Algebra Notes

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These algebra notes are written in conjunction with the reading on Serge Lang's aptly titled **Algebra**. This will be a collection of notes and solutions to certain exercises. I intend to add onto this over time, and study it thoroughly in preparation for graduate school (I hope I get admitted somewhere!). These are meant to be read for people who already know a few definitions and canonical examples of what a monoid, group, ring, field, vector space, etc. are. This will not be self-contained as you should read the book itself in that case.

0 Preliminaries

The notation for multiplication and addition should be obvious from context. In the case it is arbitrary, I will denote the law of composition for two elements a and b as ab. Some books like to use $a * b$ (Fraleigh), some books like to define groups with addition $a + b$ and denote multiplication as ab (Hungerford does this). Notation should not be confusing and I will always give context.

Remark 0.1. A notion we will find useful throughout all math is commutativity. If we find that a sequence of maps:

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} A_n$$

$$A_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \dots \overset{g_{m-1}}{A} {}_n$$

Then we call the mappings **commutative** if:

$$f_{n-1} \circ \cdots \circ f_1 = g_{m-1} \circ \cdots \circ g_1$$

1 Chapter 1 (Lang): Groups

1.1 Monoids

We begin from the ground up with monoids. These are the most basic algebraic structures we will regard in our study of algebra. Despite how basic they are, we still find that monoids are rich with structure and give us a majority of what is meaningful in algebra.

Definition 1.1. Monoids are a nonempty set G, equipped with a composition law that is associative and has a unit element (identity).

Definition 1.2. A **composition law** (or binary operation in more modern terms) is a map:

$$S \times S \longrightarrow S$$
$$(x, y) \longmapsto xy$$

Definition 1.3. A composition is associative in that:

$$\forall x, y, z \in S \quad (xy)z = x(yz)$$

Definition 1.4. A **unit element** is an element $e \in S$ such that:

$$\forall x \in S \quad xe = ex = x$$

It is trivial to verify that e is unique if it exists.

Definition 1.5. Using **Remark 0.1**, and **Definition 1.1**, the composition law is **commutative** if $\forall x, y \in S$, $xy = yx$.

In this case, the monoid S is called **abelian**.

Note that **Definition 1.2, 1.5** may be extended to finitely many monoids, meaning that our composition law is well-defined for finite operations.

Remark 1.1. Consider when S, S' are two subsets of a monoid G. Then we define:

$$SS' = \{xy \mid x \in S \ y \in S'\}$$

More noticeably, we consider:

$$SG = \{sg \mid s \in S \ g \in G\}$$

And we may also consider for $x \in S$:

$$xG = \{x\}G = \{xg \mid x \in S \ g \in G\}$$

This will be important in our definition of a **normal subgroup**.

Proposition 1. ***Power Associativity***

Let $x \in G$ for a monoid G. Then the following is true for finite $n, m \in \mathbb{N}$.

$$x^{n+m} = x^n x^m \qquad (x^n)^m = x^{nm}$$

Definition 1.6. A **submonoid** of G is a subset $H \subseteq G$ such that H contains the unit (identity) element of G and $\forall x, y \in H$, $xy \in H$ where we use the composition law of G for H.

Example 1.1. A special submonoid of any monoid G is the monoid generated by $x \in G$, i.e.

$$\{x^m \mid x \in G \; m \in \mathbb{N}\}$$

Example 1.2. This is relevant later, but for R a commutative ring, we will refer to a **multiplicative subset** of R, S, as monoids under multiplication of R (with the multiplicative unit element of R)

Example 1.3. The natural numbers $\mathbb{N}$, the set of nonnegative integers, if a monoid.

1.2 Groups

We will now build on the idea of a monoid.

Definition 1.7. A **group**, G, is a monoid G such that $\forall x \in G, \exists y \in G : xy = yx = e$. We often say $y = x^{-1}$ (or $-x$ if we are considering additive groups, it should be obvious from context).

Proposition 2. *For a set G, if G only has left inverses, i.e. $y \in G : yx = e$, then G must have two-sided inverses, hence G must be a group.*

Proof. This is a trivial computation.

Let $b \in G$ so that $\forall a \in G$, $ba = e$, then $bab = eb = b$ implying $ab = e$. One must simply for the same thing starting with the right inverse, and conclude that it must also be a left inverse. $\square$

Definition 1.8. Group of Maps Let G be a group and S a nonempty set. We call the set of maps from S to G as:

$$M(S, G) = \{f \mid f : S \longrightarrow G\}$$

This is a group, with the composition law inherited from G (if G is additive, so is $M(S, G)$ and etc.). One may easily deduce what the inverse and identity elements are.

Definition 1.9. Permutation Groups Let S be a nonempty set. Let G be the set of bijections of S to itself. Then G is a group under the compositions of these bijections. The unit element is the identity map of S and inverses are just inverse maps (i.e. the function that returns the preimage of an element in the codomain of f).

This group has a special name and is called the group of permutations $Perm(S)$. We will extensively study the case for when S has finite cardinality, these are called the **Symmetric Groups on n-letters**.

Example 1.4. Linear Maps and General Linear Group Consider the group in **Definition 1.8**. Now let $S = G = V$ where V is a vector space over a field k. Then:

$$GL(V) := M(V, V)$$

As these are linear maps of V, this is a group under composition of maps (which is really just multiplication as in the group $M(V, V)$).

Specifically, if we consider the case where V is finite-dimensional (n-dimensional in fact) and our linear maps are $n \times n$ matrices acting on V, over k, then we obtain the **General Linear Group over** k:

$$GL(n, k) := GL(V) = \{A \in M_{n\times n}(k) \mid \forall A \exists A^{-1} : AA^{-1} = A^{-1}A = I\}$$

For $n > 1$, this group is noncommutative.

Definition 1.10. Automorphism Groups

Consider an object A in any category. The set of all automorphisms (isomorphisms of $A \longrightarrow A$) is denoted:

$$Aut(A) = \{f : A \longrightarrow A\}$$

The binary operation is clear depending on context, but it will usually be the binary operation associated with objects of the category.

This setting is currently too general but it will be useful in the future.

Example 1.5. Cyclic Groups and Abelian Groups

We will start by saying that the set of integers $\mathbb{Z}$, forms a group, and it is an **abelian group** (i.e. an abelian monoid with an inverse under composition law of addition).

Furthermore, $\mathbb{Z}$ is known as a **cyclic group**, a group that is generated by an element of itself (refer to **Example 1.1**). The generator of $\mathbb{Z}$ is ± 1.

We particularly would like to focus on a finite cyclic group. Naively, we define the **cyclic group of order** n as:

$$\mathbb{Z}_n = \{k \bmod n \mid kin\mathbb{Z}\}$$

Please note that this notation will change later as this is easily confused with n-adic numbers.

Example 1.6. Direct Product of Groups

Consider the Cartesian Product of two sets, $G_1 \times G_2$. If these are groups, then the resulting product is a group itself:

The composition is defined as:

$$(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2)$$

The unit (identity) element is:

$$(e_1, e_2) \quad e_1 \in G_1\ , e_2 \in G_2$$

And the inverses are the inverses of each component:

$$(x_1, x_2)(x_1^{-1}, x_2^{-1}) = (e_1, e_2)$$

In general, we can take an product of n groups, $\prod_{i=1}^{n} G_i$, whose elements are called families, denoted as the collection $(x_i)_{i\in I} \quad x_i \in G_i$. The composition law of these families is given by:

$$(x_i)_{i\in I}(y_i)_{i\in I} = (x_i y_i)_{i\in I}$$

The inverse is $(x_i^{-1})_{i\in I}$ and the unit element is $(e_i)_{i\in I}$.

This group is called the **Direct Product** of the family, and if one orders these families into n-tuples, they will obtain the usual direct product of groups.

Definition 1.11. A **subgroup** of a group G, denoted as H, is a submonoid of G such that H contains all of its inverses, inherited from G.

Proposition 3. *The arbitrary intersection of subgroups is a subgroup itself.*

Proof. Let X, Y be subgroups of G. Firstly, we have that $X \cap Y \subseteq X, Y$. Then $\forall x, y \in X \cap Y$, we have that $x, y \in X$ so $xy \in X$. $x, y \in Y$, so $xy \in Y$. Hence, $xy \in X \cap Y$.

Likewise, since $e \in X$, and $e \in Y$ (we know this is the same e since the unit element is unique), we know that $e \in X \cap Y$. Hence, $X \cap Y$ is a submonoid of G.

Since X and Y contain inverses of all their elements, any element of $X \cap Y$ must inherit these as well. $\square$

Definition 1.12. Generated Groups

Let G be a group, and S a subset of G. S is the set of **generators** for G is every element of G may be expressed as a composition of elements of S, or inverses of elements of S. By definition, the set of all such products $x_1 \cdots x_n$ is the **smallest** subgroup of G that contains S (this idea will be explored and expanded upon later in our construction of the free group).

If S generates G, then we say that $G = \langle S \rangle$

Example 1.7. A cyclic group is a group such that the generating set S has one element.

Example 1.8. The two non-abelian groups of order 8 are:

Dihedral Group of order 8:

This is the symmetries of the square, generated by:

$$\langle \sigma, \tau \rangle \quad \sigma^4 = \tau^2 = e\ \tau\sigma\tau^{-1} = \sigma^3$$

Quaternion Group:

This is the group generated by

$$\langle i, j \rangle \quad i^2 = j^2 = k^2 \; ij = i^2 ji$$

Definition 1.13. For two monoids G, G', a **(Monoid) Homomorphism** $f : G \longrightarrow G'$ is a map such that $f(xy) = f(x)f(y)$ for any $x, y \in G$.

Definition 1.14. The **Kernel** of a map $f : G \longrightarrow G'$ is the subset of G:

$$\ker(f) := \{x \in G \mid f(x) = e'\}$$

Proposition 4. *The Kernel of a group homomorphism is a subgroup of the domain.*

Definition 1.15. The **Image** of a map $f : G \longrightarrow G'$ is the subset of G':

$$\mathrm{Im}(f) := \{f(x) \in G' \mid x \in G\}$$

Proposition 5. *Noting that $f : G \longrightarrow G'$ is a homomorphism, we can prove the following:*

1. $f(x^{-1}) = f(x)^{-1}$
2. $f(e) = e'$ *where* $e \in G$, $e' \in G'$
3. $\ker(f) = \{e\}$ *if and only if* f *is injective.*

Proof. 1. and 2. are all a matter of using the definition of a homomorphism. 3 is trivial. Assuming that $\ker(f) = \{e\}$. Let $f(x) = f(y)$, then that implies that $f(x)f(y^{-1}) = e'$, hence $xy^{-1} = e$, hence $x = y$.

To prove the opposite implication, start with the fact that f is injective. We know that $e \in \ker(f)$. Suppose that there were $x \in \ker(f)$ such that $x \neq e$. Then we would have $e' = f(e) = f(x)$. By injectivity of f, we obtain that $x = e$. Thus, this contradicts our assumption that the kernel has a nontrivial element. □

Proposition 6. *G is a group, and H, K are two subgroups so that $H \cap K = \{e\}$, $HK = G$, and $xy = yx \; \forall x \in H$ and $\forall y \in K$. Then, the map:*

$$F : H \times K \longrightarrow G$$

$$F : (x, y) \longmapsto xy$$

Is an isomorphism.

Proof. It is easy to check that F is a homomorphism.

$$F((a, b)(c, d)) = acbd \quad F(a, b)F(c, d) = abcd$$

By commutativity in HK, we see this is a homomorphism.

$\text{Im}(F) = HK = G$ by assumption, hence F is surjective.

Let $(x, y) \in \ker(F)$. Then $F(x, y) = xy = e'$. Hence, this implies that $x = y^{-1}$. This means that $x \in K$ and $y \in H$, hence $x \in H \cap K = \{e\}$. Hence, since $x = e$ by uniqueness of the unit element.

F is an isomorphism. $\square$

Remark 1.2. Note that this is true for any finite number of subgroups: $H_1, \ldots, H_n$ such that (i) all of their elements commute with one another (ii) $H_1 \cdots H_n = G$ (iii) $H_{i+1} \cap H_1 \ldots H_i = e$. The implication is that $G \cong H_1 \times \cdots \times H_n$.

Definition 1.16. Let G be a group and H a subgroup. A **left coset** of H in G is a subset of G, aH where $a \in G$ (see **Remark 1.1**). Any element of aH is called a **coset representative** of aH.

Proposition 7. *All cosets have equal cardinality.*

Proof. The map $H \longrightarrow aH$ is $x \longmapsto ax$ for $x \in H$. This is clearly surjective, and, in fact, it is even more clearly injective. Hence this is a bijection of sets, and, therefore, have equal cardinality. $\square$

Proposition 8. *All left cosets of H in G are disjoint.*

Proof. Let $a, b \in G$, and let aH, bH be left cosets sharing one element. Then $ax = by$ for $x, y \in H$, not necessarily equal. This implies that $a = byx^{-1}$, hence $aH = byx^{-1}H = b(yx^{-1})H = bH$. Hence, this suffices to show that two cosets are equal if and only if they share at least one element. Hence, they are disjoint. $\square$

Remark 1.3. The consequence of the above proposition is that $G = \coprod_{a \in G} aH$, i.e. the disjoint union of all left cosets of H in G.

Remark 1.4. We can extend the exact same proofs and ideas to the idea of the **right coset**, Ha.

Definition 1.17. The number of left cosets of H in G is known as the **(left) index** of H in G, and denoted by $(G : H)$.

Proposition 9.

$$(G : H)(H : 1) = (G : 1)$$

Furthermore, if H, K are subgroups of G, and $K \subset H$, for a set of (left) coset representatives $\{x_i\}$ of K in H, and (left) coset representatives $\{y_j\}$ of H in G, $\{y_j x_i\}$ forms the set of coset representatives of K in G.

Proof. Using **Remark 1.3**, we see that:

$$H = \coprod_i x_i K \quad G = \coprod_j y_j H$$

We may multiply these together:

$$G = \bigcup_{i,j} y_j x_i K$$

To see that this union is actually disjoint, we assume that:

$$y_j x_i K = y_{j'} x_{i'} K$$

Then

$$y_j x_i K H = y_{j'} x_{i'} K H$$

As $K \subset H$ and $x_i \in H$, we see that:

$$y_j H = y_{j'} H$$

Therefore, $y_j = y_{j'}$ and $x_i = x_{i'}$ as the cosets of K and H must have distinct elements, whence they not be disjoint.

This proves that $(G : H)(H : K) = (G : K)$. Just take K to be the trivial subgroup of G and the result follows. □

1.3 Normal Subgroups

Definition 1.18. Let G be a group and H be a subgroup such that $\forall x \in G$, $xH = Hx$, or $xHx^{-1} = H$. Then we say that H is a **normal subgroup** of G.

There's special notation for the partial ordering by inclusion of H in G, $H \triangleleft G$

Proposition 10. *For a homomorphism $f : G \longrightarrow G'$, the kernel, $\ker(f)$, is a normal subgroup of G.*

Proof. Let $x \in G$. Let $H = \ker(f)$. We know by **Proposition 4** that H is a subgroup of G. We must only verify normality.

Let $y \in H$.

$$xH = \{xy \mid x \in G\ y \in H\}$$

$f(xy) = f(x)e' = f(x)$. Likewise, we can do $f(xy) = e'f(x) = f(y)f(x)$, hence, $xy = yx\ \forall y \in H$ and any $x \in G$. This is equivalent to saying that $xH = Hx$. □

Definition 1.19. Composition Law for Cosets

Let G' be a set of cosets of H in G. Taking two cosets, xH and yH, we may define the following composition law:

$$G' \times G' \longrightarrow G'$$

$$(xH, yH) \longmapsto xyH$$

Thus, the composition law if inherited from the composition law of the group G.

This leads us to the following result.

Proposition 11. *The set of cosets of H in G, G', is a group.*

Definition 1.20. Quotient (Factor) Groups

The group G' defined beforehand is called a **Quotient Group** and is denoted as G/H.

Definition 1.21. Let $\pi : G \longrightarrow G/H$ be defined as:

$$\pi : x \longrightarrow xH$$

If $\pi(x) = H$, this means that $x \in H$ by definition of cosets. Therefore, $H = \ker(\pi)$.

This map π is called the **canonical map** and is surjective.

Proposition 12. *G/H is a group if and only if $H = \ker(\pi)$.*

Proposition 13. *Let $(H_i)_{i \in I}$ be a family of normal subgroups of G. The intersection of this family is itself a normal subgroup of G, i.e. $H = \bigcap_{i \in I} H_i \triangleleft G$.*

Proof. The intersection of subgroups is itself, a subgroup. Hence, we must only check normality.

Let $x \in G$, $y \in H$. By normality of each H_i, we have that $xyx^{-1} \in H_i$ for any i. As $y \in H$, meaning $y \in H_i$ $\forall i \in I$, it immediately follows that $xyx^{-1} \in H$, hence, $xy = yx$ for any $y \in H$. $\square$

Definition 1.22. Normalizer, Centralizer, and Center

Let S be a subset of G, and let $N = N_S$ such that:

$$N_S := \{x \in G \mid xSx^{-1} = S\}$$

N_S is called the **Normalizer of S in G**.

Let $S = \{a\}$, then N is called the **Centralizer of a in G**.

Denote:

$$Z_S := \{x \in G \mid xyx^{-1} = y \ \forall y \in S\}$$

This is called the **Centralizer of S in G**.

When we consider $S = G$, we called Z_G the **Center of the group G**.

Remark 1.5. Take care not to confuse the **Normalizer** and **Centralizer**. They seem to be the same at first glance, but they are not!

The normalizer N_S consists of elements $x \in G$ such that $xSx^{-1} = S$. This is a *set equality*, meaning that for some **not necessarily equal** $a, b \in S$, $xax^{-1} = b$. For the centralizer of S, Z_S, it is a special case of the normalizer, where $a = b$, so that $xax^{-1} = a$ for any a in S. Therefore, we see that $Z_S \subseteq N_S$, i.e. the Normalizer is more general than the Centralizer.

Example 1.9. Examples of Normal Subgroups

1. First consider the group of $n \times n$ matrices, $M_{n\times n}(k)$ with entries in a field k. We use the determinant homomorphism, $\det : M_{n\times n}(k) \longrightarrow k^*$ to find that $\ker(\det) = SL_n(k)$, the **Special Linear Group**.

2. For a more detailed example. Consider G as the set of all maps $T_{a,b} : \mathbb{R} \longrightarrow \mathbb{R}$ such that:
$$T_{a,b}(x) = ax + b \quad a \neq 0$$
G is a group under composition of maps. Now let A be the multiplicative group of maps $T_{a,0}$, $A \cong \mathbb{R}^*$. Let N be a group of translations $T_{1,b}$. Then clearly, for the homomorphism $T_{a,b} \longmapsto a$, $G \longrightarrow A$, $N = \ker(T_{a,b} \longmapsto a)$.

   We also have that $G = AN = NA$ and $N \cap A = \{id\}$. This is known as the **Semidirect Product of** A **and** N.

Proposition 14. *If K is any subgroup of G containing H such that H is normal in K, then $K \subset N_H$.*

Proposition 15. *If K is a subgroup of N_H, then KH is a group, and H is normal in KH. The normalizer of H is the largest subgroup of G in which H is normal.*

Remark 1.6. If G is a group and H a normal subgroup of G, then for x,$y \in G$, we denote the equivalence of x and y by:
$$x \equiv y \pmod{H}$$
We may equivalently state this as:
$$xy^{-1} \equiv e \pmod{H}$$
In our language, we read this as *x and y are congruent modulo H.*

Note that we may adapt this using additive notation when context is obvious.

1.3.1 Isomorphism Theorems

We can define normal subgroups through short exact sequences. We first construct the diagram for the first isomorphism theorem, and it will give us a lot of useful results through careful choices of homomorphisms.

Definition 1.23. Consider the following sequence of homomorphisms:
$$G' \xrightarrow{f} G \xrightarrow{g} G''$$
We say that this sequence is **exact** if $\text{Im}(f) = \text{Ker}(g)$.

Example 1.10. Any normal subgroup H of G is described by the exact sequence:
$$H \xrightarrow{j} G \xrightarrow{\varphi} G/H$$
This is because H must be the kernel of j, and we proved in **Proposition 10**, that the kernel is a normal subgroup of the domain of a homomorphism.

Definition 1.24. Short Exact Sequence

We have a special instance of an exact sequence known as the **short exact sequence**:

$$0 \longrightarrow G' \xrightarrow{f} G \xrightarrow{g} G'' \longrightarrow 0$$

Proposition 16. *In the **short exact sequence** above, the following are true:*

1. *f is injective*
2. *g is surjective*

Proof. The proof is quick and easy.

1. We recognize that $0 \longrightarrow G$ maps only to 0. Therefore, $\text{Im}(0 \longrightarrow G) = 0 = \text{Ker}(f)$.

   Hence, we see that f is injective.

2. Recognize that $\text{Ker}(G'' \longrightarrow 0) = G'' = \text{Im}(g)$.

   Therefore g is surjective.

□

Remark 1.7. The exact sequence

$$0 \longrightarrow H \xrightarrow{j} G \xrightarrow{\varphi} G/H \longrightarrow 0$$

gives us a canonical construction of quotient groups.

Example 1.11. Let G and G' be groups, and $f : G \longrightarrow G'$ a homomorphism whose kernel is G. Let $\varphi : G \longrightarrow G/H$ be the canonical surjection. There exists a unique homomorphism $f_* : G/H \longrightarrow G'$ such that:

$$f = f_* \circ \varphi$$

This f_* is injective.

To construct this f_*, we do the following:

Let xH be a coset of H. Since $f(xy) = f(x)$ for any $y \in H$, we see that:

$$f(xy) = f_*(\varphi(x)) = f(x)$$

We know that $\varphi(x) = xH$, hence we see that:

$$f_*(xH) = f(x)$$

Therefore, f_* is a homomorphism and it is well-defined. f_* is the injection induced by f.

We can illustrate this with the following diagram.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "\varphi"] & G' \\ & G/H \arrow[u, "f_*"] \end{tikzcd}$$

This leads us to the following theorem

Theorem 17. ***First Isomorphism Theorem***

Following the developments in ***Example 1.11****, if f is a surjective homomorphism, then the following hold true, and are equivalent.*

1. *f_* is surjective.*
2. *f_* is an isomorphism.*

More generally, f_* induces the following sequence:

$$G \xrightarrow{\varphi} G/H \xrightarrow{\lambda} \mathrm{Im}(f) \xrightarrow{i} G'$$

Where $f_* = i \circ \lambda$, i being the inclusion of $\mathrm{Im}(f)$ *into G'.*

Proof. Our developments in **Remark 1.7** and **Example 1.11** give us most of this proof. Just notice that if f is surjective, then since $f = f_* \circ \varphi$, and φ is surjective, this must mean that f_* is surjective as well. But f_* is an injective homomorphism, hence it is an isomorphism.

The more general case is trivial as inclusions are universal and f_* is injective, meaning that λ must be injective as well, but it must also be surjective by exactness of the induced sequence. □

Example 1.12. Let G be a group and H a subgroup of G. Let N be the intersection of ALL normal subgroups containing H. Then by **Proposition 13**, N is normal and the smallest normal subgroup of G containing H. Let $f : G \longrightarrow G'$ be a homomorphism such that $\mathrm{Ker}(f) \supset H$. We also see that $\mathrm{Ker}(f) \supset N$. Therefore we have the following order by inclusion: $H \subset N \subset \mathrm{Ker}(f)$. We assert that $\exists! f_* : G/N \longrightarrow G'$ such that the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "\varphi"] & G' \\ & G/N \arrow[u, "f_*"] \end{tikzcd}$$

f_* is defined exactly in **Example 1.11**.

Theorem 18. ***Third Isomorphism Theorem***[1]

Let G be a group and H, K two normal subgroups of G, where $H \supset K$. Then:

$$(G/K)/(H/K) \cong G/H$$

[1]We call the isomorphism theorems according to popular literature's reference to them. The name doesn't matter aside from being the standard.

Proof. Proof is quick and simple. Define

$$G/K \longrightarrow G/H$$

$$xK \longmapsto xH$$

It is very trivial that this is a surjective homomorphism (because composition of cosets involves the representatives) with a kernel consisting of any xK where $x \in H$, (this is H/K). The surjective part is trivial as $K \subset H$. Applying the **First Isomorphism Theorem** (**Theorem 17**) to the above map between G/K and G/H, we establish the desired correspondence. $\square$

Theorem 19. ***Second Isomorphism Theorem***

*Let G be a group, H, K be two subgroups of G. Assume that H is contained in the normalizer of K (look at **Definition 1.22**). We have the following correspondence:*

$$H/(H \cap K) \cong HK/K$$

Proof. Since $H \subset N_K$, we have that $xHx^{-1} = H$ for any $x \in N_K$. Hence, $H \cap K \triangleleft H$. We see that $HK = KH$ is a subgroup of G. We can define the surjective homomorphism:

$$H \longrightarrow HK/K$$

$$x \longmapsto xK$$

We only need to verify the kernel of this homomorphism is $x \in K$, but since $x \in H$ also, the kernel must be $H \cap K$. Using the **First Isomorphism Theorem** (**Theorem 17**), we can immediately establish our correspondence. $\square$

Proposition 20. *Let $f : G \longrightarrow G'$ be a homomorphism. Let H' be a normal subgroup of G'. Let $H = f^{-1}(H')$. Then H is normal in G.*

Proof. Let $x \in G$ and $h \in H$. Then $f(xhx^{-1}) = f(x)f(h)f(x)^{-1} \in H'$. Hence, $xhx^{-1} \in H \forall x \in G \forall h \in H$. Our conclusion follows. $\square$

Remark 1.8. Get the homomorphism by composing f with the canonical map φ.

$$G \xrightarrow{f} G' \xrightarrow{\varphi} G'/H'$$

$\mathrm{Ker}(\varphi \circ f) = H$ (by **Proposition 20**). The result is a homomorphism $\bar{f}$ according to the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi\circ f"] \arrow[dr, "\pi"] & G'/H' \\ & G/H \arrow[u, "\bar{f}"] \end{tikzcd}$$

Notice that this is simply the diagram in **Example 1.12** translated with our choice of $\varphi \circ f$.

As with before, if f is surjective, then $\varphi \circ f$ is surjective, hence $\bar{f}$ is an isomorphism.

Definition 1.25. The sequence of subgroups:

$$G = G_0 \supset G_1 \supset \cdots \supset G_m$$

is callled a **tower** of subgroups of G.

We call this tower **normal** if each $G_{i+1} \triangleleft G_i$ (G_{i+1} normal in G_i for each i).

We call this tower **abelian** if it is normal and each G_i/G_{i+1} is abelian.

We call this tower **cyclic** if it is normal and each G_i/G_{i+1} is cyclic.

Proposition 21. *Let $f : G \longrightarrow G'$ be a homomorphism and let*

$$G' = G'_0 \supset G'_1 \supset \cdots \supset G'_m$$

be a normal tower. Let $G_i = f^{-1}(G'_i)$. If the G'_i form an abelian (cyclic) tower, then the G_i form an abelian (cyclic) tower.

Proof. The fact that

$$G = G_0 \supset G_1 \supset \cdots \supset G_m$$

is a normal tower is trivial, using **Proposition 20**. It suffices to prove that each factor group G_i/G_{i+1} is abelian/cyclic if G'_i/G'_{i+1} is.

This follows easily because, by **Remark 1.8**, we have an injective map:

$$\bar{f}_i : G_i/G_{i+1} \longrightarrow G'_i/G'_{i+1}$$

for each i.

(This is because to prove that a factor group G_i/G_{i+1} is abelian/cyclic, it only requires that you prove that G_i is abelian/cyclic since our composition law only relies on the composition law of the representative in G_i).

Let x, $y \in G_i/G_{i+1}$. Then if G'_i/G'_{i+1} is abelian, we see that:

$$f_i(xy) = f_i(x)f(y) = f_i(y)f(x) = f_i(yx)$$

Therefore, as f_i is injective, we see that $xy = yx$.

Likewise, let $y = f_i(x)$ be a generator of G'_i/G'_{i+1}. $f_i(x)\cdots f_i(x) = f_i(x)^k = f_i(x^k)$ as f_i is a homomorphism. This means that if y is a generator of G'_i/G'_{i+1}, then x is a generator of G_i/G_{i+1}. □

Definition 1.26. A **refinement** of a tower

$$G = G_0 \supset G_1 \supset \cdots \supset G_m$$

is a tower that can be obtained by inserting a finite number of subgroups in the given tower.

A group is **solvable** if it has an abelian tower such that $G_m = \{e\}$.

Lemma 22. *Let G be a finite group. Then an abelian tower of G admits a cyclic refinement.*

Proof. Look up the **Fundamental Theorem of Finitely Generated Abelian Groups**. For an abelian tower of G:

$$G = G_0 \supset G_1 \supset \cdots \supset G_k$$

Then, for each G_i, let $G_i' = G_i/G_{i+1}$. As G_i' is a finite abelian group for any i, we see that by the **Fundamental Theorem of Finitely Generated Abelian Groups**, any finite abelian group is factorizable into a direct sum of cyclic groups of finite order. Thus, there exists a decomposition of G_i' such that they are elements in a cyclic tower of G_i'. □

Proposition 23. *Let G be a finite group. An abelian tower of G admits a cyclic refinement. Let G be a finite solvable group, then G admits a cyclic tower whose last element is $\{e\}$.*

Proof. The assumptions are justified due to the previous lemma. Take G be a finite solvable group.

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_m = \{e\}$$

Notice that there is a canonical surjection:

$$\pi : G_i \longrightarrow G_i/G_{i+1} = G_i' \quad \forall i$$

Notice that for each G_i', it admits a cyclic refinement as follows:

$$G_i' = G_i/G_{i+1} \supset A_1' \supset A_2' \supset \cdots \supset A_{k(i)}' \supset \{G_{i+1}\}$$

Then, taking the preimage of this cyclic tower under π, we obtain:

$$G_i \supset A_1 \supset A_2 \supset \cdots \supset A_{k(i)} \supset G_{i+1}$$

We now do this for every i. Then we see that by repeatedly applying the previous lemma, and building a cyclic refinement for every G_i', we can build a cyclic tower for G by taking the preimage at every node in the tower for G_i', i.e.

$$G' \supset \cdots \supset G_1' \supset \cdots \supset \cdots \supset G_{m-1}' \supset \cdots \supset G_m$$

Which, by taking the preimage results in:

$$G \supset \cdots \supset G_1 \supset \cdots \supset G_{m-1} \supset \cdots \supset G_m \supset \{e\}$$

As G is solvable, and we have built a cyclic tower, we may simply insert a trivial group into the end of the cyclic tower in the latter tower, hence we are done. □

Example 1.13. P-groups

For a group such that $|G| = p^n$ for a prime p, G is solvable.

Example 1.14. Feit-Thompson Theorem

All finite groups of odd order are solvable.

Theorem 24. *For G a group and $H \lhd G$, G is solvable if and only if H and G/H are solvable.*

Proof. **Note: We will do the complete proof here, as Lang leaves the rest of this proof as an exercise.**

Assume that G is solvable. Then G admits an abelian tower of the following form:

$$G = G_0 \supset G_1 \supset \cdots \supset G_m \supset \{e\}$$

Then we can take $H_i = H \cap G_i$. Then $H_{i+1} \lhd H_i$. We can map by inclusion:

$$H_i/H_i \longrightarrow G_i/G_{i+1}$$

By **Proposition 21**, we have that H_i/H_{i+1} is abelian, hence H must have an abelian tower, hence solvable.

Now assume that G is solvable. Then we can consider a tower for G/H by applying the canonical map $\pi_i : G_i \longrightarrow G_i/H_i$ to every node in the tower for G resulting in the following tower:

$$G/H = G_0/H_0 \supset G_1/H_1 \supset \cdots \supset G_m/H_m \supset \{e\}$$

Note, by **Theorem 19** (**Second Isomorphism Theorem**), we may express this tower as:

$$G/H \supset G_1H/H \supset G_2H/H \supset \cdots \supset G_mH/H \supset \{e\}$$

Now by **Theorem 18** (**Third Isomorphism Theorem**), we obtain that:

$$(G_iH/H)/(G_{i+1}H/H) \cong G_iH/G_{i+1}H$$

Then we claim that the map

$$h : G_iH/G_{i+1}H \longrightarrow G_i/G_{i+1}$$

is actually an isomorphism. This is more easily seen if we claim that:

$$G_iH/G_{i+1}H \cong \pi(G_i)/\pi(G_{i+1}) \cong G_i/G_{i+1}$$

The last equivalence is achieved through **Theorem 19** (**Third Isomorphism Theorem**). To prove the former equivalence, we use **Theorem 17** (**First Isomorphism Theorem**) on the homomorphism[2]:

$$\psi : G_iH \longrightarrow \pi(G_i)/\pi(G_{i+1})$$

[2] I obtained this exact homomorphism through an exercise in Fraleigh's *A First Course in Abstract Algebra*, **Section 36**, page 321, exercise 28.

$$\psi : g_i h \longmapsto \pi(g_i h)\pi(G_{i+1})$$

Therefore, $\mathrm{Ker}(\psi) = G_{i+1}H$.

Now we find that $G_iH/G_{i+1}H \cong \pi(G_i)/\pi(G_{i+1})$, giving us that h is an isomorphism. We are done once we use that G_i/G_{i+1} is abelian, as our string of isomorphisms directly imply that $G_i/H_i \cong G_i/G_{i+1}$.

The opposite implications are trivial given all of our computation for the case of the solvability of both H and G/H. $\square$

Definition 1.27. For a group G, a **commutator** in G is an element of the form $xyx^{-1}y^{-1}$ for any x, $y \in G$. Denote the subgroup of G generated by such elements as G^c. We call this the **commutator subgroup** of G.

Proposition 25. *The following are true for any group G:*

1. *The commutator subgroup of G, G^c is normal in G.*

2. *Every homomorphism $f : G \longrightarrow G'$ (where G' is an abelian group) is such that* $\mathrm{Ker}(f) \supset G^c$.

Proof. This is a very important theorem, let us be careful to understand it.

1. Let G be a group with elements x, y. We first prove that the set of all elements generated by $xyx^{-1}y^{-1}$ is a subgroup of G.

   Clearly, $e = xxx^{-1}x^{-1}$. Clearly, $\forall xyx^{-1}y^{-1} \in G^c$, $yxy^{-1}x^{-1}$ serves as an inverse element and is contained in G^c. Closure is trivial given the composition law inherited from G. Let $a = x_1x_2\cdots x_k$, $b = y_1y_2\cdots y_m$. Where each x_i and y_j is a commutator generated by x and y. Then, clearly, under the binary operation of G, we have that:

   $$ab = x_1x_2\cdots x_ky_1y_2\cdots y_m$$

   The normality is not difficult. Let $x \in G$ and let $u \in G^c$: $xux^{-1} = xux^{-1}u^{-1}u = (xux^{-1}u^{-1})u$. Since this is a subgroup, the composition of these two elements is contained in G^c.

2. Let $f : G \longrightarrow G'$ be a homomorphism with its codomain in an abelian group. Then that means that for any $g, h \in G$, $f(gh) = f(g)f(h) = f(h)f(g) = f(hg)$. Now, as G' is a group, we can manipulate this equality in the following way:

   $$f(g)f(h) = f(h)f(g) \Longrightarrow f(g)f(h)f(g)^{-1}f(g)^{-1} = e$$

   The second equality tells us that:

   $$f(ghg^{-1}h^{-1}) = e$$

   meaning that the commutator subgroup generated by fixed g and h, G^c, is contained in the kernel of f.

Note that implication 1 and 2 gives us that the quotient, G/G^c, is, in fact, a group. $\square$

Remark 1.9. The group, G/G^c, is abelian. Using the canonical map $\pi : G \longrightarrow G/G^c$, we see that for any $g, h \in G$, that $\pi(ghg^{-1}h^{-1}) = e$, therefore, for $\bar{x} = \pi(x)$ and $\bar{y} = \pi(y)$, we see that $\bar{x}\bar{y} = \bar{y}\bar{x}$, hence all elements of G/G^c commute.

This corresponds with something called **abelianization**, and this is actually a functor from the category of groups to the category of abelian groups. The functoriality is established through the universal property of abelianization, which is derived from the universal property of quotient groups (we will recast most of what we know into category theoretic language later).

Definition 1.28. A group G is **simple** if it is nontrivial and has no normal subgroups aside from $\{e\}$ and G itself.

Remark 1.10. This tells us that, in principle, we can establish the simplicity of groups based on if the we can find nontrivial commutators of that group.

Example 1.15. An abelian group is simple if and only if it is of prime order.

Proof. Assume that G is finite-dimensional and simple, but assume that G does not have prime order. Then $|G| = m = rn$ and we have that $g^m = e$, meaning that $g^r \neq e$. This means that g^r generates a unique cyclic (hence normal) subgroup of G, however, this is a contradiction to the simplicity of G.

Now assume that G is infinite-dimensional cyclic. Then take any integer $m > 1$. Recognize that g^m will generate a unique (cyclic) subgroup of G, hence contradiction to the simplcity of G. Therefore, only cyclic groups of prime order will be simple. $\square$

Lemma 26. ***Butterfly (Zassenhaus) Lemma***

Let U, V be subgroups of some group G. Let $u \triangleleft U$ and $v \triangleleft V$. Then:

1. $u(U \cap v) \triangleleft u(U \cap V)$
2. $(u \cap V)v \triangleleft (U \cap V)v$
3. *The quotient groups are isomorphic:*

$$u(U \cap V)/u(U \cap v) \cong (U \cap V)v/(u \cap V)v$$

Proof. Proof omitted. We will return to this at a later time, the technicalities are not extremely necessary and are simply an application of the isomorphism theorems. $\square$

Definition 1.29. Let G be a group and let:

$$G = G_1 \supset G_2 \supset \cdots \supset G_r = \{e\}$$

$$G = H_1 \supset H_2 \supset \cdots \supset H_s = \{e\}$$

be normal towers of subgroups that terminate at a trivial group. These towers are **equivalent** if $r = s$ and if there exists a permutation of the indices $i = 1, \ldots, r-1$ written $i \mapsto i'$ such that:

$$G_i/G_{i+1} \cong H_{i'}/H_{i'+1}$$

Theorem 27. ***Schreier Theorem***

Let G be a group. Two normal towers of subsgroups ending with the trivial group have equivalent refinements.

Proof. This proof is constructive.

Use the towers in **Definition 1.29**. Define $G_{ij} = G_{i+1}(H_j \cap G_i)$. Note that $G_{is} = G_{i+1}$. Then, we obtain the following refinement for $i = 1, \ldots, r-1$ and $j = 1, \ldots, s$:

$$G = G_{11} \supset G_{12} \supset \cdots \supset G_{1,s-1} \supset G_2 = G_{21} \supset \cdots \supset G_{r-1,s-1} \supset \{e\}$$

Define something similar for the tower in H_j: $H_{ji} = H_{j+1}(G_i \cap H_j)$ for the indices $j = 1, \ldots, s-1$ and $i = 1, \ldots, r$:

$$H = H_{11} \supset H_{21} \supset \cdots \supset H_{r-1,1} \supset H_2 = H_{12} \supset \cdots \supset H_{r-1,s-1} \supset \{e\}$$

Lemma 26 gives us the necessary isomorphism:

$$G_{ij}/G_{i,j+1} \cong H_{ji}/H_{j,i+1}$$

This is sufficient to prove the equivalence of the two refined towers. □

Theorem 28. ***Jordan-Holder Theorem***

Let G be a group, and let G have the following normal tower:

$$G = G_1 \supset G_2 \supset \cdots \supset G_r = \{e\}$$

Where each G_i/G_{i+1} is simple, and $G_i \neq G_{i+1}$ for every $i = 1, \ldots, r-1$. This is unique tower of G, i.e. any other tower of G having these properties is equivalent to this one.

Proof. Let $\{G_{ij}\}$ form a refinement for the given tower of G. By the Schreier Theorem, we have that for every i, there exists only one j such that $G_i/G_{i+1} = G_{ij}/G_{i,j+1}$. So the refined tower and the original tower is equivalent. □

1.4 Cyclic Groups

Example 1.16. $\mathbb{Z}$ is a cyclic group.

Proof. Pick ± 1. Then ± 1 generates every integer through addition. Hence, $\mathbb{Z}$ is cyclic. □

Proposition 29. *Any subgroup of $\mathbb{Z}$ (under addition) is of the form $n\mathbb{Z}$.*

Proof. Let H be any subgroup of $\mathbb{Z}$. If H is nontrivial, let a be the smallest positive integer in H. Let $y \in H$, then $\exists n, r > 0$ with $0 \leq r < a$ such that:

$$y = na + r$$

Because H is a subgroup and $r = y - na$, $r \in H$, meaning that $r = a$ or $r = 0$, but $r < a$ by assumption, hence $r = 0$. And every $y \in H$ is of the form $y = na$. $\square$

Definition 1.30. G is a cyclic group if $\exists a \in G : \forall x \in G\ x = a^n$ for some $n \in \mathbb{Z}$.

We can also reformulate this definition as saying that for some $a \in G$, the map:

$$f : \mathbb{Z} \longrightarrow G$$

$$n \longmapsto a^n$$

is surjective.

This a is called a **generator** of G.

Definition 1.31. Let G be a group, and $a \in G$. The subset of all elements a^n for $n \in \mathbb{Z}$ is called a **cyclic subgroup** of G. If $m \in \mathbb{Z}$ such that $a^m = e$, then we call m an **exponent** of a. We say that $m > 0$ is an exponent of G if $x^m = e\ \forall x \in G$.

Remark 1.11. Let G be a group and fix $a \in G$. Let $f : \mathbb{Z} \longrightarrow G$ be a homomorphism:

$$f : n \longmapsto a^n$$

let $H := \mathrm{Ker}(f)$. We have two distinct cases:

1. If H is trivial, then f is an isomorphism of $\mathbb{Z}$ onto the cyclic subgroup of G generated by a. This subgroup is infinite cyclic. In the case that a generates G, G itself is cyclic. a has **infinite period**.

2. If H is nontrivial. Let d be the smallest positive integer in the kernel. d is called the period of a. If m is an integer such that $a^m = e$ then $m = ds$ for some integer s. The set of all elements:

   $$\{e, a, \dots, a^{d-1}\}$$

   is distinct. If $a^r = a^s$ for $0 \leq r, s \leq d - 1$, and without loss of generality $r \leq s$, then $a^{s-r} = e$. As $0 \leq s - r \leq d$, $s - r = 0$. The cyclic subgroup generated by a has order d.

Proposition 30. *Let G be a finite group of order $n > 1$. Let a be an element of G, $a \neq e$. Then the period of a divides n. If the order of G is a prime, then G is cyclic and the period of any generator is prime.*

Proposition 31. *Let G be a cyclic group. Then every subgroup of G is cyclic. If g is a homomorphism of G, then $f(G)$ is cyclic.*

Proof. **Proof of Proposition 30**

The assertion about the prime order group follows from the first statement about the divisibility of the period of a and the order of G. Let G be a nontrivial finite group of order n. Let a be a nontrivial element of G with period m.

Let H be the cyclic subgroup generated by a, by assumption, $|H| = m$. Then, using **Proposition 9**, we obtain that $(G : H)(H : 1) = (G : 1)$. Then, this means that:

$$\frac{n}{m}m = n$$

Hence, m must divide n.

Using this implication, let $|G| = p$. Then we see that for any $x \neq e \in G$, that the subgroup, H, generated by x must divide the order of G. However, as the order of G is prime, H must either be order 1 or order p. Only the latter holds as $x \neq e$. So any G of prime order is cyclic and any nontrivial element has prime period. □

Proof. **Proof of Proposition 31**

The latter proposition is actually more fundamental. Let f be any homomorphism, and G be a cyclic group. Take a generator $a \in G$ with a period n. As f is a homomorphism, $f(a^n) = f(e) = e' = f(a)^n$. Therefore, the image of G is obviously a cyclic group itself.

Now we can take the homomorphism:

$$f(n) = a^n$$

for a generator a of G. As $f^{-1}(H)$ is a subgroup of $\mathbb{Z}$, it is of the form $m\mathbb{Z}$ for some m. Then, as f is surjective, we have that:

$$f(m\mathbb{Z}) = H$$

By the previous implication, a cyclic group's image under a homomorphism is itself cyclic. Hence, our desired conclusions follow. □

Proposition 32. ***A Bunch of Facts About Cyclic Groups***

1. *An infinite cyclic group has exactly two generators (if a is a generator, then a^{-1} is the only other generator).*

2. *Let G be a finite cyclic group of order n, and let x be a generator. The set of generators of G consists of those powers x^ν of x such that ν is relatively prime to n.*

3. *Let G be a cyclic group, and let a, b be two generators. Then there exists an automorphism of G mapping a onto b. Conversely, any automorphism of G maps a on some generator of G.*

4. *Let G be a cyclic group of order n. Let d be a positive integer dividing n. Then there exists a unique subgroup of G of order d.*

5. *Let G_1, G_2 be cyclic of orders m and n, respectively. If m and n are relatively prime, then $G_1 \times G_2$ is cyclic.*

6. *Let G be a finite abelian group. If G is not cyclic, then there exists a prime p and a subgroup of G isomorphic to $C \times C$, where C is cyclic of order p.*

Lemma 33. *In a finite cyclic group, G, of order n generated by g. Then the order of an element (i.e. the order of the cyclic subgroup generated by) g^m is given by:*

$$|g^m| = \frac{|G|}{\gcd(|G|, m)}$$

Proof. Let $k = \gcd(n, m)$. Then, we let $m = ks$, and $n = kt$. Then we have:

$$(g^m)^{n/k} = g^{mn/k} = g^{ksn/k} = g^{sn} = (g^n)^s = e$$

Therefore, n/k is a period of the element g^m. It is easily provable that this is the smallest such period. □

Proof. **Proof of Proposition 32**

There is a lot to prove here! We will make it short and quick.

1. Let G be an infinite cyclic group. We show that G has at least two generators. Fix a as one generator of G so that a^k is any element of G for any $k \in \mathbb{Z}$. Likewise, we see that a^{-1} also generates G because we can take $(a^{-1})^{-k}$ as $-k \in \mathbb{Z}$.

   We now show that G cannot have more than two generators. Suppose b was another generator for G. Then we see that there exists $m, n \in \mathbb{Z}$ such that $a = b^m$ and $b = a^n$. This further implies that $a = b^m = a^{nm}$. We solve the equation $nm = 1$ in the integers, which only has the solutions $(n, m) = (\pm 1, \pm 1)$. Therefore, the only generators of G can be $b = a$ and $b = a^{-1}$.

2. G is a finite cyclic group of order n with a generator x.

   We use the fact that the order of the element x^ν is given by $|x^\nu| = n/\gcd(\nu, n)$ (this is **Lemma 33**). Using this fact, we see that $|g^\nu| = n$ if and only if $n/\gcd(\nu, n) = n$, which is true if and only if $\gcd(\nu, n) = 1$.

3. Let G be a cyclic group (note not necessarily finite!!!) with two generators a and b. Let $f(a) = b$. Then $f(a^k) = f(a)^k = b^k$. Trivially, this endomorphism is surjective. If we let $a^k \in \mathrm{Ker}(f)$, then $f(a^k) = b^k = e$. b has

order k, hence a must have order k. And this shows us that $\text{Ker}(f) = \{e\}$, hence, f is both injective and surjective. That it is a homomorphism is easily seen by our definition.

Conversely, assume that we have an automorphism $f : G \longrightarrow G$. There is a unique $x \in G$ so that $f(x) = a$. For any $y \in G$, we have that $y = f(x)^k = f(x^k)$. Likewise, take the inverse map $f^{-1}(y) = x^k$. So for any $g \in G$, we have that $f^{-1}(y) = g = x^k$, and x is a generator of G as well.

4. We will leave the next three blank because Lang proves it himself.

5.

6.

$\square$

1.5 Group Actions

Definition 1.32. Definition of Group Action 1

Let G be a group and S a set. A **group action** of G on S is a homomorphism:

$$\pi : G \longrightarrow \text{Perm}(S)$$

This S is called a **G-set**. The permutation associated with the element $x \in G$ is called π_x. Therefore:

$$\pi : x \longmapsto \pi_x$$

For any $s \in S$, the image of s under the permutation associated with $x \in G$ is $\pi_x(s)$. This operation gives us a mapping:

$$G \times S \longrightarrow S$$

$$(x, s) \longmapsto \pi_x(s)$$

We denote $\pi_x(s) \equiv xs$.

Definition 1.33. Definition of Group Action 2

For any $x, y \in G$, and $s \in S$, we have that $\pi_x(\pi_y(s)) = \pi_{xy}(s)$. If e is the unit element of G, then $\pi_e(s) = s$ for any $s \in S$.

Example 1.17. Examples of Group Actions

The following are examples of **groups of permutations of** S

1. For each $x \in G$, we denote by $c_x : G \longrightarrow G$ the map:

$$c_x(y) = xyx^{-1}$$

The map $x \longmapsto c_x$ is a homomorphism $G \longrightarrow \text{Aut}(G)$.

The kernel of this map is the set of all $x \in G$ so that $xyx^{-1} = y$. This is called the **center** of G (see **Definition 1.22**). The group action c_x is called **conjugation**, and automorphisms of the form c_x are called **inner automorphisms**.

We find it worthwhile to bring up the special case where S is the **set of subsets of** G. Let $A \in S$, then xAx^{-1} is a subset of G, and is the image of the permutation c_x. The map:

$$(x, A) \longmapsto xAx^{-1}$$

is a group action of G on S. This is easily seen since $c_x(c_y(A)) = xyA(xy)^{-1}$ and it is equivalent to c_{xy}. And likewise, $\pi_e(A) = A$.

Two subsets are **conjugate** if there exists $x \in G$ such that $B = c_x(A)$.

2. For each $x \in G$, define the translation $T_x : G \longrightarrow G$ by $T_x(y) = xy$. The map:

$$(x, y) \longmapsto xy = T_x(y)$$

is a group action of G onto itself.

As with conjugation, we can perform a group action of G on the set of subsets of G, S. Let A be an element of S (i.e. a subset of G), then we have that $T_x(A) = xA$. But note that $T_x(H) = xH$, is not a subgroup, but actually a coset of H in G. Therefore, the translation group action of G onto the set of subgroups of G is actually a translation on the set of cosets of H. H is not necessarily normal!

Group actions are the basis of group theory, and they will appear everywhere! Be mindful of the above two examples.

Remark 1.12. Let S, S' be two G-sets. Let $f : S \longrightarrow S'$ be a map. f is a **morphism of G-sets**, or a G-map if $f(xs) = xf(s)$ for any $x \in G$ and $s \in S$. In this way, G-sets form a **category of G-sets**.

Definition 1.34. Let G be a group acting on a set S. Fix $s \in S$. The set of elements $x \in G$ such that $\pi_x(s) = s$ is called the **isotropy subgroup of** s **in** G. i.e.

$$G_s = \{x \in G \mid \pi_x(s) = s\}$$

The isotropy subgroup of s in G is denoted G_s.

Remark 1.13. If the group action of G on a G-set S is conjugation, then for $s \in S$, $G_s := N_s = \{x \in G \mid xsx^{-1} = s\}$, i.e. the isotropy group is the normalizer of s. Likewise when the G-set is a subgroup of G.

Remark 1.14. Let G act on a set S. Let s, s' be elements of S, and y an element of G so that $\pi_y(s) = s'$. We see that if π_y is conjugation, then:

$$G_{s'} = yG_sy^{-1}$$

We see that if

$$\pi_{x'}(s') = s'$$

then, $\pi_{x'}\pi_y(s) = \pi_y(s)$. The isotropy groups of s and s' are conjugate.

Definition 1.35. Let K be a kernel of the representation $G \longrightarrow \text{Perm}(S)$, then, the kernel is going to be any element $x \in G$ such that $\pi_x(s) = s$, or $\pi_x \equiv id$. Then we obtain:

$$K = \bigcap_{s \in S} G_s$$

Definition 1.36. A group action of G on S is said to be **faithful** if $K = \{e\}$,i.e. if the homomorphism $G \longrightarrow \text{Perm}(S)$ is injective.

A **fixed point** of G is an element $s \in S$ such that:

$$\pi_x(s) = s \quad \forall x \in G$$

An equivalent way to state that $s \in S$ is a fixed point of G is that the isotropy subgroup of s, G_s is the whole group G, i.e. $G_s = G$.

The subset of S consisting of all $\pi_x(s)$, denoted by Gs, is called the **orbit of s under G**. For convenience, we will adopt the common notation of $\text{Orb}(s)$ to denote the orbit[3] of $s \in S$.

Remark 1.15. We can construct a map:

$$f : G/H \longrightarrow S$$

$$xH \longmapsto \pi_x(s)$$

This map induces a bijection of G/H onto the orbit Gs. Therefore, we see that if $x, y \in H$, then $f(xH) = \pi_x(s) = \pi_y(s) = f(yH)$.

Proposition 34. *If G is a group acting on a set S, then $|Gs| = (G : G_s)$.*

Proof. Fix $s \in S$. The orbit of s under G is:

$$Gs = \{t \in S \mid t = \pi_x(s)\}$$

The index $(G : G_s)$ is the order of the group:

$$G/G_s = \{xG_s \mid x \in G\}$$

By **Remark 1.15**, we see that $f : G/G_s \longrightarrow S$ gives a bijection of G/G_s onto the orbit Gs. $\square$

Proposition 35. *The number of conjugate subgroups to H is equal to the index of the normalizer of H.*

[3] I do this purely because the notation Gs becomes confusing with right cosets when we start talking about the set S as being subgroups of G.

Proof. Let G act on a subgroup H. Then by **Remark 1.13**, $G_h = N_h$, i.e. the isotropy subgroup is exactly the normalizer of $h \in H$. By **Proposition 34**, the number of conjugate subgroups to H, i.e. the number of distinct orbits of $h \in H$, is equal to the index of G/G_h, but $G_h = N_h$. Hence, we obtain that:

$$(G : N_h) = |Gh|$$

$\square$

Example 1.18. Let G be a group, if H is a subgroup of index 2, then $H \triangleleft G$.

Proof. **Proof of Example 1.18**

Let G be a group with a subgroup of index 2 by assumption. We know, by **Proposition 35**, that the number of conjugate subgroups of H is equal to the index of the normalizer of H:

$$(G : N_H) = |\mathrm{Orb}(H)|$$

Where $\mathrm{Orb}(H) = \{xHx^{-1} \mid x \in G\}$. Note that $H \in \mathrm{Orb}(H)$, therefore, as $(G : H) = 2$, we see that $|\mathrm{Orb}(H)| \leq (G : H)$ meaning that $(G : N_H) = 1$ or 2. Then we consider the case when $(G : N_H) = 2$.

That means that for some $x \in G$, $\mathrm{Orb}(H) = \{H, xHx^{-1}\}$ according to the bijection in **Remark 1.15**: $xN_H \longmapsto xHx^{-1}$.

Recall that (by **Proposition 9**):

$$(G : H)(H : 1) = (G : 1)$$

We also have the following sequence:

$$H \hookrightarrow \mathrm{Ker}(\pi) \longrightarrow G \xrightarrow{\pi} \mathrm{Perm}(\mathrm{Orb}(H))$$

The action, π, is nontrivial, so $\mathrm{Ker}(\pi) \subset G$, and $(G : \mathrm{Ker}(\pi)) = 2$. Therefore $(G : H) = (G : \mathrm{Ker}(\pi))(\mathrm{Ker}(\pi) : H) = 2$. This means that $1 < (G : \mathrm{Ker}(\pi)) \leq 2$, consequently, $(G : \mathrm{Ker}(\pi)) = 2$, and this implies that $(\mathrm{Ker}(\pi) : H) = 1$. Hence, $\mathrm{Ker}(\pi) = H$. This is clearly a contradiction as $H \subset \mathrm{Ker}(\pi)$ but $\mathrm{Ker}(\pi) \not\subset H$.

Now, we must have that $|\mathrm{Orb}(H)| = 1$, which implies that $xHx^{-1} = H \; \forall x \in G$, so that H is clearly normal in G. $\square$

Definition 1.37. A group action of G on S is said to be **transitive** if there is only one orbit.

Remark 1.16. We can restate **Example 1.18** by saying that conjugation on an index 2 subgroup, H, of G by G is transitive.

Example 1.19. Let $S = \{1, \dots, n\}$, and $S_n = \mathrm{Perm}(S)$. Then S_n acts transitively on S.

Remark 1.17. Any two orbits of G on S are always disjoint or equal.

Proof. Let $\mathrm{Orb}(s_1)$ and $\mathrm{Orb}(s_2)$ be two orbits with an element s in common. Then we see that there exists some $g \in G$ so that:

$$\pi_g(s_1) = s \in \mathrm{Orb}(s_1)$$

Likewise for s_2. Therefore, $\mathrm{Orb}(s_1) = \mathrm{Orb}(s) = \mathrm{Orb}(s_2)$. So two orbits are always disjoint or equal. $\square$

Remark 1.18. Any G-set S is a disjoint union of distinct orbits of G. We denote it as $S = \coprod_{i \in I} \mathrm{Orb(s_i)}$, where s_i are elements in distinct orbits.

Proposition 36. ***Orbit Decomposition Formula***

When S is finite, we obtain:

$$|S| = \sum_{i \in I} (G : G_{s_i})$$

Proof. Using **Remark 1.18**, this is obvious as S has $\mathrm{Orb}(s_i) + |S| - \mathrm{Orb}(s_i)$ elements. $\square$

Remark 1.19. For G a group, $Z(G) := \{x \in G \mid xyx^{-1} = y \forall y \in G\}$ is the center of G. For G acting on itself, $y \in Z(G)$ if and only if $\mathrm{Orb}(y) = \{y\}$. Therefore, if G is a finite group, we see that it admits a decomposition into $Z(G)$ and the other orbits of G. Therefore,

$$|G| - |Z(G)| = \sum_{x \notin Z(G)} \mathrm{Orb}(x)$$

Proposition 37. ***Class Formula***

A ***conjugacy class*** *is just another name for* ***orbits of*** G ***under the group action of*** G ***onto*** G ***by conjugation****.*

Let C be a set of representatives of distinct conjugacy classes, then the extension of the result in ***Remark 1.19*** *is given, for arbitrary order groups, by:*

$$(G : 1) = \sum_{x \in C} (G : G_x)$$

Proof. The proof is simply a restatement of **Remark 1.18**, as the indexing set I is not necessarily finite. $\square$

The Symmetric Group is so ubiquitous that we give it its own treatment.

Definition 1.38. Let $S = \{1, \ldots, n\}$ be a G-set. Let G be $\mathrm{Perm}(S)$. Then we call G the **Symmetric Group** on n letters, it is denoted as S_n.

We see here that if $G := S_n$, then the homomorphism (by **Definition 1.32**):

$$\pi : G \longrightarrow \mathrm{Perm}(S)$$

$$\pi : x \longmapsto \pi_x$$

is defined by $\pi_x s = xs$, where we regard $x \in \text{Hom}(S, S)$, i.e. a bijection of S to itself.

Remark 1.20. For S_n, let $\sigma \in S_n$, and let $1 \leq i \leq n$. The orbit of i under the cyclic group generated by σ is called a cycle for σ. We write:

$$[i_1 i_2 \dots i_r] \quad \sigma(i_k) = i_{k+1} \quad \sigma(i_r) = i_1$$

We can then decompose $\{1, \dots, n\}$ into a disjoint union of orbits generated by σ.

Example 1.20. The cycle [132] is a permutation σ such that:

$$\sigma(1) = 3 \quad \sigma(3) = 2 \quad \sigma(2) = 1$$

We call $\{1, 3, 2\}$ the orbit of 1 under the cyclic group generated by σ.

Note: We will skip the sections on the symmetric group and come back at a later time. There are more computational references on the symmetric group in other resources[4].

1.6 Sylow Subgroups

Let p be a prime number.

Definition 1.39. A **p-group** is a finite group whose order is prime power, i.e. $|G| = p^k$ $k \geq 0$.

For G a group, H a subgroup, H is a p-subgroup if H is of prime power order.

H is a **Sylow p-subgroup** if the order of H is p^k and p^k is the highest power of p that divides the order of G.

Lemma 38. *Let G be a finite abelian group of order m. Let p be a prime dividing m. The G has a subgroup of order p.*

Proof. Assume G has exponent n so that it divides any power of n. Let $b \in G$ be a nontrivial element. Let H be a cyclic subgroup generated by b. $|H|$ must divide n as $b^n = e$ since n is an exponent for G/H. Therefore as,

$$(G : 1) = (G : H)(H : 1)$$

If G has order divisible by p, by what we just saw above, there exists $x \in G$ with a period divisible by p, and H is a cyclic subgroup generated by x. Then $x^{pk} = e$, but $x^k \neq e$, so x^k has period p. Then x^k generates a subgroup of order p. $\square$

[4] Look into *Fraleigh* and *Dummit and Foote* for more indepth calculations into permutations.

Theorem 39. *Let G be a finite group and p a prime dividing the order of G. There exists a Sylow p-subgroup of G.*

Note: We will treat the Sylow Theorems later after we finish this section on groups. This is purely out of desire to see something new on my part.

1.7 Direct Sums and Free Abelian Groups

Definition 1.40. Let $\{A_i\}_{i\in I}$ be a family of abelian groups. The **Direct Sum** of this family is denoted as:

$$A = \bigoplus_{i\in I} A_i$$

and is a subset of the direct product $\prod A_i$ consisting of all families $(x_i)_{i\in I}$ with $x_i \in A_i$ such that $x_i = 0$ for all but a finite number of indices i. In other literature, they say that $A_i = 0$ with the exception of a finite number of indices i. These are equivalent.

For every $j \in I$, we have the inclusion map

$$\lambda_j : A_j \longrightarrow A$$

$$\lambda_j : x \longmapsto X \quad X = (0, \ldots, x_j, \ldots, 0)\ x_j = x$$

Or equivalently, we can say:

$$\lambda_j : x_j \longmapsto (x_j)_{j\in I}$$

Where $x_j = 0$ with the exception of finitely many j. λ_j is an injective homomorphism.

We give the following proposition its own name, and we give it its own name. It falls into a special class of abstract objects in algebra.

Proposition 40. ***Universal Property of the Direct Sum***

Let $\{f_i : A_i \longrightarrow B\}$ be a family of homomorphisms into an abelian group B. Let $A = \oplus A_i$. There exists a unique homomorphism:

$$f : A \longrightarrow B$$

such that $f \circ \lambda_j = f_j \quad \forall j$.

Proof. Consider $f_j : A_j \longrightarrow B$. Considering that A comes equipped with a family of inclusions $\{\lambda_j : A_j \longrightarrow A\}$, we can draw the following diagram.

$$\begin{tikzcd} A_j \arrow[r, "\lambda_j"] \arrow[dr, "f_j"'] & \bigoplus_{j\in I} A_j \arrow[d, "f"] \\ & B \end{tikzcd}$$

We simply define a map that fits this commutative diagram and performs the role of the f_j. Define f by:

$$f\left((x)_{j\in I}\right) = \sum_{j\in I} f_j(x_j)$$

Then, as $x_j \in A$, we see that all $x_j = 0$ except for finitely many j, therefore, this sum is well-defined. Additionally, since $(x_j)_{j\in I} = \lambda_j(x_j)$ for arbitrary $x_j \in A_j$, we see that this choice of f is, indeed, the unique choice of f that makes this diagram commute, i.e. $f_j(x_j) = f(\lambda_j(x_j))$. $\square$

Remark 1.21. It must be proven, but things with the universal property are called **functors** or **functorial**. They are essentially homomorphisms of categories. In this case, the direct sum is actually a functor from the category of abelian groups to itself.

Example 1.21. Direct Sum vs. Direct Product for Abelian Groups

For an abelian group A, and a family $\{A_i\}_{i\in I}$ of subgroups, we have a homomorphism:

$$\bigoplus_{i\in I} A_i \longrightarrow A$$
$$(x_i) \longmapsto \sum x_i$$

Let A be an abelian group and B, C subgroups of A. If $B + C = A$ and $B \cap C = \{0\}$, the map:

$$B \times C \longrightarrow A$$
$$(x, y) \longmapsto x + y$$

is an isomorphism. Naively, we can say that $A \cong B \times C$, but in this special case, we denote the **direct sum of abelian groups**, $A = B \oplus C$, and because the two coincide, $A \cong B \times C$.

More generally, if we have a finite number of subgroups of A, $\{B_i\}_{i\in K}$ such that:

$$B_1 + \cdots + B_n = A$$

and

$$B_{i+1} \cap (B_1 + \cdots + B_i) = 0 \quad \forall i$$

Then the direct sum of abelian groups is:

$$A = B_1 \oplus \cdots \oplus B_n$$

Definition 1.41. For the abelian group $A = B_1 \oplus \cdots \oplus B_n$, we let $\{e_i\}_{i\in K}$ be a family of elements of A. This family is a **basis** for A if the family is (i) nonempty (ii) if every element of A has a unique expression as a linear combination, i.e.

$$x = \sum x_i e_i \quad \forall x \in A$$

with $x_i \in \mathbb{Z}$ and $x_i = 0$ for all but finitely many $i \in K$.

An abelian group equipped with such a family (i.e. a basis) is called **free**.

Remark 1.22. If we let $Z_i = \mathbb{Z} \quad \forall i$, we see that **if A is a free abelian group**, then is isomorphic to the direct sum:

$$A \cong \bigoplus_{i \in K} Z_i$$

Note: When we talk about modules over a ring, we will know more about this, but all free abelian groups are called $\mathbb{Z}$-modules, meaning that free abelian groups are modules over the integers.

Proof. **Proof of Remark 1.22**

A is free abelian, thus it is endowed with a family of elements $(e_i)_{i \in K}$ such that $\forall x \in A$:

$$x = \sum_{i \in K} x_i e_i \quad x_i \in Z_i := \mathbb{Z}$$

We simply need to verify that the direct sum above has the same universal property as described in **Proposition 40**. This is trivial by seeing that the inclusion of Z_i into $\bigoplus Z_i$ defines any element of the form:

$$x = \sum_{i \in K} x_i e_i$$

In fact, any such x can be represented as a tuple (look at **Definition 1.40**):

$$x := (x_1, \dots, x_n)$$

Therefore, the basis elements e_i are *essentially* the inclusion maps λ_i. □

Remark 1.23. Let S be a set. We construct the free abelian group generated by S as follows.

Let $\mathbb{Z}\langle S \rangle$ be the set of all maps $\varphi : S \to \mathbb{Z}$ such that $\varphi(x) = 0$ for all but finitely many $x \in S$. By **Definition 1.8**, if the codomain of maps from $S \to G$ is abelian, then $Hom(S, G)$ must be abelian. Therefore, $\mathbb{Z}\langle S \rangle$ is abelian. If k is an integer and $x \in S$, we denote by $k \cdot x$ the map φ so that $\varphi(x) = k$ and $\varphi(y) = 0$ if $y \neq x$.

Every element φ of $\mathbb{Z}\langle S \rangle$ can be written in the form:

$$\varphi = k_1 \cdot x_1 + \cdots + k_n \cdot x_n$$

for integers k_i and elements $x_i \in S$, with all the x_i distinct. It is trivial to verify that this expression of φ is unique.

Map S into $\mathbb{Z}\langle S \rangle$ by the map $f_S = f$ where $f(x) = 1 \cdot x$. It is trivial that f is injective and that $f(S) = \mathbb{Z}\langle S \rangle$. If $g : S \to B$ is a map of S into some abelian group B, then we can define the pullback:

$$g_* : \mathbb{Z}\langle S \rangle \longrightarrow B$$

$$g_*\left(\sum_{x\in S} k_x \cdot x\right) = \sum_{x\in S} k_x g(x)$$

Note my choice of words of why it is a **pullback**: Noting that $\varphi = \sum_{x\in S} k_x \cdot x$, the above definition for g_* stipulates that

$$g_* \circ \varphi = \varphi \circ g$$

Furthermore, with the f defined as before, we see that:

$$g_* \circ f = g$$

This is the **unique** homomorphism with this property as this property only holds in the instance where $g_*(1 \cdot x) = g(x)$.

Proposition 41. *If $\lambda : S \to S'$ is a map of sets, there exists a unique homomorphism $\bar{\lambda}$ making the following diagram commute:*

$$\begin{tikzcd} S \arrow[r, "f_S"] \arrow[d, "\lambda"] & \mathbb{Z}\langle S\rangle \arrow[d, "\bar{\lambda}"] \\ S' \arrow[r, "f_{S'}"] & \mathbb{Z}\langle S\rangle \end{tikzcd}$$

Proof. We will prove this as Lang leaves it as an exercise. We take f_S as defined in **Remark 1.23**. We follow the diagram. Define $\bar{\lambda} = (f_{S'} \circ \lambda)_*$. Using that $f_S(x) = 1x \quad \forall x \in S$, we see that:

$$(f_{S'} \circ \lambda)_* \circ f_S = f_{S'} \circ \lambda$$

□

Definition 1.42. Denote $\mathbb{Z}\langle S\rangle$ also by $F_{ab}(S)$. It is called the **free abelian group generated by** S. The elements of S are the **free generators**. In fact, **Proposition 41** and **Remark 1.23** actually lead us to the following proposition.

Proposition 42. ***Universal Property of Free Abelian Groups***

Let $i : S \to F(S)$ be the inclusion of S into $F(S)$. For a free abelian group generated by a set S, $F(S)$, and an abelian group G, for any $f : S \longrightarrow G$, there exists a unique $\phi : F(S) \longrightarrow G$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "i"'] & G \\ & F(S) \arrow[u, "\exists!\phi"'] \end{tikzcd}$$

Proof. We note that $i : S \hookrightarrow F(S)$ is the inclusion of S into $F(S)$ defined by:

$$i(x) = 1x \quad \forall x \in S$$

Define ϕ as the pullback of f, i.e $\phi := f_*$ as described in **Remark 1.23**. Then we obtain[5] by composing the pullback of $f : F(S) \to G$ with the inclusion of S into $F(S)$:

$$\phi \circ i = f_* \circ i = f$$

Hence, our diagram commutes. $\square$

Proposition 43. *Every abelian group A is the quotient group of a free abelian group F. Furthermore, if A is finitely generated, then we can select F to have a finite number of generators.*

Proof. Lang leaves this as an exercise. Let us solve it.

For the first part, by the Universal Property of Free Abelian Groups (**Proposition 42**), we see that $\exists! \phi : F(A) \to A$:

$$\phi : g \longmapsto g \quad \forall g \in A$$

As ϕ is clearly surjective, we can use the First Isomorphism Theorem (**Theorem 17**), to conclude that $A \cong F(A)/\mathrm{Ker}(\phi)$.

For the second part, we can refer to the Universal Property of Free Abelian Groups, and realize that if S is the underlying set of some A (i.e. the map $S \to A$ is a strict inclusion), and A is of finite rank, then $F(S)$ is of finite rank as well, and this is uniquely determined as $\phi : F(S) \to A$ is unique.

Note: For the second implication, a more obvious but clearly unenlightening proof is that if A is finitely generated, then by the fundamental theorem of finitely generated abelian groups, $F(A)$ must be of finite rank (have finite number of basis elements). $\square$

Remark 1.24. If A is an abelian group, and if S is the basis for A, then $A \cong F_{ab}(S)$. If this is the case, then A is called a **Free Abelian Group**.

Proof. **Proof of Remark 1.24**

By **Proposition 43**, for any abelian group A, $A \cong F(S)/\mathrm{Ker}(\phi)$. Because S generates A itself, the homomorphism $\phi : F(S) \to A$ is actually injective because A has the same rank as $F(S)$, and this homomorphism exists uniquely. This proves that $A \cong F(S)$. $\square$

Remark 1.25. Compare the definitions of **Free** in **Remark 1.24** and **Remark 1.22**. So that the reader does not have to go looking for it:

[5] Look at **Remark 1.23**. There, we say that the inclusion i is $f = f_S$ and the map $g : \mathbb{Z}\langle S \rangle \to B$ is actually f in our current case.

1. **Remark 1.22**: A is **Free Abelian** if
$$A \cong \bigoplus_{i \in K} Z_i \quad Z_i = \mathbb{Z} \; \forall i \in K$$

2. **Remark 1.24**: A is **Free Abelian** if A is of rank Card(S)[6] and has a basis S, then:
$$A \cong F_{ab}(S)$$

These two definitions are equivalent.

Proof. The easy way to see this is to compare the universal properties of the free abelian groups (**Proposition 42**) and the direct sums (**Proposition 40**). Equate G in 42 with B in 40. There is an inclusion from S into $F(S)$ in 42. The analogous inclusion in 40 is from A_i into $\bigoplus_{i \in I} A_i$. The unique homomorphism in 42 and 40 are both pullbacks of an arbitrary map in $M(S, G)$[7], where S is a set and G an abelian group. To see this, look directly at the construction of g_* in **Remark 1.23**, and the construction of f in **Proposition 40**. This suffices as a thorough sketch of the equivalence of these definitions, as universal properties uniquely define these structures. □

Remark 1.26. For notation, if A is an abelian group, and T a subset of elements of A, $\langle T \rangle$ is the subgroup generated by the elements of T. Consequently, **this is the smallest subgroup** of A containing T.

We are led to the following example of an interesting free abelian group.

Proposition 44. ***Universal Property of Grothendieck Group***

Let M be a commutative (additive) monoid. There exists an abelian group $K(M)$ and a monoid homomorphism
$$\gamma : M \longrightarrow K(M)$$

Having the following universal property.

If $f : M \to A$ is a homomorphism into an abelian group A, then $\exists! f_* : K(M) \to A$ making the following diagram commutative:

$$\begin{tikzcd} M \arrow[r, "\gamma"] \arrow[dr, "f"'] & K(M) \arrow[d, "f_*"] \\ & A \end{tikzcd}$$

*This universal group, $K(M)$, is called the **Grothendieck Group**.*

[6] This denotes the cardinality of a set, the abuse of notation $|S|$ is admittedly confusing, as it may imply that S is a group, which is not true.

[7] See **Definition 1.8**.

Proof. Let $F(M)$ be the free abelian group generated by M. We denote the generator of $F(M)$ corresponding to an element $x \in M$ by $[x]$. Let B be the subgroup generated by all elements of type $[x+y]-[x]-[y]$, for $x, y \in M$. Let $K(M) := F(M)/B$. Let γ be the map obtained in the following way:

Take the inclusion of M into $F(M)$, $i : M \hookrightarrow F(M)$, defined by $i : x \longmapsto [x]$. Take the canonical surjection $\pi : F(M) \to F(M)/B$. Then $\gamma := \pi \circ i$.

Let us define f_* as follows:

$$f_*([x]) = f(x)$$

We know that this is valid because if we plug in:

$$f_*([x+y]-[x]-[y]) = f_*([x+y]) - f_*([x]) - f_*([y]) = f(x+y) - f(x) - f(y) = 0$$

Hence, as f_* is a homomorphism, it means that $[x+y]-[x]-[y]$ is the identity element of $K(M)$.

Note that there is some **abuse of notation** as we say that $[x] \in F(M)$ is $[x] \in K(M)$. We identify the coset with its representative directly by saying that $\gamma([x]) = [x] \quad \forall x \in M$.

Regardless, the diagram commutes. ☐

Remark 1.27. The **cancellation law** holds in M if, whenever $x, y, z \in M$, and $x + z = y + z$, we have $x = y$.

If the cancellation law holds in M, then the map $\gamma : M \longrightarrow K(M)$ is injective.

Proof. Lang guides us through a proof of this so we will not copy it. Note that in cancellative monoids (not necessarily groups), the Grothendieck Group becomes the typical negative integers. ☐

We now consider when we are given an abelian group A and a subgroup B. We want to find a subgroup C such that A decomposes into a direct sum of B and C (recall a similar idea in linear algebra where a vector space decomposes into a subspace and the orthogonal complement of that subspace).

Lemma 45. *Let $A \xrightarrow{f} A'$ be a surjective homomorphism of abelian groups and assume that A' is free. Let $B := \mathrm{Ker}(f)$. There exists a subgroup C of A such that the restriction of f to C induces an isomorphism of C with A', and such that $A = B \oplus C$.*

Proof. Lang outlines a proof but it is worth seeing why this is true.

Let $\{x'_i\}_{i \in I}$ form a basis of A', and for each $i \in I$, let x_i be an element of A such that $f(x_i) = x'_i$.

Let C be a subgroup of A generated by all elements x_i. If we have a relation:

$$\sum_{i \in I} n_i x_i = 0$$

with integers n_i, of which all but finitely many are 0, then applying f to this linear combination yields:

$$0 = \sum_{i \in I} n_i f(x_i) = \sum_{i \in I} n_i x_i'$$

Therefore, $\{x_i\}_{i \in I}$ is a basis for C.

If $z \in C$, and $f(z) = 0$, then $z = 0$. Therefore, $B \cap C = 0$. Now let $x \in A$, and since $f(x) \in A'$, there exist integers n_i so that:

$$f(x) = \sum_{i \in I} n_i x_i'$$

Therefore, $x - \sum_{i \in I} n_i x_i = b \in B$. Hence, rearranging this we see that:

$$x = b + \sum_{i \in I} n_i x_i \in B + C$$

Therefore, $A = B + C$ and $B \cap C = 0$, therefore, $A = B \oplus C$. $\square$

Theorem 46. *Let A be a free abelian group, and let B be a subgroup. Then B is also a free abelian group, and the rank of $B \leq$ rank of A. Any two bases of B have the same cardinality.*

Proof. Proof omitted. $\square$

Definition 1.43. The **Rank** of a free abelian group is the number of elements in its basis. Hence, finitely generated abelian groups will have **finite rank**.

1.8 Finitely Generated Abelian Groups

Definition 1.44. Let A be an abelian group. An element $a \in A$ is said to be a **torsion** element if it has finite period, i.e. $\exists m : a^m = e$.

The subset of all torsion elements of A is a subgroup of A (proof is trivial) called the **torsion subgroup** of A.

We denote the torsion subgroup of A as A_{tor}. We call an abelian group A a **torsion group** if $A_{tor} = A$. This means that every element of A has finite order.

Remark 1.28. Let A be an abelian group and p a prime. We denote by $A(p)$ the subgroup of all elements $x \in A$ whose period is a power of p. Clearly $A(p)$ is a torsion group and is a p-group, if finite.

Theorem 47. *Let A be a torsion abelian group. Then A is a direct sum of all of its subgroups $A(p)$ for all primes p such that $A(p) \neq 0$.*

Proof. Set the homomorphism:

$$\phi : \bigoplus_p A(p) \longrightarrow A$$

$$\phi : (x_p) \longmapsto \sum x_p$$

We must simply prove that this is a bijection.

Let $x \in \mathrm{Ker}(\phi)$. Then $\sum x_p = 0$. Let q be a prime number. Then

$$x_q = \sum_{p \neq q} (-x_p)$$

Let m be the least common multiple of the periods of elements x_p on the right side, with $x_q \neq 0$ and $p \neq q$. Then $mx_q = 0$. As $x_q \in A(q)$, we see that for some positive integer r, $q^r x_q = 0$. If $d = gcd(m, q^r)$, then $dx_q = 0$. But we also see that $d = 1$, then $x_q = 0$. So, without a doubt, this map is injective as $x = 0$.

For each positive integer m, denote by A_m the kernel of the map which is multiplication by m, i.e. the subgroup of $x \in A$ so that $mx = 0$. We show that if $m = rs$ for r, s positive relatively prime integers, then $A_m = A_r + A_s$.

Let $x \in A_m$. There exist integers u, v such that $ur + vs = 1$ (by Bezout's Theorem). Then we see that $x = urx + vsx$ so that $urx \in A_r$ and $vsx \in A_s$.

If we continue this, we obtain:

$$m = \prod_{p|m} p^{e(p)} \Longrightarrow A_m = \sum_{p|m} A_{p^{e(p)}}$$

So the map is surjective. □

Example 1.22. Let $A = \mathbb{Q}/\mathbb{Z}$. Clearly, A is a torsion abelian group and

$$A \cong \bigoplus_p (\mathbb{Q}/\mathbb{Z})(p)$$

Where

$$(\mathbb{Q}/\mathbb{Z})(p) = \{a/p^k \mid a \in \mathbb{Z},\ p \text{ prime},\ k \in \mathbb{Z}\}$$

Since the group $\mathbb{Q}/\mathbb{Z}$ will take any integer to the identity.

Definition 1.45. Let $r_q, \ldots, r_s \geq 1$, and a finite p-group A is of **type** $(\mathbf{p^{r_1}}, \ldots, \mathbf{p^{r_s}})$ if A is isomorphic to the product of cyclic groups of orders p^{r_i} $i = 1, \ldots, s$.

Remark 1.29. Let A be a finite abelian p-group. Let b be an element of A, $b \neq 0$. Let k be a non-negative integer such that $p^k b \neq 0$, and let p^m be the period of $p^k b$. Then b has period p^{k+m}.

Proof. This fact is rather obvious. The proof is direct. □

Theorem 48. *Every finite abelian p-group is isomorphic to a product of cyclic p-groups. If it is of type* $(p^{r_1}, \ldots, p^{r_s})$ *with* $r_1 \geq r_2 \geq \cdots \geq r_s \geq 1$*, then the sequence of integers* $(r_1, \ldots, r_s)$ *is uniquely determined.*

Proof. We will not go into the proof as Lang lays it out in extensive detail. □

Definition 1.46. A group G is said to be **torsion-free** if whenever $x \in G$ has finite period, then x is the unit (identity) element.

Theorem 49. *Let* A *be a finitely generated torsion-free abelian group. Then* A *is free.*

Proof. Assume $A \neq 0$. Let S be a finite set of generators for A. Let $x_1, \ldots, x_n$ be a maximal subset of S having the property that whenever $\nu_1, \ldots, \nu_n$ are integers such that:

$$\nu_1 x_1 + \cdots + \nu_n x_n = 0$$

then $\nu_j = 0 \quad \forall j$. Let B be the subgroup generated by $x_1, \ldots, x_n$. B is obviously free. Because of the maximality of $x_1, \ldots, x_n$, given $y \in A$, there exists integers $m_1, \ldots, m_n$, m not all zero so that:

$$my + m_1 x_1 + \cdots + m_n x_n = 0$$

In fact, $m \neq 0$ as this would imply that $m_i = 0 \quad \forall i$, therefore, $my \in B$. If y is a generator of A, this holds for every generator.

The left-multiplication map:

$$x \longmapsto mx$$

is a surjective homomorphism (trivial). This has trivial kernel because A is torsion-free, hence only the identity element has finite period. Therefore, this is an isomorphism of A onto a subgroup of B. By **Theorem 46**, mA is a free abelian group for any nonzero m. Hence, A is free. □

Theorem 50. *Let* A *be a finitely generated abelian group, and let* A_{tor} *be a subgroup consisting of all elements of* A *having finite period. Then* A_{tor} *is finite, and* A/A_{tor} *is free. There exists a free subgroup* B *of* A *such that* A *is the direct sum of* A_{tor} *and* B*.*

Proof. Let us take the fact that a finitely generated torsion abelian group is finite. Let A be finitely generated by n elements, and let F be the free abelian group on n generators. Due to the **Universal Property of Free Abelian Groups** (**Proposition 42**), there exists a unique surjective homomorphism $\phi : F \longrightarrow A$.

$\phi^{-1}(A_{tor})$ of F is a subgroup, and is finitely generated by **Theorem 46**. So A_{tor} is finitely generated, and finite.

A/A_{tor} is torsion-free, and this is apparent because if we let $\bar{x}$ be such that $m\bar{x} = 0$ for nonzero m. Then we have that $mx \in A_{tor}$, as it is a torsion element, and $qmx = 0$ for an integer q. As $x \in A_{tor}$, we have that $\bar{x} = 0$, meaning that for any $\bar{x} \in A/A_{tor}$, $\bar{x}$ is torsion if and only if $\bar{x} = 0$. Hence, A/A_{tor} is torsion-free. A/A_{tor} is torsion-free finitely generated abelian, hence, by **Theorem 49**, it is free.

We now use **Lemma 45** and **Theorem 46** to deduce the existence of B and give the direct sum relation, and to deduce that it is free. □

1.9 The Dual Group

Definition 1.47. Let A be an abelian group of exponent $m \geq 1$. For each element $x \in A$, we have $mx = 0$. Let Z_m be a cyclic group of order m. Denote by $A^\wedge$, or $\mathrm{Hom}(A, Z_m)$, the group of homomorphisms of A into Z_m, and call it the **dual** of A.

Remark 1.30. Let $f : A \to B$ be a homomorphism of abelian groups, and assume both have exponent m. Then f induces a homomorphism:

$$f^\wedge : B^\wedge \longrightarrow A^\wedge$$

We can illustrate this in a commutative diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "H()"] & B \arrow[d, "H()"] \\ A^\wedge & B^\wedge \arrow[l, "f^\wedge"] \end{tikzcd}$$

Where $H = \mathrm{Hom}(\cdot, Z_m)$. In the language of category theory, this H is called a **functor**, and specifically, this functor is called **contravariant** (as it reverses the direction of any morphism between objects in the domain category).

To demonstrate what $f^\wedge : \mathrm{Hom}(B, Z_m) \to \mathrm{Hom}(A, Z_m)$ does, we can just draw another diagram. Let $\phi \in \mathrm{Hom}(A, Z_m)$ and $\psi \in \mathrm{Hom}(B, Z_m)$.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "f"] & Z_m \\ & B \arrow[u, "\psi"] \end{tikzcd} \qquad f^\wedge : \psi \longmapsto \phi = \psi \circ f$$

Therefore, for each $\psi \in B^\wedge$, we see that $f^\wedge$ acts in the following way:

$$f^\wedge(\psi) = \psi \circ f$$

Noting this, it is clear that $f^\wedge$ is a homomorphism as:

$$f^\wedge(\psi_1 + \psi_2) = (\psi_1 + \psi_2) \circ f = \psi_1 \circ f + \psi_2 \circ f$$

Furthermore, it is clear that:

$$id^\wedge = id$$

as $id^\wedge(\psi) = \psi \circ id = \psi$

And it is also clear (draw the commutative diagram) that for $f : B \to C$ and $g : A \to B$ that:

$$(f \circ g)^\wedge = g^\wedge \circ f^\wedge$$

Theorem 51. *If A is a finite abelian group, expressed as a product $A = B \times C$, then $A^\wedge \cong B^\wedge \times C^\wedge$. A finite abelian group is isomorphic to its own dual.*

Proof. Note that the product of two abelian groups has the following diagram:

$$\begin{tikzcd} & B \times C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B & & C \end{tikzcd}$$

Where π_i denotes the projection map of the ith component.

If we apply the functor $\mathrm{Hom}(\cdot, Z_m)$, then we obtain:

$$\begin{tikzcd} & (B \times C)^\wedge & \\ B^\wedge \arrow[ur, "\pi_1^\wedge"] & & C^\wedge \arrow[ul, "\pi_2^\wedge"'] \end{tikzcd}$$

Let $\phi \in B^\wedge$, $\psi \in C^\wedge$. We see that $(\phi, \psi) \in B^\wedge \times C^\wedge$. We have a corresponding element of $(B \times C)^\wedge$ by defining:

$$(\phi, \psi)(x, y) = \phi(x) + \psi(y) \qquad (x, y) \in B \times C$$

We obtain a homomorphism:

$$B^\wedge \times C^\wedge \to (B \times C)^\wedge$$

Let $\eta \in (B \times C)^\wedge$. Then we see:

$$\eta(x, y) = \eta(x, 0) + \eta(0, y)$$

Now $\phi(x) = \eta(x, 0) \in B^\wedge$ and $\psi(y) = \eta(0, y) \in C^\wedge$. This is a map $(B \times C)^\wedge \to B^\wedge \times C^\wedge$.

We see that if we compose our maps, then we obtain the identity in both directions. We verify this:

$$B^\wedge \times C^\wedge \to (B \times C)^\wedge$$

$$(\phi, \psi)(x, y) \longmapsto (\phi, \psi)(x, 0) + (\phi, \psi)(0, y) = \phi(x) + \psi(y) = (\phi, \psi)(x, y)$$

$$(B \times C)^\wedge \to B^\wedge \times C^\wedge$$

$$\eta(x, y) \longmapsto \eta(x, 0) + \eta(0, y) = \eta(x) + \eta(y) = (\eta, \eta)(x, y) = \eta(x, y)$$

Hence, these are mutual inverses, and we establish the isomorphism:

$$(B \times C)^\wedge \cong B^\wedge \times C^\wedge$$

By the structure theorem for abelian groups (**Theorem 47**), every finite abelian group is expressible as a product of cyclic groups. Note that every morphism in the Hom sets to Z_m are isomorphisms. Lang goes into more detail, but it simply requires us to recognize that for any summand in the direct sum of cyclic groups, a homomorphism on the generator will have the same exponent a the period of the generator. This will uniquely determine the order of each summand. More specifically, if x has period k, then by linearity of homomorphisms, we see that for $\psi \in \mathrm{Hom}(A_1, Z_m)$, $\psi(kx) = k\psi(x)$. Hence, we can identify the exponent of element $\psi(x)$ as being an exponent of ψ itself. This is a rough sketch but once, we realize this, it is not hard to see why finite abelian groups are isomorphic to their duals. □

Definition 1.48. Let A, A' be two abelian groups, let C be abelian. A **bilinear map** $A \times A' \to C$ is a map:

$$A \times A' \to C$$
$$(x, x') \longmapsto \langle x, x' \rangle$$

With the property that the maps:

$$x \longmapsto \langle x, x' \rangle \qquad x' \longmapsto \langle x, x' \rangle$$

are both homomorphisms ("linear") in their respective arguments.

We also may refer to a bilinear map as a **pairing**.

Example 1.23. The map we just discussed is bilinear.

$$A \times \mathrm{Hom}(A, C) \to C$$
$$(x, f) \longmapsto f(x)$$

We can think of this as an evaluation map.

Definition 1.49. An element $x \in A$ is said to be **orthogonal** (or any synonyms of it) to a subset S' of A' if $\langle x, x' \rangle = 0 \quad \forall x' \in S'$.

Proposition 52. *The set of all $x \in A$ orthogonal to S' is a subgroup of A.*

Proof. Proof is trivial. Linear combinations of orthogonal elements are still orthogonal. □

Proposition 53. *The **left** kernel is the kernel of the map:*

$$L : x \longmapsto \langle x, x' \rangle$$

And consists of all elements:

$$\mathrm{Ker}(L) := \{(t, e') \in A \times A' \mid L(t, e') \equiv \langle e, e' \rangle\}$$

*Clearly this is the set of all elements in A orthogonal to A'. The **right** kernel is the kernel of the map:*

$$R : x' \longmapsto \langle x, x' \rangle$$

And consists of all elements:

$$\mathrm{Ker}(R) := \{(e,t') \in A \times A' \mid L(e,t') \equiv \langle e,e' \rangle\}$$

i.e. the set of all elements $x' \in A'$ orthogonal to A.

Remark 1.31. For a given bilinear map:

$$A \times A' \to C$$

We let $B := \mathrm{Ker}(L)$ and $B' := \mathrm{Ker}(R)$. An element $x' \in A'$ gives rise to an element of $\mathrm{Hom}(A,C)$ given by $\psi_{x'} : x \longmapsto \langle x, x' \rangle$. As $\psi_{x'}$ vanishes on B, $\psi_{x'} : A/B \to C$. $\psi_{x'} = \psi_{y'}$ if x' and y' are elements of A' such that:

$$x' \equiv y' \mod B'$$

Therefore $\psi : A'/B' \to \mathrm{Hom}(A/B, C)$, $\psi : x' \longmapsto \psi_{x'}$:

$$0 \to A'/B' \to \mathrm{Hom}(A/B, C)$$

This is injective as B' is the group orthogonal to A, and any element that gets mapped to the kernel of ψ will be equivalent to $0 \mod B'$.

There is a similar injective homomorphism:

$$0 \to A/B \to \mathrm{Hom}(A'/B', C)$$

If C is cyclic of order m, then for any $x' \in A'$, we have $m\psi_{x'} = \psi_{mx'} = 0$.

Therefore, A'/B' has exponent m. Likewise, A/B has exponent m.

Theorem 54. *Let $A \times A' \to C$ be a bilinear map of two abelian groups into a cyclic group C of order m. Let B, B' be the left/right kernels, respectively. Assume that A'/B' is finite. Then A/B is finite, and A'/B' is isomorphic to the dual group of A/B under ψ.*

Proof. The injection of A/B into $\mathrm{Hom}(A'/B', C)$ tells us that (i) A/B is finite, since A'/B' is finite and (ii) $|A/B| \le |(A'/B')^\wedge| = |A'/B'|$ and $|A'/B'| \le |(A/B)^\wedge| = |A/B|$. Hence, $|A'/B'| = |A/B|$ and the map ψ is surjective, hence an isomorphism. $\square$

Corollary 55. *Let A be a finite abelian group, B a subgroup, $A^\wedge$ the dual group, and $B^\perp$ the set of all $\phi \in A^\wedge$ such that $\phi(B) = 0$. We then have a natural isomorphism of $A^\wedge/B^\perp$ with $B^\wedge$.*

Proof. For the bilinear map $A \times A' \to C$ in **Theorem 54**, we let $A' := A^\wedge$, $A := B$, $C := C$, so that the bilinear map in question is:

$$B \times A^\wedge \to C$$

We see that under the maps given in **Remark 1.31**, $B' := \mathrm{Ker}(R) := B^\perp$, $B := \mathrm{Ker}(L) := 0$. Then we obtain:

$$B^\wedge \cong A^\wedge/B^\perp$$

$\square$

1.10 Inverse Limit and Completion

Consider a sequence of groups $\{G_n\}$, and suppose for all $n \geq 1$ homomorphisms:

$$f_n : G_n \to G_{n-1}$$

Assume that these are all surjective homomorphisms, and form infinite sequences:

$$x = (x_0, x_1, \dots) \quad x_{n-1} = f_n(x_n)$$

Remark 1.32. As f_n is **surjective**, for any $x_{n+1} \in G_{n+1}$, there exists x_n such that $f_{n+1}(x_{n+1}) = x_n$.

As a recurring term, we say that given $x_n \in G_n$, we can **lift** x_n to G_{n+1} via f_{n+1}. We call f_{n+1} a **lift**. Therefore, in this context, **lifting** is just a way to say that there exists $x_{n+1} \in G_{n+1}$ so that a surjective map relates x_{n+1} and x_n in some way.

Note: This idea will be very important in the future when we learn about **diagram chasing**.

Definition 1.50. Such an infinite sequence described above always exists given surjective homomorphisms as there exists a lift from G_k to G_{k+1}.

We can define multiplication of these sequences component-wise. Using this as our binary operation, the set of all such sequences is a group called an **Inverse Limit** of the family $\{(G_n, f_n)\}$. We denote this as:

$$\varprojlim (G_n, f_n)$$

or when it is clear the sequence of maps we refer to, $\lim G_n$.

Example 1.24. Tate Group

Let A be an additive abelian group. Let p be a prime. Let $p_A : A \to A$ denote multiplication by p. A is **p-divisible** if p_A is surjective. Form an inverse limit by taking $A_n = A \quad \forall n$ and $f_n = p_A$ for all n. We denote this inverse limit by $V_p(A)$. Let $T_p(A)$ be the subset of $V_p(A)$ consisting of the aformentioned sequences, so that $x_0 = 0$. Let $A[p^n] := \text{Ker}(p_A^n)$.

We immediately see that since $x_0 = 0$, that for the liftings f_n:

$$0 = f_1(x_1) \ \dots \ 0 = f_n(x_n)$$

Hence, any element contained in $T_p(A)$ will vanish under the maps, hence given that $x_k \in G_k$ is contained in the sequences of $T_p(A)$

$$G_k := \text{Ker}(f_{k-1}) \quad \forall k$$

Then clearly

$$T_p(A) = \lim A[p^{n+1}]$$

$T_p(A)$ is called the **Tate group** associated with the p-divisible group A.

Example 1.25. Tate Group Continued

We have already seen p-divisible groups before. Recall in **Example 1.22**, the torsion abelian group $\mathbb{Q}/\mathbb{Z}$. The subgroups, $(\mathbb{Q}/\mathbb{Z})(p)$ are a p-divisible group.

If we let $\mu[p^n]$ be the group of pth roots of unity in $\mathbb{C}$, i.e.

$$\mu[p^n] = \{e^{2\pi ik/p^n} \mid n \in \mathbb{Z}_n \quad k \in \mathbb{Z} \text{ (fixed)}\}$$

Let

$$\mu[p^\infty] := \bigcup_{n\in\mathbb{N}} \mu[p^n]$$

Furthermore, $\mu[p^\infty] \cong (\mathbb{Q}/\mathbb{Z})(p)$ (this is prove directly, but readily apparent from inspecting the elements in these groups).

We see that, according to the surjective homomorphisms (the multiplicative version of the one shown above):

$$f_n = a^p$$

The inverse limit of the roots of unity μ is:

$$T_p(\mu) = \lim \mu[p^n]$$

Example 1.26. Take a group G. Let $\{H_n\}$ be a sequence of normal subgroups so that $H_n \supset H_{n+1}$ for all n. Let

$$f_n : G/H_n \to G/H_{n-1}$$

be the canonical homomorphisms (recall by the **Third Isomorphism Theorem** (**Theorem 18**) that $(G/H_n)/(H_n/H_{n-1}) \cong G/H_{n-1}$, the canonical map being referred to here is the map $G/H_n \to (G/H_n)/(H_n/H_{n-1})$). Then form the inverse limit:

$$\varprojlim G/H_n$$

There is a natural homomorphism out of G to this inverse limit:

$$g : G \to \varprojlim G/H_n$$

$$g : x \longmapsto (\dots, x_n, \dots) \qquad x_n = f_n(x)$$

Example 1.27. p-adic integers

Let $G_n = \mathbb{Z}/p^{n+1}\mathbb{Z}$ for every $n \geq 0$. Let:

$$f_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$$

be the canonical surjections described in **Example 1.26** (recall that all abelian subgroups are normal). The inverse limit

$$\varprojlim (\mathbb{Z}/p^n\mathbb{Z}, f_n) = \mathbb{Z}_p$$

and we call $\mathbb{Z}_p$ the **p-adic** integers.

Note: We dealt with inverse limits of **abelian groups**. Those have a very simple description as normality and the types of surjections admissible are very simple in abelian groups (the types of abelian groups themselves are very simple due to the structure theorem). We now deal with **groups in general**.

Definition 1.51. We describe the **Inverse Limit** (in general) as follows:

Let I be a set of indices, and suppose that we have a relation with partial ordering in I defined as follows:

For all (i, j), we have a relation $i \leq j$ satisfying (i) $\forall i, j, k \in I$, $i \leq i$ (ii) if $i \leq j$ and $j \leq k$, then $i \leq k$ (iii) if $i \leq j$ and $j \leq i$, then $i = j$.

We call I **directed** if given $i, j \in I$, there exists k so that $i \leq k$ and $j \leq k$.

Assume that I is directed, by an **inversely directed family of groups**, we mean a family $\{G_i\}_{i \in I}$ and for each pair $i \leq j$, a homomorphism $f_i^j : G_j \to G_i$ such that whenever $k \leq i \leq j$, we have:

$$f_k^i \circ f_i^j = f_k^j \qquad f_i^i = id$$

Let $G = \prod G_i$ be the product of the family. Let Γ be a subset of G consisting of all (x_i) with $x_i \in G_i$ such that for any i and $j \geq i$, we have:

$$f_i^j(x_j) = x_i$$

Remark 1.33. Such Γ forms a subgroup of G, and it is called the **Inverse Limit of the family**. We denote it as

$$\Gamma = \varprojlim G_i$$

Proof. The binary operation is component-wise as defined in **Example 1.6**.

As G_i is a group for all i, there's an identity element $e_i \in G_i$. Furthermore, G must have an identity element (e_i), making $(x_i)(e_i) = (x_i e_i) = (x_i) \quad \forall (x_i) \in \Gamma$. Furthermore, the homomorphisms f_i^j preserve this identity element.

Likewise with the inverse operation of the group.

Clearly, this binary operation is closed, and furthermore, remains closed when we consider all elements to be images of some family of homomorphisms. $\square$

Example 1.28. Let G be a group. Let $\mathcal{F}$ be the family of normal subgroups of finite index. If H, K are normal of finite index, then so is $H \cap K$ (this is trivial by **Proposition 9**). Therefore $\mathcal{F}$ is a directed family, and we can form the inverse limit

$$\varprojlim G/H \qquad H \in \mathcal{F}$$

Take $\mathcal{F}_p$ to be the family of normal subgroups with prime-power index. Any $H \in \mathcal{F}_p$ can be used to form an inverse limit. This is because if H, K are normal and of finite p-power index, we have by **Proposition 9**:

$$(G : H)(H : K)(K : H \cap K) = p^k p^l p^m = (G : H \cap K)$$

A group that is an inverse limit of finite groups is called **profinite**.

Definition 1.52. Assume that G is a group, and suppose that given a sequence of normal subgroups $\{H_r\}$, with $H_r \supset H_{r+1}$, and such that these are all of finite index. A sequence $\{x_n\}$ in G will be called a **Cauchy Sequence** if given H_r, there exists N such that $\forall m, n \geq N$, $x_n x_m^{-1} \in H_r$.

We call $\{x_n\}$ a **null sequence** if given r, there exists N such that for $n \geq N$, $x_n \in H_r$.

Proposition 56. *Cauchy Sequences form a group under the component-wise product. Furthermore, the null sequences are a normal subgroup of the group of Cauchy Sequences.*

Proof. Let us first verify that the set of all Cauchy Sequences in G is a group.

$\{x_n\}$, $\{y_n\}$ be two Cauchy Sequences. Then, clearly, we see that for a given H_r, there exists N so that for any $m, n \geq N$, $x_n x_m^{-1} \in H_{r_1}$. Likewise, there exists M so that for any $m, n \geq M$, $y_n y_m^{-1} \in H_{r_2}$. We take $r := \min(r_1, r_2)$, $K := \max(M, N)$, so that for a given H_r, $\exists K$ so that $\forall m, n \geq K$:

$$(x_n y_n y_m^{-1} x_m^{-1}) = x_n (y_n y_m^{-1}) x_m^{-1} \in H_r$$

Clearly, this is true as $y_n y_m^{-1} \in H_{r_2}$, but by construction $H_r \supset H_{r_2}$, so $y_n y_m^{-1} \in H_r$. Furthermore, as $y_n y_m^{-1} \in H_r$, $x_n y_n y_m^{-1} \in H_r$ (again, as $H_r \supset H_{r_1}$), and this proves that $x_n (y_n y_m^{-1}) x_m^{-1} \in H_r$. Hence, the set of all cauchy sequences under the component-wise product $\{x_n\}\{y_n\} = \{(xy)_n\}$ is **closed**.

The associativity of the component-wise product is trivial, given the above constructions we made.

The assertions of inverses and the unit element are trivial as every H_r is a normal subgroup of G. More specifically, we can take the sequence of identity elements $\{e\}$, and this is clearly a null sequence, hence a cauchy sequence. Likewise, as $x_n x_m^{-1} \in H_r$, $x_m x_n^{-1} \in H_r$. Hence, the set of Cauchy Sequences of a group G forms a group itself.

Let us prove that the null sequences form a normal subgroup.

For the above reasons, we see that the set of all null sequences are a subgroup of the group of Cauchy Sequences. It will suffice to only prove normality.

Let $\{z_n\}$ be a null sequence. And assume that we have H_r as a normal subgroup for some r. We have that for any $n \geq N$, $z_n \in H_r$. Then let $\{x_n\}$ be a Cauchy Sequence, fix H_k to be the normal subgroup (of G) so that there exists M so that $\forall n, m \geq M$, $x_n x_m^{-1} \in H_k$. Then, likewise as before, take $K := \max(M, N)$, $l := \min(k, r)$. Consequently, $H_l \supseteq H_k$ and $H_l \supseteq H_r$. Therefore, we see that, since H_l is a subgroup of G, it is closed, meaning that $x_n z_n \in H_l$ for $n \geq K$. Furthermore, for any $n, m \geq K$, it means that $x_n z_n x_m^{-1} \in H_l$. Therefore, for any $z_n \in H_l$, we see that $x_n z_n$ is in the group of Cauchy Sequences. Therefore, $x_n z_n x_m^{-1} \in H_l$. This is the definition of normality. $\square$

Definition 1.53. Let C denote the Cauchy Sequences in G. Let N denote the Null Sequences in G. By**Proposition 56**, we see that C is a group, and N is a normal subgroup of C. We then call C/N the **completion** of G with respect to $\{H_r\}$.

Remark 1.34. G has a natural homomorphism of G into C/N in the following way:

$$i : G \longrightarrow C/N$$

$$i : x \longmapsto (x, x, x, \dots) \mod N$$

Then, since N consists of all elements in the normal subgroups $\{H_r\}$ given some $n \geq N$, we see that $\text{Ker}(i) = \bigcap H_r$. If this intersection is trivial, then i is an embedding.

Theorem 57. *The completion and the inverse limit*

$$\varprojlim G/H_r$$

are isomorphic under natural mappings.

Proof. We will sketch a proof. Let $x = \{x_n\}$ be a Cauchy Sequence in C. For a given r and for n sufficiently large, the equivalence classes of $x_n \mod H_r$ is independent of the choice of n. Namely, this equivalence class is the set of all $z \in G$ so that:

$$x_n z^{-1} \in H_r$$

This class is $x(r)$. Now define the sequence:

$$(\bar{x}_1, \bar{x}_2, \dots) \in \varprojlim G/H_r$$

This is clearly an element of the inverse limit, given the family $\{H_r\}$, and the canonical maps $f_i^j : G/H_j \to G/H_i$ for $i \leq j$ (recall that $H_i \supset H_j$. Notice that this sequence is defined by $f_i^j(x(j)) = x(i)$, and since $H_i \supset H_j$, we have that $\bar{x}_j = \bar{x}_i \mod H_i$.

Likewise, let $(\bar{x}_1, \bar{x}_2, \dots)$ be an element of the inverse limit, $\lim G/H_r$, where $\bar{x}_n \in G/H_n$. Let x_n be a representative in G. We can see $\{x_n\}$ is a Cauchy Sequence as follows:

Let $f_i^j : G/H_j \longrightarrow G/H_i$ be the canonical surjection defined by the **Third Isomorphism Theorem** (Theorem 18):

$$f_i^j(\bar{x}_j) = \bar{x}_i$$

Then, we see that, since f_i^j is the canonical surjection, that $x_j = x_i \mod H_i$ (since $\bar{x}_j$ is technically in $(G/H_i)/(H_i/H_j) \cong G/H_j$ by definition of the canonical map). Therefore, we see that:

$$\bar{x}_j = \bar{x}_i \Longrightarrow \overline{x_j x_i^{-1}} = e \in G/H_i$$

Therefore, $x_j x_i^{-1} \in H_r$, as when we are given some H_r where $i, j \geq r$, $x_j x_i^{-1} \in H_i \subset H_r$ (by definition of the family $\{H_i\}$). Therefore, such an integer r exists, and it makes any element of the inverse limit Cauchy.

We will not give a formal proof of the invertibility (hence isomorphism), but we have essentially verified that Cauchy Sequences are inverse limits, and inverse limits are Cauchy Sequences, when we take the family of normal subgroups $\{H_k\}$ and the family of canonical homomorphisms $\{f_i^j\}$ where:

$$f_i^j : G/H_i \longrightarrow G/H_j \quad i \geq j$$

We may brush up on this at a later time, but for now, we will convince ourself that this correspondence is direct. $\square$

1.11 A Brief Introduction to Category Theory

We will introduce everything necessary here. It is to get a good idea of how category theory relates to algebra.

1.11.1 What is a Category? Group Actions on Objects of Arbitrary Categories.

Definition 1.54. A **category** C consists of a collection of **objects**, and a collection of **morphisms**. Context will make it clear what the objects are. We give a special notation to the set of morphisms, $\text{Mor}(A, B)$. There is a composition law on $\text{Mor}(A, B)$ for any three objects, A, B, C in C:

$$\text{Mor}(B, C) \times \text{Mor}(A, B) \to \text{Mor}(A, C)$$

In a concrete category where our morphisms are mappings, we see that this results in:

$$(g, f) \longmapsto g \circ f$$

Note that this may not always be true. Categories are very broad!!!

The composition law obeys the following axioms:

1. $\text{Mor}(A, B)$ and $\text{Mor}(A', B')$ are disjoint unless $A = A'$, $B = B'$, in which case, they are equal.

2. For every object A in C, there exists a two-sided identity element $id_A \in \text{Mor}(A, A)$. It is the left identity for $\text{Mor}(A, B)$, and it is the right identity for $\text{Mor}(B, A)$.

3. The composition law is associative (this should be no surprise as it is the usual composition of maps).

Remark 1.35. I will often refer to an object as belonging to a category (i.e. for a category C, and object in C, A, will be denoted as $A \in C$, although this is not technically true as "elements" of a category include both the object and morphisms out of that object; it is meant to be clear whether we are referring to morphisms or objects).

Remark 1.36. A morphism $f \in \mathrm{Mor}(A, B)$ is written $f : A \to B$.

Definition 1.55. A morphism $f : A \to B$ is an **isomorphism** if there exists $g : B \to A$ such that:

$$g \circ f = id_A \in \mathrm{Mor}(A, A)$$

$$f \circ g = id_B \in \mathrm{Mor}(B, B)$$

In the instance where $A = B$, we call the isomorphism an **automorphism**.

A morphism $A \to A$ is called an **endomorphism**. $\mathrm{End}(A) := \mathrm{Mor}(A, A)$ is referred to as the **set of endomorphisms**.

An automorphism is just an endomorphism with a two-sided identity element, we denote the set of automorphisms of the object $A \in C$ as $\mathrm{Aut}(A)$.

Remark 1.37. $\mathrm{End}(A)$ is a monoid under composition defined in **definition 1.54**. The proof of this is trivial, and follows from the axioms of a category.

Likewise, the automorphisms $\mathrm{Aut}(A)$ form a group (it's just the monoid of endomorphisms in which every element is invertible on both sides).

Example 1.29. Lots of Examples of Categories

It is impossible to list every category, I'd die before accomplishing that task. Here are a bunch of relevant ones:

1. Sets
2. Monoids
3. Groups
4. G-sets
5. Abelian Groups
6. Rings (with unit element)
7. Vector Spaces over a field
8. Rngs (rings without unit element)
9. Holomorphic maps
10. Topological Spaces
11. Smooth/Topological Manifolds

Definition 1.56. Group Actions on Categories

Let G be a group, C be a category, and A be an object in C. A **group action** of G on A means a homomorphism (see **Definition 1.32**):

$$\rho : G \to \mathrm{Aut}(A)$$

Then, we see that for every $x \in G$, we have an automorphism $\rho_x : A \to A$. If A is a set, then we obtain the **group action on a set** (see **Section 1.5**).

We call ρ a **representation** of G on A. We say G is **represented** as an automorphism group of A.

Remark 1.38. Let $A, B \in C$. Let $\mathrm{Iso}(A, B)$ be the set of isomorphisms of A with B. The group $\mathrm{Aut}(B)$ operates on $\mathrm{Iso}(A, B)$ by (left) composition.

$$(v, u) \longmapsto v \circ u \qquad v \in \mathrm{Aut}(B) \quad u \in \mathrm{Iso}(A, B)$$

If u_0 is an element of $\mathrm{Iso}(A, B)$, then the orbit of u_0 is all of $\mathrm{Iso}(A, B)$. Therefore, $v \longmapsto v \circ u_0$ is a bijection $\mathrm{Aut}(B) \to \mathrm{Iso}(A, B)$.

We can also consider the group $\mathrm{Aut}(A)$ acting on $\mathrm{Iso}(A, B)$ by (right) composition. Moreover, if $u : A \to B$ is an isomorphism, then $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ under conjugation:

$$w \longmapsto uwu^{-1}$$

This is easily visualized through the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "w"] & B \arrow[d, "uwu^{-1}"] \\
A \arrow[r, "u"] & B
\end{tikzcd}$$

Remark 1.39. Let $\rho : G \to \mathrm{Aut}(A)$, and $\rho' : G \to \mathrm{Aut}(A')$ be two representations of a group G on two objects $A, A' \in C$. A **morphism of representations** $\rho \longmapsto \rho'$ is a **morphism** $h : A \to A'$ such that the following diagram commutes for any $x \in G$:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "\rho_x"] & A' \arrow[d, "\rho'_x"] \\
A \arrow[r, "h"] & A'
\end{tikzcd}$$

Representations of a group G in the objects of a category C **form a category** themselves.

An **isomorphism of representations** is an **isomorphism** $h : A \to A'$ making the diagram above commute. If $h : A \to A'$ is an isomorphism of representations, ρ, ρ', we can let $\{h\}$ denote conjugation by h, and then use the equivalent diagram:

$$\begin{tikzcd}
G \arrow[r, "\rho"] \arrow[dr, "\rho'"] & Aut(A') \arrow[d, "\{h\}"] \\
 & Aut(A')
\end{tikzcd}$$

Remark 1.40. Let Ab denote the category of abelian groups. Let A be an abelian group, and G a group. For an action of G on A:

$$\rho : G \to \mathrm{Aut}(A)$$

$$x \longmapsto x \cdot a \equiv \rho_x(a) \quad a \in A$$

By properties of the action (see **Section 1.5**), we obtain the following things:

1. Since $\rho_x \rho_y = \rho_{xy}$:
$$x \cdot (y \cdot a) = (xy) \cdot a$$

2. Since $\rho_x \in \mathrm{Aut}(A)$, it must be a homomorphism, hence $\rho_x(a+b) = \rho_x(a) + \rho_x(b)$, leading us to:
$$x \cdot (a + b) = x \cdot a + x \cdot b$$

3. Since $\rho_e(a) = a$:
$$e \cdot a = a$$

4. Since ρ_x becomes a homomorphism on A:
$$x \cdot 0 = 0$$

Thus, when the group action is extended to a group, we see that the permutations are actually group automorphisms.

This generalized notion of a group action will be extremely useful later on.

Definition 1.57. Fix a category C with the objects and morphisms obvious from context. We can speak of a **category of morphisms of** C by considering the morphisms of every object in C, as the objects in our new category of morphisms of C, denoted by CC.

Let A, A', B, B' be objects in C, and let $f : A \to B$ and $f' : A' \to B'$ be morphisms in C (objects of CC). We define a morphism in CC:

$$f \to f'$$

by the pair (ϕ, ψ) making the following diagram (in C) commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"] & B \arrow[d, "\psi"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

It is easy to verify that the composition laws, and identity morphisms all exist in this category CC.

1.11.2 Introduction to Universal Objects

Definition 1.58. Let C be a category. Any object P of C is called **universally attracting** if $\exists!$ morphism of each object of C into P.

Likewise, P is called **universally repelling** if for every object of C, $\exists!$ morphism of P into the other objects.

Visually, we could remember this in the following way:

(i) $P \in C$ is **universally attracting** if for every $A \in C$, $\exists! f$:

$$A \xrightarrow{f} P$$

(ii) $P \in C$ is **universally repelling** if for every $A \in C$, $\exists! f$:

$$P \xrightarrow{f} A$$

For ease of use, when it is obvious (and it usually is in the simpler categories) if it's attracting or repelling, we call these objects P **universal**.

Proposition 58. *Universal objects in a category C are unique up to a unique isomorphism.*

Proof. Let P, P' be two universal objects in C. Regardless of if it is attracting or repelling, P and P' both admit the identity morphism into themselves:

$$id_P : P \to P \qquad id_{P'} : P' \to P'$$

Therefore, we see that $\exists! f$:

$$f : P \to P'$$

But $\exists! g$:

$$g : P' \to P$$

We compose these appropriately to find that $fg = id_{P'}$ and $gf = id_P$. Therefore, f and g are unique isomorphisms (mutual two-sided inverses of one another). $\square$

Example 1.30. The trivial group $\{e\}$ is universal (both repels and attracts). We see that it attracts because for any $A \in \text{Grp}$,

$$\exists! f : A \to \{e\}$$

And $A := \text{Ker}(f)$.

It repels because for any $A \in \text{Grp}$,

$$\exists! g : \{e\} \to A$$

g is the inclusion of $\{e\}$ into A.

Definition 1.59. Products

Let C be a category, and let $A, B \in C$ be objects. A **product** of A, B in C means a triple (P, f, g) consisting of an object $P \in C$ and two morphisms:

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

satisfying the following condition:

Given two morphisms $\phi : C \to A$ and $\psi : C \to B$ in C, there exists a *unique* morphism $h : C \to P$ which makes the following diagram commutative:

$$\begin{tikzcd}
 & C \arrow[dl, "\phi"'] \arrow[d, "h"] \arrow[dr, "\psi"] & \\
A & P \arrow[l, "f"] \arrow[r, "g"'] & B
\end{tikzcd}$$

So that: $\phi = f \circ h$, $\psi = g \circ h$.

More generally, we can take a family of objects $\{A_i\}$ in C, the **product** for this family consists of the pair $(P, \{f_i\})$, where $P \in C$ and $\{f_i\}$ is a family of morphisms where:

$$f_i : P \to A_i$$

satisfying the following condition:

Given a family of morphisms:

$$g_i : C \to A_i$$

$\exists! h : C \to P$ such that $f_i \circ h = g_i$ for any i.

Example 1.31. Let Set be the category of sets, and let $\{A_i\}$ be a family of sets. Then the product is:

$$A = \prod_i A_i$$

which is the usual Cartesian product, and let $p_i : A \to A_i$ the the projection map on the ith object. $(A, \{p_i\})$ is a universal pair.

Example 1.32. Let $\{G_i\}$ be a family of groups. Let $G = \prod G_i$ be the direct product. Let $p_i : G \to G_i$ be the projection homomorphism. This constitutes a product of the family in Grp. The unique homomorphism is the homomorphism:

$$g : G' \to \prod G_i$$

$$g(x')_i = g_i(x') \qquad \forall i$$

Definition 1.60. Coproducts

Let $\{A_i\}$ be a family of objects in a category C. By the **coproduct**, we mean a pair $(S, \{f_i\})$ consisting of an object S, and a family of morphisms:

$$\{f_i : A_i \to S\}$$

(Notice how the family of morphisms goes out of the family of objects)

Satisfying the following property:

Given a family of morphisms $\{g_i : A_i \to C\}$, $\exists! h : S \to C$ so that $h \circ f_i = g_i \quad \forall i$.

The morphism h is said to be the morphism induced by the family $\{g_i\}$.

For the case of a coproduct of two objects, A, B, given an object C, and morphisms $\phi : A \to C$, $\psi : B \to C$, $\exists! h : S \to C$ so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "f"'] & C & B \arrow[l, "\psi"'] \arrow[dl, "g"] \\ & S \arrow[u, "h"] & \end{tikzcd}$$

So that $\phi = h \circ f$ and $\psi = h \circ g$.

Example 1.33. Let Ring be the category of commutative rings with unit elements. Given two rings A, B, we may form the tensor product. So we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "i_1"'] & C & B \arrow[l, "\psi"'] \arrow[dl, "i_2"] \\ & A \otimes B \arrow[u, "\exists! h"] & \end{tikzcd}$$

Then we see that for $a \in A$, $b \in B$, we can give the following inclusion maps:

$$i_1 : A \to A \otimes B$$

$$i_1 : a \longmapsto a \otimes 1$$

$$i_2 : B \to A \otimes B$$

$$i_2 : b \longmapsto 1 \otimes b$$

Since tensor product is defined as $a \otimes b = (a \otimes 1)(1 \otimes b)$, we see that:

$$h(a \otimes 1) = \phi(a) \qquad h(1 \otimes b) = \psi(b)$$

Therefore, the tensor product is a coproduct in the category of commutative rings with units.

Definition 1.61. Fibered Products

Let C be a category, and let Z be an object in C. We have a new category of objects over Z, denote it as C_Z. The objects of C_Z are all morphisms (refer to **Definition 1.57**) that end in Z in C:

$$f : X \to Z \in C$$

A morphism from $f : X \to Z$ to $g : Y \to Z$ is a morphism $h : X \to Y$ so that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

A product in C_Z is called a **fiber product** of f and g in C. We denote it by $X \times_Z Y$. It also comes with natural morphisms on X, Y over Z, which we denote as p_1 and p_2.

$$\begin{tikzcd}
& X \times_Z Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

Note that this is just the product defined on morphisms over Z. This diagram can be taken to be the definition of the fiber product.

In the above diagram, we call p_1 the **pull-back** of g by f, and p_2 the **pull-back** of f by g.

We take the following "property" as a definition of the fibered product over $Z \in C$.

Proposition 59. ***Universal Property of the Fiber Product***

Given any object $T \in C$, and morphisms making the following diagram commutative:

$$\begin{tikzcd}
& T \arrow[dl] \arrow[dr] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

There exists a unique morphism $T \to X \times_Z Y$ making the following diagram commutative:

$$\begin{tikzcd}
& X \times_Z Y \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\
X & T \arrow[l] \arrow[r] \arrow[u] & Y
\end{tikzcd}$$

In this way, we may think of ***products*** *as being* ***Universally Attracting****.*

Definition 1.62. Fibered Coproduct

We denote the **fiber coproduct** over Z as C^Z. C^Z is the category of all morphisms that start from Z, i.e. $f : Z \to X$ for any $X \in C$.

Therefore, given $f : Z \to X$ and $g : Z \to Y$, a morphism of f to g is clearly a morphism $h : X \to Y$, and we can take the coproduct of f and g in C^Z. This is denoted as $X \coprod_Z Y$ with morphisms q_1, q_2 as in the following commutative diagram:

$$\begin{tikzcd}
& X \coprod_Z Y & \\
X \arrow[ur, "q_1"] & & Y \arrow[ul, "q_2"'] \\
& Z \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

We call q_1 the **push out** of g by f. Likewise, q_1 is the **push out** of f by g.

We can define the coproduct using the following universal property.

Proposition 60. ***Universal Property of the Fibered Coproduct***

Given any object $T \in C$, and morphisms making the following diagram commute:

$$\begin{tikzcd}
& T & \\
X \arrow[ur] & & Y \arrow[ul] \\
& Z \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

There exists a unique morphism $X \coprod_Z Y \to Z$ so that the following diagram commutes

$$\begin{tikzcd}
& T & \\
X \arrow[ur] \arrow[r, "q_1"'] & X \coprod_Z Y \arrow[u] & Y \arrow[ul] \arrow[l, "q_2"]
\end{tikzcd}$$

*In this sense, we can think of coproducts as being **Universally Repelling**.*

Remark 1.41. Fibered Products and Coproducts in the Category of Abelian Groups

The **product** of two homomorphisms over Z, $f : X \to Z$ and $g : Y \to Z$ exists in the category of abelian groups, Ab.

It consists of all pairs (x, y) so that:

$$f(x) = g(y)$$

The **coproduct** of two homomorphisms over Z, $f : X \to Z$ and $g : Y \to Z$ is the quotient group:

$$(X \oplus Y)/W$$

Where W is the subgroup of $X \oplus Y$ generated by the elements $(f(z), -g(z))$ for $z \in Z$.

Proof. To show this, we must simply verify that this satisfies the Universal Properties, **Proposition 59, 60**, for products/coproducts, respectively.

Take $X \times_Z Y$ to be the **Universally Attracting Object** in Ab_Z, such that it is the subgroup of $X \times Y$ with the pairs (x, y) so that $f(x) = g(y)$. Let T be an object with a morphism $\tau : T \to Z$. Draw the following diagram:

$$\begin{tikzcd}
 & X \times_Z Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & T \arrow[l, "\phi"'] \arrow[r, "\psi"] \arrow[u, "\exists! h"] \arrow[d, "\tau"] & Y \arrow[dl, "g"] \\
 & Z &
\end{tikzcd}$$

$f \circ p_1 : X \times_Z Y$ so that $f(p_1(x, y)) = f(x)$. Likewise for $g \circ p_1$. By commutativity, for any $t \in T$, we can define τ as:

$$\tau(t) = z = f(\phi(t)) = g(\psi(t))$$

Hence:

$$\tau = f \circ \phi = g \circ \psi$$

Thus, we may define $h(t) = (\phi(t), \psi(t))$. We can see that $\phi = p_1 \circ h$ and $\psi = p_2 \circ h$ for all such t. Hence, we can write the above as:

$$\tau = f \circ (p_1 \circ h) = g \circ (p_2 \circ h)$$

Which means

$$\tau = (f \circ p_1) \circ h = (g \circ p_2) \circ h$$

Therefore, this h exists (by construction) and is the unique morphism that f and g factors through, and it satisfies the property that $f(x) = g(y)$ for any x, y able to be lifted to some $t \in T$. The specified product satisfies the universal property of **Proposition 59**.

For the coproduct, assume that we are given T so that the following diagram commutes:

$$\begin{tikzcd}
 & T & \\
X \arrow[ur, "\phi"] \arrow[r, "q_1"] & X \coprod_Z Y \arrow[u, "\exists! h"] & Y \arrow[ul, "\psi"'] \arrow[l, "q_2"'] \\
 & Z \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

We set the **universally repelling object** in Ab^Z as:

$$X \coprod_Z Y = (X \oplus Y) / \langle (f(z), -g(z)) \rangle$$

Note that we have a canonical surjection $\pi : X \oplus Y \to X \coprod_Z Y$.

We require $h \circ \pi \circ q_1 = \phi$ and $h \circ \pi \circ q_2 = \psi$. If we list this all to an arbitrary $z \in Z$, we get that $\phi(f(z)) = \psi(g(z))$, and by commutativity, this forces:

$$h(\pi(q_1(f(z)))) = h(\pi(q_2(g(z))))$$

So that (assuming additive abelian groups):

$$h(\pi(q_1(f(z)) - q_2(g(z)))) = 0$$

If we ignore the π, we have:

$$h(q_1(f(z)) - q_2(g(z))) = 0$$

Hence, $q_1(f(z)) - q_2(g(z)) \in \mathrm{Ker}(h)$. By definition of the inclusion $q_1(f(z)) = (f(z), 0)$, likewise, $q_2(g(z)) = (0, g(z))$, so that $(f(z), -g(z)) \in \mathrm{Ker}(h)$.

Define h then, $\forall \overline{(x, y)} \in X \coprod_Z Y$:

$$h(x, y) = \phi(x) + \psi(y)$$

With some nuance[8]. Just to check, by commutativity of the diagram, we have $\phi(f(z)) = \psi(g(z))$, therefore:

$$h(f(z), -g(z)) = \phi(f(z)) - \psi(g(z)) = 0$$

This implies that:

$$h \circ (q_1 \circ f) = h \circ (q_2 \circ g)$$

Hence, the homomorphisms f and g factor uniquely through h once again. Thus, our specified coproduct has the desired universal property in **Proposition 60**. $\square$

Remark 1.42. This is how most "Universal Property" verifications go. Depending on whether the object is attracting or repelling, we choose an arbitrary object in the category, assume that we are given some morphisms and universal morphisms, and we construct a unique morphism from our universal object to an arbitrary object in C, using the universal morphisms and any given morphisms. This is precisely what makes universal properties so powerful! The verification is relatively simple, and it works for ANY object in that category. This latter point is particularly important as it implies that any object with a **universal property** is a **functor**!

Definition 1.63. Let A and B be two categories. A **covariant functor** $F : A \to B$ is a mapping rule which does the following:

[8] Note that our map defined is actually $h : X \oplus Y \to T$. Technically, h should be acting on equivalence classes with the subgroup W, generated by $(f(z), -g(z))$ equivalent to identity. We are really referring to a map $h(\pi(x, y))$, but we consider (x, y) as a representative of W and exclusively act h on our representatives, with the correct identity element in mind.

1. Assigns each object $a \in A$ to $F(a) \in B$.
2. Assigns to each morphism $f : x \to y \in \text{Mor}(x, y)$ for any $x, y \in A$, the morphism $F(f) : F(x) \to F(y) \in B$.

   Furthermore, the action of a functor on a morphism satisfies the following properties:

   (a) For any object $a \in A$, we have $F(id_a) = id_{F(a)}$.

   (b) If $f : a \to b$ and $g : b \to c$, where a, b, c are objects in A and the morphisms are in their respective morphism sets, then we have:

   $$F(g \circ f) = F(g) \circ F(f)$$

Likewise, a **contravariant functor** $F : A \to B$ is a mapping rule which satisfies everything above, except for 2b:

For $f : a \to b$, where a, b are objects in A, we have:

$$F(f) : F(b) \to F(a)$$

Likewise, it reverses the composition:

$$F(g \circ f) = F(f) \circ F(g)$$

Example 1.34. Forgetful Functor

There (uniquely) exists a functor $F : \text{Grp} \to \text{Set}$ such that for any $G \in \text{Grp}$, $F(G) = S_G$, where S_G denotes G as a set.

Hence, this functor "forgets" the group structure of a group and turns it into a set of elements of G, without the binary operation. Furthermore, for any morphism $f \in \text{Mor}(G, G')$, we have that $F(f) : S_G \to S_{G'}$. Hence, it turns group homomorphisms into a map of sets.

Remark 1.43. Sometimes a covariant functor applied to a morphism f is written as f_* instead of $F(f)$. Contravariant functors applied to f are denoted as f^* instead of $F(f)$. It should be clear when this arises and we will make note of it.

Example 1.35. This is more of a proposition than an example, but we take time to say this: **anything with a universal property can be considered as a functor**. For example, the **Universal Property of Free Abelian Groups** (**Proposition 42**) indicates that the free abelian group generated by S, $F_{ab}(S)$ is actually a functor from Set to Ab. It is clear that F_{ab} takes a set S to a free abelian group generated by S. Likewise, its actions on maps of sets is that it turns them into homomorphisms.

Definition 1.64. Representation Functors

The following functors are so important, they get their own definition.

Let C be a category, and A a fixed object in C. We obtain a **covariant functor**:

$$M_A : C \to \text{Set}$$

$$M_A = \text{Mor(A, X)}$$

Where X is any object of C. If $\phi : X \to X'$ is a morphism, we let:

$$M_A(\phi) : \text{Mor}(A, X) \to \text{Mor}(A, X')$$

$$g \longmapsto \phi \circ g \qquad g \in \text{Mor}(A, X)$$

so that

$$A \xrightarrow{g} X \xrightarrow{\phi} X'$$

The fact that this is a functor is trivial, just set $\phi = id_X$ and see its action on any $g \in \text{Mor}(A, X)$. Furthermore, just compose $g : A \to X$ with some $f : A' \to A$. It is clear that this shows functoriality.

Likewise, we can take the **contravariant** case:

$$M^B : C \to \text{Set}$$

Where for any object $Y \in C$

$$M^B(Y) = \text{Mor}(Y, B)$$

And for any morphism $\psi : Y' \to Y$:

$$M^B(\psi) : \text{Mor(Y, B)} \to \text{Mor}(Y', B)$$

$$f \longmapsto f \circ \psi$$

where $f \in \text{Mor}(Y, B)$, i.e.

$$Y' \xrightarrow{\psi} Y \xrightarrow{f} B$$

We leave the trivial verification up to the reader, that this is a functor.

We call these two functors the (covariant and contravariant, respectively) **representation functors**.

Example 1.36. We have a ubiquitous example of a representation functor:

Let Ab denote the category of abelian groups. Then if we fix an abelian group A, the functor:

$$X \longmapsto \text{Hom}(A, X)$$

is a **covariant hom functor** from Ab into Ab.

The hom functor:

$$X \longmapsto \text{Hom}(X, A)$$

is a **contravariant hom functor** from Ab into itself.

Example 1.37. We will prove this more indepth later, but let R be a commutative ring and M be an R-module:

$$X \longmapsto M \otimes X$$

is a covariant functor from the category Mod_R into itself.

Remark 1.44. The representation functors in **Definition 1.64** are compatible with the product and coproduct.

For a product P, and two objects A, B, for every object X, $\mathrm{Mor}(X, P)$ is a product of the sets $\mathrm{Mor}(X, A) \times \mathrm{Mor}(X, B)$ in the category of sets.

Proof. We will prove this as a more general rule. But for now, we take it that:

$$\mathrm{Mor}(X, A) \times \mathrm{Mor}(X, B) \cong \mathrm{Mor}\,(X, A \times B)$$

□

Remark 1.45. Basic Notions of (Strictly) Higher Categories

Let us organize our thoughts on a bit of higher category theory. Let C be an arbitrary category.

We define an **n-morphism** as a morphism of (**n-1**)-morphisms.

We define a **0-morphisms** in C as the *objects* of C. We define a **1-morphism** in C as a morphism between **0-morphisms** (i.e. morphisms between objects of C).

A (strict) **n-category** is a category necessarily consisting of all k-morphisms where $k \in \{0, 1, \ldots, n\}$. We see that if C is an n-category, the **0-morphisms** are (n-1)-categories, and the **1-morphisms** are morphisms between (n-1)-categories.

Note that **every n-category may be viewed as an (n-1)-category by just omitting the n-morphisms**. Hence, it is possible to view any category as a 1-category, by repeatedly omitting higher morphisms.

Example 1.38. As an application of the above remark, we may consider the category of categories. The 0-morphisms are objects (i.e. categories), and the 1-morphisms are functors. The 2-morphisms are morphisms of functors.

Definition 1.65. Let A and B be two 1-categories, i.e. elements of a 2-category, C. The functors of A into B (covariant for the sake of simplicity) are 1-morphisms of A and B.

Let L and M be two such functors (1-morphisms). A **2-morphism** $H : L \to M$, in C, is called a **Natural Transformation**. It is a mapping rule to which each object X of the 1-category A associates a morphism:

$$H_X : L(X) \to M(X)$$

such that for any morphism of X in A, $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc} L(X) & \xrightarrow{H_X} & M(X) \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle M(f)} \\ L(Y) & \xrightarrow[H_Y]{} & M(Y) \end{array}$$

Remark 1.46. Representation functors can be used to transport notions of structures on sets to different categories.

Let A be a category, and G an object of A. G is a **Group Object** if for each object $X \in A$, we have group structure on the set $\text{Mor}(X, G)$ so that:

$$X \longmapsto \text{Mor}(X, G)$$

is a functor from A into the category of groups, Grp.

1.12 Free Groups

We now look into the category of groups, Grp.

Remark 1.47. Let $G = \prod G_i$ be the direct product of groups. Each G_j admits an injective map onto the product on the jth component:

$$\lambda_j : G_j \to \prod G_i$$

such that for $x \in G_j$, the ith component of $\lambda_j(x)$ is the unit element of G_i if $i \neq j$ (this would be zero in an additive group).

Definition 1.66. For the embedding in **Remark 1.47**, we call it the **canonical embedding**.

This describes the product of groups completely. Clearly, the pair $(\prod G_i, \{\lambda_j\})$ is universal.

Remark 1.48. Let G be a group and S a subset of G. Recall that G is **generated** by S if every element of G can be written as a finite product of elements of S and their inverses (the empty product is the unit element of G). S is called the **generating set**, and its elements **generators**. If there exists a *finite* set of generators of G (i.e. S is finite), then G is **finitely generated**. For S a set, and $\phi : S \to G$ a map, ϕ **generates** G if its image, $\phi(S)$, generates G.

NOTE: We will deviate from Lang for a little while here. This is sufficient for continuing on with the rest of the book, as we only need notions of representability and functors and morphisms, and universal objects. We are continuing our study of category theory here. The reference is Awodey's Category Theory.

2 Category Theory

We will go in depth into category theory here. We will first introduce some notions useful in algebra, and then we will go off the deep end.

2.1 Abstractions of Things We Already Know

Here, we will start to introduce more category-theoretic fomralism and deviate away from our application-based approach earlier.

2.1.1 Free Monoid

This section will make the idea of a free (monoidal objects) easier to understand, as a free monoid underlies every free object.

Definition 2.1. We call a set A an **alphabet**, and the elements of A **letters**. We write $-$ for the empty word (don't confuse this with subtraction! it just means no word). The **Kleene Closure** of A is defined to be the set

$$A^* := \{\text{words with letters in } A\}$$

The binary operation on A^* is called **concatenation** and we denote it as $*$:

$$w * w' = ww' \quad \forall w, w' \in A^*$$

Note that $w * - = w \quad \forall w \in A^*$, hence, $-$ is a unit element of A^*. Clearly, these satisfy the definition of a monoid, and A^* is called a **free monoid** on A. This free monoid comes with an inclusion map $i : A \hookrightarrow A^*$, and clearly, this is an inclusion of generators in the free monoid, i.e.

$$i(a) = a$$

Definition 2.2. In general, a monoid $(M, *)$ is freely generated by a subset, $A \subset M$, if:

(i) Any element $m \in M$ can be written as a product of elements of A:

$$m = a_1 * \cdots * a_k \qquad a_i \in A$$

(ii) Only the most fundamental relations that will make M a monoid, hold. To state it more formally, if $a_1 \ldots a_j = a'_1 \ldots a'_j$, then this is precisely required to make M a monoid.

We denote the free monoid with generators A as $M(A)$.

In some sense, a free monoid is the most "minimal" object in the category of monoids. We now give a complete characterization of a free monoid.

Definition 2.3. Universal Property of Free Monoids

Let N be any object in the category of monoids, and A the subset of $M(A)$

generating $M(A)$. Recall that for a free monoid, there is map from sets to monoids called an inclusion of generators $i : A \hookrightarrow M(A)$:

$$i(a) = a$$

Such that $i(a) * i(b) = ab \qquad \forall a, b \in M(A)$.

Given a function $f : A \to N$, there exists a unique morphism $\bar{f} : M(A) \to N$ so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & M(A) \arrow[d, "\bar{f}"] \\ & N \end{tikzcd}$$

We will show that the Kleene Closure is the unique free monoid in the category of monoids.

Proposition 61. $(A^*, *)$ *has the universal property described above.*

Proof. Assume $f : A \to N$. Define $\bar{f} : M(A) \to N$ by:

$$\bar{f}(m) = f(m)$$

so that

$$\bar{f}(m_1 \cdots m_k) = f(m_1) \cdots f(m_k)$$

Clearly, this means that $\bar{f}(-) = e_N$, where e_N denotes the identity element of N.

We need only check that this makes the diagram commute. Without loss of generality, let:

$$m = a_1 * \cdots * a_k \qquad \forall m \in M(A)$$

Then, we see that for any $m \in M(A)$:

$$m = i(a_1) * \cdots * i(a_k)$$

And $\bar{f}(m) = f(m)$, so that, by definition of $\bar{f}$:

$$\bar{f}(m) = \bar{f}(i(a_1) * \cdots * i(a_k)) = f(i(a_1)) * \cdots * f(i(a_k))$$

But also recognize that because $i(a_i) = a_i \quad \forall i$, and $\bar{f}$ is a morphism of monoids (i.e. has the homomorphism property):

$$\bar{f}(m) = \bar{f}(a_1 * \cdots * a_k) = \bar{f}(a_1) * \cdots * \bar{f}(a_k) = f(a_1) * \cdots * f(a_k)$$

Hence, $f(a_i) = \bar{f}(i(a_i)) \quad \forall a_i \in A$. So that $f = \bar{f} \circ i$.

Repeating the calculation for some other $g : M(A) \to N$ that satisfies the property of $\bar{f}$ shows that this is unique as well. □

Remark 2.1. It is quite easy to show that for any two free monoids, M and N, there exists a unique isomorphism $M \to N$ and, thus, there is only one free monoid in the category of monoids.

2.1.2 Epi and Mono

Remark 2.2. For a morphism, f, in a category C, $f : A \to B$, we denote the underlying morphism on the underlying set of the objects as:

$$|f| : |A| \to |B|$$

Definition 2.4. In a category C, the morphism:

$$f : A \to B$$

is called a:

(i) **monomorphism**: given any two morphisms $g, h : C \to A$, if $fg = fh$, then it implies that $g = h$. i.e.

$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

(ii) **epimorphism**: given any two morphisms $i, j : B \to D$, if $if = jf$, then it implies that $i = j$. i.e.

$$A \xrightarrow{f} B \underset{j}{\overset{i}{\rightrightarrows}} D$$

We will denote a monomorphism $f : A \hookrightarrow B$, and an epimorphism $f : A \twoheadrightarrow B$. We call monomorphisms, *mono* or *monic*, and epimorphisms, *epi* or *epic*. We call isomorphisms, *iso*.

Proposition 62. *Let $f : A \to B$ be a morphism in the category of sets. f is injective if and only if f is monic (i.e. a monomorphism).*

Proof. Let us prove the opposite conclusion. If $a, a' \in A$, and let $\{x\}$ be a singleton set. We abuse notation and say that $a : x \longmapsto a$ and $a' : x \longmapsto a'$. As f is monic, we see that if $a \neq a'$, then $fa \neq fa'$ (take the contrapositive of the definition of a monomorphism). Thus, this map if injective.

For the original direction, assume that f is injective, and let $g, h : C \to A$ be functions so that $g \neq h$ for some $c \in C$. Then, we see that by injectivity of f:

$$g(c) \neq h(c) \Longrightarrow f(g(c)) \neq f(h(c))$$

for some $c \in C$.

Therefore, $g \neq h$ implies that $fg \neq fh$ (a contraposition of the definition of monic). $\square$

In fact, we have an even stronger result. In any monoidal category (monoids, groups, rings, modules, vector spaces, etc.), the monomorphisms are exactly the injective maps of the objects of that category.

Proposition 63. *Let C be a monoidal category, then a morphism of objects in C is monic if and only if it is an injective map of the underlying set of those objects (i.e. the objects of C under the forgetful functor).*

Proof. Let M and N be two monoids, and $h : M \to N$ a monoid morphism.

Let h be monic and take two different elements $m, m' \in M$ so that:

$$m : \{*\} \longmapsto M$$

$$m(*) = m \qquad m'(*) = m'$$

By the **Universal Property of the Free Monoid**, we obtain that:

$$\exists! \bar{m}, \bar{m}' : M(*) \to M$$

so that the following diagram commutes:

$$\begin{tikzcd}
\{*\} \arrow[d, hook, "i"] \arrow[dr, shift left, "m"] \arrow[dr, shift right, "m'"'] & & \\
M(*) \arrow[r, shift left, "\bar{m}"] \arrow[r, shift right, "\bar{m}'"'] & M \arrow[r, "h"] & N
\end{tikzcd}$$

If we map the underlying sets, $|M|$ to $|N|$, we automatically see that we can reduce this to the following diagram:

$$\begin{tikzcd}
\{*\} \arrow[r, shift left, "|m|"] \arrow[r, shift right, "|m|'"'] & |M| \arrow[r, "|h|"] & |N|
\end{tikzcd}$$

As h is monic, and $\bar{m} \neq \bar{m}'$, we conclude that $m \neq m'$, so that $hm \neq hm'$, hence h is not injective. So monic in the category implies injective as sets.

Assume $|h| : |M| \to |N|$ is injective on sets. Let $f, g : X \to M$ be two distinct monoid morphisms. We see that:

$$|f|, |g| : |X| \to |M|$$

are also distinct morphisms of sets, so $|h \circ f| = |h| \circ |f| \neq |h| \circ |g| = |h \circ g|$. Therefore, $h \circ f \neq h \circ g$. So injective in sets implies monic in the category of monoids. $\square$

This above proposition is a much deeper fact.

Definition 2.5. Let C be some monoidal category. Let F be the **forgetful functor** from $C \to$ Sets. Then F does the following:

(i) for any objects $A \in C$, $F(A) = |A|$

(ii) for any morphisms between objects $f : A \to B$, $F(f) = |f| : |A| \to |B|$

Clearly, these two characteristics of the functor F satisfy the properties of a functor (composition and identity functor), as compositions of morphisms are equivalently the compositions of morphisms of sets. The identity morphism leaves an element of an object unchanged in the category C, so it must do the same as a set.

Remark 2.3. The above proposition (**Proposition 63**) is a much deeper fact regarding the forgetful functor!

Proposition 64. *The forgetful functor F* ***preserves monomorphisms*** *in monoidal categories.*

Proof. The proof, with what we know, is an arduous task. However, consider that every object of a monoidal category is always a set, and for anything more complicated than a monoid, a monoid, hence, considering only monoids and above, the monoidal category C will have the universal property of the free monoid by default. Thus, using **Proposition 63**, we can see that it is a direct statement that F preserves monic morphisms. $\square$

Remark 2.4. We will get to this idea later, but this is an even deeper fact.

Theorem 65. *The* ***forgetful functor*** *is* ***left adjoint*** *to the "free" constructions of monoidal categories.*

Proof. We don't know what adjunctions are yet, but the if and only if correspondence between monomorphisms in Sets and a monoidal category C, is not a coincidence. $\square$

Now back to more manageable ideas:

Proposition 66. *Every isomorphism (in a category) is both a monomorphism and epimorphism. We could very well take this as a definition of an isomorphism in a category, but it is provable.*

Proof. This is easily provable through a clever diagram. Look at the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & B \arrow[r, "m"] \arrow[dr, "id"] & C \arrow[d, "e"] \\ & & B \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & E \end{tikzcd}$$

Just chase this diagram.

Assume that m is an isomorphism with an inverse e. Then $mx = my$ implies that $x = emx = emy = y$. Therefore, $mx = my$ implies that $x = y$, and m is monic. Similarly, assume that e is an isomorphism with an inverse m, then $fe = ge$ implies that $f = fem = gem = g$, so that $fe = ge$ implies that $f = g$. Therefore, e is an epimorphism, or epic. $\square$

Remark 2.5. The converse is not true in general! (only in Sets).

2.1.3 Initial and Terminal Objects

Definition 2.6. In a category C, an object:

(i) 0 is **initial** if for any object $A \in C$, there is a unique morphism:

$$0 \to C$$

(ii) 1 is **terminal** if for any object $A \in C$, there is a unique morphism:

$$C \to 1$$

These are the same thing as in **Definition 1.58**.

Remark 2.6. Note that terminal and initial objects have trivial universal properties, hence they are unique up to isomorphism (we proved this in **Proposition 58**, but we will do it again).

Proof. Let 0 and $0'$ be both initial. Then let us show that there is a unique isomorphism $0 \to 0'$. This is trivial when presented with the following diagram:

$$\begin{tikzcd}
0 \arrow[r, "u"] \arrow[dr, "id_0"'] & 0' \arrow[d, "v"] \arrow[dr, "id_{0'}"] & \\
 & 0 \arrow[r, "u"'] & 0'
\end{tikzcd}$$

We made a judicious choice of object (we chose $0'$ as the codomain for 0 and likewise the other way). Clearly, v and u are mutually inverses of one another (this is obvious as $uv = id_0$ and $vu = id_{0'}$), therefore, $\exists!$ isomorphism $v : 0' \to 0$.

Likewise, letting both 1 and $1'$ being terminal objects. Let us show that there is a unique isomorphism $1' \to 1$.

$$\begin{tikzcd}
1' \arrow[r, "u"] \arrow[dr, "id_{1'}"'] & 1 \arrow[d, "v"] \arrow[dr, "id_1"] & \\
 & 1' \arrow[r, "u"'] & 1
\end{tikzcd}$$

Once again, we see that $uv = id_{1'}$ and $vu = id_1$, v is the unique isomorphism between the two terminal objects, 1 and $1'$. $\square$

Remark 2.7. A terminal object in C is the initial object in the **opposite category**, C^{op}.

Example 2.1. Examples of Initial and Terminal Objects

(i) In Sets, $\varnothing$ is the initial object, any singleton set is terminal (note that all singletons are isomorphic to one another, but they are different).

(ii) In Grp, the group with one element is both initial and terminal (we often denote this in homological algebra as 0).

(iii) In Ring, the integers, $\mathbb{Z}$, is initial. The one element ring in which $0 = 1$ is the terminal object.

2.1.4 Sections and Retractions

Proposition 67. *Let C be a category and $f : A \to B$ be a morphism of objects $A, B \in C$. If f has a left inverse $g : B \to A$ so that $gf = id_A$, then f is monic and g is epic.*

Proof. By assumption, $gf = id_A$. $f : A \to B$ and $g : B \to A$. Let $h, h' : A \to X$ be distinct and $j, j' : Y \to A$ be distinct. Consider:

(i)

$$B \xrightarrow{g} A \xrightarrow{f} B \xrightarrow{g} A \underset{h}{\overset{h'}{\rightrightarrows}} X$$

(ii)

$$Y \underset{j}{\overset{j'}{\rightrightarrows}} A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{f} B$$

By (i), we see that if $h' \neq h$, then $h' id_A \neq h id_A$, meaning that $h'g \neq hg$. Therefore, g is epic.

By (ii), we see that if $j' \neq j$, then $id_A j \neq id_A j' \implies fj \neq fj'$. Therefore, f is monic. $\square$

Definition 2.7. A **split mono (epi)** is an arrow with a left (right) inverse.

Given arrows $e : X \to A$ and $s : A \to X$ so that $es = id_A$, s is called the **section of** e, and e is called the **retraction of** s. The object A is called a **retract** of X.

Proposition 68. *Functors preserve split epis and split monos.*

Proof. Let F be a functor. Then $F(id_A) = id_{F(A)}$. Indeed, $F(es) = F(e)F(s) = id_{F(A)}$. $\square$

Proposition 69. *Every epimorphism splitting is equivalent to the axiom of choice.*

Proof. Let $e : R \twoheadrightarrow X$ be an epi. Then we have a family of non-empty subsets of E, $E_x := e^{-1}\{x\} \quad x \in X$. Consider the section of e, $s : X \to E$ so that $es = id_X$. This implies that $s(x) \in E_x \quad \forall x \in X$. Therefore, the section s of e is a choice function for the family $(E_x)_{x \in X}$.

Now, given a family of non-empty sets, $(E_x)_{x \in X}$, we take:

$$E = \{(x, y) \mid x \in X,\ y \in E_x\}$$

and define the epimorphism $e : E \twoheadrightarrow X$ by the projection:

$$e : (x, y) \longmapsto x$$

The section of e, denoted as s, is the choice function for the family $(E_x)_{x \in X}$. $\square$

Definition 2.8. An object P is said to be **projective** if for any epimorphism $e : E \twoheadrightarrow X$, and a morphism $f : P \to X$, there exists an arrow $\bar{f} : P \to E$ so that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "e"] \\ P \arrow[ur, "\exists! \bar{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

i.e. $e \circ \bar{f} = f$.

In this case, we say that f **lifts across** e. Note that since all surjective maps are epi (the converse is not true), surjective maps are guaranteed to lift elements of X to an element of E (this is exactly **Remark 1.32**).

Remark 2.8. The axiom of choice implies that all sets are projective objects in Sets.

Proposition 70. *Any retract of a projective object is, itself, projective.*

Proof. Let P be projective by assumption. For every morphism $f : P \to X$, we can lift over an epimorphism $e : E \to X$. Then consider a retract of P, i.e. an object Z such that there is a section of r, $s : Z \to P$ and a retraction of s, $r : P \to Z$. Let us draw the diagram with all that we are given:

$$\begin{tikzcd} & & E \arrow[d, "e"] \\ P \arrow[urr, "\exists! \bar{f}"] \arrow[r, shift right, "r"'] & Z \arrow[l, shift right, "s"'] \arrow[ur] \arrow[r] & X \end{tikzcd}$$

Clearly, we see that Z is a retract of P, as $rs = id_Z$. If we follow this diagram, we see that the morphism $Z \to X$ must necessarily exist as we are given the morphism $f : P \to X$ by assumption. Thus, $\exists h : Z \to X$ so that $f = hr$. Furthermore, since Z is a retract of P, $\exists! \bar{h} : Z \to E$ as $\bar{h} = \bar{f}s$. Therefore, given a morphism out of a retract of a projective object P, Z, $h : Z \to X$, and given an epimorphism $e : E \to X$, we see that $\exists! \bar{h} : Z \to E$ so that $e\bar{h} = h$.

Hence, the retract (Z) of a projective object (P) is, itself, projective.

Our above proof is equivalent to completing the remainder of the above diagram as follows:

$$\begin{tikzcd} & & E \arrow[d, "e"] \\ P \arrow[urr, "\exists! \bar{f}"] \arrow[r, shift right, "r"'] & Z \arrow[l, shift right, "s"'] \arrow[ur, "\exists! \bar{h}"'] \arrow[r, "\exists h"] & X \end{tikzcd}$$

∎

2.1.5 Products

We developed fibered products and coproducts earlier, but we will discuss these notions in more generality here.

Definition 2.9. Universal Property of the Product

In a category C, a product, P, of objects A and B is the universal triplet (P, p_1, p_2) such that it satisfies the following universal property:

Given an object X, and morphisms x_1 and x_2 so that $x_2 : X \to B$ and $x_1 : X \to A$, $\exists! u : X \to P$ making the following diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, "\exists! u"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

i.e. so that $x_1 = p_1 u$ and $x_2 = p_2 u$.

Proposition 71. *Products in a category are unique up to (unique) isomorphism.*

Proof. We will not prove this but we simply assume that there is an object Q so that $q_1 : Q \to A$ and $q_2 : Q \to B$ and assume that there exists a unique morphism $v : Q \to P$. We then must prove that there exists $i : Q \to U$ and $j : U \to Q$, both mutually inverses of one another. Reader should fill in the rest (**Proposition 58**). $\square$

2.1.6 Products in Various Categories

Fill this section in with examples of products in various categories.

2.1.7 Categories with Products

Let C be a category that has a product diagram for every pair of objects as follows:

$$\begin{tikzcd} A \arrow[d, "f"] & A \times A' \arrow[l, "p_1"] \arrow[r, "p_2"] & A' \arrow[d, "f'"] \\ B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"] & B' \end{tikzcd}$$

Then, we can write $fp_1 \times f'p_2 : A \times A' \to B \times B'$ so that the above square commutes:

$$\begin{tikzcd} A \arrow[d, "f"] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "f \times f'"] & A' \arrow[d, "f'"] \\ B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B' \end{tikzcd}$$

Remark 2.9. Choosing a product for each pair of objects in C yields a functor:

$$\times : C \times C \to C$$

This has a universal property which we will not check.

We call a category with such a functor as *having binary products*.

We can speak of ternary products (three products), n-ary products.

Any category with binary products has all finite n-ary products! This is easily seen as $A \times B \in C$ by the universal property of the product. Then, we can take $A \times B$ and D and form a product: $(A \times B) \times D$. Furthermore, by the universal property, we see that:

$$(A \times B) \times D \cong A \times (B \times D)$$

Remark 2.10. A terminal object is a product of no objects as follows:

Given no objects, there's an object 1 with no maps, and given any other object X and no maps, there is a unique morphism:

$$! : X \to 1$$

Remark 2.11. We can define a product for a family of objects indexed by a set I, $(A_i)_{i \in I}$, and it has a universal property very similar to the original product defined in **Section 2.1.5**.

Definition 2.10. A category C is said to have all finite products if it has a terminal object and all binary products.

2.1.8 Hom-Sets

Recall a bunch of basic definitions once again. Assume our categories are locally small.

Definition 2.11. In a category C, given any objects A and B, we define the **Hom-Set** as:

$$\mathrm{Hom}(A, B) = \{f \in C \mid f : A \to B\}$$

i.e. it is the set of all morphisms between objects A and B.

Any morphism $g : B \to B'$ in C induces a function:

$$\mathrm{Hom}(A, g) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$$

$$\mathrm{Hom}(A, g) : (f : A \to B) \longmapsto (g \circ f : A \to B \to B')$$

Instead of $\mathrm{Hom}(A, g)$, we sometimes write g_* instead (the Hom notation is cumbersome). Therefore, we write this as:

$$g_* : f \longmapsto g \circ f$$

It is quite easy to verify that

$$\mathrm{Hom}(A,-) : C \to \mathbf{Sets}$$

is a functor. We call it the **covariant representable functor of** A. The proof of this is trivial (and it is in **Definition 1.64**).

We now use Hom functors to give another definition of products.

Remark 2.12. An object P with arrows $p_1 : P \to A$ and $p_2 : P \to B$ is an element:

$$(p_1, p_2) \in \mathrm{Hom}(P, A) \times \mathrm{Hom}(P, B)$$

Now let X be an arbitrary object and let $x : X \to P$ be a morphism.

Compose x with p_1 and p_2. We get $x_1 = p_1 \circ x$ and $x_2 = p_2 \circ x$. This is indicated in the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, "\exists! u"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Therefore, we have a function:

$$\nu_X := (\mathrm{Hom}(X, p_1), \mathrm{Hom}(X, p_2)) : \mathrm{Hom}(X, P) \to \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

$$\nu_X : x \longmapsto (x_1, x_2)$$

Proposition 72. ***Corollary to the Universal Property of the Product***

A diagram

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

is a product of A *and* B *if and only if for every object* X*, the canonical map* ν_X *is an isomorphism:*

$$\nu_X : \mathrm{Hom}(X, P) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

Proof. By the **Universal Property of the Product** (we will give the diagram again below:

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, "\exists! x"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Therefore, as $\exists! x : X \to P$ so that $(p_1, p_2)(x) = (x_1, x_2)$ for every $(x_1, x_2) \in \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$. Hence, this map is a bijection! As these are morphisms, we can keep composing as necessary, so this is actually an isomorphism. $\square$

Definition 2.12. Let C and D be categories with binary products. A functor $F : C \to D$ is said to preserve binary products if for every product diagram in C:

$$A \xleftarrow[p_1]{} A \times B \xrightarrow[p_2]{} B \quad \in C$$

We have a product diagram in D:

$$F(A) \xleftarrow[F(p_1)]{} F(A \times B) \xrightarrow{F(p_2)} F(B) \quad \in D$$

Such that the product diagram in D satisfies the universal property if the product diagram in C satisfies the universal property.

(i) Equivalently, let X be some object in C. Then F preserves products if and only if $F(A \times B) \cong F(A) \times F(B)$.

We can recall the previous proposition and formulate this as:

(ii) F preserves binary products if and only if $\nu_{F(X)}$, as follows:

$$\nu_{F(X)} : \mathrm{Hom}(F(X), F(P)) \cong \mathrm{Hom}(F(X), F(A)) \times \mathrm{Hom}(F(X), F(B))$$

is an isomorphism in D.

(iii) Alternatively, F preserves products if and only if

$$(F(p_1), F(p_2)) : F(A \times B) \to F(A) \times F(B)$$

is an isomorphism.

Proposition 73. *For any object X in C with products. The covariant representable functor (the Hom-functor if C is locally small):*

$$\mathrm{Hom}_C(X, -) : C \to \mathbf{Sets}$$

preserves products.

Proof. The canonical map defined in the Corollary to the Universal Property of the Product says that:

$$\mathrm{Hom}_C(X, A \times B) \cong \mathrm{Hom}_C(X, A) \times \mathrm{Hom}_C(X, B)$$

As C is locally small (by assumption) and $\mathrm{Hom}_C(X, K)$ is a set for any object K, the conclusion follows immediately. □

2.2 Duality

We describe duality in a precise manner and we describe its implications on category theory.

2.2.1 Duality Principle

Remark 2.13. As a rule of thumb, in category theory, the dual statement of something adheres to the following rules:

(i) If $f \circ g$, then the dual statement is $g \circ f$

(ii) The domain of a morphism, $\mathrm{dom}(f)$, turns into $\mathrm{cod}(f)$

(iii) Likewise for the codomain of a morphism.

If Σ is some statement, then the dual statement is denoted as Σ^*.

For statement with an implication:

$$\Sigma \implies \Delta$$

The dual statement and implication is:

$$\Sigma^* \implies \Delta^*$$

However, see that the axioms in category are self-dual (i.e. they remain the same in either the dual category or the original one).

Proposition 74. ***Formal Duality***

For any statement Σ in category theory, if Σ follows from the axioms of categories, then so does Σ^*.

Remark 2.14. For example, if Σ is a statement that involves the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

Then Σ^* would involves the following diagram:

$$\begin{tikzcd}
A & B \arrow[l, "f"'] \\
& C \arrow[u, "g"'] \arrow[ul, "f \circ g"]
\end{tikzcd}$$

Therefore, if Σ were a statement in the category C, then Σ^* would be logically equivalent in the opposite category C^{op}.

Proposition 75. ***Conceptual Duality***

For any statement Σ about categories, if Σ holds for all categories, then so does the dual statement Σ^*:

$$\Sigma \implies \Sigma^*$$

Proof. It's obvious that Σ holding for all categories C means that it holds in all categories C^{op}. But Σ^* holds in all C^{op} and in all $(C^{op})^{op}$. Hence, in all categories C. $\square$

2.2.2 Coproducts

We will give a more appropriate definition of a coproduct.

Definition 2.13. Definition of the Product Again

In a category C, a **product** of two objects A and B is the diagram:

$$A \xleftarrow[p_1]{} P \xrightarrow{p_2} B$$

such that for a diagram:

$$A \xleftarrow[z_1]{} Z \xrightarrow{z_2} B$$

$\exists! u : Z \to P$ so that $p_i \circ u = z_i$. This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
 & Z \arrow[dl, "z_1"'] \arrow[d, "\exists! u"'] \arrow[dr, "z_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

Definition 2.14. Coproduct

Now, let us take the above ideas in the previous section, and apply it to the definition of the product.

A **coproduct** is a diagram:

$$A \xrightarrow{q_1} Q \xleftarrow[q_2]{} B$$

such that for a diagram:

$$A \xrightarrow{z_1} Z \xleftarrow[z_2]{} B$$

$\exists! u : Q \to Z$ so that $u \circ q_i = z_i$. This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "z_1"] \arrow[r, "q_1"] & Q \arrow[u, "\exists! u"] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"]
\end{tikzcd}$$

The typical notation for the coproduct is $A+B$ (as opposed to $A \times B$). We denote the unique morphism u as $[z_1, z_2]$ (following the above diagram's notation). We call i_1 and i_2 the inclusions, or coprojections[9].

Therefore, we have come full circle. Where as in **Section 1.11**, we defined coproducts and products as separate entities, we see now that the coproduct is quite literally the dual notion in the opposite category.

Example 2.2. Coproduct in the Category of Sets

In **Sets**, the coproduct $A + B$ is the disjoint union of two sets:

$$A + B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\}$$

Where the coprojections i_1, i_2 are:

$$i_1(a) = (a, 1) \qquad i_2(b) = (b, 2)$$

We define the unique morphism u by:

$$u := [f_1, f_2](x, \delta)$$

Where $[f_1, f_2](x, \delta) = f_1(x)$ if $\delta = 1$ and $g_1(x)$ if $\delta = 2$. We may easily check that:

$$u \circ i_1 = f_1 \qquad u \circ i_2 = f_2$$

Proposition 76. *Every finite object in* **Sets** *(a finite set) is a coproduct.*

Proof. This is actually the definition of the cardinal numbers.

For $n = \mathrm{Card}(A)$, we have:

$$A \cong 1 + 1 + \cdots + 1 \quad (\mathrm{n-times})$$

Let $f : A \to Z$ be a function:

$$f : a \longmapsto a_n$$

Therefore, f is a bijection establishing the isomorphism:

$$A \cong \{a_1\} + \{a_2\} + \cdots + \{a_n\}$$

Where each $\{a_k\} \cong 1$ itself.

We see that we can define cardinal numbers this way as any singleton can be defined as 1 (up to a natural isomorphism of course), so the cardinality of a finite object in Sets is exactly the definition of a cardinal number. ☐

[9] I have personally never used this term before but it is cool to slap on "co-" to the beginning of everything.

Example 2.3. Coproducts in the Category of Monoids

We can consider coproducts in **Mon** by considering the coproduct of their free monoids generated by (sets) A and B.

$$M(A) + M(B) \cong M(A + B)$$

This easily follows from the previous proposition about the coproduct of sets.

$$\begin{tikzcd}
& N & \\
M(A) \arrow[ur, "f_A"] \arrow[r, "h_A"'] & M(A+B) \arrow[u, "\exists! u"] & M(B) \arrow[ul, "f_B"'] \arrow[l, "h_B"] \\
A \arrow[u, "\eta_A"'] \arrow[r, "j_1"'] & A+B \arrow[u, "\eta_{A+B}"] & B \arrow[u, "\eta_B"] \arrow[l, "j_2"]
\end{tikzcd}$$

Note that j_1 and j_2 are the coprojections on the coproduct $A + B$. η_A, η_B, and η_{A+B} are all the inclusion maps (recall that this is the inclusion of generators described in **Definition 2.1**). By commutativity of each square, h_A and h_B must necessarily exist. And by the **universal property of the coproduct**, given N and morphisms f_A and f_B, $\exists! u : M(A + B) \to N$ so that the "roof" of this diagram commutes. This is the same exact universal property of $M(A) + M(B)$, hence, $M(A + B) \cong M(A) + M(B)$.

Remark 2.15. Note that the free monoid functor M : **Sets** $\to$ **Mon**, preserves coproducts. Keep this in mind as we talk about adjunctions later.

Example 2.4. Coproducts in the Category of Topological Spaces

For two topological spaces in **Top**, their coproduct $A + B$ is given by:

$$A + B = A \coprod B$$

Where the topology (the set of open sets of a topological space, A, $O(A)$) is:

$$O(A + B) \cong O(A) \times O(B)$$

Hence, the coproduct of two topological spaces inherits the product topology, but the morphisms of the product in the opposite category $\textbf{Top}^{op}$.

Example 2.5. Coproducts in the Category of Monoids (cont.)

Now consider two monoids A and B. The coproduct $A + B$ can be described as:

$$A + B = M(|A| + |B|)/ \sim$$

Where $\sim$ is an equivalence relation, and $|A|$ denotes the underlying set of the monoid A.

We describe the equivalence relation $v \sim w$ as the least one containing the following expressions:

$$u_A = (-) = u_B$$

(where u_A denotes the unit in A, and $(-)$ the unit in $M(|A| + |B|)$)

$$(\dots aa' \dots) = (\dots a \cdot a' \dots)$$

$$(\dots bb' \dots) = (\dots b \cdot b' \dots)$$

Where $\cdot$ denotes the multiplication of equivalence classes of the element (a is in an equivalence class $[a]$, for example).

We have equivalence class multiplication too:

$$[x \dots y] \cdot [x' \dots y'] = [x \dots yx' \dots y']$$

The coprojections $i_A : A \to A + B$ and $i_B : B \to A + B$ are defined by:

$$i_A(a) = [a] \qquad i_B(b) = [b]$$

Given some morphism $f_A : A \to M$ and $f_B : B \to M$, and a monoid M, we have the unique morphism:

$$u = A + B \to M$$

where we define u as follows:

Draw the following commutative diagram:

$$\begin{tikzcd}
{|A|} \arrow[r, "{|i_A|}"] \arrow[d, "\eta_A"] & {|A| + |B|} \arrow[d, "\eta_{A+B}"] & {|B|} \arrow[l, "{|i_B|}"'] \arrow[d, "\eta_B"] \\
M(|A|) \arrow[r, "h_A"] \arrow[ddr, "g_A"'] & M(|A| + |B|) \arrow[d, "\pi_\sim"] & M(|B|) \arrow[l, "h_B"'] \arrow[ddl, "g_B"] \\
 & \frac{M(|A|+|B|)}{\sim} \arrow[d, "\exists ! h"] & \\
 & M &
\end{tikzcd}$$

To ease confusion, we can consider this commutative diagram by considering the coproduct on the free monoid (**Example 2.3**), by seeing that $f_A : A \to M$ and

$f_B : B \to M$ are restrictions of the g_A and g_B indicated above (i.e. $f_A = g_A|_A$ and $f_B = g_B|_B$). In fact, we can illustrate this notion:

$$\begin{tikzcd}
{|A|} \arrow[r, "{|i_A|}"] \arrow[d, "\eta_A"] & {|A| + |B|} \arrow[d, "\eta_{A+B}"] & {|B|} \arrow[l, "{|i_B|}"'] \arrow[d, "\eta_B"] \\
M(A) \arrow[r, "h_A"] \arrow[d, "\pi_A"] & M(|A| + |B|) \arrow[d, "\pi_\sim"] & M(B) \arrow[l, "h_B"'] \arrow[d, "\pi_B"] \\
\frac{M(|A|)}{\sim} \cong A \arrow[r, "i_A"] \arrow[dr, "f_A"'] & \frac{M(|A|+|B|)}{\sim} \arrow[d, "\exists! h"] & \frac{M(|B|)}{\sim} \cong B \arrow[l, "i_B"'] \arrow[dl, "f_B"] \\
& M &
\end{tikzcd}$$

To unpack what this diagram says, we start with the underlying sets of the monoids A and B. Then we can map into the coproduct of the underlying sets of these monoids. Then due to the **universal property of free monoids** (**Definition 2.3**), we can map by **inclusion of generators** into their respective free monoids ($\eta_{A,B}$, η_{A+B}). Then, we can apply the **canonical projection into the quotient** of the free monoids. The equivalence relation $\sim$ is as defined above. This will give us every element of A as a monoid. Likewise with B. Then we may safely map this with f_A (f_B likewise). Since every map here exists (the **quotient maps are, in fact, universal**!), this automatically means that $h : M(|A| + |B|)/\sim \to M$ must uniquely exist. Furthermore, $h \circ i_{A,B} = f_{A,B}$.

Example 2.6. Coproducts in the Category of Groups

In the category **Grp** (category of groups), we denote the coproduct of groups A and B as the **free product** of A and B, denoted $A \oplus B$.

For abelian groups, we can define the coproduct through a universal property, but there is an even deeper way to define it.

Consider that the free product in $A \oplus B$ must be forced to satisfy the commutativity condition:

$$(a_1 b_1 b_2 a_2 \dots) \sim (a_1 a_2 \dots b_1 b_2 \dots)$$

Therefore, we can move all a's to the front, and b's to the back. Furthermore, we have:

$$(a_1 a_2 \dots b_1 b_2 \dots) \sim (a_1 + a_2 + \cdots, b_1 + b_2 + \cdots)$$

Therefore, we can think of all elements in abelian groups as being pairs of a's and b's, (a, b). We can take:

$$|A \times B| = |A + B|$$

As inclusion maps (the coprojections) we use:

$$i_A(a) = (a, 0_B) \qquad i_B(b) = (0_A, b)$$

And then, for any group X with the following diagram:

$$A \xrightarrow{\quad f \quad} X \xleftarrow{\quad g \quad} B$$

we let the unique morphism $u := [f, g] : A + B \to X$ be:

$$u(a, b) = f(a) +_X +g(b)$$

For a similar example, look at **Remark 1.41**.[10]

We can also work it out quickly. We need to show that:

$$f = u \circ i_A \qquad g = u \circ i_B$$

We immediately see that:

$$u \circ i_A(a) = u(a, 0) = f(a) \qquad u \circ i_B(b) = u(0, b) = g(b)$$

Hence, we immediately verify that $A \oplus B/ \sim \cong A + B$ (the direct sum).

We now happen upon a very important theorem about abelian groups, and one that characterizes **abelian categories**.

Proposition 77. *In the category* **Ab***, of abelian groups, there is a canonical isomorphism between the binary coproduct, and product.*

$$A \times B \cong A + B$$

Proof. Let us define morphisms as follows:

Our goal is to define $\nu : A \times B \to A + B$. To do this, let us take the following:

$$h_A : A \to A \times B \qquad h_B : B \to A \times B$$

$$id_A : A \to A \qquad id_B : B \to B$$

$$0_B : A \to B \qquad 0_A : B \to A$$

[10] This exact remark was regarding the fibered coproduct over some object. If we choose the kernel in that example as being trivial, then we obtain our map immediately.

i.e. we need to draw the following diagram:

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "0_B"] & A \times B \arrow[d, "\nu"] & B \arrow[l, "h_B"'] \arrow[d, "0_A"] \\
B \arrow[r, "i_B"] & A + B & A \arrow[l, "i_A"']
\end{tikzcd}$$

Now we describe ν as follows:

$$\nu = [\langle id_A, 0_B\rangle, \langle 0_A, id_B\rangle] : A + B \to A \times B$$

We follow the diagram twice and create two maps: $\langle id_A, 0_B\rangle$ and $\langle 0_A, id_B\rangle$ and we add them because we are in the category of abelian groups.

$$\begin{aligned}
\nu(a, b) = [\langle id_A, 0_B\rangle, \langle 0_A, id_B\rangle]\,(a, b) &= \langle id_A, 0_B\rangle(a), \langle 0_A, id_B\rangle(b) \\
&= (id_A(a), 0_B(a)) + (0_A(b), id_B(b)) \\
&= (a, 0) + (0, b) = (a, b)
\end{aligned}$$

This is sufficient to prove the isomorphism as the other direct is almost identical (take the statement in $\mathbf{Ab}^{op}$ and see that we get the exact same statement!). $\square$

Remark 2.16. Note that the addition of morphisms that we used in the above proof is indicative of something called **Abelian Categories**. Formally, these are any categories whose underlying structure can be seen as that of an abelian group, such as modules over a commutative ring R ($\mathbf{Mod}_R$), vector spaces over a field k ($\mathbf{Vec}_k$) , algebras over a commutative ring R ($\mathbf{Alg}_R$), etc.)

Proposition 78. *Coproducts are unique up to isomorphism.*

Proof. Take the proof in the opposite category, and notice that even if we reverse all the morphisms, the most difference we get is a switching of the order of the factors in the coproduct. Switching factors is a trivial isomorphism (since we do not care about orientation). $\square$

Proposition 79. *The Binary coproducts are associative.*

$$(A + B) + C \cong A + (B + C)$$

Proof. This follows from taking a coproduct, shifting to its dual, and repeating with a factor of both $A + B$ and/or $B + C$. $\square$

2.2.3 Equalizers

Definition 2.15. In a category C, given the arrows

$$\begin{tikzcd}
A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B
\end{tikzcd}$$

an **equalizer** of f and g consists of a universal pair (E, e), consisting of an object E and a morphism $e : E \to A$ so that $f \circ e = g \circ e$.

Definition 2.16. Universal Property of the Equalizer

Let us be in a category C. Given an arbitrary object Z in C, and a morphism $z : Z \to A$ so that $f \circ z = g \circ z$, $\exists! u : Z \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \\ Z \arrow[u, "\exists! u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

i.e. such that $z = e \circ u$

Example 2.7. Equalizer in the Category of Sets

In **Sets**, the equalizer of the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

is the **inclusion of the following subset** into A:

$$i : E := \{x \in A \mid f(x) = g(x)\} \hookrightarrow A$$

This is easy to verify as we may take an arbitrary set Z and an arbitrary map $z : Z \to A$. Then we see that for any $y \in Z$:

$$E = \{y \in Z \mid f(z(y)) = g(z(y))\}$$

Hence, $x = z(y)$ for any $x \in A$ and any $y \in Z$. Then we take a function:

$$u : Z \to E$$

$$u = i^{-1} \circ z$$

Note, that we can do this as the inclusion is injective map of sets, hence it is monic. This uniquely determines u and it proves that $z = i \circ u$ (note that i is right-cancellable as the equalizer is epi by default). This proves the universal property desired as the equalizer object, E, is the same whether we describe it through $f \circ i \circ u = g \circ i \circ u$ or $f \circ z = g \circ z$.

Remark 2.17. We call subsets of the form E above, an **equational subset** (it has an element described as satisfying some relations of functions). More specifically, we can denote an "equational subset" as a **variety**.

Proposition 80. *Any subset $U \subseteq A$ is a variety (takes this equational form).*

Proof. Let $2 = \{\uparrow, \downarrow\}$. Consider the characteristic function:

$$\chi_U : A \to 2$$

$$\chi_U = \begin{cases} \downarrow & x \in U \\ \uparrow & x \notin U \end{cases}$$

So $U = \{x \in A \mid \chi_U = \downarrow\}$

The following is an equalizer:

$$\begin{tikzcd} U \arrow[r, "f"] & A \arrow[r, shift left, "\chi_U"] \arrow[r, shift right, "\downarrow"'] & 2 \end{tikzcd}$$

Where $\downarrow f : U \to 1 \to 2$. For any morphism of sets:

$$\phi : A \to 2$$

We may form the variety:

$$V_\phi = \{x \in A \mid \phi(x) = \downarrow\}$$

which is an equalizer.

If we use the case where $\chi_U = \phi$, we obtain that:

$$V_{\chi_U} = \{x \in A \mid \chi_U(x) = \downarrow\} = \{x \in A \mid x \in U\} = U$$

And likewise, $\phi : A \to 2$ results in:

$$\chi_{V_\phi}(x) = \begin{cases} \downarrow & x \in V_\phi \\ \uparrow & x \notin V_\phi \end{cases} = \begin{cases} \downarrow & \phi(x) = \downarrow \\ \uparrow & \phi(x) = \uparrow \end{cases} = \phi(x)$$

So, we have established an isomorphism:

$$\mathrm{Hom}(A, 2) \cong P(A)$$

Where $P(A)$ denotes the set of all subsets of A. □

Proposition 81. *In any category, if $e : E \to A$ is an equalizer of some pair of morphisms, then e is monic.*

Proof.

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \\ Z \arrow[u, shift left, "y"] \arrow[u, shift right, "x"'] \arrow[ur, "z"] & & \end{tikzcd}$$

Consider the above diagram, assume e is the equalizer of f and g. Suppose $ex = ey$. Set $z = ex = ey$. Then, we have that

$$fz = fex = gex = gz$$

By the universal property of the equalizer, $\exists! u : Z \to E$ so that $eu = z$. As $z = ex = ey = eu$, by uniqueness of u, $x = u = y$. Hence, $ex = ey$ implies $x = y$, hence, e is monic. □

Remark 2.18. Oftentimes, we will denote the equalizer of a pair of morphisms, corresponding to an object A, as:

$$A(f = g)$$

Example 2.8. Consider the category **Ab**. Consider the equalizer for a pair of (abelian) group morphisms $f, g : A \to B$:

$$E = \{x \in A \mid f(x) = g(x)\}$$

By properties of group morphisms, we obtain:

$$f(x) = g(x) \implies (f - g)(x) = 0$$

Therefore, the equalizer of f and g is equivalent to the equalizer:

$$\begin{tikzcd} E \arrow[r] & A \arrow[r, shift left, "0"] \arrow[r, shift right, "f-g"'] & B \end{tikzcd}$$

Therefore, we can consider equalizers of the form:

$$E = \{x \in A \mid h(x) = 0\}$$

This is, in fact, a subgroup of A and it is the **kernel** of the morphism h. Thus, we have a categorical definition of the kernel:

$$\begin{tikzcd} \mathrm{Ker}(f-g) \arrow[r, hook, "i"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

2.2.4 Coequalizers

A coequalizer is a generalization of a quotient object (quotient group, quotienting by relations, etc.). We talk about the act of "modding out" by an equivalence relation here.

Definition 2.17. A quick primer (I assume that you know what an equivalence relation is), assume that we are in some category C. Then we have an object X, and take an element of that object X, x. We define the **equivalence class** of $x \in X$, $[x]$, by:

$$[x] = \{y \in X \mid x \sim y\}$$

Where $\sim$ is some equivalence relation on X. It is a provable (but it is unnecessary to do so here) fact that all equivalence classes of distinct ("distinct" meaning $x \not\sim y$) elements are *disjoint*.

Definition 2.18. Let X be an object in a category C. We call the **quotient** of X by $\sim$ as the set of all equivalence classes:

$$X/\sim = \{[x] \mid x \in X\}$$

The differences between all $x \sim y$ are reduced to identity, leaving only distinct equivalence classes remaining.

Definition 2.19. Universal Property of the Coequalizer

For any pair of morphisms $f, g : A \to B$ in a category C, a coequalizer is a universal pair (Q, q), where Q is the object (the coequalizer) and q is the canonical map, so that $qf = qg$.

Furthermore, it has the following universal property:

For a pair of morphisms $f, g : A \to B$, given an arbitrary object, Z, in C, and a morphism $z : B \to Z$, $\exists! u : Q \to Z$ so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, "\exists! u"] \\ & & Z \end{tikzcd}$$

i.e. $\exists! u : Q \to Z$ such that $uq = z$.

Note: This is just all the morphisms in the diagram for the equalizer reversed, hence, we may consider the coequalizer in C as the equalizer in C^{op}.

Proposition 82. *If $q : B \to Q$ is a coequalizer for a pair of morphisms, then q is epic.*

Proof. There are two possible proofs for this:

1. **Using Duality**:

   Recognize that the proposition before **Example 2.8** above was a statement about the equalizer of a pair of morphisms in C being monic. Thus, in C^{op}, this clearly becomes a statement about the coequalizer for the same pair of morphisms being epic.

2. **Direct Proof Using the Universal Property of Coequalizers**:

   Consider the following diagram (also given above):

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, shift right, "x"'] \arrow[d, shift left, "y"] \\ & & Z \end{tikzcd}$$

   Suppose that $xq = yq$. Then, by commutativity, of the diagram, we may set $z = xq = yq$. By the **Universal Property of Coequalizers**, $\exists! u : Q \to Z$ so that $uq = z$. Therefore, $xq = uq = yq$. By uniqueness of u, we can conclude that $x = u = y$. Therefore, $xq = yq$ implies that $x = y$. Hence q is epic.

□

Example 2.9. Coequalizer in the Category of Sets

Let $R \subseteq X \times X$ be an equivalence relation on a set X, and consider the diagram:

$$\begin{tikzcd} R \arrow[r, shift left, "r_2"] \arrow[r, shift right, "r_1"'] & X \end{tikzcd}$$

Where the r_k is defined in the following diagram:

$$\begin{tikzcd} R \arrow[r, hook, "i"] \arrow[dr, "r_k"] & X \times X \arrow[d, "p_k"] \\ & X \end{tikzcd}$$

We define the **canonical projection**:

$$\pi : X \to X/R$$

$$\pi : x \longmapsto [x]$$

π **is a coequalizer of** r_1**,** r_2 as follows.

Given an $f : X \to Y$, $\exists! \bar{f} : X/R \to Y$ so that the following commutes:

$$\begin{tikzcd} X \times X \arrow[dr, shift left, "p_2"] \arrow[dr, shift right, "p_1"'] & & \\ R \arrow[u, hook, "i"] \arrow[r, shift left, "r_2"] \arrow[r, shift right, "r_1"'] & X \arrow[r, "\pi"] \arrow[dr, "f"] & X/R \arrow[d, "\bar{f}"] \\ & & Y \end{tikzcd}$$

We can trivially verify this by setting

$$\bar{f}([x]) = f(x) \qquad \forall x \in X$$

We see that $(x, x') \in R \implies f(x) = f(x')$. This is because $fr_1 = fr_2$ (by the diagram before this diagram), and since $fr_1(x, x') = f(x)$ and $fr_2(x, x') = f(x')$. Hence, our choice of $\bar{f}$ preserves the equivalence relation, as necessary. This choice of $\bar{f}$ is unique as $f = \bar{f}\pi$, and π is an epimorphism, hence has a right inverse and $\bar{f} = f\pi^{-1}$.

Example 2.10. Coequalizer in the Category of Sets (cont.)

In **Example 2.9**, we constructed the coequalizer of projections from an equivalence relation, R, into our set X, which resulted in the universal object X/R.

We can pursue a more general description of the coequalizer in **Sets**, in which we consider the coequalizer of an arbitrary pair of morphisms $f, g : A \to B$. We do this as follows:

By the **universal property of the coequalizer**, for a pair of morphisms $f, g : A \to B$, given an arbitrary object Z, and a morphism $h : B \to Z$, $\exists! u : B/\sim \to Z$ so that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \arrow[r, "\pi"] \arrow[dr, "h"'] & B/\sim \arrow[d, "\exists! u"] \\
& & Z
\end{tikzcd}$$

To construct this u, let us consider a *sensible* equivalence relation $\sim$ defined as follows:

It is the least equivalence relation such that $f(x) \sim g(x) \quad \forall x \in A$. Therefore, $b \in [b]$ if and only if $b = f(x) = g(x)$.

Once again, define $u([b]) = h(b)$. Clearly, this preserves the commutativity of the diagram ($h = u\pi$), and because π is the coequalizer of a pair of morphisms f, g, it is epi. Hence u is uniquely defined, $u = h\pi^{-1}$. We only have to check that it preserves the equivalence relation described above.

Assume $f(x) \sim g(x)$ so that $f(x) = g(x)$. Then $\pi f = \pi g \quad \forall x \in A$. And because the choice of u is unique, it follows that $u\pi f = u\pi g$, using the explicit form of u, it follows that $hf = hg$, and hence, it preserves the equivalence relation, meaning that our maps are well-defined and that our equivalence is suitable.

Example 2.11. Using Coequalizers to deduce Finitely Presented Algebras

We shall apply the ideas of **Example 2.10** here. Let us consider a category of algebras (like monoids or groups) which has free algebras and coequalizers for any parallel pairs of morphisms. We can speak of presenting algebras through *generators and relations.*

Suppose we are given three generators x, y, z, and the relations: $xy = z \quad y^2 = 1$.

To build an algebra that satisfies these generators and relations, first start with the free algebra on the 3 generators:

$$F(3) = F(x, y, z)$$

Let us now impose the relations through the coequalizer as follows:

$$\begin{tikzcd}
F(1) \arrow[r, shift left, "z"] \arrow[r, shift right, "xy"'] & F(3) \arrow[r, "q"] & Q
\end{tikzcd}$$

We consider maps $F(1) \to F(3)$ as "generalized elements" where the maps are the elements of $F(3)$ acting on the single generator of $F(1)$, i.e. a map of the form $v \longmapsto a$ for v generating $F(1)$ and a an element in $F(3)$.

Furthermore, we take another coequalizer for the next relation $y^2 = 1$:

$$\begin{tikzcd} F(1) \arrow[r, shift left, "q(1)"] \arrow[r, shift right, "q(y^2)"'] & Q & & Q' \end{tikzcd}$$

Note that, while we did these separately, we can actually do these simultaneously:

Note that by **Example 2.3**, we see that:

$$F(2) = F(1) + F(1)$$

And we can construct a coequalizer of $F(2)$:

$$\begin{tikzcd} F(2) \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & F(3) \end{tikzcd}$$

Where $f = [xy, y^2]$ and $g = [z, 1]$. If we break it down into the individual coequalizers we constructed above, we get something which really looks like this:

$$\begin{tikzcd} F(1) + F(1) \arrow[r, shift left, "z"] \arrow[r, shift right, "xy"'] \arrow[d, shift right, "y^2"'] \arrow[d, shift left, "1"] & F(3) \\ F(3) \arrow[ur, "id_{F(3)}"'] & \end{tikzcd}$$

We can now take the coequalizer of f and g above to be the morphism:

$$q : F(3) \to Q$$

$$q(xy) = q(z) \qquad q(y)^2 = q(1) = 1$$

Now, by the **Universal Property of the Coequalizer** (**Definition 2.19**), if we take some arbitrary object A with some morphism $h : F(3) \to A$, we have that $\exists! u : Q \to A$ so that $h = uq$.

Given any three elements $a, b, c \in A$ such that $ab = c$ and $b^2 = 1$, take the morphism u:

$$u(x) = a \qquad u(y) = b \qquad u(z) = c \qquad x, y, z \in Q$$

This morphism is unique, and because A was arbitrary, this ensures that Q is the universal algebra such that these relations hold (as the morphisms preserve the relations in both the image and the domain, for example, since $q(x)q(y) = q(z)$, the factorization by the Universal Property tells us that $u(q(x)q(y)) = u(q(z)) \Longrightarrow ab = c$, by the property of homomorphisms).

Therefore, the resulting free algebra that satisfies the relations we originally imposed is given by:

$$Q \cong F(x, y, z)/(xy = z,\ y^2 = 1)$$

Where we think of the canonical projection into Q from $F(x, y, z)$ as projecting the generators to the equivalence classes of the original relations.

We can describe the full extent of our construction above as the following:

$$F(2) \underset{f}{\overset{g}{\rightrightarrows}} F(3) \xrightarrow{\quad q \quad} Q \cong F(3)/(f = g)$$

Where $f = g \implies xy = z\ , y^2 = 1$.

This is essentially the same idea with a more general idea:

Given

$$\text{Generators}: \ g_1, \ldots, g_m$$

$$\text{Relations}: \ l_1 = r_1, \ldots, l_m = r_m$$

As a short hand, we define the the tuples $l = (l_1, \ldots, l_m)$ and $r = (r_1, \ldots, r_m)$, so the shorthand for the relations is $l = r$.

The algebra (more specifically, the **universal object in a monoidal category** with a base structure of a monoid) with the presentation that has the above generators and relations is exactly the **coequalizer of the free algebra on** m **generators**:

$$F(m) \underset{l}{\overset{r}{\rightrightarrows}} F(n) \xrightarrow{\quad q \quad} Q \cong F(n)/(l = r)$$

Where by $F(m)$, we mean the free algebra generated by m elements, also denoted as $F(x_1, \ldots, x_m)$, likewise with $F(n)$.

These free algebras with a finite number of generators and relations are called **finitely presented**.

Remark 2.19. One important point to make, these presentations are **not unique!** We can have multiple different presentations of the same algebra, as long as the relations that we use to impose our equivalences result in the same elements, i.e. for $l = r$ and $l' = r'$, $n \neq n'$, where $l \neq l'$ and $r \neq r'$, we may have:

$$F(n)/(l = r) \cong F(n')/(l' = r')$$

In fact, it is easy enough to purposefully create more generators and impose the most trivial relations on them:

For $F(n)/(l = r)$, add in another generator, g_{n+1}, and impose the relation $g_n = g_{n+1}$. This is a completely trivial relation, but the presentations are NOT the same.

Remark 2.20. Let G be a set of generators and R be a set of relations. Any set of each will give rise to some algebra (the possibilities are quite literally endless). G and R need not be finite either! We used a finitely presented algebra in the preceding examples, but there are no logical dependencies that follow from having a finitely presented algebra. The coequalizer will look the same[11]:

$$F(R) \underset{r_1}{\overset{r_2}{\rightrightarrows}} F(G) \longrightarrow F(G)/(r_1 = r_2)$$

i.e. *any algebra may be presented through a coequalizer of relations and a free algebra generated by a set of generators.*

The following proposition is for monoids for the sake of generality, but it holds for any monoidal category in which the base substructure is a monoid.

Proposition 83. *For all objects M in* **Mon**, *there are sets G and R (generators and relations) as a coequalizer:*

$$F(R) \underset{r_1}{\overset{r_2}{\rightrightarrows}} F(G) \longrightarrow M$$

with $F(G)$ and $F(R)$ free. And therefore, we see that:

$$M \cong F(G)/(r_1 = r_2)$$

Proof. Let N be a monoid, and denote the free monoid functor as:

$$TN = M(|N|)$$

where M acts on the underlying set of N.

Now we denote the homomorphism:

$$\pi : TN \to N$$

$$\pi : (x_1, \dots, x_n) \longmapsto x_1 \cdot \dots \cdot x_n$$

Apply this construction twice for a monoid M.

We obtain the following diagram:

$$TTM \underset{\epsilon}{\overset{\mu}{\rightrightarrows}} TM \xrightarrow{\pi} M$$

We denote $TTM = T^2M$. $\mu = T\pi$. This is a coequalizer of monoids. To see this, we simply have to verify that this has the universal property.

[11] Note that $F(R) \cong F(\mathrm{Card}(R))$, so it is a free algebra with as many generators as there are relations in R.

Assume an arbitrary monoid N, and a morphism $h : TM \to N$ with $h\epsilon = h\mu$, i.e. the following diagram:

$$\begin{tikzcd}
TTM \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\epsilon"'] & TM \arrow[r, "\pi"] \arrow[dr, "h"] & M \arrow[d, "\exists! u"] \\
& & N
\end{tikzcd}$$

Denote the multiplication in TM as $*$ and the one in M as $\cdot$. We have the following morphisms:

$$\epsilon : (x_1, \ldots, x_n), \ldots, (z_1, \ldots, z_m) \longmapsto (x_1 * \cdots * x_n, \ldots, z_1 * \cdots * z_m)$$

$$\mu : (x_1, \ldots, x_n), \ldots, (z_1, \ldots, z_m) \longmapsto (x_1 \cdot \cdots \cdot x_n, \ldots, z_1 \cdot \cdots \cdot z_m)$$

For any tuple $(x, \ldots, z) \in TM$, since $h\epsilon = h\mu$, we have:

$$h(x * \cdots * z) = h\epsilon((x, \ldots, z)) = h\mu((x, \ldots, z)) = h(x \cdot \cdots \cdot z)$$

Define $\bar{h} = h \circ i$ where i is the *inclusion of generators* $|M| \to |TM|$:

$$\begin{tikzcd}
TTM \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\epsilon"'] & TM \arrow[r, shift right, "\pi"'] \arrow[dr, "h"'] & M \arrow[l, shift right, "i"'] \arrow[d, "\exists! \bar{h}"] \\
& & N
\end{tikzcd}$$

$$\bar{h}\pi(x * \cdots * z) = hi\pi(x, \ldots, z) = hi(x \cdot\ \ldots\ z)$$

Note that if we map $(x \cdot\ \ldots\ \cdot z)$ back into TM by inclusion, it will remain unchanged:

$$hi(x \cdot\ \ldots\ z) = h(x \cdot\ \ldots\ \cdot z)$$

Since $h\epsilon = h\mu$, we obtain that:

$$h(x \cdot\ \ldots\ \cdot z) = h(x * \cdots * z)$$

It is almost trivial that $\bar{h}$ is a homomorphism, as we define the binary operations $*$ and $\cdot$ component-wise. If we start in M with a binary operation of the form:

$$(x \cdot\ \ldots\ \cdot z) \cdot (x' \cdot\ \ldots\ \cdot z')$$

then, pulling this back to TM through the inclusion of generators, $i : M \to TM$, gives us

$$(x \cdot\ \ldots\ \cdot z) * (x' \cdot\ \ldots\ \cdot z')$$

Which, due to the condition that $h\epsilon = h\mu$, gives us:

$$(x * \cdots * z) * (x' * \cdots * z')$$

Thus, as h is a morphism of monoids, it immediately follows that $\hbar$ is a morphism of monoids. $\square$

2.3 Groups and Categories

We lay out the connections between groups and the notions we have developed.

2.3.1 Groups in a Category

A group G consists of certain morphisms $g : X \to X$ for an object X in a category C.

$$G \subseteq \mathrm{Hom}_C(X, X)$$

Let C be a category with products. We generalize a notion of a group in a **Sets**.

Definition 2.20. A **group** in C consists of objects and morphisms as such:

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
 & 1 \arrow[u, "u"] &
\end{tikzcd}$$

We note that m is the "binary operation", u is the "unit" (identity) element, and i is the "inverse" element. These satisfy the following conditions:

1. m **is associative**, i.e. the following diagram commutes:

$$\begin{tikzcd}
(G \times G) \times G \arrow[rr, "\cong"] \arrow[d, "m\times 1"'] & & G \times (G \times G) \arrow[d, "1\times m"] \\
G \times G \arrow[dr, "m"] & & G \times G \arrow[dl, "m"'] \\
 & G &
\end{tikzcd}$$

   where $\cong$ denotes the canonical isomorphism of associativity for products

2. u is a unit for m

$$\begin{tikzcd}
G \arrow[r, "\langle u, id_G \rangle"] \arrow[d, "\langle id_G, u \rangle"'] \arrow[dr, "id_G"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

   Where we write u for the constant morphism

$$uf : G \xrightarrow{f} 1 \xrightarrow{u} G$$

3. i is an inverse with respect to m, so that the following diagram commutes:

$$\begin{tikzcd}
G \times G \arrow[d, "id_G \times i"'] & G \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "u"] & G \times G \arrow[d, "i \times id_G"] \\
G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"]
\end{tikzcd}$$

where $\Delta = \langle id_G, id_G \rangle$

Requirement 3 is the following:

$$m(m(x, y), z) = m(x, m(y, z))$$

$$m(x, u) = x = m(u, x)$$

$$m(x, ix) = u = m(ix, x)$$

Definition 2.21. A homomorphism $h : G \to H$ of groups in C consists of morphisms in C.

$$h : G \to H$$

which meet the following conditions:

1. h preserves m (i.e. the following commutes):

$$\begin{tikzcd}
G \times G \arrow[r, "h \times h"] \arrow[d, "m"] & H \times H \arrow[d, "m"] \\
G \arrow[r, "h"] & H
\end{tikzcd}$$

2. h preserves u (the diagram commutes):

$$\begin{tikzcd}
G \arrow[r, "h"] & H \\
1 \arrow[u, "u"] \arrow[ur, "u"] &
\end{tikzcd}$$

3. h preserves i (the diagram commutes):

$$\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, "i"] & H \arrow[d, "i"] \\
G \arrow[r, "h"] & H
\end{tikzcd}$$

We have a category of groups about C, denoted as Group(C).

2.3.2 The Category of Groups

Here, we will describe all the concepts and results we've discussed, in the category of groups, **Grp**, i.e. Group (**Sets**).

Let G and H be objects in **Grp**, and let $h : G \to H$ be a morphism.

Definition 2.22. The **kernel** of h is defined as:

$$\mathrm{Ker}(h) \xhookrightarrow{\quad i \quad} G \underset{h}{\overset{u}{\rightrightarrows}} H$$

Where $\mathrm{Ker}(h) = \{g \in G \mid h(g) = u\}$ (where u denotes the constant morphism [12]). The constant morphism is analogous to the 0-map and is defined as:

$$u := G \xrightarrow{c} 1 \xrightarrow{u} H$$

Recall from **Example 2.8** that the inclusion i and $\mathrm{Ker}(h)$ **forms an equalizer** of the morphisms u and h and the object G.

Proposition 84. $\mathrm{Ker}(h)$ *is a normal subgroup.*

Proof. This is **Proposition 10**. ☐

Remark 2.21. If $N \hookrightarrow G$ is an arbitrary **normal** subgroup of G, then we may construct a **coequalizer** as follows:

$$N \underset{i}{\overset{u}{\rightrightarrows}} G \xrightarrow{\quad \pi \quad} G/N$$

Where $g \in G$ is mapped to u (the constant map) if and only if $g \in N$. We define the equivalence relation $\sim$ as follows:

$$g \sim h \text{ iff } gh^{-1} \in N$$

It is trivial that this is an equivalence relation.

We denote $G/\sim$ as G/N, and G/N is the set of all equivalence classes resulting from our equivalence relation $\sim$. We denote the equivalence classes of an element g in G/N as $[g]$ (it is also denoted commonly as gN). The binary operation of the equivalence classes in G/N is defined as:

$$[g][g'] = [gg']$$

Hence, the binary operation on G/N is induced from the binary operation on G.

We may easily show that the following diagram:

$$N \underset{i}{\overset{u}{\rightrightarrows}} G \xrightarrow{\quad \pi \quad} G/N$$

is a coequalizer.

[12]We switch from e to u for the unit element because the previous discussions of the group over a category used the u as the unit morphism, and because e will be used to denote epimorphisms.

Proposition 85. *$(G/N, \pi)$ in the following diagram:*

$$\begin{tikzcd} N \arrow[r, shift left, "u"] \arrow[r, shift right, "i"'] & G \arrow[r, "\pi"] & G/N \end{tikzcd}$$

is a coequalizer.

Proof. We must simply verify that it has the universal property of a coequalizer. Given an arbitrary object H, and a morphism $h : G \to H$, we construct the following diagram:

$$\begin{tikzcd} N \arrow[r, shift left, "u"] \arrow[r, shift right, "i"'] & G \arrow[r, "\pi"] \arrow[dr, "h"'] & G/N \arrow[d, "\exists! \bar{h}"] \\ & & H \end{tikzcd}$$

We define $\bar{h}(gN) = h(g)$, which, naively, makes this diagram commute. To check that it makes the equivalence relation hold, suppose that $h : G \to H$ reduces N to the kernel of h,i.e.

$$\forall n \in N \quad h(n) = u$$

Then it is clear that our choice of $\bar{h}$ makes the diagram commute (in principle) AND preserves our equivalence relation as:

$$x \sim y \Longrightarrow h(x) = h(y)$$

since

$$xy^{-1} \in N \Longrightarrow h(xy^{-1}) = h(x)h(y)^{-1} = u \Longrightarrow h(x) = h(y)$$

We see that $\bar{h} = h\pi$ and it is unique because π is an epimorphism, hence $h = \bar{h}\pi^{-1}$. $\square$

Remark 2.22. This is the **First Isomorphism Theorem** for groups (**Theorem 17**).

Theorem 86. ***First Isomorphism Theorem (Revisited)***

Every group homomorphism $h : G \to H$ has a kernel $\mathrm{Ker}(h) = h^{-1}(u)$, which is a normal subgroup of G wiht the property that for any normal subgroup $N \subseteq G$:

$$N \subseteq \mathrm{Ker}(h)$$

if and only if $\exists! \bar{h} : G/N \to H$ such that $\bar{h}\pi = h$ as indicated by the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "\pi"'] & H \\ & G/N \arrow[u, "\bar{h}"'] \end{tikzcd}$$

Proof. The first direction is completely obvious, as that is what we proved by showing that $(G/N, \pi)$ is a coequalizer of $i, u : N \to G$.

Assume that $\bar{h}$ exists uniquely. $\pi(N) = \{u_G\}$. Hence, $h(n) = \bar{h}\pi(n) = \bar{h}(nN) = \bar{h}(u_G) = u_H$. Hence, $N \subseteq \mathrm{Ker}(h)$. $\square$

Corollary 87. *$\bar{h}$ is injective.*

Proof. Set $N = \mathrm{Ker}(h)$, and take any $xN, yN \in G/N$. We see that:

$$\bar{h}(xN) = \bar{h}(yN) \Longrightarrow h(x) = h(y) \Longrightarrow h(xy^{-1}) = u$$

so that

$$xy^{-1} \in \mathrm{Ker}(h) \Longrightarrow x \sim y \Longrightarrow xN = yN$$

Hence, if $\bar{h}(xN) = \bar{h}(yN)$, we get that $xN = yN$, hence, $\bar{h}$ is injective. $\square$

Corollary 88. *A group homomorphism h is injective if and only if* $\mathrm{Ker}(h) = \{u\}$.

Proof. We took this as a fact in the earlier sections, but it is quite clear here that this is the case. $\square$

2.3.3 Groups as Categories

We now speak about groups as categories themselves.

Definition 2.23. Groups as Categories

A group is a category with one object (we can call it the identity object but it is inconsequential what this is), denote it as $\{*\}$. Every morphism of this object is an isomorphism.

Under this definition of a group, we see that if G and H are two groups (as categories themselves), then the **functor** $f : G \to H$ is the exact same thing as a group homomorphism (this is trivial to see once you compare the axioms of a category and the properties of a functor).

Remark 2.23. Note that we do not have to map only between groups! We can map to any other category.

More formally, we can have a functor:

$$R : G \to C$$

Where G is a group (as described in **Definition 2.23**) and C is another category.

This functor R is called a representation of G in C.

Example 2.12. If we choose $R : G \to C$, and $C = \mathbf{Vec}_k$ (the category of vector spaces), then R is called a **linear representation** of G.

We will now introduce notions of kernels and factor groups and quotients, congruences of a category, analogously to groups in the previous section.

Definition 2.24. A **congruence on a category** C is an equivalence relation on morphisms of C, $f \sim g$ such that:

1. $f \sim g$ implies that $\mathrm{dom}(f) = \mathrm{g}$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$. We can depict this as the following:

$$\begin{tikzcd} * \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & * \end{tikzcd}$$

2. $f \sim g$ implies that $bfa \sim bga$ for all morphisms $a : A \to X$ and $b : Y \to B$, where $\mathrm{dom}(f) = \mathrm{dom}(g) = X$ and $\mathrm{cod}(f) = \mathrm{cod}(g) = Y$. We can depict this as the following:

$$\begin{tikzcd} * \arrow[r, "a"] & * \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & * \arrow[r, "b"] & * \end{tikzcd}$$

Remark 2.24. Notation

It is helpful to distinguish between the objects of a category and the morphisms in a category.

(i) **Objects** of a category C will be denoted C_0

(ii) **Morphisms** of a category C will be denoted C_1

Definition 2.25. Congruence Category

Let $\sim$ be a congruence on a category C. Define the **congruence category**, denoted as $C^\sim$ by the following properties:

1. $$(C^\sim)_0 = C_0$$

2. $$(C^\sim)_1 = \{\langle f, g\rangle \mid f \sim g\}$$

3. $$\tilde{id}_C = \langle id_C, id_C\rangle$$

4. $$\langle f', g'\rangle \circ \langle f, g\rangle = \langle f'f, g'g\rangle$$

It is obvious that the composition law for morphisms is well-defined under the congruence condition.

Furthermore, we have two **projection functors**:

$$\begin{tikzcd} C^\sim \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & C \end{tikzcd}$$

Note that the projection functors leave the objects unchanged (by the first property).

Definition 2.26. Quotient Category

We can build the **quotient category** of C, denoted $C/\sim$, as having the following properties:

1. $$(C/\sim)_0 = C_0$$
2. $$(C/\sim)_1 = C_1/\sim$$

By (2), the morphisms of $C/\sim$ have the form $[f]$, where $f \in C_1$. Furthermore, notice that $id_{[C]} = [id_C]$, and $[g] \circ [f] = [g \circ f]$, these properties arise the functorial nature of the following:

We have a **quotient functor** $\pi : C \to C/\sim$, so that it **becomes a coequalizer** of $C^{\sim}$ and the projection functors, p_1, p_2:

$$\begin{tikzcd} C^{\sim} \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & C \arrow[r, "\pi"] & C/\sim \end{tikzcd}$$

As suspected, we have a **First Isomorphism Theorem** for categories!

Definition 2.27. Suppose that we have categories C and D, and a functor F:

$$F : C \to D$$

F determines a congruence $\sim_F$ on C by setting:

$$f \sim_F g \text{ iff } \operatorname{dom}(f) = \operatorname{dom}(g),\ \operatorname{cod}(f) = \operatorname{cod}(g),\ F(f) = F(g)$$

This is obviously a congruence in C, as the domains and codomains match, while F being a functor ensures that compositions preserve equivalence.

Let us write:

$$\begin{tikzcd} \operatorname{Ker}(F) = C^{\sim_F} \arrow[r, shift left] \arrow[r, shift right] & C \end{tikzcd}$$

We call this the **Kernel Category of** F.

As expected, we can now construct the quotient category:

$$C/\sim_F$$

Proposition 89. ***Universal Property of the Quotient Category***

Every functor $F : C \to D$ has a kernel category $\operatorname{Ker}(F)$, determined by a congruence $\sim_F$ on C, such that, given any congruence $\sim$ on C, we have:

$$f \sim g \Longrightarrow f \sim_F g$$

if and only if $\exists \tilde{F} : C/\sim \to D$ such that the following commutes:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[dr, "\pi"'] & D \\ & C/\sim \arrow[u, "\tilde{F}"'] \end{tikzcd}$$

Proof. There's two directions to this proof.

Assume that every functor $F : C \to D$ has a kernel category $\mathrm{Ker}(F)$, determined by a congruence $\sim_F$ on C so that for any given congruence $\sim$ on C, we have:

$$f \sim g \Longrightarrow f \sim_F g$$

Thus, $F(f) = F(g)$ for any morphisms f and g such that $\mathrm{dom}(f) = \mathrm{dom}(g)$, and $\mathrm{cod}(f) = \mathrm{cod}(g)$.

As we have a congruence $f \sim_F g$ on C, there is a coequalizer of C as indicated above in **Definition 2.26**. Thus the projection functor, $\pi : C \to C/\sim$, exists.

Define $\tilde{F} : C/\sim \to D$ as:

$$\tilde{F}([f]) = F(f)$$

Naively, this makes the following diagram commute:

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[drr, "\pi"'] & & D \\ & & C/\sim \arrow[u, "\exists \tilde{F}"'] \end{tikzcd}$$

To check that it preserves the equivalence relation, we can simply see that:

$$f \sim g \Longrightarrow F(f) = F(g) \Longrightarrow \tilde{F}([f]) = \tilde{F}([g]) \quad \forall f, g \in C_1$$

Therefore, $\tilde{F}$ exists, makes the diagram commute, and preserves the equivalence relation.

Likewise, the opposite direction is trivial. Let us comment on it.

The commutativity of the above diagram holds if for morphisms $f, g \in C_1$, if $f \sim g$, then $[f] = [g]$ and, as a result, $\tilde{F}([f]) = \tilde{F}([g])$. By our construction of $\tilde{F}$, this implies that $F(f) = F(g)$.

Therefore, the existence of $\tilde{F}$ (defined as by $\tilde{F}\pi = F$) (which is equivalent to the commutativity of the above diagram) implies that if $f \sim g$, then $F(f) = F(g)$.

Both directions are proved, and we can take this as a definition of the **Quotient Category**. □

Corollary 90. *Every functor $F : C \to D$ factors as $F = \tilde{F}\pi$.*

Where π is bijective on objects and surjective on Hom-sets, and $\tilde{F}$ is injective on Hom-sets (i.e. it is "faithful"). i.e. The following diagram commutes:

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[drr, "\pi"'] & & D \\ & & C/\mathrm{Ker}(F) \arrow[u, "\tilde{F}"'] \end{tikzcd}$$

Note: *$\tilde{F}$ faithful means that:*

$$\tilde{F}_{A,B} : \mathrm{Hom}(A,B) \to \mathrm{Hom}(FA,FB) \quad \forall A,B \in (C/\mathrm{Ker}(F))_0$$

Proof. This proof follows directly from the previous statement. If we choose the equivalence relation as $\sim_F$, then for the quotient category $C/\sim_F$, the previous proposition establishes that $f \sim g$ if and only if $f \sim_F g$. Therefore, assuming that $F(f) = F(g)$, this implies that $f \sim_F g$, which implies that $[f] = [g] \quad \forall [f],[g] \in C/\sim_F$. Hence, $\tilde{F}([f]) = \tilde{F}([g]) \implies [f] = [g]$, hence, injective on morphisms.

Furthermore, the action of F on the objects of $C/\sim_F$, is obvious:

$$\tilde{F}_{A,B} : \mathrm{Hom}(A,B) \to \mathrm{Hom}(FA,FB) \quad \forall A,B \in (C/\mathrm{Ker}(F))_0$$

as it is the definition of the congruence, $\sim_F$, in the Kernel Category, $\mathrm{Ker}(F)$. □

2.3.4 Free Categories and Finitely Presented Categories

We now introduce the notion of a free category (much like a free monoid, group, abelian group, etc.).

Definition 2.28. A **directed graph** consists of nodes and edges, so that each arrow (morphism) has a source and a target node.

A **graph**, therefore, consists of two sets: (i) E (edges) (ii) V (vertices), and two functions $s : E \to V$ (source), and $t : E \to V$ (target).

Every graph G generates a category $C(G)$ as follows:

1. Take the vertices (or nodes) as the objects in $C(G)_0$.

2. Take the paths between nodes (vertices) as the morphisms.

   A **path** between nodes is a sequence of edges so that $t(e_i) = s(e_{i+1})$ for any i. We write the morphisms of $C(G)$ as:

   $$e_n e_{n-1} \cdots e_1$$

   Clearly, the composition is concatenation of paths.

   $$\mathrm{dom}(e_n \cdots e_1) = s(e_1)$$

   $$\mathrm{cod}(e_n \cdots e_1) = t(e_n)$$

   For each vertex v, we have an empty path, denoted as 1_v, which is the identity morphism at v.

2.4 Limits and Colimits

We will build on the existing definitions and ideas we have already used previously here.

2.4.1 Subobjects

Example 2.13. We previously saw that every subset $U \subseteq X$ of a set X occurs as an equailizer, which are always monic. Therefore, we regard monos a "generalized subset".

A mono in **Grp** is a subgroup of the group X.

A mono in **Top** is a subspace of the topological space X.

Remark 2.25. For a rough idea, we lay out the following:

Given a monomorphism

$$m : M \hookrightarrow X$$

in a category G, of structured sets (we will call these **gadgets**), the **image subset**

$$m(M) := \{m(y) \mid y \in M\} \subseteq X$$

is, oftentimes, a sub-gadget of X, to which M is isomorphic via m as follows:

$$m : M \overset{\cong}{\to} m(M) \subseteq X$$

Before we continue, let us recall a particular category:

Definition 2.29. Let C be a category. Let A be an object of C. Then the **slice category** of C over A (denoted C/A) is the category whose:

1. Objects are the morphisms of C, f such that $\text{cod}(f) = A$.

2. Morphisms are all morphisms $g : f \longmapsto f'$, (i.e. a morphism $g : X \to X'$) where $f : X \to A$ and $f' : X' \to A$, such that $f'g = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & A & \end{tikzcd}$$

The identity morphism in C/A is the morphism $id_X : X \to X$ such that for any $f : X \to A$, $id_X : f \longmapsto f$. The following diagram illustrates it:

$$\begin{tikzcd} X \arrow[rr, "id_X"] \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

And for three objects $f : X \to A$, $f' : X' \to A$, $f'' : X'' \to A$, we have that $g : f \longmapsto f'$ and $g' : f' \longmapsto f''$ as follows:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X'' \arrow[dl, "f''"] \\ & A & \end{tikzcd}$$

Thus the composition of objects is given by $g' \circ g$.

Definition 2.30. A **subobject** of an object X in a category C is a monomorphism:

$$m : M \hookrightarrow X$$

Given subobjects m, m' of X, a morphism $f : m \to m'$ is a morphism in the slice category over X, C/X as in:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "m"'] & M' \arrow[d, "m'"] \\ & X \end{tikzcd}$$

Thus, the **category of subobjects of** X is denoted:

$$\mathrm{Sub}_C(X)$$

This is technically the **Slice Category over** X, C/X.

Definition 2.31. Define the relation of **inclusion of subobjects** by:

$$m \subseteq m' \text{ iff } \exists f : m \to m'$$

We say that m and m' are equivalent (denoted as $m \equiv m'$) if and only if they are isomorphic as subobjects, i.e. $m \subseteq m'$ and $m' \subseteq m$. This holds only if there are f, f' making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, shift right, "f"'] \arrow[dr, "m"] & M' \arrow[l, shift right, "f'"'] \arrow[d, "m'"] \\ & X \end{tikzcd}$$

In the above diagram, $m = m'f = mf'f$. Since m is monic, we have that $f'f = id_M$, and $ff' = id_{M'}$. therefore, $M \cong M'$ via f.

Therefore, equivalent subobjects have isomorphic domains.

Remark 2.26. We will abuse notation and call M the subobject when the monomorphism $m : M \hookrightarrow X$ is obvious.

Remark 2.27. Note that if $M \subseteq M'$, then the morphism f which makes this occur:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr] & M' \arrow[d, "i"] \\ & X \end{tikzcd}$$

is monic, so that M is a subobject of M' (or technically $m \subseteq m'$, but look at the above abuse of notation for this).

We have a functor:

$$i_* : \mathrm{Sub}(M') \to \mathrm{Sub}(X)$$

defined by the composition:

$$i_*(f) = if$$

The generalized elements of an object X are:

$$z : Z \to X$$

and we can define a **local membership relation**:

$$z \in_X M$$

between the generalized elements on X, and the subobjects $m : M \hookrightarrow X$ by:

$$z \in_X M \text{ iff } \exists f : Z \to M \text{ such that } z = mf$$

as follows:

$$\begin{tikzcd} Z \arrow[r, "\exists f"] \arrow[dr, "z"'] & M \arrow[d, "m"] \\ & X \end{tikzcd}$$

Because m is a monomorphism, $z = mf$ implies that $f = m^{-1}z$, thus the **factorization is unique**.

Therefore, this is a relation by inclusion of subobjects $z \subseteq m$.

Example 2.14. An equalizer:

$$\begin{tikzcd} E \arrow[r] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

is a **subobject** of A with the property:

$$z \in_A E \text{ iff } f(z) = g(z)$$

This encapsulates the **Universal Property of the Equalizer** (**Definition 2.16**), which is described by the following diagram:

$$\begin{tikzcd}
E \arrow[r] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, "\exists! u"] \arrow[ur, "z"] & &
\end{tikzcd}$$

Therefore, E (or more precisely $E \to A$) is the subobject of generalized elements $z : Z \to A$ such that $fz = gz$:

$$E = \{z \in Z \mid f(z) = g(z)\} \subseteq A$$

2.4.2 Pullbacks

Note: This is very important!

Definition 2.32. Pullbacks

In a category C, a **pullback** of morphisms f, g with $\mathrm{cod}(f) = \mathrm{cod}(g)$:

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

consists of an object P, and morphisms $p_1 : P \to Z$, $p_2 : Y \to Z$:

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\
X &
\end{tikzcd}$$

such that the following diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

The **pullback** of morphisms f, g actually has a **Universal Property**.

Definition 2.33. Universal Property of the Pullback

In a category C, the **pullback**, (the triple (P, p_1, p_2)) of morphisms $f : X \to Z$ and $g : Y \to Z$ has the following universal property:

Given any object A and morphisms $a_1 : A \to X$, $a_2 : A \to Y$ such that $fa_1 = ga_2$, $\exists! u : A \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr, "\exists! u"] \arrow[drr, bend left, "a_2"] \arrow[ddr, bend right, "a_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

i.e. $a_1 = p_1 u$ and $a_2 = p_2 u$.

Remark 2.28. Note: This is exactly the fibered product in **Proposition 59**, with the obvious exception that the fibered product in **Proposition 59** is in the category of abelian groups **Ab**.

Much like the case in **Ab**, the pullback is unique up to isomorphism.

Definition 2.34. Another Characterization of Pullbacks

We may take another viewpoint of the pullback.

Let z be a generalized element (where we view an element of an object as a map whose terminal object is the object itself), so that for any $z \in P$, we have that:

$$z = \langle a_1, a_2 \rangle$$

such that $fa_1 = ga_2$, making the following diagram commute:

$$\begin{tikzcd}
A \arrow[dr, "\exists! u"] \arrow[drr, bend left, "a_2"] \arrow[ddr, bend right, "a_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Therefore, this means that:

$$P = \{(a_1, a_2) \mid fa_1 = ga_2\}$$

Hence, the pullback of f and g is actually an **equalizer of** fp_1 **and** gp_2. We may collapse the above diagram to be:

$$\begin{tikzcd} A \arrow[r, "z"] & P \arrow[r, shift left, "gp_2"] \arrow[r, shift right, "fp_1"'] & Z \end{tikzcd}$$

Now the connection between the equalizer and the **Universal Property of the Pullback** is readily apparent.

Remark 2.29. This gives us an easy proof of **Proposition 59** (the **Universal Property of the Fibered Product**), as we may invoke it by using the **Universal Property of the Equalizer** (**Definition 2.1.6**) instead.

Proposition 91. *In a category with products and equalizers, given a corner of morphisms:*

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Consider the diagram:

$$\begin{tikzcd} E \arrow[dr, "\exists! e"] \arrow[drr, "p_2"] \arrow[ddr, "p_1"'] & & \\ & X \times Y \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

in which e is an equalizer of $f\pi_1$ and $g\pi_2$, and $p_1 = \pi_1 e$ and $p_2 = \pi_2 e$. Then (E, p_1, p_2) is a pullback of f and g.

Likewise, if (E, p_1, p_2) are given as this pullback, then the morphism:

$$e := \langle p_1, p_2 \rangle : E \to X \times Y$$

is the equalizer of $f\pi_1$ and $g\pi_2$.

Proof. For the forward implication, assume that we are given a corner of morphisms:

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Complete this diagram by taking:

$$\begin{tikzcd} P \arrow[r, "h_2"] \arrow[d, "h_1"] & Y \\ X & \end{tikzcd}$$

so that $fh_1 = gh_2$. Now, we have that $z := \langle h_1, h_2 \rangle : P \to X \times Y$ therefore:

$$f\pi_1\langle h_1, h_2\rangle = g\pi_2\langle h_1, h_2\rangle$$

Which can be illustrated by:

$$\begin{tikzcd} X \times Y \arrow[drr, "\pi_1"'] \arrow[drrr, "\pi_2"] & & & \\ & & P \arrow[ull, "z"'] \arrow[r, "h_2"] \arrow[d, "h_1"] & Y \arrow[d, "g"] \\ & & X \arrow[r, "f"] & Z \end{tikzcd}$$

Therefore, as we are given the original diagram, there is a $u : P \to E$ so that the equalizer composes with this morphism to give us the generalized element z:

$$eu = z$$

Therefore,

$$p_1u = \pi_1eu = \pi_1z = h_1 \qquad p_2u = \pi_2eu = \pi_2z = h_2$$

The uniqueness of this is obvious through construction.

The converse is also obvious, as it is the relationship between **Definition 2.33** and **Definition 2.34**. □

Corollary 92. *If a category C has binary products and equalizers, then it must have pullbacks.*

Proof. As C has binary products, we can take products of the objects in C. C also has equalizers necessarily. By the **Universal Property of the Product** and the **Universal Property of the Equalizer**, the pullback is forced to exist. □

Example 2.15. Pullbacks in the Category of Sets

We construct the pullback in **Sets**, as a subset of the product:

$$\{(a, b) \mid fa = gb\} = A \times_Z B \hookrightarrow A \times B$$

Take a function $f : A \to B$, and a subset $V \subseteq B$. Recall the preimage of V under f:

$$f^{-1}(V) = \{a \in A \mid f(a) \in V\} \subseteq A$$

Consider the following diagram:

$$\begin{tikzcd}
f^{-1}(V) \arrow[r, "\bar{f}"] \arrow[d, "j"] & V \arrow[d, "i"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Where i and j are the canonical inclusions from the respective sets, and $\bar{f} = f|_{f^{-1}(V)}$.

To show that this diagram is, in fact, a pullback of the preimage, we must simply verify the universal property, i.e. that we can construct z such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, "\exists! z"] \arrow[drr, "z_2"] \arrow[ddr, "z_1"'] & & \\
& f^{-1}(V) \arrow[r, "\bar{f}"'] \arrow[d, "j"] & V \arrow[d, "i"] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Take $z = \langle z_1, z_2 \rangle$. So that:

$$\bar{f}z = z_2 \qquad jz = z_1$$

Taking z as a generalized element of $f^{-1}(V)$, we see that this, indeed, does commute. This is because for any $z \in f^{-1}(V)$, $fz \in V$. Meanwhile, the inclusion of z into A is obvious as the preimage is a subset of A.

By nature of $\bar{f}$ and j being monic, we see that:

$$\bar{f}z = z_2 \implies z = \bar{f}^{-1}z_2$$

Likewise for the other morphism. Hence, it is unique.

Proposition 93. *For any pullback in a category:*

$$\begin{tikzcd}
A \times_B M \arrow[r, "\bar{f}"] \arrow[d, "m'"] & M \arrow[d, "m"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

If m is monic, then m' is monic.

Proof. This is a corollary of the universal property of the pullback, but we will elaborate as an exercise (cover it and try it yourself if you're a curious reader).

We can create the following diagram:

$$\begin{tikzcd}
X \arrow[dr, shift left, "y"] \arrow[dr, shift right, "x"'] \arrow[drr, "u_1"] \arrow[ddr, "u_2"'] & & \\
& A \times_B M \arrow[r, "g"'] \arrow[d, "m'"] & M \arrow[d, "m"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

Now take $u_2 = m'x = m'y$ and $u_1 = gx = gy$. By assumption that m is monic, we have that $mgx = mgy$ implies that $gx = gy$.

Now, by the universal property of the pullback, $\exists! u : X \to A \times_B M$ such that for any given u_1, u_2, $gu = u_1 = gx = gy$ and $m'u = u_2 = m'x = m'y$ (this is just a restatement of the commutativity). As m is monic, we obtain the commutativity of the whole square as $m^{-1} f u_2 = u_1 = gx = gy$.

Furthermore, as u is unique, $x = u = y$. Hence, $m'x = m'y$ implies that $x = y$, so that m' is monic. $\square$

Remark 2.30. An immediate corollary of the above proposition is that for any fixed $f : A \to B$ (where A and B are objects in the category C), the pullback induces a map:

$$f^{-1} : \text{Subobjects}(B) \to \text{Subobjects}(A)$$
$$f^{-1} : m \longmapsto m'$$

Furthermore, f^{-1} is actually well-defined as a functor as the following is true (and will be proven):

$$M \equiv N \implies f^{-1}(M) = f^{-1}(N)$$

2.4.3 Properties of Pullbacks

Lemma 94. ***Double Pullbacks***

Let us be in a category that admits pullbacks, where the following diagram commutes.

$$\begin{tikzcd}
F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

The following are true:

1. *If the two squares above are pullbacks, so is the larger rectangle. This is encapsulated formally by saying:*

$$A \times_B (B \times_C D) \cong A \times_C D$$

2. *If the right square and the outer rectangle are pullbacks, so is the left square.*

Proof. This is an exercise in diagram chasing. It is the first time we have done that in these notes so we will go over **how** to diagram chase.

1. Assume that the two squares given are pullbacks. Then we see that:

$$F = A \times_B E \qquad E = B \times_C D$$

   So that:

$$F = A \times_B (B \times_C D)$$

   Invoking the **Universal Property of the Pullback** onto both squares, we see that:

   Given morphisms u_1, u_2 and v_1, v_2 and objects U, V we see that $\exists! u, v$ so that the following commutes:

$$\begin{tikzcd}
U \arrow[dr, "\exists! u"] \arrow[drr, "u_1"] \arrow[ddr, "u_2"'] & & & & V \arrow[dll, "\exists! v"'] \arrow[dl, "v_1"] \arrow[ddll, bend left, "v_2"] \\
& F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] & \\
& A \arrow[r, "f"] & B \arrow[r, "g"] & C &
\end{tikzcd}$$

   With all the excruciating detail given in the diagram above, because we are given commutativity of the initial rectangle, and the universal property gives us the commutativity of the additional morphisms, we can conclude that there must be a unique way to complete the above diagram as follows.

$$
\begin{tikzcd}
U \arrow[dr, "\exists! u"] \arrow[drr, "u_1"] \arrow[ddr, "u_2"'] & & & & V \arrow[llll, "\exists!\Delta"'] \arrow[dll, "\exists! v"'] \arrow[dl, "v_1"] \arrow[ddll, bend left, "v_2"] \\
& F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] & \\
& A \arrow[r, "f"] & B \arrow[r, "g"] & C &
\end{tikzcd}
$$

Because v exists uniquely, this $\Delta : V \to U$ must exists uniquely as well since the factorization of a unique morphism should, itself, be unique.

Therefore, we have just shown that given that $F = A \times_B E$ has the universal property of a pullback, and so does $E = B \times_C D$, this implies that $A \times_B (B \times_C D)$ has this universal property as well.

By commutativity, if we compose $g'f' = j'$ and $gf = j$, then using the existence and uniqueness of Δ, we obtain commutativity of the equivalent diagram:

$$
\begin{tikzcd}
F \arrow[r, "j'"] \arrow[d, "h''"] & D \arrow[d, "h"] \\
A \arrow[r, "j"] & C
\end{tikzcd}
$$

Invoking the above universal properties, we obtain that this diagram commutes:

$$
\begin{tikzcd}
V \arrow[dr, "\exists! u\Delta"] \arrow[drr, bend left, "v_1"] \arrow[ddr, bend right, "u_2\Delta"'] & & \\
& F \arrow[r, "j'"] \arrow[d, "h''"] & D \arrow[d, "h"] \\
& A \arrow[r, "j"] & C
\end{tikzcd}
$$

Hence, this not only proves that $F = A \times_B (B \times_C D)$ has the universal property, but that it is actually equal to $A \times_C D$, as $A \times_C (B \times_C D)$ and $A \times_C D$ actually satisfy the **identical universal property**.

2. This involves a similar invoking of a universal property and diagram chasing as previously.

   Assume that $E = B \times_C D$ is a pullback, and that $F = A \times_C D$ are pullbacks. After the previous proof, the conclusion should be obvious, but we will draw the commutative diagram in detail.

   $F = A \times_C D$ has the universal property, thus for any given object U and morphisms u_1, u_2, $\exists! u : U \to F$ so that the rectangle commutes. Likewise, $E = B \times_C D$ has the universal property, thus for any given object V and morphisms v_1, v_2, $\exists! v : V \to E$ such that the right square commutes. This is easily seen in the following commutative diagram.

$$\begin{tikzcd}
U \arrow[dr, "\exists! u"] \arrow[drrr, bend left=15, "u_1"] \arrow[ddr, "u_2"'] & & & & V \arrow[dll, "\exists! v"'] \arrow[dl, "v_1"] \arrow[ddll, bend left=60, "v_2"] \\
& F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] & \\
& A \arrow[r, "f"] & B \arrow[r, "g"] & C &
\end{tikzcd}$$

   As the diagram commutes, there must exist a $\Delta : U \to V$ such that $f'u = v\Delta$, i.e. the following must commute:

$$\begin{tikzcd}
U \arrow[rrrr, "\Delta"] \arrow[dr, "\exists! u"] \arrow[drrr, bend left=15, "u_1"] \arrow[ddr, "u_2"'] & & & & V \arrow[dll, "\exists! v"'] \arrow[dl, "v_1"] \arrow[ddll, bend left=60, "v_2"] \\
& F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] & \\
& A \arrow[r, "f"] & B \arrow[r, "g"] & C &
\end{tikzcd}$$

Thus, we are given that for any $v\Delta$, u_2, the following diagram commutes:

$$\begin{tikzcd}
U \arrow[drr, "\exists! u"] \arrow[drrr, "v\Delta"] \arrow[ddrr, "u_2"'] & & & \\
& & F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[d, "h'"] \\
& & A \arrow[r, "f"] & B
\end{tikzcd}$$

Hence, the left square has the universal property of the pullback, hence it is a pullback.

□

Corollary 95. ***Prism Pullback***

The pullback of a commutative triangle is a commutative triangle. Specifically, for any $h : C' \to C$, if we are given α', β' as depicted, then $\exists! \gamma'$ as depicted, making the following diagram commute.

$$\begin{tikzcd}
A' \arrow[rr, "h_\alpha"] \arrow[dr, "\exists!\gamma'"] \arrow[dd, "\alpha'"'] & & A \arrow[dr, "\gamma"] \arrow[dd, "\alpha"] & \\
& B' \arrow[rr, "h_\beta"] \arrow[dl, "\beta'"] & & B \arrow[dl, "\beta"] \\
C' \arrow[rr, "h"] & & C &
\end{tikzcd}$$

Proof. This is an obvious application of the Double Pullbacks lemma (can you see it?). Left as exercise to the reader, I already did all the hard work. □

Proposition 96. *Pullback is a functor.*

For a fixed $h : C' \to C$ in a category $\mathbf{C}$ with pullbacks, there is a functor:

$$h^* : \mathbf{C}/C \to \mathbf{C}/C'$$

defined by:

$$(\alpha : A \to C) \longmapsto (\alpha' : C' \times_C A \to C')$$

Where we define α' as the pullback of α along h, and the action of some $\gamma : \alpha \to \beta$ is given by the Prism Pullback corollary.

Proof. We will first illustrate what h^* is supposed to do.

$$
\begin{tikzcd}
C' \times_C A \arrow[r] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\longleftarrow"] & A \arrow[d, "\alpha"] \\
C' \arrow[r, "h"'] & C
\end{tikzcd}
$$

This demonstrates what it means for α' to be the pullback of α along h.

Let us check that $h^*(id_X) = id_{h^*X}$ and $h^*(gf) = h^*(g)h^*(f)$.

We can kill two birds with one stone and consider the following double pullback diagram:

$$
\begin{tikzcd}
A' \arrow[r, "h'"] \arrow[d, "id_{A'}"] & A \arrow[d, "id_A"] \\
A' \arrow[r, "h'"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\
C' \arrow[r, "h"] & C
\end{tikzcd}
$$

Clearly, we have that:

$$h^*(id_A) = id_{h^*A} = id_{A'}$$

Since composition yields the following:

$$h^*(\alpha id_A) = h^*(\alpha) = \alpha' = \alpha' id_{A'} = h^*(\alpha)h^*(id_A)$$

By the double pullback lemma, if the bottom square is a pullback, then the whole rectangle is a pullback, and so is the top square. Hence, identity maps are preserved by the pullback functor.

Likewise, the composition holds by the exact same computation as above (replace the identity map and the objects with suitable replacements).

Hence, h^* is a functor from one pullback to another. □

Corollary 97. *Let* $\mathbf{C}$ *be a category with pullbacks. For any morphism* $f : A \to B$*, we have that the following square*

$$
\begin{tikzcd}
\mathrm{Sub}(A) \arrow[d] & \mathrm{Sub} \arrow[l, "f^{-1}"] \arrow[d] \\
\mathbf{C}/A & \mathbf{C}/B \arrow[l, "f^*"]
\end{tikzcd}
$$

commutes because $f^{-1} = f^*|_{\mathrm{Sub}(B)}$.

Thus, f^{-1} is actually functorial in the following manner:

$$M \subseteq N \Longrightarrow f^{-1}(M) \subseteq f^{-1}(N)$$

So it follows that if $M \equiv N$, then $f^{-1}(M) \equiv f^{-1}(N)$, where $\equiv$ denotes an equivalence relation.

Therefore, f^{-1} is defined on equivalence classes:

$$f^{-1}/\equiv : \mathrm{Sub}(B)/\equiv \to \mathrm{Sub}(A)/\equiv$$

Proof. This is only a sketch of a proof, but recall that f^{-1} is a pullback (the canonical example is in **Sets**, but it applies to more general categories using the same conditions), and because pullbacks are functorial by the previous proposition, the conclusion follows. $\square$

Example 2.16. Let I be an indexing set, and consider a I-indexed family of sets:

$$(A_i)_{i \in I}$$

Given some function $\alpha : J \to I$, there is a J-indexed family:

$$\left(A_{\alpha(j)}\right)_{j \in J}$$

obtained by reindexing along α, i.e. this is a pullback.

For more detaiil, for every set A_i, take a constant, i-valued function:

$$p_i : A_i \to I$$

and consider the map on the coproduct:

$$p = [p_i] : \coprod_{i \in I} A_i \to I$$

Then we have the clear pullback:

$$\begin{tikzcd}
\coprod_{j \in J} A_{\alpha(j)} \arrow[r] \arrow[d, "q"] & \coprod_{i \in I} A_i \arrow[d, "p"] \\
J \arrow[r, "\alpha"] & I
\end{tikzcd}$$

2.4.4 Limits

The followin gives a necessary and sufficient condition for pullbacks.

Proposition 98. *A category has finite products and equalizers if and only if it has pullbacks and a terminal object.*

Proof. The forward implication has been developed already in **Definition 2.34**, a pullback is indeed an equalizer of some morphisms f, g into a terminal object. Terminal objects exist as the empty product, which is, itself, a finite product.

Now, suppose a category $\mathbf{C}$ has pullbacks and a terminal object 1. For arbitrary objects A and B, we see that:

$$A \times B \cong A \times_1 B$$

because of the following:

Let the following be a pullback over 1:

$$\begin{tikzcd}
A \times B \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & 1
\end{tikzcd}$$

Omitting the terminal object gives us the following product diagram:

$$\begin{tikzcd}
A \times B \arrow[r] \arrow[d] & B \\
A &
\end{tikzcd}$$

The **universal property** on the **first** pullback diagram can be seen as the following commutative diagram:

$$\begin{tikzcd}
U \arrow[dr, "\exists!"] \arrow[drr] \arrow[ddr] & & \\
& A \times B \arrow[r] \arrow[d] & B \arrow[d] \\
& A \arrow[r] & 1
\end{tikzcd}$$

If we disregard the terminal object, we obtain:

$$\begin{tikzcd}
U \arrow[dr, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \times B \arrow[r] \arrow[d] & B \\
& A &
\end{tikzcd}$$

The commutativity of this diagram is simply the **universal property of the product**! This establishes the isomorphism $A \times_1 B \cong A \times B$.

Also, for any morphisms $f, g : A \to B$, the equalizer $e : E \to A$ is constructable as follows:

$$\begin{tikzcd}
E \arrow[r, "h"] \arrow[d, "e"] & B \arrow[d, "\Delta = \langle 1_B, 1_B \rangle"] \\
A \arrow[r, "\langle f, g \rangle"] & B \times B
\end{tikzcd}$$

Where $E = \{(a, b) \mid \langle f, g\rangle(a) = \Delta b\}$. Where $\langle f, g\rangle(a) = \langle fa, ga\rangle$, and $\Delta b = \langle b, b\rangle$. Therefore, we see that:

$$E = \{\langle a, b\rangle \mid f(a) = b = g(a)\} \cong \{a \mid f(a) = g(a)\}$$

We can now perform a simple diagram chase: we see that since $a \in A$ is the image of some $a \in E$ (i.e. $e(a) = a$), the following

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

is indeed an equalizer as e is defined as the morphism such that $fe = ge$.

Therefore, a pullback and a terminal object imply the existence of a finite product, which, in turn, implies the existence of an equalizer. $\square$

Definition 2.35. Let **J** and **C** be categories. A **diagram of type J** in **C** is a functor:

$$D : \mathbf{J} \to \mathbf{C}$$

In the index category **J**, we write objects as lower case letters: $i, j, \ldots$ and the values of the functor $D : \mathbf{J} :\to \mathbf{C}$ in the form $D_i, D_j, \ldots$. We can summarize the action on the objects and morphisms of **J** by saying:

$$D(i) = D_i \quad \forall i \in \mathbf{J}_0$$

$$D(\alpha : i \to j) = D_\alpha : D_i \to D_j \quad \forall \alpha \in \mathbf{J}_1$$

Definition 2.36. A **cone** to a diagram D consists of an object C in $\mathbf{C}$ and a family of arrows in $\mathbf{C}$:

$$c_j : C \to D_j \qquad \forall j \in \mathbf{J}$$

such that for each $\alpha : i \to j$ in $\mathbf{J}$, the follow triangle commutes:

$$\begin{tikzcd} C \arrow[r, "c_j"] \arrow[d, "c_i"'] & D_j \\ D_i \arrow[ur, "D_\alpha"'] & \end{tikzcd}$$

A **morphism of cones**, denoted:

$$\nu : (C, c_j) \to (C', c_j')$$

is a morphism ν in $\mathbf{C}$ making each triangle:

$$\begin{tikzcd} C \arrow[r, "\nu"] \arrow[dr, "c_j"'] & C' \arrow[d, "c_j'"] \\ & D_j \end{tikzcd}$$

i.e. $c_j = c_j' \circ \nu$ for all $j \in \mathbf{J}$. Therefore, we have a new category $\mathbf{Cone}(D)$.

The morphisms are all such ν, and the objects are the $c_j : C \to D_j$.

Remark 2.31. The diagram D is a "flat" picture of $\mathbf{J}$ in $\mathbf{C}$. A cone to a diagram D is a many-sided pyramind over the flat base D. A morphism of cones is an arrow between the tops of each pyramid (each cone).

Definition 2.37. A **limit** for a diagram $D : \mathbf{J} \to \mathbf{C}$ is a terminal object in $\mathbf{Cone}(D)$. A **finite limit** is a limit for a diagram on a finite index category $\mathbf{J}$ (so there are finitely many indices).

A limit is often denoted:

$$p_i : \lim_{\leftarrow j} D_j \to D_i$$

Definition 2.38. Universal Property of the Limit

Given any cone (C, c_j) to the diagram D

$$\exists! u : C \to \lim_{\leftarrow j} D_j$$

such that for all j,

$$p_j \circ u = c_j$$

NOTE: Unless we are speaking about finite limits, it is impossible to draw a diagram for this. We will show that this is a general case of other universal properties we have discussed.

Proposition 99. *If a category has all finite limits, then it has finite products and equalizers.*

Proof. The proof is given in the following examples (**Examples 2.17, 2.18, 2.19, 2.20**). ☐

Example 2.17. Products are Limits

Take $\mathbf{J} = \{1, 2\}$ as the discrete category with two objects and no arrows besides the identity arrows. A diagram $D : \mathbf{J} \to \mathbf{C}$ is a pair of objects $D_1, D_2 \in \mathbf{C}$. A cone on D is an object of $\mathbf{C}$ equipped with morphisms

$$D_1 \xleftarrow[c_1]{} C \xrightarrow{c_2} D_2$$

A limit of D is a terminal cone:

$$D_1 \xleftarrow[p_1]{} D_1 \times D_2 \xrightarrow{p_2} D_2$$

We know that this is the terminal such object because the **Universal Property of the Limit** says that

$$\exists! u : C \to \varprojlim_j D_j$$

such that

$$p_i \circ u = c_i$$

i.e.

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[d, "\exists! u"] \arrow[dr, "c_2"] & \\ D_1 & D_1 \times D_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_2 \end{tikzcd}$$

This is the **identical** universal property to the binary product!

Note that taking the limit in a discrete category of n objects with no non-identity morphisms gives us an **n-ary product**! This is easy to see.

Hence, in $\mathbf{J} = (\{1, \ldots, n\}, id)$, the limit is:

$$\varprojlim_j D_j \cong D_1 \times \cdots \times D_n$$

Example 2.18. Equalizers are Limits

Take $\mathbf{J}$ as the category:

$$\cdot \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \cdot$$

i.e. the category with 2 non-identity morphisms, and two objects.

A diagram of type $\mathbf{J}$ looks like:

$$D_1 \overset{D_\alpha}{\underset{D_\beta}{\rightrightarrows}} D_2$$

And here, a cone is an object C and a pair of morphisms c_1, c_2 out of C. A limit of the diagram is given by the following commutative diagram:

$$\begin{tikzcd} D_1 \arrow[rr, shift left, "D_\alpha"] \arrow[rr, shift right, "D_\beta"'] & & D_2 \\ & C \arrow[ul, "c_1"] \arrow[ur, "c_2"'] & \end{tikzcd}$$

The **Universal Property of the Equalizer** says that given a morphism $z : C \to D_1$, $\exists! u : C \to E$ such that:

$$\begin{tikzcd} E \arrow[r, "e"] & D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2 \\ C \arrow[u, "\exists! u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

We can verify that the limit for D is an equalizer for D_α, D_β by showing that the above universal property is identical to the universal property for a limit.

Namely, we may regard $c_1 := e \circ u$. Then, doing a quick diagram chase, we see that we may set $c_2 := D_\beta \circ z$ so that $c_2 = D_\beta \circ c_1$. Then, we obtain exactly the diagram:

$$\begin{tikzcd} D_1 \arrow[rr, shift left, "D_\alpha"] \arrow[rr, shift right, "D_\beta"'] & & D_2 \\ & C \arrow[ul, "c_1"] \arrow[ur, "c_2"'] & \end{tikzcd}$$

Hence, the **Universal Property of the Limit**, that

Given any cone (C, c_j) to the diagram D

$$\exists! u : C \to \lim_{\leftarrow j} D_j$$

is satisfied, as if we set the limit of D as:

$$p_i : \lim_{\leftarrow j} D_j \to D_i$$

then we see that $p_1 = e$ and $p_2 = D_\beta \circ e$. And because $c_1 = e \circ u$ and $c_2 = D_\beta \circ c_1$, we see that:

$$p_1 \circ u = c_1 \qquad p_2 \circ u = c_2$$

We illustrate this in the commutative diagram below, it should be immediately obvious that this has all the components of the universal property of the limit:

$$
\begin{tikzcd}
E \arrow[r, "p_1"] \arrow[rr, bend left, "p_2"] & D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2 \\
C \arrow[u, "\exists! u"] \arrow[ur, "z"'] & &
\end{tikzcd}
$$

Thus, we have proven that the limit of the diagram

$$
\begin{tikzcd}
D_1 \arrow[r, shift left, "D_\alpha"] \arrow[r, shift right, "D_\beta"'] & D_2
\end{tikzcd}
$$

is actually the equalizer of D_α and D_β.

Example 2.19. Terminal Objects are Limits

Let $\mathbf{J}$ be an empty category. There is one unique diagram $D : \mathbf{J} \to \mathbf{C}$. The limit of this is the terminal object of $\mathbf{C}$.

$$\lim_{\leftarrow j \in \mathbf{0}} D_j \cong 1$$

To verify this, simply recognize that the empty diagram is simply the absence of diagrams. A cone **over the empty diagram** is simply any object $C \in \mathbf{C}$. Therefore, if we invoke the universal property of the limit, it states:

Given a cone (C, c_j)

$$\exists! u : C \to \lim_{\leftarrow j} D_j$$

Since a cone for the empty diagram is just C itself, we just have that:

$$\exists! u : C \to 1$$

This is identical to the **Universal Property of the Terminal Object**.

Example 2.20. Pullbacks are Limits

Let $\mathbf{J}$ be the finite category:

$$
\begin{tikzcd}
 & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
$$

The diagram $D : \mathbf{J} \to \mathbf{C}$ is of the form:

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

The limit of this diagram is the pullback of f and g, i.e.

$$\lim_{\leftarrow j} D_j \cong A \times_C B$$

To verify this, let us simply recall the **Universal Property of the Pullback**:

Given an object U and morphisms $u_1 : U \to A$ and $u_2 : U \to B$, we have that $\exists! u : U \to A \times_C B$ such that the following commutes:

$$\begin{tikzcd}
U \arrow[dr, "\exists! u"] \arrow[drr, bend left, "c_1"] \arrow[ddr, bend right, "c_2"'] & & \\
& A \times_C B \arrow[r, "p_1"] \arrow[d, "p_2"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

It is immediately obvious now, that:

$$\exists! u : U \to \lim_{\leftarrow j} D_i$$

so that:

$$c_1 = p_1 \circ u \qquad c_2 = p_2 \circ u$$

Hence, the pullback and the limit of the diagram:

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

share **identical universal properties**, hence they are the same.

Proposition 100. *In fact, a category has all finite limits if and only if it has finite products and equalizers.*

Note: *A category* $\mathbf{C}$ *is said to have all finite limits if every finite diagram,* $D : \mathbf{J} \to \mathbf{C}$*, in* $\mathbf{C}$ *has a limit in* $\mathbf{C}$*.*

Proof. We proved the forward implication (that finite limits implies that a category has all finite products and equalizers) in **Examples 2.17, 2.18, 2.19, 2.20**.

For the opposite implication, we assume that we have finite products

$$\prod_{i \in J_0} D_i \qquad \prod_{(\alpha : i \to j) \in J_1} D_j$$

Define a pair of morphisms:

$$\prod_i D_i \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{\alpha : i \to j} D_j$$

Take their composites with the projections π_α from the second product:

$$\pi_\alpha \circ \phi = \phi_\alpha = \pi_{\mathrm{cod}(\alpha)}$$

$$\pi_\alpha \circ \psi = \psi_\alpha = D_\alpha \circ \pi_{\mathrm{dom}(\alpha)}$$

Where $\pi_{\mathrm{dom}(\alpha),\mathrm{cod}(\alpha)}$ are projections from the first product.

Take the equalizer:

$$E \xrightarrow{\quad e \quad} \prod_i D_i \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{\alpha : i \to j} D_j$$

Take some morphism:

$$c : C \to \prod_i D_i$$

Write $x = \langle c_i \rangle$ and $c_i = \pi_i \circ c$. See that $c_i : C \to D_i$ is a cone to D iff $\phi c = \psi c$ (this is obvious by the universal property of the equalizer).

$$\phi\langle c_i \rangle = \psi \langle c_i \rangle$$

iff

$$\pi_\alpha \phi \langle c_i \rangle = \pi_\alpha \psi \langle c_i \rangle$$

Notice, however, that $\pi_\alpha \phi \langle c_i \rangle = \phi_\alpha \langle c_i \rangle = \pi_{\mathrm{cod}(\alpha)} \langle c_i \rangle = c_j$.

Likewise,

$$\pi_\alpha \psi \langle c_i \rangle = \psi_\alpha \langle c_i \rangle = D_\alpha \circ \pi_{\mathrm{dom}(\alpha)} \langle c_i \rangle = D_\alpha \circ c_i$$

Clearly, this is a cone.

Clearly, (E, e_i) is a cone (where $e_i = \pi_i \circ e$), and any cone c_i gives a morphism $c = \langle c_i \rangle : C \to \prod_i D_i$ with $\phi c = \psi c$, therefore, $\exists! u : C \to E$ such that $c = e_i \circ u$ (as E is a cone). This u is clearly a morphism of cones and we see that a terminal object exists in $\mathrm{Cone}(D)$. Hence, the finite limit exists. $\square$

Corollary 101. *A category has all limits of some cardinality if and only if it has all products and equalizers of that cardinality, where* $\mathbf{C}$ *has limits (resp. products) of cardinality* κ *iff* C *has a limit for every diagram* $D : \mathrm{J} \to \mathrm{C}$ *where* $\mathrm{Card}(J_1) \leq \kappa$ *(resp.* $\mathbf{C}$ *has all products of* κ *many objects).*

Obviously, we are working in a category, so we implicitly obtain the statement in the dual category.

Theorem 102. ***Everything, Except in the Opposite Category***

A category $\mathbf{C}$ *has finite colimits if and only if it has finite coproducts and coequalizers (respectively, if and only if it has all pushouts and an initial object).* $\mathbf{C}$ *has all colimits of size* κ *if and only if it has coequalizers and coproducts of size* κ*.*

Proof. Redo the entirety of this section (**Section 2.4.4**), except reverse all the morphisms! ☐

2.4.5 Preservation of Limits

Definition 2.39. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to **preserve limits of type J** if, whenever $p_j : L \to D_j$ is a limit for a diagram $D : \mathbf{J} \to \mathbf{C}$, the cone

$$Fp_j : FL \to FD_j$$

is then a limit for the diagram $FD : \mathbf{J} \to \mathbf{D}$. Namely:

$$F\left(\lim_{\leftarrow} D_j\right) \cong \lim_{\leftarrow} F(D_j)$$

A functor that preserves all limits is said to be **continuous**[13].

Remark 2.32. Let $\mathbf{C}$ be locally small, and recall the representable functor:

$$\mathrm{Hom}_{\mathbf{C}}(D, -) : \mathbf{C} \to \mathbf{Sets}$$

For any object $C \in \mathbf{C}$, we can take a morphism $f : X \to Y$ to

$$f_* : \mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$$

where $f_*(g : C \to X) = f \circ g$.

This leads us to the next very important fact.

Proposition 103. *Representable functors preserve all limits.*

Proof. This is an easy verification, as we must simply prove that $\mathrm{Hom}_{\mathbf{C}}(C, -)$ preserves products and equalizers (or terminal objects and pullbacks, the former is much easier however).

(i) **On Terminal Objects**:

Suppose that $\mathbf{C}$ has a terminal object 1. Then we see that:

$$\mathrm{Hom}(C, 1) = \{*_C\}$$

[13]Note that the category-theoretic limit (**Definition 2.37, 2.38**) is a stark generalization of what it means for a function to be continuous at a point, such as in the calculus of real functions in one real variable.

where $*_C : C \to 1$ (we proved in **Example 2.19** that a cone over an empty diagram was just an arbitrary object in the category $\mathbf{C}$, hence there must only be one morphism up to strict equality). Therefore,

$$\mathrm{Hom}_{\mathbf{C}}(C, 1) \cong 1$$

(ii) **On Binary Products**:

Consider a binary product $X \times Y$ in $\mathbf{C}$. We proved that:

$$\mathrm{Hom}(C, X \times Y) \cong \mathrm{Hom}(C, X) \times \mathrm{Hom}(C, Y)$$

in **Remark 2.12** (and the proposition following it) and **Definition 2.12**.

(iii) **Arbitrary Products**:

For arbitrary products $\prod_{i \in I} X_i$, we have that:

$$\mathrm{Hom}_{\mathbf{C}}\left(C, \prod_i X_i\right) \cong \prod_i \mathrm{Hom}_{\mathbf{C}}(C, X_i)$$

This follows easily from the same statement on binary products, as we may keep taking products, and use the associativity of the product to obtain this result.

(iv) **Equalizers**:

Given an equalizer in $\mathbf{C}$,

$$E \xrightarrow{\quad e \quad} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

Consider the image of the representable functor on the above diagram:

$$\mathrm{Hom}(C, E) \xrightarrow{\quad e_* \quad} \mathrm{Hom}(C, X) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \mathrm{Hom}(C, Y)$$

Let us verify that this is an equalizer in **Sets**, let $h : C \to X \in \mathrm{Hom}(C, X)$, such that:

$$f_* h = g_* h$$

We know that $fh = gh$. And by the universal property of the equalizer, $\exists! u : C \to E$ such that $eu = h$. Therefore, we have a unique $u \in \mathrm{Hom}(C, E)$ such that:

$$e_* u = eu = h$$

Therefore, e_* and $\mathrm{Hom}(C, E)$ satisfy the universal property of an equalizer for f_*, g_*, and thus, forms an equalizer. $\square$

Definition 2.40. A functor of the form

$$F : \mathbf{C}^{op} \to \mathbf{D}$$

is called a **contravariant functor** on $\mathbf{C}$. It is defined by:

$$F(f : A \to B) = F(f) : F(B) \to F(A)$$

$$F(g \circ f) = F(f) \circ F(g)$$

Remark 2.33. The typical example of a contravariant functor is a representable functor of the form:

$$\mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{op} \to \mathbf{Sets}$$

for any $C \in \mathbf{C}$ ($\mathbf{C}$ is locally small!!!).

A contravariant representable functor takes $f : X \to Y$ to:

$$f^* : \mathrm{Hom}(Y, C) \to \mathrm{Hom}(X, C)$$

so that

$$f^*(g : Y \to C) = g \circ f$$

This remark and the previous proposition gives us the following result:

Corollary 104. *Contravariant representable functors map all colimits to limits.*

Proof. For a limit in $\mathbf{C}$, it becomes a colimit in $\mathbf{C}^{op}$. Furthermore, as representable functors preserve limits (or colimits in the opposite category), it must map to the respective limit in the target category. □

Example 2.21. Given a coproduct $X + Y$ in a locally small category, $\mathbf{C}$, there is a canonical isomorphism

$$\mathrm{Hom}(X + Y, C) \cong \mathrm{Hom}(X, C)) \times \mathrm{Hom}(Y, C)$$

This is quite apparent by looking at the universal property of the product, reversing the morphisms, and then using the contravariant hom functor to return it to a product in the category of sets, **Sets**.

Remark 2.34. For sets, we see exponents for sets.

$$C^{X+Y} \cong C^X \times C^Y$$

Example 2.22. Pullbacks in an Index Category

2.4.6 Colimits

Remark 2.35. Pushouts in Sets

Suppose that we have two functions

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & C \\ B & \end{tikzcd}$$

To construct the pushout of f and g:

Start with the coproduct $B + C$,

$$B \longrightarrow B + C \longleftarrow C$$

Identify the elements $b \in B$ and $c \in C$ so that for some $a \in A$:

$$f(a) = b \qquad g(a) = c$$

Take the equivalence relation $\sim$ on $B+C$ generated by the condition $f(a) \sim g(a)$ for all $a \in A$.

Take the quotient by $\sim$ to get the pushout:

$$B +_A C \cong (B + C) / \sim$$

This amounts to taking the dual of the construction for pullbacks in **Sets** (**Example 2.15**).

Definition 2.41. A colimit for a diagram $D : \mathbf{J} \to \mathbf{C}$ is an initial object in the **category of cocones**. Recall **Definition 2.35**, this is the same definition of a **diagram**. A **cocone** from a base D consists of an object C, and morphisms $c_j : D_j \to C$ for each $j \in \mathbf{J}$, such that for all $\alpha : i \to j$ in $\mathbf{J}$:

$$c_j \circ D(\alpha) = c_i$$

i.e. the following diagram commutes

$$\begin{tikzcd}
C & D_j \arrow[l, "c_j"'] \\
D_i \arrow[u, "c_i"] \arrow[ur, "D(\alpha)"']
\end{tikzcd}$$

A **morphism of cocones** $f : (C, c_j) \to (C', c'_j)$ is a morphism $f : C \to C'$ in $\mathbf{C}$ such that $f \circ c_j = c'_j$, i.e. one that makes the following diagram commute

$$\begin{tikzcd}
C \arrow[r, "f"] & C' \\
& D_j \arrow[ul, "c_j"] \arrow[u, "c'_j"']
\end{tikzcd}$$

Note: The above two diagrams are the morphisms in **Definition 2.36** reversed.

Definition 2.42. We denote the **colimit** both as the following morphism and following object:

$$q_i : D_i \to \lim_{\to j \in \mathbf{J}} D_j$$

More precisely, a colimit is characterized by the following universal property.

Definition 2.43. Universal Property of the Colimit

Given any cocone (C, c_j) to the diagram D

$$\exists! v : \lim_{\to j} D_j \to C$$

such that for all j,

$$u \circ q_j = c_j$$

Note: This is just the dual statement of the **Universal Property of the Limit** (**Definition 2.38**).

Remark 2.36. Theoretically, colimits are exactly like limits, so we will not comment much on results regarding colimits. Quite literally, just flip the morphisms.

Example 2.23. Direct Limit of Groups

So the direct limit is actually a colimit.

Suppose we are given a sequence of groups:

$$G_0 \xrightarrow{\quad g_0 \quad} G_1 \xrightarrow{\quad g_1 \quad} G_2 \xrightarrow{\quad g_2 \quad} \dots$$

and morphisms of groups. We want a colimiting group, G_∞, with morphisms (colimits):

$$u_n : G_n \to G_\infty$$

such that $u_{n+1} \circ g_n = u_n$, i.e. the following commutes:

$$\begin{tikzcd}
\dots \arrow[r, "g_{n-1}"] & G_n \arrow[r, "g_n"] \arrow[dr, "u_n"'] & G_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "u_{n+1}"] & \dots \\
 & & G_\infty &
\end{tikzcd}$$

Thus, we see that G_∞ is the "universal" colimit.

1. The index category $\mathbf{J}$ is the ordinal numbers $\omega = (\mathbb{N}, \leq)$ regarded as the poset category.

2. The sequence

   $$G_0 \overset{g_0}{\to} G_1 \overset{g_1}{\to} G_2 \overset{g_2}{\to} \dots$$

   is a diagram of type $\mathbf{J} = \omega$ in the category of Grps.

3. The colimiting group is the colimit of the sequence

   $$\lim_{\to n \in \omega} G_n \cong G_\infty$$

We may construct this as follows:

Take the coproduct of sets:

$$\coprod_{n\in\omega} G_n$$

Then set the equivalence relation $x_n \sim y_m$ where $x_n \in G_n$ and $y_m \in G_m$. In particular, we enforce that

$$x_n \sim g_n(x_n)$$

where

$$g_n : G_n \to G_{n+1}$$

The elements of G_∞ are equivalence classes of the form:

$$[x_n] \quad x_n \in G_n \quad \forall n$$

and $[x_n] = [y_m]$ if and only if for some $k \geq m, n$, we have

$$g_{n,k}(x_n) = g_{m,k}(y_m)$$

where if $i \leq j$, we define

$$g_{i.j} : G_i \to \cdots \to G_j$$

Remark 2.37. The fact that for $[x_n] = [y_m]$ if and only if for some $k \geq m, n$, we have

$$g_{n,k}(x_n) = g_{m,k}(y_m)$$

is equivalent to the fact that $x_n \sim g_n(x_n)$. Namely, each of these equivalence conditions generate the other one.

Proof. Without loss of generality, let $n \geq m$, and we see that $[x_n] = [y_m]$ if and only if

$$g_{n,k}(x_n) = g_{m,k}(y_m) \quad \forall k \geq m, n$$

Pick $k := n + 1$, then we see that:

$$g_{n,n+1}(x_n) = g_n(x_n) = g_n \cdots g_m(y_m)$$

Let us assume that for $y_m \in G_m$, that $y_m \sim g_m(y_m)$. Then, as $y_m \sim g_m(y_m)$, we see that if we set $y_{m+1} = g_m(y_m)$, then we obtain the following string of equivalences:

$$y_m \sim g_m(y_m) \sim g_{m+1}g_m(y_m) \sim \cdots \sim g_n \cdots g_m(y_m) = g_n(x_n) \sim x_n$$

Hence, we see that:

$$y_m \sim x_n$$

The opposite implication is essentially the same computation. Without loss of generality assume that $n \geq m$, lift every $y_{m+1} \in G_{m+1}$ and lift every $x_n \in G_n$

by their respective morphisms (let's say $y_{m+1} = g_m(y_m)$, $x_n = g_{n-1}(x_{n-1})$). By assumption, we have that $[x_n] = [y_m]$ (equivalently $x_n \sim y_m$) implies that:

$$\exists k \geq n \geq m : \; g_{n,k}(x_n) = g_{m,k}(y_m)$$

Choose $k := n$ (as the above equality holds for ANY $k \geq m, n$). By lifting, we obtain that:

$$g_{n,n}(x_n) = x_{n-1} = g_{m,n}(y_m) = g_{n-1}(y_{n-1})$$

Where $y_{n-1} = g_{m,n-1}(y_m)$. Therefore, we obtain that:

$$x_{n-1} = g_{n-1}g_{m,n-1}(y_m)$$

Meaning that $x_{n-1} \sim g_{m,n-1}(y_m)$. Besides $m \leq n$, m is arbitrary. Choose $m = n-2$, and recall that $x_{n-1} \sim y_{n-1}$, so that:

$$x_{n-1} \sim g_{n-1}(y_{n-1}) \sim g_{n-1}(x_{n-1})$$

This establishes the equivalence of the two definitions of the equivalence relation.

□

We leave it as an exercise to the reader that the above equivalence relation is, indeed, an equivalence relation.

Example 2.24. Direct Limit of Groups cont.

Now that we have checked that our equivalence is valid. The elements of G_∞ are equivalence classes of the form

$$[x_n] \quad x_n \in G_n$$

The binary operation on G_∞ is defined as:

$$[x] \cdot [y] = [x' \cdot y']$$

Where $x \sim x'$, $y \sim y'$, and $x', y' \in G_n$ for some n such that $n \geq N$. The unit element is $[u_0]$ (the unit element in G_0). Furthermore, we take $[x^{-1}] = [x]^{-1}$.

This is clearly well-defined under the equivalence relation discussed above, and this gives a group on G_∞.

Evidently, we have that all $u_n : G_n \to G_\infty$, defined by $u_n(x) = [x]$, are homomorphisms of groups.

Note: We will refer to the underlying set of G_∞ as G_ω.

Remark 2.38. Direct Limit of Groups, Universal Property

Aside from the explicit construction, we can immediately identify the direct limit of groups as a colimit in **Grps**, as it has the following universal property:

Given a cocone (a group and homomorphisms):

$$(H, h_n) \qquad h_n : G_n \to H$$

with $h_{n+1} \circ g_n = h_n$, define $h_\infty : G_\infty \to H$ by:

$$h_\infty([x_n]) = h_n(x_n)$$

This clearly satisfies the equivalence relation (the verification is tedious, but simple enough if you understood the proof of **Remark 2.37**).

Hence, we have verified the universal property of the colimit for the direct limit of groups.

Remark 2.39. Direct Limit of Groups, Universal Property, cont.

The above universal property (the direct limit of groups) is actually the same description of a colimit in **Sets**. The underlying set of G_∞ is G_ω, so replace all ∞ with ω, and we get the direct limit of sets for free.

This is a result of the **forgetful functor** U : **Grps** $\to$ **Sets** preserving colimits. We say that the forgetful functor creates ω-colimits.

Definition 2.44. Creating Limits

A functor $F : \mathbf{C} \to \mathbf{D}$ is said to **create limits of type J** if for every diagram $C : \mathbf{J} \to \mathbf{C}$, and limit $p_j : L \to FC_j$ in $\mathbf{D}$, $\exists!$ cone $\bar{p}_j : \bar{L} \to C_j$ in $\mathbf{C}$ with:

$$F(\bar{p}_j) = p_j \qquad F(\bar{L}) = L$$

the latter of which is a limit for the diagram C.

For **Creating Colimits**, just reverse all the morphisms here.

Proposition 105. *The forgetful functor U :* **Grps** $\to$ **Sets** *creates ω-colimits. It also creates all limits.*

Example 2.25. Cumulative Hierarchy of Sets

Remark 2.40. The main point that we encapsulated with the discussion above is that common constructions that satisfy universal properties are special cases of a general construction known as a **limit** and **colimit**. This is a large part of our discussion of categories.

Next, we will talk about universal properties that arise from things that are NOT (co)limits.

2.5 Exponentials

2.5.1 Exponential in a Category

Definition 2.45. Let us begin by considering a function of sets

$$f(x, y) : A \times B \to C$$

Hold $a \in A$ fixed, we have a function

$$f(a, y) : B \to C$$

We denote the image of this function f as $f(a, y) \in C^B$. C^B denotes the set of all functions whose image is in C, but the domain is in B.

Furthermore, letting a be the variable gives a map:

$$\tilde{f} : A \to C^B$$

$$\tilde{f} : a \longmapsto f(a, y)$$

Therefore, we see that the action of $\tilde{f}$ is to take $a \longmapsto f_a(y) := f(a, y) : B \to C$.

Note: To ease confusion, we can reconsider the above so that $\tilde{f}$ maps a to a *function* (or morphism) $B \to C$, and not a *value* (or image of f) in C:

$$\tilde{f} : a \longmapsto f(a, -)$$

Thus, $\tilde{f}(a)$ is still a function:

$$\tilde{f}(a) : C \to B$$

This $\tilde{f}$ is called the **transpose** of f.

We may express this resulting function (the image of the transpose) as:

$$\tilde{f}(a)(b) = f(a, b)$$

This operation is called **evaluation**.

Remark 2.41. Any map

$$\phi : A \to C^B$$

is uniquely of the form

$$\phi = \tilde{f}$$

for

$$f : A \times B \to C$$

We set

$$f(a, b) := \phi(a)(b)$$

Therefore, we have an isomorphism of sets

$$\mathrm{Sets}(A \times B, C) \cong \mathrm{Sets}(A, C^B)$$

In **Sets**, the evaluation is defined as:

$$\mathrm{eval} : C^B \times B \to C$$

$$\mathrm{eval} : (g, b) \longmapsto g(b)$$

Definition 2.46. Universal Property of the Evaluation Map

Given any set A, and any function $f : A \times B \to C$:

$$\exists! \tilde{f} : A \to C^B$$

such that:

$$\text{eval} \circ (\tilde{f} \times id_B) = f$$

i.e.

$$\text{eval}(\tilde{f}(a), b) = f(a, b)$$

This can be given by the following commutative diagram:

$$\begin{tikzcd}
C^B & C^B \times B \arrow[r, "\text{eval}"] & C \\
A \arrow[u, "\exists! \tilde{f}"] & A \times B \arrow[u, "\tilde{f} \times id_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Given the definition of the evaluation map, eval, a quick diagram chase shows that the above map eval, satisfies the commutativity of this diagram.

The above definition of the evaluation map will make sense in any category!

Definition 2.47. Universal Property of the Exponential

Let **C**, a category, have binary products. An **exponential** of objects B and C of **C** consists of an object:

$$C^B$$

and a morphism

$$\epsilon : C^B \times B \to C$$

such that for any object Z, and a morphism

$$f : Z \times B \to C$$

such that $\exists! \tilde{f} : Z \to C^B$ so that:

$$\epsilon \circ \left(\tilde{f} \times id_B\right) = f$$

This is illustrated as the commutativity of the following diagram:

$$\begin{tikzcd}
C^B & C^B \times B \arrow[r, "\epsilon"] & C \\
Z \arrow[u, "\exists! \tilde{f}"] & Z \times B \arrow[u, "\tilde{f} \times id_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We call the map

$$\epsilon : C^B \times B \to C$$

the **evaluation**.

The map

$$\tilde{f} : Z \to C^B$$

is called the (exponential) **tranpose** of f.

Given some morphism $g : Z \to C^B$, we write:

$$\bar{g} := \epsilon(g \times id_B) : Z \times B \to C$$

And we call $\bar{g}$ the **transpose** of g. Therefore, as the transpose is unique, we have that:

$$\bar{\tilde{g}} = g$$

and for any $f : Z \times B \to C$, we have:

$$\bar{\tilde{f}} = f$$

Tldr, the transpose of the transpose is the identity.

Proposition 106.

$$\mathrm{Hom}_{\mathbf{C}}(Z \times B, C) \cong \mathrm{Hom}_{\mathbf{C}}(Z, C^B)$$

Proof. The transpose gives the isomorphism:

$$\mathrm{Hom}_{\mathbf{C}}(Z \times B, C) \cong \mathrm{Hom}_{\mathbf{C}}(Z, C^B)$$

Where for $f \in \mathrm{Hom}_{\mathbf{C}}(Z \times B, C)$, we have $f \longmapsto \tilde{f}$, and likewise, for $g \in \mathrm{Hom}_{\mathbf{C}}(Z, C^B)$, we have $g \longmapsto \bar{g}$. By the **Universal Property of the Exponential**, these maps are unique, hence the mutual inverses are unique, giving a unique (canonical) isomorphism. $\square$

2.5.2 Cartesian Closed Categories

Definition 2.48. A category is called **Cartesian-closed** if it has all *finite* products and exponentials.

Example 2.26. Category of finite Sets is Cartesian-Closed

$\mathbf{Sets}_{fin}$, the category of finite sets, is cartesian-closed as $\mathrm{Hom}_{\mathbf{Sets}}(M, N)$ has cardinality:

$$|\mathrm{Hom}_{\mathbf{Sets}}(M, N)| = |N|^{|M|}$$

Furthermore, as $\mathbf{Sets}_{fin}$ has an embedding into $\mathbf{Sets}$, $\mathbf{Sets}$ is cartesian-closed.

Example 2.27. Category of Partially-Ordered Sets is Cartesian-Closed

The category, **Pos** has morphisms $f : P \to Q$, the monotone functions, satisfying:

$$p \le p' \implies fp \le fp'$$

Given posets P, Q, the poset $P \times Q$ has pairs (p, q) as elements. It is partially ordered by:

$$(p, q) \le (p', q') \text{ iff } p \le p',\ q \le q'$$

Therefore, the projections:

$$P \xleftarrow[\pi_1]{} P \times Q \xrightarrow[\pi_2]{} Q$$

are monotone functions. The pairing

$$\langle f, g \rangle : X \to P \times Q$$

is monotone if we enforce that $f : X \to P$ and $g : X \to Q$ are also monotone.

The exponential Q^P consists of the following:

A set of monotone functions:

$$Q^P = \{f : P \to Q \mid f \text{ monotone}\}$$

which is ordered **pointwise**, i.e.

$$f \le g \text{ iff } fp \le gp \quad \forall p \in P$$

The evaluation:

$$\epsilon : Q^P \times P \to Q$$

The transposition:

$$\tilde{f} : X \to Q^P$$

where f is defined by:

$$f : X \times P \to Q$$

And the action of ϵ is defined by:

$$\epsilon(\tilde{f} \times id_P) : X \times P \to Q$$

Remark 2.42. Evaluations are montone functions in Pos.

Proof. Let $(f, p) \le (f', p')$ in $Q^P \times P$. Then:

$$\epsilon(f, p) = f(p) \le f(p')$$

as Q^P are partially ordered pointwise. Then,

$$f(p) \le f(p') \le f'(p') = \epsilon(f', p')$$

Therefore, $\epsilon(f, p) \le \epsilon(f', p')$. $\square$

Remark 2.43. Transpositions are monotone functions in Pos.

Proof. Take $f : X \times P \to Q$ monotone. Let $x \leq x'$. We see that:

$$\tilde{f}(x) \leq \tilde{f}(x') \Longrightarrow \tilde{f}(x)(p) \leq \tilde{f}(x')(p) \quad \forall p \in P$$

$\tilde{f}(x)(p) = f(x,p) \leq f(x',p) = \tilde{f}(x')(p)$. Therefore, $\tilde{f}(x) \leq \tilde{x'}$ if $x \leq x'$. $\square$

Proposition 107. *In any cartesian-closed category,* $\mathbf{C}$*, exponentiation by a fixed object* A *is a functor.*

$$-^A : \mathbf{C} \to \mathbf{C}$$

Example 2.28. In the Category of Sets

We can check the above proposition in the category of sets:

$$\beta : B \to C$$

Now the exponentiation functor is

$$\beta^A : B^A \to C^A$$

$$\beta^A : f \longmapsto \beta \circ f$$

We can draw the action of this by:

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "\beta^A = \beta \circ f"] \arrow[d, phantom, shift left=3ex, "\xrightarrow{\beta^A}"] & \\
B \arrow[r, "\beta"'] & C
\end{tikzcd}$$

β^A is a functor as we may take $\alpha : C \to D$:

$$(\alpha \circ \beta)^A(f) = \alpha \circ \beta \circ f = \alpha \circ \beta^A(f) = \alpha^A \circ \beta^A(f)$$

Therefore, $(\alpha \circ \beta)^A = \alpha^A \circ \beta^A$.

Furthermore, we have that:

$$(id_B)^A(f) = id_B \circ f = f = id_{B^A}(f)$$

Therefore, $-^A$ is a functor.

Proof. **Proof of Above Proposition**

We may adopt the proof of the canonical example of Sets here.

Take $\beta : B \to C$ and define

$$\beta^A : B^A \to C^A$$

$$\beta^A := \beta \,\tilde{\circ}\, \epsilon$$

Where the action of β^A can be thought of as (fixing $a \in A$):

$$\beta^A : g \longmapsto \beta\epsilon(g, a) \quad \forall g \in B^A$$

This makes sense as we hold the "variable" a fixed, as A is fixed. Thus, the "evaluation" will occur by placing the morphism on the fixed variable.

Thus, we take the transpose of the map:

$$B^A \times A \xrightarrow{\epsilon} B \xrightarrow{\beta} C$$

$$\beta^A : B^A \to C^A$$

To illustrate this, we draw the following commutative diagram:

$$\begin{tikzcd}
C^A & C^A \times A \arrow[r, "\epsilon"] & C \\
B^A \arrow[u, "\beta^A"] & B^A \times A \arrow[u, "\beta^A \times id_A"] \arrow[r, "\epsilon"] & B \arrow[u, "\beta"']
\end{tikzcd}$$

Notice that we may diagram chase this to verify the commutativity of this diagram.

Take $g \in B^A$ and fix $a \in A$. Then going up the left side of the square:

$$(\beta^A \times id_A)(g, a) = \beta^A(g) \times a = \tilde{\beta}\epsilon(g) \times a = \beta\epsilon(g, -) \times a$$

So that

$$\epsilon \circ \beta^A \times id_A = \epsilon(\beta\epsilon(g, -), a) = \beta g(a)$$

Now going up the right side of the square, it is trivial to see that for any $g \times a \in B^A \times A$, $\epsilon(g \times a) = g(a)$ and $\beta\epsilon(g \times a) = \beta g(a)$.

Hence, this diagram really is commutative, ensuring that the definition of β^A is correct.

We will now only verify that composition is preserved (the identity map is trivial):

Take $\alpha \circ \beta$ so that:

$$\begin{tikzcd}
D^A & D^A \times A \arrow[r, "\epsilon"] & D \\
C^A \arrow[u, "\alpha^A"] & C^A \times A \arrow[u, "\alpha^A \times id_A"] \arrow[r, "\epsilon"] & C \arrow[u, "\alpha"'] \\
B^A \arrow[u, "\beta^A"] & B^A \times A \arrow[u, "\beta^A \times id_A"] \arrow[r, "\epsilon"] & B \arrow[u, "\beta"']
\end{tikzcd}$$

Using that $\left(\alpha^A \times id_A\right) \circ \left(\beta^A \times id_A\right) = (\alpha \circ \beta)^A \circ id_A$, we perform a diagram chase up the right side of the above rectangle, for any $g \in B^A$:

$$\beta^A(g) = \beta\epsilon(g, a)$$

$$\alpha^A\beta^A(g) = \alpha^A(\beta\epsilon(g, a)) = \alpha\beta\epsilon(g, a)$$

Now performing a diagram chase on the left side:

$$\epsilon \circ \left(\alpha^A \times id_A\right) \circ \left(\beta^A \times id_A\right)$$

implies that

$$\epsilon(\alpha^A \circ \beta^A, a) = \alpha^A\beta^A(g) = \alpha\beta\epsilon(g, a) \quad \forall g \in B^A$$

Thus, this diagram commutes. □

Remark 2.44. We can now look at the transpose of the identity map:

$$id_{A\times B} : A \times B \to A \times B$$

$$\tilde{id}_{A\times B} : A \to (A \times B)^B$$

We can think of this in the above way that we thought about the transpose in **Definition 2.45**:

Let $a \in A$:

$$\tilde{id}_{A\times B} : a \longmapsto id_{A\times B}(a, -)$$

so that for any $b \in B$:

$$\tilde{id}_{A\times B}(a) : b \longmapsto id_{A\times B}(a, b)$$

As a shorthand, we set $\eta := \tilde{id}_{A\times B}(-, -)$.

Example 2.29. Computing the Transpose in the Category of Sets

Consider the above in **Sets**:

$$\eta(a)(b) = id_{A\times B}(a, b) = (a, b)$$

Therefore, we see that given some $\tilde{f}$ where $f : Z \times A \to B$:

$$f^A : (Z \times A)^A \to B^A$$

Where, by functoriality of $-^A$:

$$f^A = (f \tilde{\circ} \epsilon)$$

So that:

$$f^A \circ \eta = \tilde{f}$$

To check this, we can simply chase the following diagram:

$$\begin{tikzcd}
(Z \times A)^A \arrow[r, "f^A"] & B^A \\
Z \arrow[u, "\eta"] \arrow[ur, "\tilde{f}"']
\end{tikzcd}$$

Let $z \in Z$. Then, we see that $\eta : z \longmapsto \eta(z,-)$. Apply f^A:

$f^A\eta z = f^A\eta(z,-) = f \tilde{\circ}\, \epsilon\eta(z,-) = f\epsilon(\eta(z,-) \times id_A) = f\eta(z,-)id_A \equiv f\eta(z,-) = f(z,-)$

Where[14] $f(z,-) : A \to B$.

Now chase the slanted arrow, trivially resulting in:

$$\tilde{f} : z \longmapsto f(z,-)$$

This follows by **Definition 2.45** on the **transpose**.

Therefore, we see that this diagram commutes, so our transpose, itself, has a universal property in **Sets**.

Example 2.29 holds in any CCC (Cartesian-Closed Category).

Definition 2.49. Universal Property of the Transpose

Given an arbitrary object Z in a CCC (Cartesian-Closed Category), and a morphism $f : Z \times A \to B$, $\exists! \eta : Z \to (Z \times A)^A$ such that the following diagram commutes:

$$\begin{tikzcd}
(Z \times A)^A \arrow[r, "f^A"] & B^A \\
Z \arrow[u, "\exists!\eta"] \arrow[ur, "\tilde{f}"']
\end{tikzcd}$$

i.e. $\tilde{f} = f^A \circ \eta$

Proof. Although we gave the Universal Property as a definition, it is somewhat a nontrivial fact that requires some proof.

We must prove that this η uniquely exists. So fix $a \in A$ and define η in the following way:

$$\eta : Z \to (Z \times A)^A$$

$$\eta := id_{Z \times A}(-,-)$$

[14]Note that there is some abuse of notation going on here. Technically, we fix some $a \in A$ and say that $f \tilde{\circ}\, \epsilon\eta(z,-) = f\epsilon(\eta(z,-),a) = f(z,a)$ for any $a \in A$, but it is more notationally clear that our resulting element is a function in B^A the way we wrote it.

So that:

$$\eta(z) = id_{Z\times A}(z, -) \quad \forall z \in Z$$

The diagram chase in **Example 2.29** is true for any CCC, therefore, this η uniquely exists and makes this diagram commute. $\square$

Remark 2.45. **Definition 2.49** tells us that the transpose of a morphism $f : Z \times A \to B$ is computable using a universal morphism η, and the exponentiation functor $f^A : (Z \times A)^A \to B^A$.

2.5.3 Equational Definition

We can describe CCC's completely with the following proposition.

Proposition 108. *A category* $\mathbf{C}$ *is a CCC if and only if it has the following structure:*

1. *A distinguished object* 1*, and for each object* C*, there is a given morphism:*
$$!_C : C \to 1$$
such that for any morphism $f : C \to 1$*:*
$$f =!_C$$
2. *For each pair of objects* A, B*, there is a given object* $A \times B$ *and morphisms:*
$$p_1 : A \times B \to A \qquad p_2 : A \times B \to B$$
and for each pair of morphisms $f : Z \to A$ *and* $g : Z \to B$*, there is a given morphism*
$$\langle f, g\rangle : Z \to A \times B$$
such that:
$$p_1\langle f, g\rangle = f$$
$$p_2\langle f, g\rangle = g$$
$$\langle p_1h, p_2h\rangle = h \quad \forall h : Z \to A \times B$$
3. *For each pair of objects* A, B*, there is a given object* B^A *and a morphism*
$$\epsilon : B^A \times A \to B$$
and for each morphism $f : Z \times A \to B$*, there is a given morphism*
$$\tilde{f} : Z \to B^A$$
such that
$$\epsilon \circ (\tilde{f} \times id_A) = f$$
and $\forall g : Z \to B^A$ *where*
$$g \times id_A = \langle gp_1, p_2\rangle : Z \times A \to B^A \times A$$
we have that
$$(\epsilon \circ (g \tilde{\times} id_A)) = g$$

We can check these equational conditions rather than the universal properties.

Proof. The first two follow by the **Universal Property of Limits** (namely of the product and terminal object).

The only nontrivial assertion is the last one. However, clearly, the first equational condition follows immediately from the **Universal Property of the Exponential**. The only one that remains to check is the last one.

Take

$$\epsilon \circ (\tilde{f} \times id_A) = f$$

Recall by the **Universal Property of the Transpose**, we have that for $f : Z \times A \to B$:

$$\tilde{f} = f^A \eta$$

If we take the transpose of this:

$$\tilde{\tilde{f}} = \tilde{f^A}\eta = (f \circ \tilde{\epsilon} \circ \eta)$$

If we take any $z \in Z$, $a \in A$, we have that:

$$\tilde{\tilde{f}}z = (f \circ \tilde{\epsilon} \circ \eta)z = f\epsilon(\eta(z,-) \times id_A) = f\eta(z,a) = f(z,a)$$

By uniqueness of the transpose (**Definition 2.47**), we have that $\tilde{\tilde{f}} = f$. Hence, it immediately follows that:

$$\tilde{f} = (\epsilon \circ (f \tilde{\times} id_A))$$

as desired. □

Definition 2.50. Therefore, we may summarize a **Cartesian-Closed Category** (a CCC) as a category C that has the following:

(i) Has all finite limits

(ii) Has all exponentials

2.6 Functors and Naturality

We will be introduced to the first notion of higher categories here. We will also see the general theory of functors.

2.6.1 Category of Categories

Definition 2.51. Category of Categories

Consider the category of categories and functors, **Cat**. **Cat** has finite coproducts **0** and **C** + **D**, and all finite products **1** and **C** × **D**.

We can show that **Cat** has all limits by constructing equalizers as follows:

Consider a pair of parallel functors F, G, and categories $\mathbf{C}, \mathbf{D}$. We define the equalizer category, $\mathbf{E}$ and the functor E as follows:

$$\begin{tikzcd} \mathbf{E} \arrow[r, "E"] & \mathbf{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbf{D} \end{tikzcd}$$

Where

$$\mathbf{E}_0 = \{C \in \mathbf{C}_0 \mid F(C) = G(C)\}$$
$$\mathbf{E}_1 = \{f \in \mathbf{C}_1 \mid F(f) = G(f)\}$$

And $E : \mathbf{E} \to \mathbf{C}$ is the obvious inclusion.

This satisfies the **Universal Property of the Equalizer**. For a given category $\mathbf{Z}$, and a given morphism $H : \mathbf{Z} \to \mathbf{C}$, $\exists! U : \mathbf{Z} \to \mathbf{E}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{E} \arrow[r, "E"] & \mathbf{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "F"'] & \mathbf{D} \\ \mathbf{Z} \arrow[u, "\exists! U"] \arrow[ur, "H"'] & & \end{tikzcd}$$

As the inclusion, E, is monic, we see that:

$$H = EU \implies U = E^{-1}H$$

Therefore, the unique existence follows immediately.

Definition 2.52. Category of Categories (cont.)

The equalizer (object) $\mathbf{E}$, is an example of a **subcategory**, a monomorphism in **Cat**. A **subcategory** of a category $\mathbf{C}$ is a collection $\mathbf{U}$ of some objects and morphisms of $\mathbf{C}$ that is closed under composition, contains the identity morphism, and contains the codomain and domain of all of its morphisms. The subcategory is uniquely equipped with an **inclusion functor**

$$i : \mathbf{U} \to \mathbf{C}$$

which is monic.

Remark 2.46. Coequalizers of categories are not as well-defined analogously in arbitrary categories of categories.

Definition 2.53. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be:

1. **injective on objects** if the object part $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ is injective. Likewise for **surjective on objects**.
2. **injective on morphisms** if the morphism part $F_1 : \mathbf{C}_1 \to \mathbf{D}_1$ is injective. Likewise for **surjective on morphisms**.

3. **faithful** if $\forall A, B \in \mathbf{C}_0$, the map

$$F_{A,B} : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(FA, FB)$$

$$F_{A,B} : f \longmapsto F(f)$$

   is always injective for every pair of objects $A, B \in \mathbf{C}$.

4. **full** if $F_{A,B}$ is always surjective for every pair of objects $A, B \in \mathbf{C}$.

5. **fully faithful** if $F_{A,B}$ is both full and faithful (i.e. injective and surjective for every pair of objects $A, B \in \mathbf{C}$).

Remark 2.47. Note that there is a difference between being **full/faithful** and being **surjective/injective** on morphisms.

The important point here is that being full and faithful are both conditions of surjectivity/injectivity for **every pair of objects** in the domain of the functor $F : \mathbf{C} \to \mathbf{D}$.

Example 2.30. Faithful but Not Injective

An example to show this is the following diagram for the *codiagonal functor*, $\nabla : \mathbf{C} + \mathbf{C} \to \mathbf{C}$

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "id_C"] & C + C \arrow[d, "\nabla"] & C \arrow[l] \arrow[dl, "id_C"'] \\
& C &
\end{tikzcd}$$

Here, ∇ is faithful, but not injective on morphisms.

Example 2.31. Faithful but Not Injective (cont.)

Another simpler example to show that faithful does not mean injective is the following[15]:

Consider a four-object category $\mathbf{C}$:

$$\mathbf{C}_0 = \{A, B, C, D\}$$

with the morphisms given by:

$$\mathrm{Hom}_{\mathbf{C}}(A, B) = \{f_i\}$$

$$\mathrm{Hom}_{\mathbf{C}}(C, D) = \{g_i\}$$

$$\mathrm{Hom}_{\mathbf{C}}(A, D) = \{id_A, id_D\}$$

$$\mathrm{Hom}_{\mathbf{C}}(B, C) = \{id_B, id_C\}$$

[15] Credit to Josh Chen on Math Stackexchange for the example. Look up "Difference between being faithful and being injective on arrows".

And likewise for the other pairs of objects.

Now consider the two-object category $\mathbf{D}$, where:

$$\mathbf{D}_0 = \{P, Q\}$$

and

$$\mathrm{Hom}_{\mathbf{D}}(P, Q) = \{h_i\}$$

Now consider a functor $F : \mathbf{C} \to \mathbf{D}$ where F maps the objects of $\mathbf{C}$ in the following way

$$A \longmapsto P$$

$$B \longmapsto Q$$

$$C \longmapsto P$$

$$D \longmapsto Q$$

where

$$f_i \longmapsto h_i$$

$$g_i \longmapsto h_i$$

Then, clearly, it is true that for every pair of objects in $\mathbf{C}$, the mapping:

$$F_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}(P, Q)$$

is faithful as the only possibilities are:

$$f_i \longmapsto h_i$$

$$g_i \longmapsto h_i$$

$$id_{X,Y} \longmapsto id_{P,Q}$$

Where $id_{X,Y}$ denotes an identity map of either X or Y. Each assignment is a unique one, hence $F_{X,Y}$ is injective, so F is faithful.

F is no injective on objects or morphisms because the assignments of morphisms is not unique in general.

Definition 2.54. A **full subcategory**

$$\mathbf{U} \twoheadrightarrow \mathbf{C}$$

consists of some objects of $\mathbf{C}$ and **all of the morphisms** between them.

Example 2.32. Examples of Full Subcategories

The inclusion functor

$$\mathbf{Sets}_{fin} \rightarrowtail \mathbf{Sets}$$

is full and faithful.

Whereas, the forgetful functor

$$\mathbf{Grps} \rightarrowtail \mathbf{Sets}$$

is faithful, but not full.

Example 2.33. Forgetful Functors

There exists a forgetful functor

$$G : \mathbf{Grps} \to \mathbf{Cat}$$

G is obviously full. Furthermore, it is faithful because for every pair of groups, A, B, we have that:

$$G_{A,B} : \mathrm{Hom}_{\mathbf{Grps}}(A, B) \to \mathrm{Hom}_{\mathbf{Cat}}(G(A), G(B))$$

However, for any group homomorphism $f : A \to B$, we have that $G(f) : G(A) \to G(B)$ is itself a group homomorphism. Furthermore, every functor between objects $G(A), G(B) \in \mathbf{Cat}$ is actually just a group homomorphism. Therefore, the assignment by $G_{A,B}$ uniquely identifies group homomorphisms in **Grps** with group homomorphisms in **Cat**, hence G is faithful.

This is exactly true for the functor:

$$M : \mathbf{Mon} \to \mathbf{Cat}$$

Even for the functor:

$$P : \mathbf{Pos} \to \mathbf{Cat}$$

this holds again, as the map:

$$P_{A,B} : \mathrm{Hom}_{\mathbf{Pos}}(A, B) \to \mathrm{Hom}_{\mathbf{Cat}}(P(A), P(B))$$

dictates that for any monotone map $f : A \to B$, that:

$$P(f) : P(A) \to P(B)$$

is again a monotone map, therefore, the exact assignment of f is unique, and it is faithful.

Remark 2.48. Notice that **Cat** is very useful for comparing structures because the notion of a category is extremely broad.

Example 2.34. Functors in Cat

Let us consider the functors in two instances: (i) $G \to P$ where G is a group and P is a poset (ii) $P \to G$.

(i) A poset, considered as a category where every object is an element with a morphism regarded as $\leq$ which indicates the partial order between elements. A group is considered as a 1-object category where every morphism is an isomorphism. With these in mind, we see that a functor

$$p : G \to P$$

$$p : * \longmapsto ?$$

$$p : g \longmapsto ??$$

? indicates the object in P, and ?? indicates the morphism in P. The action of p on the objects of G is not well-defined if we map to more than one object of P, therefore, we see that the action of p on the objects of G is **trivial**, i.e. $p(*) = p$ where $p \in P$ is some fixed element. For some morphism in G, we have that it corresponds to at most one partial order between any two objects of P, however, since p only maps to one object, p, we see that the only morphism in P is $=$. Therefore, this functor p is trivial.

(ii) The other direction is a lot more interesting. For $g : P \to G$, we have, since G only has one object, that

$$g(p) = * = g(q) \quad \forall p, q \in P$$

Therefore, for every $p \leq q$, we pick an element $g_{p,q}$ so that

$$g_{p,p} = u$$

$$g_{p,r} g_{q,p} = g_{q \leq r}$$

This is the **poset representation of a group**.

We can recreate what we know with this, consider:

$$g : (\mathbb{R}, \leq) \to (\mathbb{R}, +)$$

defined by

$$g_{x,y} = (y - x) \in (\mathbb{R}, +) \quad \forall x \leq y \in (\mathbb{R}, \leq)$$

so that

$$g_{x,x} = 0$$

$$g_{y,z} g_{x,y} = (z - y) + (y - x) = (z - x) = g_{x,z}$$

Therefore, subtraction is a functor between the poset of real numbers and the group of real numbers under addition. This identical construction works for any additive group with a total ordering (only $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ satisfy this, up to isomorphism).

2.6.2 Representable Structure

Definition 2.55. Let $\mathbf{C}$ be a locally small category. Consider the representable functor

$$\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

for all objects $C \in \mathbf{C}$.

This functor is faithful if the object C has the following property:

For any objects X, Y and morphisms $f, g : X \rightrightarrows$, if $f \neq g$, then there is a morphism $x : C \to X$ so that $fx \neq gx$.

This C is called a **generator**[16] for $\mathbf{C}$.

[16] Other literature calls this a **separator**, we will use generator to denote this idea.

Alternatively, a generator is defined as an object C such that for all $x : C \to X$, if $fx = gx$, then $f = g$. This implies that C is the object that ensures that x is an epimorphism.

Example 2.35. Terminal Objects in (many) Concrete Categories are Generators

In **Sets**, the terminal object 1 is a generator because

$$1 = \{*\} \in \mathbf{Sets}$$

Therefore, any pair of morphisms acting on $x \in 1$

$$f, g : X \rightrightarrows Y \quad \forall X \in \mathbf{Sets}$$

such that $fx = gx$, must satisfy $f = g$ as $x = * \quad \forall x \in 1$. Therefore, we have $f* = g*$, and the fact that $f = g$ follows by well-definedness.

Example 2.36. Free Groups Generated by a Terminal Object are Generators

The free group on one element is a generator. This follows by the fact that the one-element group (a terminal object in **Sets**) generates $F(1)$. And because of the preceding example[17], we see that $\forall f \neq g : X \to Y$, there is a $x : 1 \to X$ so that $fx \neq gx$. This implies that all distinct group homomorphisms result in different elements. Hence, as this holds on the generator of $F(1)$, this must also be true on $F(1)$. The set of all such $x : F(1) \to X$ is denoted by

$$\mathrm{Hom}_{\mathbf{Grps}}(F(1), X)$$

As all maps from $F(1) \to X$ form an element in X, we see that the following isomorphism is clear:

$$\mathrm{Hom}_{\mathbf{Grps}}(F(1), G) \cong U(g)$$

Where G is an object in **Grps** and U is a forgetful functor $U : \mathbf{Grps} \to \mathbf{Sets}$.

Example 2.37. Naturality of the Preceding Isomorphism

This isomorphism holds for every group G, but it respects group homomorphisms in the following way:

Given any $h : G \to H$ (H and G are groups), the following diagram commutes:

$$\begin{tikzcd}
G \arrow[d, "h"] & \mathrm{Hom}(F(1), G) \arrow[r, "\cong"] \arrow[d, "h_*"] & U(G) \arrow[d, "U(h)"] \\
H & \mathrm{Hom}(F(1), H) \arrow[r, "\cong"] & U(H)
\end{tikzcd}$$

Where h_* refers to the map of Hom sets induced by h.

We refer to the isomorphism established above as being **natural in** G.

[17] That 1 is a generator (separator).

Remark 2.49. The commutativity of the above square and the isomorphism

$$\mathrm{Hom}_{\mathbf{Grps}}(F(1), G) \cong U(G)$$

gives a barebones explanation of why the *forgetful functor preserves all limits*. Recall that in **Proposition 103**, we concluded that **representable functors preserve all limits**. Since we have established an isomorphism between the forgetful functor and the representable functor (over a locally small category), we see that this is necessarily true.

The faithfulness of the forgetful functor captures the idea that a category like **Grps** is a **concrete category**.

Remark 2.50. Let $f : A \to B$, and recall the **contravariant** representable functors.

$$\mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{op} \to \mathbf{Sets}$$

$$f \longmapsto f^*$$

Where $f^* : \mathrm{Hom}_{\mathbf{C}}(B, C) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$ is defined by

$$f^* h = h \circ f \quad \forall h : B \to C$$

Example 2.38. Contravariant Representable Functors in Action

Let G be a group in a locally small category $\mathbf{C}$. The contravariant representable functor $\mathrm{Hom}_{\mathbf{C}}(-, G)$ has a group structure, resulting in a functor:

$$\mathrm{Hom}_{\mathbf{C}}(-, G) : \mathbf{C}^{op} \to \mathbf{Grps}$$

For $\mathbf{C} = \mathbf{Sets}$, for every set X, we define the following operations on the group $\mathrm{Hom}_{\mathbf{Sets}}(X, G)$:

$$u(x) = u$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$f^{-1}(x) = f(x)^{-1}$$

Where $x \in X$.

As G consists of all the elements dictated by an image of the morphism $f : X \to G$, we see that the set of all morphisms from $X \to G$ is dictated by

$$\prod_{x \in X} G$$

Hence,

$$\mathrm{Hom}_{\mathrm{Sets}}(X, G) \cong \prod_{x \in X} G$$

This is easily seen in finite groups, where the cardinality of the set of all maps $X \to G$ is seen as

$$|\mathrm{Hom}_{\mathrm{Sets}}(X, G)| = |G|^{|X|}$$

The notion of an exponential also illustrates that

$$\mathrm{Hom}_{\mathrm{Sets}}(X, G) = G^X \cong \prod_{x \in X} G$$

Given any $h : Y \to X$, the functor h^* has the action:

$$\begin{aligned} h^*(f \cdot g)(y) &= (f \cdot g)(h(y)) = f(h(y)) \cdot g(h(y)) \\ &= h^*(f)(y) \cdot h^*(g)(y) = (h^*(f) \cdot h^*(g))(y) \end{aligned}$$

The functoriality also holds for inverses and units.

This works just as well in any monoidal category, although with the absence of inverses and units as necessary.

Example 2.39. Previous Example cont.

Consider the case where $\mathbf{C} = \mathbf{Top}$. For any object $X \in \mathbf{Top}$, the ring

$$\mathcal{C}(X) = \mathrm{Hom}_{\mathbf{Top}}(X, \mathbb{R})$$

of all continuous, real-valued functions on X.

Now take any continuous function $h : Y \to X$. And the contravariant representable functor $\mathrm{Hom}_{\mathbf{Top}}(-, \mathbb{R})$ induces a ring homomorphism

$$h^* : \mathcal{C}(X) \to \mathcal{C}(Y)$$

by precomposition (as usual).

The "Ring of Real-Valued Functions" is, therefore a functor:

$$\mathcal{C} : \mathbf{Top}^{op} \to \mathbf{Rings}$$

When we pass $\mathbb{R}$ to $\mathrm{Hom}_{\mathbf{Top}}(X, \mathbb{R})$, the algebraic structure of $\mathbb{R}$ is retained **but any properties that are not specified in target category are not necessarily preserved**. Note that $\mathbb{R}$ is a field, so, namely, it has multiplicative inverses well-defined. But notice that the target category of $\mathcal{C}$ is **Rings**, therefore, $\mathcal{C}$ does not necessarily preserve the field structure of $\mathbb{R}$, only the ring structure of it.

To see this, consider that:

$$f(x) = x^2 \quad f \in \mathcal{C}(\mathbb{R})$$

The multiplicative inverse is defined by

$$g(x) = 1/x^2$$

But, g is not continuous at $x = 0$, hence, it is not continuous $\forall x \in \mathbb{R}$, so $g \notin \mathcal{C}(\mathbb{R})$ and $g \notin \mathcal{C}(\mathbb{R})$. So the field structure is not preserved. It is very easy to show that the set of continuous real-valued functions is a ring however; that is one thing that it does preserve.

Example 2.40. Boolean Algebras

Denote the category of Boolean Algebras as **BA**. Given the boolean algebra **2**, with the operations (given by a truth table):

$$\{\wedge, \vee, \neg, 0, 1\}$$

we make the set

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathbf{2})$$

into a boolean algebra with the following operations

$$0(x) = 0$$

$$1(x) = 1$$

$$(f \wedge g)(x) = f(x) \wedge g(x)$$

$$(f \vee g)(x) = f(x) \vee g(x)$$

$$f(\neg x) = \neg f(x)$$

The precomposition by the functor

$$\mathrm{Hom}_{\mathbf{Sets}}(-, \mathbf{2}) : \mathbf{Sets}^{op} \to \mathbf{BA}$$

is contravariant in nature.

For any set X, we see that the isomorphism

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathbf{2}) \cong \mathcal{P}(X)$$

between the characteristic functions

$$\phi : X \to \mathbf{2}$$

and the subsets

$$V_\phi = \phi^{-1}(1) \subseteq X$$

gives a correspondence between the boolean operations in $\mathrm{Hom}_{\mathbf{Sets}}(X, \mathbf{2})$ to the typical subset operations:

$$V_{\phi \wedge \psi} = V_\phi \cap V_\psi$$

$$V_{\phi \vee \psi} = V_\phi \cup V_\psi$$

$$V_{\neg \phi} = X - V_\phi$$

$$V_1 = X$$

$$V_0 = \varnothing$$

We can also denote the contravariant representable functor into the category of boolean algebras as the powerset functor $\mathcal{P}^{BA}$. Therefore, the set-theoretic operations on $\mathcal{P}(X)$ are induced by the operations on the boolean algebra **2**, and this, specifically, gives the relation between de-Morgan's Laws and the logic of set theory.

2.6.3 Stone Duality Theorem

We will come back to this, regard the theorem/proposition numbers ahead as tentative.

2.6.4 Naturality

We will just throw out the definition of a natural transformation and elaborate on it.

Definition 2.56. A **natural transformation** is a morphism of functors.

Remark 2.51. Fix categories $\mathbf{C}, \mathbf{D}$. Any functor $\mathbf{C} \to \mathbf{D}$ can be regarded as objects in their own category! The morphisms between these objects are precisely the natural transformations.

Example 2.41. Bare Minimum Working Example

We are going to assume that we have two *constructions* on a category $\mathbf{C}$. We want to relate these *constructions* in a way independent of the morphisms and objects that are involved.

A simple working example is the product of three objects in $\mathbf{C}$.

$$(A \times B) \times C \cong A \times (B \times C)$$

We know that in a category that has all limits, this is clearly true. Therefore, we can establish the isomorphism

$$h : (A \times B) \times C \overset{\cong}{\to} A \times (B \times C)$$

However, to show that it is the case that the product is independent of choice of object and morphism, we can think of it like this:

Given a $f : A \to A'$, we get a commutative diagram

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, "h_A"] \arrow[d] & A \times (B \times C) \arrow[d] \\
(A' \times B) \times C \arrow[r, "h_{A'}"] & A' \times (B \times C)
\end{tikzcd}$$

So, in lieu of the objects A, A', we see that the isomorphism h is between the *constructions*

$$(- \times B) \times C \cong - \times (B \times C)$$

$-$ denotes an argument which takes an object or a morphism.

Notice that in our discussion on the product, a *construction* just means a functor, and a "relation between constructions" is a morphism of functors. The morphism of functors in the example is the isomorphism given above.

In fact, we can get rid of every object completely and define the functor:

$$F = (-_1 \times -_2) \times -_3 : \mathbf{C}^3 \to \mathbf{C}$$

$$G = -_1 \times (-_2 \times -_3) : \mathbf{C}^3 \to \mathbf{C}$$

And it is true that $F \cong G$.

Definition 2.57. For categories $\mathbf{C}, \mathbf{D}$, and functors

$$F, G : \mathbf{C} \to \mathbf{D}$$

a **natural transformation**, $\nu : F \to G$, is a family of morphisms in $\mathbf{D}$

$$(\nu_C : FC \to GC)_{C \in \mathbf{C}_0}$$

with the following property:

For any $f : C \to C'$ in $\mathbf{C}$, we have that $\nu_{C'} \circ F(f) = G(f) \circ \nu_C$

i.e. the following diagram commutes

$$\begin{tikzcd} FC \arrow[r, "\nu_C"] \arrow[d, "Ff"] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\nu_{C'}"] & GC' \end{tikzcd}$$

Given a natural transformation $\nu : F \to G$, the $\mathbf{D}$-morphism $\nu_C : FC \to GC$ is called the **component of ν at** C.

2.6.5 Examples of Natural Transformations

We will look at some examples of Natural Transformations.

Example 2.42. Take the free monoid $M(X)$ where X is a set. The natural transformation $\eta : 1_{\mathbf{Sets}} \to UM$ (where U is the forgetful functor $\mathbf{Mon} \to \mathbf{Sets}$) is defined as the family $\{\eta_X\}_{X \in \mathbf{Sets}}$ where each η_X is defined as

$$\eta_X : X \to UM(X)$$

Where each η_X is the *insertion of generators*, where an element $x \in X$ is taken to itself in the underlying set of $M(X)$. Given any $f : X \to Y$ (in $\mathbf{Sets}$), we have that the following commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"] & UM(X) \arrow[d, "UM(f)"] \\ Y \arrow[r, "\eta_Y"] & UM(Y) \end{tikzcd}$$

The "naturality" here comes from the fact that f is arbitrary, and completely defines $M(f)$.

Remark 2.52. Note the similarity in the idea of "universal properties" and the idea of "naturality".

In the former, we assume that we are given an arbitrary object and morphism so that it allows a diagram to commute. This is rigorously encapsulated by the idea of a *limit* and a *colimit* for certain universal constructions, which gives an arbitrary object and family of morphisms so that the "(co)limit" is the "best" way to complete the diagram.

In the latter, for a functor $F : \mathbf{C} \to \mathbf{D}$, we simply assume an arbitrary morphism between two arbitrary objects in $\mathbf{C}$, and the "natural transformation" is the family of morphisms such that we yield a commutative square indicated above.

Example 2.43. Products as Functors

Let $\mathbf{C}$ be a category with products. Fix $A \in \mathbf{C}$. A natural transformation, η, between the functors

$$A \times - : \mathbf{C} \to \mathbf{C}$$

$$1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$$

is given by

$$\eta_C := \pi_2 : A \times C \to C$$

We can now take the pairing operation $\langle -, - \rangle$ so that we can get the isomorphism

$$h : (A \times B) \times C \overset{\cong}{\to} A \times (B \times C)$$

by placing η in either argument of the pairing operation.

Consider the functor

$$\times : \mathbf{C}^2 \to \mathbf{C}$$

Whose action on objects $(A, B) \in \mathbf{C}^2$ is:

$$(A, B) \to A \times B$$

and whose action on morphisms, f, g, is:

$$(f, g) \to f \times g$$

Likewise, consider the functor

$$\bar{\times} : \mathbf{C}^2 \to \mathbf{C}$$

Whose action on objects (A, B) is given by

$$A \bar{\times} B = B \times A$$

and whose action on morphisms f, g, is:

$$(f, g) \to g \times f$$

We define a "twisting" natural transformation $t : \times \to \bar{\times}$ by

$$t_{(A,B)}\langle a, b\rangle = \langle b, a\rangle$$

Let us check that the following commutes

$$\begin{tikzcd} A \times B \arrow[r, "t_{(A,B)}"] \arrow[d, "\alpha\times\beta"] & A\bar{\times}B \arrow[d, "\alpha\bar{\times}\beta"] \\ A' \times B' \arrow[r, "t_{(A',B')}"] & A'\bar{\times}B' \end{tikzcd}$$

We diagram chase this. Start on the left side from the top left hand corner. Let Z be an arbitrary object in $\mathbf{C}$, and let $a : Z \to A$ and $b : Z \to B$ be generalized elements.

$$(\beta \times \alpha)t_{(A,B)}\langle a, b\rangle = (\beta \times \alpha)\langle b, a\rangle = \langle \beta b, \alpha a\rangle$$
$$= t_{(A',B')}\langle \alpha a, \beta b\rangle = t_{(A',B')}(\alpha \times \beta)\langle a, b\rangle$$

Therefore, we see that the above square commutes, and that $t : \times \to \bar{\times}$ is, indeed, a natural transformation.

For the family $\{t_{(A,B)}\}_{(A,B)\in\mathbf{C}^2}$, every $t_{(A,B)}$ has an inverse $t_{(B,A)}$, thus, it is an isomorphism for every object (A, B). Therefore, t is actually a *natural isomorphism.*

Definition 2.58. The **functor category**, $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$, has

Objects:

The objects are functors $F : \mathbf{C} \to \mathbf{D}$.

Morphisms:

The morphisms are natural transformations $\nu : F \to G$, where F,G, H are functors $\mathbf{C} \to \mathbf{D}$, and let C be an object of $\mathbf{C}$.

For every object F, id_F has components:

$$(id_F)_C = id_{FC} : FC \to FC$$

The composition $F \xrightarrow{\nu} G \xrightarrow{\phi} H$ has components:

$$(\phi \circ \nu)_C = \phi_C \circ \nu_C$$

Definition 2.59. A **natural isomorphism** is a natural transformation

$$\nu : F \to G$$

which is an isomorphism in the functor category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$.

Lemma 109. *A natural transformation $\nu : F \to G$ is a natural isomorphism if and only if each component $\nu_C : FC \to GC$ is an isomorphism.*

Proof. This is a fundamental fact, so let us prove it.

Assume that $\nu : F \to G$ is a natural isomorphism. Clearly it satisfies naturality, and as it is an isomorphism in Fun$(\mathbf{C}, \mathbf{D})$, it has a well-defined mutual inverse $\phi : G \to F$, so that $\forall C \in \mathbf{C}_0$

$$(\nu \circ \phi)_C = \nu_C \circ \phi_C = (id_G)_C$$

$$(\phi \circ \nu)_C = \phi_C \circ \nu_C = (id_F)_C$$

Given this, it is now clear that $\nu_C : FC \to FC$ has a mutual inverse $\phi_C : GC \to FC$, hence an isomorphism of objects in $\mathbf{D}$.

Assume that $\nu_C : FC \to GC$ is an isomorphism in $\mathbf{D}$. So it has a mutual inverse $\phi_C : GC \to FC$. We see that for an arbitrary morphism in $\mathbf{C}$, $f : C \to C'$, we see that:

$$Gf \circ \nu_C : FC \to GC'$$

Likewise, given $\nu_{C'} : FC' \to GC'$, we have that

$$\nu_{C'} \circ Ff : FC \to GC'$$

As f is arbitrary, this composition uniquely defines the equivalence of $Gf \circ \nu_C = \nu_{C'} \circ Ff$, as $Gf = \nu_{C'} \circ Ff \circ \nu_C^{-1}$, and likewise for Ff. Hence the following commutes uniquely

$$\begin{tikzcd}
FC \arrow[r, "\nu_C"] \arrow[d, "Ff"] & GC \arrow[d, "Gf"] \\
FC' \arrow[r, "\nu_{C'}"] & GC'
\end{tikzcd}$$

and naturality is satisfied.

The isomorphism of objects in Fun$(\mathbf{C}, \mathbf{D})$ is now obvious. □

Remark 2.53. Recall that in **Example 2.43**, the isomorphism

$$\nu_A : (A \times B) \times C \cong A \times (B \times C)$$

is *natural* in A. This means that the functors:

$$F \cong G$$

Where $F(A) = (A \times B) \times C$ and $G(A) = A \times (B \times C)$.

Example 2.44. Naturality and Dual Spaces

We work in the category, $\mathbf{Vec}_k$, category of vector spaces over a field k. Choose a real vector space, so $k = \mathbb{R}$. The morphisms are linear transformations

$$f : V \to W$$

To every object V in $\mathbf{Vec}_{\mathbb{R}}$, we have the vector space of linear functionals

$$V^* := \mathrm{Hom}(V, \mathbb{R})$$

Clearly, this identification of V with V^* is a contravariant representable functor, one which gives rise to a dual linear transformation:

$$f^* : W^* \to V^*$$

defined by the precomposition $f^*(A) = A \circ f$ for $A : W \to \mathbb{R}$. In summary:

$$(-)^* : \mathrm{Hom}_{\mathbf{Vec}_{\mathbb{R}}}(-, \mathbb{R}) : \mathbf{Vec}_{\mathbb{R}}^{op} \to \mathbf{Vec}_{\mathbb{R}}$$

is the contravariant representable functor, and as the target space is a vector space category, it is one which gives vector space structure.

We have a natural transformation

$$\eta : 1_{\mathbf{Vec}} \to **$$

whose components are defined by

$$\eta_V : V \to V^{**}$$

$$\eta_V : x \longmapsto (\mathrm{ev}_x : V^* \to \mathbb{R})$$

Where $\mathrm{ev}_x(A) = A(x)$ is the evaluation (**Definition 2.46**).

Note, η_V is the transpose of the evaluation (**Definition 2.45**).

Note that the terminal object $1_{\mathbf{Vec}} = 0$, where 0 just means the zero vector space[18].

Given some $f : V \to W$, we see that the following commutes:

$$\begin{tikzcd}
V \arrow[r, "\eta_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\
W \arrow[r, "\eta_W"'] & W^{**}
\end{tikzcd}$$

To verify this, let us do a quick diagram chase.

Assume that we have some $v \in V$ and $A : W \to \mathbb{R}$ in W^*. We have

$$(f^{**} \circ \eta_V)(v)(A) = f^{**}(\mathrm{ev}_v)(A)$$

As the dual space functor is contravariant, we see that

$$f^{**}(\mathrm{ev}_v)(A) = \mathrm{ev}_v(f^*(A)) = \mathrm{ev}_v(A \circ f)$$

[18]This is very plausible as the empty set $\varnothing$ is the terminal object in **Sets**, and the terminal object in **Sets** is precisely the basis for the zero-dimensional vector space 0. It follows that the zero vector space is unique because of the universal property of the terminal object.

By definition of evaluation

$$\mathrm{ev}_v(A \circ f) = (A \circ f)(v)$$

$$= A(fv) = \mathrm{ev}_{fv}(A)$$

We now see that

$$(\eta_W \circ f)(v)(A) = \mathrm{ev}_{fv}(A)$$

Therefore, this diagram commutes, and η is a natural transformation.

Consider, now, when V is finite dimensional. As the rank-nullity theorem necessitates that any vector spaces with the same dimension over the same field are isomorphic, we see that $V \cong V^*$. However, as shown by our choice of η above, we only have a *natural* transformation between V and V^{**}. There is no such natural transformation between V and V^*.

This explains why there is no canonical isomorphism between finite dimensional vector spaces and their duals. Because we can't find a natural transformation from $1_{\mathbf{Vec}} \to *$, our components, η_V, and the linear maps, f, depend on a choice of element in V. As all linear transformations are uniquely determined by their action on the basis of the domain, we see that any morphisms between V and V^* are dependent on the choice of basis, hence, not canonical.

2.6.6 Exponentials of Categories

We will now look at **Cat**, the category of (small) categories with the morphisms as functors. Our main purpose here is to show that **Cat** is a CCC (look at **Definition 2.50**). **Cat** is a ubiquitous example of what we call a **2-category**.

Proposition 110. **Cat** *is a CCC, with the exponential:*

$$\mathbf{D}^{\mathbf{C}} = \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

Remark 2.54. Note that an alternative definition of a **Natural Transformation** is that it makes $\mathrm{Hom}(\mathbf{C}, \mathbf{D})$ into an exponential category (a natural transformation is a morphism of functors, defined as being independent of the choice of object and morphisms from said objects).

Lemma 111. ***Bifunctor Lemma***

Given categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$*, a map of objects and morphisms given by*

$$F_0 : \mathbf{A}_0 \times \mathbf{B}_0 \to \mathbf{C}_0$$

$$F_1 : \mathbf{A}_1 \times \mathbf{B}_1 \to \mathbf{C}_1$$

respectively, forms a functor

$$F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$$

if and only if

1. *F is functorial in each argument:*

$$F(A, -) : \mathbf{B} \to \mathbf{C}$$

$$F(-, B) : \mathbf{A} \to \mathbf{C}$$

are both functors for all $A \in \mathbf{A}_0$, $B \in \mathbf{B}_0$.

2. *F satisfies the following interchange law:*

   Given $\alpha : A \to A'$ and $\beta : B \to B'$ ($A, A' \in \mathbf{A}$, $B, B' \in \mathbf{B}$), the following diagram commutes:

$$\begin{tikzcd}
F(A,B) \arrow[r, "F(A,\beta)"] \arrow[d, "F(\alpha,B)"'] & F(A,B') \arrow[d, "F(\alpha,B')"] \\
F(A',B) \arrow[r, "F(A',\beta)"'] & F(A',B')
\end{tikzcd}$$

Proof. **Proof of Lemma**

Let us prove both directions.

Let $F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ be a functor on objects and morphisms as follows:

$$F_0 : \mathbf{A}_0 \times \mathbf{B}_0 \to \mathbf{C}_0$$

$$F_1 : \mathbf{A}_1 \times \mathbf{B}_1 \to \mathbf{C}_1$$

We can take an arbitrary morphism in $\mathbf{A} \times \mathbf{B}$:

$$\langle \alpha, \beta \rangle : \langle A, B \rangle \to \langle A', B' \rangle$$

Given component-wise composition, we see that $\langle \alpha, \beta \rangle$ factors as follows:

$$\begin{tikzcd}
\langle A,B \rangle \arrow[r, "\langle id_A,\beta \rangle"] \arrow[d, "\langle \alpha,id_B \rangle"'] & \langle A,B' \rangle \arrow[d, "\langle \alpha,id_{B'} \rangle"] \\
\langle A',B \rangle \arrow[r, "\langle id_{A'},\beta \rangle"'] & \langle A',B' \rangle
\end{tikzcd}$$

If we take $\beta : B \to B'$ and $\alpha : A \to A'$, and using that F is a functor:

$$F(\langle \alpha, \beta \rangle)$$

$$= F(\langle \alpha, id_{B'} \rangle \circ \langle id_A, \beta \rangle) = F(\langle \alpha, id_{B'} \rangle) \circ F(\langle id_A, \beta \rangle) = F(\alpha, B') \circ F(A, \beta)$$

$$= F(\langle id_{A'}, \beta \rangle \circ \langle \alpha, id_B \rangle) = F(\langle id_{A'}, \beta \rangle) \circ F(\langle \alpha, id_B \rangle) = F(A', \beta) \circ F(\alpha, B)$$

Therefore, we see that:

$$F(A', \beta) \circ F(\alpha, B) = F(\alpha, B') \circ F(A, \beta)$$

i.e. the following diagram commutes

$$\begin{tikzcd}
F(A,B) \arrow[r, "F(A,\beta)"] \arrow[d, "F(\alpha,B)"'] & F(A,B') \arrow[d, "F(\alpha,B')"] \\
F(A',B) \arrow[r, "F(A',\beta)"'] & F(A',B')
\end{tikzcd}$$

This also shows that the action of F on objects is functorial

$$F(A, -) : \mathbf{B} \to \mathbf{C}$$

$$F(A, \beta) : F(A, B) \to F(A, B')$$
$$F(-, B) : \mathbf{A} \to \mathbf{C}$$
$$F(\alpha, B) : F(A, B) \to F(A', B)$$

The opposite direction is similar in principle, we can define the functor F with a similar decomposition as before:

$$F(\langle A, B \rangle) = F(A, B)$$

$$F(\langle \alpha, \beta \rangle) = F(A', \beta) \circ F(\alpha, B)$$

For $\alpha' : A' \to A''$, $\beta' : B' \to B''$, functoriality of F gives that:

$$F(\alpha', \beta') \circ F(\alpha, \beta) = F(\langle \alpha', \beta' \rangle \circ \langle \alpha, \beta \rangle)$$

This can be accomplished by using the interchange law:

$$\begin{tikzcd}
F(A,B) \arrow[d, "F(\alpha,B)"'] \arrow[dr, "F(\alpha,\beta)"] \arrow[ddrr, bend left, "F(\langle \alpha',\beta' \rangle \circ \langle \alpha,\beta \rangle)"] & & \\
F(A',B) \arrow[r, "F(A',\beta)"] \arrow[d, "F(\alpha',B)"'] & F(A',B') \arrow[d, "F(\alpha,B')"] \arrow[dr, "F(\alpha',\beta')"] & \\
F(A'',B) \arrow[r, "F(A'',\beta)"'] & F(A'',B') \arrow[r, "F(A'',\beta')"'] & F(A'',B'')
\end{tikzcd}$$

This diagram is equivalent to applying the functor F on the diagram:

$$\begin{tikzcd}
\langle A, B\rangle \arrow[d, "\langle\alpha, id_B\rangle"'] \arrow[dr, "\langle\alpha,\beta\rangle"] \arrow[ddrr, bend left, "\langle\alpha'\circ\alpha,\beta'\circ\beta\rangle"] & & \\
\langle A', B\rangle \arrow[r, "\langle id_{A'},\beta\rangle"] \arrow[d, "\langle\alpha', id_B\rangle"'] & \langle A', B'\rangle \arrow[d, "\langle\alpha', id_{B'}\rangle"] \arrow[dr, "\langle\alpha',\beta'\rangle"] & \\
\langle A'', B\rangle \arrow[r, "\langle id_{A''},\beta\rangle"'] & \langle A'', B'\rangle \arrow[r, "\langle id_{A''},\beta'\rangle"'] & \langle A'', B''\rangle
\end{tikzcd}$$

As the objects used here are arbitrary, this, indeed, ensures that F is a functor.

□

Proof. **Proof of Proposition Preceding the Bifunctor Lemma**

We need to show the following:

1. $\epsilon = \text{eval} : \text{Fun}(\mathbf{C}, \mathbf{D}) \times \mathbf{C} \to \mathbf{D}$ is a functor

2. For any category $\mathbf{X}$, and a functor

$$F : \mathbf{X} \times \mathbf{C} \to \mathbf{D}$$

   There is a functor

$$\tilde{F} : \mathbf{X} \to \text{Fun}(\mathbf{C}, \mathbf{D})$$

   so that

$$F = \epsilon \circ \left(\tilde{F} \times id_{\mathbf{C}}\right)$$

3. Given any functor

$$G : \mathbf{X} \to \text{Fun}(\mathbf{C}, \mathbf{D})$$

   we have that:

$$G = (\epsilon \circ (\tilde{G} \times id_{\mathbf{C}}))$$

   i.e. the transpose gives an isomorphism of hom sets as indicated in **Proposition 106**.

1. Fix $F : \mathbf{C} \to \mathbf{D}$ and consider ϵ:

$$\epsilon(F, -) = F \in \mathbf{D}^{\mathbf{C}}$$

   Now fix some $C \in \mathbf{C}_0$, and consider

$$\epsilon(-, C) : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$$

   defined by

$$(\nu : F \to G) \longmapsto (\nu_C : FC \to GC)$$

Take it for granted that this is a functor, although it is entirely obvious by just composing objects of $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$.

The interchange law is satisfied. Consider $\nu : F \to G \in \mathrm{Fun}(\mathbf{C}, \mathbf{D})$, and consider $f : C \to C' \in \mathbf{C}$. Because $\mathrm{eval}(F, C) = F(C)$, and ν is a natural transformation, the following commutes

$$\begin{tikzcd}
\mathrm{eval}(F,C) \arrow[r, "\nu_C"] \arrow[d, "F(f)"] & \mathrm{eval}(G,C) \arrow[d, "G(f)"] \\
\mathrm{eval}(F,C') \arrow[r, "\nu_{C'}"] & \mathrm{eval}(G,C')
\end{tikzcd}$$

By the **Bifunctor Lemma**, we have that ϵ is a functor.

2. Given a functor

$$F : \mathbf{X} \times \mathbf{C} \to \mathbf{D}$$

Let the transpose be defined by

$$\tilde{F} : \mathbf{X} \to \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

$$\tilde{F}(X)(-) = F(X, -)$$

Where $\forall C \in \mathbf{C}_0$:

$$\tilde{F}(X)(C) = F(X, C)$$

3. For this, refer to **Proposition 108**, condition 3.

Therefore, as **Cat** has all finite limits, and the evaluation map and the transpose are well-defined (by the Bifunctor Lemma), we see that the exponential exists in **Cat**. Therefore, this is a CCC. □

2.6.7 Functor Categories

We will now look at some functor categories. Note, we will use exponential notation from here on out.

$$\mathbf{D}^{\mathbf{C}} := \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

Example 2.45. Discrete Categories

$$\mathbf{C}^{\mathbf{1}} = \mathbf{C}$$

Where **1** is the terminal category[19] This is also known as a discrete category[20]. This is clear because the only objects in **1** are $\{*\}$, and the only morphism in **1**

[19] The terminal category is one with one object and no non-identity morphisms.

[20] A **discrete category** is any category that has no non-identity morphisms between objects.

is id_*. Any functor out of $\mathbf{1}$ (let's say $\mathcal{C}$) forms a generalized element in $\mathbf{C}$, i.e. $\forall C \in \mathbf{C}_0$, the action on objects is defined by

$$\mathcal{C} : * \to C \quad \forall C \in \mathbf{C}_0$$

And the action on morphisms of $\mathbf{1}$ is defined by

$$\mathcal{C} : id_* \to id_C \quad \forall C \in \mathcal{C}_0$$

The morphisms of $\mathbf{C^1}$ are the natural transformations $\eta_{\mathcal{C},\mathcal{D}} : \mathcal{C} \to \mathcal{D}$. In this context, as every object is a generalized element, it follows that the natural transformations are just the morphisms between the objects $C, D \in \mathbf{C}_0$. It follows that

$$\mathbf{C^1} = \mathbf{C}$$

Example 2.46. Interval Categories

What is $\mathbf{C^2}$, where $\mathbf{2}$ is the two-object category with exactly one non-identity morphism between the objects[21]: $* \to *$. Realize that the action of functors on objects of $\mathbf{2}$ is given as follows:

$$\mathcal{F}(\text{dom}(\to)) = A = \text{dom}(f) \quad \mathcal{F}(\text{cod}(\to)) = B = \text{cod}(f)$$

While the action of functors on morphisms is

$$F(\to) = f : A \to B$$

We see that the objects of the $\mathbf{C^2}$ are, therefore, the morphisms f. The morphisms of $\mathbf{C^2}$ are the morphisms between $\text{dom}(f)$ and $\text{dom}(g)$ and between $\text{cod}(f)$ and $\text{cod}(g)$, $\mathbf{C}$, $f : A \to B$, $g : C \to D$ (which are, consequently, the natural transformations between all functors $F, G : \mathbf{2} \to \mathbf{C}$). We can describe this all in a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\eta_{\text{dom}}"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "\eta_{\text{cod}}"'] & D
\end{tikzcd}$$

Very clearly, this is the **morphism category** of $\mathbf{C}$, $\mathbf{C}^{\to}$, the category with objects and morphisms described above.

Example 2.47. Discrete Categories (cont.)

Instead of the interval category, consider the discrete category $2 = \{0, 1\}$. The category has two objects, and no non-identity morphisms between the objects.

[21] This has no special name, we can call it the **Interval Category**, according to a suggestion by StackExchange user: goblin GONE.

Take some functor $F : 2 \to \mathbf{C}$. Then, referring to **Example 2.45**, we see that the action of the functor F on objects is

$$F(0) = \mathbf{C} \quad F(1) = \mathbf{C}$$

While the action of F on morphisms of 2 is trivial (as the only morphisms are identities of the objects given). Very clearly, we see that we have two copies of $\mathbf{C}$ along with all the morphisms of $\mathbf{C}^1$ associating every object of $\mathbf{C}^1$.

$$\mathbf{C}^2 \cong \mathbf{C} \times \mathbf{C}$$

In general, for any indexing set I (regarded as a discrete category), we have

$$\mathbf{C}^I \cong \prod_{i \in I} \mathbf{C}$$

This is trivial to see given the example of $\mathbf{C}^1$ and $\mathbf{C}^2$.

Example 2.48. Deducing Functor Categories

If we are given an arbitrary functor category $\mathbf{D}^{\mathbf{C}}$, we can determine the objects and morphisms easily:

Objects: The objects correspond to functors of the form

$$\mathbf{1} \to \mathbf{D}^{\mathbf{C}}$$

Using **Example 2.45**, we see that

$$\left(\mathbf{D}^{\mathbf{C}}\right)^{\mathbf{1}} = \mathbf{D}^{\mathbf{C}}$$

Therefore, all objects are functors

$$\mathbf{C} \to \mathbf{D}$$

Morphisms: We follow **Example 2.46** and see that, given the **single morphism category**, **2**, we see that the morphisms are of the form

$$\mathbf{1} \to \left(\mathbf{D}^{\mathbf{C}}\right)^{\mathbf{2}}$$

Therefore, to functors of the form

$$\mathbf{2} \to \mathbf{D}^{\mathbf{C}}$$

Using the evaluation map, we see that this becomes a functor

$$\mathbf{C} \times \mathbf{2} \to \mathbf{D}$$

And by the transpose, we see that this can be

$$\mathbf{C} \to \mathbf{D}^{\mathbf{2}}$$

Example 2.49. Category of Directed Graphs in terms of Sets

A *directed graph* is a pair of sets and a pair of functions

$$\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \end{tikzcd}$$

where G_1 is the set of edges and G_0 is the set of vertices

Example 2.50. Posets and Category of Categories

Let P and Q be posets (as categories) and consider the functor category

$$P^Q$$

Functors $Q \to P$ are monotone maps. The natural transformations

$$\nu : f \to g$$

are given by the following:

For each $p \in P$, we have

$$\nu_p : fp \leq gp$$

for any $p \leq q$, we have that the following diagram commutes:

$$\begin{tikzcd} fp \arrow[r, "\nu_p"] \arrow[d, "f\leq"'] & gp \arrow[d, "g\leq"] \\ fq \arrow[r, "\nu_q"'] & gq \end{tikzcd}$$

Proposition 112. *The inclusion functor*

$$i : \mathbf{Pos} \hookrightarrow \mathbf{Cat}$$

preserves the CCC structure

Proof. Look at **Example 2.50**. Regarding Posets as categories, and using the example, for any two posets P and Q, the functor category P^Q, is the category of all monotone maps $Q \to P$. We proved that **Cat** is a CCC. Therefore, **Pos**, the category with objects as posets (the categories P and Q), and morphisms as the monotone functions (i.e. objects of the functor category P^Q). Therefore, **Pos** is really a 2-category consisting of all monotone functions and posets. Since we can define exponentials and **Pos** has all finite limits, the inclusion of **Pos** into **Cat** is one that preserves the structures of being CCC. □

Example 2.51. Functor Categories of Two Groups

Take two groups G and H. Does the functor category

$$H^G$$

form an exponential of groups?

Take two group homomorphisms, $f, g : G \to H$. Define a map

$$\nu : f \to g$$

$$\nu := h$$

where for any $x \in H$:

$$\nu_x = h^{-1} f(x) h$$

such that (for a binary operation $\cdot$ on a group)

$$f(x) \cdot h = h \cdot g(x)$$

whence

$$g(x) = h^{-1} \cdot f(x) \cdot h$$

This is a natural transformation $\nu : f \to g$. $\nu : f \to g$ is called the **inner automorphism**.

Proposition 113. *Considering groups G and H as categories, and group homomorphisms $f, g : G \to H$ as functors between categories G and H, we have that the* ***inner automorphisms*** *are natural transformations between all such f and g.*

Proof. Take G and H to be groups, $f, g : G \to H$ as group homomorphisms. Define

$$\nu : f \to g$$

Where $\forall x \in G, h \in H$

$$\nu_x : f(x) \to h^{-1} f(x) h$$

As G is a one-object category with the object $*$, and morphisms $x : * \longmapsto x$ $\forall x \in G$, we see that ν_x, as defined above, satisfies the following "naturality square":

$$\begin{tikzcd} f(*) \arrow[r, "\nu_*"] \arrow[d, "f(x)"'] & g(*) \arrow[d, "g(x)"] \\ f(x) \arrow[r, "\nu_{x*}"'] & g(x) \end{tikzcd}$$

Where we define

$$f(*) := f(u)$$

Where u is the unit element of G. In fact, this ensures that:

$$\nu_x f(x) = h^{-1} f(x) h$$

as when $x = u$ (so that $*$ is the object in consideration), we have that:

$$\nu_* f(*) = h^{-1} f(u) h = h^{-1} h = u$$

Using the fact that

$$id_* * = *$$

and that

$$x = xid_* * \quad \forall x \in G$$

we can quickly diagram chase to verify that:

$$\nu_{x*} f(x) = h^{-1} f(x) h = h^{-1} \left(hg(x)h^{-1} \right) h = g(x) = g(x)u = g(x)\nu_*$$

So the above naturality square commutes. □

Example 2.52. Functor Categories of Two Groups (cont.)

We have seen above that the inner automorphisms of G are the natural transformations for homomorphisms of groups (viewed as functors on groups viewed as categories themselves).

Note that every natural transformation of group homomorphisms is invertible, i.e. $\nu : f \to g$ has an inverse $\nu^{-1} : g \to f$ for any f, g. However, the functor category H^G is not usually a group because there are many different homomorphisms $G \to H$, so the functor category H^G has more than one object[22].

Therefore, we may *generalize* the idea of a group to include all categories that contain more than one object, but in which every morphism is an isomorphism of objects.

Definition 2.60. A **groupoid** is a category with *at least* one object, and in which every morphism is an isomorphism of objects.

Lemma 114. *The Category of Groupoids,* **Grpd***, has an exponential object.*

Proof. Look at **Example 2.51, 2.52** along with the proposition above. This establishes that the functor category, H^G (or $\mathrm{Fun}(G, H)$), is, in fact, an exponential object for any groups G,H. □

Proposition 115. **Grpd** *is a Cartesian-Closed Category. Furthermore, the inclusion functor*

$$i : \mathbf{Grpd} \hookrightarrow \mathbf{Cat}$$

preserves the Cartesian-Closed structure.

Proof. The above lemma establishes that **Grpd** has an exponential. The important (and long) part is proving that the category of groupoids has all finite limits (hence colimits).

[22]Let us see why this means that H^G is not a group (hence, not an exponential in the category of groups). Recall that a *group*, categorically, is defined as a one-object category whose morphisms are all $x \in G$. Thus, we view all elements of G, x, as *generalized elements*. This is precisely what gives group structure, the fact that all elements of the group are really morphisms acting on a single object. This is a necessary and sufficient definition of a group, hence, if this is not satisfied, it is not a group. This tells us that if there is only one homomorphism $G \to H$, then H^G IS a group.

Use the proposition (**TEMPORARILY**: It is Proposition 100) that says that a category has all finite limits iff it has (i) finite products and (ii) equalizers.

(i) **Constructing Finite Products**:

We will construct finite products by the use of the product functor discussed in **Example 2.43**. It is sufficient to discuss only binary products, as we may inductively extend the number of products we take.

(ii) **Constructing the Equalizer**:

We will construct equalizers for groupoids by taking some inspiration of equalizers for groups (**Section 2.3.2**, pg. 95).

Recall that in **Grp**, the equalizer for the constant map and a morphism h is the inclusion map and the kernel of h.

It is clear that **Grpd** is a 2-category because it is the functor category, H^G, of two groups, G, H, both viewed as categories. As **Grpd** is a subcategory of **Cat**, and a CCC, it is a CCC in **Cat**. □

2.6.8 Equivalence of Categories

Example 2.53. Natural Isomorphism in Sets

Let $\mathbf{Ord}_{fin}$ be a category of finite ordinal numbers. The objects are the set $0, 1, 2, \ldots,$ where $0 = \varnothing$, and $n = \{0, \ldots, n-1\}$. The morphisms are **all** functions between these sets. Suppose that for each finite set A, we select an ordinal $|A|$ that is its cardinal, and an isomorphism

$$A \cong |A|$$

We now have $\forall f : A \to B$, A and B finite sets, we have a function $|f|$ defined by:

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "f"] & {|A|} \arrow[d, "|f|"] \\ B \arrow[r, "\cong"'] & {|B|} \end{tikzcd}$$

We see that $\cong$ is defined by the **inclusion functor**:

$$i : \mathbf{Ord}_{fin} \to \mathbf{Sets}_{fin}$$

So that the isomorphism above is defined by the component of a natural transformation:

$$\nu_A : A \overset{\cong}{\to} i\,|A|$$

We see that this is a naturality square, and that the following is a functor

$$|\cdot| : \mathbf{Sets}_{fin} \to \mathbf{Ord}_{fin}$$

Therefore, by the naturality square above that:

$$i(|f|) \circ \nu_A = \nu_B \circ f$$

Therefore, we have a natural isomorphism

$$\nu : 1_{\mathbf{Sets}_{fin}} \to i \circ |\cdot|$$

Where $1_{\mathbf{Sets}_{fin}}$ is identity functor on $\mathbf{Sets}_{fin}$.

However, maybe obvious by following the diagram above, taking the ordinal of the ordinal is meaningless:

$$|\cdot| \circ i = 1_{\mathbf{Ord}_{fin}} : \mathbf{Ord}_{fin} \to \mathbf{Ord}_{fin}$$

Take any cardinal n:

$$|i(n)| = ||n|| = n$$

We see that the category of finite ordinals and finite sets are very similar. However, they are not isomorphic.

Definition 2.61. An **equivalence of categories** consists of a pair of functors

$$E : \mathbf{C} \to \mathbf{D}$$

$$F : \mathbf{D} \to \mathbf{C}$$

and a **pair of natural isomorphisms**

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\simeq} F \circ E \quad \mathbf{C}^{\mathbf{C}}$$

$$\beta : 1_{\mathbf{D}} \xrightarrow{\simeq} E \circ F \quad \mathbf{D}^{\mathbf{D}}$$

The functor F is called the **pseudo-inverse** of E. If these functors exist, these categories $\mathbf{C}$ and $\mathbf{D}$ are said to be **equivalent**, and it is written $\mathbf{C} \simeq \mathbf{D}$.

Remark 2.55. Note that **equivalence** is a generalization of isomorphism. Two categories $\mathbf{C}$, $\mathbf{D}$ are **isomorphic** if there are functors

$$E : \mathbf{C} \to \mathbf{D}$$

$$F : \mathbf{D} \to \mathbf{C}$$

such that

$$1_{\mathbf{C}} = F \circ E$$

$$1_{\mathbf{D}} = E \circ F$$

Where we see that

$$\alpha := id_{\mathbf{C}^{\mathbf{C}}} : 1_{\mathbf{C}} \to F \circ E$$

$$\beta := id_{\mathbf{D}^{\mathbf{D}}} : 1_{\mathbf{D}} \to E \circ F$$

Where $id_{\mathbf{C}}$ means the **identity natural transformation**.

Example 2.54. Natural Isomorphism in Sets (cont.)

In **Example 2.53**, we demonstrated that $\mathbf{Sets}_{fin} \cong \mathbf{Ord}_{fin}$ (equivalence of categories), namely, the two functors

$$i : \mathbf{Ord}_{fin} \to \mathbf{Sets}_{fin}$$

$$|\cdot| : \mathbf{Sets}_{fin} \to \mathbf{Ord}_{fin}$$

are mutually pseudoinverses, i.e.

Inclusion of an Ordinal

$$\nu : 1_{\mathbf{Sets}_{fin}} \to i \circ |\cdot|$$

Ordinal of an Ordinal

$$\nu' : 1_{\mathbf{Ord}_{fin}} \to |\cdot| \circ i$$

Therefore, the inclusion functor i and the ordinal functor $|\cdot|$ give an equivalence of categories $\mathbf{Sets}_{fin} \cong \mathbf{Ord}_{fin}$.

Therefore, we see that

1. For any finite set A, there is some ordinal n such that $A \cong i(n)$ (this is the equivalence on objects)

2. For any ordinals n, m, we have an isomorphism of Hom-sets

$$\mathrm{Hom}_{\mathbf{Ord}_{fin}}(n, m) \cong \mathrm{Hom}_{\mathbf{Sets}_{fin}}(i(n), i(m))$$

   (this is the equivalence on morphisms)

In fact, these two observations give us necessary and sufficient conditions for equivalence.

Proposition 116. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. The following are equivalent (TFAE):*

1. *F is (part of)*[23] *an equivalence of categories.*

2. *F is full and faithful and "essentially" surjective on objects, i.e. for every $D \in \mathbf{D}_0$, there's some $C \in \mathbf{C}_0$ such that $FC \cong D$*

Proof. We prove both directions:

$(1 \to 2)$:

Let $E : \mathbf{D} \to \mathbf{C}$ be a pseudoinverse of F (as above) and consider the natural isomorphisms

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\sim} EF$$

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE$$

[23] By "part of", we mean that it is a pseudoinverse of some other functor which forms an equivalence on the source and target categories.

Now, in $\mathbf{C}$, for any C, we have that:

$$\alpha_C : C \xrightarrow{\sim} EF(C)$$

so that the following commutes for any morphism $f : C \to C'$ (this follows by naturality of α):

$$\begin{tikzcd} C \arrow[r, "\alpha_C"] \arrow[d, "f"] & EF(C) \arrow[d, "EF(f)"] \\ C' \arrow[r, "\alpha_{C'}"] & EF(C') \end{tikzcd}$$

If $F(f) = F(f')$, then $EF(f) = EF(f')$, so that $f = f'$. Therefore, F is a **faithful** functor. Analogously, looking at β_D for all $D \in \mathbf{D}_0$, we obtain that E is also faithful.

Take a morphism in $\mathbf{D}$:

$$h : F(C) \to F(C')$$

and consider the naturality square

$$\begin{tikzcd} C \arrow[r, "\cong"] \arrow[d, "f"] & EF(C) \arrow[d, "E(h)"] \\ C' \arrow[r, "\cong"] & EF(C') \end{tikzcd}$$

where $f = (\alpha_{C'})^{-1} \circ E(h) \circ \alpha_C$ by the previous diagram.

We also have that, because F is faithful, E is also faithful, and

$$EF(f) = E(h) \Longrightarrow F(f) = h$$

Therefore, F is **full**.

For any object $D \in \mathbf{D}_0$, we have

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE$$

so that

$$\beta_D : D \cong F(E(D)) \quad E(D) \in \mathbf{C}_0$$

So that F is "essentially" surjective on objects of $\mathbf{D}$.

$(2 \to 1)$:

By assumption, $F : \mathbf{C} \to \mathbf{D}$ is essentially surjective on objects and fully faithful. As F is essentially surjective, for every $D \in \mathbf{D}_0$, we pick some $E(D) \in \mathbf{C}_0$ so that there's some $\beta_D : D \xrightarrow{\sim} FE(D)$. This β_D forms the components of a natural transformation $\beta : 1_{\mathbf{D}} \to FE$.

Now, given $h : D \to D'$ for $D, D' \in \mathbf{D}$, consider the following naturality square:

$$\begin{tikzcd} D \arrow[r, "\beta_D"] \arrow[d, "h"] & FE(D) \arrow[d, "\beta_{D'} \circ h \circ \beta_D^{-1}"] \\ D' \arrow[r, "\beta_{D'}"] & FE(D') \end{tikzcd}$$

As F is fully faithful by assumption, and β is a natural transformation, $\exists! E(h) : E(D) \to E(D')$ with

$$FE(h) = \beta_{D'} \circ h \circ \beta_D^{-1}$$

$E : \mathbf{D} \to \mathbf{C}$ is a functor, and $\beta : 1_{\mathbf{D}} \overset{\sim}{\to} FE$ is a natural transformation. Clearly it is an isomorphism between functors because F is fully faithful, so its action on pairs of objects and morphisms between those objects is an isomorphism, meaning that E is the pseudoinverse of F.

Now apply F to C and consider the object $FC \in \mathbf{D}$, and consider the component of β on FC so that

$$\beta_{FC} : 1_{\mathbf{D}}(FC) \to FE(FC)$$

or

$$\beta_{FC} : F(C) \to FEF(C)$$

As F is fully faithful with a pseudoinverse E, $F^{-1}(\beta_{FC})$ is an isomorphism as well, meaning that

$$\alpha_C = F^{-1}(\beta_{FC})$$

As β_{FC} is the component of a natural transformation for any $C \in \mathbf{C}_0$, so is α_C, hence, α is a natural transformation.

Therefore, we conclude that we have the following:

$$\alpha : 1_{\mathbf{C}} \overset{\sim}{\to} EF$$

$$\beta : 1_{\mathbf{D}} \overset{\sim}{\to} FE$$

Where $F : \mathbf{C} \to \mathbf{D}$ is the pseudoinverse of $E : \mathbf{D} \to \mathbf{C}$. Hence, F is a functor in the equivalence of categories $\mathbf{C}$ and $\mathbf{D}$. $\square$

2.6.9 Examples of Categorical Equivalence

We return to this section later.

2.7 Categories of Diagrams

2.7.1 Set-Valued Functor Categories

Let us turn our attention to functor categories of the form:

$$\mathbf{Sets}^{\mathbf{C}}$$

where $\mathbf{C}$ is locally small.

The objects are set-valued functors

$$P, Q : \mathbf{C} \to \mathbf{Sets}$$

The morphisms are natural transformations

$$\alpha, \beta : P \to Q$$

Recall from **Example 2.38** the contravariant representable functor

$$\mathrm{Hom}_{\mathbf{C}}(-, S) : \mathbf{C}^{op} \to \mathbf{Sets}$$

As $\mathbf{Sets}^{\mathbf{C}}$ is a functor category, we see that for any commutative diagram in $\mathbf{Sets}^{\mathbf{C}^{op}}$

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[dr, "\beta\alpha"'] & Q \arrow[d, "\beta"] \\
& R
\end{tikzcd}$$

We can evaluate it at any object C to obtain the following commutative diagram

$$\begin{tikzcd}
PC \arrow[r, "\alpha_C"] \arrow[dr, "(\beta\alpha)_C"'] & QC \arrow[d, "\beta_C"] \\
& RC
\end{tikzcd}$$

Therefore, for every object $C \in \mathbf{C}_0$, we have an **evaluation functor**:

$$\mathrm{ev}_C : \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathbf{Sets}$$

Due to naturality, if we have any morphism $f : D \to C$, where $D \in \mathbf{D}_0$ and $C \in \mathbf{C}_0$, we obtain the following "cylinder" diagram in **Sets**:

$$\begin{tikzcd}
& PD \arrow[r, "\alpha_D"] \arrow[dl, "Pf"] \arrow[dr, bend left, "(\beta\alpha)_D"'] & QD \arrow[dl, "Qf"'] \arrow[d, "\beta_D"] \\
PC \arrow[r, "\alpha_C"] \arrow[dr, bend right, "(\beta\alpha)_C"'] & QC \arrow[d, "\beta_C"] & RD \arrow[dl, "Rf"] \\
& RC &
\end{tikzcd}$$

Remark 2.56. If we consider functors $G : \Gamma \to \mathbf{Sets}$ as a graph, where Γ is a category

$$1 \rightrightarrows 0$$

and a natural transformation $\alpha : G \to H$ as a graph homomorphism, then we obtain:

$$\mathbf{Sets}^{\Gamma} = \mathbf{Graphs}$$

Therefore, we may think of categories of the form

$$\mathbf{Sets}^{\mathbf{C}}$$

as a generalized category of sets with certain structures, and homomorphisms.

We now approach the two most important sections in Section 2 of our notes. Pay Attention Here!!

2.7.2 Yoneda Embedding

Remark 2.57. We have certain special objects in the category $\mathbf{Sets}^{\mathbf{C}}$, namely the *covariant* representable functors:

$$\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

Now for every $h : C \to D$ in $\mathbf{C}$, we have the following natural transformation

$$\mathrm{Hom}_{\mathbf{C}}(h, -) : \mathrm{Hom}_{\mathbf{C}}(D, -) \to \mathrm{Hom}_{\mathbf{C}}(C, -)$$

Note that the direction of the natural transformation reverses the direction of the morphism because for $\mathrm{Hom}_{\mathbf{C}}(h, -)$, given any morphism $f : G \to S$, we have that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(C, G) \arrow[d, "\mathrm{Hom}_{\mathbf{C}}(C, f)"'] & \mathrm{Hom}_{\mathbf{C}}(D, G) \arrow[l, "\mathrm{Hom}_{\mathbf{C}}(h, -)"'] \arrow[d, "\mathrm{Hom}_{\mathbf{C}}(D, f)"] \\
\mathrm{Hom}_{\mathbf{C}}(C, S) & \mathrm{Hom}_{\mathbf{C}}(D, S) \arrow[l, "\mathrm{Hom}_{\mathbf{C}}(h, -)"]
\end{tikzcd}$$

And recall that by **Remark 2.50** (contravariant representable functors) and **Definition 2.55** (covariant representable functor), we can rewrite this as:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(C, G) \arrow[d, "f_*"'] & \mathrm{Hom}_{\mathbf{C}}(D, G) \arrow[l, "h^*"'] \arrow[d, "f_*"] \\
\mathrm{Hom}_{\mathbf{C}}(C, S) & \mathrm{Hom}_{\mathbf{C}}(D, S) \arrow[l, "h^*"]
\end{tikzcd}$$

Where the rows (natural transformations) are contravariant, and the columns (morphisms) are covariant.

We see that the action of the natural transformation at a component G is defined by:

$$\mathrm{Hom}_{\mathbf{C}}(h, -) : (g : D \to G) \longmapsto (g \circ h : C \to G)$$

Therefore, we obtain a contravariant functor

$$k : \mathbf{C}^{op} \to \mathbf{Sets}^{\mathbf{C}}$$

$$k : C \longmapsto \mathrm{Hom}_{\mathbf{C}}(C, -)$$

k is the exponential transpose of the bifunctor

$$\mathrm{Hom}_{\mathbf{C}} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Sets}$$

k is contravariant because if we put in any morphism $h : C \to D$ into the first argument of $\mathrm{Hom}(-, -)$, we obtain that:

$$\mathrm{Hom}(h, -) : \mathrm{Hom}(D, -) \to \mathrm{Hom}(C, -)$$

Remark 2.58. We may tranpose $\mathrm{Hom}_{\mathbf{C}}$ with respect to the other argument, we obtain the covariant functor

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

$$y : C \longmapsto \mathrm{Hom}_{\mathbf{C}}(-, C)$$

And likewise, this is covariant because for any morphism $h : C \to D$, putting into the second argument of $\mathrm{Hom}(-, -)$, we obtain

$$\mathrm{Hom}(-, h) : \mathrm{Hom}(-, C) \to \mathrm{Hom}(-, D)$$

Definition 2.62. Recognize that k (the contravariant exponential tranpose of $\mathrm{Hom}_{\mathbf{C}}$) maps into $\mathbf{Sets}^{\mathbf{C}}$, which is the space of **covariant** functors. Likewise, y (the covariant exponential transpose of $\mathrm{Hom}_{\mathbf{C}}$) maps into $\mathbf{Sets}^{\mathbf{C}^{op}}$, which is the space of **contravariant** functors.

$\mathbf{Sets}^{\mathbf{C}^{op}}$, the category of *contravariant* set-valued functors, is also known as the category of presheaves. The contravariant set-valued functors are known as **presheaves**.

Definition 2.63. A functor $F : \mathbf{C} \to \mathbf{D}$ is called an **embedding** if it is full, faithful, and injective on objects.

Definition 2.64. Summary about the Hom Functor (Representable Functors)

Let $\mathbf{C}$ be a locally small category. Then, we denote the hom-set, $\mathrm{Hom}_{\mathbf{C}}(C, C')$ as $\mathrm{Hom}(C, C')$, where the category we are working over is implicitly defined.

Hom is a bifunctor (Recall **Lemma 111**, the Bifunctor Lemma), therefore it is functorial in both arguments, each are covariant/contravariant and we will end the confusion of which is which once and for all.

We denote $-$ as an arbitrary argument.

$$\mathrm{Hom}_{\mathbf{C}}(-,-) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Sets}$$

is the **Hom Functor**. We describe the covariance and contravariance of the Hom Functor as follows:

For any morphism $h : C \to D$, we have that

$$\mathrm{Hom}(h, -)$$

reverses the morphism h (i.e. $\mathrm{Hom}(h,-) : \mathrm{Hom}(D,-) \to \mathrm{Hom}(C,-)$), so that the map

$$C \longmapsto \mathrm{Hom}(C, -)$$

is **contravariant**.

Likewise, if we do

$$\mathrm{Hom}(-, h)$$

it **preserves** the direction of h (i.e. $\mathrm{Hom}(-,h) : \mathrm{Hom}(-,C) \to \mathrm{Hom}(-,D)$), so that the map

$$C \longmapsto \mathrm{Hom}(-, C)$$

is **covariant**.

Therefore, the Hom Functor is **contra**variant in the **first** argument, **co**variant in the **second** argument.

Note the action of the hom functor:

For $\mathrm{Hom}(-, C') \in \mathbf{Sets}^{\mathbf{C}^{op}}$, we see that for a given morphism $f \in \mathbf{C}_1$, that (Recall **Remark 2.50**)

$$\mathrm{Hom}(h, C')(f) := h^* f = f \circ h$$

Likewise, for $\mathrm{Hom}(C, -) \in \mathbf{Sets}^{\mathbf{C}}$, we see that for a given morphism $f \in \mathbf{C}_1$, that (Recall **Definition 2.55**)

$$\mathrm{Hom}(C, h)(f) = h_* f = h \circ f$$

Knowing these will be essential to our computations in proving the Yoneda Lemma.

Definition 2.65. Yoneda Embedding

There is a **covariant** functor (described in **Remark 2.58**)

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

whose action on objects C is

$$y : C \longmapsto (\mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{op} \to \mathbf{Sets})$$

so that

$$yC = \mathrm{Hom}_{\mathbf{C}}(-, C) \in \mathbf{Sets}^{\mathbf{C}^{op}}$$

whose action on any morphism $f : C \to D$ is

$$y : f \longmapsto (\mathrm{Hom}_{\mathbf{C}}(-, f) : \mathrm{Hom}_{\mathbf{C}}(-, C) \to \mathrm{Hom}_{\mathbf{C}}(-, D))$$

so that

$$yf = \mathrm{Hom}_{\mathbf{C}}(-, f) \in \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(\mathrm{Hom}(-, C), \mathrm{Hom}(-, D))$$

This functor, y, is called the **Yoneda Embedding**.

Remark 2.59. The Yoneda Embedding y is a "representation" of $\mathbf{C}$ in a category of presheaves and natural transformations of these presheaves. In fact, this map y is *full*, meaning that any map $\nu : yC \to yD$ comes from a unique map $h : C \to D$ so that $yh = \nu$.

2.7.3 The Yoneda Lemma

Lemma 117. ***Yoneda Lemma***

Let $\mathbf{C}$ *be a locally small category. For any object* $C \in \mathbf{C}_0$*, and a functor* $F \in \mathbf{Sets}^{\mathbf{C}^{op}}$*, there's an isomorphism*

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC, F) \cong FC$$

which is, furthermore, natural in both F *and* C*.*

Remark 2.60. Some Remarks about Yoneda Lemma

We note the following here:

1. The Hom is over the presheaf category $\mathbf{Sets}^{\mathbf{C}^{op}}$.
2. Naturality in F implies that, given any $\nu : F \to G$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(yC, F) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(yC,\nu)"'] & FC \arrow[d, "\nu_C"] \\ \mathrm{Hom}(yC, G) \arrow[r, "\cong"'] & GC \end{tikzcd}$$

3. Naturality in C means that, given any $h : C \to D$, the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Hom}(yC, F) \arrow[r, "\cong"] & FC \\ \mathrm{Hom}(yD, F) \arrow[r, "\cong"'] \arrow[u, "\mathrm{Hom}(yh,F)"] & FD \arrow[u, "Fh"'] \end{tikzcd}$$

Note which argument of Hom that the naturality will be in.

Proof. **Proof of Yoneda Lemma**

We first **prove the isomorphism**:

Let

$$\eta_{C,F} : \mathrm{Hom}(yC, F) \overset{\cong}{\to} FC$$

be defined by (where $\nu : yC \to F$):

$$\eta_{C,F}(\nu) = \nu_C(1_C)$$

which we will write as

$$x_\nu = \nu_C(1_C)$$

Where the component ν_C is defined as

$$\nu_C : \mathbf{C}(C, C) \to FC$$

so that $\nu_C(1_C) \in FC$

For the converse, given any $a \in FC$, define the natural transformation $\nu_a : yC \to F$ as follows

Given any C', we define the component of the natural transformation

$$(\nu_a)_{C'} : \mathrm{Hom}(C', C) \to FC'$$

as

$$(\nu_a)_{C'}(h) = F(h)(a)$$

where $h : C' \to C$

We now prove naturality of the ν_a, take any $f : C'' \to C'$, and consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}(C'', C) \arrow[r, "(\nu_a)_{C''}"] & FC'' \\
\mathrm{Hom}(C', C) \arrow[u, "\mathrm{Hom}(f,C)"] \arrow[r, "(\nu_a)_{C'}"'] & FC' \arrow[u, "F(f)"']
\end{tikzcd}$$

Take $h \in yC(C') = \mathrm{Hom}(C', C)$:

$$(\nu_a)_{C''} \circ \mathrm{Hom}(f, C)(h) = (\nu_a)_{C''}(h \circ f) = F(h \circ f)(a)$$

$$= F(f) \circ F(h)(a) = F(f)(\nu_a)_{C'}(h)$$

So ν_a is natural.

We now show that ν_a and x_ν are mutually inverse.

Given $\nu : yC \to F$, by definition, for any $h : C' \to C$

$$(\nu_{(x_\nu)})_{C'}(h) = F(h)(\nu_C(1_C))$$

As ν is natural, we see that the following diagram commutes

$$\begin{tikzcd}
yC(C) \arrow[r, "\nu_C"] \arrow[d, "yC(h)"'] & FC \arrow[d, "Fh"] \\
yC(C') \arrow[r, "\nu_{C'}"] & FC'
\end{tikzcd}$$

Therefore, by commutativity

$$(\nu_{x_\nu})_{C'}(h) = F(h)(\nu_C(1_C)) = \nu_{C'} \circ yC(h)(1_C)$$

We see that $yC(h)(1_C) = \mathrm{Hom}(h, C)(1_C) = 1_C \circ h = h$, therefore

$$\nu_{C'} \circ yC(h)(1_C) = \nu_{C'}(h)$$

Therefore, we see that $\nu_{x_\nu} = \nu$, which implies that $\nu_{x_{(-)}} = id_{yC}$.

For the converse, for any $a \in FC$, we can take

$$x_{\nu_a} = (\nu_a)_C(1_C) = F(1_C)(a) = 1_{FC}(a) = a$$

This implies that $x_{\nu_{(-)}} = id_F$. Therefore, we obtain that

$$\mathrm{Hom}(yC, F) \cong FC$$

With the isomorphism proven, we simply need to **prove the naturality of the isomorphism** in *both C and F*.

We first **prove the naturality in F**:

Fix $C \in \mathbf{C}_0$. Given some $\phi : F \to F'$, taking $\nu \in \mathrm{Hom}(yC, F)$, and recalling that $\eta_{C,F}(\nu) := x_\nu := \nu_C(1_C)$, we can chase the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(yC, F) \arrow[r, "\eta_{C,F}"] \arrow[d, "\mathrm{Hom}(yC,\phi)"'] & FC \arrow[d, "\phi_C"] \\
\mathrm{Hom}(yC, F') \arrow[r, "\eta_{C,F'}"'] & F'C
\end{tikzcd}$$

$$\begin{aligned}
\phi_C \circ \eta_{C,F}(\nu) := \phi_C(x_\nu) := \phi_C(\nu_C(1_C)) = (\phi\nu)_C(1_C) = x_{\phi\nu} \\
= \eta_{C,F'}(\mathrm{Hom}(yC, \phi)(\nu))
\end{aligned}$$

So therefore,

$$\phi_C \circ \eta_{C,F} = \eta_{C,F'} \circ \mathrm{Hom}(yC, \phi)$$

Now we **prove the naturality in** C:

Fix $F \in \mathbf{Sets}^{\mathbf{C}^{op}}$. Given a morphism $f : C' \to C$, we chase the following diagram:

$$\begin{tikzcd} \mathrm{Hom}(yC', F) \arrow[r, "\eta_{C',F}"] & FC' \\ \mathrm{Hom}(yC, F) \arrow[u, "\mathrm{Hom}(yf,F)"] \arrow[r, "\eta_{C,F}"] & FC \arrow[u, "Ff"'] \end{tikzcd}$$

$$\eta_{C',F} \circ \mathrm{Hom}(yf, F)(\nu) = \eta_{C'}(\nu \circ yf)$$

This follows because it holds that[24]:

$$\mathrm{Hom}(yf, F)(\nu) = \nu \circ yf$$

Then it follows that

$$\eta_{C'}(\nu \circ yf) = (\nu \circ yf)_{C'}(1_{C'}) = \nu_{C'} \circ (yf)_{C'}(1_{C'})$$

As $(yf)_{C'} = \mathrm{Hom}(C', f)$, we see that

$$\mathrm{Hom}(C', f)(1_{C'}) = f \circ 1_{C'}$$

therefore, we obtain the next string of equalities

$$\nu_{C'} \circ (yf)_{C'}(1_{C'}) = \nu_{C'}(f \circ 1_{C'}) = \nu_{C'}(f) = \nu_{C'}(1_C \circ f)$$

$$= \nu_{C'} \circ \mathrm{Hom}(f, C)(1_C) = \nu_{C'} \circ yC(f)(1_C)$$

Now by naturality of ν earlier, we see that

$$\nu_{C'} \circ yC(f)(1_C) = F(f) \circ \nu_C(1_C) := F(f) \circ \eta_C(\nu)$$

Therefore, we see that

$$\eta_{C',F} \circ \mathrm{Hom}(yf, C) = F(f) \circ \eta_{C,F}$$

Hence, the naturality square above commutes in C.

Therefore, we have proven the desired isomorphism:

$$\mathrm{Hom}(yC, F) \cong FC$$

and the naturality of the isomorphism in both C and F. □

Theorem 118. ***Yoneda Embedding***

The Yoneda Embedding $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ *is fully faithful and injective on objects.*

[24]Note at what we are feeding into $\mathrm{Hom}(yf, F)$, ν is a natural transformation, hence, as $\mathrm{Hom}(yf, F)$ are morphisms between functors, i.e. natural transformations, it is well-defined to say that $\mathrm{Hom}(yf, F)(\nu) = \nu \circ yf$

Proof. This is a simple corollary of the Yoneda Lemma.

Let $C, D \in \mathbf{C}_0$. For any pairs of such objects, we can construct an isomorphism as follows. Let F in **Lemma 117** (**Yoneda Lemma**) be

$$F = yD$$

Then apply the Yoneda Lemma:

$$yD(C) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC, yD)$$

Noting that $yD(C) = \mathrm{Hom}_{\mathbf{C}}(C, D)$, we see that

$$\mathrm{Hom}_{\mathbf{C}}(C, D) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC, yD)$$

In fact, with ν_a as defined in the proof of **Lemma 117** (Yoneda Lemma), we may even prove that the Yoneda Embedding, y, is **full** in the following way:

Let $f : C' \to C$ and $h \in \mathrm{Hom}_{\mathbf{C}}(C, D)$, then we can do a simple computation:

$$(\nu_h)_{C'}(f) = yD(f)(h) = \mathrm{Hom}_{\mathbf{C}}(f, D)(h) = h \circ f = (yh)_{C'}(f)$$

As f is arbitrary, we see that

$$\nu_h = y(h) \quad \forall h \in \mathrm{Hom}_{\mathbf{C}}(C, D)$$

Proving that y is injective on objects is simple. Let $C, D \in \mathbf{C}_0$. Then we see that if $yC = yD$, then for $1_C \in \mathrm{Hom}(C, C) = yC(C) = yD(C) = \mathrm{Hom}(C, D)$. Because the identity morphism on an object must be unique, we see that $C = D$. $\square$

2.7.4 Application of the Yoneda Lemma

Remark 2.61. Why the Yoneda Lemma is Useful

The Yoneda Lemma is useful because we know that for any fully faithful functor, F, if $FA \cong FB$, then $A \cong B$.

We just discovered that the Yoneda Embedding into the presheaf category, $\mathbf{Sets}^{\mathbf{C}^{op}}$, is fully faithful. Therefore, we are able to show that $A \cong B$ in $\mathbf{C}$ only if $yA \cong yB$ in $\mathbf{Sets}^{\mathbf{C}^{op}}$.

Corollary 119. *Let* $\mathbf{C}$ *be a locally small category. Given objects* $A, B \in \mathbf{C}_0$*,*

$$yA \cong yB \Longrightarrow A \cong B$$

Proof. This is immediate by **Theorem 118**, namely, the injectivity of the Yoneda Embedding on objects guarantees that this will hold. $\square$

Proposition 120. *In any Cartesian-Closed Category,* $\mathbf{C}$*, there is a natural isomorphism*

$$\left(A^B\right)^C \cong A^{(B \times C)}$$

Proof. Applying **Corollary 119**, it suffices to prove that

$$y\left(\left(A^B\right)^C\right) \cong y\left(A^{(B\times C)}\right)$$

Take any object $X \in \mathbf{C}$, and we can apply **Proposition 106** to obtain the following:

$$\mathrm{Hom}(X, \left(A^B\right)^C) \cong \mathrm{Hom}(X \times C, A^B)$$

Apply **Proposition 106** again:

$$\mathrm{Hom}(X \times C, A^B) \cong \mathrm{Hom}((X \times C) \times B, A)$$

Using the associativity of the product, and **Example 2.43** (the "**twisting**" natural isomorphism), we see that

$$(X \times C) \times B = X \times (C \times B) \cong X \times (B \times C)$$

Where the isomorphism $C \times B \cong B \times C$ is natural. Then, we obtain the following isomorphism

$$\mathrm{Hom}((X \times C) \times B, A) \cong \mathrm{Hom}(X \times (B \times C), A)$$

Finally, applying **Proposition 106** again, we obtain the final result:

$$\mathrm{Hom}(X \times (B \times C), A) \cong \mathrm{Hom}(X, A^{B\times C})$$

Leaving us to conclude that

$$\mathrm{Hom}(X, \left(A^B\right)^C) \cong \mathrm{Hom}(X, A^{B\times C})$$

Furthermore, this isomorphism is *natural* because every isomorphism involved in the above string of isomorphisms was natural. Namely, the transpose is a natural transformation. A composition of natural transformations is, itself, a natural transformation, hence, we have proven that these isomorphisms are natural. □

Proposition 121. *If the Cartesian-Closed Category* $\mathbf{C}$ *has coproducts, then* $\mathbf{C}$ *is* ***distributive****, i.e. there is a canonical isomorphism*

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

Proof. We can repeatedly apply **Proposition 106**, with the implication that the transpose is a natural isomorphism in X.

$$\mathrm{Hom}(A \times (B + C), X) \cong \mathrm{Hom}(B + C, X^A)$$

Recall that **Representable Functors Preserve Limits** (**Proposition 103**), and **Contravariant Representable Functors map all colimits to limits** (**Corollary 104**), therefore we have the isomorphism

$$\mathrm{Hom}(B + C, X^A) \cong \mathrm{Hom}(B, X^A) \times \mathrm{Hom}(C, X^A)$$

Which we will continue below

$$\cong \mathrm{Hom}(A \times B, X) \times \mathrm{Hom}(A \times C, X) \cong \mathrm{Hom}((A \times B) + (A \times C), X)$$

Since

$$k((A \times B) + (A \times C)) \cong k(A \times (B + C))$$

Using the dual of the Yoneda Lemma for $k : \mathbf{C}^{op} \to \mathbf{Sets}^{\mathbf{C}}$, we can apply **Corollary 119**, to obtain that

$$(A \times B) + (A \times C) \cong A \times (B + C)$$

□

Remark 2.62. In general, given objects $A, B \in \mathbf{C}_0$, where $\mathbf{C}$ is locally small, to find a morphism $g : A \to B$, it suffices to give a natural transformation $\nu : yA \to yB$ in $\mathbf{Sets}^{\mathbf{C}^{op}}$ (where $yA, yB \in \mathbf{Sets}_0^{\mathbf{C}^{op}}$, and $\nu \in \mathbf{Sets}_1^{\mathbf{C}^{op}}$) since, by **Theorem 118**, the Yoneda Embedding is fully faithful, meaning that

$$\exists! g : \nu = yg$$

There may be instances where working in the presheaf category may be significantly easier than working in the original category.

2.7.5 Limits in Categories of Diagrams

Definition 2.66. The **Constant Functor** is defined as a functor

$$\Delta : \epsilon \to \epsilon^J$$

which is the transpose of the projection (look at **Proposition 106**).

Definition 2.67. A category ϵ is said to be **complete** if it has all *small* limits, i.e. for any small category J, and a functor $F : J \to \epsilon$, we have a limit

$$L = \varprojlim_{j \in J} Fj \in \epsilon$$

and a cone

$$\eta : \Delta L \to F \in \epsilon^J$$

that is universal among morphisms from constant functors ΔE.

Proposition 122. *For any locally small category $\mathbf{C}$, the functor category $\mathbf{Sets}^{\mathbf{C}^{op}}$ is complete. Furthermore, for every object $C \in \mathbf{C}_0$, the evaluation functor*

$$\mathrm{ev}_C : \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathbf{Sets}$$

(which evaluates all presheaves to a set) preserves all limits.

Remark 2.63. We make some remarks about how we obtain this proposition. Assume that J is small and that we have a functor

$$F : J \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

The limit of F, **if it exists**, should be a presheaf (object of $\mathbf{Sets}^{\mathbf{C}^{op}}$)

$$\lim_{\leftarrow i \in J} F_i : \mathbf{C}^{op} \to \mathbf{Sets}$$

Applying the Yoneda Lemma (**Lemma 117**), if such a presheaf were to exist, then we would have:

$$\left(\lim_{\leftarrow i \in J} F_i\right)(C) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}\left(yC, \lim_{\leftarrow i \in J} F_i\right)$$

As Representable Functors Preserve Limits (**Proposition 103**), we see that:

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}\left(yC, \lim_{\leftarrow i \in J} F_i\right) \cong \lim_{\leftarrow i \in J} \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC, F_i)$$

And by the Yoneda Lemma, we see that:

$$\lim_{\leftarrow i \in J} \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC, F_i) \cong \lim_{\leftarrow i \in J} F_i(C)$$

We may define the following limit due to the evaluation functor acting on any presheaf (object) in $\mathbf{Sets}^{\mathbf{C}^{op}}$

$$\left(\lim_{\leftarrow i \in J} F_i\right)(C) = \lim_{\leftarrow i \in J}(F_i C) \quad \forall C \in \mathbf{C}_0$$

Proof. Now to prove the proposition.

Let $F : J \to \mathbf{Sets}^{\mathbf{C}^{op}}$ be a presheaf (contravariant set-valued functor), where we can see that F is actually a bifunctor $F : \mathbf{C}^{op} \times J \to \mathbf{Sets}$. Then F is a diagram of the form:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & F(C_2)(J_1) \arrow[r] \arrow[d] & F(C_2)(J_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F(C_1)(J_1) \arrow[r] \arrow[d] & F(C_1)(J_2) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

Notice how the first argument of $F(-)(-)$ is contravariant and the second argument is covariant. On this note, the rows form a presheaf on $\mathbf{C}$, because vertical composition (on the diagram) is a contravariant functor.

Notice that for any cone (C, c_i) over any row of the above diagram, a morphism $h_i : C_i \to C_{i+1}$ induces a cone over the rows below it, since we can move down a row by just composing $F(h)(J_k) \circ c_k$ (where k runs along the rows). We can illustrate this as follows:

$$\begin{tikzcd}
& C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\
F(C_2)(J_1) \arrow[rr] \arrow[d, "F(h)(J_1)"'] & & F(C_2)(J_2) \arrow[d, "F(h)(J_2)"] \\
F(C_1)(J_1) \arrow[rr] & & F(C_1)(J_2)
\end{tikzcd}$$

Letting I be an index category, if we now consider the limit

$$p_{k,i} : \lim_{\leftarrow k \in I} F(C_i)(J_k) \to F(C_i)(J_k) \quad k \in I$$

then we see that due to the Universal Property of Limits (**Definition 2.38**) in the category **Sets**, we have that for all of our pointwise limits

$$\exists! u : C \to \lim_{\leftarrow j \in J} F(C_k)(j)$$

so that

$$p_{k,i} \circ u = c_k$$

Furthermore, due to the commutativity of the diagrams, we see that

$$\exists \mu : h_i \longmapsto \mu_i = \left[\lim_{\leftarrow j \in J} F(C_{i+1})(j) \to \lim_{\leftarrow j \in J} F(C_i)(j) \right]$$

where

$$\mu = \lim_{\leftarrow} F(-)(J_k)$$

$$\mu_i := \mu(h_i) = \lim_{\leftarrow} F(h)(J_k)$$

so that

$$p_{k,i} \circ \mu_i = F(h_i)(J_k) \circ p_{k,i+1}$$

We must simply show that μ, as defined, is the universal cone in the presheaf category. This is simple as the Universal Property of the Limit in the **Presheaf Category** is **induced by** the Universal Property of the Limit in the **Category of Sets**.

By commutativity, we have the following cone over F for each $j \in J$:

$$\Delta \lim_{\leftarrow j \in J} F(-)(j) \to F(-)$$

Now let I be a cone in $\mathbf{Sets}^{\mathbf{C}^{op}}$ (the presheaf category). By the universal property of limits in **Sets**, for each $C \in \mathbf{C}_0$, $\exists! \nu$ so that

$$\nu_C : I(C) \to \lim_{\leftarrow} F(C)(J_k)$$

Where ν is the natural transformation of presheaves that exists due to the Universal Properties of the Limit satisfied in **Sets**. The naturality is easily seen in the commutativity of the diagram below:

$$\begin{array}{ccc} I(C_2) & \xrightarrow{\exists! \nu_{C_2}} & \lim_{\leftarrow} F(C_2)(J_k) \\ \downarrow & & \downarrow \mu_1 \\ I(C_1) & \xrightarrow{\exists! \nu_{C_1}} & \lim_{\leftarrow} F(C_1)(J_k) \end{array}$$

Therefore, $\mu := \lim_{\leftarrow j \in J} F(-)(j)$ is the limit in $\mathbf{Sets}^{\mathbf{C}^{op}}$.

Furthermore, it follows by the Yoneda Lemma (look at **Remark 2.63**) that

$$\left(\lim_{\leftarrow} F_i(-)(J_k) \right)(C) \cong \lim_{\leftarrow} F_i(C)(J_k)$$

So that the evaluation functor preserves all limits.

$\mathbf{Sets}^{\mathbf{C}^{op}}$ is, therefore, a complete category. □

2.7.6 Colimits in Categories of Diagrams

Definition 2.68. Cocompleteness is the dual notion of completeness. A category is **cocomplete** if it has all (small) colimits.

Proposition 123. *Given categories* $\mathbf{C}$ *and* $\mathbf{D}$, *if* $\mathbf{D}$ *is cocomplete, then so is the functor category* $\mathbf{D}^{\mathbf{C}}$, *and the colimits in* $\mathbf{D}^{\mathbf{C}}$ *are computed pointwise, i.e.* $\forall C \in \mathbf{C}_0$, *the evaluation functor*

$$\mathrm{ev}_C : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$$

preserves colimits. For any small index category $\mathbf{J}$, *and a functor* $A : \mathbf{J} \to \mathbf{D}^{\mathbf{C}}$, *for each* $C \in \mathbf{C}_0$, *there is a canonical isomorphism*

$$\lim_{\rightarrow}(A_j C) \cong \left(\lim_{\rightarrow} A_j \right)(C)$$

Proof. There's two ways we can proceed with this proof: (i) we may invoke formal duality on **Proposition 122** (ii) we may prove this directly noting the formal duality.

We can immediately prove that

$$\left(\lim_{\to j\in\mathbf{J}} A_jC\right) \cong \left(\lim_{\to j\in\mathbf{J}} A_j\right)(C)$$

By the *Contravariant Yoneda Lemma* (dual of **Lemma 117**), we obtain

$$\left(\lim_{\to j\in\mathbf{J}} A_j\right)(C) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}}}(\lim_{\to j\in\mathbf{J}}, kC)$$

The statement of **Proposition 103** (Representable Functors Preserve Limits) says that the second argument (covariant) of the hom-functor preserves limits. Taking the dual says that the first argument (contravariant) preserves colimits.

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}}}\left(\lim_{\to j\in\mathbf{J}} A_j, kC\right) \cong \lim_{\to j\in\mathbf{J}} \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}}}(A_j, kC)$$

Apply the *Contravariant* Yoneda Lemma again to obtain:

$$\lim_{\to j\in\mathbf{J}} \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}}}(A_j, kC) \cong \left(\lim_{\to j\in\mathbf{J}} A_jC\right)$$

We now see that colimits in $\mathbf{Sets}^{\mathbf{C}}$ acting on an object C may be computed pointwise:

$$\lim_{\to j\in\mathbf{J}}(A_jC) \cong \left(\lim_{\to j\in\mathbf{J}} A_j\right)(C)$$

For the remainder of the proof, it follows almost exactly as in **Proposition 122** as, by assumption, we have that $\mathbf{D}$ is cocomplete, meaning that the constant functor of the limit in $\mathbf{D}^{\mathbf{C}}$ is actually a universal cone over a functor $F \in \mathbf{D}^{\mathbf{C}}$. Reverse all the arrows in the above diagrams and replace the notion of a limit with a colimit (we may come back to this and do it in more detail later). □

Corollary 124. *For any locally small* $\mathbf{C}$*, the presheaf category*

$$\mathbf{Sets}^{\mathbf{C}^{op}}$$

is cocomplete, and all the colimits are computed pointwise.

Proof. It is trivial. **Sets** is a cocomplete category, so apply **Proposition 123**. □

Definition 2.69. Let P be an object in the presheaf category $\mathbf{Sets}^{\mathbf{C}^{op}}$. The **Category of Elements** of P is denoted as

$$\int_{\mathbf{C}} P$$

and is an **Index Category** defined as follows:

The *objects* consist of pairs (x, C), where $C \in \mathbf{C}_0$ and $x \in PC$

The *morphisms* are $h : (x', C') \to (x, C)$ where x' is defined as the image of a presheaf

$$P(h)(x) = x'$$

Moreover, we can say that the morphisms are triples, $(h, (x', C'), (x, C))$, that satisfy the above relation.

The *identity morphism* is the following

$$id_{\int_{\mathbf{C}} P} : (x, C) \to (x, C)$$

or in the relational form:

$$P(id)(x) = x$$

The compositions of morphisms is defined as follows:

$$h : (x', C') \to (x, C) \qquad g : (x'', C'') \to (x', C')$$

so that

$$P(g)(x') = x'' \qquad P(h)(x) = x'$$

We now see that the composition is definable through the functorial nature of P:

$$P(g \circ h)(x) = P(g) \circ P(h)(x) = P(g)(P(h)(x)) = P(g)(x') = x''$$

Therefore, the *composition of two morphisms* is defined as follows:

For $g \circ h : (x'', C'') \to (x, C)$, it satisfies

$$P(g \circ h)(x) = \left(P(g) \circ P(h)\right)(x)$$

Remark 2.64. As $\mathbf{C}$ is small, we see that $\int_{\mathbf{C}} P$ is small as well.

Furthermore, there is a projection functor

$$\pi : \int_{\mathbf{C}} P \to \mathbf{C}$$

in the obvious way:

On objects of $\int_{\mathbf{C}} P$

$$\pi(x, C) = C$$

For $h : C' \to C \in \mathbf{C}_1$, we have that morphisms of $\int_{\mathbf{C}} P$ are

$$\pi(h : (x', C') \to (x, C)) = \pi(h) := h : C' \to C$$

Proposition 125. *For any small category* $\mathbf{C}$*, every object* P *in the presheaf category* $\mathbf{Sets}^{\mathbf{C}^{op}}$ *is a colimit of representable functors*

$$\lim_{\to j \in \mathbf{J}} yA_j \cong P$$

More specifically, there is a canonical choice of an index category $\mathbf{J}$ *and a functor* $A : \mathbf{J} \to \mathbf{C}$ *such that there is a natural isomorphism*

$$\varinjlim_{j \in \mathbf{J}} y \circ A \cong P$$

where y *is the Yoneda Embedding (**Definition 2.65** and **Theorem 118**).*

Proof. Refer to **Definition 2.69** and **Remark 2.64** for the necessary index category.

Let us define a cocone of the form

$$y \circ \pi \to P$$

Take an object $(x, C) \in \int_{\mathbf{C}} P$, and by the Yoneda Lemma, we have

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC, P) \cong P(C)$$

Therefore, there is a natural correspondence between objects of the form

$$x \in P(C) \qquad x : yC \to P$$

Furthermore, given any morphism

$$h : (X', C') \to (x, C)$$

Naturality in C implies that we have a commutative triangle

$$\begin{tikzcd} yC' \arrow[rr, "yh"] \arrow[dr, "x'"'] & & yC \arrow[dl, "x"] \\ & P & \end{tikzcd}$$

Hence, we see that $\int_{\mathbf{C}} P$ is actually equivalent to a **Slice Category** over P, i.e.

$$y/P \to \mathbf{Sets}^{\mathbf{C}^{op}}/P$$

We take the component of the cocone

$$y\pi \to P$$

at each (x, C) to be $x : yC \to P$.

This is a colimiting cocone, as if we take any other cocone $y\pi \to Q$ with components

$$\nu_{(x,C)} : yC \to Q$$

and we require a natural transformation $\nu : P \to Q$ indicated by the following diagram

$$\begin{tikzcd}
yC' \arrow[rr, "yh"] \arrow[dr, "x'"] \arrow[ddr, "\nu_{(x',C')}"'] & & yC \arrow[dl, "x"'] \arrow[ddl, "\nu_{(x,C)}"] \\
& P \arrow[d, "\nu"] & \\
& Q &
\end{tikzcd}$$

Define the components

$$\nu_C : PC \to QC$$

as follows:

$$\nu_C(x) = \nu_{(x,C)}$$

We can identify the elements $\nu_{(x,C)} \in Q(C)$ and $\nu_{x,C} : yC \to Q$ according to the Yoneda Lemma as discussed above. The naturality is clear by the Naturality in both arguments of the representable functor in the Yoneda Lemma. We take uniqueness for granted. □

We leave the proof of this next proposition for a later section (will edit later!), but it is worth noting here.

Proposition 126. *For any small category* $\mathbf{C}$*, the Yoneda Embedding*

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

is the ***free cocompletion*** *of* $\mathbf{C}$ *in the following sense:*

Given any cocomplete category ϵ*, and a functor* $F : \mathbf{C} \to \epsilon$*, there is a colimit-preserving functor*

$$F_! : \mathbf{Sets}^{\mathbf{C}^{op}} \to \epsilon$$

unique up to natural isomorphism, with the property that the following diagram commutes

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{op}} \arrow[r, "F_!"] & \epsilon \\
\mathbf{C} \arrow[u, "y"] \arrow[ur, "F"'] &
\end{tikzcd}$$

i.e. $F_! \circ y = F$

Proof. Proof omitted until later. □

2.7.7 Exponentials in Diagram Categories

Let $\mathbf{C}$ be small. Let us consider exponentials for presheaf categories $\mathbf{Sets}^{\mathbf{C}^{op}}$.

Lemma 127. *Let $\mathbf{J}$ be a small index category. Then for the functor $A : \mathbf{J} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ and a diagram $B \in \mathbf{Sets}^{\mathbf{C}^{op}}$, there is a natural isomorphism*

$$\varinjlim_{j \in \mathbf{J}} (A_j \times B) \cong \left(\varinjlim_{j \in \mathbf{J}} A_j \right) \times B$$

i.e. the functor $- \times B : \mathbf{Sets}^{\mathbf{C}^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ preserves colimits.

Proof. Start with the cocone

$$\nu_j : A_j \to \varinjlim_{j} A_j$$

Apply the product functor

$$\nu_j \times B : A_j \times B \to \left(\varinjlim_{j} A_j \right) \times B$$

This is still a cocone.

We have the following natural isomorphism

$$\nu : \varinjlim_{j \in \mathbf{J}} (A_j \times B) \to \left(\varinjlim_{j \in \mathbf{J}} A_j \right) \times B$$

By **Proposition 123**, it is sufficient to show that this is an isomorphism at every component.

$$\nu_C : \left(\varinjlim_{j \in \mathbf{J}} A_j \times B \right) (C) \to \left(\left(\varinjlim_{j \in \mathbf{J}} A_j \right) \times B \right) (C)$$

By abuse of notation, we may just regard B and all A_j as sets, as there are just presheaves being evaluated on an object, C, that will become a set anyways. Now let X be an arbitrary set.

By **Corollary 104**, **contravariant representable functors take colimits to limits**, therefore

$$\mathrm{Hom}\left(\varinjlim (A_j \times B), X \right) \cong \varprojlim \mathrm{Hom}\left(A_j \times B, X \right)$$

Because **Sets** is a Cartesian-Closed Category, the following isomorphism is natural in X

$$\varprojlim \mathrm{Hom}\left(A_j \times B, X \right) \cong \varprojlim \mathrm{Hom}\left(A_j, X^B \right)$$

Using **Corollary 104** again, and applying that **Sets** is a CCC, we see

$$\varprojlim \mathrm{Hom}\left(A_j, X^B \right) \cong \mathrm{Hom}\left(\varinjlim A_j, X^B \right) \cong \mathrm{Hom}\left(\left(\varinjlim A_j \right) \times B, X \right)$$

Therefore, we see that

$$\text{Hom}\left(\varinjlim (A_j \times B), X\right) \cong \text{Hom}(\left(\varinjlim A_j\right) \times B, X)$$

Applying the (Contravariant) Yoneda Lemma with **Corollary 119**, we see that

$$k\left(\varinjlim (A_j \times B)\right) \cong k\left(\left(\varinjlim A_j\right) \times B\right) \Longrightarrow \varinjlim (A_j \times B) \cong \left(\varinjlim A_j\right) \times B$$

This last isomorphism is natural by the Yoneda Lemma, hence we are done. $\square$

Definition 2.70. Let $P, Q : \mathbf{C} \to \mathbf{Sets}$ be two functors in a functor category $\mathbf{Sets}^{\mathbf{C}}$. In a functor category in which P and Q are objects of, we may define the action of the **Exponential of Functors** on objects of as follows:

$$Q^P(C) = \text{Hom}_{\mathbf{D}^{\mathbf{C}}}(yC \times P, Q)$$

Where the action on morphisms $h : C' \to C$ is

$$Q^P(h) = \text{Hom}(yh \times 1_P, Q)$$

This is a presheaf on $\mathbf{C}$.

Proposition 128. *For any presheaves X, P, Q in $\mathbf{Sets}^{\mathbf{C}^{op}}$, there is a natural isomorphism (in X)*

$$\text{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(X, Q^P) \cong \text{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(X \times P, Q)$$

Proof. Let $\mathbf{J}$ be a small category and let $C : \mathbf{J} \to \mathbf{C}$ be a functor. By **Proposition 125**, every presheaf is a colimit of representable functors.

$$X \cong \varinjlim_{j \in \mathbf{J}} yC_j$$

Now we can give the following string of isomorphisms

$$\text{Hom}(X, Q^P) \cong \text{Hom}\left(\varinjlim yC_j, Q^P\right) \cong \varprojlim \text{Hom}(yC_j, Q^P)$$

$$\cong \varprojlim Q^P(C_j) \cong \varprojlim \text{Hom}(yC_j \times P, Q) \cong \text{Hom}\left(\varinjlim (yC_j \times P), Q\right)$$

$$\cong \text{Hom}\left(\varinjlim (yC_j) \times P, Q\right) \cong \text{Hom}(X \times P, Q)$$

Where the proofs for each isomorphism (and their naturality) are as follows:

(1) By assumption, (2) By **Corollary 104**, *contravariant representable functors take limits to colimits*, (3) By Yoneda Lemma (**Lemma 117**), (4) By **Definition 2.70**, (5) By **Corollary 104** again, (6) By **Lemma 127**, (7) By assumption.

Therefore, we conclude that

$$\text{Hom}(X, Q^P) \cong \text{Hom}(X \times P, Q)$$

$\square$

Theorem 129. *For any small category* $\mathbf{C}$*, the presheaf category (i.e. the diagrams) is a Cartesian-Closed Category. Furthermore, the Yoneda Embedding*

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

$$C \longmapsto \mathrm{Hom}(-, C)$$

$$h : C \to C' \longmapsto \mathrm{Hom}(-, h : C \to C')$$

preserves all products and exponentials that exist in $\mathbf{C}$*.*

Proof. **Proposition 128**, **Proposition 122** shows that $\mathbf{Sets}^{\mathbf{C}^{op}}$ is a CCC.

We now prove that the Yoneda Embedding preserves products and exponentials. To this end, fix X as an arbitrary object in $\mathbf{C}$, and consider any objects A, B.

We first prove that the products are preserved by the Yoneda Embedding:

$$y(A \times B) = \mathrm{Hom}(-, A \times B)$$

Because **Representable Functors Preserve Limits** (**Proposition 103**), and the **Products are Limits** (**Example 2.17**), we see that

$$y(A \times B) = \mathrm{Hom}(-, A \times B) = \mathrm{Hom}(-, A) \times \mathrm{Hom}(-, B) = y(A) \times y(B)$$

Hence, the Yoneda Embedding preserves products.

Now we do a similar computation to prove that the Yoneda Embedding preserves exponentials

$$y(B^A)(X) = \mathrm{Hom}(X, B^A) \cong \mathrm{Hom}(X \times A, B) = y(B)(X \times A)$$

$$\cong \mathrm{Hom}(y(X \times A), y(B)) \cong \mathrm{Hom}(yX \times yA, yB) \cong \mathrm{Hom}(yX, (yB)^{yA})$$

$$\cong (yB)^{yA}(X)$$

Where the proof of each equality/isomorphism is as follows:

(1) Definition of Yoneda Embedding (**Definition 2.65**, (2) **Sets** is a Cartesian-Closed Category, (3) By definition of Yoneda Embedding (**Definition 2.65**), (4) Yoneda Lemma (**Lemma 117**), (5) By the previous proof that the Yoneda Embedding preserves products, (6) **Sets** is a Cartesian-Closed Category, (7) By Yoneda Lemma.

Therefore, we conclude that

$$y\left(B^A\right) = (yB)^{yA}$$

Hence the Yoneda Embedding preserves exponentials. $\square$

2.7.8 Topoi

A **topos** is a vast generalization of everything we know. We must first characterize a vast generalization of characteristic functions of subsets.

Definition 2.71. Let ε be a category with all finite limits. A **Sub-Object Classifier** in ε consists of an object Ω together with a morphism $t : 1 \to \Omega$ that is a *Universal Subobject* in the following sense:

Given any object E and any subobject $U \to E$, $\exists! u : E \to \Omega$ such that the following diagram is a pullback

$$\begin{tikzcd} U \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\ E \arrow[r, "u"] & \Omega \end{tikzcd}$$

The morphism u is called the **classifiying arrow** of the subobject $U \to E$.

Remark 2.65. We can think of u as taking the part of E that is exactly U (as any subobject $U \to E$ is a monomorphism), to the point t in Ω, where we regard t as a *generalized element*.

Definition 2.72. The **Subobject Presheaf** is denoted as the following:

$$\mathrm{Sub}_\varepsilon : \varepsilon^{op} \to \mathbf{Sets}$$

$$E \longmapsto (U \hookrightarrow E)/\sim$$

Where $\sim$ is the equivalence relation that includes only the elements of E that share the same preimage in U (as in **Remark 2.65**).

For the following, see our previous section on pullbacks on pages 105-118.

Proposition 130. *In **Definition 2.72**, if we take the subobject (see **Definition 2.30**)*[25] *$u : U \to E$, and we take the subobject classifier, Ω, as the representing object, we obtain*

$$\mathrm{Sub}_\varepsilon(E) \cong \mathrm{Hom}_\varepsilon(E, \Omega)$$

In other words, we see that if Sub_ε *is representable, then in ε, there is a subobject classifier, in other words,* $\mathrm{Sub}(E) \cong \mathrm{Hom}(E, \Omega)$ *implies **Definition 2.71**.*

Proof. To prove that 2.72 implies 2.71, we see that representability of the Sub presheaf means that there is an object Ω with a subobject $t : T \to \Omega$ that is universal. So that for any subobject $i : S \hookrightarrow C$, $\exists! f : C \to \Omega$ so that i is the pullback of t along f, i.e.

$$t = i^* f = f \circ i$$

[25]See page 103.

or that

$$\mathrm{Sub}(i)(f) = i^* f = f \circ i$$

so that the following commutes

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook, "i"] & T \arrow[d, hook, "t=i^* f"] \\ C \arrow[r, "\exists! f"] & \Omega \end{tikzcd}$$

□

Definition 2.73. A **Topos** is a category ε such that

1. ε has all finite limits
2. ε has a subobject classifier
3. ε has all exponentials

i.e. A Topos is a Cartesian-Closed Category with a Subobject Classifier.

Definition 2.74. The following are true of topoi:

1. Topoi have all finite colimits
2. The slice category of a topos is, again, a topos

We shall not prove this however.

We will now prove that the category of presheaves is a topos. To prove this important result, we now need the following definition.

Definition 2.75. Let $\mathbf{C}$ be a small category. A **sieve** on an object C to be any set $S \subseteq (\mathbf{C}/C)_1$ (see **Slice Category**) containing morphisms $f : \cdot \to C$ (arbitrary domain) such that it is closed under precomposition:

Given a morphism $f : D \to C \in (\mathbf{C}/C)_1$, for any morphism $g : E \to D \in \mathbf{C}_1$, we have that

$$f \circ g : E \to D \to C \in (\mathbf{C}/C)$$

Note: A sieve is a generalization of a right ideal in the theory of rings and algebras over rings.

Proposition 131. *For any small category $\mathbf{C}$, the category of presheaves (diagrams), $\mathbf{Sets}^{\mathbf{C}^{op}}$ is a topos.*

Proof. By **Theorem 129**, $\mathbf{Sets}^{\mathbf{C}^{op}}$ is a Cartesian-Closed Category, therefore, we only need to construct the subobject classifier for this category.

For convenience, let us **give a bunch of convenient definitions**, and then confirm that these are sufficient to form a subobject classifier for $\mathbf{Sets}^{\mathbf{C}^{op}}$.

Let
$$\Omega(C) = \{S \subseteq \mathbf{C}_1 \mid S \text{ is a sieve on } C\}$$
Now given $h : D \to C \in (\mathbf{C}/C)_1$, let
$$h^* : \Omega(C) \to \Omega(D)$$
be defined by
$$h^*(S) = \{g : \cdot \to D \mid h \circ g \in S\}$$
This is a presheaf $\Omega : \mathbf{C}^{op} \to \mathbf{Sets}$, with a distinguished point
$$t : 1 \to \Omega$$
At each C, the "total sieve" is defined as
$$t_C = \{f : \cdot \to C\}$$
This definition of t is a subobject classifier for presheaves as follows:

Given some object E and a subobject $U \hookrightarrow E$, define $u : E \to \Omega$ at any $C \in \mathbf{C}_0$ by
$$u_C(e) = \{f : D \to C \mid f^*(e) \in U(D) \hookrightarrow E(D)\}$$
Where $e \in E(C)$. $u_C(e)$ is the sieve of morphisms into C taking $e \in E(C)$ back into the subobject U.

With all of our definitions in place, we will verify that it is the correct choice to show that t is a subobject classifier.

We **first check** that $u_C(e)$ is actually a sieve over C:

Take $g \in \mathbf{C}_1$ so that $g : H \to D$. Because E (a presheaf) is a colimit of representable functors (see **Proposition 125**), take
$$E = \mathrm{Hom}(-, X)$$
so that
$$e \in E(C) = \mathrm{Hom}(C, X)$$
Then we see that $u_C(e)$ contains the set of all $f^*(e)$ so that
$$D \xrightarrow{f} C \xrightarrow{e} X$$
Take $g : E \to D$ as above, and we see that:
$$f \circ g \Longrightarrow (f \circ g)^*(e) = e \circ f \circ g$$
Therefore, we see that for any $f : D \to C$ such that
$$D \xrightarrow{f} C \xrightarrow{e} X \in U(D) \hookrightarrow E(D)$$

That $f \circ g : H \to C$ results in

$$H \to D \to C \to X \in U(H) \hookrightarrow E(D)$$

Therefore, we see that $f \circ g$ induces a map that takes $e \in E(C)$ back into the subobject U. Hence, we conclude that $f \circ g \in u_C(e)$, and that $u_C(e)$ is, indeed, a sieve on C.

Next, we verify that $u : E \to \Omega$ is, indeed, the correct choice of classifying morphism for the subobject classifier $t : 1 \to \Omega$.

Given an arbitrary subobject $U \hookrightarrow E$ (remember that U and E are presheaves!) and some $f : D \to C$, we see that the diagram on the left induces the diagram below on the right

$$\begin{tikzcd}
U \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook, "t"] \\
E \arrow[r] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
U(D) \arrow[rr] \arrow[d, hook] & & 1 \arrow[dll, "e"'] \arrow[d, hook, "t_C"] \arrow[dd, bend left, "t_D"] \\
E(C) \arrow[rr, "u_C"] \arrow[d, "f^*:=E(f)"'] & & \Omega(C) \arrow[d, "f^*"] \\
E(D) \arrow[rr, "u_D"] & & \Omega(D)
\end{tikzcd}$$

Because $u : E \to \Omega$ is a natural transformation, the bottom square commutes, and by our definitions, the top square commutes. Hence, the following square must commute:

$$\begin{tikzcd}
U(D) \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook, "t_D"] \\
E(D) \arrow[r, "u_D"] & \Omega(D)
\end{tikzcd}$$

The choice of components C, D are arbitrary, hence, this is true for all C, D. Therefore, $t : 1 \to \Omega$ is our subobject classifier, where $u_C(e : 1 \to E(C))$, as defined above, is our sieve over C, and $u : E \to \Omega$ is the correct choice of classifying morphism.

Therefore, the category of presheaves ($\mathbf{Sets}^{\mathbf{C}^{op}}$) is a topos. $\square$

2.8 Adjoint Functors

Here is the most important section in these notes. The notion of adjointness is ubiquitous in mathematics as a whole (whereas the Yoneda Embedding and Yoneda Lemma are only of consequence for presheaves, which encapsulates almost everything as well).

2.8.1 Preliminaries

Recall the **Universal Property of Free Monoids** (see **Definition 2.3**). Then consider the following

Definition 2.76. Universal Property of Something (?) for Monoids With every monoid M, we can regard the underlying set (sans binary operations) as $U(M)$. Every set X has a free monoid $F(X)$, there there is a function:

$$i_X : X \to UF(X)$$

with the following **Universal Property**:

For every monoid M, and every map $f : X \to U(M)$, $\exists! g : F(X) \to M$ so that the following diagram commutes:

$$F(X) \xrightarrow{\quad g \quad} M \qquad \begin{tikzcd} UF(X) \arrow[r, "U(g)"] & U(M) \\ X \arrow[u, "i_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

i.e. so that $f = U(g) \circ i_X$

Remark 2.66. Now we consider the following map

$$\phi : \mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \to \mathrm{Hom}_{\mathbf{Sets}}(X, U(M))$$

$$\phi : g \longmapsto U(g) \circ i_X$$

This ϕ is an isomorphism by the Universal Property above (**Definition 2.76**).

Remark 2.67. Some Important Notation

We write bijections (one-one correspondences or isomorphisms) in the following way:

$$\frac{F(X) \to M}{X \to U(M)}$$

Where we go from a morphism g on the top form to $\phi(g)$ by the recipe

$$\phi(g) = U(g) \circ i_X$$

Note: We can write objects and categories with this notation. As long as we define a recipe for give the correspondences.

Definition 2.77. Preliminary Definition of Adjunctions

An **adjunction** between categories **C** and **D** consists of functors:

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : U$$

and a natural transformation

$$\eta : 1_{\mathbf{C}} \to U \circ F$$

with the property:

(*) For any $C \in \mathbf{C}_0$, $D \in \mathbf{D}_0$ and $f : C \to U(D)$, $\exists! g : FC \to D$ such that

$$f = U(g) \circ \eta_C$$

as indicated by

$$F(C) \xrightarrow{\quad g \quad} D \qquad \begin{tikzcd} UF(C) \arrow[r, "U(g)"] & U(D) \\ C \arrow[u, "\eta_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

For some *terminology*:

1. F is called the **left adjoint**, U is called the **right adjoint**, and η is called the **unit** of the adjunction.

2. Sometimes, we denote the left and right adjoint as:

$$F \dashv U$$

3. Statement (*) is the **Universal Property of the Unit** η.

Remark 2.68. Recall *Categorical Equivalence* (see **Definition 2.61**).

The situation where for $F : \mathbf{C} \rightleftharpoons \mathbf{D} : U$, $F \dashv U$ is a generalization of an equivalence. A **pseudoinverse** is an adjoint.

On this note, always remember that **equivalences** concern **categories**. **Adjunctions** concern **functors**.

We now discuss adjunctions in a more clear manner.

Remark 2.69. Suppose that we have an adjunction

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : U$$

And as in the example for monoids (**Definition 2.76**), take $C \in \mathbf{C}_0$ and $D \in \mathbf{D}_0$ and consider the operation

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \to \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

$$\phi : g \longmapsto U(g) \circ \eta_C$$

By the **Universal Property of the Unit** (see **Definition 2.77**, statement (*)), for any $f : C \to UD$, $\exists! g$ so that $\phi(g)$ is as given. Furthermore, ϕ is an isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

This is also presentable as the rule

$$\frac{FC \to D}{C \to UD}$$

Example 2.55. Right Adjoint of the Diagonal Functor

Consider the diagonal functor

$$\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$$

whose action on objects is

$$\Delta(C) = (C, C)$$

whose action on morphisms is

$$\Delta(f : C \to C') = (f, f) : (C, C) \to (C', C')$$

A right adjoint for this functor is a functor

$$R : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

such that $\forall C \in \mathbf{C}_0$, and $(X, Y) \in \mathbf{C} \times \mathbf{C}$, there is a bijection

$$\frac{\Delta C \to (X, Y)}{C \to R(X, Y)}$$

i.e. we would have

$$\mathrm{Hom}_{\mathbf{C}}(C, R(X, Y)) \cong \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta C, (X, Y)) \cong \mathrm{Hom}_{\mathbf{C}}(C, X) \times \mathrm{Hom}_{\mathbf{C}}(C, Y)$$

Therefore, we have that

$$R(X, Y) \cong X \times Y$$

indicating that Δ is the left adjoint to the product functor $\times$

$$\Delta \dashv \times$$

To verify this, let us verify that we can define an **appropriate unit**:

$$\eta : 1_{\mathbf{C}} \to \times \circ \Delta$$

$$\eta_C = \langle 1_C, 1_C \rangle$$

And check that this satisfies the Universal Property (see **Definition 2.77**, statement (*)).

$$(C, C) \xrightarrow{(f_1, f_2)} (X, Y) \qquad \begin{tikzcd} C \times C \arrow[r, "f_1 \times f_2"] & X \times Y \\ C \arrow[u, "\eta_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

Starting with $f_1 : C \to X, f_2 : C \to Y$, assume that we are given any $f : C \to X \times Y$. We have unique f_1, f_2 with $f = \langle f_1, f_2 \rangle$ so that we have

$$f_1 \times f_2 \circ \eta_C = \langle f_1 \pi_1 \eta_C, f_2 \pi_2 \eta_C \rangle$$

Where $\pi_1 : (X, Y) \to X$ and $\pi_2 : (X, Y) \to Y$.

Now if we chase the above diagram, starting with $C \in \mathbf{C}_0$, we get:

$$\langle f_1 \pi_1 \eta_C, f_2 \pi_2 \eta_C \rangle C = \langle f_1 \pi_1 \eta_C(C), f_2 \pi_2 \eta_C(C) \rangle$$

$$= \langle f_1 \pi_1(C, C), f_2 \pi_2(C, C) \rangle = \langle f_1 C, f_2 C \rangle = \langle f_1, f_2 \rangle C$$

Therefore,

$$f_1 \times f_2 \circ \eta_C = \langle f_1 \pi_1 \eta_C, f_2 \pi_2 \eta_C \rangle = f$$

And the unit $\eta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is a unit of the adjunction

$$\Delta : \mathbf{C} \rightleftharpoons \mathbf{C} \times \mathbf{C} : \times$$

Remark 2.70. Warning!!!

In our notation, for any given $f_1 : C \to X,\ f_2 : C \to Y$, we have

$$\langle f_1, f_2 \rangle : C \to X \times Y$$

$$\langle f_1, f_2 \rangle : C \longmapsto \langle f_1 C, f_2 C \rangle$$

whereas

$$f_1 \times f_2 : C \times C \to X \times Y$$

$$f_1 \times f_2 := (f_1, f_2) : (C, C) \longmapsto (f_1 C, f_2 C)$$

Therefore the bracket $\langle \cdot, \cdot \rangle$, and the product $(\cdot, \cdot) := \cdot \times \cdot$ are **NOT THE SAME**.

Proposition 132. *The diagonal functor* $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ *has a right adjoint if and only if* $\mathbf{C}$ *has binary products.*

Proof. Both implications of the proposition are proven through verifying the universal property of the unit η, in **Example 2.55** $\square$

2.8.2 Hom-Set Definition

The isomorphism, ϕ (see **Remark 2.69**) is, in fact, a natural isomorphism, in both arguments of the hom functor.

Proposition 133. *Given categories and functors*

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : U$$

TFAE (the following are equivalent)

1. *F is left adjoint to U, i.e. there is a natural transformation*

$$\eta : 1_{\mathbf{C}} \to U \circ F$$

that has the ***Universal Property of the Unit*** *(see* ***Definition 2.77****).*

2. *For any $C \in \mathbf{C}_0$ and $D \in \mathbf{D}_0$, there is an isomorphism*

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

where ϕ is natural in both arguments, C and D.

Moreover, condition (1) and (2) are related by the formulas:

$$\phi(g) = U(g) \circ \eta_C$$

$$\eta_C = \phi(1_{FC})$$

Proof. We first prove that **1 implies 2**.

Now, recall in **Remark 2.69**, that the map

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \to \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

$$\phi : g \longmapsto U(g) \circ \eta_C$$

is an isomorphism, given the Universal Property of the Unit (see **Definition 2.77**). All we need to do is prove the naturality.

To that end, assume that $\eta : 1_{\mathbf{C}} \to U \circ F$ is a natural transformation, and take any morphism $h : C' \to C$ and consider the following:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(FC, D) \arrow[r, "\phi_{C,D}"] \arrow[d, "(Fh)^*"'] & \mathrm{Hom}_{\mathbf{C}}(C, UD) \arrow[d, "h^*"] \\
\mathrm{Hom}_{\mathbf{D}}(FC', D) \arrow[r, "\phi_{C',D}"'] & \mathrm{Hom}_{\mathbf{C}}(C', UD)
\end{tikzcd}$$

Let us take any $f : FC \to D$ and diagram chase:

$$h^*\phi_{C,D}(f) = h^*(U(f) \circ \eta_C) = U(f) \circ \eta_C \circ h$$

By naturality of $\eta : 1_{\mathbf{C}} \to U \circ F$, we see that $\eta_C \circ h = UF(h) \circ \eta_{C'}$ so that:

$$U(f) \circ \eta_C \circ h = U(f) \circ UF(h) \circ \eta_{C'} = U(f \circ F(h)) \circ \eta_{C'}$$

By definition of ϕ, and as $f \circ F(h) : FC' \to D$:

$$U(f \circ F(h)) \circ \eta_{C'} = \phi_{C',D}(f \circ F(h)) = \phi_{C',D}(Fh)^*(f)$$

Therefore, we obtain that $\forall f \in \mathrm{Hom}_{\mathbf{D}}(FC, D)$:

$$h^*\phi_{C,D}(f) = \phi_{C',D}(Fh)^*(f)$$

so that

$$h^* \circ \phi_{C,D} = \phi_{C',D} \circ (Fh)^*$$

Therefore, ϕ is a *natural* isomorphism in C.

We must also prove naturality in D.

To that end, take $h : D \to D'$ and we will verify that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(FC, D) \arrow[r, "\phi_{C,D}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathbf{C}}(C, UD) \arrow[d, "(Uh)_*"] \\
\mathrm{Hom}_{\mathbf{D}}(FC, D') \arrow[r, "\phi_{C,D'}"'] & \mathrm{Hom}_{\mathbf{C}}(C, UD')
\end{tikzcd}$$

Take $g : FC \to D$, and consider:

$$(Uh)_* \circ \phi_{C,D}(g) = U(h) \circ U(g) \circ \eta_C = U(h \circ g) \circ \eta_C$$

$$\phi_{C,D'}(h \circ g) = \phi_{C,D'}(h_* g)$$

Therefore, $\forall g \in \mathrm{Hom}_{\mathbf{D}}(FC, D)$ we have that

$$(Uh)_*\phi_{C,D}(g) = \phi_{C,D'}(h_* g)$$

meaning that

$$(Uh)_* \circ \phi_{C,D} = \phi_{C,D'} \circ h_*$$

Therefore, ϕ is a *natural* isomorphism in D.

We now prove that **2 implies 1**. All that we have to do is show that the Universal Property of the Unit (**Definition 2.77**) follows as a result of assuming statement 2.

Assume that there is a natural isomorphism

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

where for $g : FC \to D$

$$\phi(g) = U(g) \circ \eta_C$$

$$\eta_C = \phi(1_{FC})$$

Where we lay this bijection out as:

$$\frac{FC \to D}{C \to UD}$$

Now, assume that we have an element $f \in \mathrm{Hom}_{\mathbf{D}}(FC, D)$. We see that given the commutative diagram

$$\begin{tikzcd} FC \arrow[r, "f"] \arrow[dr, "g\circ f"'] & D \arrow[d, "g"] \\ & D' \end{tikzcd}$$

We see that the following commutes also

$$\begin{tikzcd} C \arrow[r, "\phi(f)"] \arrow[dr, "\phi(g\circ f)"'] & UD \arrow[d, "Ug"] \\ & UD' \end{tikzcd}$$

by naturality in D.

Now assume that we have $f \in C \to UD$. We see that given the following commutative diagram:

$$\begin{tikzcd} C' \arrow[d, "h"'] \arrow[dr, "f\circ h"] & \\ C \arrow[r, "f"'] & UD \end{tikzcd}$$

As ϕ is a bijection, we may consider $\psi = \phi^{-1}$ and see that the following commutes by naturality in C:

$$\begin{tikzcd} FC' \arrow[d, "Fh"'] \arrow[dr, "\psi(f\circ h)"] & \\ FC \arrow[r, "\psi(f)"'] & D \end{tikzcd}$$

Given ϕ, a natural isomorphism, we are looking for a natural transformation

$$\eta : 1_{\mathbf{C}} \to U \circ F$$

It suffices to consider the components

$$\eta_C : C \to UFC$$

And we may put $FC = D$, so that our bijection is

$$\frac{1_{FC} : FC \to FC}{\eta_C : C \to UFC}\phi$$

So we have defined

$$\eta_C = \phi(1_{FC})$$

Let us now verify that this η has the universal property of the unit.

Given $g \in \mathrm{Hom}_{\mathbf{D}}(FC, D)$, we compute $U(g) \circ \eta_C$ to make sure that our definition of η_C yields the correct universal property:

$$U(g) \circ \eta_C = U(g) \circ \phi(1_{FC})$$

To proceed with this computation, recall that due to naturality of ϕ in D, we have that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "\phi(f)"] \arrow[dr, "\phi(g\circ f)"'] & UD \arrow[d, "Ug"] \\ & UD' \end{tikzcd}$$

Replacing D with FC and $f = 1_{FC}$, we obtain that:

$$\phi(g \circ 1_{FC}) = U(g) \circ \phi(1_{FC})$$

So we complete the computation:

$$U(g) \circ \phi(1_{FC}) = \phi(g \circ 1_{FC}) = \phi(g)$$

What we have concluded is that, given the following diagram (which is clearly commutative)

$$\begin{tikzcd} FC \arrow[r, "g"] & D \\ FC \arrow[u, "1_{FC}"] \arrow[ur, "g\circ 1_{FC}"'] & \end{tikzcd}$$

We can apply the recipe of ϕ to obtain that:

Given some $g : FC \to D$, $\exists! U(g) : UFC \to UD$ so that the following commutes

$$\begin{tikzcd} UFC \arrow[r, "U(g)"] & UD \\ C \arrow[u, "\eta_C"] \arrow[ur, "\phi(g)"'] & \end{tikzcd}$$

And as ϕ is an isomorphism, we see that this diagram commutes if and only if the previous diagram commutes. Therefore, $\eta : 1_{\mathbf{C}} \to U \circ F$ satisfies the Universal Property of the Unit (see **Definition 2.77**).

This concludes our proof. $\square$

Remark 2.71. Invoking formal duality gives us that the dual of statement 2 of **Proposition 133** is symmetric, but statement 1 is not. This is why **Proposition 133** is important, because it gives us a definition of an adjunction that is dual-invariant.

Note that **Proposition 133** yields the following corollary which is easily provable by reversing the direction of every morphism involved:

Corollary 134. *Given the functors and categories*

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : U$$

TFAE:

1. *There is a natural transformation*

$$\varepsilon : F \circ U \to 1_{\mathbf{D}}$$

   with the following ***Universal Property****:*

   For any $C \in \mathbf{C}_0$, $D \in \mathbf{D}_0$ and $g : FC \to D$, $\exists! f : C \to UD$ so that

$$g = \varepsilon_D \circ F(f)$$

   i.e. so that the following diagram commutes:

$$C \xrightarrow{f} UD \qquad \begin{tikzcd} FC \arrow[r, "F(f)"] \arrow[dr, "g"'] & FUD \arrow[d, "\varepsilon_D"] \\ & D \end{tikzcd}$$

   Where, denoting $\psi := \phi^{-1}$, we see that

$$\psi(f) = \varepsilon_D \circ F(f)$$

$$\varepsilon_D = \psi(1_{UD})$$

2. *For any $C \in \mathbf{C}_0$, $D \in \mathbf{D}_0$, there is an isomorphism*

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

   that is natural in C and D.

Proof.

1. Reverse every morphism and natural transformation involved in the proof of the previous proposition.

2. The Hom functor is invariant under formal duality, and the isomorphism is invariant under formal duality.

In any case, this follows as a direct result of **Proposition 133**. $\square$

Definition 2.78. Official Definition of an Adjunction

An **adjunction** is a pair of functors

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : U$$

and a natural isomorphism

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD) : \psi$$

also denoted by the schematics/recipes

$$\frac{FC \to D}{C \to UD}\phi \qquad \frac{C \to UD}{FC \to D}\psi$$

See **Remark 2.71** for why we should use this definition over **Definition 2.77**.

The **unit**, $\eta : 1_{\mathbf{C}} \to U \circ F$, and the **counit**, $\varepsilon : F \circ U \to 1_{\mathbf{D}}$ are determined (component-wise) as:

$$\eta_C = \phi(1_{FC})$$

$$\varepsilon_D = \psi(1_{UD})$$

2.8.3 Examples of Adjoints

Example 2.56. Products

Suppose $\mathbf{C}$ has binary products. Fix an object $A \in \mathbf{C}_0$ and consider the product functor

$$P(-) = - \times A : \mathbf{C} \to \mathbf{C}$$

defined on objects by

$$X \longmapsto X \times A$$

and defined on morphisms by

$$(h : X \to Y) \longmapsto (h \times 1_A : X \times A \to Y \times A)$$

or just by

$$h \longmapsto h \times 1_A$$

Let us construct a right adjoint to P:

$$U : \mathbf{C} \to \mathbf{C}$$

such that for any $X, Y \in \mathbf{C}$, there is a natural bijection given by the recipe

$$\frac{P(X) \to Y}{X \to U(Y)}$$

Define U by defining the action on objects as

$$U(Y) = Y^A$$

and defining the action on morphisms by

$$U(g : Y \to Z) = g^A : Y^A \to Z^A$$

Substituting $X = U(Y)$ in the recipe above, we obtain the recipe for the **counit** $\varepsilon : P \circ U \to 1_{\mathbf{C}}$ as:

$$\frac{P(U(Y)) \xrightarrow{\varepsilon_Y} Y}{U(Y) \xrightarrow{1} U(Y)}$$

We can now guarantee that $P : \mathbf{C} \rightleftharpoons \mathbf{C} : U$ is an adjunction if ε has the **Universal Property of the Counit** (see **Corollary 134**):

For any $f : P(X) \to Y$, $\exists! \bar{f} : X \to U(Y)$ so that

$$f = \varepsilon_Y \circ P(\bar{f})$$

However, this is exactly the **Universal Property of the Exponential** (see **Definition 2.47**). This is manifestly so if we write it out in full form

$$f = \varepsilon_Y \circ \bar{f} \times 1_A$$

Where the Universal Property for the Counit becomes

$$\begin{tikzcd} X \times A \arrow[r, "\exists! \bar{f}"] \arrow[dr, "f"'] & X \times Y^A \arrow[d, "\varepsilon_Y"] \\ & Y \end{tikzcd}$$

Therefore,

$$U(-) = -^A$$

is the right adjoint to the product functor

$$P(-) = - \times A$$

Example 2.57. Terminal Categories

Let $\mathbf{C}$ be any category. Consider the unique functor to the terminal category

$$! : \mathbf{C} \to \mathbf{1}$$

We want an adjunction $! : \mathbf{C} \rightleftharpoons \mathbf{1} : U$, so our requirement is that we have a bijection given by the recipe

$$\frac{!C \to *}{C \to U(*)}$$

This U is a terminal object in $\mathbf{C}$ (which depends on what $\mathbf{C}$ is of course).

Proposition 135. *Adjoints are unique up to isomorphism.*

More specifically, given a functor $F : \mathbf{C} \to \mathbf{D}$, and right adjoints $U, V : \mathbf{D} \to \mathbf{C}$ so that $F \dashv U$ and $F \dashv V$, then $U \cong V$.

Proof. Fix C and set D as an arbitrary object in $\mathbf{D}$. By adjointness, we have the following string of isomorphisms

$$\mathrm{Hom}_{\mathbf{C}}(C, VD) \cong \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

Clearly, this means that

$$\mathrm{Hom}_{\mathbf{C}}(C, VD) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

Which is equivalent to saying that:

$$y(VD) \cong y(UD)$$

(see Yoneda Embedding **Definition 2.66**)

By the Yoneda Lemma (see **Lemma 117**), this implies that there is a natural isomorphism

$$VD \cong UD \quad \forall D \in \mathbf{D}_0$$

$\square$

Example 2.58. Left Adjoint of the Diagonal Functor

Recall that in **Example 2.55**, we looked at the right adjoint of the diagonal functor:

$$\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$$

with the adjunction given by

$$\Delta : \mathbf{C} \rightleftharpoons \mathbf{C} \times \mathbf{C} : R(-,-)$$

Where we determined that for any category $\mathbf{C}$ with binary products, that $R : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ was the *product bifunctor*

$$- \times - : ((C, C) \in \mathbf{C} \times \mathbf{C}) \longmapsto ((C, C) \in \mathbf{C})$$

We now consider the **left adjoint** of the diagonal functor. This is an adjunction

$$L : \mathbf{C} \times \mathbf{C} \rightleftharpoons \mathbf{C} : \Delta$$

We then need to find a suitable functor L so that the isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(L(C), C') \cong \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(C, \Delta(D')$$

Whose recipe looks like

$$\frac{LC \to C'}{C \to \Delta C'}$$

gives the correct unit (counit).

Let $C' = C$ and $C = (X, Y)$ above. Then we need to find the correspondence

$$\frac{L(X, Y) \to C}{(X, Y) \to (C, C)}$$

We claim that $L(X, Y) = X + Y$, the **binary coproduct**. Although we may attempt to verify this directly, we can use some powerful ideas we developed before. Start with

$$\mathrm{Hom}_{\mathbf{C}}(L(X, Y), C) = \mathrm{Hom}_{\mathbf{C}}(X + Y, C)$$

Then recall that **Contravariant Representable Functors take Colimits to Limits** (see **Corollary 104**), so that

$$\mathrm{Hom}_{\mathbf{C}}(X + Y, C) \cong \mathrm{Hom}_{\mathbf{C}}(X, C) \times \mathrm{Hom}_{\mathbf{C}}(Y, C)$$

Then recognize that

$$\mathrm{Hom}_{\mathbf{C}}(X, C) \times \mathrm{Hom}_{\mathbf{C}}(Y, C) = \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((X, Y), (C, C))$$

This is because

$$f \times g : (X, Y) \to (C, C) \in \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((X, Y), (C, C))$$

implies that

$$f \in \mathrm{Hom}_{\mathbf{C}}(X, C) \qquad g \in \mathrm{Hom}_{\mathbf{C}}(Y, C)$$

Then, by definition of the diagonal functor, it is clear that

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((X, Y), (C, C)) = \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((X, Y), \Delta C)$$

Therefore, we end up with a natural isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(X + Y, C) \cong \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((X, Y), \Delta C)$$

Hence, $+ \dashv \Delta$.

Remark 2.72. Note that **Example 2.55, 2.58** imply that

$$+ \dashv \Delta \dashv \times$$

Where $+$, Δ, $\times$ denote the coproduct, diagonal, and product, respectively.

This is a special case of adjunctions between the diagonal functor and limits/colimits.

Remark 2.73. Relationship Between (Co)Limits and the Diagonal Functor

Recall that $\mathbf{C} \times \mathbf{C} \cong \mathbf{C}^{\mathbf{2}}$ (see **Example 2.47**). Therefore, $\Delta(C)$ is a constant C-valued functor $\forall C \in \mathbf{C}_0$.

Instead, now consider $\mathbf{J}$, a small index category. Where we define the diagonal functor as

$$\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$$
$$\Delta(C)(j) = C^j \qquad \forall C \in \mathbf{C}_0 \forall j \in \mathbf{J}$$

We have the following adjunctions

$$\varinjlim \dashv \Delta \dashv \varprojlim$$

if and only if $\mathbf{C}$ has colimits and limits, of type $\mathbf{J}$.

Definition 2.79. Fix an object $D \in \mathbf{D}_0$. The **Constant Functor**, denoted $\Delta D : \mathbf{C} \to \mathbf{D}$, is a functor whose action on objects of $\mathbf{C}$ is

$$\Delta D(C) = D$$

and whose action on morphisms of $\mathbf{C}$, f, is

$$\Delta D(f) = 1_D$$

Remark 2.74. Everything About the Constant Functor

We will lay out the significance of the constant functor. We use the same notation for the constant functor and the diagonal functor because we can write the constant functor as the following composition:

$$\mathbf{1} \xrightarrow{D} \mathbf{D} \xrightarrow{\Delta} \mathbf{D}^{\mathbf{C}}$$

Where $\mathbf{1}$ is the *terminal category*. Here, we clearly regard the fixed object D as a generalized element $D : 1 \to D$. This suggests that

$$\mathbf{D} \cong \mathbf{D}^{\mathbf{1}}$$

Clearly, $\Delta : \mathbf{D} \to \mathbf{D}^{\mathbf{C}}$ is the diagonal functor, and $\Delta D \in (\mathbf{D}^{\mathbf{C}})_0$.

Clearly, we can write the diagonal as a bifunctor:

$$\Delta : \mathbf{D} \times \mathbf{C} \to \mathbf{D}$$

Where

$$\Delta(D)(C) = D \qquad \forall D \in \mathbf{D}_0 \quad \forall C \in \mathbf{C}_0$$

Proposition 136. *Let* $\mathbf{C}$ *be a category and* $\mathbf{J}$ *a small index category. Let* $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ *be the diagonal functor where the action on objects (see* ***Definition 2.79*** *and* ***Remark 2.74****) is*

$$\Delta(C)(j) = C$$

and the action on morphisms is

$$\Delta(f)(j) = 1_C$$

Then we have the following adjoints:

$$\varinjlim \dashv \Delta \dashv \varprojlim$$

if and only if $\mathbf{C}$ *has colimits and limits of type* $\mathbf{J}$ *(see* ***Definition 2.37*** *for a limit of type* $\mathbf{J}$*)*[26].

[26] A limit of type $\mathbf{J}$ is referring to a limit of a diagram D in a category $\mathbf{C}$ is

$$D : \mathbf{J} \to \mathbf{C}$$

Where type $\mathbf{J}$ is simply referring to the fact that the source category of the diagram D is in $\mathbf{J}$. In other words, a limit is of type $\mathbf{J}$ if the limit is indexed over $\mathbf{J}$. Analogously for colimits.

Proof. One direction is completely obvious. Assume that $\mathbf{C}$ does not have limits or colimits, then we do not have an adjunction, therefore, if we have an adjunction, then those limits and colimits must exist.

Now assume that we have limits and colimits of type $\mathbf{J}$. We must simply prove that

1. Limits are right adjoints of the diagonal.

$$\mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(\Delta(\cdot), \cdot) \cong \mathrm{Hom}_{\mathbf{C}}(\cdot, \varprojlim(\cdot))$$

2. Colimits are left adjoints of the diagonal.

$$\mathrm{Hom}_{\mathbf{C}}(\varinjlim(\cdot), \cdot) \cong \mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(\cdot, \Delta(\cdot))$$

We can prove (2) by directly using **Definition 2.77**. If we can define an appropriate unit

$$\eta : 1_{\mathbf{C}} \to \Delta \circ \varinjlim$$

Whose universal property is appropriately verified, we are done. To that end, consider

$$\begin{tikzcd} \lim_{\to} C \arrow[r, "\exists! g"] & D \end{tikzcd} \qquad \begin{tikzcd} \Delta \lim_{\to}(C) \arrow[r, "\Delta(g)"] & \Delta D \\ C \arrow[u, "\eta_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

We see that the unit

$$\eta_C : C \to \Delta \varinjlim C$$

is the colimiting cocone (see **Definition 2.67** and take the dual). The colimiting cocone is the natural transformation (and the object) that takes the functor C to the constant functor of the colimit of C. Recall that this is a universal cocone. The counit of this adjunction is

$$\varepsilon : \varinjlim \Delta \to 1_{\mathbf{C}}$$

Whose components are just the identity morphism, since the colimit of the constant functor is just the constant value, i.e.

$$\varepsilon_C = \varinjlim \Delta(C) = \varinjlim C = 1_C$$

The universality of both of these is easily verified (in terms of the unit, which suffices to prove this, the colimiting cocone is the universal such cocone by definition).

(1) follows by showing that the unit of the adjunction satisfies a universal property (so the following diagram commutes):

$$\Delta C \xrightarrow{\exists! g} D \qquad \begin{tikzcd} \lim_{\leftarrow} \Delta(C) \arrow[r, "\lim_{\leftarrow}(g)"] & \lim_{\leftarrow} D \\ C \arrow[u, "\eta_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

However, this is obvious as

$$\lim_{\leftarrow} \Delta(C) = \lim_{\leftarrow} C = C$$

since the limit of a fixed object is just the fixed object.

Therefore,

$$\eta_C : C \to \lim_{\leftarrow} \Delta(C)$$

is just the identity map, 1_C, which trivially satisfies a universal property.

If you couldn't see by now, the counit of this adjunction is dictated by the following (commutative) diagram

$$C \xrightarrow{\exists! f} \lim_{\leftarrow} D \qquad \begin{tikzcd} \Delta C \arrow[r, "\Delta(f)"] \arrow[dr, "g"'] & \Delta \lim_{\leftarrow}(D) \arrow[d, "\varepsilon_D"] \\ & D \end{tikzcd}$$

Where, clearly

$$\varepsilon_D : \Delta \lim_{\leftarrow}(D) \to D$$

is clearly the **limiting cone** (see **Definition 2.67**), which, by definition, is universal among all morphisms emerging from constant functors. $\square$

Remark 2.75. Reiteration of Definition 2.67

Recall that a category $\mathbf{C}$ is **complete** if, for any small category $\mathbf{J}$, and a functor $F : \mathbf{J} \to \mathbf{C}$, $\mathbf{C}$ has a limit, i.e.

$$\lim_{\leftarrow j \in \mathbf{J}} Fj \in \mathbf{C}$$

and a universal cone

$$\eta : \Delta \lim_{\leftarrow j \in \mathbf{J}} Fj \to F \in \mathbf{C}^{\mathbf{J}}$$

With our machinery on adjunctions now, we recognize that a category $\mathbf{C}$ is **complete** if, for all F whose target category is $\mathbf{C}$, that the constant functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ at F admits the limit as a right adjoint.

Definition 2.80. Let $\mathbf{J}$ be a small category. A category $\mathbf{C}$ is **complete** if $\forall F \in \mathrm{Fun}(\mathbf{J}, \mathbf{C})$, the constant functor, Δ, at F, admits the limit as a right adjoint, i.e.

$$\Delta \dashv \lim_{\leftarrow \mathbf{J}}$$

Example 2.59. Universal Property of the Polynomial Ring

Let R be a commutative ring. $R[x]$ is the ring of polynomials in one indeterminate x, and whose coefficients are in R. Formally, the elements of $R[x]$ look like this

$$r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n \qquad r_i \in R$$

There is a morphism $\eta : R \to R[x]$ that "promotes" elements of a ring to the constant term in a polynomial, i.e.

$$\eta(r) = r_0$$

It has the following **Universal Property**:

Given any ring A, a morphism $\alpha : R \to A$, and an element $a \in A$, $\exists! a^*$

$$a^* : R[x] \to A$$

So that the following commutes

$$\begin{tikzcd}
R[x] \arrow[r, "a^*"] & A \\
R \arrow[u, "\eta"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

If we set r_i as the coefficients of the polynomial $r(x)$, we define a^* explicitly to be the **evaluation homomorphism**:

$$a^* : r(x) \longmapsto \sum_i \alpha(r_i) a^i$$

Where, more specifically,

$$a^* \left(r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n \right)$$

$$= \alpha(r_0) + \alpha(r_1)x + \alpha(r_2)x^2 + \cdots + \alpha(r_n)a^n = \alpha(r_0) + \alpha(r_1)a + \alpha(r_2)a^2 + \cdots + \alpha(r_n)a^n$$

Clearly this is a homomorphism as given $r(x)$ and $s(x)$:

$$a^*(r(x) + s(x)) = \alpha(r_0 + s_0) + \alpha(r_1 + s_1)a + \cdots + \alpha(r_n + s_n)a^n$$

$$= \sum_i \left[\alpha(r_i)a^i + \alpha(s_i)a^i \right] = a^*(r(x)) + a^*(s(x))$$

Therefore, a^* is a homomorphism.

We now define the category of pointed rings, $\mathbf{Rings}_*$, where

Objects: Pairs where A is a ring and $a \in A$

$$(A, a)$$

Morphisms: For any ring homomorphism $h : A \to B$

$$h : (A, a) \to (B, b)$$

such that

$$h(a) = b$$

The **Universal Property** above says that the **forgetful functor**

$$U : \mathbf{Rings}_* \to \mathbf{Rings}$$

whose action on objects is

$$U(A, a) = A$$

and whose action on morphisms is

$$U(h : (A, a) \to (B, b)) = h : A \to B$$

has a left adjoint, the functor

$$[x] : \mathbf{Rings} \to \mathbf{Rings}_*$$

that adjoins the indeterminate x

$$[x](R) = (R[x], x)$$

and every morphism of rings $h : R \to S$ becomes

$$[x](h : R \to S) : (R[x], x) \to (S[x], x)$$

Therefore,

$$\mathrm{Hom}_{\mathbf{Rings}_*}([x]R, (R[x], x)) \cong \mathrm{Hom}_{\mathbf{Rings}}(R, U(R[x], x))$$

The unit of this adjunction must be

$$\eta : 1_{\mathbf{Rings}} \to U \circ [x]$$

Then the $\eta_R : R \to R[x]$ is the component of this unit, and we see that the η defined above satisfies this definition. This leads us to the following proposition.

Proposition 137. ***Intrinsic Characterization of Polynomial Rings***

Let R be any (unital) commutative ring. Then the functors

Adjoining an Indeterminate:

$$[x] : \mathbf{Rings} \to \mathbf{Rings}_*$$

Forgetful Functor:

$$U : \mathbf{Rings}_* \to \mathbf{Rings}$$

are, respectively, left and right adjoints.

Proof. Look to **Example 2.59** for the main idea of the proof. We can also do it formally here.

We simply need to show that the unit, given in the previous example, satisfies the universal property of the unit, i.e. that the following diagram commutes

$$(R[x], x) \xrightarrow{\exists! h} (A, a) \qquad \begin{tikzcd} U(R[x], x) \arrow[r, "U(h)"] & U(A, a) \\ R \arrow[u, "\eta_R"] \arrow[ur, "f"'] & \end{tikzcd}$$

Given that $U(A, a) = A$ and $U(R[x], x) = R[x]$, we see that the above diagram becomes:

$$\begin{tikzcd} R[x] \arrow[r, "U(h)"] & A \\ R \arrow[u, "\eta_R"] \arrow[ur, "f"'] & \end{tikzcd}$$

However, this is exactly the **Universal Property of the Polynomial Ring**. Therefore, the map η

$$\eta(r) = r_0 \quad r \in R \quad r_0 \in R[x]$$

is actually a component of a unit

$$\eta : 1_{\mathbf{Rings}} \to U \circ [x]$$

that satisfies the universal property of the unit. □

Remark 2.76. Proposition 137 and **Example 2.59** are useful because it defines the polynomial ring $R[x]$ concretely instead of just "the ring of formal sums of powers of x with coefficients in R". Among other things, it shows that these notions did not necessarily "appear out of thin air".

2.8.4 Order Adjoints

Definition 2.81. Let $\mathbf{C}$ be a category. A **skeleton** of $\mathbf{C}$ is an equivalent category (see **Proposition 116** and **Definition 2.61**) $\mathbf{D}$ in which no two distinct objects are isomorphic.

Moreso, $\mathbf{D} = \text{Skel}(\mathbf{C})$ if:

1. $\mathbf{D}$ is a subcategory of $\mathbf{C}$, i.e. $\mathbf{D}_0 \subseteq \mathbf{C}_0$.
2. $\forall D_1, D_2 \in \mathbf{D}_0$, the morphisms in $\mathbf{D}$ are morphisms in $\mathbf{C}$, i.e.

$$\text{Hom}_{\mathbf{D}}(D_1, D_2) \subseteq \text{Hom}_{\mathbf{C}}(D_1, D_2)$$

The compositions are identity morphism are inherited from those of $\mathbf{C}$.

Moreover, because $\mathbf{D}$ is equivalent to $\mathbf{C}$ as a category, the pseudoinverses that form the equivalence are fully faithful, and $\mathbf{D}$ is dense in $\mathbf{D}$. This promotes (2) to an equality

$$\mathrm{Hom}_{\mathbf{D}}(D_1, D_2) = \mathrm{Hom}_{\mathbf{C}}(D_1, D_2)$$

We call **D skeletal** if $\mathbf{D} = \mathrm{Skel}(\mathbf{C})$.

Remark 2.77. Adjunctions on Preorders

A *preordered set* is a category in which there is **at most** one morphism $x \to y$ between two objects $x, y \in P$. A poset is a skeleton of a preordered set.

Define an order on the objects of P by

$$x \leq y \text{ iff } \exists x \to y$$

Given another preorder Q with the same order relation, suppose that we have an adjunction, $F \dashv U$:

$$F : P \rightleftharpoons Q : U$$

Then by **Proposition 133**, we have

$$\mathrm{Hom}_Q(Fa, x) \cong \mathrm{Hom}_P(a, Ux)$$

This correspondence can be restated as

$$Fa \leq x \text{ iff } a \leq Ux$$

Adjunctions on preorders, therefore, are simply order-preserving maps that satisfy the following bijection

$$\frac{Fa \leq x}{a \leq Ux}$$

$\forall p \in P$, we define the unit of this adjunction as

$$p \leq UFp$$

that is the least among all x satisfying $p \leq Ux$.

Likewise, the counit is, for each $q \in Q$, the element

$$FUq \leq q$$

that is the greatest among all y satisfying

$$Fy \leq q$$

This adjunction on preorders is called a **Galois Connection**.

Example 2.60. Adjunction in Topology

Let X be a topological space. Let $\mathcal{O}(X)$ be the set of open subsets of X (the topology of X), and consider the following:

$$\mathrm{inc} : \mathcal{O}(X) \to \mathcal{P}(X)$$

$$\text{int} : \mathcal{P}(X) \to \mathcal{O}(X)$$

Where $\mathcal{P}$ is the power set functor (which returns the set of all subsets of an object put into it). Let A be any subset and U be any open subset. Then, we get the condition that:

$$\frac{U \subseteq A}{U \subseteq \text{int}(A)}$$

Therefore, finding the interior is right adjoint to the inclusion of open subsets among the power set.

The following is clearly true for all subsets A:

$$\text{int}(A) \subseteq A$$

This is another way to state that the counit is:

$$\text{inc}(\text{int}A)) \to A \quad \forall A$$

2.8.5 RAPL - Right Adjoint Preserves Limits

We consider the preservation of limits by adjoints. The primary question we will attempt to answer in this section is: **Given a functor, does it have an adjoint? If it does, when does it have one?**

Example 2.61. There are a string of four adjoint functors between **Cat** and **Set**:

$$V \dashv F \dashv U \dashv R$$

Where U is the obvious forgetful functor.

Proposition 138. ***RAPL - Right Adjoints Preserve Limits***

As it says in the title, right adjoints preserve limits.

By formal duality, left adjoints preserve colimits.

Proof. With our existing machinery, this is very simple.

Let

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : U$$

be a pair of adjoint functors. By definition of $F \dashv U$ (see **Proposition 133**), we have that:

$$\text{Hom}_{\mathbf{C}}(X, U(Y)) \cong \text{Hom}_{\mathbf{D}}(FX, Y)$$

Then assume that $\mathbf{D}$ has limits. Let

$$D : \mathbf{J} \to \mathbf{D}$$

be a diagram, so that

$$\lim_{\leftarrow j \mathbf{J}} D_j$$

exists in $\mathbf{D}$.

We see that

$$\mathrm{Hom}_{\mathbf{C}}(X, U\left(\lim_{\leftarrow} D_i\right)) \cong \mathrm{Hom}_{\mathbf{D}}(FX, \lim_{\leftarrow} D_j) \cong \lim_{\leftarrow} \mathrm{Hom}_{\mathbf{D}}(FX, D_j)$$

The second natural isomorphism follows from **Proposition 104** (**Covariant Representable Functors Preserve Limits**). Then we may continue this:

$$\lim_{\leftarrow} \mathrm{Hom}_{\mathbf{D}}(FX, D_j) \cong \lim_{\leftarrow} \mathrm{Hom}_{\mathbf{C}}(X, UD_j) \cong \mathrm{Hom}_{\mathbf{C}}(X, \lim_{\leftarrow}(UD_j))$$

The first isomorphism follows from the definition of an adjunction, the second follows from **Corollary 104**. We conclude that:

$$\mathrm{Hom}_{\mathbf{C}}(X, U\left(\lim_{\leftarrow} D_j\right)) \cong \mathrm{Hom}_{\mathbf{C}}(X, \lim_{\leftarrow} UD_j)$$

We see that the above isomorphism is the same as:

$$yU\left(\lim_{\leftarrow} D_j\right) \cong y \lim_{\leftarrow} UD_j$$

And using **Corollary 119**, we see that this implies that

$$U\left(\lim_{\leftarrow} D_j\right) \cong \lim_{\leftarrow} UD_j$$

Hence, right adjoints preserve limits.

The case for colimits follows by duality (although you could reproduce the proof easily just like we did here). $\square$

Example 2.62. Demonstrating RAPL

We may demonstrate RAPL for a simple case of a limit. X is an object in $\mathbf{D}$, A and B are objects in $\mathbf{D}$, and consider the diagram for the following adjunction:

$$\mathrm{Hom}_{\mathbf{C}}(X, U(A \times B)) \cong \mathrm{Hom}_{\mathbf{D}}(FX, A \times B)$$

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, "\exists! t"] \arrow[dr, "g"] & \\
UA & U(A \times B) \arrow[l] \arrow[r] & UB
\end{tikzcd}
\qquad
\begin{tikzcd}
 & FX \arrow[dl, "\bar{f}"'] \arrow[d, "\exists! \bar{t}"] \arrow[dr, "\bar{g}"] & \\
A & A \times B \arrow[l] \arrow[r] & B
\end{tikzcd}$$

The isomorphism is present in the fact that:

$$\bar{\bar{t}} = t$$

Clearly, this isomorphism is natural.

Example 2.63. Recall that we proved that the presheaf category, $\mathbf{Sets}^{\mathbf{C}^{op}}$ has exponentials, and we proved it by using **Lemma 127**.

We had to jump through some hoops to show **Lemma 127**, however, now, we see that it is an obvious consequence of RAPL. We show the following easily:

$$\left(\lim_{\to} X_i\right) \times A \cong \lim_{\to}(X_i \times A)$$

We showed earlier (in **Example 2.56**) that

$$- \times A \dashv -^A$$

Therefore, using RAPL (or more technically LAPC), we see that

$$\left(\lim_{\to} X_i\right) \times A \cong \lim_{\to}(X_i \times A)$$

Remark 2.78. How do we use RAPL?

We may use RAPL to show that a functor does NOT have an adjoint by showing that it doesn't preserve (co)limits.

We may also show that a functor preserves all (co)limits by showing that it has an adjoint.

Example 2.64. All free constructions and forgetful functors are adjunctions of one another. Namely, the "free" construction is a left adjoint and the forgetful functor is the right adjoint.

Definition 2.82. A functor is **(co)continuous** if it preserves (co)limits.

Example 2.65. Yoneda Embedding

An important example of RAPL is the universal property in the presheaf category $\mathbf{Sets}^{\mathbf{C}^{op}}$. For $\mathbf{C}$ a small category, a contravariant, set-valued functor $P : \mathbf{C}^{op} \to \mathbf{Sets}$ is called a presheaf on $\mathbf{C}$. Sometimes, we denote $\mathbf{Sets}^{\mathbf{C}^{op}} := \hat{\mathbf{C}}$. We call $\hat{\mathbf{C}}$ the "free cocompletion" of $\mathbf{C}$ in the following way:

Proposition 139. ***Universal Property of the Yoneda Embedding***

For any small category $\mathbf{C}$, the (covariant) Yoneda Embedding:

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

has the following universal property:

Given an arbitrary cocomplete category ε, and a functor $F : \mathbf{C} \to \varepsilon$, $\exists ! F_! : \mathbf{Sets}^{\mathbf{C}^{op}} \to \varepsilon$ such that:

$$F_! \circ y \cong F$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{op}} \arrow[r, "\exists ! F_!"] & \varepsilon \\
\mathbf{C} \arrow[u, "y"] \arrow[ur, "F"'] &
\end{tikzcd}$$

Furthermore, $F_!$ is cocontinuous.

Proof. We must show that there is an adjunction

$$F_! \dashv F^*$$

so that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{op}} \arrow[rr, shift left, "F_!"] & & \varepsilon \arrow[ll, shift left, "F^*"] \\
\mathbf{C} \arrow[u, "y"] \arrow[urr, "F"'] & &
\end{tikzcd}$$

We define $F_!$ as follows:

Take a presheaf $P \in \mathbf{Sets}^{\mathbf{C}^{op}}$, and write it as a colimit of representable functors (see **Proposition 125**):

$$\varinjlim_{j \in \mathbf{J}} yC_j \cong P$$

Where $\mathbf{J} = \int_{\mathbf{C}} P$, the category of elements of P (see **Definition 2.69**). We can now set

$$F_!(P) = \varinjlim_{j \in \mathbf{J}} FC_j$$

Where the colimit is taken in ε. For a morphism of presheaves, we see that the **action on presheaves** must be:

$$f : P \to P'$$

$$F_!(f) : F_!(P') \to F_!(P)$$

We can write this as:

$$F_!(f) : \varinjlim_{j \in \mathbf{J}} FD_j \to \varinjlim_{j \in \mathbf{J}} FC_j$$

Therefore, letting $f_j : C_j \to D_j$ for a morphism of presheaves $f : P \to P'$, where

$$\varinjlim_{j \in \mathbf{J}} FD_j = P' \qquad \varinjlim_{j \in \mathbf{J}} FC_j = P$$

The morphism of presheaves f is related to f_j by:

$$f = \varinjlim_{j \in \mathbf{J}} Ff_j$$

Then it is obvious now that the **action** of $F_!$ **on morphisms** of presheaves f are:

$$F_!(f : P \to P') : \varinjlim F(f_j : C_j \to D_j)$$

If we want $F_!$ to be cocontinuous AND satisfy commutativity, we see that the only value that $F_!$ applied onto P can take on is:

$$F_!(\varinjlim_{j \in \mathbf{J}} C_j) = \varinjlim_{j \in \mathbf{J}} FC_j$$

Now, for an adjunction of $F_!$, $F^* : \varepsilon \to \mathbf{Sets}^{\mathbf{C}^{op}}$, we see that, for E in ε, and C in $\mathbf{C}$, that

$$F(E)(C) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC, F^*(E)) \cong \mathrm{Hom}_\varepsilon(F_!(yC), E)$$

Where the first isomorphism follows by Yoneda Lemma (**Lemma 117**), and the second isomorphism follows by our desired adjunction. By **Proposition 125** and the action of $F_!$ on presheaves, we see that

$$F_!(yC) = \lim_{\to j} FC_j \cong FC$$

Because the category of elements, $\mathbf{J}$ for a representable yC has, a terminal object, the element $1_C \in \mathrm{Hom}_{\mathbf{C}}(C, C)$ Therefore, we obtain that:

$$\mathrm{Hom}_\varepsilon(F_!(yC), E) \cong \mathrm{Hom}_\varepsilon(FC, E)$$

Therefore, we simply define the action of F^* on **objects** by saying:

$$F^*(E)(C) = \mathrm{Hom}_\varepsilon(FC, E)$$

The **action on morphisms** should be obvious through computation, we give it here but the reader should thoroughly verify:

$$F^*(E)(f : C \to C') = \mathrm{Hom}_\varepsilon(Ff_*, E)$$

Where $f_* = \mathrm{Hom}_{\mathbf{C}}(-, f)$.

Now, we prove that $F_!$ and F^* are adjunctions.

$$\mathrm{Hom}_\varepsilon(F_!(P), E) \cong \mathrm{Hom}_\varepsilon(\lim_{\to j} FC_j, E)$$

By **Corollary 104** (contravariant representable functors take colimits to limits), we see that:

$$\mathrm{Hom}_\varepsilon(\lim_{\to j} FC_j, E) \cong \lim_{\leftarrow j} \mathrm{Hom}_\varepsilon(FC_j, E) \cong \lim_{\leftarrow j} F^*(E)(C_j)$$

Now apply the Yoneda Lemma (**Lemma 117**) and **Corollary 104** to continue the isomorphism

$$\lim_{\leftarrow j} F^*(E)(C_j) \cong \lim_{\leftarrow j} \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(yC_j, F^*(E)) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(\lim_{\to j} yC_j, F^*(E))$$

$$\cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(P, F^*(E))$$

Therefore, we have established the adjunction $F_! \dashv F^*$, i.e.

$$\mathrm{Hom}_\varepsilon(F_!(P), E) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}}(P, F^*(E))$$

This proves, with certainty, that $F_!$ exists, and because RAPL (so that LAPC), and $F_! \dashv F^*$, we have that $F_!$ is cocontinuous. Because adjunctions are unique (see **Proposition 135**), it follows that $F_!$ is the **unique such cocontinuous left adjoint** of F^*. $\square$

Corollary 140. *Let* $\mathbf{C}$, $\mathbf{D}$ *be small categories and let* $f \in \mathrm{Fun}(\mathbf{C}, \mathbf{D})$ *be a functor between those categories. The precomposition functor*

$$f^* : \mathbf{Sets}^{\mathbf{D}^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

$$f^*(Q)(C) = Q(fC)$$

has left and right adjoints

$$f_! \dashv f^* \dashv f_*$$

Furthermore, there is a natural isomorphism

$$f_! \circ y_{\mathbf{C}} \cong y_{\mathbf{D}} \circ f$$

indicated by the following naturality square:

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{op}} \arrow[r, shift left=2, "f_*"] \arrow[r, leftarrow] \arrow[r, shift right=2, "f_!"'] & \mathbf{Sets}^{\mathbf{D}^{op}} \\
\mathbf{C} \arrow[u, "y_{\mathbf{C}}"] \arrow[r, "f"'] & \mathbf{D} \arrow[u, "y_{\mathbf{D}}"']
\end{tikzcd}$$

The induced left and right adjoints, $f_!$ *and* f_**, are called (left and right)* ***Kan Extensions****, respectively.*

Proof. We first define the following

$$F = y_{\mathbf{D}} \circ f : \mathbf{C} \to \mathbf{Sets}^{\mathbf{D}^{op}}$$

By the **Universal Property of the Yoneda Embedding**, the following commutes

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{op}} \arrow[r, shift right=1, "F_!"'] & \mathbf{Sets}^{\mathbf{D}^{op}} \arrow[l, shift right=1, "F^*"'] \\
\mathbf{C} \arrow[u, "y_{\mathbf{C}}"] \arrow[r, "f"'] & \mathbf{D} \arrow[u, "y_{\mathbf{D}}"']
\end{tikzcd}$$

Leading us to the fact that

$$F_! \circ y_{\mathbf{C}} \cong y_{\mathbf{D}} \circ f$$

We now see that

$$F^*(Q)(C) = \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{D}^{op}}}(FC, Q) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{D}^{op}}}(y(fC), Q) \cong Q(fC) = f^*(Q)(C)$$

The first equality is by definition of F^*, the first isomorphism follows by the definition of F, the second isomorphism follows by Yoneda Lemma (**Lemma 117**), and the final equality follows by definition of f^*.

By the **Universal Property of the Yoneda Embedding** applied to F^*, combined with the fact that

$$F_! \dashv F^* \cong f^*$$

This gives that $\exists! f_! \dashv f^*$. Furthermore, this says that $f_! \cong F_!$ due to uniqueness of adjoints. We obtain that

$$f_! \circ y_{\mathbf{C}} = y_{\mathbf{D}} \circ f$$

Similary, in the context of **Proposition 139**, we let $F := f^* \circ y_{\mathbf{D}} : \mathbf{D} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ and $\varepsilon := \mathbf{Sets}^{\mathbf{C}^{op}}$, so that the following commutes

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{D}^{op}} \arrow[r, "\exists! F_! := (f^* \circ y_{\mathbf{D}})_!"] & \mathbf{Sets}^{\mathbf{C}^{op}} \\
\mathbf{D} \arrow[u, "y"] \arrow[ur, "F := f^* \circ y_{\mathbf{D}}"'] &
\end{tikzcd}$$

Then we have that

$$(f^* \circ y_{\mathbf{D}})_! \dashv (f^* \circ y_{\mathbf{D}})^*$$

We must now show that

$$(f^* \circ y_{\mathbf{D}})_! \cong f^*$$

and it will give us that $f^* \dashv f_*$ for free (as the Universal Property of the Yoneda Embedding necessitates that a right adjoint of f^* exist).

Notice that $(f^* \circ y_{\mathbf{D}})_! : \mathbf{Sets}^{\mathbf{D}^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ is a cocontinuous functor that admits a right adjoint that uniquely exists due to **Proposition 139**. This is important because to show that $(f^* \circ y_{\mathbf{D}})_! \cong f^*$, we must show that $f^* : \mathbf{Sets}^{\mathbf{D}^{op}} \to \mathbf{Sets}^{\mathbf{C}^{op}}$ is another such functor that is cocontinuous on presheaves over $\mathbf{D}$. As adjunctions are unique up to isomorphism, this will guarantee that $f^* \cong (f^* \circ y_{\mathbf{D}})_!$. Moreover, because $F^* \cong f^*$, it is sufficient to show that F^* is cocontinuous on presheaves over $\mathbf{D}$.

$$\left(F^*(\lim_{\to j} Q_j)\right)(C) \cong (\lim_{\to j} Q_j)(fC) \cong \lim_{\to j}(Q_j(fC)) \cong \lim_{\to j}(F^* Q_j(C))$$

$$\cong \left(\lim_{\to j} F^* Q_j\right)(C)$$

Because F^* is cocontinuous, this implies that f^* is cocontinuous, and by **Proposition 135** (**Adjunctions are unique up to isomorphism**) and **Proposition 139** (**Universal Property of Yoneda Embedding**), this implies that $(f^* \circ y_{\mathbf{D}})_! \cong f^*$, and that $f^* \dashv (f^* \circ y_{\mathbf{D}})^*$. This forces the existence of a right adjoint, f_* such that $f^* \dashv f_*$. Because $f^* \dashv (f^* \circ y_{\mathbf{D}})^*$ and $f^* \dashv f_*$, by uniqueness of adjoints (**Proposition 135**), this forces $f_* \cong (f^* \circ y_{\mathbf{D}})^*$.

The conclusion is that

$$f_! \dashv f^* \dashv f_*$$

$\square$

Remark 2.79. Foundations

We can naively interpret this corollary as saying that EVERY functor has an adjoint!

Namely, for ANY $f : \mathbf{C} \to \mathbf{D}$, there is a right adjoint:

$$f^* \circ y_{\mathbf{D}} : \mathbf{D} \to \mathbf{Sets}^{\mathbf{C}^{op}}$$

The image of this right adjoint are select elements of the cocompletion of $\mathbf{C}$.

2.8.6 Locally Cartesian Closed Categories

Remark 2.80. Recap of Pullbacks

Recall that pullbacks are defined by the following commutative square.

$$\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

And namely a **Universal Property** as follows:

For an arbitrary W, and $b : W \to Y$ $a : W \to X$, $\exists ! u : W \to P$ so that the following commutes:

$$\begin{tikzcd}
W \arrow[dr, "u"] \arrow[drr, "b"] \arrow[ddr, "a"'] & & \\
& P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

We already know all the universal properties of this, however, we will clarify the terminology.

We refer to the **pullback** of g **along** f to be π_1, and it is denoted as $f^*g = \pi_1$. Likewise, the **pullback** of f **along** g to be π_2, and it is denoted $g^*f = \pi_2$.

In fact, we obtain the following important result:

Proposition 141. ***Left Adjoints of Pullbacks***

Consider the pullback in the sense of ***Remark 2.80****. Using the notation in that remark, let us regard the pullback of f under g, $g^*(f)$. Let $g_!$ denote the postcomposition functor, i.e. $g_!(-) = g \circ (-)$. Then it's true that $g_! \dashv g^*$.*

Proof. **Note: Awodey claims that this is a simple corollary of the Universal Property of Pullbacks, let us verify this**.

Consider the diagram of the universal property of the pullback in the remark above, and assume that it holds in a category $\mathbf{C}$. It is evident that $g^*(f) = \pi_2$. We claim that:

$$\mathrm{Hom}_{\mathbf{C}/Z}(g_!(b), f) \cong \mathrm{Hom}_{\mathbf{C}/Y}(b, g^*(f))$$

Where $g : Y \to Z$.

To verify this, simply note that, due to the commutativity given by the universal property, we have that:

$$a : g \circ b \to f$$

$$u : b \to \pi_2 := g^*(f)$$

For a visual aid, the universal property gives us the commutativity of the following diagrams:

$$\begin{tikzcd} W \arrow[rr, "a"] \arrow[dr, "g\circ b"'] & & X \arrow[dl, "f"] \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} W \arrow[rr, "u"] \arrow[dr, "b"'] & & P \arrow[dl, "\pi_2"] \\ & Y & \end{tikzcd}$$

To see how these two diagrams imply the desired adjunction, let us apply g^* to a, we obtain:

$$g^* \circ g_! b \overset{g^*a}{\to} g^* f$$

Clearly, we can draw a diagram in $\mathbf{C}/Y$:

$$\begin{tikzcd} g^* \circ g_! b \arrow[r, "g^*(a)"] & g^* f \\ b \arrow[u, "\eta_b"] \arrow[ur, "\exists! u"'] & \end{tikzcd}$$

Notice that due to the **Universal Property of the Pullback**, u necessarily "pushes" b to g^*f! Because this exists uniquely, the unit of this adjunction at $b \in \mathbf{C}/Y$, η_b, must necessarily exist uniquely! We adopted the suggestive notation because this latter diagram is the **Universal Property of the Unit** (see **Definition 2.77**) for the adjunction $g_! \dashv g^*$. Therefore, it follows that for any $b \in \mathbf{C}/Y$ and for any $f \in \mathbf{C}/Z$, and $g : Y \to Z$, that:

$$\mathrm{Hom}_{\mathbf{C}/Z}(g_!(b), f) \cong \mathrm{Hom}_{\mathbf{C}/Y}(b, g^*(f))$$

$\square$

We now look at a special case of **Corollary 140**.

Example 2.66. Base Change (simple case)

A **change of base** for indexed families of sets by a reindexing function $\alpha : \mathbf{J} \to \mathbf{I}$ gives rise to a triplet of adjoint functors described in **Corollary 140**. We can look at:

$$\begin{tikzcd} \mathbf{Sets}^{\mathbf{J}} \arrow[r, shift left=4, "\alpha_*"] \arrow[r, shift right=4, "\alpha_!"'] & \mathbf{Sets}^{\mathbf{I}} \arrow[l, "\alpha^*"'] \end{tikzcd}$$

So that

$$\alpha_! \dashv \alpha^* \dashv \alpha_*$$

Now, we see that an object A in $\mathbf{Sets}^{\mathbf{I}}$, is a family of sets indexed by $\mathbf{I}$:

$$(A_i)_{i \in \mathbf{I}}$$

We can further interpret A as being a diagram of type $\mathbf{I}$:

$$A : \mathbf{I} \to \mathbf{Sets}$$

$$A : i \longmapsto A_i$$

Let $p_i : A_i \to \mathbf{I}$ We see that the pullback of p_i along α is given by the following commutative diagram:

$$\begin{tikzcd} A_j \arrow[r, "(A\circ\alpha)_i"] \arrow[d, "p_j"'] & A_i \arrow[d, "p_i"] \arrow[dl, phantom, "\xleftarrow{\alpha^*}"] \\ \mathbf{J} \arrow[r, "\alpha"'] & \mathbf{I} \end{tikzcd}$$

And it is denoted as α^* where:

$$\alpha^*(A) = A \circ \alpha$$

is the reindexing of A along α into a $\mathbf{J}$-indexed family:

$$\alpha^*(A) = \left(A_{\alpha(j)}\right)_{j \in \mathbf{J}}$$

Given $\mathbf{J}$-indexed family B, we now compute the other two adjoints.

We consider the case $I = 1$, $\alpha = !_{\mathbf{J}} : \mathbf{J} \to 1$.

$$(!_{\mathbf{J}})^* : \mathbf{Sets} \to \mathbf{Sets}^{\mathbf{J}}$$

$(!_{\mathbf{J}})^*$ is, then, the constant family, which we know to be the diagonal functor (see **Remark 2.74**)

$$\Delta(A)(j) = A$$

By **Proposition 136**, we know that

$$\begin{tikzcd} \mathbf{Sets}^{\mathbf{J}} \arrow[r, shift left=3, "\prod"] \arrow[r, shift right=3, "\coprod"'] & \mathbf{Sets} \arrow[l, "\Delta"'] \end{tikzcd}$$

So that

$$\coprod \dashv \Delta \dashv \prod$$

For a **J**-indexed family B, we

$$\coprod_{j \in \mathbf{J}} B_j \qquad \prod_{j \in \mathbf{J}} B_j$$

By uniqueness of adjunctions, it follows that

$$(!_{\mathbf{J}})_! \cong \coprod \qquad (!_{\mathbf{J}})_* \cong \prod$$

A general reindexing $\alpha : \mathbf{J} \to \mathbf{I}$ gives way to a generalized coproduct and product along α:

$$\coprod_{\alpha} \dashv \alpha^* \dashv \prod_{\alpha}$$

defined on **J**-indexed families B_j by:

$$\alpha_!(B)_i = \left(\coprod_{\alpha} (B_j) \right)_i = \coprod_{\alpha(j)=i} B_j$$

$$\alpha_*(B)_i = \left(\prod_{\alpha} B_j \right)_i = \prod_{\alpha(j)=i} B_j$$

We can interpret these products and coproducts as being taken over indices j that are in the preimage of i under the reindexing function α.

Example 2.67. Base Change (General Case)

There is an equivalence of categories:

$$\mathbf{Sets}^{\mathbf{J}} \cong \mathbf{Sets}/\mathbf{J}$$

$$\{A_j\}_{j \in \mathbf{J}} \to (p : \sum_{j \in \mathbf{J}} A_j \to \mathbf{J})$$

$$(\pi : A \to \mathbf{J}) \to \{\pi^{-1}(j)\}_{j \in \mathbf{J}}$$

It is provable that given any $\alpha : \mathbf{J} \to \mathbf{I}$, the following commutes:

$$\begin{tikzcd} \mathbf{Sets}/\mathbf{J} \arrow[r, "\cong"] & \mathbf{Sets}^{\mathbf{J}} \\ \mathbf{Sets}/\mathbf{I} \arrow[u, "\alpha^{\#}"] \arrow[r, "\cong"'] & \mathbf{Sets}^{\mathbf{I}} \arrow[u, "\alpha^*"'] \end{tikzcd}$$

Where $\alpha^{\#}$ is the pullback functor along α. By the last example, since α^* has left and right adjoints, we may lay out a (convoluted) diagram of all the adjunctions:

$$\begin{tikzcd}
\mathbf{Sets/J} \arrow[r, "\cong"] \arrow[d, shift right=2ex, "\alpha_L"'] \arrow[d, shift left=2ex, "\alpha_{\#}"] & \mathbf{Sets^J} \arrow[d, shift right=2ex, "\alpha_!"'] \arrow[d, shift left=2ex, "\alpha_*"] \\
\mathbf{Sets/I} \arrow[u] \arrow[r, "\cong"'] & \mathbf{Sets^I} \arrow[u]
\end{tikzcd}$$

Proposition 142. *For any function on index categories*

$$\alpha : \mathbf{J} \to \mathbf{I}$$

the pullback functor

$$\alpha^{\#} : \mathbf{Sets/I} \to \mathbf{Sets/J}$$

has both left and right adjoints:

$$\alpha_L \dashv \alpha^{\#} \dashv \alpha_{\#}$$

Moreover, $\alpha^{\#}$ preserves all limits and colimits.

Proof. **Sketch of Proof**

We will exhibit these functors explicitly:

Refer to **Example 2.66**. Given $\pi : A \to \mathbf{J}$, we let $A_j = \pi^{-1}(j)$:

$$\{\alpha_!(A)\}_i = \coprod_{i=\alpha(j)} A_i = \coprod_{j \in \alpha^{-1}(i)} A_j = \coprod_{j \in \alpha^{-1}(i)} \pi^{-1}(j) = \pi^{-1} \circ \alpha^{-1}(i) = (\alpha \circ \pi)^{-1}(i)$$

We can now look for the left adjoint of $\alpha^{\#}$. We see by **Proposition 141**, that the following is a left adjoint to the pullback $\alpha^{\#}$:

$$\alpha_L(\pi : A \to \mathbf{J}) = \alpha \circ \pi$$

The right adjoint of $\alpha^{\#}$ can be described as:

$$\alpha_{\#} : \mathbf{Sets/J} \to \mathbf{Sets/I}$$

$$\alpha_{\#}(\pi) : \alpha_{\#}(A) \to I$$

$$\alpha_{\#}(\{A_i\}_i) = \{s : \alpha^{-1}(i) \to A \mid \mathrm{UP}\}$$

Where UP means that s satisfies the following commutative diagram:

$$\begin{tikzcd}
 & A \arrow[d, "\pi"] \\
\alpha^{-1}(i) \arrow[ur, "s"] \arrow[r, hook] & \mathbf{J}
\end{tikzcd}$$

To show that this demonstrates $\alpha_\#$ is an adjoint, we can simply write the definition of $\alpha_\#$ down:

$$\alpha_\#(\pi) := \prod_{j \in \alpha^{-1}(i)} \pi^{-1}(j)$$

Then this implies that

$$\alpha_\#(\pi) = \prod_i (\alpha \circ \pi)^{-1}(i)$$

Then we can take:

$$\mathrm{Hom}_{\mathbf{Sets/I}}(f, \alpha_\#(\pi)) \cong \mathrm{Hom}_{\mathbf{Sets/I}}(f, \prod_i (\alpha \circ \pi)^{-1}(i)) \cong (?)^{27} \prod_{j \in (\alpha)^{-1}(i)} \mathrm{Hom}_{\mathbf{Sets/I}}(f, \alpha \circ \pi)$$

$$\cong \mathrm{Hom}_{\mathbf{Sets/J}}(\coprod_{j \in \alpha^{-1}(i)} f^{-1}(j), \pi) = \mathrm{Hom}_{\mathbf{Sets/J}}(\alpha^\#(f), \pi)$$

Thus, $f^\# \dashv f_\#$.

As $f^\#$ has both left and right adjoints, by RAPL and LAPC, we obtain that it must preserve limits and colimits. □

In the end, we may take the change of base adjoints as the right leg of the natural isomorphism square above.

Remark 2.81. Even moreso, take $\mathbf{I} = 1$. Then all of the adjoints are of this form:

$$\begin{tikzcd} \mathbf{Sets/J} \arrow[d, shift right=4, "\coprod_J"'] \arrow[d, shift left=4, "\Pi_J"] \\ \mathbf{Sets} \arrow[u, "J^*"'] \end{tikzcd}$$

Where $! : \mathbf{J} \to 1$ (map onto the terminal object). In this simple case, we obtain:

$$\coprod_J (\pi : A \to J) = A$$

$$J^*(A) = \{p_1 : J \times A \to J\}$$

$$\prod_J (\pi : A \to J) = \{s : J \to A \mid \pi \circ s = 1\}$$

(just check these with the definitions given above).

We also obtain

$$\coprod_J J^*(A) = J \times A$$

---

[27] I am unsure if this is valid or not so we put an asterisk here for now.

$$\prod_J J^*(A) = A^J$$

Therefore, this highlights the origin of our product-exponential adjunction. Namely, the product-exponential adjunction can be decomposed into smaller adjunctions according to the above diagram:

$$\begin{tikzcd}
\mathbf{Sets} \arrow[rr, shift left, "J\times(-)"] \arrow[d] & & \mathbf{Sets} \arrow[ll, shift left, "(-)^J"] \arrow[d] \\
\mathbf{Sets} \arrow[r, shift left, "J^*"] & \mathbf{Sets/J} \arrow[l, shift left, "\prod_J"] \arrow[r, shift left, "\coprod_J"] & \mathbf{Sets} \arrow[l, shift left, "J^*"]
\end{tikzcd}$$

Note that these notions hold given any index category and a family of diagrams.

Definition 2.83. A category $\mathbf{C}$ is called **Locally Cartesian Closed** if $\mathbf{C}$ has a terminal object, and for every morphism $f : A \to B$, the composition functor

$$\coprod_f : \mathbf{C}/A \to \mathbf{C}/B$$

has a right adjoint f^* (defined as the post-composition functor in **Proposition 139**). This f^* has a right adjoint $\prod_f$, where we think of $\prod_f$ similary to the products in the change of base above.

Or in shorter words, a locally Cartesian Closed Category, $\mathbf{C}$, is one that has a terminal object and that, for every $f : A \to B$ in $\mathbf{C}$, induces the adjunction

$$f_! \dashv f^* \dashv f_*$$

Proposition 143. *Let* $\mathbf{C}$ *be a category and* A, B, *objects of* $\mathbf{C}$. *And let* $f : X \to A$ *Then the following is true:*

$$(\mathbf{C}/A)/f \cong \mathbf{C}/X$$

Proof. Recall that a slice category $\mathbf{C}/A$ is the following:

Objects:

$$f : X \to A$$

Morphisms:

$$g : X \to X'$$

such that the following commutes

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& A &
\end{tikzcd}$$

Now we can essentially transcribe this definition:

A slice category $(\mathbf{C}/A)/f$ is the category with:

Objects:

$$a : (h : Z \to A) \to (h' : Z' \to A)$$

Morphisms:

$$g : h \to h'$$

such that the following commutes:

$$\begin{tikzcd}
(h : Z \to A) \arrow[rr, "g"] \arrow[dr, "a"'] & & (h' : Z' \to A) \arrow[dl, "a'"] \\
& (f : X \to A) &
\end{tikzcd}$$

The characterizing diagram for $(\mathbf{C}/A)/f$ can be drawn as the commutativity of the following diagram:

$$\begin{tikzcd}
Z \arrow[rr, "g"] \arrow[dr, "h"] \arrow[ddr, "a"'] & & Z' \arrow[dl, "h'"'] \arrow[ddl, "a'"] \\
& A & \\
& X \arrow[u, "f"] &
\end{tikzcd}$$

The commutativity of this diagram, however, means that this is a slice category of $\mathbf{C}$ over X (just follow the edges on the outside, and recognize that A is simply the colimit of the diagram that results in the edges of the triangle, whose universal property dictates that this diagram must commute at every leg).

□

Proposition 144. *For any category* $\mathbf{C}$ *with a terminal object, TFAE:*

1. $\mathbf{C}$ *is locally cartesian closed*

2. *Every slice category* $\mathbf{C}/A$ *for all* A *in* $\mathbf{C}$ *is cartesian closed.*

Proof. $(1 \to 2)$:

If $\mathbf{C}$ is locally cartesian closed, since $\mathbf{C}$ has a terminal object, by **Definition 2.48**, we must prove that it simply has a product and exponential. As $\mathbf{C}$ necessarily has a pullback, and there is a base change functor on slice categories of $\mathbf{C}$ by **Proposition 142**, $\mathbf{C}$ has left and right adjoints, we simply refer to

Remark 2.81 (the general case follows analogously) to see that we can obtain the following product and exponential:

$$A \times B = \coprod_B B^* A$$

$$B^A = \prod_B B^* A$$

Therefore, $\mathbf{C}$ is Cartesian Closed. Then, as it follows (see **Proposition 143**) that for any $f : Z \to A$, that

$$(\mathbf{C}/A)/f \cong \mathbf{C}/Z$$

we see that the existence of base change functors between every slice of $\mathbf{C}$ means that any slice $\mathbf{C}/Z$ also has a product (the dependent sum/coproduct) and exponential (dependent product) (both given as the left and right adjoints of the base change f^* in **Proposition 142**), so that every slice is Cartesian Closed.

$(2 \to 1)$:

Now, assume every slice of $\mathbf{C}$ is Cartesian Closed. This means that every slice has a product, hence $\mathbf{C}$ has pullbacks given by products on slice categories (see **Remark 2.81**). We take for granted the left adjoint of the pullback because it is simply just postcomposition (see **Proposition 142**). Therefore, we just need the dependent product functor so that we obtain a right adjoint for the pullback.

Let $f : A \to B$, and let $p : E \to A$ so that:

$$\prod_f : \mathbf{C}/A \to \mathbf{C}/B$$

Now consider the following diagram in $\mathbf{C}/B$ (this is what Awodey gave us):

$$\begin{tikzcd}
(P \xrightarrow{h} B) \arrow[r, "\phi"] \arrow[d, "\pi"'] & (E \xrightarrow{f \circ p} B)^{(A \xrightarrow{f} B)} \arrow[d, "p^f"] \\
(B \xrightarrow{1_B} B) \arrow[r] & (A \xrightarrow{f} B)^{(A \xrightarrow{f} B)}
\end{tikzcd}$$

Now apply the adjunction $(-) \times A \dashv (-)^A$ to obtain the following diagram:

$$\begin{tikzcd}
(P \xrightarrow{h} B) \times (A \xrightarrow{f} B) \arrow[r, "\psi"] \arrow[d, "\pi \times 1_f"'] & (E \xrightarrow{f \circ p} B) \arrow[d, "p"] \\
(B \xrightarrow{1_B} B) \times (A \xrightarrow{f} B) \arrow[r] & (A \xrightarrow{f} B)
\end{tikzcd}$$

We now recognize that because 1_B is a terminal object in $\mathbf{C}/B$, the bottom leg of this diagram is trivial, and we can collapse the diagram as follows:

$$\begin{tikzcd} (P \xrightarrow{h} B) \times (A \xrightarrow{f} B) \arrow[rr, "\psi"] \arrow[dr, "f^*h"'] & & (E \xrightarrow{f \circ p} B) \arrow[dl, "p"] \\ & (A \xrightarrow{f} B) & \end{tikzcd}$$

Now this is a slice category over $(A \xrightarrow{f} B)$, and using **Proposition 143**, we see the canonical equivalence:

$$(\mathbf{C}/B)/(A \xrightarrow{f} B) \cong \mathbf{C}/A$$

Therefore, ψ is a morphism in $\mathbf{C}/A$.

Now, we know that

$$\mathrm{Hom}_{\mathbf{C}/A}(h \times f, f \circ p) \cong \mathrm{Hom}_{\mathbf{C}/B}(h, (f \circ p)^f)$$

as this is just a statement of the product-exponential adjunction. This means that for each $\psi \in \mathrm{Hom}_{\mathbf{C}/A}(h \times f, f \circ p)$, there is a $\phi \in \mathrm{Hom}_{\mathbf{C}/B}(h, (f \circ p)^f)$ so that $\psi \cong \phi$. Moreover, notice that:

$$\psi : f^*h \to p$$
$$\phi : h \to \prod_f p$$

Hence, we have that:

$$\mathrm{Hom}_{\mathbf{C}/A}(f^*h, p) \cong \mathrm{Hom}_{\mathbf{C}/B}(h, \prod_f p)$$

□

Example 2.68. Presheaves

For a small category $\mathbf{C}$, the presheaf category, $\mathbf{Sets}^{\mathbf{C}^{op}}$ is locally cartesian closed.

Lemma 145. *For any presheaf P over $\mathbf{C}$, there is a small category $\mathbf{D}$ and an equivalence:*

$$\mathbf{Sets}^{\mathbf{D}^{op}} \cong \mathbf{Sets}^{\mathbf{C}^{op}}/P$$

Furthermore, there is a functor $p : \mathbf{D} \to \mathbf{C}$ so that the following commutes:

$$\begin{tikzcd} \mathbf{Sets}^{\mathbf{D}^{op}} \arrow[rr, "\cong"] \arrow[dr, "p_!"'] & & \mathbf{Sets}^{\mathbf{C}^{op}}/P \arrow[dl, "\coprod_P"] \\ & \mathbf{Sets}^{\mathbf{C}^{op}} & \end{tikzcd}$$

Proof. We take $\mathbf{D} = \int_{\mathbf{C}} P$ where $p := \pi : \int_{\mathbf{C}} P \to \mathbf{C}$. By Yoneda Lemma, we can write the end, $\int_{\mathbf{C}} P$ isomorphically as y/P which may be described as follows:

Objects: pairs (C, x) where C is an object of $\mathbf{C}$ and $x : yC \to P$ in $\mathbf{Sets}^{\mathbf{C}^{op}}$

Morphisms: all morphisms between objects in the slice category over P:

$$\begin{tikzcd} yC \arrow[rr, "\nu"] \arrow[dr, "x"'] & & yC' \arrow[dl, "x'"] \\ & P & \end{tikzcd}$$

In fact, **Theorem 118** says that every morphism is of the form $\nu = yh$ for a unique $h : C \to C'$ such that $P(h)(x') = x$.

We let $I : y/P \to \mathbf{Sets}^{\mathbf{C}^{op}}$ be the inclusion functor and define a functor

$$\phi : \mathbf{Sets}^{\mathbf{C}^{op}}/P \to \mathbf{Sets}^{(y/P)^{op}}$$

$$\phi(q : Q \to P)(C, x) = \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{op}}/P}(x, q)$$

Therefore, we see that ϕ and I are a pair of pseudoinverses (we take it for granted for now, but we shall prove a little bit later). $\square$

Corollary 146. *For any small category* $\mathbf{C}$*, the category of presheaves* $\mathbf{Sets}^{\mathbf{C}^{op}}$ *is locally cartesian closed.*

Proof. Apply **Lemma 145** and **Proposition 144** together to obtain that a $\mathbf{Sets}^{\mathbf{C}^{op}}$ is locally cartesian closed if and only if $\mathbf{Sets}^{\mathbf{C}^{op}}/P$ is cartesian closed. Use **Remark 2.83** to see that the functor p has a pullback that is right adjoint to $p_!$ and left adjoint to $\coprod_P$. $\square$

2.8.7 Adjoint Functor Theorem

The question we want to know: **When does a functor have an adjoint?**

Theorem 147. ***Adjoint Functor Theorem (Freyd)***

Let $\mathbf{C}$ *be locally small and complete. Given any category* $\mathbf{X}$ *and functor*

$$U : \mathbf{C} \to \mathbf{X}$$

TFAE:

1. *U has a left adjoint*
2. *U preserves limits, and for each X in $\mathbf{X}$, the functor U satisfies the following **solution set condition**:*

$\exists \{C_i\}_{i\in I}$ in $\mathbf{C}$ such that for any object $C \in \mathbf{C}$, and morphism $f : X \to UC$, $\exists i \in I$ and morphisms $\phi : X \to UC$, and $\bar{f} : C_i \to C$ such that:

$$f = U(\bar{f}) \circ \phi$$

i.e.

$$\begin{tikzcd} C_i \arrow[d, "\bar{f}"'] \\ C \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "f"'] & UC_i \arrow[d, "U\bar{f}"] \\ & UC \end{tikzcd}$$

Lemma 148.