omega$.

(b) Ω is an open subset of $L(\mathbb{R}^n)$, and the mapping $A \longmapsto A^{-1}$ is continuous on Ω.

Proof. Let $\left\|A^{-1}\right\| = 1/\alpha$, put $\|B - A\| = \beta$, and the condition given in the hypothesis may be rewritten as:

$$\|B - A\| \cdot \left\|A^{-1}\right\| < 1$$

Is equivalent to

$$\frac{\beta}{\alpha} < 1$$

Or

$$\beta < \alpha$$

Consider the first step:

$$\alpha\,|\boldsymbol{x}| = \alpha\left|A^{-1}A\boldsymbol{x}\right| \leq \alpha\left\|A^{-1}\right\||A\boldsymbol{x}| = |A\boldsymbol{x}|$$

$$\leq |(A - B)\boldsymbol{x} + B\boldsymbol{x}| \leq |(A - B)\boldsymbol{x}| + |B\boldsymbol{x}| \leq \beta\,|\boldsymbol{x}| + |B\boldsymbol{x}|$$

This implies that:

$$(\alpha - \beta)\boldsymbol{x} \leq |B\boldsymbol{x}|$$

$\alpha - \beta > 0$, therefore, we have that $|B\boldsymbol{x}| = 0$ if and only if $\boldsymbol{x} = 0$, so that B is necessarily injective. By the **Rank-Nullity Theorem**, we have that B must also be surjective, so that $B \in \Omega$. We must now prove continuity. Take $\boldsymbol{x} = B^{-1}\boldsymbol{y}$ and substitute into the above inequality.

$$(\alpha - \beta)\left|B^{-1}\boldsymbol{y}\right| \leq \left|BB^{-1}\boldsymbol{y}\right| = |\boldsymbol{y}|$$

$$(\alpha - \beta)\left\|B^{-1}\right\||\boldsymbol{y}| \leq |\boldsymbol{y}|$$

Therefore, we have that:

$$\left\|B^{-1}\right\| \leq \frac{1}{\alpha - \beta}$$

Then we recognize that $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$ and we substitute:

$$\left\|B^{-1} - A^{-1}\right\| \leq \left\|B^{-1}\right\|\|A - B\|\left\|A^{-1}\right\| \leq \frac{1}{\alpha - \beta}\beta(1/\alpha)$$

$$\left\|B^{-1} - A^{-1}\right\| \leq \frac{\beta}{\alpha(\alpha - \beta)}$$

By definition of β, if we take $B \longmapsto A$, then we obtain that $\beta \longmapsto 0$ meaning that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that:

$$\|B - A\| < \delta$$

implies that

$$\left\|B^{-1} - A^{-1}\right\| < \varepsilon$$

Where δ is fixed, and our choice of ε is $\varepsilon = \frac{\delta}{\alpha(\alpha-\delta)}$. Therefore, the mapping $A \longmapsto A^{-1}$ is uniformly continuous. $\square$

We will skip Rudin's discussion of matrices as we presume that whoever is following this already has the satisfactory linear algebra knowledge. All we need to know is that the matrix representation of a linear transformation $A : V \longrightarrow W$ is an $dim(W) \times dim(V)$ matrix. We will state the fact below:

Definition 1.2. If S is a metric space, and if $a_{11}, ..., a_{mn}$ are real continuous functions on S, and if, for each $p \in S$, A_p is the linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$, then the mapping $p \longmapsto A_p$ is a continuous mapping of S into $L(\mathbb{R}^n, \mathbb{R}^m$.

We will take this as a definition of a correspondence between matrices and linear transformations. If one wants to prove this, we simply need to show that for any $|p - p_0| < \delta$, it implies $\|A_p - A_{p_0}\| < \varepsilon$ This amounts to using the continuity of each entry of A_p and showing that this epsilon-delta definition holds.

We will now discuss derivatives of a function of multiple variables.

Recall that derivatives of a function defined on an open subset $U \subset \mathbb{R}$ are defined as follows:

Definition 1.3. Definition of a Derivative of single-variable function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We may refine this definition slightly more, as the **best linear approximation** to a function at a point x.

Definition 1.4. Definition of a Derivative of single-variable function

$$f(x+h) - f(x) = f'(x) + r(h)$$

Where

$$\lim_{h \to 0} \frac{r(h)}{h} = 0$$

Notice that our second definition expresses the idea that the derivative is the best linear approximation to a function $f(x)$. We may further refine our definition so that it generalizes to higher variables.

Definition 1.5. Definition of a Derivative of single-variable function

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - hf'(x)}{h} = 0$$

We may also write:

$$\lim_{h \to 0} \left| \frac{f(x+h) - f(x) - hf'(x)}{h} \right| = 0$$

Definitions 1.3, 1.4, 1.5 are all equivalent as verified through quick computations. Definition 1.5 is the definition we will opt for, as this will clearly generalize to higher derivatives.

Definition 1.6. Derivative of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

A function $\boldsymbol{f} : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that there exists a **Linear Transformation** $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$:

$$\lim_{h \to 0} \frac{|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - A\boldsymbol{h}|}{|\boldsymbol{h}|}$$

is called **differentiable** at $\boldsymbol{x}$, and the linear transformation A is called the **Derivative of $\boldsymbol{f}$ at the point $\boldsymbol{x}$**:

$$A = \boldsymbol{f}'(\boldsymbol{x})$$

Notice that the numerator of our quotient is in the codomain $\mathbb{R}^m$, and the denominator is in the domain $\mathbb{R}^n$.

It is not trivial to see that derivatives are unique (up to a constant).

Theorem 3. *Suppose that E and $\boldsymbol{f}$ are as in the previous definition. If we have two linear transformations A_1, A_2 such that they are both the derivative of the function $\boldsymbol{f}$, then $A_1 = A_2$.*

Proof. If $B = A_1 - A_2$, let us recognize that:

$$B\boldsymbol{h} = f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_1\boldsymbol{h} - (f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_2\boldsymbol{h})$$

If we take the absolute value on both sides (the norm of $\mathbb{R}^n$):

$$|B\boldsymbol{h}| = |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_1\boldsymbol{h} - (f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_2\boldsymbol{h})|$$

Invoke the triangle inequality:

$$\leq |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_1\boldsymbol{h}| + |-(f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_2\boldsymbol{h})|$$

$$= |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_1\boldsymbol{h}| + |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_2\boldsymbol{h}|$$

So we have:

$$|B\boldsymbol{h}| \leq |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_1\boldsymbol{h}| + |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_2\boldsymbol{h}|$$

Then divide both sides of this inequality by $|\boldsymbol{h}|$. Then we obtain:

$$\frac{|B\boldsymbol{h}|}{|\boldsymbol{h}|} \leq \frac{|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_1\boldsymbol{h}|}{|\boldsymbol{h}|} + \frac{|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - A_2\boldsymbol{h}|}{|\boldsymbol{h}|}$$

As both A_1 and A_2 are derivatives of $\boldsymbol{f}$, it follows that the right hand side of the inequality goes to 0 as $\boldsymbol{h} \to 0$ and that $\frac{|B\boldsymbol{h}|}{\boldsymbol{h}} \to 0$.

Likewise, if we have that for $t \to 0$:

$$\frac{|B(t\boldsymbol{h})|}{|t\boldsymbol{h}|} \to 0$$

for a nonzero fixed $\boldsymbol{h}$, then by linearity, we may pull out the t variable and it will cancel:

$$\frac{|B(t\boldsymbol{h})|}{|t\boldsymbol{h}|} = \frac{|tB(\boldsymbol{h})|}{|t\boldsymbol{h}|} = \frac{|B(\boldsymbol{h})|}{|\boldsymbol{h}|} \to 0$$

Because this expression is independent of t, and $\boldsymbol{h} \neq 0$, $B = 0$, and that proves that $A_1 = A_2$, hence the derivative of a function is unique. $\square$

The A in our above discussion is what is called a **Total Derivative** (or the Jacobian if we translate it into matrices), and we may equally just denote it as $\boldsymbol{f}'(\boldsymbol{x})$. To ease confusion, there are two ways to look at the **derivative of a function**: (i) We may view the derivative as a linear map $\boldsymbol{f}' : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (ii) We may view the derivative as a function $\boldsymbol{f}' : E \subset \mathbb{R}^n \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$. That is, it maps a vector in E into a linear map in $L(\mathbb{R}^n, \mathbb{R}^m)$.

As quick side note, a derivative of A is easily found to be:

$$A'(\boldsymbol{x}) = A$$

for any $\boldsymbol{x} \in \mathbb{R}^n$. This is because:

$$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{x})\boldsymbol{h} + r(\boldsymbol{h})$$

If we evaluate the left hand side for $\boldsymbol{f} = A$, we see that:

$$A(\boldsymbol{x}+\boldsymbol{h}) - A\boldsymbol{x} = A\boldsymbol{h} = \boldsymbol{f}'(\boldsymbol{x})\boldsymbol{h} + r(\boldsymbol{h})$$

Because $r(\boldsymbol{h}) = 0$ as $h = 0$, we have that:

$$\lim_{h \to 0} \frac{|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x})|}{|\boldsymbol{h}|} = 0$$

Exactly from plugging in $A = \boldsymbol{f}$. So the derivative of A is itself.

We now see what the chain rule is for the total derivative of a function $\boldsymbol{f} : E \longrightarrow \mathbb{R}^m$.

Theorem 4. *Chain Rule (Total Derivatives)*

Suppose that E is an open set in $\mathbb{R}^n$, $\boldsymbol{f} : E \longrightarrow \mathbb{R}^m$, $\boldsymbol{f}$ is differentiable at $\boldsymbol{x}_0 \in E$, $\boldsymbol{g} : \boldsymbol{f}(E) \subset U \longrightarrow \mathbb{R}^k$, and $\boldsymbol{g}$ is differentiable at $\boldsymbol{f}(\boldsymbol{x}_0)$. then the mapping $\boldsymbol{F} : E \longrightarrow \mathbb{R}^k$ defined by:

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$$

is differentiable at $\boldsymbol{x}_0$, and

$$\boldsymbol{F}'(\boldsymbol{x}_0) = \boldsymbol{g}'(\boldsymbol{f}(\boldsymbol{x}_0)\boldsymbol{f}'(\boldsymbol{x}_0)$$

Proof. This is a long proof with lots of intricate parts. Let us see what it's all about!

Let us follow Rudin's proof and define the following:

$$\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$$
$$A = \boldsymbol{f}'(\boldsymbol{x}_0)$$
$$B = \boldsymbol{g}'(\boldsymbol{y}_0)$$
$$u(\boldsymbol{h}) = \boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}_0) - A\boldsymbol{h}$$
$$v(\boldsymbol{k}) = \boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}_0) - B\boldsymbol{k}$$

Where $\boldsymbol{h} \in \mathbb{R}^n$ and $\boldsymbol{k} \in \mathbb{R}^m$. Then we define:

$$\varepsilon(\boldsymbol{h}) = \frac{|u(\boldsymbol{h})|}{|\boldsymbol{h}|} = \frac{|\boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}_0) - A\boldsymbol{h}|}{|\boldsymbol{h}|}$$
$$\eta(\boldsymbol{k}) = \frac{|v(\boldsymbol{k})|}{|\boldsymbol{k}|} = \frac{|\boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}_0) - B\boldsymbol{k}|}{|\boldsymbol{k}|}$$

As $\boldsymbol{f}$ and $\boldsymbol{g}$ are both differentiable, we see that ε and η must both go to 0 as the respective parameters $\boldsymbol{h}, \boldsymbol{k}$ go to zero.

Now, fix $\boldsymbol{h}$ and let $\boldsymbol{k} = \boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}_0)$. Recognize that:

$$\boldsymbol{k} = u(\boldsymbol{h}) + A\boldsymbol{h}$$
$$|\boldsymbol{k}| = |u(\boldsymbol{h}) + A\boldsymbol{h}| = |\varepsilon(\boldsymbol{h})\boldsymbol{h} + A\boldsymbol{h}| \leq |\varepsilon(\boldsymbol{h}) + A|\,|\boldsymbol{h}|$$
$$\leq (\varepsilon(\boldsymbol{h}) + \|A\|)\,|\boldsymbol{h}|$$

With everything given now, we compute the composite:

$$\boldsymbol{F}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x}_0) - BA\boldsymbol{h} = \boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}_0) - BA\boldsymbol{h} = B(\boldsymbol{k} - A\boldsymbol{h}) + v(\boldsymbol{k})$$

Recognize that

$$\boldsymbol{k} = u(\boldsymbol{h}) + A\boldsymbol{h}$$

So that the above composition becomes:

$$\boldsymbol{F}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x}_0) - BA\boldsymbol{h} = Bu(\boldsymbol{h}) + v(\boldsymbol{k})$$

If we take the norm of both sides:

$$|\boldsymbol{F}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x}_0) - BA\boldsymbol{h}| = |Bu(\boldsymbol{h}) + v(\boldsymbol{k})| \leq |Bu(\boldsymbol{h})| + |v(\boldsymbol{k})|$$
$$\leq \|B\|\,|u(\boldsymbol{h})| + \eta(\boldsymbol{k})\,|\boldsymbol{k}| \leq \|B\|\,\varepsilon(\boldsymbol{h})\,|\boldsymbol{h}| + \eta(\boldsymbol{k})(\varepsilon(\boldsymbol{h}) + \|A\|)\,|\boldsymbol{h}|$$

Carry the $|\boldsymbol{h}|$ to the other side, and we obtain:

$$\frac{|\boldsymbol{F}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x}_0) - BA\boldsymbol{h}|}{|\boldsymbol{h}|} \leq \|B\|\,\varepsilon(\boldsymbol{h}) + \eta(\boldsymbol{k})(\varepsilon(\boldsymbol{h}) + \|A\|)$$

Everything on the right hand side goes to 0 as we take $\boldsymbol{h} \to 0$, meaning:

$$\lim_{h \to 0} \frac{|\boldsymbol{F}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x}_0) - BA\boldsymbol{h}|}{|\boldsymbol{h}|} = 0$$

Therefore, this implies that $\boldsymbol{F} = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))$, with a derivative $\boldsymbol{F}'(\boldsymbol{x}_0) = BA$. We are done. $\square$

We have discussed all the preliminaries for the total derivative of a function. We now explore partial derivatives in detail. NOTE: We are going to stop bold facing vectors and vector functions, it should be clear from context what is being used.

Definition 1.7. Partial Derivative

Consider $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. $\mathbb{R}^n$ has a standard basis $\{e_1, ..., e_n\}$, $\mathbb{R}^m$ has a standard basis $\{u_1, ..., u_m\}$.

$$f(x) = \sum_{j=1}^{m} f_j(x) u_j$$

The **Partial Derivative** is defined as:

$$(D_j f_i)(x) = \lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t}$$

Where $1 \leq i \leq m$, $1 \leq j \leq n$ (we can remember these indices by recalling that the number of D_j is based on the number of coordinates in the domain, and the number of f_i is based on the number of basis vectors in the codomain.

We can also use the following notation:

$$\frac{\partial f_i}{\partial x_j} \equiv D_j f_i$$

We need to connect the derivative and the partial derivative together, we give a theorem that connects them.

Theorem 5. *Derivative Theorem*

Suppose $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, and f is differentiable at $x \in E$. Then the partial derivatives $(D_j f_i)(x)$ exist and:

$$f'(x) e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i$$

Proof. If we fix j, we will look at f and its derivative at x.

$$f(x + te_j) - f(x) = f'(x)(te_j) + r(te_j)$$

We have that:

$$|r(te_j)| / t \to 0$$

as $t \to 0$. We now rearrange the above expression:

$$\frac{f(x + te_j) - f(x)}{t} = f'(x) e_j$$

If we write f in components, we see that each component of f is of the form:

$$\lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t} u_i$$

The left hand side for each j and i is exactly the definition of $D_j f_i(x)$. So if we do:

$$\lim_{t \to 0} \sum_{i=1}^{m} \frac{f_i(x + te_j) - f_i(x)}{t} u_i = f'(x)e_j$$

Therefore, we obtain that:

$$f'(x)e_j = \sum_{i=1}^{m} D_j f_i(x) u_i$$

Note that $f'(x)e_j$ is the jth component of the derivative of f (recall that $f'(x)$ is a linear transformation of the best linear approximation of f at a point x). $\square$

We can show the matrix of the linear transformation $f'(x)$:

$$f'(x) = \begin{pmatrix} (D_1 f_1)(x) & \dots & (D_n f_1)(x) \\ \dots & \dots & \dots \\ (D_1 f_m)(x) & \dots & (D_n f_m)(x) \end{pmatrix}$$

We see that this is the matrix given as A_{ij} where i is the indices ranging the dimension of the codomain, and the j ranges the dimensions of the domain (we know this because the j is always attached to the partial derivatives, which depends on the number of coordinates in the domain).

If we want to act the derivative on any vector in $\mathbb{R}^n$, we see that it occurs as:

$$f'(x)h = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (D_j f_i)(x) h_j \right) u_i$$

We now look at partial derivatives in further detail. Consider a regular, piecewise continuous curve $\gamma : (a, b) \subset \mathbb{R} \longrightarrow E \subset \mathbb{R}^n$. And let $f : E \longrightarrow \mathbb{R}$. We define the composition as:

$$g(t) = f(\gamma(t))$$

Where $t \in (a, b)$. By the chain rule, we have that:

$$g'(t) = f'(\gamma(t))\gamma'(t)$$

To compute $g'(t)$, we need to make use of partial derivatives (notice that $\gamma'(t) \in L(\mathbb{R}, \mathbb{R}^n)$, and $f'(\gamma(t)) \in L(\mathbb{R}^n, \mathbb{R})$, so we can't simply compose these derivatives without taking into account the partial derivatives). We generalize this:

$$(\nabla f)(x) = \sum_{i=1}^{n} (D_i f)(x) e_i$$

We also have that

$$\gamma'(t) = \sum_{i=1}^{n} \gamma_i'(t) e_i$$

$$g'(t) = (\nabla f)(\gamma'(t)) \cdot \gamma'(t)$$

Let u be a unit vector in $\mathbb{R}^n$ so that:

$$\gamma(t) = x + tu$$

Then, we see that:

$$g'(0) = (\nabla f)(x) \cdot u$$

$$g(t) - g(0) = f(x + tu) - f(x)$$

Then, we see that by the definition of a derivative:

$$\lim_{t \to 0} \frac{g(t) - g(0)}{t} = g'(0)$$

Alternatively, with what we have said above:

$$\lim_{t \to 0} \frac{f(x + tu) - f(x)}{t} = (\nabla f)(x) \cdot u$$

Definition 1.8. Directional Derivative

The **Directional Derivative** is defined as:

$$\lim_{t \to 0} \frac{f(x + tu) - f(x)}{t} = (\nabla f)(x) \cdot u$$

This limit will often be denoted as:

$$(D_u f)(x) = \sum_{i=1}^{n} (D_i f)(x) u_i$$

Theorem 6. *Suppose $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, where E is a convex, open subset of $\mathbb{R}^n$. And suppose that f is differentiable in E, and there exists a real constant M such that:*

$$|f'(x)| \leq M$$

For every $x \in E$

Then:

$$|f(b) - f(a)| \leq M\,|b - a|$$

for all $a, b \in E$

Proof. Define $a, b \in E$, so that

$$\gamma(t) = (1 - t)a + tb$$

By convexity of E, this is guaranteed to be an element of E as long as $t \in [0, 1]$. Now, we can set:

$$g(t) = f(\gamma(t))$$

Then,

$$g'(t) = f'(\gamma(t))\gamma'(t)$$

Then, we obtain:

$$\gamma'(t) = b - a$$

Therefore,

$$g'(t) = f'(\gamma(t))(b - a)$$

Then, we obtain that:

$$|g'(t)| = |f'(\gamma(t))|\,|b - a|$$

We use the fact that the derivative of f is bounded (given by assumption) to write:

$$|g'(t)| \leq M\,|b - a|$$

And this holds for any $t \in [0, 1]$. We can invoke **Theorem 5.19** in Rudin, to give the last step:

$$|g(1) - g(0)| \leq |1 - 0|\,|g'(t)| = |g'(t)|$$

This shows that:

$$|g(1) - g(0)| \leq M\,|b - a|$$

And by definition of $\gamma(t)$, we have that:

$$|f(b) - f(a)| \leq M\,|b - a|$$

□

Corollary 7. *If we have that $f'(x) = 0$ for all $x \in E$, then f is constant.*

Proof. $M = 0$ in this case. The conclusion follows immediately. □

Definition 1.9. A differentiable mapping $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is C^1 if the derivative is continuous in E. Any function that is **continuously differentiable** is said **to be** C^1.

We will now present a string of theorems (and their proofs) that will be exceptionally important to the discussion of multivariable, real-valued functions. Listing them out, it will be in the following order: **Differentiability and Partial Differentiability, Banach Fixed Point Theorem (General Case), Inverse Function Theorem, Implicit Function Theorem**.

Theorem 8. *Necessity and Sufficiency of Being C^1*

Suppose $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. f is C^1 if and only if the partial derivatives $D_j f_i$ ($1 \leq i \leq m$, $1 \leq j \leq n$) exist and are continuous.

Proof. ($\longrightarrow$):

We assume that $f \in C^1$. Then, we have that:

$$(D_j f_i)(x) = (f'(x)e_j) \cdot u_i$$

Then we compute:

$$(D_j f_i)(x) - (D_j f_i)(y) = [f'(x) - f'(y)]\, e_j \cdot u_i$$

We have:

$$|(D_j f_i)(x) - (D_j f_i)(y)| \leq |[f'(x) - f('(y)]\, e_j| \leq |[f'(x) - f'(y)]|$$

As f is C^1 by assumption, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$|x - y| < \delta$$

Implies

$$|f'(x) - f'(y)| < \varepsilon$$

Therefore, this means that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$|x - y| < \delta$$

Implies

$$|(D_j f_i)(x) - (D_j f_i)(y)| \leq |[f'(x) - f'(y)]| < \varepsilon$$

($\longleftarrow$):

We now assume that the partial derivatives $D_j f_i$ exist and are continuous. We complete Rudin's proof, it suffices to consider only $m = 1$ as f is only continuously differentiable if each component is continuously differentiable, so we need only consider one component. Consider an open ball B centered at x in E, with radius r. We can choose $r = \varepsilon/n$ so that:

$$|(D_j f)(y) - (D_j f)(x)| < \frac{\varepsilon}{n}$$

Now we let $h = \sum h_j e_j$, and $|h| < r$. Let $v_0 = 0$, and $v_k = h_1 e_1 + ... + h_k e_k$ for $1 \leq k \leq n$.Then we have:

$$f(x + h) - f(x) = \sum_{j=1}^{n} (f(x + v_j) - f(x + v_{j-1}))$$

Since $|v_k| < r$ for any k, and by convexity of B, the segments $x+v_{j-1}, x+v_j \in B$. As $v_j = v_{j-1} + h_j e_j$, we have that the **Mean Value Theorem** applies to give us:

$$h_j (D_j f)(x + v_{j-1} + \theta_j h_j e_j)$$

for some $\theta_j \in (0,1)$.

$$\left| f(x+h) - f(x) - \sum_{i=1}^{n} h_j (D_j f)(x) \right| \leq \frac{1}{n} \sum_{i=1}^{n} |h_j| \, \varepsilon \leq |h| \, \varepsilon$$

This implies that $f'(x) = \sum_{i=1}^{n} (D_j f)(x)$. Since all the partial derivatives are continuous, we obtain that $f \in C^1$.

□

Definition 1.10. Let X be a metric space with metric d. If $\varphi : X \longrightarrow X$, and if there exists $c < 1$ such that:

$$d(\varphi(x), \varphi(y)) \leq cd(x,y)$$

Then φ is called a **contraction** of X into X.

Theorem 9. *Banach Fixed Point Theorem*

If X is a complete metric space, and if φ is a contraction of X into itself, then there exists a unique $x \in X$ such that $\varphi(x) = x$.

Proof. Existence and Uniqueness of Fixed Points of Contraction

The uniqueness is trivial. Let $\varphi(x) = x$ and $\varphi(y) = y$. Then,

$$d(\varphi(x), \varphi(y)) = d(x,y) \leq cd(x,y)$$

This can happen if and only if $d(x,y) = 0$, so $x = y$.

For existence, we pick $x_0 \in X$. Define $\{x_n\}$ by:

$$\varphi(x_n) = x_{n+1}$$

Choose any $c < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq cd(x,y)$$

Then:

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq cd(x_n, x_{n-1})$$

Applying this recursively gives us that:

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$$

For any $n < m$, it follows:

$$d(x_n, x_m) \leq \sum_{k=n+1}^{m} d(x_k, x_{k-1}) \leq (c^n + c^{n+1} + ... + c^{m-1}) d(x_1, x_0)$$

$$\leq \frac{1}{1-c} c^n d(x_1, x_0)$$

So that x_n forms a Cauchy sequence. By completeness of X, we can say that $\{x_n\} \longmapsto x$. As φ is a contraction, it is uniformly continuous by definition (so we may interchange the limit and the argument of the function). So we can easily say:

$$\varphi(x) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

$\square$

Theorem 10. *Inverse Function Theorem*

Suppose $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is C^1, and that $f'(a)$ is invertible for some $a \in E$, and $b = f(a)$. Then:

(a) There exists open sets $U, V \subset \mathbb{R}^n$ such that $a \in U$, $b \in V$, f is injective on U, and $f(U) = V$.

*(b) If g is the inverse of f (which exists by **Rank-Nullity Theorem** and the previous part), defined in V by:*

$$g(f(x)) = x$$

for all $x \in U$,

then $g \in C^1(V)$ (i.e. g is continuously differentiable on V).

Proof. (a) Put $f'(a) = A$, and choose a constant λ such that:

$$2\lambda \left\|A^{-1}\right\| = 1$$

Because f' is continuous at a, we may find an open ball $U \subset E$ centered at a such that, for any $x \in U$:

$$\|f'(x) - A\| < \lambda$$

We associate to each $y \in \mathbb{R}^n$, a function φ:

$$\varphi(x) = x + A^{-1}(y - f(x))$$

Where $x \in E$. Note that $f(x) = y$ if and only if x is a fixed point of φ, this is by definition of how we constructed this function, and not some special insight.

We have that the derivative of this is:

$$\varphi'(x) = I - A^{-1}f'(x) = A^{-1}(A - f'(x))$$

If we take the operator norm of this function, we obtain, for any $x \in U$:

$$\|\varphi'(x)\| = \left\|A^{-1}(A - f'(x))\right\| \le \left\|A^{-1}\right\| \|A - f'(x)\| < \frac{1}{2\lambda}\lambda = \frac{1}{2}$$

Thus, as U is an open ball and is automatically convex, we may apply **Theorem 6** (Theorem 9.19 in Rudin) to obtain that, for any $x_1, x_2 \in U$:

$$|\varphi(x_1) - \varphi(x_2)| \leq |x_2 - x_1| \, \|\varphi'(x)\| < \frac{1}{2} |x_1 - x_2|$$

Notice that this inequality implies that φ has, at most, **one fixed point**. Why? (This is not trivial to see). It is because, assume that we had two fixed points (any more fixed points we see that the following reasoning would equivalently apply), x_1, x_2 so that $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$:

$$|\varphi(x_1) - \varphi(x_2)| = |x_1 - x_2| \leq \frac{1}{2} |x_1 - x_2|$$

This is clearly a contradiction, to the inequality, and **Theorem 6**, so we must have, at most, one fixed point.

Therefore, we have that $f(x) = y$ for at most one point $x \in U$. This is to each $y \in \mathbb{R}^n$, so $f(x)$ and y are arbitrary, and we conclude that f is injective on U. (There was a lot of arbitrary-ness there! A lot of guessing games had to be played!).

Now, to prove the second part of the part (a) of the theorem, let us do the following:

We then put $V = f(U)$, and pick some $y_0 \in V$. Then we have that $y_0 = f(x_0)$ for some $x_0 \in U$. Let B be the open ball centered at x_0 with radius $r > 0$, so that the closure of B, $\overline{B}$, lies in U.

Fix y, then let

$$|y - y_0| < \lambda r$$

And let $\varphi(x) = x + A^{-1}(y - f(x))$. Then, we have that:

$$\varphi(x_0) - x_0 = A^{-1}(y_0 - f(x))$$

$$|\varphi(x_0) - x_0| = \left|A^{-1}(y_0 - f(x))\right| < \left\|A^{-1}\right\| \lambda r = \frac{r}{2}$$

Let $x \in \overline{B}$, and we may use **Theorem 6** again to show that:

$$|\varphi(x_0) - x_0| \leq |\varphi(x) - \varphi(x_0)| + |\varphi(x_0) - x_0| < \frac{1}{2} |x - x_0| + \frac{r}{2} \leq r$$

Therefore, $\varphi(x) \in B$.

By definition of φ, it is a contraction of $\overline{B}$ into itself. And $\overline{B}$ is a closed subset of $\mathbb{R}^n$, thus, it is complete. We may apply **Banach Fixed Point Theorem**, and set x as a fixed point of φ in $\overline{B}$. For this x, we have that $f(x) = y$ by definition of φ. Therefore, $y \in f(\overline{B}) \subset f(U) = V$

(b) Pick $y \in V$, and $y + k \in V$. Then there exists $x \in U$, $x + h \in U$, so that $y = f(x)$, $y + k = f(x + h)$.

$$\varphi(x + h) - \varphi(x) = h + A^{-1}(f(x) - f(x + h)) = h - A^{-1}k$$

By the inequality derived from **Theorem 6**, we have that:

$$|\varphi(x+h) - \varphi(x)| = \left|h - A^{-1}k\right| \leq \frac{1}{2}|h|$$

If we apply the **reverse triangle inequality**, we obtain that:

$$|h| - \left|A^{-1}k\right| \leq \frac{1}{2}|h|$$

We see that:

$$2\left|A^{-1}k\right| \geq |h|$$

$$|h| \leq 2\left|A^{-1}k\right| \leq 2\left\|A^{-1}\right\||k| = \frac{1}{\lambda}|k|$$

Recall $y + k = f(x+h)$, $y = f(x)$ (we picked these). Then, noting that g is the inverse mapping of f, we may compute:

$$g(y+k) = x + h$$

$$g(y) = x$$

Then we can use **Theorem 2** to give us the existence of the inverse of $f'(X)$, which we will call T:

$$g(y+k) - g(y) - Tk$$

$$= h - Tk$$

And recall that:

$$h - A^{-1}k = h - A^{-1}(f(x+h) - f(x)) = -A^{-1}(f(x+h) - f(x) - Ah)$$

Noting that $A = f'(x)$, we obtain:

$$g(y+k) - g(y) - Tk = h - Tk = -T(f(x+h) - f(x) - f'(x)h)$$

We now norm both sides and use the bound on $|h|$:

$$\frac{|g(y+k) - g(y) - Tk|}{|k|} \leq \|T\| \frac{1}{\lambda} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|}$$

As k approaches 0, we have that h approaches 0 as well, therefore, both sides of the above inequality vanish at once, and we have proved that:

$$g'(y) = T$$

Where

$$T = [g'(g(y))]^{-1}$$

Then, as g is a continuous map of V onto U, f' is a continuous mapping of U into Ω of all invertible elements of $L(\mathbb{R}^n)$, and by **Theorem 2**, the inversion of a linear operator in $L(\mathbb{R}^n)$ is a continuous map. Therefore, g is not only differentiable, but has a continuous derivative. Thus $g \in C^1(V)$. □

This was a long proof full of unintuitive computation! But we have a powerful tool for knowing when an inverse exists!

Theorem 11. *Open Mapping Theorem*

If f is C^1 and maps $E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and if $f'(x)$ is invertible for every $x \in E$, then $f(W)$ is an open subset of $\mathbb{R}^n$ for every open set $W \subset E$.

Proof. This is immediate by part a of the **Inverse Function Theorem**. Anytime $f \in L(\mathbb{R}^n)$ maps an open subset of $\mathbb{R}^n$, it will be some open subset of $\mathbb{R}^n$ once again. $\square$

There are two cases of the **Implicit Function Theorem**, a linear case, and a nonlinear case. In any case, for any $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$, we may split up the A into two parts:

$$A_x \in L(\mathbb{R}^n, \mathbb{R}^n)$$

$$A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$$

So that:

$$A(h, k) = A_x h + A_y k$$

Theorem 12. *Implicit Function Theorem (Linear Version)*

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$, and if A_x is invertible, then there corresponds to every $k \in \mathbb{R}^m$, a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$.

Proof.

$$A(h, k) = 0$$

if and only if

$$A_x h + A_y k = 0$$

Then as A_x is invertible, we obtain:

$$h = -A_x^{-1} A_y k$$

$\square$

Theorem 13. *Implicit Function Theorem (General Version)*

Let $f : E \subset \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ be C^1, and $f(a, b) = 0$ for some point $(a, b) \in E$. Put $A = f'(a, b)$ and assume that A_x is invertible.

Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$, with $(a, b) \in U$ and $b \in W$ with the following property:

To every $y \in W$ corresponds a unique x such that $(x, y) \in U$ and $f(x, y) = 0$.

If this x is defined to be $g(y)$, then $g : W \longrightarrow \mathbb{R}^n$ is C^1, $g(b) = a$ and:

$$f(g(y), y) = 0$$

and

$$g'(b) = -A_x^{-1} A_y$$

Proof. We define F:
$$F(x,y) = (f(x,y), y)$$
$(x,y) \in E$

F is C^1 and maps $E \longrightarrow \mathbb{R}^{n+m}$. We want to show that $F'(a,b)$ is an invertible element of $L(\mathbb{R}^{n+m})$.

As $f(a,b) = 0$, we obtain: $f(a+h, b+k) - f(a,b) = f(a+h, b+k) = A(h,k) + r(h,k)$ Since

$$F(a+h, b+k) - F(a,b) = (f(a+h, b+k), k) = (A(h,k), k) + (r(h,k), 0)$$

$F(a,b)$ is the linear operator that sends (h,k) to $(A(h,k), k)$. The kernel of this linear operator is $(0,0)$, therefore it is injective, and hence, bijective, and invertible. The **Inverse Function Theorem** applies here and tells us that there exist open subsets of $\mathbb{R}^{n+m}$ such that: $(a,b) \in U$, $(0,b) \in V$, so that $F : U \longrightarrow V$ is injective.

$$W = \{y \in \mathbb{R}^m : (0,y) \in V\}$$

Clearly, $b \in W$, and W is also open. If $y \in W$, then $(0,y) = F(x,y)$ for some $(x,y) \in U$. $f(x,y) = 0$ for this x. And we now suppose that with this y, that $(x', y) \in U$, and $f(x', y) = 0$.

$$F(x', y) = (f(x', y), y) = (f(x,y), y) = F(x,y)$$

By injectivity of F, $x' = x$.

Now define $g(y)$. $y \in W$, so that $(g(y), y) \in U$ and $f(g(y), y) = 0$.

Then, we obtain:
$$F(g(y), y) = (0, y)$$

Let $G : V \longrightarrow U$ that inverts F, then G is C^1, and by the **Inverse Function Theorem**, we obtain that:

$$(g(y), y) = G(0, y)$$

As G is C^1 (by **Inverse Function Theorem**), we obtain that g is also C^1.

Let us put $(g(y), y) = \phi(y)$, and we obtain:

$$\phi'(y)k = (g'(y)k, k)$$

Where $y \in W$, $k \in \mathbb{R}^m$.

$f(\phi(y)) = 0$ in W, and the chain rule gives us that:

$$f'(\phi(y))\phi'(y) = 0$$

Set $y = b$, then $\phi(y) = (a,b)$ and $f(\phi(y)) = A$. Therefore,

$$A\phi'(b) = 0$$

We now obtain that:

$$A_x g'(b)k + A_y k = A(g'(b)k, k) = A\phi'(b)k = 0$$

for every $k \in \mathbb{R}^m$. Therefore, we have:

$$A_x g'(b) + A_y = 0$$

Just invert to obtain that:

$$g'(b) = -A_x^{-1} A_y$$

□

We will skip most of Rudin's presentations of additional results. We know sufficiently enough to move onto differential forms.

2 Differential Forms (Rudin)

We will pay close attention to every aspect of what goes on in here. We focus on differential forms for integration theory, in Rudin's perspective. We will consider taking another view with multilinear maps once we have explored this geometric viewpoint.

We present our first definition:

Definition 2.1. Suppose I^k is a k-cell in $\mathbb{R}^k$, consisting of all k-tuples $(x_1, ..., x_k)$, such that:

$$a_i \leq x_i \leq b_i$$

I^j is the jth cell in $\mathbb{R}^j$ defined by the above inequality. f is a real continuous function on I^k. Put $f = f_k$ and define f_{k-1} on I^{k-1} by:

$$f_{k-1}(x_1, ..., x_{k-1}) = \int_{a_k}^{b_k} f_k(x_1, ..., x_k) dx_k$$

By uniform continuity of f_k on I^k, it is apparent that every f_j $(j \leq k)$ is also uniformly continuous on I^j. After integrating this for k steps, we obtain a real constant, f_0, which is denoted as the **Integral of f over I^k**.

$$\int_{I^k} f(x) dx$$

or

$$\int_{I^k} f$$

(in the latter notation, the integration measure is given by context). Let us define $L(f)$ as the integral of f over the k-cell and let $L'(k)$ be a rearrangement of the order of integraton. Clearly, we're going to need this to be invariant under some conditions.

Theorem 14. *For every $f \in C^1(I^k)$, $L(f) = L'(f)$.*

Proof. We will leave off the proof for later. $\square$

Definition 2.2. Suppose $E \subset \mathbb{R}^n$ is an open set. A **k-surface** in E is a C^1 mapping ϕ from a compact set $D \subset \mathbb{R}^n$ into a set E. We call D the **parameter domain** of ϕ. Points of D will be denoted as $u = (u_1, ..., u_n)$.

We will only consider when D is a k-simplex of a k-cell as we can generalize accordingly because every surface can be written in terms of either of those.

Definition 2.3. Suppose $E \subset \mathbb{R}^n$ is an open set. A **k-form** in E is a function ω, symbolically represented by:

$$\omega = \sum a_{i_1,...,i_k}(x) dx_{i_1} \wedge ... \wedge dx_{i_k}$$

We may assign to each k-surface, a number $\omega(\phi)$:

$$\int_\phi \omega = \int_D \sum a_{i_1,...,i_k}(\phi(u)) \frac{\partial(x_{i_1}, ..., x_{i_k})}{\partial(u_1, ..., u_k)} du$$

Here, D is the parameter domain of ϕ.

Note that a k-form ω is of class C^1 if the component functions $a_{i_1,...,i_k}(x)$ are C^1 themselves.

3 Differential Forms (An Introduction)

Here, we will redo differential forms, except from a top-down perspective. I will not do many theorems, just calculations and short proofs to get used to the formalism. I will copy some questions from supplemental material and (more importantly) answer it. Our goal is to get familiar with forms on $\mathbb{R}^n$ and generalize it to manifolds.

We will talk about differential forms on $\mathbb{R}^n$. We will list a few basic properties of differential forms.

3.1 Formalism and Notation

Definition 3.1. A (differential) n-form is a tensor field of the form:

$$\omega = \sum_k W_{i_1,...,i_k}(x) dx^1 \wedge \cdots \wedge dx^k$$

Where $W_{i_1,...,i_k}(x)$ is a continuous function.

Proposition 15. *The n-form ω is of differentiability class C^k if and only if $W_{i_1,...,i_k}(x)$ is C^k.*

The differentiability class of ω usually depends on the manifold that we are working over. We usually assume smoothness, but it does not have to be the case.

Definition 3.2. The **set of n-forms** is denoted as $\Omega^n(M)$ where M is our manifold. We will cover its algebraic properties later, but for now, we put out there that it is a free module over the ring of functions in a differentiability class of the manifold M.

Definition 3.3. Wedge Product

Let α, β, $\gamma \in \Omega^n(M)$. The **Wedge Product** is a binary operator on the space of n-forms, $\Omega^n(M)$:

1.
$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

2.
$$(\alpha + \beta) \wedge \gamma = (\alpha \wedge \gamma) + (\beta \wedge \gamma)$$

3.
$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$$

4. Let $c \in \mathbb{R}$
$$(c\alpha) \wedge \beta = c(\alpha \wedge \beta)$$

We automatically see that if any index repeats, i.e.

$$dx^i \wedge dx^j, i = j$$

Then

$$dx^i \wedge dx^j = 0$$

Notation: We have the contraction of indices defined in the following manner

$$I = \{i_1, i_2, \ldots, i_n\}$$
$$dx^I = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

From now on, all n-forms will be written as:

$$\omega = \sum_I W_I(x) dx^I$$

We also note that the wedge product is compatible between elements from $\Omega^k(M)$ and $\Omega^l(M)$ where $l \neq k$ necessarily. Let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, then:

$$\alpha \wedge \beta = \sum_{K,L} a_K b_L dx^K \wedge dx^L$$

Clearly, if K and L have intersecting indices i.e. one of $dx^l = dx^k$ where $k \in K$ and $l \in L$, then

$$\alpha \wedge \beta = 0$$

By induction, we see the following result:

Proposition 16. *With $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ as above, we see that:*

$$\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta$$

Proof. We give a quick proof. Note that if we expand out dx^K and dx^L, and interchange both full sets of indices, then we obtain:

$$dx^L \wedge dx^K = (-1)^{kl} dx^K \wedge dx^L$$

The result is obvious. □

3.2 Exterior Derivatives

We want to do calculus with differential forms.

Definition 3.4. Let us define the map (we will elaborate on this later in our theory section):

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

Where d maps:

$$d : \alpha \in \Omega^k(M) \longmapsto \sum_{I,j} \frac{\partial \alpha_I(x)}{\partial x^j} dx^j \wedge dx^I$$

We call this map d, the **Exterior Derivative** of a k-form.

We pull out an example computation:

Example 3.1. Let $\alpha = xydx + e^x dx \in \Omega^1(\mathbb{R}^2)$. Let us compute the exterior derivative of this. We just follow the definition of the exterior derivative given above:

$$d\alpha = \frac{\partial \alpha_1}{\partial x^1} dx^1 \wedge dx^1 + \frac{\partial \alpha_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \alpha_2}{\partial x^1} dx^1 \wedge dx^2 + \frac{\partial \alpha_2}{\partial x^2} dx^2 \wedge dx^2$$

We see that only half of the components are nonvanishing. Anything proportional to $dx^i \wedge dx^i$ will vanish. We see that:

$$\frac{\partial \alpha_1}{\partial x^2} = xdy \wedge dx$$

$$\frac{\partial \alpha_2}{\partial x^1} = e^x dx \wedge dy$$

The final result is:

$$d\alpha = (e^x - x) dx \wedge dy$$

We show a simple corollary to **Theorem 5** in this document (it was the theorem regarding the total derivative in terms of the partial derivatives) in the case where f is a 0-form:

Corollary 17. ***Corollary to Theorem 5 in the Case of a Single Variable Function***

If f is a 0-form (i.e. $f \in \Omega^0(\mathbb{R}^n)$)

$$df(x) = \sum_j \frac{\partial f(x)}{\partial x^j} dx^j$$

Proof. It is trivial with the machinery of forms we have. Just apply the formula and make the contracted indices the empty set, which gives us a scalar function. This is alternatively the definition of a **Gradient of a Scalar Field**. □

We give our first real theorem here:

Theorem 18. *These are properties of the exterior derivative:*

1. *If $\alpha \in \Omega^k(\mathbb{R}^n)$, and $\beta \in \Omega^l(\mathbb{R}^n)$, then*

   $$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$$

   This is the "forms" version of the product rule.

2. *$d(d\alpha) = 0$ (or $d^2 = 0$)*

   ***NOTE**: This has profound implications on the theory behind differential forms and deRham Cohomology.*

Proof. We only consider one permutation of indices I. We can easily extend this argument as the wedge product and exterior derivative are both linear.

$$\alpha = \alpha_I(x) dx^I$$

$$\beta = \beta_K(x) dx^K$$

$$\alpha \wedge \beta = \alpha_I(x)\beta_K(x) dx^I \wedge dx^K$$

Let us compute the exterior derivative carefully:

$$d(\alpha \wedge \beta) = \sum_j \partial_j(\alpha_I(x)\beta_K(x)) dx^j \wedge dx^I \wedge dx^K$$

We simply use the product rule to expand the derivative:

$$= \sum_j \left([\partial_j \alpha_I(x)]\, \beta_K(x) + \alpha_I(x)\, [\partial_j \beta_K(x)]\, dx^j \wedge dx^I \wedge dx^K \right)$$

$$= \sum_j \left(\left[\partial_j \alpha_I(x) dx^j \wedge dx^I \wedge \beta_K(x) dx^K\right] + \left[\partial_j \beta_K(x) dx^j \wedge \alpha_I(x) dx^I \wedge dx^K\right] \right)$$

We interchange the dx^j and dx^I in the second summand and pull out the appropriate sign of the permutation.

$$= \sum_j \left(\left[\partial_j \alpha_I(x) dx^j \wedge dx^I \wedge \beta_K(x) dx^K \right] + (-1)^k \left[\alpha_I(x) dx^I \wedge \partial_j \beta_K(x) dx^j \wedge dx^K \right] \right)$$

We then group the terms suggestively to give us the final form:

$$\sum_j \left(\left[\partial_j \alpha_I(x) dx^j \wedge dx^I \right] \wedge \beta_K(x) dx^K + (-1)^k \alpha_I(x) dx^I \wedge \left[\partial_j \beta_K(x) dx^j \wedge dx^K \right] \right)$$

This is more clear if we pull in the summation:

$$\left[\sum_j \partial_j \alpha_I(x) dx^j \wedge dx^I \right] \wedge \beta_K(x) dx^K + (-1)^k \alpha_I(x) dx^I \wedge \left[\sum_j \partial_j \beta_K(x) dx^j \wedge dx^K \right]$$

The terms in the square brackets resolve as follows:

$$[d\alpha] \wedge \beta_K(x) dx^K + (-1)^k \alpha_I(x) dx^I \wedge [d\beta]$$

This gives us our desired result. The general case follows if we attach a summation in the respective index contractions:

$$d\alpha \wedge \sum_K \beta_K(x) dx^K + (-1)^k \sum_I \alpha_I(x) dx^I \wedge d\beta$$

In any case:

$$d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

We will now prove the second part, which is slightly more difficult. We first note that our form of the k-form α is:

$$\alpha = \alpha_I(x) dx^I$$

Note that by **Corollary 17**, we see that the exterior derivative for our α is of the form:

$$d\alpha = d\alpha_I(x) \wedge dx^I$$

Then we act on this with the exterior derivative:

$$d(d\alpha) = d(d\alpha_I(x) \wedge dx^I)$$

Notice that $\alpha_I(x)$ is a 0-form (a function). Use the previous assertion to resolve this exterior derivative.

$$d(d\alpha) = d(d\alpha_I(x)) \wedge dx^I - d\alpha_I(x) \wedge d(dx^I)$$

We pick up the negative sign because $d\alpha_I(x)$ is a 1-form so we will pick up a factor of just -1 when we interchange the wedge product order.

Now we must only prove that $d(df) = 0$ for any 0-form f.

$$d(df) = d\left(\sum_j \partial_j f dx^j\right) = \sum_{j,m} \partial_m \partial_j f dx^j \wedge dx^m$$

$$= -\sum_{j,m} \partial_m \partial_j f dx^m \wedge dx^j = -d(df)$$

Therefore, we see that $d^2 f = 0$ for any 0-form f. Then we return to our original k-form $\alpha_I(x)$.

$$d(d\alpha) = d(d\alpha_I(x)) \wedge dx^I - d\alpha_I(x) \wedge d(dx^I)$$

$$= 0 - d\alpha_I(x) \wedge d(dx^I)$$

For the second term, just recognize that dx^I is a k-form of the following form:

$$g = dx^I = 1dx^I$$

Then we easily see that:

$$dg = d(1) \wedge dx^I = 0$$

Thus, $d\alpha_I(x) \wedge d(dx^I) = 0$ And we see that:

$$d^2\alpha = 0$$

For any k-form α. □

This is a little ahead of the scope of our notes here, but $d^2\alpha = 0$ is a necessary condition for a sequence of homomorphisms on modules in an abelian category to be exact. This notion will be explored in deRham Cohomology.

Problems:

Exercise 1.1: On $\mathbb{R}^3$, there are interesting 1-forms and 2-forms associated with every vector field $\vec{v}(x) = (v_1(x), v_2(x), v_3(x))$. If we let $\omega^1_{\vec{v}} = v_1 dx + v_2 dy + v_3 dz$, let $\omega^2_{\vec{v}} = v_1 dy \wedge dz + v_2 dz \wedge dx + v_3 dx \wedge dy$. Let f be any function (any 0-form). Show that:

(a)
$$df = \omega^1_{\nabla f}$$

(b)
$$d\omega^1_{\vec{v}} = \omega^2_{\nabla \times \vec{v}}$$

(c)
$$d\omega^2_{\vec{v}} = (\nabla \cdot \vec{v})\, dx \wedge dy \wedge dz$$

Give the significance of each quantity.

Solution to Exercise 1.1:

We do this in full specificity for the sake of practice with computing exterior derivatives.

(a)

$$df = \sum_{I,j} \partial_j f dx^j \wedge dx^I$$

As f is a 0-form, we see that $I =$, so we can ignore the summation over I.

$$df = \sum_j \partial_j f dx^j = \partial_1 v_1 dx^1 + \partial_2 v_2 dx^2 + \partial_3 v_3 dx^3$$

Transcribing notation, we see that:

$$df = \omega^1_{\nabla f}$$

(b)

$$d\omega^1_{\vec{v}} = \sum_{I,j} \partial_j v_I dx^j \wedge dx^I$$

dx^I contains every possible permutation of wedge products that are possible for the given degree of the form. It suffices to sum over only the wedge products that appear in our form as all the other ones can be considered as having vanishing component functions.

$$\sum_j \partial_j \left[v_1 dx^j \wedge dx^1 + v_2 dx^j \wedge dx^2 + v_3 dx^j \wedge dx^3 \right]$$

Explicitly sum over j

$$= \partial_1 v_1 dx^1 \wedge dx^1 + \partial_1 v_2 dx^1 \wedge dx^2 + \partial_1 v_3 dx^1 \wedge dx^3$$
$$+ \partial_2 v_1 dx^2 \wedge dx^1 + \partial_2 v_2 dx^2 \wedge dx^2 + \partial_2 v_3 dx^2 \wedge dx^3$$
$$+ \partial_3 v_1 dx^3 \wedge dx^1 + \partial_3 v_2 dx^3 \wedge dx^2 + \partial_3 v_3 dx^3 \wedge dx^3$$

The "diagonal" terms vanish by antisymmetry of the wedge product. We may work out the rest, and realize that this is exactly the curl of a vector in $\mathbb{R}^3$.

$$(\partial_2 v_3 - \partial_3 v_2) dx^2 \wedge dx^3 - (\partial_3 v_1 - \partial_1 v_3) dx^3 \wedge dx^1 + (\partial_1 v_2 - \partial_2 v_1) dx^1 \wedge dx^2$$

Transcribing notation:

$$d\omega^1_{\vec{v}} = \omega^2_{\nabla \times \vec{v}}$$

(c)

$$d\omega^2_{\vec{v}} = \sum_{I,j} \partial_j v_I dx^j \wedge dx^I$$

$dx^I = dx^n \wedge dx^m$ for any $n, m = 1, 2, 3$. Sum over all available wedge products.

$$\sum_j \left[\partial_j v_1 dx^j \wedge dx^2 \wedge dx^3 + \partial_j v_2 dx^j \wedge dx^3 \wedge dx^1 + \partial_j v_3 dx^j \wedge dx^1 \wedge dx^2\right]$$

We will sum over all available j:

$$\partial_1 v_1 dx^1 \wedge dx^2 \wedge dx^3 + \partial_1 v_2 dx^1 \wedge dx^3 \wedge dx^1 + \partial_1 v_3 dx^1 \wedge dx^1 \wedge dx^2$$

$$+\partial_2 v_1 dx^2 \wedge dx^2 \wedge dx^3 + \partial_2 v_2 dx^2 \wedge dx^3 \wedge dx^1 + \partial_2 v_3 dx^2 \wedge dx^1 \wedge dx^2$$

$$+\partial_3 v_1 dx^3 \wedge dx^2 \wedge dx^3 + \partial_3 v_2 dx^3 \wedge dx^3 \wedge dx^1 + \partial_3 v_3 dx^3 \wedge dx^1 \wedge dx^2$$

Only the "diagonal" terms are nonvanishing. Furthermore, every wedge product of differentials here are a cyclic permutation of the default wedge product $dx^1 \wedge dx^2 \wedge dx^3$.

We obtain:

$$(\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3) dx^1 \wedge dx^2 \wedge dx^3 = (\nabla \cdot \vec{v})\, dx^1 \wedge dx^2 \wedge dx^3$$

These are significant because, given that we only admit orientable surfaces as subsets of $\mathbb{R}^3$, these are rigorous (coordinate-independent) formulations of the Gradient Theorem, Stokes' Theorem, and Divergence Theorem in $\mathbb{R}^3$. We know this because the wedge products define volume forms on $\mathbb{R}^3$ if we restrict to orientable manifolds only.

Exercise 1.2: $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$, and ω is exact if there exists $\nu \in \Omega^{k-1}(M)$ such that $\omega = d\nu$. In general, all **exact forms are closed** since $d^2 = 0$. In $\mathbb{R}^n$, exactness and closedness are necessary and sufficient conditions, namely, **any closed form is exact** (this is a simple statement of **Poincare's Lemma**). This is not true on subsets of $\mathbb{R}^n$. To demonstrate this, consider the following 1-form on the punctured plane $M = \mathbb{R}^2 \backslash \{0\}$.

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

Is it closed or not? Is it exact?

Solution to Exercise 1.2:

We just verify this.

$$\omega = \frac{-ydx + xdy}{x^2 + y^2} \in \Omega^1(M)$$

Take the exterior derivative of ω:

$$d\omega = \sum_{I,j} \partial_j \omega_I dx^j \wedge dx^I$$

$$\omega_2 = \frac{x}{x^2 + y^2}$$

$$\omega_1 = \frac{-y}{x^2 + y^2}$$

Sum over the I:

$$d\omega = \sum_j \left(\partial_j \omega_1 dx^j \wedge dx^1 + \partial_j \omega_2 dx^j \wedge dx^2\right)$$

We disregard all the terms where the $\wedge$ will trivially vanish.

$$\partial_1 \omega_2 = \frac{y^2 - x^2}{x^2 + y^2}$$

$$\partial_2 \omega_1 = \frac{y^2 - x^2}{x^2 + y^2}$$

We see that summing over the j:

$$d\omega = \partial_1 \omega_1 dx^1 \wedge dx^1 + \partial_2 \omega_1 dx^2 \wedge dx^1 + \partial_1 \omega_2 dx^1 \wedge dx^2 + \partial_2 \omega_2 dx^2 \wedge dx^2$$

$$= (\partial_1 \omega_2 - \partial_2 \omega_1) dx^1 \wedge dx^2 = 0$$

Thus, ω is a closed 1-form.

3.3 Pullbacks (over $\mathbb{R}^n$)

We will define the pullback of differential forms. The intuition behind the pullback is that functions and anything that is like a function (forms) are defined as a composition with the function that induces the pullback. Hence, instead of getting sent to the codomain, functions and forms remain stuck in the space of forms. We will make these more clear below.

Theorem 19. *Pullback of Differential Forms*

For a smooth function $g : X \longrightarrow Y$, $\exists! g^* : \Omega^k(Y) \longrightarrow \Omega^l(X)$, that is linear, with the following properties:

If $f : Y \longrightarrow \mathbb{R}$ is a 0-form on Y, then $g^* f = f \circ g$

If α and ω are forms on Y (any degree), then $g^*(\alpha \wedge \omega) = (g^*\alpha) \wedge (g^*\omega)$

If ω is a form on Y, then $g^*(d\omega) = d(g^*\omega)$

Exercise 1.3: Prove this theorem.

Solution to Exercise 1.3:

Proof. We will prove it in sequential order. Once we prove the first two points, the last one will follow directly.

1. We take this as definition of a pullback of a 0-form. We may return to show this with a commutative diagram that will break down every part of this, but we will take it as fact and build on it for higher degree forms.

$$g^* f = f \circ g$$

For any $f \in \Omega^0(Y) = C^\infty(Y)$.

2. We first look at the action of g^* on the basis forms.

$$g^*(dy^j)$$

We may choose to look at dy^j as a form, or we can choose to look at it as a derivation of a function y^j, which we choose as the coordinates of Y. Note that it quite literally follows that:

$$dy^j = d(y^j)$$

So if we act on this by the pullback g^*. We can use fact (1) to obtain:

$$g^*(dy^j)(x) = d(g^* y^j)(x) = d(y^j \circ g)(x) = dg^j(x)$$

Where $y^j \circ g$ picks out the jth coordinate of $g : X \longrightarrow Y$. We can extend this to obtain the general form of α:

$$\alpha = \alpha_I dy^I$$

$$g^*\alpha(x) = \sum_I \alpha_I(g(x)) dg^I$$

Then we take another form ω:

$$\omega = \sum_J \omega_J dx^J$$

$$\alpha \wedge \omega = \sum_{I,J} \alpha_I(x)\omega_J(x) dx^I \wedge dx^J$$

$$g^*(\alpha \wedge \omega) = \sum_{I,J} g^*(\alpha_I(x)\omega_J(x)) dg^I \wedge dg^J = \sum_{I,J} \alpha_I(g(x))\omega_J(g(x)) dg^I \wedge dg^J$$

$$= \sum_{I,J} \alpha_I(g(x)) dg^I \wedge \omega_J(g(x)) dx^J$$

$$= \left[\sum_I \alpha_I(g(x)) dg^I\right] \wedge \left[\sum_J \omega_J(g(x)) dx^J\right]$$

The terms in the brackets are $g^*\alpha$ and $g^*\omega$, respectively. Hence

$$= g^*\alpha \wedge g^*\omega$$

3. If we let α be as before. Then we obtain:

$$g^*(d\alpha) = g^*\left(\sum_{I,j} \partial_j \alpha_I dx^j \wedge dx^I\right)$$

Use linearity of g^* to bring it inside the sum:

$$\sum_{I,j} g^*\left[\partial_j \alpha_I dx^j \wedge dx^I\right] = \sum_{I,j} \partial_j g^* \alpha_I(x) dg^j \wedge dg^I$$

Note that $\partial_j \alpha_I(x)$ is a 0-form, so we may use fact (1) again, it brings us to:

$$\sum_{I,j} \partial_j \alpha_I(g(x)) dg^j \wedge dg^I$$

$$= \sum_{I,j} \partial_j \alpha_I(g(x)) dg^j \wedge dg^I = d(g^*\alpha)$$

□

In physical situations, the important case of forms arises when the vector space dimension of the codomain, domain, and the degree of the form all coincide. Elements of $\Omega^k(\mathbb{R}^n)$ of the form $dx^1 \wedge \cdots \wedge dx^n$ are known as **volume forms** in this case. The name is obvious, it represents an infinitesimal volume element.

Exercise 1.4: Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be $C^\infty(\mathbb{R}^n)$. Let $\omega \in \Omega^n(\mathbb{R}^n)$ be the volume form of $\mathbb{R}^n$. Show that $g^*\omega$ evaluated at a point $x \in \mathbb{R}^n$, is $det(dg_x)$ times the volume form $dx^1 \wedge \cdots \wedge dx^n$.

Solution to Exercise 1.4:

As ω is a volume form on $\mathbb{R}^n$, we may assume it is something of the form:

$$\omega = dx^1 \wedge \cdots \wedge dx^n$$

Where x^i are coordinates in $\mathbb{R}^n$. Then, let us just pull our volume form back

$$g^*(\omega) = g^*(dx^1 \wedge \cdots \wedge dx^n) = g^*dx^1 \wedge \cdots \wedge g^*dx^n$$

$$= dg^1 \wedge \cdots \wedge dg^n$$

As each g^1 is a 0-form, we have that the exterior derivative looks like:

$$dg^j = \sum_i \frac{\partial g^j}{\partial x^i} dx^i$$

Then, we see that if we take the product over all the available component functions (j) and sum over all the i, then we obtain:

$$g^*(\omega) = dg^1 \wedge \cdots \wedge dg^n = \left(\prod_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial g^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n$$

We see that:

$$det(dg_x) = \left(\prod_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial g^j}{\partial x^i} \right)$$

We end up with:

$$g^*\omega = det(dg_x) dx^1 \wedge \cdots \wedge dx^n$$

As desired.

Exercise 1.5: Prove that pullbacks are natural. So let $g : U \longrightarrow V$, $h : V \longrightarrow W$. U, V, W are open subsets of Euclidean space of various dimensions. $h \circ g : U \longrightarrow W$. Show that:

$$(h \circ g)^* = g^* \circ h^*$$

Solution to Exercise 1.5:

This seems quite obvious if we look at a diagram of maps:

$$U \xrightarrow{g} V \xrightarrow{h} W$$

Which induces:

$$\Omega^0(U) \xleftarrow{g^*} \Omega^0(V) \xleftarrow{h^*} \Omega^0(W)$$

We note that $\Omega^0(X) = C^\infty(X)$ by definition of a 0-form.

To demonstrate the above pictures requires a quick computation. Just pick an arbitrary function $f \in \Omega^0(W)$, where $f : W \longrightarrow \mathbb{R}$, and act on it by $(h \circ g)^*$.

$$(h \circ g)^* f = f \circ (h \circ g) = (f \circ h) \circ g = (h^* f) \circ g = g^*(h^* f) = (g^* h^*) f = (g^* \circ h^*) f$$

Therefore, it follows that:

$$(h \circ g)^* = g^* \circ h^*$$

Exercise 1.6: Let $U = (0, \infty) \times (0, 2\pi)$, and let V be $\mathbb{R}^2 \backslash \{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\}$. U has coordinates (r, θ), and V has coordinates (x, y). Let $g(r, \theta) = (r\cos(\theta), r\sin(\theta))$, we let $h = g^{-1}$. On V, let our 2-form be $\alpha = e^{-(x^2+y^2)} dx \wedge dy$.

(a) Compute $g^*(x)$, $g^*(y)$, $g^*(dx)$, $g^*(dy)$, $g^*(dx \wedge dy)$, and $g^*\alpha$.

(b) Compute $h^*(r)$, $h^*(\theta)$, $h^*(dr)$, $h^*(d\theta)$.

Solution to Exercise 1.6:

These are routine computations but it is incredibly important to realize the connection here.

On U and V, the map:

$$g : (r, \theta) \longmapsto (r\cos(\theta), r\sin(\theta))$$

is a smooth mapping (notice the branch cut at where 0 and 2π coincide).

(a)

$$g^*(x) = x \circ g = g_1$$

Where g_1 denotes the first (x) component of g.

$$g^*(x) = r\cos(\theta)$$

Similarly,

$$g^*(y) = r\sin(\theta)$$

We start using our properties from here on out.

$$g^*(dx) = d(g^*(x)) = d(r\cos(\theta))$$

Recall that for 0-forms, the exterior derivative is:

$$df = \sum_j \frac{\partial f}{\partial x^j} dx^j$$

Then, we find:

$$d(r\cos(\theta)) = \frac{\partial(r\cos(\theta))}{\partial r} dr + \frac{\partial(r\cos(\theta))}{\partial \theta} d\theta$$

Evaluating these, we find

$$g^*(dx) = \cos(\theta)dr - r\sin(\theta)d\theta$$

With a similar computation for dy, we obtain:

$$g^*(dy) = \sin(\theta)dr + r\cos(\theta)d\theta$$

Then, we may also compute $dx \wedge dy$ easily:

$$g^*(dx \wedge dy) = g^*(dx) \wedge g^*(dy)$$

$$= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta)$$

Using antisymmetry, we obtain:

$$r\cos^2(\theta)dr \wedge d\theta - r\sin^2(\theta)d\theta \wedge dr = r\left(\cos^2(\theta) + \sin^2(\theta)\right) dr \wedge d\theta$$

Therefore,

$$g^*(dx \wedge dy) = r dr \wedge d\theta$$

Now if we want to pull back our 2-form:

$$\alpha = e^{-\left(x^2+y^2\right)} dx \wedge dy$$

Just apply the pullback:

$$g^*\left(e^{-\left(x^2+y^2\right)} dx \wedge dy\right) = g^*\left(e^{-\left(x^2+y^2\right)}\right) r dr \wedge d\theta$$

To make our pullback more apparent, define:

$$f(x,y) = x^2 + y^2$$

and our Gaussian becomes:

$$e^{-f(x,y)}$$

Pull back:

$$g^*\left(e^{-f(x,y)}\right) = e^{-f(g(x),g(y))}$$
$$= e^{-r^2\cos^2(\theta)+r^2\sin^2(\theta)}$$

We are done now:

$$g^*\alpha = e^{-(r^2)} r dr \wedge d\theta$$

(b) This is slightly less obvious but it is simple enough given our domains.

$$h = g^{-1}$$

Meaning that

$$h : V \longrightarrow U$$
$$h : (x,y) \longrightarrow (r(x,y), \theta(x,y))$$

Recall that in polar coordinates:

$$r = \sqrt{x^2+y^2} \quad \theta = \arctan\left(\frac{x}{y}\right)$$

So

$$h : (x,y) \longmapsto \left(\sqrt{x^2+y^2}, \arctan\left(\frac{x}{y}\right)\right)$$

$$h^*(r) = r \circ h = \sqrt{x^2+y^2} \quad h^*(\theta) = \theta \circ h = \arctan\left(\frac{x}{y}\right)$$

The differentials are easily evaluated given this:

$$h^*(dr) = d(h^*r) = \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x}dx + \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial y}dy$$

$$h^*(d\theta) = d(h^*d\theta) = \frac{\partial\left(\arctan\left(\frac{x}{y}\right)\right)}{\partial x}dx + \frac{\partial\left(\arctan\left(\frac{x}{y}\right)\right)}{\partial y}dy$$

We see that:

$$h^*(dr) = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$$
$$h^*(d\theta) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

We see that when we perform a change of variables, it was just a pullback on our forms!

3.4 Integration of Forms

Just for now, we assume that $\alpha \in \Omega^n(\mathbb{R}^n)$ and that α is compactly supported. If we want to avoid this restrictive condition, we must use partitions of unity, which we will develop later. We see that an integral over all $\mathbb{R}^n$ is of the form:

$$\int_{\mathbb{R}^n} \alpha \equiv \int_{\mathbb{R}^n} \alpha_I(x) \left| dx^1 \cdots dx^n \right|$$

Note that we may also integrate over any open subset, U, of $\mathbb{R}^n$ as long as α_I restricted to U is still compactly supported. We also assume the standard orientation over $\mathbb{R}^n$.

Exercise 1.7: Suppose g is an orientation-preserving diffeomorphism from an open subset $U \subseteq \mathbb{R}^n :\longrightarrow V \subseteq \mathbb{R}^n$. Let α be a compactly supported n-form on V. Show that:

$$\int_U g^*\alpha = \int_V \alpha$$

How would this change if g were orientation-reversing?

Solution to Exercise 1.7:

We state that:

$$g : U \longrightarrow V \quad \alpha \in \Omega^n(V)$$

$$g^* : \Omega^n(V) \longrightarrow \Omega^n(U)$$

We now have that:

$$\alpha = \alpha_I(x) dx^1 \wedge \cdots \wedge dx^n$$

Then we integrate over the pullback of α:

$$\int_U g^*\alpha = \int_U g^* \left(\alpha_I(x) dx^1 \wedge \cdots \wedge dx^n\right)$$

$$= \int_U g^*\alpha_I(x) dg^1 \wedge \cdots \wedge dg^n$$

We left off of pulling $\alpha_I(x)$ back to make a subtle point.

As $\alpha_I : V \longrightarrow \mathbb{R}$ presumably, we when we pull it back by g^*, we have that it composes with g i.e.

$$g^*\alpha = \alpha \circ g \equiv \alpha(g(x))$$

Therefore, if we consider that $y = g(x) \forall x \in U, \forall y \in V$, then we have that:

$$\alpha \circ : U \longrightarrow \mathbb{R}$$

turns into:

$$\alpha : V \longrightarrow \mathbb{R}$$

Hence, our pullback actually ”pushes” our domain forward. We obtain:

$$\int_U g^*\alpha_I dg^1 \wedge \cdots \wedge dg^n = \int_V \alpha_I(y) dy^1 \wedge \cdots \wedge dy^n$$

$$= \int_V \alpha$$

We note that:

$$dy^1 \wedge \cdots \wedge dy^n = det(dg_x) dx^1 \wedge \cdots \wedge dx^n$$

We used the assumption that g was **orientation preserving** by leaving the Jacobian alone. If we have that g is **orientation-reversing**, then we just tack on a negative sign onto the Jacobian resulting in:

$$\int_U g^*\alpha = -\int_V \alpha$$

We want to extend our results to functions that are not compactly supported. To achieve this, we bring in something called a Partition of Unity.

Theorem 20. *Existence of the Partition of Unity*

Suppose that K is a compact subset of $\mathbb{R}^n$, and $\{V_j\}$ is an open cover of K. There exist functions $\rho_i \in C^0(\mathbb{R}^n)$ such that:

(a)

$$0 \leq \rho_i \leq 1$$

(b) Each ρ_i has its support in some v_j

(c)

$$\sum_i \rho_i = 1$$

We will opt to not prove this right now. All that is important is that it exists.

If we define:

$$\int \alpha = \sum \int \rho_j \alpha$$

If the integral:

$$\int_{\mathbb{R}^n} |\alpha_I(x)| dx^1 \cdots dx^n$$

converges, then it will be independent of the choice of the partition of unity. We will prove this as a theorem.

Theorem 21. *If $\int_{\mathbb{R}^n} |\alpha_I(x)| dx^1 \cdots dx^n$ converges, then the integral:*

$$\int_{U \subseteq \mathbb{R}^n} \alpha$$

is independent of our choice of partition of unity $\{\rho_j\}$

Proof. To show this, we will stack arbitrarily many partitions of unity. Let $\{\rho'_j\}$ be our partition of unity, subordinate to another partition of unity $\{\rho_j\}$. Also note that our integration domain will be an open subset $U \subseteq \mathbb{R}^n$ such that $U \supset K$, where K is a compact subset of $\mathbb{R}^n$ such that the partition of unity exists on it.

$$\int \alpha = \sum_j \int \rho'_j \alpha = \sum_j \int \left(\sum_i \rho_i \right) \rho'_j \alpha$$

$$= \sum_j \int \left(\sum_i \rho_i \right) \rho'_j \alpha = \sum_j \int \sum_i (\rho_i \rho'_j) \alpha = \sum_j \int \sum_i (\rho'_j \rho_i) \alpha$$

By absolute convergence of $\alpha_I(x)$, we are allowed to interchange the sum and the integral (which is a limit in all technicality). Let us do exactly that.

$$\sum_i \int \left[\sum_j \rho'_j \right] \rho_i \alpha \quad \sum_j \rho'_j = 1$$

Therefore,

$$= \sum_i \int \rho_i \alpha$$

Thus,

$$\sum_j \int \rho'_j \alpha = \sum_i \int \rho_i \alpha$$

So our integral is independent of our choice of partition of unity. □

3.5 Differential Forms on Manifolds

We assume that our n-manifold is a Hausdorff Space that locally looks like $\mathbb{R}^n$.

Assume that $\psi_{1,2} : U_{1,2} \longrightarrow X$ are parametrizations of the same neighborhood of X. Then p is associated with both $\psi_1^{-1}(p) \in U_1$, and $\psi_2^{-1}(p) \in U_2$. If we have an atlas of parametrizations $\psi_i : U_i \longrightarrow X$, and if $g_{ij} = \psi_j^{-1} \circ \psi_i$ is the transition function from ψ_i coordinates to ψ_j coordinates on their overlap, then X is an abstract manifold defined as:

$$X = \coprod U_i / \sim$$

Where $x \in U_i \sim g_{ij} \in U_j$ is the equivalence relation.

We will do a similar thing, where we mod out the space of n-forms by an equivalence relation to obtain a space where we have no overlapping elements.

Let $\Omega^k(U)$ denote the k-forms on $U \subseteq \mathbb{R}^n$. Let V be a coordinate neighborhood of p in X.

$$\Omega^k(V) = \coprod \Omega^k(U_j / \sim \quad \alpha \in \Omega^k(U_j) \sim g_{ij}^* \alpha \in \Omega^k(U_j)$$

Here, as $g_{ij} : U_i \longrightarrow U_j$, we have that the pullback is:

$$g_{ij}^* : \Omega^k(U_j) \longrightarrow \Omega^k(U_i)$$

Now, we let $\nu \in \Omega^k(V)$, represented by a form $\alpha \in \Omega^k(U_j)$. We write:

$$\alpha = \psi_j^*(\nu)$$

We take a moment to understand what has been done here (I want this to be as self-contained as possible for any future reference). We have a smooth, n-dimensional Hausdorff manifold X. We construct it by cutting it up into a bunch of open sets that don't coincide (this is guaranteed to exist as X is a Hausdorff Manifold) as follows:

$$U_i / \sim$$

We define these open sets as "coinciding" when there are elements that are related by a transition map (which is a diffeomorphism), and we proceed to kill these elements to retain uniqueness. Elements $x \in X$ coinciding is represented by our equivalence relation $\sim$:

$$x \in U_i \sim g_{ij}(x) \in U_j$$

We now glue these pieces together by using a disjoint union, which is guaranteed to exist by the **universal property of the coproduct in the category of sets**.

$$X = \coprod_i U_i / \sim$$

It is now natural to speak of the space of forms defined in the same manner, as our **pullback** on the **space of forms** is induced by any diffeomorphism, in particular, **our transition map is a diffeomorphism**. We define our space of forms in a neighborhood of a point $p \in X$ in the same manner, we kill any forms that are related by the pullback of the transition map as follows:

$$\Omega^k(U_j) / \sim'$$

Where

$$\alpha \in \Omega^k(U_j) \sim' g_{ij}^* \in \Omega^k(U_i)$$

Thus, all the forms defined on every neighborhood of every point $p \in X$, denoted as V, are guaranteed to be unique. As usual, we glue this together by taking the disjoint union (it exists in the category of sets, so it must exist in any abelian category):

$$\Omega^k(V) = \coprod \Omega^k(U_j) / \sim'$$

In particular, note the utility of the transition map ($g_{ij} = \psi_j^{-1} \circ \psi_i$) and the parametrizations (ψ_i, ψ_j). We can simply do calculus with forms on manifolds as long as we pay attention to the parametrizations.

1. **We retain the properties of the pullback in relation to the wedge product:**

   Let $\mu \in \Omega^k(V)$, $\nu \in \Omega^l(V)$ be forms on an n-manifold X, then:

   The pullback of a form $\mu \wedge \nu$, in some coordinate neighborhood, V of X, corresponds to a form on U_i which is:

   $$\psi_i^* \mu \wedge \psi_i^* \nu$$

   So that:

   $$\psi_i^*(\mu \wedge \nu) = \psi_i^* \mu \wedge \psi_i^* \nu$$

2. **We also retain the exterior derivative properties in relation to the pullback:**

   If $d\mu \in \Omega^{k+1}(V)$, then we have that:

   $$\psi_i^*(d\mu) = d(\psi_i^* \mu)$$

We see that our forms are retained perfectly and present little trouble.

Exercise 1.8: Show that $\mu \wedge \nu$ and $d\mu$ are well-defined.

Solution to Exercise 1.8:

Let $\mu \in \Omega^k(V)$, and $\nu \in \Omega^l(V)$. Then it is trivial to see that if $\mu' = \mu$, and $\nu' = \nu$, then both:

$$\mu' \wedge \nu' = \mu \wedge \nu \quad d\mu' = d\mu$$

Let us relax our assumptions a bit. Use the equivalence relation $\sim$, and assume that

$$\mu' = g_{ij}^* \mu \quad \nu' = g_{ij}^* \nu$$

So that:

$$\mu' \sim \mu \quad \nu' \sim \nu$$

Simply plug this in:

$$\mu' \wedge \nu' = g_{ij}^* \mu \wedge g_{ij}^* \nu = g_{ij}^* \left(\mu \wedge \nu\right)$$

$$d\mu' = dg_{ij}^* \mu = g_{ij}^* \left(d\mu\right)$$

Therefore, we conclude that if $\mu \sim \mu'$, $\nu \sim \nu'$, then

$$\mu' \wedge \nu' \sim \mu \wedge \nu \quad d\mu' \sim d\mu$$

We may stop here, as we can accept that our pullback does not change our domain.

For the readers who look more closely, our operations work formally, but does it preserve the domain of our two operations?

The answer is yes, given the domains of our two operations, d and $\wedge$, we have that g_{ij}^* does not change their domain. Consider that the forms on X, μ, ν, that are defined in a neighborhood of points $x \in X$, $\forall x \in X$. We denote the neighborhoods of every x in X as V. We also see that every x is mapped to V from a collection of open sets $\{U_i\}$ by the use of a collection of atlases (or parametrizations), $\{\psi_i\}$. Therefore, we see that the atlases on our manifold are all mappings of the form $\psi_i : U_i \longrightarrow V$.

$$\wedge : \Omega^k(V) \times \Omega^l(V) \longrightarrow \Omega^{k+l}(V)$$

$$d : \Omega^k(V) \longrightarrow \Omega^{k+1}(V)$$

Then we see that as $g_{ij} : U_i \longrightarrow U_j$. Then **regarding g_{ij}^* as a functor**:

$$g_{ij}^* \left(\psi_j\right) = \psi_i$$

Therefore, we get that:

$$g_{ij}^* \left(\psi_j : U_j \longrightarrow V\right) = \psi_i : U_i \longrightarrow V$$

Therefore, it is well defined to assume that:

$$g_{ij}^* : \Omega^k(U_j) \longrightarrow \Omega^k(U_i)$$

Since we defined:

$$\Omega^k(V) = \coprod \Omega^k(U_j)/\sim$$

$$g_{ij}^* \left(\Omega^k(V)\right) = \coprod \Omega^k(U_i)/\sim = \Omega^k(V)$$

This holds for any k-form over V, so we have that d and $\wedge$ are well-defined under the equivalence relation $\sim$.

Furthermore, if we want to translate between forms on two separate manifolds, we can do the following.

Let $f : X \longrightarrow Y$ be a map between two smooth manifolds. And let $\alpha \in \Omega^k(Y)$. Let $\phi : U \subset \mathbb{R}^n \longrightarrow X$, $\psi : V \subset \mathbb{R}^m \longrightarrow Y$. Then, we can draw the following commutative diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "\phi"'] & V \arrow[r, "\psi"] & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

We see that by commutativity of this diagram, that a mapping $h : U \longrightarrow V$ must necessarily exist. It is also apparent from the diagram that if we reverse all the arrows, that

$$\phi^*(f^*\alpha) = h^*(\psi^*\alpha)$$

This is how we define the form $f^*\alpha$ on X.

A special case occurs when f is an inclusion of $X \subset Y$, so that X is a submanifold of Y. f^* is the restriction of α, a form in Y, to X.

3.6 Integration on Oriented Manifolds

Let X be an oriented k-manifold. Let ν be a k-form on X whose compact support is a subset of a coordinate chart $V = \psi_i(U_i)$. Where $U_i \subseteq \mathbb{R}^k$. Since X is oriented, we require the atlases ψ_i to be orientation-preserving. Then define:

$$\int_X \nu = \int_{U_i} \psi_i^* \nu$$

We present another exercise:

Exercise 1.9: Show that this definition does not depend on the choice of coordinates. That is, if $\psi_{1,2} : U_{1,2} \longrightarrow V$ are two sets of coordinates for V, both orientation-preserving, that:

$$\int_{U_1} \psi_1^* \nu = \int_{U_2} \psi_2^* \nu$$

i.e. This will show that our integral is well-defined.

Solution to Exercise 1.9:

We take as definition the definition of the integral:

$$\int_X \nu = \int_{U_i} \psi_i^* \nu$$

We must show that this is independent of the choice of coordinates, to show that this is well-defined. We note that, for the sake of consistency, that $\psi_i^* \nu$ is a k-form on U_i; as $\psi_i^* : \Omega^k(V) \longrightarrow \Omega^k(U_i)$. Therefore, we have that:

$$\int_X \nu = \int_{U_j} \psi_j^* \nu$$

Then, recall the equivalence relation $\sim$:

$$\nu \in \Omega^k(U_j) \sim g_{ij}^* \nu \Omega^k(U_i)$$

If we invoke the equivalence relation, then we may compute:

$$\psi_j^* \nu \in \Omega^k(U_j) \sim g_{ij}^* \left(\psi_j^* \nu \right) \in \Omega^k(U_i)$$

Thus, we see that:

$$\int_{U_j} \psi_j^* \nu = \int_{U_i} g_{ij}^* \left(\psi_j^* \nu \right)$$

The next step is to look at the space of k-forms in the open set V: $\Omega^k(V)$. Previously, we defined the space of k-forms in a coordinate neighborhood $V \subset X$ as:

$$\coprod U_i / \sim$$

With $\sim$ defined as above.

Therefore, we see that acting on our form by an orientation-preserving pullback does not introduce any sign changes into our integrals. Furthermore, under our equivalence relation $\sim$, we see that our k-form ν, still remains in $\Omega^k(V)$, as any transition mapping only changes the domain between various U_i, which are all included in the disjoint union that defines our space of k-forms, $\Omega^k(V)$. We ultimately obtain the latter equality presented here, namely that:

$$\int_{U_i} g_{ij}^* \left(\psi_j^* \nu\right) = \int_{U_i} \psi_i^* \nu = \int_X \nu$$

Thus, our integrals are well-defined under pullbacks of charts.

If a form is not supported in any coordinate chart ψ_i. Pick an open cover of X which contains coordinate neighborhoods of X, and we pick a partition of unity whose compact supports are contained inside the open cover. Like in the Euclidean case, we define:

$$\int_X \nu = \sum \int_X \rho_i \nu$$

Where our partition of unity is $\{\rho_i\}$.

For a bit of notation, $\alpha = \alpha_I(x) dx^1 \wedge \cdots \wedge dx^k$ is a k-form on $\mathbb{R}^n$, then $|\alpha| = |\alpha_I(x)| dx^1 \wedge \cdots \wedge dx^k$.

We denote ν as **absolutely integrable** if every $|\psi^*(\rho_i \nu)|$ is integrable over $U_i \subset \mathbb{R}^k$, and if the sum of these integrals converges. We proved before that integrals of forms are independent of the choice of partition of unity (in Euclidean Space). We cite **Theorem 21** for this purpose, only making note that we have an extra step now, because we must pull back with the pullback induced by the charts from each open subset of $\mathbb{R}^k$ to a coordinate neighborhood of X. The functoriality (acts like a functor) of these pullbacks guarantees that we reduce integration of k-forms on manifolds, to integration on an open subset of $\mathbb{R}^k$.

In these conditions, the integral of our k-form

$$\int_X \nu$$

is unambiguous and well-defined, as the partition of unity is guaranteed to exist.

Furthermore, when X is compact and ν is C^∞ (recall what it means for a form to be C^k), then the above condition of **absolute integrability** is implied, as the partition of unity will exist throughout the entire manifold (by compactness), and we will have diffeomorphic pullbacks by smoothness.

If X is 0-dimensional Euclidean space, then the integral of a form on X is just an integral of a function:

$$\int_X \alpha = \sum_{x \in X} \pm \alpha(x)$$

The orientation is defined as which sign you pick up on α at a point x.

We have done a comprehensive (if not brief treatment of forms on manifolds, we will explore Stokes' Theorem soon). We present various problems to demonstrate our understanding of forms on manifolds, and solutions to these problems:

Exercise 1.10: Let $X = S^1 \subset \mathbb{R}^2$ be the unit circle, oriented at the boundary of the disk. Compute

$$\int_X xdy - ydx$$

using a pullback to $\mathbb{R}$ with an orientation-preserving chart (note that this doesn't require a use of a partition of unity).

Solution to Exercise 1.10:

The solution to this relies on our choice of chart, so let us be judicious and pick the simplest one.

Let $U = (0, 2\pi)$. Define our chart as:

$$\psi : U \longrightarrow S^1 \quad t \longmapsto (\cos(t), \sin(t))$$

Then, the induced pullback is:

$$\psi^* : \Omega^1(S^1) \longrightarrow \Omega^1(U)$$

Since we are in local coordinates, all the work with forms we did in the previous exercises is valid.

$$\psi^* \left(xdy - ydx\right) = \psi^*(x)\psi^*(dy) - \psi^*(y)\psi^*(dx)$$

We compute each quantity (refer to **Exercise 2.6** for a step-by-step).

$$\psi^* x = x \circ \psi = \cos(t) \quad \psi^* y = y \circ \psi = \sin(t)$$

$$\psi^* dx = d(\psi^* x) = d(\cos(t)) \quad \psi^* dy = d(\sin(t))$$

As sin and cos are 0-forms, this just amounts to computing their differential:

$$\psi^* dx = -\sin(t)dt \quad \psi^* dy = \cos(t)dt$$

Plug it all in:

$$\psi^*(xdy - ydx) = \cos^2(t)dt + \sin^2(t)dt = dt$$

Thus, our pullback results in:

$$\int_{S^1} (xdy - ydx) = \int_U dt = \int_0^{2\pi} dt = 2\pi$$

NOTE: This 1-form is the definition of a winding number. It (effectively) calculates how many times we traverse a path homeomorphic to a closed loop.

Exercise 1.11: Now do the same thing one dimension up. Let $X = S^2 \subset \mathbb{R}^3$ be the unit sphere, oriented as the boundary of the unit ball. Let the 2-form in consideration be:

$$\omega = xdy \wedge dx + ydz \wedge dx + zdx \wedge dy$$

Compute

$$\int_X \omega$$

using an orientation-preserving pullback induced by a chart mapping from a subset of $\mathbb{R}^2$ to S^2.

Solution to Exercise 1.11:

We immediately pick a coordinate patch.

$$U = (0, \pi) \times (0, 2\pi)$$

So that our chart is:

$$h : U \longrightarrow S^2$$

$$h : (\varphi, \theta) \longmapsto (\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi))$$

Then the induced pullback is:

$$h^* : \Omega^2(S^2) \longrightarrow \Omega^2(U)$$

Then computation of our pullback gives us:

$$h^*\omega = h^*(x)h^*(dy) \wedge h^*(dx) + h^*(y)h^*(dz) \wedge h^*(dx) + h^*(z)h^*(dx) \wedge h^*(dy)$$

We calculate all the necessary results of our pullback below:

$$h^*x = \sin(\varphi)\cos(\theta) \quad h^*y = \sin(\varphi)\sin(\theta) \quad h^*z = \cos(\theta)$$

$$h^*dx = \cos(\varphi)\cos(\theta)d\varphi - \sin(\varphi)\sin(\theta)d\theta \quad h^*dy = \cos(\varphi)\sin(\theta)d\varphi + \sin(\varphi)\cos(\theta)d\theta$$

$$h^*dz = -\sin(\varphi)d\varphi$$

Then, plugging into the pullback of ω (it's a long computation, just use anti-symmetry and collect coefficients of $d\varphi \wedge d\theta$), we obtain:

$$h^*\omega = \left(\sin^3(\varphi) + \cos^2(\varphi)\sin(\varphi)\right) d\varphi \wedge d\theta$$

Now, putting it into our integral, we have that:

$$\int_U = \int_0^{2\pi} \int_0^{\pi} \left(\sin^3(\varphi) + \cos^2(\varphi)\sin(\varphi)\right) d\varphi d\theta$$

We evaluate this:

$$\int_X \omega = 4\pi$$

Therefore, we see that the cyclic permutation of the n-form on S^n represents the **area** (we could easily verify this with a computation of the totally antisymmetric 3-form on S^3, and so on...).

4 Stokes' Theorem

Let $X = H^n$ denote the **upper half plane** in $\mathbb{R}^n$ (we do this to fix the orientation). Then, ∂X is the boundary of X and is the embedding of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$. The orientation is ambiguous, however. It is **not necessarily the same orientation as** $\mathbb{R}^{n-1}$. It is $(-1)^n$ times the standard orienation on $\mathbb{R}^{n-1}$, as it takes $n-1$ flips to change the standard basis to be oriented negatively.

Theorem 22. ***Stokes' 1*** *(Over* $\mathbb{R}^n$*)*

Let ω *be a compactly-supported and in* $\Omega^{n-1}(X)$*. Then (with* X *denoted as above):*

$$\int_X d\omega = \int_{\partial X} \omega$$

Proof. We will use the fact that ω is compactly supported extensively.

Let us consider the case where $\omega = \omega_j(x) dx^1 \wedge \cdots dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$ (notice how we deleted the jth coordinate). As a shorthand, let us denote these forms as:

$$dx^{I_j} = dx^1 \wedge \cdots dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

Then the exterior derivative is

$$d\omega = \partial_j \omega_j dx^j \wedge dx^{I_j} = (-1)^{j-1} \partial_j \omega_j dx^I$$

Consider 2 cases:

(i) $j < n$

In this case, then $dx^n = 0$, so that:

$$\int_X \omega = 0$$

trivially. Then

$$\int_{H^n} \partial_j \omega_j dx^I = \int_{\mathbb{R}^n} \left(\int_{-\infty}^{\infty} \partial_j \omega_j(x) dx^j \right) dx^{I_j}$$

We apply the **fundamental theorem of calculus** on the bracketed integral. Namely:

$$\int_{-\infty}^{\infty} \omega_j(x) dx^j = \omega|_{-\infty}^{\infty}$$

As ω_j is compactly supported, it should vanish in as we go outside of a finite interval, therefore, at the endpoints, $\omega_j \equiv 0$. Therefore, the whole integral vanishes. And we have that:

$$\int_X d\omega = \int_{\partial X} \omega \equiv 0$$

(ii) $j = n$

Compute

$$\int_X d\omega = \int_{H^n} (-1)^{n-1} \partial_n \omega_n(x) d^n x$$

$$= \int_{\mathbb{R}^{n-1}} \left((-1)^{n-1} \partial_n \omega_n(x_1, \cdots, x_n) dx^n \right) dx^1 \cdots dx^{n-1}$$

We, again, look at the bracketed integral, evaluate it by **fundamental theorem of calculus**:

$$(-1)^{n-1} \int_0^\infty \partial_n \omega_n(x) dx^n = (-1)^{n-1} \omega_n(x_1, \cdots, x_n)|_0^\infty$$

We again use that $\omega_n(x)$ should be compactly supported, so it will vanish when we evaluate at $x_n = \infty$.

$$(-1)^{n-1} \int_0^\infty \partial_n \omega_n(x) dx^n$$

$$= (-1)^{n-1}(-1)\omega_n(x_1, \cdots, x_{n-1}, 0) = (-1)^n \omega_n(x_1, \cdots, x_{n-1}, 0)$$

Put it into the rest of the integral:

$$\int_{\mathbb{R}^{n-1}} \left((-1)^{n-1} \partial_n \omega_n(x_1, \cdots, x_n) dx^n \right) dx^1 \cdots dx^{n-1}$$

$$= \int_{\mathbb{R}^{n-1}} (-1)^n \omega_n(x_1, \cdots, x_{n-1}, 0) dx^1 \cdots dx^{n-1}$$

We have that $\omega \in \Omega^n(H^n)$ transformed into $\Omega^{n-1}(\mathbb{R}^{n-1})$, which is the main idea of Stokes' Theorem.

To complete the proof, recognize that:

$$\int_{\mathbb{R}^{n-1}} (-1)^n \omega_n dx^1 \wedge \cdots \wedge dx^{n-1} = \int_{\mathbb{R}^{n-1}} \omega_n dx^1 \cdots dx^{n-1}$$

As we may interchange the forms in the wedge product n times to return to the canonical ordering, therefore, we have $(-1)^n = 1$

$$\int_X d\omega = \int_{\partial X} \omega$$

Note that we only assumed that $\omega = \omega_j(x) dx^{I_j}$, and this is sufficient as we can write any (n-1)-form as a finite sum of ω_j, and we may manipulate linearity to trivially identify this general case. As it is true for each j, the whole sum must obey the same idea. Therefore, Stokes' Theorem follows. □

We now extend this to any smooth manifold.

Theorem 23. ***Stokes' 1*** *(Any smooth n-manifold)*

Let X be a compact, oriented n-manifold with the boundary. Let $\omega \in \Omega^{n-1}(X)$. Then

$$\int_X d\omega = \int_{\partial X} \omega$$

Proof. Let $\omega_i = \rho_i \omega$. Let us explain why our assumption that X is compact. By the topological definition of compactness, we are given that X is covered by a finite union of open sets $\{V_j\}$. This is necessary as it guarantees the existence of our partition of unity, as we are now able to construct a finite set of functions $\{\rho_i\}$ such that their compact support is a subset of some V_j. For the remainder of this proof, we denote:

$$\omega_i = \rho_i \omega$$

Let $\psi_i : U_i \longrightarrow X$ be a coordinate chart, where $U_i \subseteq H^n$ to guarantee orientability. Then, we have by definition of integration on a manifold:

$$\int_X d\omega_i = \int_{U_i} \psi^*(d\omega_i)$$

then, we use exterior derivative's commutativity with the pullback, and the extension of the coordinate chart $U_i \subset \mathbb{R}^n$ to all of H^n (we can discuss this quickly, simply denote $\psi_i : U_i \longrightarrow X$ as a restriction of $\psi_i : H^n \longrightarrow X$, this follows as we can embed U_i in H^n without changing the integration region) to obtain:

$$\int_{U_i} \psi^*(d\omega_i) = \int_{U_i} d(\psi^*\omega_i) = \int_{H^n} d(\psi^*\omega_i)$$

As we are working in Euclidean space now, we may apply **Stokes' 1** to obtain:

$$\int_{H^n} d(\psi^*\omega_i) = \int_{\partial H^n} \psi^*\omega_i$$

By definition of integration on manifolds:

$$\int_{\partial H^n} \psi^*\omega_i = \int_{\partial X} \omega_i$$

So we have proved it for a particular component of our (n-1)-form ω, we need to incorporate our partition of unity.

$$\int_X d\omega = \int_X d\left(\sum_i \omega_i\right) = \sum_i \int_X d\omega_i$$

We now perform the following computations:

$$\int_X d\omega = \int_X d\left(\sum_i \omega_i\right) = \sum_i \int_X d\omega_i$$

By Stokes' theorem for ω_i (which we proved just now):

$$\sum_i \int_X d\omega_i = \sum_i \int_{\partial X} \omega_i = \int_{\partial X} \sum_i \omega_i = \int_{\partial X} \omega$$

Therefore, we are done. $\square$

We now offer exercises to test our understanding of forms and integration of forms.

Exercise 2.1: Let γ be a regular, oriented, unit-speed path in $\mathbb{R}^3$, and let $v(x)$ be a vector field in $\mathbb{R}^3$. Show that:

$$\int_\gamma \omega_v^1 = \int v \cdot T ds$$

Where $\omega_v^1 = v_1 dx^1 + v_2 dx^2 + v_3 dx^3$ (as in **Exercise 1.1**), and T denotes the unit tangent vector to γ.

Solution to Exercise 2.1:

Let us break down the definition of T, and we will almost immediately get our solution.

$$T = \gamma'(s) \quad |\gamma'(s)| = 1$$

As γ is a path in $\mathbb{R}^3$, we have that:

$$\gamma(s) = (x(s), y(s), z(s)) \quad \gamma'(s) = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$T ds = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds = (dx, dy, dz)$$

Therefore, we immediately see that

$$\omega_v^1 = v \cdot T ds$$

$$\int_\gamma \omega_v^1 = \int v \cdot T ds$$

We progress forward in our next exercise:

Exercise 2.2: If S is an oriented surface in $\mathbb{R}^3$ and v is a vector field over $\mathbb{R}^3$, show that:

$$\int_S \omega_v^2 = \int_S v \cdot d\vec{S}$$

Where we define ω_v^2 as in **Exercise 1.1**:

$$\omega_v^2 = v_1 dx^2 \wedge dx^3 + v_2 dx^3 \wedge dx^1 + v_3 dx^1 \wedge dx^2$$

This is a surface integral written in terms of differential forms.

Solution to Exercise 2.2:

We assume that the surface has a parameterization so that the normal vector is unit magnitude. We use the fact that the integral of a form is invariant under a pullback to compute this.

$$h : U \longrightarrow S$$

Where U is some open subset of $\mathbb{R}^2$. Explicitly:

$$h : (s,t) \longmapsto (x(s,t), y(s,t), z(s,t)) = \gamma(s,t)$$

Here, γ is technically the **parametrization of our surface** S.

$$h^* : \Omega^k(S) \longrightarrow \Omega^k(U)$$

Just compute the pullback.

$$h^*x = x \circ h = x(s,t) \quad h^*y = y \circ h = y(s,t) \quad h^*z = z \circ h = z(s,t)$$

$$h^*dx = \frac{\partial x(s,t)}{\partial s} ds + \frac{\partial x(s,t)}{\partial t} dt$$
$$h^*dy = \frac{\partial y(s,t)}{\partial s} ds + \frac{\partial y(s,t)}{\partial t} dt$$
$$h^*dz = \frac{\partial z(s,t)}{\partial s} ds + \frac{\partial z(s,t)}{\partial t} dt$$
$$h^*\left(dy \wedge dz\right) = \left(\frac{\partial y(s,t)}{\partial s} ds + \frac{\partial y(s,t)}{\partial t} dt\right) \wedge \left(\frac{\partial z(s,t)}{\partial s} ds + \frac{\partial z(s,t)}{\partial t} dt\right)$$
$$h^*\left(dz \wedge dx\right) = \left(\frac{\partial z(s,t)}{\partial s} ds + \frac{\partial z(s,t)}{\partial t} dt\right) \wedge \left(\frac{\partial x(s,t)}{\partial s} ds + \frac{\partial x(s,t)}{\partial t} dt\right)$$
$$h^*\left(dy \wedge dz\right) = \left(\frac{\partial y(s,t)}{\partial s} ds + \frac{\partial y(s,t)}{\partial t} dt\right) \wedge \left(\frac{\partial z(s,t)}{\partial s} ds + \frac{\partial z(s,t)}{\partial t} dt\right)$$

If we compute these in full, we obtain the following expression for the pullback of our form ω_v^2:

$$h^*\left(\omega_v^2\right) = \left(\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}\right) v_1(h(s,t)) ds \wedge dt + \left(\frac{\partial z}{\partial s}\frac{\partial x}{\partial t} - \frac{\partial z}{\partial t}\frac{\partial x}{\partial s}\right) v_2(h(s,t)) ds \wedge dt$$

$$+ \left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right) v_3(h(s,t)) ds \wedge dt$$
$$= v \cdot \left(\frac{\partial \gamma}{\partial s} \times \frac{\partial \gamma}{\partial t}\right) dx \wedge dt$$

Now it is immediate:

$$\int_S \omega_v^2 = \int v \cdot \left(\frac{\partial \gamma}{\partial s} \times \frac{\partial \gamma}{\partial t}\right) dx \wedge dt$$

If we define the unit normal vector as:

$$\left(\frac{\partial \gamma}{\partial s} \times \frac{\partial \gamma}{\partial t}\right) = n$$

Then we simplify our integral:

$$\int_S \omega_v^2 = \int v \cdot n ds \wedge dt \equiv \int v \cdot n dS \equiv \int v \cdot d\vec{S}$$

Thus we have defined our surface integral in terms of forms.

Exercise 2.3: Suppose that X is a compact connected oriented 1-manifold with boundary in $\mathbb{R}^n$. Show that Stokes' Theorem, applied to X, is essentially the Fundamental Theorem of Calculus.

Solution to Exercise 2.3:

Stokes' Theorem, on an n-manifold X, is:

$$\int_X d\omega = \int_{\partial X} \omega$$

Let ω be a 0-form. $\omega = f(x)$. We see that ω can only be a 0-form as the exterior derivative must raise the degree of the form by 1. As our manifold is 1-dimensional, any form with degree above 1 must vanish. So we may only choose ω to be a 0-form.

$$\omega = f(x) \quad d\omega = \frac{\partial f}{\partial x} dx$$

$$\int_X \frac{\partial f}{\partial x} dx = \int_{\partial X} f(x)$$

By definition of integration of a 0-form, we just sum up the points in ∂X. As ∂X is boundary of a compact oriented 1-manifold, we see that ∂X is two points (as X is compact, hence closed and bounded).

$$\int_{\partial X} f(x) = \sum_{x \in \partial X} f(x) = f(b) - f(a)$$

Therefore, we obtain:

$$\int_X \frac{\partial f}{\partial x} dx = f(b) - f(a)$$

This is the Fundamental Theorem of Calculus.

Exercise 2.4: Now suppose that X is a bounded domain in $\mathbb{R}^2$. Write down Stokes' Theorem in this setting and relate it to Green's Theorem.

Solution to Exercise 2.4:

Let ω be a 1-form (this must also be the case given the dimension of X).

$$\omega = P(x, y)dx + Q(x, y)dy$$

By definition of exterior derivatives:

$$d\omega = \sum_j \frac{\partial \omega_I(x)}{\partial x^j} dx^j \wedge dx^I$$

$$d\omega = \frac{\partial P(x,y)}{\partial x} dx \wedge dx + \frac{\partial P(x,y)}{\partial y} dy \wedge dx + \frac{\partial Q(x,y)}{\partial y} dy \wedge dy + \frac{\partial Q(x,y)}{\partial x} dx \wedge dy$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

By Stokes' Theorem:

$$\int_{\partial X} (P(x,y)dx + Q(x,y)dy) = \int_X \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

This is Green's Theorem.

Exercise 2.5: Now suppose that S is an oriented surface in $\mathbb{R}^3$ with boundary curve $C = \partial S$. Let v be a vector field. Apply Stokes' Theorem to ω_v^1 and to S, and express the result in terms of line integrals and surface integrals. This should give you the classical Stokes' Theorem.

Solution to Exercise 2.5:

Once again we transcribe Stokes' Theorem:

$$\int_S d\omega = \int_{\partial S} \omega$$

Choose $\omega = \omega_v^1$. $\omega_v^1 = v_1 dx + v_2 dy + v_3 dz$. By **Exercise 1.1**, we see that:

$$\int_{\partial S} \omega = \oint \partial S v \cdot T ds$$

Furthermore, by **Exercise 1.1**, we see that:

$$d\omega = \omega_{\nabla \times v}^2$$

$$\int_S \omega_{\nabla \times v}^2$$

By Stokes' theorem:

$$\oint \partial S v \cdot T ds = \int_S \omega_{\nabla \times v}^2$$

Now, we see that by **Exercise 2.2**, that:

$$\int_S \omega_{\nabla \times v}^2 = \oint_S \nabla \times v \cdot d\vec{S}$$

Note, that we can rederive the last step by pulling back the 2-form $\omega_{\nabla \times v}^2$, we find that it is the identical calculation as **Exercise 2.2**.

Exercise 2.6: On $\mathbb{R}^3$, let

$$\omega = \left(x^2 + y^2\right) dx \wedge dy + (x + ye^z)\, dy \wedge dz + e^x dx \wedge dz$$

Compute

$$\int_{\partial S} \omega$$

Where S is the upper hemisphere of the unit sphere. The answer depends on the orientation you pick for S. Look at it from the "top"?

Solution to Exercise 2.6:

We will work this out after.

Exercise 2.7: On $\mathbb{R}^2$ with the origin removed, let $\alpha = \frac{(xdy - ydx)}{x^2+y^2}$. We showed that $d\alpha = 0$ (so α is closed). Show that α is not exact.

Solution to Exercise 2.7:

Although the intention was probably to use Stokes' Theorem in some manner, we will show this a more direct, less illuminating way.

Let a chart be:

$$h : U \longrightarrow \mathbb{R}^2 \backslash \{0\}$$
$$h : \theta \longmapsto (\cos(\theta), \sin(theta))$$

So that $U = (0, 2\pi)$. By definition of an integral on a manifold, we have:

$$\int_{\mathbb{R}^2 \backslash \{0\}} \alpha = \int_U h^* \alpha$$

Computing the pullback is simple enough, and since we have done it before, we will simply state the result here:

$$h^* \alpha = d\theta$$

Therefore, we obtain that:

$$\int_{\mathbb{R}^2 \backslash \{0\}} \alpha = \int_0^{2\pi} d\theta = 2\pi$$

This contradicts the fact that exact 1-forms on a closed curve vanish. Alternatively, Stokes' Theorem tells us that:

$$\int_{\mathbb{R}^2 \backslash \{0\}} \alpha = \int_M d\alpha = 0$$

Where $\mathbb{R}^2 \backslash \{0\}$ is the boundary of some M. Putting a string of equalities gives us:

$$2\pi = \int_0^{2\pi} d\theta = \int_U h^* \alpha = \int_{\mathbb{R}^2 \backslash \{0\}} \alpha = \int_M d\alpha = 0$$

This, alternatively, establishes our contradiction (this actually is how we would give a proof of the fact that exact 1-forms vanish on a closed curve).

Theorem 24. *Integrals of exact 1-forms over a closed curve in $\mathbb{R}^k$ vanish.*

Proof. Let X be an orientable, compact manifold, a subset of $\mathbb{R}^k$. Then ∂X is a closed curve in $\mathbb{R}^k$. Stokes' Theorem says:

$$\int_X d\omega = \int_{\partial X} \omega$$

As ω is exact, there exists a $\mu \in \Omega^0(\mathbb{R}^2)$ such that $d\mu = \omega$. Therefore, we obtain:

$$\int_X d(d\mu) = \int_{\partial X} \omega$$

As $d^2 = 0$, we have that the integral of ω over a closed curve ∂X vanishes.

$$\int_{\partial X} \omega = 0$$

$\square$

Onto the next exercise.

Exercise 2.8: On $\mathbb{R}^3 \backslash \{0\}$, show that:

$$\beta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{5/2}}$$

is closed, but not exact.

Solution to Exercise 2.8:

Let us compute the exterior derivative first.

$$d\beta = \frac{\partial \beta_1}{\partial x} dz \wedge dy \wedge dz + \frac{\partial \beta_2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial \beta_3}{\partial z} dz \wedge dx \wedge dy$$

(All the other terms vanish by antisymmetry of forms). If we compute the derivatives, we obtain that:

$$\frac{\partial \beta_i}{\partial x^i} dx^i \wedge dx^I = \frac{(x^I)^2 - 2(x^i)^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Then, summing over all i, we have that:

$$d\beta = 0$$

Thus, β is closed.

For exactness, we may either compute it, or we may transcribe the solution of **Exercise 1.11**, as we computed the integral of β (the denominator factor goes to 1 so it does not matter). We see that:

$$\int_X \beta = 4\pi$$

By Stokes' Theorem, this contradicts the fact that exact forms must vanish on closed curves in $\mathbb{R}^k$. Therefore, this is not exact.

Exercise 2.9: Let X be a compact, oriented n-manifold (without boundary), let Y be a manifold, and let ω be a closed n-form on Y. Suppose that f_0, f_1 are homotopic maps $X \longrightarrow Y$. Show that:

$$\int_X f_0^* \omega = \int_X f_1^* \omega$$

Is this true for null-homotopic maps?

Solution to Exercise 2.9:

We define homotopy.

Definition 4.1. Homotopy

Consider two maps f_0, f_1 (both at least C^0) such that $X \longrightarrow Y$. f_0 and f_1 are **homotopic** if there exists a continuous map:

$$H : X \times [0,1] \longrightarrow Y$$

Such that:

$$H(x,0) = f_0(x) \quad H(x,1) = f_1(x) \quad \forall x \in X$$

The difficulty here is that X has no boundary. We must work around this by introducing a map that has a boundary (hint: it's the homotopy!). As f_0 an f_1 are homotopic, related by the homotopy H. Let $[0,1]$ be our unit interval. We can then apply Stokes' Theorem to obtain:

$$\int_{\partial(X\times[0,1])} H^*\omega = \int_{X\times[0,1]} d\left(H^*\omega\right) = \int_{X\times[0,1]} H^*(d\omega) \equiv 0$$

Where the last equality follows because ω is closed.

We recognize that:

$$\partial\left(X \times [0,1]\right) = \left(X \times \{0\}\right) \cup \left(X \times \{1\}\right)$$

$\{1\}$ has positive orientation and $\{0\}$ has negative orientation. Therefore:

$$\int_{\partial(X\times[0,1])} H^*\omega = \int_{X\times\{1\}} H^*\omega - \int_{X\times\{0\}} H^*\omega$$

By our above result using Stokes' Theorem, we can immediately string together all our equalities together:

$$\int_{\partial(X\times[0,1])} H^*\omega = \int_{X\times\{1\}} H^*\omega - \int_{X\times\{0\}} H^*\omega = \int_{X\times[0,1]} d\left(H^*\omega\right) = \int_{X\times[0,1]} H^*(d\omega) \equiv 0$$

We now see that:

$$\int_{X\times\{1\}} H^*\omega = \int_{X\times\{0\}} H^*\omega$$

As H is a homotopy, we see that:

$$H\big|_{X\times\{1\}} = f_1 \quad H\big|_{X\times\{0\}} = f_0$$

Therefore,

$$\int_X f_1 = \int_X f_0$$

Note that this exercise was super easy given that f_0 and f_1 are homotopic. Proving a homotopy exists is the hard part.

Exercise 2.10: Let $f : S^1 \longrightarrow \mathbb{R}^2\backslash\{0\}$ be a smooth map whose winding number around the origin is k. Show that:

$$\int_{S^1} f^*\alpha = 2\pi k$$

Where

$$\alpha = \frac{xdy - ydx}{x^2 + y^2}$$

Use the pullback of f to finish what we did in **Exercise 1.6**.

Solution to Exercise 2.10:

We may transcribe the results of **Exercise 1.6** for a simple solution.

$$f : S^1 \longrightarrow \mathbb{R}^2\backslash\{0\}$$

$$\theta \longmapsto (\cos(\theta), \sin(\theta))$$

As f has a winding number of k, we see that the domain for θ is $(0, 2\pi k)$.

$$f^*x = \cos(\theta) \quad f^*y = \sin(\theta)$$

$$f^*dx = -\sin(\theta)d\theta \quad f^*dy = \cos(\theta)d\theta$$

Then, pulling back α to S^1, we have:

$$f^*\alpha = d\theta$$

Integrate over our region:

$$\int_0^{2\pi k} d\theta = 2\pi k$$

Note: There may be another solution by invoking Stokes' Theorem, but I do not know how to invoke it, as taking the exterior derivative of α will give us 0 as it is closed. This is the simplest and most intuitive way to obtain this result as far as I know.

On a side note, we have dealt with forms in a manner where we work on a real manifold, and we are given some algebraic properties of forms. Analysis of forms only requires some topological notions of connectedness and differentiability classes. We will be exploring the algebraic nature of forms in the coming section. It is very important that we understand what is going on structurally, and how we can prove the existence of everything we have constructed before.

5 Tensors

For our purposes, let V denote a finite-dimensional (n dimensional) vector space.

5.1 Tensors Over Vector Spaces

Definition 5.1. A **Vector Space** is also a free R-module over a commutative ring. Hence, each V, over the field F, is:

$$V = \bigoplus_{i}^{n} F$$

Definition 5.2. A k-tensor is a multilinear map:

$$T : V \times \cdots \times V \longrightarrow \mathbb{R}$$

The space of k-tensors is called the **Tensor Algebra** and it is denoted as $\mathbb{T}^k(V^*)$. More simply, a k-tensor takes in k arguments and spits out a number.

Let $\{b_1, \cdots, b_n\}$ be a basis of V. Every vector $v \in V$ can be uniquely expressed as a linear combination:

$$v = \sum_i v^i b_i$$

We first develop our ideas with the following object:

Definition 5.3. The **Projection Map** is a 1-tensor of the form:

$$\psi^i(v) = v^i$$

Proposition 25. *ϕ^i forms the basis for the space of 1-tensors.*

Proof. Let α be any 1-tensor. α takes in any vector $v \in V$ and spits out a number over the base field of V.

$$\alpha(v) = \alpha\left(\sum_i v^i b_i\right) = \sum_i v^i \alpha(b_i) = \sum_i \alpha(b_i)\phi^i(v)$$

Then by linearity of ϕ^i, we have that:

$$\sum_i \alpha(b_i)\phi^i(v) = \left(\sum_i \alpha(b_i)\phi^i\right)(v)$$

Therefore, the 1-tensor α may be written as:

$$\alpha = \sum_i \alpha(b_i)\phi^i$$

$\square$

Therefore, the space of 1-tensors $T^1(V^*)$, is called the dual space of V. And we write: $T^1(V^*) = V^*$.

Definition 5.4. The **dual space** V^* has a basis:

$$\{\phi^i\}$$

and is also a vector space. We call this basis the **dual basis** of $\{b_k\}$. Note that:

$$\phi^i(b_k) = \delta^i_k$$

We can treat tensor products systematically (i.e. develop it using fundamentals of category theory), but for computational purposes, let us treat them the following way.

Let α be a k-tensor, and β an l-tensor. Then, taking the tensor product yields:

$$(\alpha \otimes \beta)(v_1, \cdots, v_{k+l}) = \alpha(v_1, \cdots, v_k)\beta(v_{k+1, \cdots, k+l})$$

Where we define the basis 1-tensors by:

$$\phi^i \otimes \phi^j(v, \omega) = \phi^i\phi^j = v^i\omega^j$$

Under this rule, we have that $\phi^i \otimes \phi^j$ form a basis for $T^2(V^*)$, the space of 2-tensors.

Exercise 3.1: For each ordered k-index $I = \{i_1, \cdots, i_k\}$ (where $1 \leq i_j \leq n$), let

$$\tilde{\phi}^I = \phi^{i_1} \otimes \phi^{i_2} \otimes \cdot \otimes \phi^{i_k}$$

Show that $\tilde{\phi}^I$ forms a basis for $T^k(V^*)$.

Solution to Exercise 3.1:

This is our first foray into Tensor Algebras, therefore, we will investigate closely and lay our foundations here. We will elaborate on the theory behind tensor products later. For now, we will teach the reader how to compute tensor products.

If we take the tensor product, $V \otimes V$, then denoting the bases of each copy of V as $\{e_i\}$ and $\{e_j\}$, respectively, we obtain that a tensor product of two vectors $v = \sum_i v^i e_i$, $w = \sum_j w^j e_j$:

$$v \otimes w = \sum_i v^i e_i \otimes \sum_j w^j e_j = \sum_{i,j} v^i w^j e_i \otimes e_j$$

Therefore, if we take a 2-tensor , $\alpha \in T^2(V^*)$, and input the vectors v and w:

$$\alpha(v,w) = \alpha\,(v \otimes w) = \alpha\left(\sum_{i,j} v^i w^j e_i \otimes e_j\right)$$

Then by linearity, we distribute the α:

$$\sum_{i,j} v^i w^j \alpha(e_i \otimes e_j)$$

Then we see that by definition of $\tilde{\phi}$, that $v^i w^j = \phi^i \otimes \phi^j(v,w)$.

$$\alpha(v,w) = \sum_{i,j} \alpha(e_i, e_j)\left(\phi^i \otimes \phi^j\right)(v,w)$$

Then we see that: If we make use of the compact notation, $\alpha(e_i, e_j) \equiv \alpha_I$, $\phi^i \otimes \phi^j \equiv \tilde{\phi}^I$, where $I = \{i,j\}$, where $1 \leq i,j \leq \dim V$. Then we see that:

$$\alpha(v,w) = \sum_I \alpha_I \tilde{\phi}^I$$

Therefore, we see that $\tilde{\phi}^I$ spans the space of 2-tensors, $T^2(V^*)$.

To prove linear independence is more subtle. We prove it by deduction. Firstly, we see that if $\tilde{\phi}^I = 0$ for some I, then we have that:

$$\tilde{\phi}^I \equiv \phi^i \otimes \phi^j(v,w) = 0$$

So that either $v = 0$ or $w = 0$. However, this is expected as $\alpha(0,w) = \alpha(v,0) = \alpha(0,0) = 0$. Therefore, this is ok. And this proves that any of the $\tilde{\phi}^I$ are not 0, and that tensors will **only vanish**(for nontrivial vectors) if $\alpha_I \equiv 0$.

This implies that:

$$\alpha_I = \alpha(e_i \otimes e_j) = 0$$

We have two cases here:

(i) $e_i \otimes e_j = 0$

(ii) $\alpha(e_i \otimes e_k) = 0$ We see that case (i) is not possible as none of our basis elements can be 0 in V. Therefore, it must be that $\alpha_I = 0$ for all I. Hence, it proves that $\tilde{\phi}^I$ forms a linearly independent, spanning set for $T^2(V^*)$.

I proved it for the case where we have tensors for two n-dimensional vector spaces. This gives us that $\dim T^2(V^*) = n^2$.

From here on, we see that our argument generalizes to k copies of V, hence to $T^k(V^*)$. We are done.

We make special note that we may ALWAYS split a k-tensor into a symmetric part, and an anti-symmetric part.

5.2 Alternating Tensors

We give a quick aside on permutation groups. The set of permutations of a set $\{1, \cdots, k\}$ is a group and is denoted S_k, and called the *Symmetric Group*. The symmetric group on k-letters always has $k!$ distinct permutations of the set $\{1, \cdots, k\}$. All permutations are either even or odd, depending on how many transpositions compose the permutation.

Definition 5.5. The sign function will return $+1$ for even permutations, and -1 for odd permutations.

Definition 5.6. A k-tensor $\alpha \in T^k(V^*)$ is **alternating** if for any $\sigma \in S_k$, and an ordered collection of vectors $\{v_i\}$ in V:

$$\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \text{sign}(\sigma)\alpha(v_1, \cdots, v_k)$$

Definition 5.7. The space of alternating k-tensors on V is denoted as $\Lambda^k(V^*)$.

If $\alpha \in T^k(V^*)$, we define:

$$Alt(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma)\alpha \circ \sigma$$

Or if we apply it to a collection of vectors.

$$Alt(\alpha)(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma)\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

We will explore various properties of Alt:

Exercise 3.2: Show the following three things

1. Show that $Alt(\alpha) \in \Lambda^k(V^*)$.
2. Show that $Alt\big|_{\Lambda^k(V^*)} = id_{\Lambda^k(V^*)}$. This implies that Alt is a projection of $T^k(V^*)$ to $\Lambda^k(V^*)$.
3. Suppose that α is a k-tensor with $Alt(\alpha) = 0$, and that β is an arbitrary l-tensor. Then:
$$Alt(\alpha \otimes \beta) = 0$$

These are very important in developing the background for differential forms.

Solution to Exercise 3.2:

1. This one requires us to simply to have an algebraic trick.

   Act on Alt by a permutation τ.

$$\tau(Alt(v_1, \cdots, v_k)) = \sum_{\sigma \in S_k} \text{sign}(\sigma)\alpha(v_{\tau(\sigma(1))}, \cdots, v_{\tau(\sigma(k))})$$

Then we simply insert a resolution by identity. As $(\text{sign}(\tau))^2 = 1$, we have that:

$$\tau(Alt(v_1, \cdots, v_k)) = \sum_{\sigma \in S_k} \text{sign}(\sigma)\alpha(v_{\tau(\sigma(1))}, \cdots, v_{\tau(\sigma(k))})$$

$$= \text{sign}(\tau) \sum_{\tau,\sigma \in S_k} \text{sign}(\tau)\text{sign}(\sigma)\alpha(v_{\tau(\sigma(1))}, \cdots, v_{\tau(\sigma(k))}) = \text{sign}(\tau) \sum_{\tau,\sigma \in S_k} \text{sign}(\tau\sigma)\alpha(v_{\tau(\sigma(1))}, \cdots, v_{\tau(\sigma(k))})$$

$$= \text{sign}(\tau)Alt(\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

This shows that our Alt tensor behaves exactly as an element of $\Lambda^k(V^*)$.

2. This is extremely simple to prove. Simply recognize that for any $\alpha \in \Lambda^k(V^*)$, we have that $\tau \circ \alpha = \text{sign}(\tau)\alpha$.

$$Alt(\alpha) = \frac{1}{k!} \sum_{\tau} \text{sign}(\tau)\alpha = \frac{1}{k!} \sum_{\tau} \tau \circ \alpha = \frac{1}{k!} \sum_{\tau} \alpha(v_{\tau(1)} \cdots, v_{\tau(k)}) = \alpha$$

So for any alternating k-tensor, Alt serves as the identity.

3. This one is rather simple too.

$$Alt(\alpha) = \sum_{\tau} \text{sign}(\tau)\alpha(v_{\tau(1)}, \cdots, v_{\tau(k)}) = 0$$

By assumption. Then, as sign $\neq 0$, we see that the only way this is true is if $\alpha(v_{\tau(1)}, \cdots, v_{\tau(k)})) = 0$ for all the vectors $\{v_1, \cdots, v_k\}$. Thus, we expand out $Alt(\alpha \otimes \beta)$:

$$Alt(\alpha \otimes \beta) = \sum_{\sigma \in S_{k+l}} \text{sign}(\tau)\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

As $Alt(\alpha) = 0$, we see that the $\alpha \equiv 0$. Therefore,

$$Alt(\alpha \otimes \beta) = 0$$

This is crucial to performing arithmetic with alternating tensors.

We can now define a product operation on alternating tensors. If $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$, we define:

$$\alpha \wedge \beta = C_{k,l}Alt(\alpha \otimes \beta)$$

We look at more exercises concerning alternating tensors.

Exercise 3.3: Suppose that $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$, and that $C_{k,l} = C_{l,k}$. Show that:

$$\beta \wedge \alpha = (-1)^{kl}\alpha \wedge \beta$$

Note that this was just stated as a property in the beginning of the exploration of forms.

Solution to Exercise 3.3:

Let us transcribe what both $\alpha \wedge \beta$ and $\beta \wedge \alpha$ looks like:

$$\beta \wedge \alpha = Alt(\beta \otimes \alpha) \sum_{\tau} \text{sign}(\tau)\beta(v_{\tau(1)}, \cdots, v_{\tau(k)})\alpha(v_{\tau(k+1)}, \cdots, v_{\tau(k+l)})$$

$$\alpha \wedge \beta = Alt(\alpha \otimes \beta) = \sum_{\sigma} \text{sign}(\sigma)\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

Then, inspecting the indices, we see that:

$$\sigma(1) = \tau(k+1), \cdots, \sigma(k) = \tau(k+l)$$

$$\sigma(k+1) = \tau(1), \cdots, \sigma(k+l) = \tau(k)$$

We see in each factor, to traverse each of the k steps in the first set of permutations, there are l possible ways to fix the other tensor's permutations, therefore, there are kl possible combinations of interchanging elements between σ and τ. So we have the following relation:

$$\text{sign}(\tau) = (-1)^{kl}\text{sign}(\sigma)$$

Therefore, we obtain:

$$\beta \wedge \alpha = (-1)^{kl}\alpha \wedge \beta$$

Side note, our choice of normalization will be that:

$$C_{k,l} = \binom{k+l}{k}$$

The advantage here is that:

$$det = \phi^1 \wedge \cdots \wedge \phi^{k+l}$$

Our determinants and wedges have a direct correspondence in our choice of normalization.

Exercise 3.4: Show that, for our convention, that:

$$C_{k,l}C_{k+l,m} = C_{l,m}C_{k,l+m}$$

All choices of normalization will have this property so its nice to know that it is subject to choice.

Solution to Exercise 3.4:

This is trivial when we consider that this is just a binomial coefficient identity.

$$C_{k,l}C_{k+l,m} = \frac{(k+l)!}{k!l!}\frac{(k+l+m)!}{(k+l)!m!} = \frac{(k+l+m)!}{k!l!m!}$$

$$C_{l,m}C_{k,l+m} = \frac{(l+m)!}{l!m!}\frac{(k+l+m)!}{k!(l+m)!}$$

It is now easy to see that both sides of the equality are, in fact, equal.

NOTE: We recall that Alt acting on a k-tensor α has a factor of $\frac{1}{k!}$ originally. The advantage of our use of $C_{k,l}$ is that when we perform wedge products, we do not have to track ANY factorials, as our $C_{k,l}$ already include these. Therefore, when we do wedge products, we do not need to keep track of these factors. Notation is powerful!

Exercise 3.5: Suppose that the constants $C_{k,l}$ are chosen so that $C_{k,l}C_{k+l,m} = C_{l,m}C_{k,l+m}$. Suppose that α, β, γ are in $\Lambda^k(V^*), \Lambda^l(V^*), \Lambda^m(V^*)$, respectively. Show that:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$

This is also another important property that we glossed over in our study of forms.

Solution to Exercise 3.5:

We complete this proof in one go:

$$\alpha\wedge\beta = C_{k,l}Alt(\alpha\otimes\beta) = C_{k,l}\sum_{\sigma\in S_{k+l}} \text{Sign}(\sigma)\alpha(v_{\sigma(1)},\cdots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\cdots,v_{\sigma(k+l)})$$

Then wedge with the last m-tensor:

$$(\alpha \wedge \beta) \wedge \gamma = C_{k+l,m}Alt(C_{k,l}Alt(\alpha \otimes \beta) \otimes \gamma)$$

The constant comes out by linearity:

$$C_{k,l}C_{k+l,m}Alt(Alt(\alpha \otimes \beta) \otimes \gamma)$$

Let us use the shorthand notation $\alpha^\sigma = \alpha(v_{\sigma(1)},\cdots,v_{\sigma(k)})$ from now on to shorten our notation.

We will also prove a lemma that applies to any tensor of finite rank.

Lemma 26. *Let Alt denote the alternating tensor as we have used here. Let $\alpha \in T^k(V^*)$ and $\beta \in T^l(V^*)$. Then, we see that:*

1.

$$Alt(Alt(\alpha) \otimes \beta) = k!Alt(\alpha \otimes \beta)$$

2.

$$Alt(\alpha \otimes Alt(\beta)) = l!Alt(\alpha \otimes \beta)$$

Proof. If we prove one of them, we have the proof for both, it's just a matter of getting the factorial correct once we prove one of them.

By definition of Alt:

$$Alt(\alpha) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\alpha^\sigma$$

$$Alt(Alt(\alpha) \otimes \beta) = Alt\left(\sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\alpha^\sigma \otimes \beta\right)$$

Then, we may extend the action of σ from a set on k letters to a set of $k+l$ indices. This is because $\sigma \in S_k$, and S_k has the property that for every $n \neq k$, $S_n \supseteq S_k$. Therefore, we lose no information about the first k indices that σ is acting on if we extend the set that the group action is taking place. Therefore, we justifiably extend σ to the whole tensor product:

$$\alpha^\sigma \otimes \beta = (\alpha \otimes \beta)^\sigma$$

Then, we obtain:

$$Alt(Alt(\alpha) \otimes \beta) = Alt\left(\sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\,(\alpha \otimes \beta)^\sigma\right)$$

Then we expand the last Alt:

$$Alt(Alt(\alpha) \otimes \beta) = \sum_{\tau \in S_{k+l}} \operatorname{sign}(\tau) \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\,(\alpha \otimes \beta)^\sigma$$

Then we see as S_{k+l} is a group, that there exists $\mu = \tau\sigma$.

$$= \left(\sum_{\mu \in S_{k+l}} \operatorname{sign}(\mu)\,(\alpha \otimes \beta)^\sigma\right)$$

We see that:

$$\sum_{\mu \in S_{k+l}} = \sum_{\tau \in S_{k+l}} \sum_{\sigma \in S_k} = |S_k| \sum_{\tau \in S_{k+l}} = k! \sum_{\tau \in S_{k+l}}$$

Therefore, we use this fact to obtain:

$$k! \left(\sum_{\tau \in S_{k+l}} \operatorname{sign}(\mu)\,(\alpha \otimes \beta)^\sigma\right)$$

The expression in parentheses is exactly $Alt(\alpha \otimes \beta)$. Therefore,

$$Alt(Alt(\alpha) \otimes \beta) = k! Alt(\alpha \otimes \beta)$$

The other case is simple too. Recognize that every step is the same as the previous computation, except that because our initial Alt is acting on β. Thus,

$$Alt(\alpha \otimes Alt(\beta)) = l! Alt(\alpha \otimes \beta)$$

Remark: When our coefficients $C_{k,l}$ are present, we may simply set both $l = k = 1$, as our coefficients contain all the information about the combinatorics of our wedge products. Please try not to be confused! Just remember, when you see a $C_{k,l}$, forget about the factorials! If you don't see one, remember the factorials! $\square$

We now apply this lemma to the expression. Because we have our coefficients $C_{k,l}$, we may drop the factorials in the implications of **Lemma 26**:

$$C_{k+l,m}Alt(C_{k,l}Alt(\alpha \otimes \beta) \otimes \gamma)$$

$\alpha \otimes \beta$ is a $k+l$-tensor. So we will take it out:

$$= C_{k,l}C_{k+l,m}Alt((\alpha \otimes \beta) \otimes \gamma)$$

Then, we compute the other side of the equality:

$$\alpha \wedge (\beta \wedge \gamma)$$

By similar observations as before, we obtain:

$$C_{l,m}C_{k,l+m}Alt(\alpha \otimes Alt(\beta \otimes \gamma))$$

We now use the lemma above again and use the identity of the coefficients. This time, we use the second conclusion of the lemma to obtain:

$$= C_{l,m}C_{k,l+m}Alt(\alpha \otimes (\beta \otimes \gamma))$$

As tensor products are associative, we have that:

$$C_{k,l}C_{k+l,m}Alt((\alpha \otimes \beta) \otimes \gamma) = C_{l,m}C_{k,l+m}Alt(\alpha \otimes (\beta \otimes \gamma))$$

Keep in mind the notation that we use! We must always include a factorial **IF** we are just using the *Alt* operator. Only when we do a **wedge product** do we NOT need to account for this factor.

Exercise 3.6: Using the convention $C_{k,l} = \frac{(k+l)!}{k!l!} = \binom{k+l}{k}$, show that

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_k} = k!Alt(\phi^{i_1} \otimes \wedge \otimes \phi^{i_k})$$

We see the advantage of our notation here.

Solution to Exercise 3.6:

We opt for induction.

$$\phi^{i_1} \wedge \phi^{i_2} = C_{1,1}Alt(\phi^{i_1} \otimes \phi^{i_2})$$

$$(\phi^{i_1} \wedge \phi^{i_2}) \wedge \phi^{i_3} = C_{2,1}Alt(C_{1,1}Alt(\phi^{i_1} \otimes \phi^{i_2}) \otimes \phi^{i_3}) = C_{2,1}C_{1,1}Alt(\phi^{i_1} \otimes \phi^{i_2} \otimes \phi^{i_3})$$

Refer to the **Remark** in our proof of **Lemma 26**. We may induct onto the k-1 case:

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_2} \wedge \phi^{i_{k-1}} = C_{k-2,1} \cdots C_{2,1} Alt(\phi^{i_1} \otimes \cdots \phi^{i_{k-2}} \otimes \phi^{i_{k-2}})$$

Then we proceed to the next step:

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_{k-1}} \wedge \phi^{i_k} = C_{k-1,1} C_{k-2,1} \cdots C_{2,1} Alt(\phi^{i_1} \otimes \cdots \otimes \phi^{i_{k-1}} \otimes \phi^{i_k})$$

If we compute the product of all of our coefficients, we see:

$$C_{k-1,1} C_{k-2,1} \cdots C_{2,1} = k!$$

Therefore,

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_k} = k! Alt(\phi^{i_1} \otimes \cdots \otimes \phi^{i_k})$$

Furthermore, we see that:

$$\phi^I = \phi^{i_1} \wedge \cdots \wedge \phi^{i_k}$$

forms a basis for $\Lambda^k(V^*)$ and it gives:

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We can prove that ϕ^I forms a basis for $\Lambda^k(V^*)$ because $\tilde{\phi}^I$ forms a basis for $T^k(V^*)$, and $\Lambda^k(V^*)$ is simply a **Quotient Algebra** of $T^k(V^*)$.

Exercise 3.7: Let $V = \mathbb{R}^3$ with standard basis. Let $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given as $\pi : (x, y, z) \longmapsto (x, y)$ be the projection onto the x-y plane. Let $\alpha(v, w)$ be the signed area of the parallelogram spanned by $\pi(v)$ and $\pi(w)$ in the xy plane. Let β and γ be the signed areas of the projections of v and w in the xz and xy planes. Express α, β, γ as linear combinations of $\phi^i \wedge \phi^j$.

Exercise 3.8: Let V be an arbitrary vector space. Show that $(\phi^{i_1} \wedge \cdots \wedge \phi^{i_k})(b_{j_1}, \cdots, b_{j_k}) = +1$ if $\{j_k\}$ is an even permutation of $\{i_m\}$. -1 if an odd permutation, and 0 if neither.

Solution to Exercise 3.8:

We can either use our existing knowledge of wedge products, or plug in directly into the definition using tensor products. We will do the latter as the former is obvious, from our study of differential forms.

For notation sake, we denote:

$$\phi^{i_1} \wedge \phi^{i_k} = \phi^I \quad \sigma(I) = \sigma(i_1), \cdots, \sigma(i_k)$$

$$\tilde{\phi}^I = \phi^{i_1} \otimes \cdots \otimes \phi^{i_k}$$

$$\{b_{j_1}, \cdots, b_{j_k}\} = \{b_J\}$$

We are ready now.

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_k} = k! Alt(\phi^{i_1} \otimes \cdots \otimes \phi^{i_k})$$

Then, we use that:

$$Alt(\phi^{i_1} \otimes \cdots \otimes \phi^{i_k}) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma) \phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}$$

Then, simplifying, we have that:

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_k} = \sum_{\sigma \in S_k} \text{sign}(\sigma) \phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}$$

Let us act on $(b_{j_1}, \cdots, b_{j_k})$:

$$\left(\phi^{i_1} \wedge \cdots \wedge \phi^{i_k}\right)(b_{j_1}, \cdots, b_{j_k}) = \sum_{\sigma \in S_k} \text{sign}(\sigma) \left(\phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}\right)(b_{j_1}, \cdots, b_{j_k})$$

Now that we have simplified our action of the wedge product of the basis k-tensors on the elements of V, we can now consider:

(i) When $\{j_1, \cdots, j_k\}$ is an even permutation of $\{i_1, \cdots, i_k\}$.

$$\left(\phi^{i_1} \wedge \cdots \wedge \phi^{i_k}\right)(b_{j_1}, \cdots, b_{j_k}) = \sum_{\sigma \in S_k} \text{sign}(\sigma) \left(\phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}\right)(b_{j_1}, \cdots, b_{j_k})$$

We see that $\exists \sigma \in S_k$ such that $\sigma(I) = J$ with $\text{sign}(\sigma) = +1$. Therefore, we only have one nontrivial summand, therefore, we have that:

$$(+1)(\sigma) \left(\phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}\right)(b_{j_1}, \cdots, b_{j_k}) = \phi^{\sigma(i_1)}(b_{j_1}) \cdots \phi^{\sigma(i_k)}(b_{j_k}) = +1$$

(ii) When $\{j_1, \cdots, j_k\}$ is an odd permutation of $\{i_1, \cdots, i_k\}$.

$$\left(\phi^{i_1} \wedge \cdots \wedge \phi^{i_k}\right)(b_{j_1}, \cdots, b_{j_k}) = \sum_{\sigma \in S_k} \text{sign}(\sigma) \left(\phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}\right)(b_{j_1}, \cdots, b_{j_k})$$

We see that $\exists \sigma \in S_k$ such that $\sigma(I) = J$ with $\text{sign}(\sigma) = -1$. Therefore, we only have one nontrivial summand, therefore, we have that:

$$(-1)(\sigma) \left(\phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}\right)(b_{j_1}, \cdots, b_{j_k}) = (-1)\phi^{\sigma(i_1)}(b_{j_1}) \cdots \phi^{\sigma(i_k)}(b_{j_k}) = -1$$

(iii) When $\{j_1, \cdots, j_k\}$ is **NEITHER** an odd or even permutation of $\{i_1, \cdots, i_k\}$.

$$\left(\phi^{i_1} \wedge \cdots \wedge \phi^{i_k}\right)(b_{j_1}, \cdots, b_{j_k}) = \sum_{\sigma \in S_k} \text{sign}(\sigma) \left(\phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}\right)(b_{j_1}, \cdots, b_{j_k})$$

There exists **NO** permutation $\sigma \in S_k$ such that, $\sigma(I) = J$, therefore, $\forall \sigma \in S_k$:

$$\phi^{\sigma(i_1)}(b_{j_1}) \cdots \phi^{\sigma(i_k)}(b_{j_k}) = 0$$

So we have all of our cases.

Exercise 3.9: Let $\alpha \in \Lambda^k(V^*)$. For each subset $I = \{i_1, \cdots, i_k\}$ written in increasing order, let $\alpha_I = \alpha(b_{i_1}, \cdots, b_{i_k})$. Show that:

$$\alpha = \sum_I \alpha_I \phi^I$$

You can either compute this or state it as a consequence of something more general.

Solution to Exercise 3.9:

Exhibiting this through explicit computation should not be too difficult, however, we can find that this is true by using the result of **Exercise 3.1**.

We know that

$$\tilde{\phi}^I = \phi^{i_1} \otimes \cdots \otimes \phi^{i_k}$$

Forms a basis of $T^k(V^*)$. Noting that $\Lambda^k(V^*) \subset T^k(V^*)$, we just need to show that every possible alternating k-tensor in $T^k(V^*)$ can be written as some permuted version of $\tilde{\phi}^I$. This is simple enough to show:

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_k} = \sum_{\sigma \in S_k} \text{sign}(\sigma) \phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_k)}$$

This is sufficient to show that ϕ^I **spans the vector space of alternating k-tensors**, $\Lambda^k(V^*)$, as we have every possible alternating combination of the basis of the **tensor algebra** $T^k(V^*)$, and as $\tilde{\phi}^I$ spans every k-tensor, every possible alternating permutation of $\tilde{\phi}^I$ must span every possible alternating k-tensor.

This is a roundabout way of showing that the wedge product of the projection maps span the space of alternating tensors, but it is more indicative of the underlying structure of:

$$\Lambda^k(V^*) = T^k(V^*)/\sim$$

Where $\sim$ is some equivalence relation that gives us equivalence classes that are collapsed to identity in $T^k(V^*)$. We will elaborate after the next exercise.

Exercise 3.10: Let $\{\alpha_1, \cdots, \alpha_k\}$ be an arbitrary ordered list of covectors, and let $\{v_1, \cdots, v_k\}$ be an arbitrary ordered list of vectors. Show that:

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \cdots, v_k) = det(A)$$

A is the $k \times k$ matrix whose i,j entry is $\alpha_i(v_j)$.

Solution to Exercise 3.10:

This is almost immediate when we plug it into the definition of the wedge product using the Alt operator. We first use induction to obtain a factor of $k!$

out in the front, look at the proof of **Lemma 26**, it is exactly what we did to reduce it to the following formula.

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \cdots, v_k) = k! \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \left(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(k)}\right)(v_1, \cdots, v_k)$$

By definition of the tensor product of k-tensors:

$$(\alpha_1 \otimes \cdots \otimes \alpha_k)(v_1, \cdots, v_k) = \alpha_{\sigma(1)}(v_1, \cdots, v_k) \cdots \alpha_{\sigma(k)}(v_1, \cdots, v_k)$$

Therefore, we obtain

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \cdots, v_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \left[\alpha_{\sigma(1)}(v_1, \cdots, v_k) \cdots \alpha_{\sigma(k)}(v_1, \cdots, v_k)\right]$$

If this is not familiar enough, let $\operatorname{sign}(\sigma) = \epsilon^{1, \cdots, k}$ (this is the proper definition of the **Levi-Civita Symbol**), then we obtain:

$$\sum_{i_1, \cdots, i_k} \epsilon^{i_1, \cdots, i_k} \alpha_{i_1} \cdots \alpha_{i_k}$$

This is the definition of the determinant of a matrix with columns:

$$\alpha_i(\vec{v})$$

or with entries:

$$\alpha_i(v_j)$$

5.3 Pullbacks

We immediately start with a definition.

Definition 5.8. Suppose that $L : V \longrightarrow W$ is a linear transformation and that $\alpha \in T^k(W^*)$. The **pullback tensor** L^* is defined by:

$$(L^*\alpha)(v_1, \cdots, v_k) = \alpha(L(v_1), \cdots, L(v_k))$$

For shorthand, we see that:

$$L^* : T^k(W^*) \longrightarrow T^k(V^*)$$

$$L^* : \alpha \longmapsto \alpha \circ L$$

This is important. We will introduce some properties through exercises, which we will subsequently do. Pick bases $\{b_1, \cdots, b_n\}$ and $\{d_1, \cdots, d_m\}$ for V and W, respectively. And pick $\{\phi^i\}$ and $\{\psi^j\}$ be the dual bases for V^* and W^*. Let A be the matrix of the linear map L relative to the bases

$$L(v)^j = \sum_j A_{ji} v^i$$

We give an exercise:

Exercise 3.11: Show that the matrix of $L^* : W^* \longrightarrow V^*$, relative to the bases $\{\psi^j\}$ and $\{\phi^i\}$ is A^T.

Exercise 3.12: If $\alpha \in T^k(W^*)$, and if $I = \{i_1, \cdots, i_k\}$, show that

$$(L^*\alpha)_I = \sum_{j_1, \cdots, j_k} A_{j_1,i_1} A_{j_2,i_2} \cdots A_{j_k,i_k} \alpha_{j_1,\cdots,jk}$$

This follows immediately by the previous exercise.

Exercise 3.13: Suppose that $\alpha \in \Lambda^k(W^*)$. Show that $L^*\alpha \in \Lambda^k(W^*)$.

Solution to Exercise 3.13:

Recall that all elements of $\Lambda^k(^*)$ are defined by:

$$\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \text{sign}(\sigma)\alpha(v_1, \cdots, v_k)$$

If we simply apply the pullback, let us see what we obtain:

$$L^*\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = (\alpha \circ L)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \alpha(L(v_{\sigma(1)}), \cdots, L(v_{\sigma(k)}))$$

Denoting each $L(v_{\sigma(j)}) = L(v)_{\sigma(j)}$, we obtain:

$$L^*\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \alpha(L(v)_{\sigma(1)}, \cdots, L(v)_{\sigma(k)})$$

As α is alternating, we see that:

$$L^*\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \text{sign}(\sigma)\alpha(L(v)_1, \cdots, L(v)_k)$$

Note that the conclusion would have been obvious as $L(v_j) \in W$ anyways, so it follows that $\alpha \circ L \in \Lambda^k(W^*)$ as $\alpha \in \Lambda^k(W^*)$ as well.

Exercise 3.14: If $\alpha, \beta \in \Lambda^k(W^*)$, show that:

$$L^*(\alpha \wedge \beta) = (L^*\alpha) \wedge (L^*\beta)$$

That is, pullbacks (in general) distribute over wedge products. Note that this was something we just accepted in our study of forms. We seek to prove it rigorously.

Solution to Exercise 3.14:

This is very straightforward to prove by definition, and really proves the power of the notation and machinery we have developed.

For generality, we let $\alpha \in \Lambda^k(W^*)$ and $\beta \in \Lambda^l(W^*)$. By definition of wedge product of two alternating tensors:

$$\alpha \wedge \beta = C_{k,l} Alt(\alpha \otimes \beta)$$

Expand the definition of Alt:

$$C_{k,l}\sum_{\sigma\in S_{k+l}}\operatorname{sign}(\sigma)\left(\alpha\otimes\beta\right)^{\sigma}$$

Now apply the pullback:

$$L^*(\alpha\wedge\beta)=L^*\left[C_{k,l}\sum_{\sigma\in S_{k+l}}\operatorname{sign}(\sigma)\left(\alpha\otimes\beta\right)^{\sigma}\right]$$

By linearity:

$$=C_{k,l}\sum_{\sigma\in S_{k+l}}\operatorname{sign}(\sigma)L^*\left(\alpha\otimes\beta\right)^{\sigma}$$

Let us consider that our tensor acts on vectors $(w_1,\ldots,w_{k+l})$, and consider what the tensor product of tensors does:

$$\alpha\otimes\beta(w_1,\ldots,w_{k+l})=(\alpha\beta)(w_1,\ldots,w_{k+l})=\alpha(w_1,\ldots,w_k)\beta(w_{k+1},\ldots,w_{k+l})$$

Furthermore, when we act on these tensors with a permutation:

$$\left(\alpha\otimes\beta\right)^{\sigma}(w_1,\ldots,w_{k+l})=\alpha(w_{\sigma(1)},\ldots,w_{\sigma(k)})\beta(w_{\sigma(k+1)},\ldots,w_{\sigma(k+l)})$$

When we apply our pullback, we obtain:

$$L^*\left(\left(\alpha\otimes\beta\right)^{\sigma}(w_1,\ldots,w_{k+l})\right)=\left(\alpha\otimes\beta\right)^{\sigma}\left(L(w_1),\ldots,L(w_{k+l})\right)=\left(\alpha\otimes\beta\right)^{\sigma}\left(L(w)_1,\ldots,L(w)_{k+1}\right)$$

$$=\left(\alpha\otimes\beta\right)\left(L(w)_{\sigma(1)},\ldots,L(w)_{\sigma(k+1)}\right)$$

Applying the definition of the tensor product:

$$\left(\alpha\otimes\beta\right)\left(L(w)_{\sigma(1)},\ldots,L(w)_{\sigma(k+1)}\right)=\alpha(L(w)_{\sigma(1)},\ldots,L(w)_{\sigma(k)})\beta(L(w)_{\sigma(k+1)},\ldots,L(w)_{\sigma(k+1)})$$

The computations tell us that:

$$L^*\left(\alpha\otimes\beta\right)=\left(\alpha\otimes\beta\right)\circ L=\left(\alpha\circ L\right)\otimes\left(\beta\circ L\right)=\left(L^*\alpha\right)\otimes\left(L^*\beta\right)$$

The permutations distribute over each operation so we do not have to worry about them.

$$L^*\left(\alpha\wedge\beta\right)=C_{k,l}\sum_{\sigma\in S_{k+l}}\operatorname{sign}(\sigma)\left[\left(L^*\alpha\right)\otimes\left(L^*\beta\right)\right]^{\sigma}$$

This is just the definition of Alt:

$$=C_{k,l}Alt\left(L^*\alpha\otimes L^*\beta\right)=L^*\alpha\wedge L^*\beta$$

Therefore, the pullback distributes over the wedge product.

5.4 Cotangent Bundles and Forms

With all the tensor machinery developed (in particular, the pullback, wedge product, and how the pullback and wedge product interact), we now define forms on manifolds.

Definition 5.9. Let X be a k-manifold. A **k-dimensional vector bundle over** X is a pair (E, π), where E is a manifold and π is a surjection $\pi : E \longrightarrow X$ so that:

1. The preimage $\pi^{-1}(p)$, for any $p \in X$, is an n-dimensional vector space over $\mathbb{R}$. This vector space is called the **fiber over** p.
2. For every $p \in X$, there exists a neighborhood U, and a diffeomorphism $\phi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$, so that for every $x \in U$, $\phi_U\big|_{\pi^{-1}(x)}$ **is a linear isomorphism** from $\pi^{-1}(x)$ to $x \times \mathbb{R}^n$.

We develop further definitions:

Definition 5.10. A **section** of a vector bundle $E \longrightarrow X$, is a smooth map $s : X \longrightarrow E$, so that:
$$\pi \circ s = id_X$$
Or alternatively, $\forall x \in X$, $s(x) \in \pi^{-1}(x)$.

Definition 5.11. A differential form of degree k is a section of $\Lambda^k(T^*(X))$ (where $T^*(X)$ denotes the **cotangent bundle over** X). The infinite dimesnional space of k-forms (the one that we're used to using) is $\Omega^k(X)$.

Definition 5.12. If $f : X \longrightarrow \mathbb{R}$ is a function, then $df_x : T_x(X) \longrightarrow T_{f(x)}(\mathbb{R}) = \mathbb{R}$ is a **covector** at X. Therefore, every (C^1) function f defines the 1-form, df.

Definition 5.13. If $f : X \longrightarrow Y$ is a smooth map between manifolds X amd Y, then df_x is a linear map $T_x(X) \longrightarrow T_{f(x)}(Y)$. This induces a pullback tensor $f^* : \Lambda^k\left(T^*_{f(x)}(Y)\right) \longrightarrow \Lambda^k\left(T^*_x(X)\right)$, and therefore, a linear map $f^* : \Omega^k(Y) \longrightarrow \Omega^k(X)$.

5.5 Construction of the Tangent Bundle and the kth Tensor Power of the Cotangent Bundle

Here, we will contruct an important example of a vector bundle, the **Tangent Bundle**.

To start, we will define the **Tangent Space at a point** $x \in M$ to be the set of all vectors that are tangent to a point x on a manifold M. We denote the Tangent Space of M at the point x as T_xM. Then, in the similar way we glued open subsets of $\mathbb{R}^n$ to form an n-manifold, we will glue together all tangent spaces to form the tangent bundle.

We want to make sure that it is true that we can define:

$$\coprod_{p \in M} T_p M$$

We see that this is well-defined as we can simply "relabel" our tangent space:

$$T_p M \cong p \times T_p M$$

We see that the above holds trivially true, nothing changes about our space, we are simply labeling each T_pM by p.

$$\coprod_{p \in M} T_p M = \bigcup_{p \in M} p \times T_p M$$

As every distinct point is disjoint in M (our space is Hausdorff by default, so this is always true), we see that our union is, in fact, a disjoint union.

$$\bigcup_{p \in M} p \times T_p M = \{(p, q) | p \in M, q \in T_p M\}$$

We denote this set as TM, and this is called the **Tangent Bundle**.

$$TM = \{(p, q) | p \in M, q \in T_p M\}$$

We can now verify that this set, with the appropriate properties, is a vector bundle.

As our set is a product, **the category of sets admits a natural projection, guaranteed to exist by the universal property of the product and coproduct.**

$$\pi : TM \longrightarrow M$$

$$\pi : (p, q) \longmapsto p$$

We must only verify the properties of the vector bundle given in **Definition 5.9**.

1. We see that the first requirement for a vector bundle is true.

   $$\pi^{-1}(p) = p \times T_p M$$

   And, by construction, $p \times T_pM \cong T_pM$, and T_pM is surely a vector space over $\mathbb{R}$. Therefore, The fiber over all $p \in M$ is a real vector space.

2. The second requirement is also trivially satisfied, although this requires more exploration.

   What we must recognize here is that T_pM acts on functions on local patches of M. i.e. in a local coordinate, T_pM acts on the coordinate functions x^j:

   $$\frac{\partial x^j}{\partial x^j} = e_j \quad e_j = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$$

Therefore, if we take the particular instance when T_pM acts on the coordinate functions defined locally in M at every neighborhood of p $\forall p \in M$, denoted as U_p, **then we obtain the standard basis of** $\mathbb{R}^n$. Therefore, we have a trivial correspondence by the (1):

$$\pi^{-1}(p) = p \times T_pM$$

Then, using the fact that the **Product Topology** admits continuous (natural) projections, we can extend this to any neighborhood of p:

$$\pi^{-1}(U_p) = U_p \times T_pM$$

which implies that if we restrict the action of T_pM to coordinate functions in a local neighborhood of p:

$$\pi^{-1}(U_p) = U_p \times \mathbb{R}^n$$

This correspondence is immediate.

Therefore, with the additional remark that T_pM is a manifold, and therefore TM is a manifold, we see that the pair:

$$(TM, \pi)$$

Is indeed a **Vector Bundle**.

Furthermore, our construction with the above works analogously with the Cotangent Space T_p^*M. We denote the **Cotangent Bundle** as T^*M. We inherit all the properties of the **Tangent Bundle** by just taking the fiber to be the **Cotangent Space** T_p^*M:

$$\pi^{-1}(p) = T_p^*M \quad \forall p \in M$$

We can continue this construction for any degree of tensors over T_p^*M, and this is called the **kth Tensor Power** and is denoted as:

$$T^k(T^*M)$$

Where the fiber of the **kth Tensor Power of the Cotangent Bundle** (or more compactly called the *kth Tensor Power*, the cotangent bundle is implied in this context) is:

$$\pi^{-1}(p) = T^k(T_p^*M) \quad \forall p \in M$$

Furthermore, we may restrict the kth tensor power of the cotangent bundle to just the *kth Exterior Power*, $\Lambda^k(T^*M)$, and we obtain the **kth Exterior Power of the Cotangent Bundle**, or more compactly named *kth Exterior Power*.

The **kth Exterior Power of the Cotangent Bundle** is a vector bundle whose fiber is simply the **kth exterior power of the Cotangent Space**, i.e.

$$\pi^{-1}(p) = \Lambda^k(T_p^*M)$$

Definition 5.14. Tangent Vectors

We will define **Tangent Vectors** rigorously here using the machinery of vector bundles.

A **Tangent Vector** is simply a section of a Tangent Bundle. This is entirely trivial when we refer to our construction of a tangent bundle, the **Fiber of a Tangent Bundle** is simply the **Tangent Space**, T_pX. Therefore, **any Tangent Vector is a section of the Tangent Bundle** TX.

We will also show, explicitly, that the Tangent Space is a vector space, with dimension matching the Euclidean dimension of the manifold.

Theorem 27. *The* ***Tangent Space****,* T_pM*, is a vector space spanned by* $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$

Proof. Recall **Definition 1.8**. There, we defined the **Directional Derivative of a function** f as:

$$\nabla f \cdot e = \sum_{i=1}^{dim(M)} \frac{\partial f}{\partial x^i} e_i$$

Therefore, we see that the **Directional Derivative** is just:

$$\nabla \cdot e = \sum_{j=1}^{dim(M)} \frac{\partial}{\partial x^j} e_j$$

Therefore, **if we denote the Tangent Space as the set of all Directional Derivatives**, then we can clearly see our theorem is true now.

Any element $v \in T_pM$ is:

$$v = \sum_{j=1}^{dim(M)} v^j \frac{\partial}{\partial x^j}$$

Therefore, the fact that our set spans the Tangent Space is obvious.

The linear independence is not obvious at all, it requires extensive proof and we are not going to do that here. We take it for granted that the set of derivations in each basis coordinate is linearly independent. $\square$

Definition 5.15. Cotangent Vectors

A cotangent vector is defined relative to the tangent vectors. Let $f \in C^\infty(M)$. A **Cotangent Vector** is a 1-form at a point p, df_p, that acts on our tangent vector at p, V_p by:

$$df_p : V_p \longmapsto V_p(f) \quad V_p \in T_pM$$

Therefore, the cotangent vector df_p's job is to evaluate the directional derivative of our smooth function f at a point p. In this sense, it is indeed dual to a tangent vector V_p.

We must prove that the cotangent vectors form a vector space.

Theorem 28. *Cotangent Space is a Vector Space over* $\mathbb{R}$.

Proof. Let T_p^*M be our Cotangent Space. Let df_p be any cotangent vector. And then we see that dx^j must form the basis for our cotangent vectors. We can indeed see this as:

$$df_p = \sum_{j=1}^{dim(M)} \frac{\partial f}{\partial x^j} dx^j$$

Then, each dx^j must clearly span the space. The linear independence can be proven easily. See **Section 5.6**. $\square$

Definition 5.16. Differential Forms

We now focus on what our notes were originally about, **Differential Forms**. A **Differential Form** is a **section** of $\Lambda^k(T^*X)$. We know this to be true as a differential form of degree k, ω, is simply:

1. A **totally antisymmetric tensor** of rank $(0, k)$ (0 denotes the dual space, k is in the original vector space). Therefore,

$$\omega \in \Lambda^k(V^*)$$

2. A differential form is, locally, a function of the following form:

$$\omega = \omega_I(x_{i_1}, \ldots, x_{i_k}) \frac{\partial x^{i_1}}{\partial x^{i_1}} dx^{i_1} \wedge \cdots \wedge \frac{\partial x^{i_k}}{\partial x^{i_k}} dx^{i_k}$$

   We see that the tangent vectors pop up in the dx^{i_j}, therefore, we have that $V = T_pX$.

The above two observations are sufficient in showing that any k-form ω is a section of $\Lambda^k(T^*X)$, or that the forms are an element of $\Lambda^k(T_p^*X)$.

NOTE: The infinite dimensional space of k-forms on X, $\omega^k(X)$ is a, free, $C^\infty(X)$-module, and $C^\infty(X)$ is an infinite-dimensional, real vector space. Therefore, $\Omega^k(X)$, itself, is an **infinite-dimensional real vector space.**

We will now offer an exercise:

Exercise 3.15: If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are smooth maps of manifolds, then $g \circ f$ is a smooth map $X \longrightarrow Z$. Show that $(g \circ f)^* = f^* \circ g^*$.

Solution to Exercise 3.15:

We offer the following commutative diagram, in the category of smooth manifolds, to illustrate our proof.

$$\begin{tikzcd} T_x(X) \arrow[r, "df_x"] \arrow[dr] & T_y(Y) \arrow[d, "dg_y"] \\ & T_z(Z) \end{tikzcd}$$

We see that because $x \longmapsto f(x)$ and $y \longmapsto g(y)$, and that $y = f(x), \forall x \in X$. We see that $z = g(f(x))$. Because of our composition rule, we see that the above diagram must be completed:

$$\begin{tikzcd} T_x(X) \arrow[r, "df_x"] \arrow[dr, "d(g\circ f)_x"] & T_{f(x)}(Y) \arrow[d, "dg_{f(x)}"] \\ & T_{g(f(x))}(Z) \end{tikzcd}$$

We see that the linear map $d(g \circ f)_x$ induces a pullback:

$$(g \circ f)^* : \Lambda^k \left(T_{g(f(x))} Z\right) \longrightarrow \Lambda^k \left(T_x X\right)$$

We also see that:

$$df_x \Longrightarrow f^* : \Lambda^k \left(T_{f(x)} Y\right) \longrightarrow \Lambda^k \left(T_x X\right)$$

$$dg_{f(x)} \Longrightarrow g^* : \Lambda^k \left(T_{g(f(x))} Z\right) \longrightarrow \Lambda^k \left(T_{f(x)} Y\right)$$

Thus, by reversing the arrows on our existing commutative diagram, we obtain the following commutative diagram:

$$\begin{tikzcd} \Lambda^k \left(T_x(X)\right) & \Lambda^k \left(T_{f(x)}(Y)\right) \arrow[l, "f^*"'] \\ & \Lambda^k \left(T_{g(f(x))}(Z)\right) \arrow[u, "g^*"] \arrow[ul, "(g\circ f)^*"] \end{tikzcd}$$

Extending to infinite dimensions, we see the following commutative diagram to also be true.

$$\begin{tikzcd} \Omega^k(X) & \Lambda^k \Omega^k(Y) \arrow[l, "f^*"'] \\ & \Omega^k(Z) \arrow[u, "g^*"] \arrow[ul, "(g\circ f)^*"] \end{tikzcd}$$

Therefore we see that $(g \circ f)^* = f^* \circ g^*$.

5.6 Old and New Views of Forms

We will now attempt to reconcile the old and new formulations of differential forms. We will not lay out the general strategy, however we will attempt to reconcile the two formulations, and show that the previous computations with arbitrary "symbols" is, indeed, equivalent to what we have developed painstakingly here.

First off, we assume that $T_p M$ has the standard basis, $\{e_1, \ldots, e_n\}$. Because:

$$dx^j = \frac{\partial x^j}{\partial x^i} = \delta_i^j$$

The covectors $d_p x^1, \ldots, d_p x^n$ form a basis for $T_p^*(\mathbb{R}^n)$, and it is dual to $\{e_1, \ldots, e_n\}$ by definition of the cotangent vectors (using tangent vectors).

Recall that the projections, $\phi^j \in T^1(V^*)$, forms a basis for $T^1(V^*)$, and taking $V = T_pM$, we obtain that our basis for the cotangent space is indeed, a basis. Notice how incredibly simple showing that $d_p x^1, \ldots, d_p x^n$ is a basis is without invoking any analytical proof, it follows from the algebraic properties of the Tensor Power.

We also need to define the exterior derivative, as to recover the old definition.

$$d(dx^i) = d^2(x^i) = 0$$

$$d(dx^i \wedge dx^j) = d(dx^i) \wedge dx^j - dx^i \wedge d(dx^j) = 0$$

This leads us to:

$$d(dx^I) = 0$$

by induction on the previous statement. Then, letting $\alpha \in \Omega^k(M)$, we see:

$$d\left(\alpha_I dx^I\right) = \sum_I d\alpha_I \wedge dx^I + (-1)^k \alpha_I d(dx^I) = \sum_I d\alpha_I dx^I$$

Then as α_I is a 0-form, then $d\alpha_I$ is a 1-form:

$$= \sum_{I,j} (\partial_j \alpha_I) dx^j \wedge dx^I$$

Therefore, we see that the definition of the cotangent vectors and the antisymmetry of the wedge product gives us the exact definition of an exterior derivative, in local coordinates on M.

Exterior derivatives work exactly as expected on manifolds. We need to find find functions f^i on a neighborhood of a point p, $V_p \subset M$, such that df^i spans T_p^*M. This is always available because of the parameterization. If we have a parametrization (our charts):

$$\psi : U \longrightarrow X$$

We pick f^i to be the pullback of a coordinate function x^i:

$$\psi^{-1^*} x^i = x^i \circ \psi^{-1}$$

Then, we check that **pullbacks** are the same. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Using the pullback as we defined in **Section 5.3**, we obtain:

$$g^*\left(dy^i\right)(v) = d\left(y^i(g(v))\right) \equiv dg^i$$

Furthermore, as we have proven that:

$$g^*(\alpha \wedge \beta) = g^*\alpha \wedge g^*\beta$$

We see that:

$$g^* \left(\sum_I (x) \alpha_I dy^I \right) = \sum_I \alpha_I (g(x)) dg^{i_1} \wedge \cdots \wedge dg^{i_k}$$

Therefore, our pullback acts on forms exactly as we expect it to.

Furthermore, by our action of g^* on the Cotangent basis, we see that it is true that, in general:

$$g^*(d\alpha) = d(g^*\alpha)$$

Therefore, we have that our pullback, rigorously defined, is exactly the pullback that we had worked with before.

Next, we check that **forms on manifolds** are the same as before. Let X be an n-manifold, and let $\psi : U \longrightarrow X$ be a chart (or parametrization), where $U \subset \mathbb{R}^n$. Assume that $a \in U$, and let $p = \psi(a)$. The standard bases for the corresponding spaces are the following:

$$T_a(\mathbb{R}^n) \Longrightarrow \{e_1, \ldots, d_n\} \quad e_j \equiv \frac{\partial}{\partial x^j}$$

$$T_a^*(\mathbb{R}^n) \Longrightarrow \left\{dx^1, \ldots, dx^n\right\}$$

Then let $b_i = dg_a(e_i)$, and $\{b_i\}$ form a basis for $T_p(X)$. Let $\{\phi^j\}$ be a dual basis. Then:

$$\psi^*(\phi^j)(e_i = \phi^j(dg_a(e_i)) = \phi^j(b_i) = \delta_i^j$$

But we additionally see that:

$$= dx^j(e_i)$$

Therefore, we conclude, by uniqueness of the cotangent vectors, that:

$$\psi^*(\phi^j) = dx^j$$

In this new formulation, a form on a manifold X is one that is automatically pulled back to an open subset of $\mathbb{R}^n$, U. The parametrization takes a basis from $T_p^*(X)$ to a basis for $T_a^*(\mathbb{R}^n)$. This automatically gives an isomorphism between forms on a neighborhood of $p \in X$ and $a \in U \subset \mathbb{R}^n$.

If ψ_1, ψ_2 are two different parametrizations on the same neighborhood of p, if $\psi_1 = \psi_2 \circ g_{12}$, then by our previous exercise (**Exercise 3.15**), we have:

$$\psi_1^* = g_{12}^* \circ \psi_2^*$$

Therefore, our pullbacks behave as expected, and we have freedom to work with forms, given that we work on a smooth manifold and have smooth forms.

The reasons for developing this formalism is that this allows us to gain an intrinsic viewpoint of forms on a manifold X i.e. we do not have to regard forms as functions in $\mathbb{R}^n$ that act on surfaces that are subsets of $\mathbb{R}^n$.

6 de Rham Cohomology

We will forego our discussion of integration until we have much more powerful machinery. We will develop **de Rham Cohomology** here and we will return to integration in the future.

6.1 Closed and Exact Forms

Let X be an n-manifold, let $\alpha \in \Omega^k(X)$.

Definition 6.1. $\alpha \in \Omega^k(X)$ is a **closed form** if $d\alpha = 0$.

$\alpha \in \Omega^k(X)$ is an **exact form** if $\exists \beta \in \Omega^{k-1}(X)$ such that $d\beta = \alpha$.

We note the following things:

(i) Every exact form is a closed form as $d(d\beta) = 0$

(ii) A 0-form is closed if and only if it is locally constant (attains a constant value on some connected component of X)

(iii) Every n-form is closed, since $d\alpha \in \Omega^{n+1}(X)$, and every n+1-form over an n-manifold identically vanishes.

Definition 6.2. Consider the following exact sequence of spaces of k-forms:

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d_1} \Omega^1(X) \xrightarrow{d_2} \cdots \xrightarrow{d_{k-1}} \Omega^{k-1}(X) \xrightarrow{d_k} \Omega^k(X) \xrightarrow{d_{k+1}} \Omega^{k+1}(X) \xrightarrow{d_{k+2}} \cdots$$

All k-forms are a subspace of closed f-forms, we define the quotient space:

$$H^k_{DR}(X) = Z^k(X)/B^k(X)$$

Where $Z^k(X) = ker(d_{k+1})$, and $B^k(X) = im(d_k)$. Then $H^k_{DR}(X)$ is the **kth de Rham Cohomology of** X.

If α is a closed form, we denote $[\alpha]$ to denote the class of α in H^k, and α is the **representative** of the cohomology class $[\alpha]$.

Definition 6.3. In the **kth de Rham Cohomology of** X, the equivalence relation in the space of exact forms is given by:

$$\alpha \sim \alpha' \quad \alpha' = \alpha + d\mu$$

For $\alpha, \alpha' \in \Omega^k(X)$, $\mu \in \Omega^{k-1}(X)$.

Proposition 29. *The relation given just now, $\sim$, is an equivalence relation.*

Proof. Let $\alpha, \alpha', \alpha'' \in \Omega^k(X)$, $\mu, \gamma \in \Omega^{k-1}(X)$.

1. Reflexive:

   This is trivial as $\alpha \sim \alpha$ as 0 is clearly closed, and it is trivially exact.

2. Symmetric:

   Let $\alpha' \sim \alpha$, then $\alpha' = \alpha + d\mu$. Note that we can also consider $\alpha = \alpha' - d\mu$. When we act on both sides of both equalities by an exterior derivative, we obtain that:
   $$d\alpha = d\alpha'$$

3. Transitive:

   Assume that $\alpha \sim \alpha'$, $\alpha' \sim \alpha''$.
   $$\alpha' = \alpha + d\mu \quad \alpha'' = \alpha' + d\gamma$$
   Then, plugging in:
   $$\alpha'' = \alpha + d\mu + d\gamma$$
   By linearity, we have that $d\mu + d\gamma = d(\mu + \gamma)$. Noting that the space of k-forms is a module over the space of smooth functions, $\mu + \gamma = \beta$ for $\beta \in \Omega^{k-1}(X)$. Therefore, we have that:
   $$\alpha'' = \alpha + d\beta$$
   And taking the exterior derivative of both sides, we get:
   $$d\alpha'' = d\alpha$$

Thus, we have established $\sim$ as an equivalence relation. □

The wedge product of forms has an extension to a product operation of Cohomologies:
$$H^k(X) \times H^l(X) \longrightarrow H^{k+l}(X)$$
If α, β are closed forms, then:
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta = 0$$
So a wedge product of two closed forms is closed. Therefore, $\alpha \wedge \beta$ are representatives of a class in H^{k+l}. Or more shortly,
$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$$
Or even more succinctly, we may operate on classes of forms instead of the forms themselves (this is similar to how we add and multiply cosets of groups and ideals of rings).

Proposition 30. *The **Wedge Product**, as a quotient space operation:*
$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$$
Is well-defined.

Proof. We must input equivalence classes into the arguments of the wedge product, and show that the resulting image of the wedge is the same i.e. suppose that:

$$[\alpha] = [\alpha'] \quad [\beta] = [\beta']$$

Then we must show:

$$[\alpha' \wedge \beta'] = [\alpha \wedge \beta]$$

If $[\alpha] = [\alpha']$, then $\alpha \sim \alpha'$. Similarly for β, β'.

$$\alpha' = \alpha + d\mu \quad \beta' = \beta + d\nu$$

And then let us plug it in:

$$\alpha' \wedge \beta' = (\alpha + d\mu) \wedge (\beta + d\nu)$$

$$= \alpha \wedge \beta + \alpha \wedge d\nu + d\mu \wedge \beta + d\mu \wedge d\nu$$

Then, using the fact that α, β are closed forms, and ν, μ are exact forms, we have:

$$d(\mu \wedge \beta) = d\mu \wedge \beta + (-1)^k \mu \wedge d\beta$$

But as β is closed, $d\beta = 0$. So

$$d(\mu \wedge \beta) = d\mu \wedge \beta$$

Likewise,

$$d(\nu \wedge \alpha) = d\nu \wedge \alpha + (-1)^k \nu \wedge d\alpha = d\nu \wedge \alpha$$

So that:

$$d(\nu \wedge \alpha) = d\nu \wedge \alpha \quad d(\alpha \wedge \nu) = (-1)^k \alpha \wedge d\nu$$

Or more suggestively,

$$\alpha \wedge d\nu = (-1)^k d(\nu \wedge \alpha)$$

Therefore, we obtain the following:

$$\alpha \wedge \beta + \alpha \wedge d\nu + d\mu \wedge \beta + d\mu \wedge d\nu = \alpha \wedge \beta + (-1)^k d(\alpha \wedge \nu) + d(\mu \wedge \beta) + d\mu \wedge d\nu$$

We see that if we take the exterior derivative, we get that:

$$d(\alpha' \wedge \beta') = d(\alpha \wedge \beta)$$

Every other term disappears by linearity of d, and $d^2 = 0$.

Therefore, our wedge product acting on equivalence classes in the de Rham Cohomology of X is a well-defined operation. $\square$

Therefore, $[\alpha' \wedge \beta'] = [\alpha \wedge \beta]$. This implies that:

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$$

Likewise, wedge products of de Rham Cohomology classes behave exactly the same way as any ordinary form does, for α a k-form and β an l-form:

$$[\beta] \wedge [\alpha] = (-1)^{kl}[\alpha] \wedge [\beta]$$

We now give some **examples** of de Rham Cohomology for various types of manifolds.

Example 1 of de Rham Cohomology:

If X is a point, then $H^0(X) = \Omega^0(X) = \mathbb{R}$. $H^k(X) = 0$ for any $k \neq 0$ because the forms that are degree 1 or higher are all trivial.

Example 2 of de Rham Cohomology:

If $X = \mathbb{R}$, then $H^0(X) = \mathbb{R}$. Closed 0-forms are the constant functions (ones whose derivative vanishes), and only the 0 function is exact (as all constant functions vanish to 0).

Furthermore, $H^k(X) = 0$ for any $k > 0$. Any 1-form is exact and closed:

$$\alpha = \alpha(x)dx$$

Then

$$\alpha = df \qquad f(x) = \int_0^x \alpha(t)dt$$

We can continue this construction, just wedge on more dx^i then we have this correspondence.

Example 3 of de Rham Cohomology:

X is any connected manifold, then $H^0(X) = \mathbb{R}$ (the previous ones were special cases of this fact).

Example 4 of de Rham Cohomology:

If $X = S^1$, then $H^0(X) = \mathbb{R}$, $H^1(X) = \mathbb{R}$. This is because:

$$[\alpha] \longmapsto \int_{S^1} \alpha$$

is an isomorphism (this is de Rham's Theorem).

6.2 Pullbacks in Cohomology

Suppose that $f : X \longrightarrow Y$, and that α is a closed form on Y, representing an equivalence class in $H^k(Y)$. Then the pullback, $f^*\alpha$ is also closed.

$$d(f^*\alpha) = f^*(d\alpha) = f^*(0) = 0$$

Therefore, $f^*\alpha$ is a class representative in $H^k(X)$.

Proposition 31. *The pullback on k-forms, $f^* : \alpha \longrightarrow f^*\alpha$, induces a well-defined pullback on the equivalence class $[\alpha]$ in the de Rham Cohomology. The induced pullback:*

$$f^*_{ind}[\alpha] \longmapsto [f^*\alpha]$$

is denoted as $f^{\#}$.

Proof. We may show that this pullback is well-defined by considering when $\alpha' \sim \alpha$, then:

$$\alpha' = \alpha + d\mu$$

$$f^*(\alpha') = f^*(\alpha) + f^*(d\mu) = f^*(\alpha) + d(f^*\mu)$$

If we apply d again on both sides, then we obtain:

$$d(f^*\alpha') = d(f^*\alpha)$$

Therefore, pullbacks on equivalence classes of the de Rham Cohomology are well-defined. ☐

Definition 6.4. The pullback on de Rham Cohomologies, $f^{\#}$ is a contravariant functor.

$$f : X \longrightarrow Y \qquad f^{\#} : H^k(Y) \longrightarrow H^k(X)$$

And it preserves composition:

$$f : X \longrightarrow Y \quad g : Y \longrightarrow Z$$

$$g^{\#} : H^k(Z) \longrightarrow H^k(Y) \quad f^{\#} : H^k(Y) \longrightarrow H^k(X)$$

As $(g \circ f)^* = f^* \circ g^*$, it must follow that:

$$(g \circ f)^{\#} = f^{\#} \circ g^{\#}$$

6.3 Integration Over a Fiber and the Poincare Lemma

We will immediately present a theorem:

Theorem 32. *Integration over Fibers*

Let X be any (real) manifold. Let the zero section:

$$s_0 : X \longrightarrow \mathbb{R} \times X$$

be given as

$$s_0 : x \longmapsto (0, x)$$

Let the projection

$$\pi : \mathbb{R} \times X \longrightarrow X$$

be given as

$$\pi : (t, x) \longmapsto x$$

Then the induced zero section:

$$s_0^{\#} : H^k(\mathbb{R} \times X) \longrightarrow H^k(X)$$

and the induced projection:

$$\pi^{\#} : H^k(X) \longrightarrow H^k(\mathbb{R} \times X)$$

are isomorphism, and are the inverses of one another.

Proof. This requires a trick that we will have to motivate later. The isomorphism follows if we prove that $\pi^{\#}$ and $s_0^{\#}$ are inverses of one another.

First off, we have that:

$$\pi \circ s_0 = id_X$$

Therefore, it follows immediately that:

$$s_0^{\#} \circ \pi^{\#} = id_{H^k(X)}$$

We prove the isomorphism by proving that $\pi^{\#} \circ s_0^{\#} - id_{H^k(\mathbb{R}\times X)}$.

The trick here is to construct something called a **Homotopy Operator** $P : \Omega^k(\mathbb{R} \times X) \longrightarrow \Omega^{k-1}(\mathbb{R} \times X)$ where:

$$(1 - \pi^* \circ s_0^*)\alpha = d(P(\alpha)) + P(d\alpha)$$

If α is closed (which it is if we are considering k-forms in the de Rham Cohomology of X), then we have a vanishing second term ($d\alpha = 0$).

$$(1 - \pi^* \circ s_0^*)\alpha = d(P(\alpha))$$

Work this out more:

$$\alpha - \pi^*(s_0^*(\alpha)) = d(P(\alpha))$$

If we take the exterior derivative on both sides, we see that:

$$d\alpha = d\left(\pi^*(s_0^*(\alpha))\right)$$

Therefore, we have that $\alpha \sim \pi^* \circ s_0^*(\alpha)$. This means that:

$$[\alpha] = [\pi^* \circ s_0^*(\alpha)]$$

And because pullbacks induce pullbacks of cohomologies, we obtain:

$$[\alpha] = \pi^{\#} \circ s_0^{\#}[\alpha]$$

Therefore, $\pi^{\#} \circ s_0^{\#} - id_{H^k(\mathbb{R}\times X)}$ □

We now aim to make the homology operator P more clear. We now see that every k-form on Y can be uniquely expressed as the product:

$$\alpha(t,x) = dt \wedge \beta(t,x) + \gamma(t,x)$$

Where $\beta \in \Omega^{k-1}(X)$, $\gamma \in \Omega^k(X)$. We have that β can be written as:

$$\beta(t,x) = \sum_J \beta_J(t,x) dx^J$$

Likewise, γ may be written as:

$$\gamma(t,x) = \sum_I \gamma_I(t,x) dx^I$$

Definition 6.5.

$$P(\alpha)(t,x) = \sum_J \left(\int_0^t \beta_J(s,x) ds \right) dx^J$$

$P(\alpha)$ is called the **integral along the fiber of** α.

If we evaluate $s_0^*\alpha$ at x, then we obtain:

$$\sum_I s_0^*(\gamma_I(t,x)) dx^I = \sum_I \gamma_I(s_0(t,x)) dx^I = \sum_I \gamma_I(0,x) dx^I$$

Therefore, we see that:

$$(1 - \pi^{\#} s_0^*)\alpha(t,x) = dt \wedge \beta(t,x) + \sum_I (\gamma_I(t,x) - \gamma_I(t,0)) dx^I$$

We will now compute $d(P(\alpha))$, and $P(d\alpha)$. Since

$$d\alpha(t,x) = -dt \wedge \sum_{j,J} \left(\partial_j \beta_J(t,x)\right) dx^j \wedge dx^J$$

$$P(d\alpha) = -\sum_{j,J} \left(\int_0^t \partial_j \beta_J(s,x) ds \right) dx^j \wedge dx^J + \sum_{i,I} \left(\partial_i \gamma_I(s,x) ds\right) dx^i \wedge dx^I$$

We may evaluate the γ integral, we recognize that:

$$\sum_i \left(\int_0^t \partial_i \gamma_I(s,x) ds \right) dx^i = \gamma_I(t,x) - \gamma_I(0,x)$$

Therefore,

$$P(d\alpha) = -\sum_{j,J} \left(\int_0^t \partial_j \beta_J(s,x) ds \right) dx^j \wedge dx^J + \sum_I \left(\gamma_I(t,x) - \gamma_I(0,x)\right) dx^I$$

$$d(P(\alpha)) = \sum_{j,J} \left(\int_0^t \partial_j \beta_J(s,x) ds \right) dx^j \wedge dx^J + \sum_J \int_0^t \partial_s \beta_J(s,x) ds \wedge dx^J$$

Evaluate the second term with the fundamental theorem of calculus:

$$\sum_J \left(\int_0^t \partial_s \beta_J(s,x) ds \right) dx^J = \beta_J(t,x) dt \wedge dx^J$$

Therefore, if we evaluate:

$$(dP + Pd)\alpha(t,x) = \sum_I (\gamma_I(t,x) - \gamma_I(0,x))\, dx^I + dt \wedge \beta(t,x) = (1 - \pi^* s_0^*)\alpha(x,t)$$

We have shown that, expanded to local coordinates, our operator P behaves exactly as given. Now we will confirm that this is well-defined to discuss it in local coordinates:

Exercise 5.1: In the proof above, P was defined relative to local coordinates on a manifold X. Show that this is well defined (if we have two parametrizations ϕ_i and ϕ_j, the computation of $P(\alpha)$ due to ϕ_j coordinates, and transitioning to ϕ_i coordinates gives us the same result as directly using the ϕ_i coordinates).

Solution to Exercise 5.1:

Let

$$\phi_i : U_i \longrightarrow X \qquad \phi_j : U_j \longrightarrow X$$

Where both $U_i, U_j \subset \mathbb{R}^n$. We can also denote the transition map:

$$g_{ij} = \phi_j^{-1} \circ \phi_i$$

Then, we have that we maps (presumably smooth) have pullbacks:

$$\phi_i^* : \Omega^k(X) \longrightarrow \Omega^k(U_i)$$

$$\phi_j^* : \Omega^k(X) \longrightarrow \Omega^k(U_j)$$

We can now compute our pullbacks on the operator P.

$$(P\alpha)(t,x) = \sum_J (\beta_H(s,x) ds)\, dx^J$$

We act on this by our pullbacks. First we pull back to U_j:

$$\phi_i^* (P(\alpha)(t,x)) = P(\alpha)(\phi_i(t), \phi_i(x)) = \sum_J \left(\int_{\phi_i(0)}^{\phi_i(t)} \beta_J(\phi_i(s), \phi_i(x)) d(\phi_i(s)) \right) d\,(\phi_i(x))^J$$

We then pull back to U_j by ϕ_j^*, and then we pull that back with g_{ij}^*.

$$g_{ij}^* \circ \phi_j^* (P(\alpha)(t,x)) = (\phi_j \circ g_{ij})^* (P(\alpha)(t,x)) = P(\alpha)(\phi_j \circ g_{ij}(t), \phi_j \circ g_{ij}(x))$$

As $\phi_j \circ g_{ij} = \phi_i$, by definition of the transition map, we have that:

$$g_{ij}^* \circ \phi_j^* \left(P(\alpha)(t,x)\right) = \phi_i^* \left(P(\alpha)(t,x)\right)$$

Therefore, we have shown that our operator P is invariant under reparametrization.

Theorem 32, (and our exercise in addition) is very important because if we choose $X = \mathbb{R}$, successively, we obtain that, under the maps $s_o^{\#} : H^k(\mathbb{R}\times X) \longrightarrow H^k(X)$ and $\pi^{\#} : H^k(X) \longrightarrow H^k(\mathbb{R} \times X)$:

$\forall k \geq 1$, we have that:

$$H^k(\mathbb{R}^n) = H^k(\mathbb{R}^{n-1}) = \cdots = H^k(\mathbb{R}^0)$$

This establishes the **Poincare Lemma**.

Theorem 33. ***Poincare Lemma***

On $\mathbb{R}^n$, or on any manifold diffeomorphic to $\mathbb{R}^n$, every closed form of degree 1 or higher is exact.

We will now give more exercises to familiarize ourself with this.

Exercise 5.2: Show that a vector field v on $\mathbb{R}^3$, is the gradient of a function if and only if $\nabla \times v = 0$ everywhere.

Solution to Exercise 5.2:

We can attempt to compute all of this, but we already did it! Recall **Problem 1.1**. We found in that problem that for any 0-form, φ, two relevant quantities:

(i)

$$d\varphi = \omega^1_{\nabla\varphi} = (\nabla\varphi)_1 dx^1 + (\nabla\varphi)_2 dx^2 + (\nabla\varphi)_3 dx^3$$

(ii)

$$d\omega^1_v = \omega^2_{\nabla\times v} = (\nabla \times v)_1 dx^2 \wedge dx^3 + (\nabla \times v)_2 d^3 \wedge dx^1 + (\nabla \times v)_3 dx^1 \wedge dx^2$$

If we first assume that v is the gradient of a 0-form (or a scalar function for the physicists), then we have that $v = \nabla\varphi$, and that:

$$v = d\varphi = \omega^1_{\nabla\varphi}$$

Then, if we take another exterior derivative, we obtain:

$$dv = d(d\varphi) = d\omega^1_{\nabla\varphi} = \omega^2_{\nabla\times\nabla\varphi} = (\nabla\times\nabla\varphi)_1 dx^2\wedge dx^3 + (\nabla\times\nabla\varphi)_2 dx^3\wedge dx^1 + (\nabla\times\nabla\varphi)_3 dx^1\wedge dx^2$$

However, we know that $d^2 = 0$, so that:

$$\omega^2_{\nabla\times\nabla\varphi} = (\nabla \times \nabla\varphi)_1 dx^2 \wedge dx^3 + (\nabla \times \nabla\varphi)_2 dx^3 \wedge dx^1 + (\nabla \times \nabla\varphi)_3 dx^1 \wedge dx^2 = 0$$

By linear independence of each $dx^j \wedge dx^k$, we have that each $(\nabla \times \nabla\varphi)_i = 0$, which means that:

$$\nabla \times \nabla\varphi = 0 \qquad v = \nabla\varphi$$

This implies that:

$$\nabla \times v = 0$$

For the opposite implication, we may calculate the opposite direction here. We start with the fact that $\omega^2_{\nabla\times v} = d\omega^1_v = d\varphi$, and work backwards. In any matter, we are done.

Because the case of $\mathbb{R}^3$ is so ubiquitous in physics and engineering, we see that **Exercise 1.1** brings to light that we following sequence is exact:

$$0 \longrightarrow \Omega^0(\mathbb{R}^3) \xrightarrow{d_1} \Omega^1(\mathbb{R}^3) \xrightarrow{d_2} \Omega^2(\mathbb{R}^3) \xrightarrow{d_3} \Omega^3(\mathbb{R}^3) \longrightarrow 0$$

We then recognize that **Exercise 1.1** gives:

$$d_1 = \nabla \quad d_2 = \nabla\times \quad d_3 = \nabla\cdot$$

Where each exterior derivative is the gradient, curl, divergence, respectively.

The exactness of the sequence implies the following two identities:

$$d_2 d_1 = 0 \Longrightarrow \nabla \times \nabla\varphi \equiv 0 \quad \forall\varphi \in \Omega^0(\mathbb{R}^3)$$

$$d_3 d_2 = 0 \Longrightarrow \nabla \cdot \nabla \times v \equiv 0 \quad \forall v \in \Omega^1(\mathbb{R}^3)$$

The next exercise will continue our investigation of this.

Exercise 5.3: Show that a vector field v on $\mathbb{R}^3$ can be written as a curl (so $v = \nabla \times u$ if and only if $\nabla \cdot v = 0$).

Solution to Exercise 5.3:

We, again, recall **Exercise 1.1**. The two relevant results from there are:

(i)

$$d\omega^1_u = \omega^2_{\nabla\times u}$$

(ii)

$$d\omega^2_v = (\nabla \cdot v)dx^1 \wedge dx^2 \wedge dx^3$$

We use (i) to see that for a vector field $u \in \Omega^1(\mathbb{R}^3)$, we have:

$$d\omega^1_u = \omega^2_{\nabla\times u}$$

Then using the assumption that $v = \nabla \times u$, we get:

$$d\omega^1_u = \omega^2_v$$

By (ii), we have:

$$d\omega^2_v = (\nabla \cdot v)dx^1 \wedge dx^2 \wedge dx^3$$

However, $d\omega_v^2 = d(d\omega_u^1) = 0$, so this means that:

$$(\nabla \cdot v) dx^1 \wedge dx^2 \wedge dx^3 = 0$$

As we are in $\mathbb{R}^3$, $dx^1 \wedge dx^2 \wedge dx^3 \neq 0$, so this means that $\nabla \cdot v = 0$. The other direction follows with the same computation in reverse order.

Exercise 5.2 and **Exercise 5.3** have important consequences in the mathematical formulations of physics, where 5.2 describes **vector fields arising from scalar potentials**, and 5.3 describes **vector potentials arising from a divergenceless fields**. This is much of classical electricity and magnetism.

Exercise 5.4: Now consider the 3-dimensional torus:

$$X = \mathbb{R}^3/\mathbb{Z}^3$$

Construct a vector field $v(x)$ whose curl vanishes but is not a gradient. Construct a vector field $w(x)$ whose divergence vanishes but is not a curl.

Solution to Exercise 5.4:

We understand the initial confusion to this exercise. First question is: why does the 3-Torus matter? To investigate this, we recall **Exercise 1.2**. Second question: how do we know that there even exists a curl-less vector field that is not the gradient of a 0-form, and that there exists a divergence-less vector field that is not the curl of some vector field? This question makes our approach clearer, we must **compute the de Rham Cohomology** of our manifold. Third Question: how do we explicitly construct this vector field? We will compute the de Rham Cohomology first and figure this out later.

First consider the fact that:

$$H^0(T^3) = Z^0(T^3)/B^0(T^3)$$

Where $Z^k(X)$ **denotes all the closed k-forms on our manifold** X, and $B^k(X)$ **denotes all the exact k-forms on our manifold** X.

Let us reason through this, we see that the space of all closed 0-forms on the torus, must still be the constant function! Why? The kernel of the gradient is just any constant over the field we are working on. Furthermore, we see that the space of exact 0-forms is precisely only the 0 function (as $df = 0$ for any closed 0-form f). Therefore, $B^0(T^3) = \{0\}$ and $Z^0(T^3) = \mathbb{R}$.

$$H^0(T^3) = \mathbb{R}$$

Now consider:

$$H^1(T^3)$$

The elements of $Z^1(T^3)$ are spanned by forms of the form:

$$d\theta_i \qquad i = \{1, 2, 3\}$$

The elements of $B^1(T^3)$ are just $d\theta_1$ right? NO! This is false. Recall in **Exercise 2.7**, that we proved that the 1-form $d\theta$ on S^1 with the origin punctured was **NOT EXACT**, as the integral in a closed loop was 2π. Thus, we see here that, for our 3-Torus, that the form $d\theta_i$ is not exact for any i! And because our 1-forms must be closed, the coefficients must all be real constants! Once again, the set of exact 1-forms is only spanned by $\{0\}$, so that our 1st de Rham Cohomology is:

$$H^1(T^3) = \mathbb{R}^3$$

Continuing with the same logic, we see that $H^2(T^3)$ is spanned by $d\theta_1 \wedge d\theta_2$, $d\theta_2 \wedge d\theta_3$, $d\theta_1 \wedge d\theta_3$. And because our forms must be closed, we see that the coefficients of these forms must be real constants (similarly to the previous ones).

$$H^2(T^3) = \mathbb{R}^3$$

Likewise, because the set of closed 3-forms must be a linear combination of $d\theta_1 \wedge d\theta_2 \wedge d\theta_3$, we have that:

$$H^3(T^3) = \mathbb{R}$$

Because all k-forms with degree higher than the dimension of the smooth manifold vanish, every other cohomology is trivial. **We have characterized the de Rham Cohomology of our 3-torus** completely! **As we have nontrivial cohomology from $\Omega^0(T^3)$ through $\Omega^3(T^3)$**, we see that the following sequence:

$$0 \longrightarrow \Omega^0(T^3) \xrightarrow{d_1} \Omega^1(T^3) \xrightarrow{d_2} \Omega^2(T^3) \xrightarrow{d_3} \Omega^3(T^3) \longrightarrow 0$$

is **not exact!**. In particular, we see that:

$$im(d_1) \subset ker(d_2) \qquad im(d_2) \subset ker(d_3)$$

We must now find these elements! We will cheat and use what we know already, recall in **Exercise 2.7**, we answered a question regarding the exactness of:

$$\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

The vector field in question here is:

$$v = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

Because we have singularity at the origin, and we saw that this vector field and the corresponding 1-form gives the winding number, hence it is not exact. However,

$$\nabla \times v \equiv 0$$

This is by direct computation. This follows as v was attached to a 1-form, ω_v^1. However, as this was not an exact form (by the reason we gave just now), we have that:

$$\omega_v^1 \neq d\varphi \qquad \varphi \in \Omega^0(T^3)$$

Meaning that $v \neq \nabla\varphi$.

Also, notice that for the divergenceless vector field that is not the curl of another field, we can use a similar trick as for the previous example:

$$h = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x, -y, z)$$

This clearly has trivial divergence, and because it is not exact (by **Exercise 2.8**), it is not the curl of another vector field. We know this to be true as this was a special 2-form:

$$\omega_h^2$$

And by non-exactness, we have:

$$\omega_h^2 \neq d\omega_v^1$$

So there does not exist a vector field such that $\nabla \times v = h$.

This shows the power of de Rham Cohomology and the winding numbers! This was a difficult exercise but necessary! We better find ways to compute de Rham Cohomologies more efficiently!

To prove **Theorem 32**, we used the fact that $s_0^\#$ is the inverse of $\pi^\#$. However, we find that we can use a 1-section, $s_1(x) = (1, x)$, and all of our results would have followed regardless. We obtain that:

$$s_1^\# = (\pi^\#)^{-1} = s_0^\#$$

Thus, we see that this is a result about homotopies.

Theorem 34. *Homotopic maps induce the same maps in cohomology. If X and Y are manifolds, and $f_{0,1} : X \longrightarrow Y$ are smooth homotopic maps, then:*

$$f_1^\# = f_0^\#$$

Proof. Recall the definition of a homotopy:

$$H : X \times [0,1] \longrightarrow Y \qquad H(x,0) = f_0(x) \quad H(x,1) = f_1(x) \quad \forall x \in X$$

Now, we assume that f_0 and f_1 are homotopic. Then we can extend our definition of H above, to be $H : \mathbb{R} \times X \longrightarrow Y$ so that:

$$H(t,x) = f_0(x) \quad \forall x \in X \quad t \leq 0$$

$$H(t,x) = f_1(x) \quad \forall x \in X \quad t \geq 1$$

Then we see that:

$$s_1 : X \longrightarrow \mathbb{R} \times X$$

$$f_1 = H \circ s_1 \qquad f_0 = H \circ s_0$$

Then

$$f_1^\# = s_1^\# \circ H^\# = s_0^\# \circ H^\# = (H \circ s_0)^\# = f_0^\#$$

Thus, homotopies induce the same maps in cohomologies. □

We now offer exercises before we move onto Mayer-Vietoris Sequences:

Exercise 5.5: Recall that if A is a submanifold of X, then a **retraction** $r : X \longrightarrow A$ is a smooth map such that $r(a) = a \quad \forall a \in A$. If such a map exists, we say that A is a **retract of** X. Suppose that $r : X \longrightarrow A$ is such a retraction, and that i_A is the inclusion of A in X. Show that:

$$r^{\#} : H^k(A) \longrightarrow H^k(X)$$

is surjective and that:

$$i_A^{\#} : H^k(X) \longrightarrow H^k(A)$$

is injective in every degree k.

Solution to Exercise 5.5:

We look at the retraction first. We first look at the 0-forms, as this is an important case. Let $f \in H^0(A)$, as discussed before, we see that this function must be **locally constant**. Therefore, by default, $\pi^*(f) = f \circ \pi = f$. So π^* returns every 0-form in $H^0(A)$. On an arbitrary k-form, written in local coordinates:

$$\alpha = \sum_I \alpha_I(x) dx^1 \wedge \cdots \wedge dx^k$$

$$r^*\alpha = \sum_I \alpha_I(r(x)) d(x^1 \circ r) \wedge \cdots \wedge d(x^k \circ r)$$

As $\alpha_I(r(x)) = \alpha_I(x) \quad \forall x \in A$ as these are closed 0-forms, hence constant. Likewise, we see that $x^1 \circ r = x^1$ given that we are mapping from A. Therefore, we see that:

$$r^*\alpha = \alpha \quad \forall \alpha \in H^k(A)$$

So our pullback of the retraction returns every k-form over A, as an element of $\Omega^k(X)$. Therefore, our pullback is surjective, and so is $r^{\#}$ (when we pull back on forms in our cohomology classes).

Likewise, for the inclusion map, we have a similar idea. For any 0-form, we see that:

$$i_A^* f = i_A^* f'$$

Then, $f \circ i_A = f' \circ i_A$, and because i_A maps injectively, we see that $f = f'$ $\forall i_A(x) \in X$. So, it is injective on the space of closed 0-forms, because all closed 0-forms are constants, therefore, $f = f'$ $\forall x \in X$.

In the case of k-forms, we see that

$$i_A^* \alpha = i_A^* \alpha'$$

In the case where we look at only the closed k-forms, we obtain that

$$\alpha = \alpha'$$

because $\alpha, \alpha' \propto dx^I$ as this is what all closed k-forms look like. As the 0-forms must be constants, this implies that $\alpha = \alpha'$. Therefore, $i_A^\#$ must also be injective on the cohomology classes in $H^k(X)$.

Therefore, the pullbacks of the retractions and inclusions on cohomology classes are surjective/injective, respectively.

Exercise 5.6: Recall that a **deformation retraction** is a retraction $f : X \longrightarrow A$ such that $i_A \circ r$ is homotopic to id_X. In this sense, we say that A is a deformation retraction of X. Suppose that A is a deformation retract of X. Show that $H^k(X) \cong H^k(A)$.

Solution to Exercise 5.6:

If $i_A \circ r$ is homotopic to the identity map, then by definition of a homotopy:

$$H : X \times \mathbb{R} \longrightarrow X \qquad H(t,x) = i_A \circ r \quad \forall t \leq 0 \quad H(t,x) = id_X \quad \forall t \geq 1$$

As A is a deformation retract of X, we may take the above definition as our statement of homotopy between $i_A \circ r$ and id_X.

By direct application of **Theorem 34**, we may conclude that $(i_A \circ r)^\# = id_X^\#$ This implies that

$$r^\# \circ i_A^\# = id_X^\#$$

Then,

$$r^\# = id_X^\# \circ (i_A^\#)^{-1} = id_X^\# \circ (i_A^{-1})^\# = (id_X \circ i_A^{-1})^\#$$

So that $r^\# = (i_A^{-1})^\#$. This implies that each of $i_A^\# : H^k(X) \longrightarrow H^k(A)$ and $r^\# : H^k(A) \longrightarrow H^k(X)$ are inverses of one another, and hence, isomorphisms.

6.4 Mayer-Vietoris Sequences 1

Consider the following sequence of vector spaces and homomorphisms.

$$0 \longrightarrow V \xrightarrow{L} W \longrightarrow 0$$

This sequence is exact if and only if L is an isomorphism. The kernel of L has to be the image of 0, and the image of L has to be the kernel of a zero map, thus it is both injective and surjective.

Exercise 5.7: A **short exact sequence** is the following:

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{j} W \longrightarrow 0$$

Show that if this sequence is exact, there must be an isomorphism $h : V \longrightarrow U \oplus W$ with:

$$h \circ i(u) = (u, 0) \qquad j \circ h^{-1}(u, w) = w$$

This short exact sequence, with the given conditions on the maps, is called a **spit exact sequence**.

Solution to Exercise 5.7:

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{j} W \longrightarrow 0$$

The above is a short exact sequence. We must be careful to note that U, V, W are **Vector Spaces** (i.e. they are free objects over the base field). This has important implications.

Let $\{w_i\}$ be a basis of W, let $\{u_i\}$ be a basis of U, let $\{v_i\}$ be a basis of V. Then as the sequence is exact, i and j are injective, surjective, respectively.

By surjectivity of j:

$$\exists v_i \in \{v_i\} : \quad jv_i = w_i \quad \forall w_i \in \{w_i\}$$

By injectivity of i:

$$\exists v_i \in \{v_i\} : \quad iu_i = v_i \quad \forall u_i \in \{u_i\}$$

As u_i is a basis for U, we see that our action $iu_i = v_i$ uniquely determines our homomorphism i. Likewise for j. We then define more important objects:

$$t : W \longrightarrow V \qquad tw_i = v_i$$

$$q : V \longrightarrow U \qquad qv_i = u_i$$

We check that these give the following results:

$$j \circ t : W \longrightarrow W \qquad (j \circ t)w_i = j(tw_i) = jv_i = w_i$$

$$q \circ i : U \longrightarrow U \qquad (q \circ i)(u_i) = q(iu_i) = qv_i = u_i$$

Therefore, we see that both t and q induce identity when composed with j and i, respectively.

$$j \circ t = id_W \qquad q \circ i = id_U$$

Therefore, it is simple enough to verify that:

$$h : U \oplus W \longrightarrow V \qquad h : (u, w) \longmapsto (iu, tw)$$

We must now prove that this is bijective. If we make $iu = 0$ and $tw = 0$, it is easy enough to see that this is injective, as i is injective, and t must be injective if it is to induce identity when composed with j (which is surjective). If U, V, W are finite-dimensional, then this is sufficient as we are guaranteed bijection by rank-nullity theorem. If we are considering infinite-dimensional spaces, let us invoke Axiom of Choice and the previous arguments made here all apply. Moreover, it is simple enough to just see that h is invertible when we see that $j \circ t = id_W$ and $q \circ i = id_U$. Therefore, we can explicitly give the inverse of h as:

$$h^{-1} : (v_1, v_2) \longmapsto (qv_1, jv_2)$$

As these are all linear maps, we see that h is indeed an isomorphism.

This is the special case of something called the **Splitting Lemma**.

Lemma 35. ***Splitting Lemma*** *A Short Exact Sequence in the category* ***R-Mod****:*

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{j} W \longrightarrow 0$$

Splits (i.e. $V \cong U \oplus W$) when any ***ONE*** *of the following conditions is met:*

1. *There exists a section of j (call it t) such that $j \circ t = id_W$.*
2. *There exists a retract of i (call it q) such that $q \circ i = id_U$*
3. *There exists an isomorphism $h : V \longrightarrow U \oplus W$.*

We see that we essentially proved this lemma for the case of vector spaces (which easily holds for **free modules**). The general case is slightly more work. We will save the proof for a later time. We leave the general case's proof as an exercise.

We now consider our manifold X to be coverable by two open submanifolds of X.

$$X = U \cup V$$

Then there exist natural inclusions:

$$i_U : U \longrightarrow X \qquad i_V : V \longrightarrow X$$

These induce pullbacks on the smooth sections of TX (i.e. forms):

$$i_U^* : \Omega^k(X) \longrightarrow \Omega^k(U) \qquad i_V^* : \Omega^k(X) \longrightarrow \Omega^k(V)$$

We note that the pullbacks of these inclusions are simply the restrictions of the forms on the submanifolds of X.

$$i_U^* \alpha = \alpha\big|_U \qquad i_V^* \alpha = \alpha\big|_V$$

Likewise, for the intersection of these two submanifolds, we are given the same idea:

$$\rho_U : U \cap V \longrightarrow U \qquad \rho_V : U \cap V \longrightarrow V$$

These inclusions induce pullbacks on forms:

$$\rho_V^* : \Omega^k(U) \longrightarrow \Omega^k(U \cap V) \qquad \rho_U^* : \Omega^k(V) \longrightarrow \Omega^k(U \cap V)$$

These ones in particular, form a sequence:

$$0 \longrightarrow \Omega^k(X) \xrightarrow{i_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_k} \Omega^k(U \cap V) \longrightarrow 0$$

Let $\alpha \in \Omega^k(X)$, $\beta \in \Omega^k(U)$, $\gamma \in \Omega^k(V)$. Then we define the following maps:

$$i_k(\alpha) = (i_U^* \alpha, i_V^* \alpha) \qquad j_k(\beta, \gamma) = \rho_U^* \beta - \rho_V^* \gamma$$

Then as the exterior derivative always raises the degree of a form, we see by the following (quick) computations

$$d(i_k\alpha) = (di_U^*\alpha, di_V^*\alpha) = (i_U^*(d\alpha), i_V^*(d\alpha)) = i_{k+1}(d\alpha)$$

$$d(j_k(\beta,\gamma)) = d(\rho_U^*\beta - \rho_V^*\gamma) = d\rho_U^*\beta - d\rho_V^*\gamma = \rho_U^* d\beta - \rho_V^* d\gamma = j_{k+1}(d\beta, d\gamma)$$

that:

$$d(i_k(\alpha)) = i_{k+1}(d\alpha) \qquad d(j_k(\beta,\gamma)) = j_{k+1}(d\beta, d\gamma)$$

Then we obtain the following extension of the above diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k(X) \arrow[r, "i_k"] \arrow[d, "d_k"] & \Omega^k(U)\oplus\Omega^k(V) \arrow[r, "j_k"] \arrow[d, "d_k"] & \Omega^k(U\cap V) \arrow[r] \arrow[d, "d_k"] & 0 \\
0 \arrow[r] & \Omega^{k+1}(X) \arrow[r, "i_{k+1}"] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \arrow[r, "j_{k+1}"] & \Omega^{k+1}(U\cap V) \arrow[r] & 0
\end{tikzcd}$$

By definition of i_k and j_k, we see that closed forms get sent to closed forms:

$$di_k(\alpha) = i_{k+1}(d\alpha) = 0 \qquad dj_k(\beta,\gamma) = j_{k+1}(d\beta, d\gamma) = 0$$

And exact forms get sent to exact forms

$$i_k(d\alpha) = d(i_{k-1}\alpha) \qquad j_k(d\beta, d\gamma) = d(j_{k-1}(\beta),\gamma))$$

The maps are induced on cohomologies:

$$i_k^\# : H^k(X) \longrightarrow H^k(U)\oplus H^k(V) \qquad j_k^\# : H^k(U)\oplus H^k(V) \longrightarrow H^k(U\cap V)$$

These are the morphisms in a sequence called the **Mayer-Vietoris Sequence**.

Theorem 36. ***Mayer-Vietoris***

There exists a map $d_k^\# : H^k(U\cap V) \longrightarrow H^{k+1}(X)$ such that the sequence:

$$\dots H^k(X) \xrightarrow{i_k^\#} H^k(U)\oplus H^k(V) \xrightarrow{j_k^\#} H^k(U\cap V) \xrightarrow{d_k^\#} H^{k+1}(X) \xrightarrow{i_{k+1}^\#} H^{k+1}(U)\oplus H^{k+1}(V) \longrightarrow \dots$$

*is **exact**.*

Proof. This is a large proof, so we will lay out the steps.

1. We show that the sequence:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k(X) \arrow[r, "i_k"] \arrow[d, "d_k"] & \Omega^k(U)\oplus\Omega^k(V) \arrow[r, "j_k"] \arrow[d, "d_k"] & \Omega^k(U\cap V) \arrow[r] \arrow[d, "d_k"] & 0 \\
0 \arrow[r] & \Omega^{k+1}(X) \arrow[r, "i_{k+1}"] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \arrow[r, "j_{k+1}"] & \Omega^{k+1}(U\cap V) \arrow[r] & 0
\end{tikzcd}$$

   is an **exact sequence**. **NOTE:** This is a corollary of the **Zig Zag Lemma**, which we will prove from scratch.

2. Using the exactness and commutativity of this diagram, we construct $d_k^{\#}$.
3. Having constructed this map, we show exactness at $H^k(U) \oplus H^k(V)$.
4. We show exactness at $H^k(U \cap V)$.
5. We show exactness at $H^{k+1}(X)$.

We follow these steps and introduce all the necessary material to build these from the ground up.

Step 1:

We look at the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^k(X) & \xrightarrow{i_k} & \Omega^k(U) \oplus \Omega^k(V) & \xrightarrow{j_k} & \Omega^k(U \cap V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} & & \\ 0 & \longrightarrow & \Omega^{k+1}(X) & \xrightarrow{i_{k+1}} & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) & \xrightarrow{j_{k+1}} & \Omega^{k+1}(U \cap V) & \longrightarrow & 0 \end{array}$$

And we focus on proving the exactness of the sequence:

$$0 \longrightarrow \Omega^k(X) \xrightarrow{i_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_k} \Omega^k(U \cap V) \longrightarrow 0$$

Showing this is exact boils down to showing that:

(i) $im(i_k) = ker(j_k)$

(ii) i_k is injective

(iii) j_k is surjective

(i) is a simple proof. Just use the definition of i_k, j_k:

$$i_k(\alpha) = (i_U^*\alpha, i_V^*\alpha)$$

$j_k(i_k\alpha) = \rho_U^* i_U^* \alpha - \rho_V^* i_V^* \alpha = \rho_U^*\alpha\big|_U - \rho_V^*\alpha\big|_V = \alpha\big|_{U\cap V} - \alpha\big|_{U\cap V} = 0 \in \Omega^k(U \cap V)$

Therefore, $i_k\alpha \in ker(j_k) \quad \forall \alpha \in \Omega^k(X)$, and $im(i_k) \subseteq ker(j_k)$

We must prove the opposite inclusion. Take any $(\alpha, \beta) \in ker(j_k)$, then $j_k(\alpha, \beta) = \rho_U^*\alpha - \rho_V^*\beta = 0$

Then we are given that

$$\rho_U^*\alpha = \rho_V^*\beta$$

Take a form γ such that:

$$\gamma\big|_U = \alpha \qquad \gamma\big|_V = \beta$$

Then this implies that $\rho_U^*\gamma = \gamma\big|_{U\cap V} = \rho_V^*\gamma$. Then we have that:

$$\gamma\big|_U = i_U^*\alpha \qquad \gamma\big|_V = i_V^*\beta$$

Thus, we have that:

$$ker(j_k) \subseteq im(i_k)$$

(ii) follows simply too as:

$$i_k(\alpha) = (i_U^*\alpha, i_V^*\alpha) = (0, 0)$$

This implies that:

$$i_U^*\alpha = \alpha\big|_U = 0 \qquad i_V^*\alpha = \alpha\big|_V = 0$$

As $\alpha\big|_U = 0 \qquad \alpha\big|_V = 0$, this implies that:

$$\alpha\big|_X = 0$$

as $X = U \cup V$. So i_k is injective.

(iii) Let μ be a form in $\Omega^k(U \cap V)$, or a form over $U \cap V$. Then, we let $\{r_U, r_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. We now consider the form:

$$(r_V + r_U)\mu = \mu \qquad r_U + r_V = 1$$

We now justify this. The function r_U is compactly supported in U, so the form is extendable to a smooth form on V by declaring that $r_U = 0$ on $V \backslash U$. This implies that $r_U\mu$ is a form on V as μ is not defined in the entirety of the support of r_U, so it is not a form on U, but it is defined in V where $r_U \neq 0$. For the same reason, where $r_V \neq 0$ in U, $r_V\mu$ is a form on U. Therefore, we see that on $U \cap V$, that we may easily take:

$$(r_U + r_V)\mu \in \Omega^k(U \cap V)$$

This means that:

$$\mu = j_k(r_V\mu, -r_U\mu) = \rho_U^* r_V\mu - (-\rho_V^* r_U\mu)$$

Thus, j_k is surjective.

Analogously, we may use the identical computations (change k to k+1), and we obtain the exact same results for the lower half of the diagram. We now need to prove the exactness for the degree-raising exterior derivatives.

We now introduce a bit of homological algebra terms into this.

Definition 6.6. Cochain Complex

A **co-chain complex** is a sequence of vector spaces (for our purpose) $\{A^0, A^1, \cdots\}$ and maps $d_k : A^k \longrightarrow A^{k+1}$. We define $A^{-1} = A^{-k}$ to be 0-dimensional vector spaces, and the respective d to be the zero map.

Definition 6.7. The **kth cohomology of the complex** is:

$$H^k(A) = Z^k(A)/B^k(A)$$

Where

$$Z^k(A) = ker(d_k) \qquad B^k(A) = im(d_{k-1})$$

Definition 6.8. A **cochain map** $i : A \longrightarrow B$ between complexes A and B is a family of maps $i_k : A^k \longrightarrow B^k$ such that:

$$d_k(ik(\alpha)) = i_{k+1}(d_k\alpha) \qquad \forall k \qquad \forall \alpha \in A^k$$

This is illustrated by the commutative diagram below:

$$\begin{array}{ccc} A^k & \xrightarrow{i_k} & B^k \\ \downarrow{\scriptstyle d_k^A} & & \downarrow{\scriptstyle d_k^B} \\ A^{k+1} & \xrightarrow{i_{k+1}} & B^{k+1} \end{array}$$

We now give the following exercise as part of our proof.

Exercise 5.8: Show that the cochain map $i : A \longrightarrow B$ induces maps in cohomology:

$$i_k^{\#} : H^k(A) \longrightarrow H^k(B) \qquad i_k^{\#} : [\alpha] \longmapsto [i_k\alpha]$$

This will be important to our proof, and specifically to the case of Mayer-Vietoris.

Solution to Exercise 5.8:

Letting $i : A \longrightarrow B$ be a cochain map, we can consider the family of mappings:

$$i_k : A^k \longrightarrow B^k$$

As i is a cochain map, we see that:

$$d_k(i_k(\alpha)) = i_{k+1}(d_k\alpha)$$

We will draw out the commutative diagram that we will examine to prove this.

$$\begin{array}{ccc} A^{k-1} & \xrightarrow{i_{k-1}} & B^{k-1} \\ \downarrow{\scriptstyle d_{k-1}^A} & & \downarrow{\scriptstyle d_{k-1}^B} \\ A^k & \xrightarrow{i_k} & B^k \\ \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k^B} \\ A^{k+1} & \xrightarrow{i_{k+1}} & B^{k+1} \end{array}$$

$$d_k^B \circ i_k = i_{k+1} \circ d_k^A \qquad \forall k$$

Let $\alpha \in A^k$. Let $\mu \in A^{k-1}$. Any $\bar{\alpha} \in H^k(A)$ is of the form:

$$\bar{\alpha} = \alpha + d_{k-1}^A\mu \qquad d_{k-1}^A\mu \in im(d_{k-1}^A)$$

As $\bar{\alpha} \in H^k(A)$, we see that:

$$d_k^B \circ i_k\alpha = i_{k+1} \circ d_k^A\alpha = 0$$

This means that $d_k^B \circ i_k\alpha = 0$, and $i_k\alpha \in ker(d_k^B)$. Furthermore, if we act on $\bar{\alpha}$:

$$i_k(\alpha + d_{k-1}^A\mu) = i_k\alpha + i_k d_{k-1}^A\mu \qquad d_{k-1}^A\mu \in im(d_{k-1}^A)$$

Clearly, $i_k\alpha \in ker(d_k^B)$. Then we see that, by the commutativity of the above diagram, that

$$i_k d_{k-1}^A\mu = d_{k-1}^B i_{k-1}\mu$$

Then, we see that the image of any $\bar{\alpha} \in H^k(A)$, by i_k is:

$$i_k\bar{\alpha} = i_k\alpha + d_{k-1}^B(i_{k-1}\mu) \qquad \forall\mu \in A^{k-1}$$

As $\mu \in A^{k-1}$ is arbitrary, we see that we can talk about an **equivalence class** of all elements **modulo** $im(d_{k-1}^B)$. So in the language of cohomology classes, we see that:

$$i_k\alpha + d_{k-1}^B i_{k-1}\mu \in [i_k\alpha]$$

So

$$i_k(\bar{\alpha}) \sim i_k\alpha \in ker(d_k^B)/im(d_{k-1}^B)$$

Where

$$\alpha + d_{k-1}^A\mu \sim \alpha \in H^k(A)$$

Thus, we see that any cochain map i, induces a map, $\forall k$:

$$i_k^{\#} : H^k(A) \longrightarrow H^k(B)$$

$$i_k^{\#} : [\alpha] \longmapsto [i_k\alpha]$$

We see the power of commutative diagrams here!

Definition 6.9. Furthermore, if A, B, C are cochain complexes, and $i : A \longrightarrow B$ and $k : B \longrightarrow C$ are cochain maps, then the sequence:

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is a **short exact sequence** of cochain complexes if, for each k,

$$0 \longrightarrow A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \longrightarrow 0$$

is a **short exact sequence** of vector spaces.

Therefore, with **Exercise 5.8**, we see that we have a short exact sequence of cochain complexes, where, in our definition, we denote:

$$A^k = \Omega^k(X) \quad B^k = \Omega^k(U) \oplus \Omega^k(V) \quad C^k = \Omega^k(U \cap V)$$

We have proved exactness at every point of our **Mayer-Vietoris Sequence**, **except at the portion:**

$$H^k(U) \oplus H^k(V) \xrightarrow{j_k^{\#}} H^k(U \cap V) \xrightarrow{d_k^{\#}} H^{k+1}(X) \xrightarrow{i_{k+1}^{\#}} H^{k+1}(U) \oplus H^{k+1}(V)$$

Our goal is to show that this $d_k^{\#}$ exists! This is a result called the **Zig Zag Lemma**.

Theorem 37. ***Zig Zag Lemma***

Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

be a short exact sequence of cochain complexes. Then there is a family of maps $d_k^{\#} : H^k(C) \longrightarrow H^{k+1}(A)$ *such that the sequence:*

$$\cdots \longrightarrow H^k(A) \xrightarrow{i_k^{\#}} H^k(B) \xrightarrow{j_k^{\#}} H^k(C) \xrightarrow{d_k^{\#}} H^{k+1}(A) \xrightarrow{i_{k+1}} H^{k+1}(B) \longrightarrow \dots$$

is exact.

Proof. We will split up the proof of the Zig Zag Lemma into various steps.

Step 1: We must connect $\gamma_k \in C^k$ to the beginning of the next sequence of the chain complex A^{k+1}.

First off, we will define the mapping:

$$d_k^{\#}[\gamma_k] \qquad [\gamma_k] \in H^k(C)$$

We will be very precise (albeit slightly wordy) in our proof.

As j_k is surjective:

$\exists \beta_k \in B^k$, such that $\forall \gamma_k \in C^k$:

$$\gamma_k = j_k(\beta_k)$$

Then by commutativity:

$$d_k^C \circ j_k = j_{k+1} \circ d_k^B$$

So applying it to β_k:

$$d_k^C(j_k\beta_k) = j_{k+1}(d_k^B\beta_k)$$
$$j_{k+1}(d_k^B\beta_k) = d_k^C(j_k\beta_k) = d_k^C\gamma_k$$

Because $[\gamma_k] \in H^k(C)$ by assumption, we have that:

$$\gamma \in ker(d_k^C)$$

Therefore,

$$d_k^C\gamma_k \equiv 0$$

Giving us a string of equalities:

$$j_{k+1}(d_k^B\beta_k) = d_k^C\gamma_k = 0$$

Therefore, $d_k^B\beta_k \in ker(j_{k+1})$. And since $ker(j_{k+1}) \subseteq im(i_{k+1})$, we see that $d_k^B\beta_k \in im(i_{k+1})$.

Let $\alpha_{k+1} \in A^{k+1}$ so that:

$$i_{k+1}\alpha_{k+1} = d_k^B\beta_k$$

As i_k is injective, α_{k+1} is unique to each value of $d_k^B \beta_k$.

Step 2: Step 1 told us that we should choose $\alpha_{k+1} \in A^{k+1}$ to continue our connection of chain complexes.

We define the map:

$$d_k^{\#}[\gamma_k] = [\alpha_{k+1}]$$

We must show it is well-defined!

(i) **(Well-definedness 1)**. We must show that our class representative, α_{k+1} is actually contained in the cohomology $H^{k+1}(A)$.

By commutativity:

$$i_{k+2}(d_{k+1}^A \alpha_{k+1}) = d_{k+1}^B(i_{k+1}\alpha_{k+1})$$

Due to the injectivity of i_{k+1} (discussed in **Step 1**), we see that:

$$i_{k+1}\alpha_{k+1} = d_k^B \beta_k$$

Then

$$i_{k+2}(d_{k+1}^A \alpha_{k+1}) = d_{k+1}^B(d_k^B \beta_k) = 0$$

Therefore, by injectivity of i_{k+2}, $d_{k+1}^A \alpha_{k+1} = 0$. Thus, $\alpha_{k+1} \in ker(d_{k+1}^A)$, so it is, indeed, contained in the cohomology $H^{k+1}(A)$.

(ii) **(Well-definedness 2)**. Now that we have shown that α_{k+1} is an appropriate class representative, we must now show that the class $[\alpha_{k+1}]$ is independent of choice of β_k, as long as $j_k\beta_k = \gamma_k$. (This could potentially be an issue if not true as α_{k+1} was defined, in **Step 1**, in relation to $d_k^B \beta_k$.

We pick a different $\beta_k' \neq \beta_k$ so that:

$$j_k\beta_k' = \gamma_k = j_k\beta_k$$

Then,

$$j_k(\beta_k' - \beta_k) = 0 \qquad \beta_k' - \beta_k \in ker(j_k) \subseteq im(i_k)$$

Therefore, $\exists \alpha_k \in A^k$ so that:

$$\beta_k' = \beta_k + i_k\alpha_k$$

We can act on this with d_k^B so that:

$$d_k^B \beta_k' = d_k^B \beta_k + d_k^B i_k \alpha_k$$

However, by **Step 1** and commutativity of the diagram, we see that:

$$d_k^B \beta_k = i_{k+1}\alpha_{k+1} \qquad d_k^B i_k \alpha_k = i_{k+1} d_k^A \alpha_k$$

So that

$$d_k^B \beta_k' = i_{k+1}(\alpha_{k+1} + d_k^A \alpha_k)$$

If we define $d_k^B\beta_k'$ analogously to the end of **Step 1**.

$$d_k^B\beta_k' = i_{k+1}(\alpha_{k+1}')$$

So that

$$i_{k+1}(\alpha_{k+1}') = i_{k+1}(\alpha_{k+1} + d_k^A\alpha_k)$$

As i_{k+1} is injective, we have that:

$$\alpha_{k+1}' = \alpha_{k+1} + d_k^A\alpha_k$$

However, as α_{k+1} is a representative of the class $[\alpha_{k+1}]$, by **Well Definedness 1**, we see that under the natural equivalence relation of $H^{k+1}(A)$:

$$\alpha_{k+1}' \sim \alpha_{k+1}$$

So that

$$[\alpha_{k+1}'] = [\alpha_{k+1}]$$

Therefore, our cohomology classes are independent of the choice of β_k, as long as $j_k\gamma_k = \beta_k$.

(iii) **(Well-definedness 3)**. Now that we have shown that $[\alpha_{k+1}]$ is well-defined, we must now show that the mapping:

$$d_k^{\#}[\gamma_k] = [\alpha_{k+1}]$$

is independent of the choice of representative for the cohomology class $[\gamma_k]$.

To start, we suppose that:

$$\gamma_k' \neq \gamma_k$$

We pick some $\beta_k' = \beta_k + d_{k-1}^B\beta_{k-1}$ so that:

$$j_k\beta_k' = \gamma_k'$$

$$j_k(\beta_k + d_{k-1}^B\beta_{k-1}) = j_k\beta_k + j_k d_{k-1}^B\beta_{k-1}$$

We have the results that:

$$j_k\beta_k = \gamma_k \qquad j_k d_{k-1}^B\beta_{k-1} = d_{k-1}^C j_{k-1}\beta_{k-1}$$

However, we have that:

$$j_{k-1}\beta_{k-1} = \gamma_{k-1}$$

$$\gamma_k' = j_k\beta_k' = \gamma_k + d_{k-1}^C\gamma_{k-1} = \gamma_k'$$

We see that $\beta_k' \sim \beta_k$, and therefore, $\gamma_k' \sim \gamma_k$.

This tells us that α_{k+1} is the correct image of $d_k^{\#}$ as it is invariant under choice of representative β_k, and γ_k is invariant under this choice of representative.

As long as $j_k\beta_k = \gamma_k$, it is well-defined to use cohomology classes, as they preserve enough information under our equivalence relation.

Now that we have proved that:

$$d_k^{\#}[\gamma_k] = [\alpha_{k+1}]$$

is well-defined, we must now prove the exactness of the sequences! To reiterate, we have proved the existence and uniqueness of the mappings $i_k^{\#}$, $j_k^{\#}$, $d_k^{\#}$. We must now prove the exactness of

$$\cdots \longrightarrow H^k(A) \xrightarrow{i_k^{\#}} H^k(B) \xrightarrow{j_k^{\#}} H^k(C) \xrightarrow{d_k^{\#}} H^{k+1}(A) \xrightarrow{i_{k+1}} H^{k+1}(B) \longrightarrow \dots$$

We will take this in various steps.

(Exactness 1): We will show that $im(i_k^{\#}) \subseteq ker(j_k^{\#})$.

We take any $\alpha_k \in A^k$ so that:

$$j_k^{\#} i_k^{\#}[\alpha_k]$$

We simply apply **Exercise 5.8**:

$$j_k^{\#} i_k^{\#}[\alpha_k] = j_k^{\#}[i_k \alpha_k] = [j_k i_k \alpha_k]$$

But $j_k \circ i_k \equiv 0$ in any complex. Thus

$$j_k^{\#}(i_k^{\#}[\alpha_k]) = 0$$

So that $i_k^{\#}[\alpha_k] \in ker(j_k^{\#})$. So

$$im(i_k^{\#}) \subseteq ker(j_k^{\#})$$

(Exactness 2): We will show that $ker(j_k^{\#}) \subseteq im(i_k^{\#})$.

Suppose that $j_k^{\#}[\beta_k]$, then:

$$[j_k \beta_k] = 0 \in H^k(C)$$

So this implies that:

$$j_k \beta_k = d_{k-1}^C(\gamma_{k-1}) \qquad \gamma_{k-1} \in C^k$$

As γ_{k-1} is the image of a surjection, j_{k-1}, $\forall \gamma_{k-1}$, $\exists \beta_{k-1}$ so that:

$$\gamma_{k-1} = j_{k-1}\beta_{k-1}$$

Then, we simply take $j_k \beta_k - d_{k-1}^C \gamma_{k-1} = 0$

$$d_{k-1}^C j_{k-1} \beta_{k-1} = j_k d_{k-1}^B \beta_{k-1}$$

Then, we see that

$$j_k \beta_k - j_k d_{k-1}^B \beta_{k-1} = 0$$

$$j_k(\beta_k - d_{k-1}^B \beta_{k-1}) = 0 \qquad \beta_k - d_{k-1}^B \beta_{k-1} \in ker(j_k)$$

As $ker(j_k) \subseteq im(i_k)$, we see that $\beta_k - d_{k-1}^B \beta_{k-1} \in im(i_k)$, and $\exists \alpha_k \in A^k$ so that:

$$\beta_k - d_{k-1}^B \beta_{k-1} = i_k(\alpha_k)$$

Then we take d_k^B on both sides:

$$d_k^B \beta_k - d_k^B d_{k-1}^B \beta_{k-1} = d_k^B i_k(\alpha_k)$$

We see that

$$d_k^B d_{k-1}^B \beta_{k-1} = 0 \qquad d_k^B i_k \alpha_k = i_{k+1} d_k^A \alpha_k$$

$$d_k^B \beta_k = i_{k+1} d_k^A \alpha_k$$

However, we assumed that $[\beta_k] \in H^k(B)$, so $\beta_k \in ker(d_k^B)$. Therefore:

$$d_k^B \beta_k = i_{k+1} d_k^A \alpha_k = 0$$

Because i_{k+1} is injective, we see that:

$$d_k^A \alpha_k = 0 \Longrightarrow \alpha_k \in ker(d_k^A) \Longrightarrow [\alpha_k] \in H^k(A)$$

Then:

$$i_k^{\#}[\alpha_k] = [i_k \alpha_k] = [\beta_k - d_{k-1}^B \beta_{k-1}] = [\beta_k]$$

Therefore, $\forall [\beta_k] \in ker(j_k^{\#})$, $[\beta_k] \in im(i_k^{\#})$

$$ker(j_k^{\#}) \subseteq im(i_k^{\#})$$

(Exactness 3): We now show $im(j_k^{\#}) \subseteq ker(d_k^{\#})$.

We take the composition:

$$d_k^{\#}(j_k^{\#}[\beta_k])$$

Then we see (by **Exercise 5.8**), that:

$$j_k^{\#}[\beta_k] = [j_k \beta_k] = [\gamma_k]$$

Then as $d_k^{\#}[\gamma_k] = [\alpha_{k+1}]$, and by **Step 1**, we see that (recall this was definition of α_{k+1})

$$i_{k+1}\alpha_{k+1} = d_k^B \beta_k$$

However, $[\beta_k] \in H^k(B)$, so $\beta_k \in ker(d_k^B)$, so:

$$i_{k+1}\alpha_{k+1} = d_k^B \beta_k = 0$$

Since i_{k+1} is injective, we see that:

$$\alpha_{k+1} = 0 \Longrightarrow [\alpha_{k+1}] = 0 \in H^{k+1}(A)$$

So that $\forall [\gamma_k] \in im(j_k^{\#})$:

$$[\gamma_k] \in ker(d_k^{\#}) \Longrightarrow im(j_k^{\#}) \subseteq ker(d_k^{\#})$$

We move onto the next step:

(Exactness 4): We show now that $ker(d_k^{\#}) \subseteq im(j_k^{\#})$.

We first consider $[\gamma_k] \in ker(d_k^{\#})$ so that:

$$d_k^{\#}[\gamma_k] = [\alpha_{k+1}] = 0$$

Then, $[\alpha_{k+1}] = 0$ implies that:

$$\alpha_{k+1} = d_k^A \alpha_k$$

(This is the definition of the equivalence relation in our cohomology). Recall how we defined α_{k+1}:

$$i_{k+1}\alpha_{k+1} = d_k^B \beta_k$$
$$i_{k+1} d_k^A \alpha_k = d_k^B \beta_k$$

By commutativity, $i_{k+1} \circ d_k^A = d_k^B \circ i_k$:

$$d_k^B i_k \alpha_k = d_k^B \beta_k \Longrightarrow d_k^B(i_k\alpha_k - \beta_k) = 0$$

So that $i_k\alpha_k - \beta_k \in ker(d_k^B)$. This implies that:

$$[i_k\alpha_k - \beta_k] \in H^k(B)$$

Then we act on this by $j_k^{\#}$:

$$j_k^{\#}[i_k\alpha_k - \beta_k] = [j_k i_k \alpha_k - j_k\beta_k] = [-\gamma_k] = [\gamma_k]$$

Therefore, we see that $\forall[\gamma_k] \in ker(d_k^{\#})$, $[\gamma_k] \in im(j_k^{\#})$.

This implies that:

$$ker(d_k^{\#}) \subseteq im(j_k^{\#})$$

(Exactness 5): We now have one more section to finish off. Consider $im(d_k^{\#}) \subseteq ker(i_{k+1})$.

This one is easy.

$$i_{k+1}^{\#} d_k^{\#}[\gamma_k] = i_{k+1}^{\#}([\alpha_{k+1}])$$

Then

$$i_{k+1}^{\#}[\alpha_{k+1}] = [i_{k+1}\alpha_{k+1}] = [d_k^B \beta_k] = 0$$

This follows as $d_k^B \beta_k = 0$ as $[\beta_k] \in H^k(B)$. Therefore,$\forall[\alpha_{k+1}] \in im(d_k^{\#})$, $[\alpha_{k+1}] \in ker(i_{k+1}^{\#})$.

This implies that:

$$im(d_k^{\#}) \subseteq ker(i_{k+1}^{\#})$$

(Exactness 6): This is the last part. We show that $ker(i_{k+1}^{\#}) \subseteq im(d_k^{\#})$.

Consider:

$$i_{k+1}^{\#}[\alpha_{k+1}] = 0$$

$$[i_{k+1}\alpha_{k+1}] = [d_k^B\beta_k]$$

Then consider

$$d_k^C j_k\beta_k = j_{k+1}d_k^B\beta_k = [j_{k+1}i_{k+1}\alpha_{k+1}] = 0$$

Thus,

$$[j_k\beta_k] \in H^k(C)$$

Act on it by $d_k^{\#}$, then we get:

$$[\alpha_{k+1}] = d_k^{\#}[j_k\beta_k]$$

Thus, this implies that $\forall[\alpha_{k+1}] \in ker(i_{k+1}^{\#})$, $[\alpha_{k+1}] \in im(d_k^{\#})$.

$$ker(i_{k+1}^{\#}) \subseteq im(d_k^{\#})$$

We are done with the **Zig Zag Lemma** now! We have diagram chased every possible part of the sequence, and we have proved exactness at every point. We have also proved well-definedness of every morphism involved. $\square$

With the **Zig Zag Lemma** concluded, we just take A^k, B^k, C^k appropriately and we proved (explicitly) that our sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^k(X) & \xrightarrow{i_k} & \Omega^k(U)\oplus\Omega^k(V) & \xrightarrow{j_k} & \Omega^k(U\cap V) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d_k} & & \\
0 & \longrightarrow & \Omega^{k+1}(X) & \xrightarrow{i_{k+1}} & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) & \xrightarrow{j_{k+1}} & \Omega^{k+1}(U\cap V) & \longrightarrow & 0
\end{array}$$

is exact in every row, and the **Zig Zag Lemma** gave us the exactness of the **Mayer-Vietoris Sequence**.

$$\dots H^k(X) \xrightarrow{i_k^{\#}} H^k(U)\oplus H^k(V) \xrightarrow{j_k^{\#}} H^k(U\cap V) \xrightarrow{d_k^{\#}} H^{k+1}(X) \xrightarrow{i_{k+1}^{\#}} H^{k+1}(U)\oplus H^{k+1}(V) \longrightarrow \dots$$

$\square$

6.5 Cohomology of the Unit Sphere

We will first determine cohomologies of various manifolds. The most important one being $H^k(S^n)$. Mayer-Vietoris will give us this immediately.

First Example: We look at the case of S^1.

Let $V = \{(x,y) \in S^1 \mid y > -1/2\}$, $U = \{(x,y) \in S^1 \mid y < 1/2\}$. The open sets U and V are both diffeomorphic to $\mathbb{R}$. Therefore,

$$H^k(U) = H^k(V) = \mathbb{R}$$

And $H^k(U) = H^k(V) = 0$ for any $k > 0$. Then $U \cap V$ consists of two intervals, one with $x > 0$ and the other with $x < 0$. So it is two-dimensional.

$$H^0(U \cap V) = \mathbb{R}^2$$

$$H^k(U \cap V) = 0 \qquad \forall k > 0$$

As S^1 is 1-dimensional.

Then, we can write the Mayer-Vietoris Sequence as

$$0 \longrightarrow H^0(S^1) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V) \longrightarrow H^1(S^1) \longrightarrow H^1(U) \oplus H^1(V) \longrightarrow \dots$$

By the above remarks, we see that this sequence simplifies tremendously:

$$0 \longrightarrow H^0(S^1) \xrightarrow{i_0^\#} \mathbb{R}^2 \xrightarrow{j_0^\#} \mathbb{R}^2 \xrightarrow{d_0^\#} H^1(S^1) \xrightarrow{j_1^\#} 0$$

Because S^1 is connected (simply connected), we have

$$H^0(S^1) = \mathbb{R}$$

Because $i_0^\#$ is injective, the image of $i_0^\#$ must be 1-dimensional.

This makes the kernel of $j_0^\#$ 1-dimensional, so $j_0^\#$ has rank 1. This also makes the kernel of $d_0^\#$ 1-dimensional (by exactness).

This makes $H^1(S^1)$ 1-dimensional by exactness. Thus, $H^1(S^1) = H^0(S^1) = \mathbb{R}$. And because S^1 is only 1-dimensional, we conclude that any higher cohomologies are trivial.

We may look at the generators for each cohomology group, we simply need to describe what the space of forms would look like. To compute the cohomologies of the n-sphere S^n, we do a similar computation of the Mayer-Vietoris Sequence.

Second Example: de Rham Cohomology of the Sphere.

Let S^n be a unit sphere embedded in $\mathbb{R}^{n+1}$. Let U and V be portions of sphere with $x_{n+1} < 1/2$, and $x_{n+1} > -1/2$. We have that:

$$U \cong \mathbb{R}^n \quad V \cong \mathbb{R}^n$$

Therefore, we see that $H^0(U) = H^0(V) = \mathbb{R}$. $H^k(U) = H^k(V) = 0$ for any $k > 0$. These facts follow because the 0th cohomology is simply the space of constant 0-forms. The rest of the cohomologies are all exact (as the space in question is simply connected).

We take it as a fact that $U \cap V$ is a strip around the equator of the sphere (bounded at $-1/2$ and $1/2$). This is diffeomorphic to $\mathbb{R} \times S^{k-1}$. It has the same cohomology as S^{n-1}. For $k > 0$, we obtain that for the following Mayer-Vietoris sequence of the cochain complex of de Rham Cohomologies:

$$H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \longrightarrow H^{k+1}(S^n) \longrightarrow H^{k+1}(U) \oplus H^{k+1}(V)$$

becomes:

$$0 \longrightarrow H^k(S^{n-1}) \longrightarrow H^{k+1}(S^n) \longrightarrow 0$$

Thus, this shows that:

$$H^{k+1}(S^n) \cong H^k(S^{n-1})$$

If we use induction, with the base case that $H^0(S^n) = \mathbb{R}$, we see that:

$$H^k(S^n) = \mathbb{R} \qquad k = 0, n$$

$$H^k(S^n) = 0 \qquad k \neq 0, n$$

We will state exercises here, which we may complete at a later time.

Exercise 5.10: Denote $T^2 = S^1 \times S^1$ as the 2-torus. Partition T into two cylinders so that $U \cap V$ is itself a disjoint union of two cylinders. Construct the Mayer-Vietoris Sequence, and compute the cohomology of T^2. Recall what the de Rham Cohomologies of T^n were. Is it consistent?

Exercise 5.11: Let K be the Klein Bottle. Find open sets U and V such that $U \cup V = K$. Compute the de Rham Cohomologies of K.

6.6 Further Application of Mayer-Vietoris Sequences

In this section, we will prove, what are effectively, corollaries of the **Mayer-Vietoris Sequence**.

Definition 6.10. A **point** is a topological space, denoted by $*$. We can use Homology to determine the de Rham Cohomology of the point, but we will give it as a definition.

The **de Rham Cohomology of a point** is denoted as $H^k(*)$, and is:

$$H^0(*) = \mathbb{R} \qquad H^k(*) = 0 \qquad \forall k \neq 0$$

Definition 6.11. A set is **contractible** if it deformation retracts to a single point, so that it has the same cohomology as a point (given above). An open contractible set is something that is diffeomorphic to $\mathbb{R}^n$. A **contractible manifold** is globally diffeomorphic to $\mathbb{R}^n$ (as Euclidean space is always contractible).

Lemma 38. *Let X and Y be two manifolds. If $X \cong Y$, then $H^k(X) \cong H^k(Y)$.*

Proof. The proof is trivial when we consider **Theorem 34**. As two homotopic maps induce the same maps in cohomology, and we have that the isomorphism, φ, of X and Y is automatically a homotopy. Therefore, the induced mapping $\varphi^\#$ of $H^k(X)$ and $H^k(Y)$ is automatically an isomorphism. $\square$

Proposition 39. *Let M be a contractible manifold. Then the de Rham Cohomology of M is as follows:*

$$H^0(M) = \mathbb{R}$$

$$H^k(M) = 0 \qquad k > 0$$

Proof. As M is a contractible manifold, we have that $M \cong \{*\}$. Therefore, we see that, by **Lemma 38**, that:

$$H^k(M) \cong H^k(*)$$

$\square$

Proposition 40. *The de Rham Cohomology is additive.*

$$H^k\left(\coprod_{i\in I} M_i\right) \cong \bigoplus_{i\in I} H^k(M_i)$$

Proof. We let j_i be the inclusion maps of M_i into M. The isomorphism $h : H^k\left(\coprod_i M_i\right) \longrightarrow \bigoplus_i H^k(M_i)$:

$$h : \omega \longmapsto (j_1^*\omega, j_2^*\omega, \cdots) = (\omega\big|_{M_1}, \omega\big|_{M_2}, \cdots)$$

We must verify that this is a bijection. Injectivity comes immediately as $ker(h)$ are all forms such that:

$$(\omega\big|_{M_1}, \omega\big|_{M_2}, \cdots) = (0, 0, \cdots)$$

This implies that $\omega\big|_{M_i} = 0$ for the restriction of the form ω on each M_i. As our entire manifold is a disjoint union of each M_i, it follows that $\omega = 0$ on all of $\coprod M_i$. So $ker(h) = \{0\}$.

The surjectivity follows immediately, as we have that $\exists \omega \in H^k\left(\coprod_i M_i\right)$ such that $\forall (\omega\big|_{M_1}, \omega\big|_{M_2}, \cdots) \in \bigoplus H^k(M_i)$ that $h(\omega) = (\omega\big|_{M_1}, \omega\big|_{M_2}, \cdots)$. Therefore, h is a bijection, hence an isomorphism. $\square$

Corollary 41. ***Corollary to Proposition 40***

The de Rham Cohomologies of a manifold M, that is (smoothly) homotopic to two points are:

$$H^0(M) = \mathbb{R} \oplus \mathbb{R}$$
$$H^k(M) = 0 \qquad k > 0$$

Proof. The proof is, really, really, obvious when we apply **Proposition 40**. $\square$

Proposition 42. *Let M be a smooth connected manifold, then:*

$$H^0(M) = \mathbb{R}$$

Proof. This is almost obvious. As on any manifold that is connected (not necessarily simply connected), we see that any closed 0-form, ω is such that:

$$d\omega = 0$$

However, there is no such (real-valued) function where the derivative vanishes except for the constant functions. The only exact form in the space of closed 0-forms is exactly the 0 function. Thus,

$$H^k(M) = \mathbb{R}/\{0\} = \mathbb{R}$$

$\square$

Lemma 43. *Let*

$$0 \longrightarrow A^0 \xrightarrow{d_0} \dots \xrightarrow{d_{m-1}} A^m \longrightarrow 0$$

be an exact sequence of (finite-dimensional) vector spaces. Then:

$$\sum_{i=0}^{m} (-1)^i \dim(A^i) = 0$$

Proof. We just apply the **Rank-Nullity Theorem** here.

$$\dim im(d_i) + \dim ker(d_i) = \dim A^i$$

By exactness, we see that:

$$\dim im(d_i) = \dim ker(d_{i+1})$$

Applying it to each successive sequence of A^i, we see that the final result follows immediately. $\square$

We now compute de Rham Cohomologies of a circle (more rigorously this time).

Proposition 44. ***de Rham Cohomology of a Circle***

$$H^k(S^1) = \mathbb{R} \qquad k = 0, 1$$

$$H^k(S^1) = 0 \qquad k \neq 0, 1$$

Proof. Denote N as the north pole, S as the south pole. Let $U = S^1 \backslash N$, $V = S^1 \backslash S$. Apply the **Mayer-Vietoris Sequence** to obtain:

$$0 \longrightarrow H^0(S^1) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V) \longrightarrow H^1(S^1) \longrightarrow H^1(U) \oplus H^1(V) \longrightarrow H^1(U \cap V) \longrightarrow 0$$

This sequence is exact by **Theorem 36**. As S^1 is connected, U and V are contractible, and $U \cap V$ is homotopy equivalent to two points. So we may fill out the sequence above with the knowledge that:

$$H^0(U) \oplus H^0(V) = \mathbb{R} \oplus \mathbb{R} \qquad H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}$$

And by dimension of S^1, we know that any cohomology with $k \geq 1$ must vanish. We obtain:

$$0 \longrightarrow H^0(S^1) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1(S^1) \longrightarrow 0$$

Therefore, applying **Lemma 43**, we see that:

$$\sum_{i=0}^{m} (-1)^i \dim A^i = -1 + 2 - 2 + \dim(H^1(S^1)) = 0$$

We see that:

$$\dim(H^1(S^1)) = 1$$

We see that because of the dimension, $H^1(S^1) = \mathbb{R}$. $\square$

Theorem 45. ***de Rham Cohomology of an n-sphere***

Our conclusion about the de Rham Cohomology of a circle is enough information to inductively find the de Rham Cohomology of any n-sphere S^n.

For any $n > 0$

$$H^k(S^n) = \mathbb{R} \qquad k = 0, n$$
$$H^k(S^n) = 0 \qquad k \neq 0, n$$

Proof. It suffices to take the result for S^1 and induct on it. Take $U = S^n \backslash S$, $V = S^n \backslash N$. By the same argument, these two cover S^n, and S^n is simply connected, therefore, contractible.

This time, however, $U \cap V$ is not homotopic to two points, but instead a homotopy equivalent to S^{n-1}.

Out of the default Mayer-Vietoris Sequence:

$$0 \to H^{n-1}(S^n) \to H^{n-1}(U) \oplus H^{n-1}(V) \to H^{n-1}(U \cap V) \to H^n(S^n) \to H^n(U) \oplus H^n(V) \to H^n(U \cap V) \to 0$$

We fill in all the portions that are trivial, and we simply obtain:

$$H^{n-1}(S^n) \quad H^{n-1}(S^{n-1}) \cong \mathbb{R} \quad H^n(S^n)$$

So our exact sequence becomes:

$$0 \to H^{n-1}(S^n) \to 0 \to \mathbb{R} \to H^n(S^n) \to 0$$

Therefore, this implies that:

$$H^n(S^{n-1}) \cong 0$$

Which, by deduction, implies that:

$$H^n(S^n) \cong \mathbb{R}$$

This is the desired result. □

We have that $\mathbb{R}^n$ has trivial de Rham cohomology, $H^k(\mathbb{R}^n)$ (by **Poincare's Lemma**) $\forall k \geq 1$. That is, only the 0th de Rham Cohomology is nontrivial, and it is always $\mathbb{R}$, by **Proposition 42**. However, we find an exception to that when we omit a point.

Proposition 46.

$$H^k(\mathbb{R}^n \backslash \{0\}) = \mathbb{R} \qquad k = 0, n-1$$
$$H^k(\mathbb{R}^n \backslash \{0\}) = 0 \qquad k \neq 0, n-1$$

Proof. We use the fact that, with a stereographic projection, we can prove that:

$$\mathbb{R}^n \backslash \{0\} \cong S^{n-1}$$

Then, we use **Lemma 38** to establish that all de Rham Cohomologies of the punctured plane must be isomorphic to the cohomologies of the n-1-Sphere. We use **Theorem 45** to achieve the conclusion.

NOTE: The (one of the many possible) maps we may use to establish the isomorphism between $\mathbb{R}^n \backslash \{0\}$ and S^{n-1} is:

$$(x, t) \longmapsto \left((1-t)\frac{1}{|x|} + t \right) x$$

This proposition shows up in complex analysis, in the construction of the extended complex numbers.

Remark: We can take out ANY point, it does not have to be zero. $\square$

We will also prove the following theorem. It is particularly useful because, as we see, most real smooth manifolds have interesting behavior only in the first few, or last few de Rham Cohomologies.

Proposition 47. *Assume that M is a real, smooth manifold such that it is coverable by k, disjoint, convex submanifolds. The 0th de Rham Cohomology of M is $\mathbb{R}^k$..*

Proof. We will use the fact that all convex submanifolds of a manifold are contractible. Assume that M is coverable by finitely many disjoint, open, submanifolds $U_i \subseteq M$ (we are guaranteed to find this as every submanifold of M is convex, and we may choose appropriate ones to glue together).

$$M = \coprod_{i \in I} U_i \qquad |I| = k$$

As each U_i is convex, hence contractible, we see that:

$$M \cong \{*_1\} \cup \{*_2\} \cup \cdots \cup \{*_k\}$$

And by **Lemma 38** (isomorphic manifolds have isomorphic cohomologies) and **Proposiiton 40** (de Rham Cohomology is additive), the de Rham Cohomology of M is:

$$H^0(M) = \bigoplus_{i \in I} H^0(U_i) = \mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R} = \mathbb{R}^k$$

$\square$

Remark: It is a work in progress (by the author) to see if we may relax the conditions presented in this proposition.

We will start delving into connections between de Rham Cohomology and Simplicial Homology. This connection is very important for particular classes of questions like the **Proposition 47**. We want to describe the topology of smooth manifolds.

7 de Rham's Theorem and Singular Homology

We have started a new section on the consequences of de Rham Cohomology. We saw that we could compute many interesting things about our smooth manifold through looking at forms. However, we will find a formal connection with Simplicial Homology through **de Rham's Theorem**.

Theorem 48. ***Top Cohomology is Isomorphic to*** $\mathbb{R}$

Let M be a smooth, connected, ***orientable, compact*** *n-manifold. Then*

$$I : H^n(M) \longrightarrow \mathbb{R}$$

$$I : [\omega] \longmapsto \int_M \omega$$

Where ω is the representative of the cohomology class $[\omega]$.

This is called the ***integration map****, and it is an isomorphism, with respect to the mentioned spaces.*

Proof. As with all things that are not manifestly obvious, we will have to verify that this is well-defined, i.e. this map is invariant under the choice of representative for the cohomology class $[\omega]$.

Let $\bar{\omega} \neq \omega$, but $[\bar{\omega}] = [\omega]$. Then, we see that:

$$\bar{\omega} = \omega + d\gamma \qquad \deg(\gamma) = \deg(\omega) - 1$$

Or more effectively, $\omega \in \Omega^n(M)$, and $\gamma \in \Omega^{n-1}(M)$.

$$I(\omega) = \int_M \omega \qquad I(\bar{\omega}) = \int_M \bar{\omega}$$

We compute the second result quickly:

$$I(\bar{\omega}) = \int_M \bar{\omega} = \int_M (\omega + d\gamma) = \int_M \omega + \int_M d\gamma$$

By Stokes' Theorem (which applies here as M is orientable and compact):

$$\int_M d\gamma = \int_{\partial M} \gamma = 0$$

This is because, by default, our manifolds have NO boundary.

$$\int_{\partial M = \varnothing} \gamma \equiv 0$$

Therefore, it follows that:

$$I(\bar{\omega}) = \int_M \bar{\omega} = \int_M (\omega + d\gamma) = \int_M \omega + \int_M d\gamma == \int_M \omega = I(\omega)$$

So our map is, in fact, well defined.

We must simply verify surjectivity and injectivity.

Injectivity is obvious because if $I(\omega) = I(\bar{\omega})$, then we see by definition of the integral (with Stokes' Theorem), that:

$$[\omega] = [\bar{\omega}]$$

Surjectivity is not obvious, but we use the fact that when M is orientable, we have the existence of a form, ω_0 (which tracks the orientation of M), such that:

$$\int_M \omega_0 = b > 0$$

Consider now,

$$I(a\omega) = aI(\omega) = ab \qquad q \in \mathbb{R}$$

We can make any real number with this map, therefore, it is surjective.

Hence it is bijective, and a homomorphism, so we have that it's an isomorphism. $\square$

If M is **NOT** orientable, we have that:

$$H^n(M) = 0$$

Therefore, we know exactly when a manifold is not orientable.

7.1 Simplicial Homology

We are going to introduce simplicial homology. It is a homology theory of simplices. We will give all the necessary definitions.

Definition 7.1. Let $\mathbb{R}^\infty$ have a basis $\{e_0, e_1, \cdots\}$. The **standard p-simplex** is denoted as

$$\Delta_p = \left\{ \sum_{i=0}^{p} \lambda_i e_i : \sum \lambda_i = 1 \quad 0 \le \lambda_i \le 1 \right\}$$

Definition 7.2. For given $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^q$:

$$[\cdot] : \Delta_n \longrightarrow \mathbb{R}^q$$

$$\sum \lambda_i e_i \longmapsto \sum \lambda_i v_i$$

This is called an **Affine singular n-simplex**.

Definition 7.3. The ith facemap is denoted as:

$$F_i^p = [\{e_0, e_1, \ldots, \hat{e_i}, \ldots e_p\}]$$

Where the hat means "leave that element out".

Definition 7.4. Let X be a topological space, a continuous map:

$$\sigma_p : \Delta_p \longrightarrow X$$

is called a **p-simplex**. The free abelian group over all p-simplices is denoted as $S_p(X)$, and is the singular p-chain group.

In this sense, a p-chain of X is a formal sum of p-simplices.

We can turn Δ_p into a chain complex by defining a degree-changing map:

$$\partial_p : \Delta_p(X) \longrightarrow \Delta_{p-1}(X)$$

Definition 7.5. Let $\partial_p : \Delta_p(X) \longrightarrow \Delta_{p-1}(X)$ be the homomorphism that acts on basis elements of $\Delta_p(X)$ in the following way:

$$\partial_p \sigma \equiv \sum_{i=0}^{p} (-1)^i \sigma^i \qquad \sigma^i = \sigma \circ F_i^p \quad \sigma^i : \Delta_{p-1} \longrightarrow X$$

Proposition 49. *The pair:*

$$(\Delta_p, \partial_p)_{p \geq 0}$$

is a chain complex.

The chain complex for singular homology concerns free $\mathbb{Z}$-modules whereas the de Rham Cohomology concerned modules over $\mathbb{R}$.

7.2 Simplicial Cohomology

In algebraic topology, the preeminent functors are $\cdot \otimes A$ and $hom(\cdot, A)$. The functors will be useful to us (although we must develop these if we want to use it in full). The *hom* functor will allow us to construct cohomology of simplices.

Lemma 50. $hom(\cdot, A) : \mathrm{Ab} \longrightarrow \mathrm{Ab}$

Definition 7.6. We define the map: $d_p : hom(S_p(X), A) \longrightarrow hom(S_{p+1}(X), A)$

$$d_p : f \longrightarrow f \circ \partial_{p+1}$$

Lemma 51. $(hom(S_p(X), A), d_p)_{p \geq 0}$ *is a cochain complex.*

The cohomologies arising from this cochain complex are called ***Singular Cohomologies****.*

7.3 Smooth Simplices

We are going to now consider smooth simplices.

Definition 7.7. $C_p(X)$ is the set of all p-simplices.

Definition 7.8. $C_p^{\infty}(X)$ is the set of all **smooth** p-simplices.

Analogously to the case of Δ_p, we can first define the **Smooth Singular Homologies**, $H_p^{\infty}(M)$.

We, amazingly, have that these singular homologies are the same as the previous singular homologies, $H_p(M)$.

This is due to **Whitney's Embedding Theorem**.

Theorem 52. ***Whitney Embedding Theorem***

For two smooth manifolds M, N and $F : M \longrightarrow N$ a continuous map, F is homotopic to a smooth map $\tilde{F} : M \longrightarrow N$.

Then we can map it from the category of abelian groups to itself with the $hom(\cdot, A)$ functor, then consider the differential map $d_p : hom(H_p^{\infty}(M), A) \longrightarrow hom(H_{p+1}^{\infty}(M), A)$, defined in the exact same way. Then we have that

$$\left(hom(H_p^{\infty}(M), A), d_p\right)_{p\geq 0}$$

is a cochain complex itself.

7.4 de Rham Theorem

Definition 7.9. Let σ be a smooth singular p-simplex, and ω is a closed p-form, we define:

$$\int_{\sigma} \omega \equiv \int_{\Delta_p} \sigma^* \omega$$

This extends to any smooth p-chain by:

$$\int_{\sum_{i=1}^{k} c_i \sigma_i} \omega \equiv \sum_{i=1}^{k} c_i \int_{\sigma_i} \omega$$

We can now do Stokes' Theorem in more generality.

Theorem 53. ***Stokes' Theorem 3***

Let c be a smooth q-chain in a smooth manifold M. Let ω be a smooth $q-1$ form on M. Then:

$$\int_{\partial c} \omega = \int_c d\omega$$

Proof. We do it for one simplex, and it will lead to the general case. We only need the definitions found through **Section 7.1** to prove this.

If σ is a q-simplex, then consider:

$$\int_{\sigma} d\omega$$

We recognize that:

$$\sigma : \Delta_q \longrightarrow M$$

$$\sigma^* : \Omega^{q-1}(M) \longrightarrow \Omega^{q-1}(\Delta_q)$$

Then:

$$\int_\sigma d\omega = \int_{\Delta_q} \sigma^* d\omega = \int_{\Delta_q} d\sigma^*\omega$$

Apply the original Stokes' Theorem and get:

$$\int_{\Delta_q} d\sigma^*\omega = \int_{\partial\Delta_q} \sigma^*\omega$$

By definition of the boundary map:

$$\partial\Delta_q = \sum_{j=0}^{q} (-1)^j \Delta_q \circ F_j^q$$

Plug it in:

$$\int_{\sum_{j=0}^{q}(-1)^j \Delta_q \circ F_j^q} \sigma^*\omega = \sum_{j=0}^{q} (-1)^j \int_{\Delta_q \circ F_j^q} \sigma^*\omega = \sum_{j=0}^{q} (-1)^j \int_{\Delta_{q-1}} F_j^{q*}\sigma^*\omega$$

$$= \sum_{j=0}^{q} (-1)^j \int_{\Delta_{q-1}} (\sigma \circ F_j^q)^*\omega$$

Then we get rid of the pullback appropriately:

$$\sum_{j=0}^{q} (-1)^j \int_{\Delta_{q-1}} (\sigma \circ F_j^q)^*\omega = \int_{\sum_{j=0}^{q}(-1)^j \sigma \circ F_j^q} \omega$$

The definition of the **boundary operator** is:

$$\partial\sigma = \sum_{j=0}^{q} (-1)^j \sigma \circ F_j^q$$

Therefore, we have that the integral resolves to:

$$\int_{\partial\sigma} \omega$$

And our conclusion is that:

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

If we want the more general proof, we note that, for any chain c:

$$c = \sum_{\sigma \in c} \sigma$$

Then we obtain:

$$\int_c d\omega = \int_{\sum_{\sigma\in c}\sigma} d\omega = \sum_{\sigma\in c}\int_\sigma d\omega = \sum_{\sigma\in c}\int_{\partial\sigma}\omega = \int_{\sum_{\sigma\in c}\partial\sigma}\omega$$

By linearity of the boundary operator, we have that:

$$\sum_{\sigma\in c}\partial\sigma = \partial\left(\sum_{\sigma\in c}\sigma\right) = \partial c$$

So that:

$$\int_c d\omega = \int_{\partial c}\omega$$

□

We now present the **de Rham Homomorphism**.

Definition 7.10. de Rham Homomorphism

Let $[\omega] \in H^p_{dR}(M)$ be a cohomology class in the de Rham Cohomology of M. Let $\tilde{c} \in [c] \in H^\infty_p(M)$ be a homology class in the smooth singular homology of M. The **de Rham Homomorphism** is

$$\mathcal{I} : H^p_{dR}(M) \longrightarrow H^p(M;\mathbb{R})$$

$$\mathcal{I}_c : [\omega] \longmapsto \int_{\tilde{c}}\omega$$

Here, $H^p(M;\mathbb{R})$ denotes the **pth Singular Cohomology of** M **over** $\mathbb{R}$.

Lemma 54. *$\mathcal{I}$ is well-defined in both $[\omega]$ and $[c]$ (i.e. they are independent of choice of ω and c as long as they are equivalent to another element in the respective (co)homologies).*

Proof. To begin, we just establish that:

$$\mathcal{I}_c([\omega]) = \int_{\tilde{c}}\omega$$

Then, we take another equivalent representative in the cohomology class $[\omega]$. $\omega' = \omega + d\nu$.

$$\mathcal{I}_c([\omega']) = \int_{\tilde{c}}\omega' = \int_{\tilde{c}}(\omega + d\nu) = \int_{\tilde{c}}\omega + \int_{\tilde{c}}d\nu$$

As our singular homology dictates that $\partial\tilde{c} = \varnothing$, we obtain by Stokes' Theorem:

$$\int_{\tilde{c}} d\nu = \int_{\partial\tilde{c}}\nu = 0$$

Therefore, we obtain that:

$$\mathcal{I}_c([\omega']) = \int_{\tilde{c}}\omega = \mathcal{I}_c([\omega])$$

So $\mathcal{I}$ is independent of our choice of representative in the cohomology class.

As for our choice of $\tilde{c}$. Choose $\tilde{c}' \neq \tilde{c}$, but $[\tilde{c}'] = [\tilde{c}]$. Then $\tilde{c}' = \tilde{c} + \partial b$, where b is some $p+1$ cycle. Let us establish:

$$\mathcal{I}_c([\omega]) = \int_{\tilde{c}} \omega$$

Then, we consider the de Rham homomorphism with the cycle c' instead:

$$\mathcal{I}_{c'}([\omega]) = \int_{\tilde{c}'} \omega = \int_{\tilde{c}+\partial b} \omega = \int_{\tilde{c}} \omega + \int_{\partial b} \omega$$

Then, because of Stokes' Theorem, we may write the second term as:

$$\int_{\partial b} \omega = \int_b d\omega = 0 \qquad \omega \in H^p_{dR}(M)$$

Thus, we have that:

$$\mathcal{I}_{c'}([\omega]) = \int_{\tilde{c}} \omega + \int_{\partial b} \omega = \int_{\tilde{c}} \omega = \mathcal{I}_c([\omega])$$

Therefore, our $\mathcal{I}$ is well-defined. □

Notation: From now on, $\mathcal{I}_c[\omega] \equiv \mathcal{I}[\omega][c]$. We also must prove naturality of the de Rham Homomorphism.

Lemma 55. *Let $F : M \longrightarrow N$ be a smooth function. The following diagram is commutative.*

$$\begin{array}{ccc} H^p_{dR}(N) & \xrightarrow{F^*} & H^p_{dR}(M) \\ \downarrow{\scriptstyle \mathcal{I}} & & \downarrow{\scriptstyle \mathcal{I}} \\ H^p(N;\mathbb{R}) & \xrightarrow{F^*} & H^p(M;\mathbb{R}) \end{array}$$

Proof. Let us just plug this into the definition.

$$\mathcal{I}F^*[\omega][\sigma] = \int_\sigma F^*\omega = \int_{\Delta_p} \sigma^* F^* = \int_{\Delta_p} (F \circ \sigma)^* \omega = \int_{F\circ\sigma} \omega = \mathcal{I}[\omega][F\circ\sigma] = \mathcal{I}[\omega]F^*[\sigma]$$

Therefore, we see that:

$$\mathcal{I}[\omega]F^*[\sigma] = \mathcal{I}(F^{\#}[\omega])[\sigma]$$

Therefore, we have proven that the diagram, is, in fact, commutative.

Note: There is abuse of notation above, we regarded F^* as a pullback on a cohomology class $[\omega]$, but we know that pullbacks of forms induce a pullback on cohomology classes. We regard it as the same thing, but it really isn't. □

We define various things so we may conveniently state and prove the de Rham Theorem.

Definition 7.11. A smooth manifold M is called a **de Rham Manifold** is the de Rham Homomorphism is an isomorphism of the manifold.

Lemma 56. *Every open convex subset of* $\mathbb{R}^n$ *is de Rham.*

Proof. Let U be any convex subset of $\mathbb{R}^n$. By the Poincare Lemma, we have:

$$H^{k+1}_{dR}(\mathbb{R}^n) = H^k_{dR}(\mathbb{R}^n) = 0 \qquad \forall k > 0$$

$$H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$$

Consider $H^0_{dR}(U)$. As U is contractible to a single point, consider a single point $\{*\}$ (any point will do, we will be as general as possible).

$$H^0_{dR}(U) \cong H^0_{dR}(\{*\}) = \mathbb{R}$$

Now we just map with the de Rham Homomorphism. Let $\Delta_p = \{*\}$. Let $\omega \in H^0_{dR}(U)$. Then, we have that:

$$\mathcal{I}[\omega][\sigma] = \int_\sigma \omega = \int_{\Delta_p} \sigma^*\omega$$

As $\omega \in H^0_{dR}(U)$, we have that ω is a constant function (as it is a closed 0-form). Therefore, we have that:

$$\int_{\{*\}} \sigma^*\omega = \omega \circ \sigma(*) = \omega$$

Where the last part follows as $\{*\} \cong U$, therefore, as $\omega \in \mathbb{R}$, we have that:

$$\mathcal{I} : H^0_{dR}(U) \longrightarrow H^0(U; \mathbb{R})$$

is an isomorphism. □

Lemma 57. *Let* $\{U_i\}$ *be a collection of open, disjoint, de Rham subsets of* M*, then:*

$$\coprod_{i \in I} U_i$$

is de Rham.

Proof. We know that the de Rham Cohomology is additive over coproducts (**Proposition 40**), therefore:

$$H^p_{dR}\left(\coprod_{i \in I} U_i\right) \cong \bigoplus_{i \in I} H^p_{dR}(U_i)$$

This is a general property so that the singular homologies are also additive:

$$H^p\left(\coprod_{i \in I} U_i\right) \cong \bigoplus_{i \in I} H^p(U_i)$$

Lemma 55 says that (in combination with **Proposition 40**, which gives the horizontal lines in the following diagram, and by the assumption that each cover is de Rham, which gives the rightmost vertical arrow):

$$\begin{array}{ccc} H^p_{dR}(\coprod_{i\in I} U_i) & \xrightarrow{\cong} & \bigoplus_{i\in I} H^p_{dR}(U_i) \\ \big\downarrow \mathcal{I} & & \big\downarrow \cong \\ H^p(N;\mathbb{R}) & \xrightarrow{\cong} & H^p(M;\mathbb{R}) \end{array}$$

By commutativity, this is sufficient to show the isomorphism required for the coproduct to be de Rham. $\square$

Lemma 58. *Let U and V be open subsets of a smooth manifold M, if U, V, $U\cap V$ are de Rham, then $U\cup V$ is also de Rham.*

Proof. As a result of the **Mayer-Vietoris Sequence**:

$$\begin{array}{ccc} \vdots & & \vdots \\ \big\downarrow & & \big\downarrow \\ H^p_{dR}(U\cup V) & \xrightarrow{\mathcal{I}} & H^p(U\cup V;\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ H^p_{dR}(U)\oplus H^p_{dR}(V) & \xrightarrow{\cong} & H^p(U;\mathbb{R})\oplus H^p(V;\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ H^p_{dR}(U\cap V) & \xrightarrow{\cong} & H^p(U\cap V;\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ H^{p+1}_{dR}(U\cup V) & \xrightarrow{\mathcal{I}} & H^{p+1}(U\cup V;\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ H^{p+1}_{dR}(U)\oplus H^{p+1}_{dR}(V) & \xrightarrow{\cong} & H^{p+1}(U;\mathbb{R})\oplus H^{p+1}(V;\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ H^{p+1}_{dR}(U\cap V) & \xrightarrow{\cong} & H^{p+1}(U\cap V;\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ \vdots & & \vdots \end{array}$$

We can consider this section of the diagram. We may resolve this with the use of the Five Lemma.

Lemma 59. ***Five Lemma***

Consider the following commutative diagram in the category of R-modules.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i_1"] & B \arrow[r, "f_2"] \arrow[d, "i_2"] & C \arrow[r, "f_3"] \arrow[d, "i_3"] & D \arrow[r, "f_4"] \arrow[d, "i_4"] & E \arrow[d, "i_5"] \\
A' \arrow[r, "g_1"] & B' \arrow[r, "g_2"] & C' \arrow[r, "g_3"] & D' \arrow[r, "g_4"] & E'
\end{tikzcd}$$

If the above is exact in each row, and i_1, surjective. i_2, i_4 are isomorphisms, i_5 is injective, then i_3 is an isomorphism.

Because the **Mayer-Vietoris Sequence is exact**, we have that the **Five Lemma** automatically applies here. We look at the following part of the sequence:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^p_{dR}(U) \oplus H^p_{dR}(V) \arrow[r, "\cong"] \arrow[d] & H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) \arrow[d] \\
H^p_{dR}(U \cap V) \arrow[r, "\cong"] \arrow[d] & H^p(U \cap V;\mathbb{R}) \arrow[d] \\
H^{p+1}_{dR}(U \cup V) \arrow[r, "\mathcal{I}"] \arrow[d] & H^{p+1}(U \cup V;\mathbb{R}) \arrow[d] \\
H^{p+1}_{dR}(U) \oplus H^{p+1}_{dR}(V) \arrow[r, "\cong"] \arrow[d] & H^{p+1}(U;\mathbb{R}) \oplus H^{p+1}(V;\mathbb{R}) \arrow[d] \\
H^{p+1}_{dR}(U \cap V) \arrow[r, "\cong"] \arrow[d] & H^{p+1}(U \cap V;\mathbb{R}) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

All the conditions of the **Five Lemma** are trivially satisfied as all the isomorphisms are injective and surjective. Therefore, by the **Five Lemma**, $\mathcal{I}$ must be an isomorphism. $\square$

Lemma 60. *Let M be a smooth n-manifold. Suppose that $P(U)$ is a statement about open subsets of M, satisfying the following three properties:*

1. *$P(U)$ is true for U diffeomorphic to a convex open subset of $\mathbb{R}^n$.*
2. *$P(U)$, $P(V)$, $P(U \cap V) \implies P(U \cup V)$.*
3. *$\{U_\alpha\}$ disjoint, and $P(U_\alpha)$ for all α, then $P\left(\bigcup_\alpha U_\alpha\right)$ Then $P(M)$ is true.*

Proof. We will not prove this right now, but it involves using convexity arguments and countability. We will return to prove this another time. This is a statement from *Topology and Geometry* by Bredon. $\square$

Theorem 61. ***de Rham's Theorem***

$$\mathcal{I} : H^p_{dR}(M) \longrightarrow H^p(M; \mathbb{R})$$

is an isomorphism.

Proof. By **Lemma 56, 57,58, 60**, all the statements about $\mathcal{I}$ being an isomorphism applies for M. $\square$

8 Compactly Supported Cohomology and Poincare Duality

Through the de Rham isomorphism, we see that the de Rham Cohomology is an invariant object.

The Challenge: However, one drawback is that the de Rham Cohomology cannot tell us the difference between two non-isomorphic objects that are contractible.

$$H^k_{dR}(\mathbb{R}^n) \cong H^k_{dR}(\mathbb{R}^m) \cong H^k(\{*\}) \cong \mathbb{R} \qquad n \neq m \neq 0$$

We will look to try to mitigate the lack of information that is given here.

8.1 Compactly Supported de Rham Cohomology

Definition 8.1. The **Support of a function** f, denoted as $supp(f)$ is defined as the following:

$$supp(f) = \{x \in dom(f) \subseteq \mathbb{R}^n : f(x) \neq 0\}$$

Definition 8.2. A function:

$$f : M \longrightarrow \mathbb{R}$$

is said to have compact support if:

$$supp(f) = \{p \in M : f(p) \neq 0\}$$

is compact.

Definition 8.3. $\Omega^n_c(M) \subset \Omega^n(M)$ is the set of all n-forms that have compact support (a form having a property of a function means that the coefficient of a form has that property).

As seen with the ordinary $\Omega^n(M)$, $\Omega^n_c(M)$ forms a cochain complex with the exterior derivative as the degree-raising map. The nth cohomology of this cochain complex is denoted as $H^n_c(M)$. It is officially known as **nth Compactly Supported de Rham Cohomology**. All exact forms will have compact support.

Remark: If M is compact, then

$$\Omega^q_c(M) = \Omega^q(M)$$

as all functions (trivially) have compact support.

Lemma 62.

$$H^1_c(\mathbb{R}) \cong \mathbb{R}$$

Proof. Let $I : H^1_c(\mathbb{R}) \longrightarrow \mathbb{R}$ be defined as:

$$\omega \longmapsto \int_{\mathbb{R}} \omega$$

AS ω is a compactly supported, closed 1-form, we have that:

$$\omega = \int_{-\infty}^{\infty} \frac{d\omega}{dx} dx = \int_{-R}^{R} \frac{d\omega}{dx} dx = \omega(R) - \omega(-R)$$

If we take $R \to \infty$, we have that:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{d\omega}{dx} dx = \omega(R) - \omega(-R) = 0$$

Meaning that $\omega(R) = \omega(-R)$ when we take R to be sufficiently large.

Surjectivity follows as we can easily normalize any $\omega \in H^1_c(\mathbb{R})$ to integrate to 1. Then, if we tack on an constant to this, by linearity of I, we have that:

$$I(\omega) = k \quad \forall k \in \mathbb{R}$$

Injectivity also follows when we consider that:

$$\int_{\mathbb{R}} \omega = 0$$

Let $\omega = \omega_I(x)dx$. Then if we consider that:

$$f(x) - f(-\infty) = f(x) = \int_{-\infty}^{x} \omega_I(t)dt$$

By differentiation, we have that:

$$df = \omega_I(x)dx = \omega$$

Then, if we take the definition of f:

$$f(R) = \int_{-\infty}^{R} \omega_I(t)dt$$

We get two cases:

(i) take $R < 0$ large enough (in magnitude), then we obtain that:

$$f(R) = \int_{-\infty}^{R} \omega_I(t)dt$$

As ω is compactly supported, and we have that $R \to -\infty$, we see that $\omega_I(t) \to 0$. Therefore,

$$f(R) = \int_{-\infty}^{R} \omega_I(t)dt \equiv \int_{-\infty}^{R} 0 = 0$$

(ii) take $R > 0$ large enough, then we clearly have that:

$$f(R) = \int_{-\infty}^{R} \omega_I(t) = \int_{-\infty}^{\infty} \omega_I(t)dt = 0$$

Thus, we have shown that f, like ω has compact support, and $df = \omega$, meaning that $ker(I) = \{df \mid \forall f \in \Omega_c^0(M)\} \equiv \{0\} \subset H_c^1(\mathbb{R})$. So that it is injective. Thus, I is a bijection, hence an isomorphism. □

This is interesting, as **Poincare Lemma** (**Theorem 33**) says that any $H^k(\mathbb{R}^n)$ such that $k > 0$ is always trivial. We see that if we consider only the compactly supported forms, we have that the first cohomology does not vanish as expected by Poincare Lemma. Let us move on and explore more.

Proposition 63. *Compactly supported de Rham Cohomologies are not homotopy invariant!*

Proof. As $\mathbb{R} \cong \{*\}$, and the point is a compact topological space,

$$H^k(\{*\}) \equiv H_c^k(\{*\})$$

However, we have that by **Lemma 62**, that $H_c^1(\mathbb{R}) \cong \mathbb{R}$. But clearly, we have that by **Definition 6.10**, that:

$$H^0(*) \cong H_c^0(*) = \mathbb{R} \qquad H^k(*) = H_c^k(*) = 0 \quad \forall k \neq 0$$

This is a counterexample to **Lemma 38**, which says that two isomorphic smooth manifolds have isomorphic de Rham Cohomologies. This also shows an explicit counterexample to **Theorem 34**, which says that two homotopic maps induce the same maps in cohomologies (in general). □

Proposition 64.

$$H_c^n(\mathbb{R}^n) \cong \mathbb{R} \qquad \forall n \geq 0$$

Proof. We will leave this without proof. $\square$

Proposition 65.

$$H_c^p(\mathbb{R}^n) = 0 \qquad \forall 0 \leq p < n$$

This proposition, combined with the previous proposition describe the top cohomologies of Euclidean space completely.

8.2 Mayer-Vietoris Sequences for Compactly Supported de Rham Cohomology

We will immediately go into a lemma.

Lemma 66. *Let us denote the inclusion map $i_\# : \Omega_c^n(U) \longrightarrow \Omega_c^n(M)$.*

The map $i_\#$ commutes with the exterior derivative.

Proof.

$$i_\# \omega = 0 \qquad \omega \in \Omega^n(M/U)$$

$$i_\# \omega = \omega \qquad \omega \in \Omega^n(U)$$

Act the exterior derivative on this:

$$d i_\# \omega = 0 \qquad \omega \in \Omega^n(M/U)$$

$$d i_\# \omega = d\omega \qquad \omega \in \Omega^n(U)$$

We immediately see that if we act on ω with the exterior derivative first, then we will obtain exactly the same thing as the exterior derivative of a compactly supported form is still compactly supported on the same domain. $\square$

Moreover, we see that: $i_\#$ induces a map $i_* : H_c^p(U) \longrightarrow H_c^p(M)$ on the de Rham Cohomologies.

We also have a Mayer-Vietoris Sequence for Compactly Supported de Rham Cohomologies!

Theorem 67. ***Mayer-Vietoris for Compactly Supported de Rham Cohomologies***

Let U, $V \subset M$ be open, so that $U \cup V = M$. Then there exists a map δ_* so that:

$$\cdots \xrightarrow{\delta_*} H_c^p(U \cap V) \xrightarrow{i_* \oplus -j_*} H_c^p(U) \oplus H_c^p(V) \xrightarrow{k_* + l + *}$$

$$H_c^p(M) \xrightarrow{\delta_*} H_c^{p+1}(U \cap V) \xrightarrow{i_* \oplus -j_*} \cdots$$

is exact. i, j, k, l are all inclusions.

Proof. We proved the original Mayer-Vietoris in full painstaking detail. We will omit it here. $\square$

8.3 Poincare Duality

Definition 8.4. We define the following map:

$$\mathcal{PD} : \Omega^p(M) \longrightarrow \Omega^{n-p}(M)^*$$

$$\mathcal{PD}(\omega)(\eta) = \int_M \omega \wedge \eta$$

$\mathcal{PD}$ commutes with the exterior derivative.

$$\mathcal{PD}(d\omega)(\eta) = \int_M d\omega \wedge \eta = \int_M d(\omega \wedge \eta) - \int_M \omega \wedge (-1)^p d\eta = d'\mathcal{PD}(\omega)(\eta)$$

Definition 8.5. A smooth (oriented) manifold is **Poincare** if $\mathcal{PD}$ is an isomorphism.

Lemma 68. *Every open ball $U \subset \mathbb{R}^n$ is Poincare.*

Proof. We omit the proof. But it involves using the fact that U is contractible, and that $\mathcal{PD}$ is always injective and that because the only nontrivial de Rham Cohomology is the 0th one, we have that surjectivity follows as $\mathcal{PD}$ is just multiplication by a constant. Hence $\mathcal{PD}$ is an isomorphism. □

Lemma 69. *Let $F : M \longrightarrow N$ be a smooth map, then the following diagram commutes:*

$$\begin{array}{ccc} H^p_{dR}(N) & \xrightarrow{F^*} & H^p_{dR}(M) \\ \big\downarrow{\scriptstyle \mathcal{PD}} & & \big\downarrow{\scriptstyle \mathcal{PD}} \\ H^{n-p}_c(N)^* & \xrightarrow{(F_*)^*} & H^{n-p}_c(M)^* \end{array}$$

Proof. The proof follows exactly like **Lemma 55**.

$$F^*_*\mathcal{PD}(\omega)(\eta) = F^* \int_M \omega \wedge \eta = \int_{F(M)} F^*(\omega \wedge \eta) = \int_N F^*\omega \wedge F^*\eta = \mathcal{PD}(F^*\omega)(F^*\eta)$$

Therefore, we have that:

$$F^*_* \circ \mathcal{PD} = \mathcal{PD} \circ F^*$$

and the diagram follows. □

Lemma 70. *Let U and V be open sets so that U,V, and $U \cap V$ are Poincare. Then $U \cup V$ is Poincare.*

Proof. This is the identical proof to **Lemma 58**. We will not write it down as the reader can just refer to that. □

Lemma 71. *Let $\{U_\alpha\}$ be a collection of disjoint open Poincare sets, then $\coprod_\alpha U_\alpha$ is Poincare.*

Proof. This should be a similar argument to the analogous de Rham isomorphism (**Lemma 57**). □

Theorem 72. ***Poincare Duality***

Let M be a smooth, orientable n-manifold. Then

$$H^p_{dR}(M) \cong H^{n-p}_c(M)^*$$

Proof. Use, **Lemma 68, 70, 71** and then apply **Lemma 60** to obtain this conclusion. □

Corollary 73. *Let M be an orientable, smooth, compact n-manifold. Then*

$$\dim H^p_{dR}(M) = \dim H^{n-p}_{dR}(M)$$

Proof. As M compact, the compact de Rham Cohomologies are the same as the normal ones. Apply **Theorem 72**. □

We are done with our discussion of Poincare Duality. Looking forward, we will explore topics that are useful in the discussion of differential forms and de Rham Cohomology. We seek to generalize some of these things with the use of algebraic topology.

9 Algebraic Constructions of Various Structures

We will construct some of the relevant structures in this section. We will first construct the **Exterior Algebra**, give the underlying algebraic structures of the **de Rham Cohomologies**, we will then explore **Clifford Algebras**, at least algebraically.

9.1 Exterior Algebras and Grassmann Algebra

For the sake of generality, we will work in the category of R-modules over a commutative ring R.

For some preliminaries, from now on, we will define all finite products of sets with the power notation, so that:

$$E^r \equiv E \times E \times \cdots \times E$$

Definition 9.1. An **r-multilinear map** is a map:

$$f : E^r \longrightarrow F$$

This r-multilinear map is **alternating** if

$$f(x_1, \ldots, x_r) = 0 \qquad x_i = x_j \quad \forall i \neq j$$

We also denote:

$$T^r(E) \equiv E \otimes E \otimes \cdots \otimes E \equiv \bigotimes_{i=1}^{r} E$$

Definition 9.2. We denote the submodule of $T^r(E)$, a_r as

$$a_r \equiv \{x_1 \otimes \cdots \otimes x_r : x_i \in E \quad x_i = x_j \quad i \neq j\}$$

Definition 9.3. The **rth Exterior Power** is defined, formally, as:

$$\bigwedge^r(E) = T^r(E)/a_r$$

We will characterize $\bigwedge^r(E)$ with more clarity in the following theorem.

Theorem 74. ***Universal Property of Exterior Powers***
If $f : E^r \longrightarrow F$ is an alternating map, $\exists! f_* : \bigwedge^r(E)$ so that the following diagram commutes.

$$\begin{tikzcd} E^r \arrow[r] \arrow[dr, "f"] & \bigwedge^r(E) \arrow[d, "\exists! f_*"] \\ & F \end{tikzcd}$$

Proof. Usually this would require a ton of effort, but we are lucky to have done so much with differential forms by now, that we can recall certain things and it makes our proof trivial.

This diagram is easy to prove once we recognize that we can redraw this diagram:

Our original diagram

$$\begin{tikzcd} E^r \arrow[r] \arrow[dr, "f"] & \bigwedge^r(E) \arrow[d, "\exists! f_*"] \\ & F \end{tikzcd}$$

can be turned into

$$\begin{tikzcd} & T^r(E) \arrow[d, "\pi"] \\ E^r \arrow[ur, "h"] \arrow[r, "g"] \arrow[dr, "f"] & \bigwedge^r(E) \arrow[d, "\exists! f_*"] \\ & F \end{tikzcd}$$

We define these as follows (from top down):

$$h : E^r \longrightarrow T^r(E)$$

$$(x_1, \ldots, x_r) \longmapsto x_1 \otimes \cdots \otimes x_r$$

$$\pi : T^r(E) \longrightarrow \bigwedge^r(E)$$

$$x_1 \otimes \cdots \otimes x_r \longmapsto x_1 \otimes \cdots \otimes x_r + A_r \quad A_r \in a_r$$

This exists, with its own universal property, this is the **canonical projection** of a monoid into its quotient, where it maps an element to a coset with the representative unchanged.

$$g : E^r \longrightarrow \bigwedge^r(E)$$
$$g = \pi \circ h$$

This is appropriately defined as:

$$g : (x_1, \ldots, x_r) \longmapsto x_1 \wedge \cdots \wedge x_r$$

This is the most important part of this proof, and it is realizing that the wedge product is just a realization of **Definition 9.2, 9.3**.

$$f : E^r \longrightarrow F$$

This is given by assumption.

Therefore, we see that, given all this, and the fact that this diagram must commute in an abelian category, that f_* must necessarily exist.

And if there exists another $f' : E^r \longrightarrow F$, then the arrows (g, h, π) must all "do the same thing", therefore, the map $F'_* : \bigwedge^r(E) \longrightarrow F$ must be the same up to an equivalence relation. Therefore, uniqueness follows. $\square$

This shows that $\bigwedge^r$ is indeed a functor, from the category of R-modules to the category of R-modules.

Proposition 75. $\bigwedge^r : \mathrm{Mod_R} \longrightarrow \mathrm{Mod_R}$ *is a functor.*

Proof. There is not much to prove here. We will simply show that it acts in a usual way that we expect functors to behave.

Let $u : E \longrightarrow F$ be a homomorphism.

$$u : x_i \longmapsto u(x_i)$$

Then, we under $\bigwedge^r$, we have that:

$$\bigwedge^r(u) : \bigwedge^r(E) \longrightarrow \bigwedge^r(F)$$
$$\bigwedge^r(u) = u(x_1) \wedge \cdots \wedge u(x_r)$$

This is sufficient to check that it's a functor, as we can deduce that it preserves identity and distributes over compostion by just, taking $u = id_E$, and invoking the universal property, respectively. $\square$

Immediate Application:

We have been working with the case where $E = T_pM^*$, or the **Cotangent Space**. We see that k-forms are simply elements of $\bigwedge^k(E)$.

Definition 9.4. The **Grassmann Algebra** (or **Exterior Algebra**, **Alternating Algebra**) is:

$$\bigwedge(E) = \bigoplus_{r=0}^{\infty} \bigwedge^{r}(E)$$

This is purely notation sake. We will develop it further.

We will perform the following construction, and we will see how it relates to constructing the **Grassmann Algebra**.

Let G be an additive monoid. Let $A = \bigoplus_{r \in G} A_r$ be a G-graded R-algebra. Suppose that for each A_r, a submodule of a_r, let $a = \bigoplus_{r \in G} a_r$.

Assume that a is an ideal of A. Then a is called a **homogeneous ideal**, and we can define a graded structure on A/a.

$$A_r \times A_s \longrightarrow A_{r+s}$$

We have a bilinear map:

$$A_r/a_r \times A_s/a_s \longrightarrow A_{r+s}/a_{r+s}$$

We see that this bilinear map is such that:

$$a_r \times A_s \longmapsto a_{r+s} \qquad A_r \times a_s \longmapsto a_{r+s}$$

Therefore, we see that we can this so that anytime the above case occurs, we get that the bilinear map collapses to identity!

If we extend this for all $r \in G$, then we obtain that:

$$A/a = \bigoplus_{r \in G} A_r / \bigoplus_{r \in G} a_r$$

We see that A/a is a G-graded R-Algebra.

It is relevant to our work with the exterior powers because we may simply take:

$$T^r(E) = A_r \quad a_r$$

Where a_r is as defined in **Definition 9.2**.

If we define the **Tensor Algebra** as:

$$T(E) = \bigoplus_{r \in G} T^r(E)/a_r$$

Then we get that a is an ideal of $T(E)$, and that:

$$\bigwedge(E) = T(E)/a$$

We define the bilinear map in $\bigwedge$ to be:

$$A_r/a_r \times A_s/a_s \longrightarrow A_{r+s}/a_{r+s}$$

$$(x_1 \wedge \cdots \wedge x_r, y_1 \wedge \cdots \wedge y_s) \longmapsto x_1 \wedge \cdots \wedge x_r \wedge y_1 \wedge \cdots \wedge y_s$$

$\wedge$ is called the wedge product, and we have used this before in the beginning sections.

By our above construction, we see that if we take:

$$(x + y) \wedge (x + y)$$

This is equivalent to looking at:

$$\wedge : A/a \times A/a \longrightarrow A/a$$

But we choose $A = a$, therefore, we see that $A/a \cong \{0\}$, and it collapses to identity. Thus,

$$x \wedge y + y \wedge x = 0 \qquad y \wedge x = -x \wedge y$$

This is where the antisymmetry of the wedge product comes from (This construction partially includes the *Alt* operator we saw earlier).

Furthermore, we see that the Grassmann Algebra is functorial.

Proposition 76. *$\bigwedge : Mod_R \longrightarrow Mod_R$ is a functor.*

Proof. We can verify this almost immediately. Take $f : E \longrightarrow F$, then:

$$f : x_i \longmapsto f(x_i)$$

Then

$$\bigwedge(f) : \bigwedge(E) \longrightarrow \bigwedge(F)$$

$$x_1 \wedge \cdots \wedge x_r \longmapsto f(x_1) \wedge \cdots \wedge f(x_r)$$

We see that this is exactly like when we proved the functoriality of $\bigwedge^r$. The only difference is that there is no restriction on the value of r as long as it is in the grading set G. ☐

Proposition 77. *Let E be a free module of dimension n over R. If $r > n$, then:*

$$\bigwedge^r(E) = 0$$

Let $\{v_1, \cdots, v_n\}$ be a basis of E over R. If $1 \leq r \leq n$, then $\bigwedge^r(E)$ is free over R. The basis of $\bigwedge^r(E)$ is:

$$v_{i_1} \wedge \cdots \wedge v_{i_r} \qquad i_1 < \cdots < i_r$$

Therefore, we have that:

$$\dim \bigwedge^r(E) = \binom{n}{r}$$

Proof. We first prove this for the top exterior product.

(i) $r = n$

Every element in E is expressible as

$$\sum_i a_i v_i$$

If we use the fact that $x \wedge y = -y \wedge x$, we have that the only possible unique combination for the basis of the top exterior product is

$$v_1 \wedge \cdots \wedge v_n$$

as any other combination of bases would result in a linear combination of this exact form. So $\bigwedge^n(E)$ is generated by $v_1 \wedge \cdots \wedge v_n$.

We know that there exists a unique multilinear alternating form called the volume form:

$$a\alpha_1 \wedge \cdots \wedge \alpha_n \qquad q \in R$$

Where:

$$a\alpha_1 \wedge \cdots \wedge \alpha_n(v_1, \ldots, v_n) = a \qquad q \in R$$

As the α_j are a linear combination of the basis for the dual of $\bigwedge^n(E)$ $((\bigwedge^n(E))^*)$.

Denote the basis for the dual as:

$$\varphi_j$$

Then we have that:

$$\alpha_1 \wedge \cdots \wedge \alpha_n = A_I \varphi_1 \wedge \cdots \wedge \varphi_n$$

Where φ_j denotes the projection of each vector into the jth component. Then if we plug this in:

$$A_I \varphi_1 \wedge \cdots \wedge \varphi_n(v_1, \ldots, v_n) = a$$

NOTE: $v_1, \ldots, v_n$ is used interchangeably with φ, as they both serve the same purpose. We did it to save the reader from confusion of having a v_i taking in an argument v_i.

If we choose the case where $a = 0$, then we have that:

$$A_I \varphi_1 \wedge \cdots \wedge \varphi_n = 0$$

By virtue of **Exercise 3.8**, we have that $A_I = 0$ in this case, therefore, we have that $v_1 \wedge \cdots \wedge v_n$ is linearly independent too.

The existence of this form is important because it carries over for the other cases.

(ii) $1 \leq r \leq n$

Suppose we have a relation:

$$\sum_I a_I v_{i_1} \wedge \cdots \wedge v_{i_n} = 0$$

where $I = \{i_1, \ldots, i_r\}$ $i_1 < \ldots i_r$, and $a_I \in R$. Select an r-tuple J defined exactly as I is except with j instead of i. Take the wedge product with some $v_{j_{r+1}}, \ldots, v_{j_n}$ with $j_{r+1}, \ldots, v_n \notin J$, then because it, presumably, exceeds the degree of the exterior product, we get:

$$a_I v_{j_1} \wedge \cdots \wedge v_{j_r} \wedge v_{j_{r+1}} \wedge \cdots \wedge v_{j_n} = 0$$

Now act on the left hand side with a permutation σ:

$$\sigma(j_k) = k$$

Then we obtain:

$$\text{sign}(\sigma) a_I v_1 \wedge \cdots \wedge v_r \wedge \cdots \wedge v_n = 0$$

Then, by what we proved above for $r = n$, we have that $a_I = 0$.

(iii) $r = 0$

It is trivial when $r = 0$, in that case, we simply have $\bigwedge^0(E) = R$ and is spanned by 1.

(iv) $r > n$

When $r > n$, we repeat the argument we made for the case (ii). We will have case (i), a basis volume form:

$$v_1 \wedge \cdots \wedge v_n$$

But we will also get $n+1, \ldots, r \notin I$, so that if we wedge product with any more $v_j \quad j \notin I$, we will get that it trivially vanishes.

The dimension is also trivial if we count the number of unique wedge products we could make. Indeed, the number of unique wedge products of length k we could make out of a list of n separate basis elements of E is given by a combinatorial argument. The answer is precisely $\binom{n}{k}$ $\square$

We will leave the proofs of the rest of the propositions here blank. We will return to prove these at a later time.

Proposition 78. *Let*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence of free R-modules of finite ranks r, n, s, respectively. Then there is a natural isomorphism

$$\varphi : \bigwedge^r E' \otimes \bigwedge^s E'' \longrightarrow \bigwedge^n E$$

This isomorphism is the unique isomorphism having the following property. For elements $v_1, \ldots, v_r \in E'$, and $w_1, \ldots, w_s \in E''$, let $u_1, \ldots, u_s$ be liftings of $w_1, \ldots, w_s$ in E. Then:

$$\varphi\left((v_1 \wedge \cdots \wedge v_r) \otimes (w_1 \wedge \cdots \wedge w_s)\right) = v_1 \wedge \cdots \wedge v_r \wedge u_1 \wedge \cdots \wedge u_s$$

Proof. □

For a free module E of rank n, the **determinant** of the module is:

$$det(E) = \bigwedge^{\max} E = \bigwedge^{n} E$$

Then our previous proposition is denoted by the isomorphism formula:

$$det(E') \otimes det(E'') = det(E)$$

If $R = k$ is a field, then we say that det is an Euler-Poincare map on the category Vec_k.

Proposition 79. *There is a natural isomorphism*

$$\bigwedge^{i} E' \otimes \bigwedge^{n-i} E'' \longrightarrow \bigwedge_{i}^{n} E / \bigwedge_{i+1}^{n} E$$

Proof. □

Proposition 80. *Let $E = E' \oplus E''$ be a direct sum of finite free modules. Then for every positive $n \in \mathbb{Z}$, we have a module isomorphism:*

$$\bigwedge^{n} E \cong \bigoplus_{p+q=n} \bigwedge^{p} E' \otimes \bigwedge^{q} E''$$

In terms of the Grassmann Algebras, we have the isomorphism:

$$\bigwedge E \cong \bigwedge E' \otimes_{su} \bigwedge E''$$

Where $\otimes_{su}$ is the ***superproduct*** *of graded algebras.*

Proof. □

Proposition 81. *Let E be free of rank n over R, for each positive integer r, we have a natural isomorphism*

$$\bigwedge^{r}(E^v) \cong \bigwedge^{r} E^v$$

Proof. □

9.2 Universal Derivations and the de Rham Complex

Let A be an R-Algebra and M and A-module. By a **derivation** $D : A \longrightarrow M$ (over R), we are referring to an R-linear map satisfying the rule:

$$D(ab) = aDb + bDa$$

This immediately gives us a property of derivations:

Proposition 82. *For any derivation D over R:*

$$D(R) = 0$$

Proof.

$$D(1) = D(1) + D(1) = 2D(1)$$

So this is true if and only if:

$$D(1) = 0$$

Since 1 is unity in R, we have that:

$$D(r) = rD(1) \qquad \forall r \in R$$

Therefore,

$$D(R) = 0$$

□

Definition 9.5. The set of derivations from $A \longrightarrow M$ is denoted:

$$\mathrm{Der}_R(A, M)$$

We will give more propositions that will allow us to prove the universal property of derivations.

Proposition 83. *Let $f_{1,2} : A \longrightarrow B$ be R-Algebra homomorphisms.*

Assume that $f_1 \equiv f_2 \mod J$. Then $D = f_2 - f_1$ is a derivation.

Proof. We simply act on elements of A.

$$f_2(ab) = f_2(a)f_2(b) = [f_1(a) + D(a)]\,[f_1(b) + D(b)] = f_1(ab)+f_1(a)D(b)+f_1(b)D(a)$$

Then we may rearrange these expressions to obtain:

$$D(ab) = f_2(ab) - f_1(ab) = f_1(a)D(b) + f_1(b)D(a)$$

As f_1 or f_2 define the module structure of J, so we obtain that the above formula is a realization of the Leibniz Rule. □

We now denote the tensor product $\otimes$ as a tensor product over R, $\otimes_R$.

Definition 9.6. This will be a list of definitions that we will take advantage of.

(i) Define the following map:

$$m_A : A \otimes A \longrightarrow A$$

$$a \otimes b \longmapsto ab$$

(ii) Let $J = ker(m_A)$. We define the module of differentials to be:

$$\Omega_{A/R} = J/J^2$$

and $\Omega_{A/R}$ is an ideal in $(A \otimes A)/J^2$.

The A-module structure will be given as the following embedding:

(iii)

$$A \longrightarrow A \otimes A$$

$$a \longmapsto a \otimes 1$$

We can then consider the direct sum decomposition of A-modules so that:

$$A \otimes A = (A \otimes 1) \oplus J$$

Therefore, we see that:

$$(A \otimes A)/J^2 = (A \otimes 1) \oplus J/J^2$$

(iv) Let

$$d : A \longrightarrow J/J^2$$

$$a \longmapsto 1 \otimes a - a \otimes 1 \quad \text{mod} J^2$$

To satisfy **definition 9.6 (iv)**, we may take (according to **Proposition 83**:

$$f_1 : a \longmapsto a \otimes 1 \qquad f_2 : a \longmapsto 1 \otimes a$$

So that

$$d = f_2 - f_1$$

Where d is a derivation when mapped into J/J^2 due to **Proposition 83**.

Proposition 84.

$$J = ker(m_A)$$

J is generated by elements of the form:

$$\sum_i x_i dy_i \qquad x_i \in J/J^2 \quad y_i \in A$$

Proof.

$$d : y \longmapsto 1 \otimes y - y \otimes 1 \mod J^2$$

Then, we see that:

$$m_A(d(y)) = y - y = 0 \qquad \forall y \in A$$

Therefore, we see that any linear combination in J/J^2 also gets mapped to the kernel of m_A. Hence, the proposition follows. □

By definition of J, we see that:

$$\sum x_i \otimes y_i \in J$$

$$m_A\left(\sum x_i \otimes y_i\right) = \sum m_A\left(x_i \otimes y_i\right) \sum x_i y_i = 0$$

Therefore, we can now prove the Existence and Uniqueness of the Universal Derivation:

Theorem 85. ***Universal Property of the Derivation***

Given a derivation $D : A \longrightarrow M$, $\exists! f : J/J^2 \longrightarrow M$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "D"'] & J/J^2 \arrow[d, "\exists! f"] \\ & M \end{tikzcd}$$

Remark: *We see that this establishes a functorial isomorphism:*

$$Der_R(A, M) \cong Hom_A\left(\Omega_{A/R}, M\right)$$

Proof. With everything developed previously, we see that this construction is simple. We need to define:

$$f : J/J^2 \longrightarrow M$$

$$x \otimes y \longmapsto xDy$$

We will take for granted that this is well-defined up to whatever equivalence relation we defined on our various structures defined before.

We can immediately verify that:

$$f \circ d(y) = f(1 \otimes y - y \otimes 1) = f(1 \otimes y) - f(y \otimes 1) = Dy - yD(1) = D(y)$$

If we take that $x \in J/J^2$, we see that:

$$xf(d(y)) = f(xdy) = xD(y)$$

As J is generated (and hence J/J^2) by elements of the form

$$\sum x_i dy_i$$

We see that our composition is valid. It is trivial now that:

$$D = f \circ d$$

We have proven existence. And we see that uniqueness follows if we consider that any $f' : J/J^2 \longrightarrow M$ must fulfill the exact same properties as f. For most generality, we consider that f and f' are equivalent up to an equivalence relation on J and J^2. $\square$

We will offer propositions which are restatements of things we have already proven for the specific case of Cotangent Spaces over Smooth Manifolds.

Definition 9.7. Let $R \longrightarrow A$ be an R-algebra of commutative rings. For $i \geq 0$, define

$$\Omega^i_{A/R} = \bigwedge^i \Omega^1_{A/R}$$

Where $\Omega^0_{A/R} = A$. Therefore, the wedge product is to be taken over A.

Theorem 86. *There exists a unique sequence of R-linear maps:*

$$d_i : \Omega^i_{A/R} \longrightarrow \Omega^{i+1}_{A/R}$$

such that for $\omega \in \Omega^i$, and $\eta \in \Omega^j$, we have:

1.

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^i \omega \wedge d\eta$$

2.

$$d_i \circ d_{i-1} = 0 \quad \forall i > 0$$

For more clarity, we see that each d_i is actually the universal derivation if we recognize that: $d \equiv d_0 : A \longrightarrow \Omega^1_{A/R}$. Where $\Omega^0_{A/R} = A$.

Proof. Refer to **Theorem 18** for a proof, where we choose a basis to explicitly compute this in.

We will offer a more algebraic proof.

(i) $\Omega^1_{A/R} \longrightarrow \Omega^2_{A/R}$

We see that in this instance, by our construction above, that da generates J/J^2 $\forall a \in A$. J/J^2 is subject to the relations:

$$\forall r \in R \quad d(r) = 0, \quad d(a' + a'') = da' + da'', \quad d(a'a'') = a'd(a'') + a''d(a')$$

We now define d as the map:

$$d\left(\sum a'_i da_i\right) = \sum da'_i \wedge da_i$$

Do not get this d confused with the inside d.

We will first show that this is well-defined under the relations we have for $\Omega^1_{A/R}$ above.

1. $\forall r \in R$, $dr = 0$

   This is trivial by **Proposition 82**.

2. for $a, a', a'' \in A$ we must show

$$d : ad(a' + a'') - ada' - ada'' \longmapsto 0$$

   We can just compute this:

$$d\left(ad(a' + a'') - ada' - ada''\right) = d(ad(a' + a'')) - d(ada') - d(ada'')$$

   We use the mapping rule above:

$$da \wedge d(a' + a'') - da \wedge da' - da \wedge da'' = da \wedge da' + da \wedge da'' - da \wedge da' - da \wedge da'' = 0$$

3. for $a, a', a'' \in A$ we must show

$$d : ad(a'a'') - aa'da'' - aa''da' \longmapsto 0$$

   We also compute this:

$$d\left(ad(a'a'') - aa'da'' - aa''da'\right) = d(ad(a'a'')) - d(aa'da'') - d(aa''da')$$

   Using the definition of d:

$$da \wedge d(a'a'') - da \wedge a'da'' - da \wedge a''da' =$$

$$da \wedge (a'da'' + a''da') - da \wedge a'da'' - da \wedge a''da' = 0$$

Thus, d as defined above is well-defined.

And clearly, if we consider $a \in A \equiv \Omega^0_{A/R}$:

$$d(d(a)) = d(1da) = d1 \wedge da = 0 \wedge da = 0$$

Hence, $d^2 = 0$ in the beginning of this complex.

(ii) $1 \leq p \leq \dots$

For this, we will continue something similar to what we did in case (i), for case (ii).

The proper generalization here is to consider a map that exists universally. We denote it $\gamma : T^p\left(\Omega^1_{A/R}\right) \longrightarrow \Omega^{p+1}_{A/R}$.

We recognize that this can be constructed as a factorization of two universal maps:

$$\pi : T^p\left(\Omega^1_{A/R}\right) \longrightarrow \Omega^{p+1}_{A/R}$$

$$\pi : \omega_1 \otimes \cdots \otimes \omega_p \longmapsto d\omega_1 \wedge \cdots \wedge d\omega_p$$

$$d : \Omega^p_{A/R} \longrightarrow \Omega^{p+1}_{A/R}$$

$$d : \sum \omega_0 d\omega_1 \wedge \cdots \wedge d\omega_p \longmapsto \sum d\omega_0 \wedge d\omega_1 \wedge \cdots \wedge d\omega_p$$

We recognize that using the fact that $\wedge$ is alternating, we can construct γ as:

$$\gamma : T^p \left(\Omega^1_{A/R}\right) \longrightarrow \Omega^{p+1}_{A/R}$$

$$\gamma : \omega_1 \otimes \cdots \otimes \omega_p \longmapsto \sum_i (-1)^{i+1} \omega_1 \wedge \cdots \wedge d(\omega_i) \wedge \cdots \wedge \omega_p$$

We must simply show that this γ is well-defined.

We note that we may simply justify this by acting on the generators and relations of $ker(\pi)$.

For some $f \in A$, $ker(\pi)$ consists of:

$$\omega_1 \otimes \cdots \otimes f\omega_i \otimes \cdots \otimes \omega_p - \omega_1 \otimes \cdots \otimes f\omega_j \otimes \cdots \otimes \omega_p$$

We simply act on this by γ. We recognize that when $f = 1$, this is trivially satisfied by definition of the exterior power as the quotient of the tensor power with the set of all elements

$$\{\omega_1 \otimes \cdots \otimes \omega_p : \omega_i = \omega_j \quad i \neq j\}$$

That case is trivial. We now consider $f \neq 1$.

Without loss of generality, we may only compute it for the case of $p = 2$. As every other case is computed exactly like this case.

$$\gamma : T^2 \left(\Omega^1_{A/R}\right) \longrightarrow \Omega^3_{A/R}$$

Let $p = 2$, $i = 1$, $j = 2$. Let $\omega_1 = bdb'$, $\omega_2 = cdc'$. Where $b, b', c, c' \in \Omega^0_{A/R}$.

We now need to verify if:

$$\gamma \left(f\omega_1 \otimes \omega_2 - \omega_1 \otimes f\omega_2\right) = 0$$

Use the mapping rule for γ and we will compute it:

$$\gamma \left(f\omega_1 \otimes \omega_2\right) = (-1)^2 d(f\omega_1) \wedge \omega_2 + (-1)^3 f\omega_1 \wedge d\omega_2$$

$$= d(f\omega_1) \wedge \omega_2 - f\omega_1 \wedge d\omega_2$$

Likewise,

$$\gamma \left(\omega_1 \otimes f\omega_2\right) = (-1)^2 d\omega_1 \wedge f\omega_2 + (-1)^3 \omega_1 \wedge d(f\omega_2)$$

$$= d\omega_1 \wedge f\omega_2 - \omega_1 \wedge d(f\omega_2)$$

Let us explicitly plug in our choice of $\omega_{1,2}$:

$$\gamma \left(f\omega_1 \otimes \omega_2\right) = d(fbdb') \wedge cdc' - fbdb' \wedge d(cdc')$$

$$= d(fb) \wedge db' \wedge cdc' - fbdb' \wedge dc \wedge dc'$$

$$\gamma(\omega_1 \otimes f\omega_2) = d(bdb') \wedge fcdc' - bdb' \wedge d(fcdc')$$
$$= db \wedge db' \wedge fcdc' - bdb' \wedge d(fc) \wedge dc'$$

We will use antisymmetry and multilinearity of the wedge product to work this out.

$$\gamma(f\omega_1 \otimes \omega_2) = d(fb)c \wedge db' \wedge dc' - (-fbdc) \wedge db' \wedge dc'$$
$$\gamma(\omega_1 \otimes f\omega_2) = (db)fc \wedge db' \wedge dc' - (-bd(fc) \wedge db' \wedge dc')$$

Then

$$\gamma(f\omega_1 \otimes \omega_2) - \gamma(\omega_1 \otimes f\omega_2) = [f(db)c + fb(dc) - (db)fc - bf(dc)]\, db' \wedge dc'$$

By commutativity, we see that $bf = fb$, $fc = cf$, therefore, we have that:

$$[f(db)c + fb(dc) - (db)fc - bf(dc)] \equiv 0$$

Therefore, we have verified that:

$$\gamma(f\omega_1 \otimes \omega_2 - \omega_1 \otimes f\omega_2) = 0$$

And it clearly shares the kernel with the map π, as we have just shown that:

$$\gamma(ker(\pi)) = 0$$

Therefore, by the construction of the map γ, we have that d necessarily exists, as we have that:

$$\gamma(\omega_0 d\omega_1 \otimes \cdots \otimes \omega_p) = d\omega_0 \wedge d\omega_1 \wedge \cdots \wedge d\omega_p$$

Therefore, the map d will necessarily exist, and it will act in the manner given by γ.

Therefore, the rule:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^i \omega \wedge d\eta$$

Necessarily holds by the construction of the de Rham Complex $(\Omega^i_{A/R}, d_i)$. We have proven that this is a free module, and that this satisfies the antisymmetry given by the map γ, due to action of permutations on each element of $\Omega^1_{A/R}$.

$d^2 = 0$ is trivial in the general case because:

$$d(\gamma(\omega_0 d\omega_1 \otimes \cdots \otimes d\omega_p)) = d(d\omega_0 \wedge \cdots \wedge d\omega_p) = d(1 d\omega_0 \wedge \cdots \wedge d\omega_p)$$
$$= d(1) \wedge d\omega_0 \wedge \cdots \wedge d\omega_p = 0$$

Therefore, we have proven the existence of $\left(\Omega^i_{A/R}, d_i\right)$.

The uniqueness follows immediately as the generators of $T^p\left(\Omega^i_{A/R}\right)$ are unique up to a constant. Due to the equivalence in $ker(\pi)$ up to a constant, all de Rham Complexes that satisfy every property we have specified above will be unique up to a factor in the base de Rham Cohomology $\Omega^0_{A/R} = A$.

We are done. $\square$

Given this theorem, the modules $\Omega^i_{A/R}$ form a complex of modules, called the **de Rham Complex**.

Theorem 87. *Let k be a characteristic 0 field. Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring in n variables. The de Rham Complex:*

$$0 \longrightarrow k \longrightarrow A \longrightarrow \Omega^1_{A/k} \longrightarrow \cdots \longrightarrow \Omega^n_{A/k} \longrightarrow 0$$

is exact.

Proof. In our present case, A is a finitely generated k-algebra.

We will take care of the edge cases and we will prove the rest easily.

By definition, it suffices to find the differentials that are closed, exact, and verify that, at each module in the sequence, that they are both closed and exact.

(i) $k \longrightarrow A$

This is the trivial case but sets an important precedent. k is the field of coefficients for the polynomials. Therefore, we see that by formal integration (an antiderivation of the indeterminates of the finitely-generated k-algebra, A)

$$\int k \prod_{i=1}^{n} dx^i$$

Likewise, we see that for any $f \in A$, $df = 0$ if and only if $f \in k$, so every closed form is an element of k, and every exact form is a result of formally integrating an element of k, therefore, the closed and exact forms coincide.

(ii) $A \longrightarrow \Omega^1_{A/k}$

We define the derivation as the usual exterior derivative, this time formally differentiating the indeterminates in our polynomial ring.

$$f \in \Omega^1_{A/k} \quad d_1 f = 0 \Longrightarrow \int \frac{\partial f}{\partial x^i} dx^i = 0$$

Integration leads us to obtain that:

$$f = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

For each $\frac{\partial}{\partial x^i}$, the kernel is exactly these f, and the exactness follows by the formal integration procedure.

(iii) $\Omega^1_{A/k} \longrightarrow \Omega^2_{A/k}$ is the last case we need, as (without loss of generality), every other case is just this one repeated with more terms.

We see that the general prescription is:

$$\sum_{J=\{i_1,i_2\}, i_{1,2} \in \{1,\ldots,n\}} \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^J = 0$$

We will leave out the case where $i \in J$, these cases indicate that:

$$f = x_i + K \quad K \in k$$

When $i \notin J$, we have that:

$$f = g(\{x_j : j \neq i_1, i_2, i\})$$

These are precisely the ones that vanish under the exterior derivative. And the formal integration indicates that they are exact. This gives the general prescription.

(iv) $\Omega^n_{A/k} \longrightarrow 0$

The top exterior power has the property that any $n+1$ wedge product will automatically vanish. So we get that any n-form will be closed. And all of these n-forms come from the fact that:

$$\gamma : \omega_0 d\omega_1 \wedge \cdots \wedge d\omega_{n-1} \longmapsto d\omega_0 \wedge d\omega_1 \wedge \cdots d\omega_{n-1}$$

Therefore, all the closed forms are the exact forms. $\square$

We have painstakingly constructed the de Rham Complex and the module of differentials over any R-algebra.

9.3 Clifford Algebras

We will give a brief construction of Clifford Algebras before we move on.

We will prove the existence of the Clifford Algebra, introduce its functoriality, and move on.

Theorem 88. ***Universal Property of Clifford Algebras***

*Let g be a symmetric bilinear, quadratic form on a finite-dimensional vector space E over k. Then the **Clifford Algebra** is the universal pair, $(C(g), \rho)$ **given by the following universal property**: Given that*

$$\rho : E \longrightarrow C(g)$$

If we have a linear map ψ that maps into an arbitrary k-algebra L,defined by:

$$\psi : E \longrightarrow L$$

$$\psi(x)^2 = g(x,x) \cdot \mathbf{1} \quad \mathbf{1} \in L \quad \forall x \in E$$

Then $\exists! \psi_* : C(g) \longrightarrow L$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\rho"] \arrow[dr, "\psi"'] & C(g) \arrow[d, "\exists! \psi_*"] \\ & L \end{tikzcd}$$

Proof. $T(E)$ is the tensor algebra over a free k-module E:

$$T(E) = \bigoplus_{i=1}^{n} T^i(E) \quad T^i(E) = \bigotimes_{j=1}^{i} E$$

Let I_g be the ideal of $T(E)$ generated by:

$$x \otimes x - g(x,x) \cdot \mathbf{1} \quad \forall x \in E$$

Then, we define:

$$C_g(E) = T(E)/I_g$$

We observe that E is naturally embedded into $T(E)$. We will illustrate this in the following commutative diagram.

$$\begin{tikzcd}
 & T(E) \arrow[d, "\pi"] \\
E \arrow[ur, "h"] \arrow[r, "\rho"] \arrow[dr, "\psi"'] & C_g(E) \arrow[d, "\exists!\psi_*"] \\
 & F
\end{tikzcd}$$

Where we have the canonical projection into the set of equivalence classes:

$$\pi : T(E) \longrightarrow T(E)/I_g$$

The natural embedding of E into $T(E)$.

$$h : E \longrightarrow T(E)$$

This next construction is not obvious, but we are allowed to do this because of the **Universal Property of the Tensor Product** and the **Universal Property of the Coproduct**. Both of these guarantee that there is a terminal object in a diagram involving $T(E)$. Therefore, we can guarantee that:

$$\psi_* : C_g(E) \longrightarrow L$$

Therefore, we are done showing the existence. Uniqueness follows from the uniqueness of Tensor Products, Coproducts, etc.

Furthermore, by the commutativity of the diagram, we have that:

$$\rho(x)^2 = g(x,x) \cdot \mathbf{1}$$

Even moreso, it follows that ρ is injective due to the correspondence with the symmetric, bilinear, quadratic form. $\square$

Proposition 89. *The Clifford Algebra has the* ***polarization identity****:*

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 2g(x,y) \cdot \mathbf{1}$$

Proof. We use the fact that:

$$\psi(x)^2 = g(x,x) \cdot \mathbf{1} \quad \mathbf{1} \in L$$

Replace x with $x + y$:

$$\psi(x+y)^2 = g(x+y, x+y) \cdot \mathbf{1}$$

As ψ is linear and g is symmetric and bilinear:

$$(\psi(x) + \psi(y))^2 = \psi(x)^2 + \psi(x)\psi(y) + \psi(y)\psi(x) + \psi(y)^2$$

$$= (g(x, x+y) + g(y, x+y) = g(x,x) + g(x,y) + g(y,x) + g(y,y)) \cdot \mathbf{1}$$

Using the definition of $\psi(x)^2$, we get cancellations on both sides and we obtain:

$$\psi(x)\psi(y) + \psi(y)\psi(x) = (g(x,y) + g(y,x)) \cdot \mathbf{1}$$

By symmetry of g, we have:

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 2g(x,y) \cdot \mathbf{1}$$

□

The existence was (for all intents and purposes) rather simple given what we have already done. We will be proving that the dimension of this Clifford Algebra, as a vector space over k,

Theorem 90. *The dimension of $C_g(E)$, as a vector space over k, is 2^n, where $n = \dim(E)$.*

Proof. For the proof, we need to explicitly compute the basis of $C_g(E)$.

Let $e_i = \psi(v_i)$. Then we let $c_i = g(v_i, v_i)$. Therefore, by definition of the **Clifford Algebra**, let v_i be an orthonormal basis of E:

$$\psi(v_i)^2 = c_i$$

By **Proposition 89**, we have that:

$$\psi(v_i)\psi(v_j) + \psi(v_j)\psi(v_i) = 2g(v_i, v_j) \cdot \mathbf{1}$$

As g is a symmetric, bilinear, non-degenerate quadratic form, we have that:

$$g(v_i, v_j) = 0 \quad \forall i \neq j$$

Hence

$$e_i e_j + e_j e_i = 0 \quad \forall i \neq j$$

This implies that any subalgebra generated by $\psi(E)$ over k is a vector space generated by elements:

$$e_1^{\nu_1} \cdots e_n^{\nu_n} \quad \nu_i = 0, 1 \quad \forall i = 1, \ldots, n$$

Therefore, there are $\leq 2^n$ of these, so we have $\dim C_g(E) \leq 2^n$.

We now need to construct L so that $\psi : E \longrightarrow L$ generates L as a subalgebra of $\psi(E)$ over k.

Definition 9.8. Let M be an R-module (two-sided), let $i, j \in \mathbb{Z}/2\mathbb{Z}$. Suppose M has a direct sum decomposition:

$$M = M_0 \oplus M_1 \quad 0, 1 \in \mathbb{Z}/2\mathbb{Z}$$

Then, M is **$\mathbb{Z}/2\mathbb{Z}$-graded**. M is an R-algebra, we call M a **$\mathbb{Z}/2\mathbb{Z}$-graded algebra** if $M_i M_j \subset M_{i+j}$ for any $i, j \in \mathbb{Z}/2\mathbb{Z}$.

Let A, B be graded modules. $A = A_0 \oplus A_1$, $B = B_0 \oplus B_1$.

The tensor product $A \otimes B$ has a direct sum decomposition:

$$A \otimes B = \bigoplus_{i,j} A_i \otimes B_j$$

We grade this by letting:

$$(A \otimes B)_0 \quad i + j = 0 \quad (A \otimes B)_1 \quad i + j = 1$$

Then if A,B are graded algebras over the ring, there is a unique bilinear endomorphism of $A \otimes B$ such that:

$$(a \otimes b)(a' \otimes b') \longmapsto (-1)^{ij} aa' \otimes bb'$$

This gives rise to a graded algebra, whose product is called the **Super Tensor Product**. The super tensor product of two graded modules is denoted as:

$$A \otimes_{su} B$$

We now suppose that E has dimension 1 over k. The polynomial ring:

$$k[x]/(x^2 - c_1)$$

is a Clifford Algebra. We let t_1 be the image of x in the factor ring, so:

$$C_g(E) = k[t_1] \quad t_1^2 = c_1$$

We are going to take the super product inductively:

$$C_g(E) = k[t_1] \otimes_{su} k[t_2] \otimes_{su} \cdots \otimes_{su} k[t_n] \quad k[t_i] = k[x]/(x^2 - c_i)$$

Each factor has dimension 2 (as the degree of the principal ideal is 2), and there are n of these in the super product. This thing has dimension 2^n. E is embedded in $C_g(E)$ by the map:

$$a_1 v_1 + \cdots + a_n v_n \longmapsto a_1 t_1 \oplus \cdots \oplus a_n t_n$$

We see that each $t_{i,j}$ satisfy the anti-commutation rules.

As $\dim(E) \geq 1$ over k, we have that:

$$\dim(C_g(E)) \geq 2^n$$

Since

$$2^n \geq \dim(C_g(E)) \leq 2^n$$

The dimension is exactly 2^n. $\square$

We can also just verify that our generating set is indeed a basis, and do combinatorics on it to give us the dimension explicitly.

Important Example: The Dirac Algebra

If we take g to be the metric tensor, given by the first fundamental form $dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$, take $k = \mathbb{R}$, $E = \mathbb{R}^4$.

$$\dim C_g(E) = 2^{\lfloor 4/2 \rfloor} = 4$$

So the generators of $C_g(E)$ are denoted by:

$$\{\gamma^\mu : \mu = 0, 1, 2, 3\}$$

With the famous anticommutators:

$$\{\gamma^\mu, \gamma^\nu\} = 2g_{\mu\nu} \cdot \mathbf{1}$$

These γ^μ serve as the linear map ρ.

We will return for a further exposition of Clifford Algebras later, there are a lot of discrepancies depending on the dimension of the Algebras.

10 Some Homological Algebra

Homological Algebra is an extremely useful tool, as we have witnessed. The proof of the existence and the exactness of the Mayer-Vietoris Sequence is trivial once we proved **Lemma 37**. We take some time to introduce various topics in homological algebra for the purposes of future use. Remember, amidst all our abstraction, **abstraction is meant to help us! It should not confuse us, but it should make problems simpler**.

For some preliminaries, we make note that we will only be working with abelian categories, as anything else is wild and not useful to us (I don't even know what to make of a category that is not abelian). Unless otherwise stated, we have that every category we work with is a category of R-modules over a commutative ring R.

We have various notions that will be useful in the study of homological algebra.

Definition 10.1. An **exact sequence** is a collection of objects $\{A^i\}$, and a collection of morphisms $\{f_i\}$ so that:

$$0 \longrightarrow A^1 \xrightarrow{f_i} A^2 \xrightarrow{f_2} \cdots \xrightarrow{f_k} A^{k+1} \xrightarrow{f_{k+1}} \cdots$$

And $im(f_i) = ker(f_{i+1})\ \forall i$.

More specifically, we have the notion of:

Definition 10.2. A **Short Exact Sequence** is an **exact sequence** that is the collection of three nontrivial objects.

$$0 \longrightarrow A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} A^3 \longrightarrow 0$$

Likewise, we have the notion of:

Definition 10.3. A **Long Exact Sequence** is an **exact sequence** that is not a **Short Exact Sequence**, but it is finite.

$$0 \longrightarrow A^1 \xrightarrow{f_i} A^2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} A^k \longrightarrow 0$$

We have another corresponding notion:

Definition 10.4. Let A^i be a sequence of R-modules. So that each A^i contains an **exact sequence** (as given by **Definition 9.1**).

d^i are homomorphisms called **boundary operators**, and they connect the sequences A^i together. A **cochain complex**, is a pair $(A^\cdot, d^\cdot)$, denoted as:

$$\cdots \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} A^3 \xrightarrow{d^3} A^4 \xrightarrow{d^4} A^5 \xrightarrow{d^5} \cdots$$

That satisfies the relation:

$$d^i \circ d^{i-1} = 0 \qquad \forall i$$

Likewise, for A_i a sequence of R-modules. So that each A_i contains an **exact sequence**.

d_i are homomorphisms called **boundary operators**, and they connect the sequences A_i together.

A **chain complex**, is a pair $(A., d.)$, denoted as:

$$\cdots \xleftarrow{d_0} A_0 \xleftarrow{d_1} A_1 \xleftarrow{d_2} A_2 \xleftarrow{d_3} A_3 \xleftarrow{d_4} A_4 \xleftarrow{d_5} A_5 \xleftarrow{d_6} \cdots$$

That satisfies the relation:

$$d_i \circ d_{i+1} = 0 \qquad \forall i$$

Notice, that the only difference in the **cochain** and **chain** complex is the direction of the morphisms. The intrinsic motivation for these two difference is the difference between **Cohomologies** and **Homologies**.

We now present various lemmas which we will find useful in our use of homological algebra. The most fundamental of which is the **Snake Lemma**.

Theorem 91. ***Snake Lemma***

11 Differential Forms (Background)

Here we will lay out the background for Differential Forms and deRham Cohomology (some commutative algebra background would be helpful ...). All manifolds discussed will be smooth (C^∞) unless otherwise denoted.

Definition 11.1. Let $0 \leq n \leq \dim M$. Then a **Differential n-form** is a rank (0,n)-Tensor Field that is **totally antisymmetric**.

i.e. if ω is an n-form, then the following holds:

$$\omega(x_{i_1}, \cdots, x_{i_n}) = \text{sgn}(\pi)\omega(x_{\pi(i_1)}, \cdots, x_{\pi(i_n)})$$

Where $\pi \in S_n$ (that is the **Symmetric Group on n Letters**), and $x_{i_k} \in \Gamma(TM)$ (i.e. the smooth sections of a Tangent Bundle of M).

Notation: Let the set of all n-forms be denoted as $\Omega^n(M)$.

We will develop the algebraic structure with our first proposition.

Proposition 92. *The set of all n-forms is a $C^\infty(M)$-module.*

Proof. **NOTE**: By $C^\infty(M)$, we mean the **Ring of Smooth Functions on the Manifold** M (we can guarantee this as M is a **smooth** manifold).

To verify this, just note that $\square$