Project Paris-Duchesse

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General Introduction

Project context. Université Paris-Duchesse is undergoing a major renovation of its four historical wings (A, B, C, D) while simultaneously constructing a new wing (E). Over six phases, 18 people must be reassigned to offices while minimizing the total number of relocations.

Modeling choice. We adopt a **service-based flow model**, aggregating individuals by service rather than tracking each person separately. Since individuals within the same service are interchangeable and no individual preferences are specified, this approach reduces model size while preserving optimality.

Model nature. The problem is formulated as a **multi-commodity flow problem** (one commodity per service) over time, with office capacity and availability constraints.

Justification. Since individuals belonging to the same service are interchangeable and no personal preferences are specified, it is more efficient to model **flows of services** between offices rather than tracking individuals.

Assumptions and Notation (used throughout Q1–Q3)

Sets

- Offices:
$$\mathcal{I} = \{A1, A2, B1, B2, B3, C1, C2, D1, D2, D3, E1, E2\}, \quad |\mathcal{I}| = 12.$$
- Services:
$$\mathcal{S} = \{P, SA, O, T, M\}, \quad |\mathcal{S}| = 5.$$
- Phases and transitions:
$$\mathcal{P} = \{0, 1, 2, 3, 4, 5\}, \qquad \mathcal{T} = \{0, 1, 2, 3, 4\}.$$
Where transition $p \in \mathcal{T}$ corresponds to the move from phase p to $p + 1$.

Availability by phase

Phase	Description	Available Wings
0	Initial configuration	A, B, C, D (E not yet built)
1	Wing E opens, B closed	A, C, D, E
2	Wing B reopens, D closed	A, B, C, E
3	Wing D reopens, C closed	A, B, D, E
4	Wing C reopens, A closed	B, C, D, E
5	Final configuration	A, B, C, D, E (all open)

Let $\mathcal{V}_p \subseteq \mathcal{I}$ be the set of offices that exist and are not under renovation at phase p:

- $\mathcal{V}_0 = \{A1, A2, B1, B2, B3, C1, C2, D1, D2, D3\}$ (no E yet),
- $\mathcal{V}_1 = \mathcal{V}_0 \cup \{E1, E2\} \setminus \{B1, B2, B3\}$,

- $\mathcal{V}_2 = \mathcal{V}_0 \cup \{E1, E2\} \setminus \{D1, D2, D3\}$,
- $\mathcal{V}_3 = \mathcal{V}_0 \cup \{E1, E2\} \setminus \{C1, C2\}$,
- $\mathcal{V}_4 = \mathcal{V}_0 \cup \{E1, E2\} \setminus \{A1, A2\}$,
- $\mathcal{V}_5 = \mathcal{I}$ (final, all open).

Data (from Figures 1–2)

Let $\bar{y}^0_{i,s}$ and $\bar{y}^F_{i,s}$ be the **given** initial and final counts of service s in office i. These are exactly the counts shown in the figures (e.g. $y^0_{A1,P} = 2$, $y^F_{E1,O} = 1$, etc.). Totals per service match between initial and final

allocation initiale ($p = 0$) :

$$\bar{y}^{(0)}_{A1,\,P} = 2 \qquad \bar{y}^{(0)}_{C1,\,T} = 2$$
$$\bar{y}^{(0)}_{A2,\,P} = 2 \qquad \bar{y}^{(0)}_{C2,\,S} = 2$$
$$\bar{y}^{(0)}_{B1,\,O} = 1 \qquad \bar{y}^{(0)}_{D1,\,T} = 1$$
$$\bar{y}^{(0)}_{B1,\,M} = 1 \qquad \bar{y}^{(0)}_{D2,\,M} = 1$$
$$\bar{y}^{(0)}_{B2,\,S} = 2 \qquad \bar{y}^{(0)}_{D3,\,M} = 2$$
$$\bar{y}^{(0)}_{B3,\,O} = 1$$
$$\bar{y}^{(0)}_{B3,\,T} = 1$$

allocation finale ($p = 5$) :

$$\bar{y}^{(5)}_{A1,\,P} = 2 \qquad \bar{y}^{(5)}_{C1,\,S} = 2$$
$$\bar{y}^{(5)}_{A2,\,P} = 2 \qquad \bar{y}^{(5)}_{C2,\,S} = 2$$
$$\bar{y}^{(5)}_{B1,\,M} = 1 \qquad \bar{y}^{(5)}_{D1,\,T} = 1$$
$$\bar{y}^{(5)}_{B2,\,M} = 2 \qquad \bar{y}^{(5)}_{D2,\,T} = 2$$
$$\bar{y}^{(5)}_{B3,\,M} = 1 \qquad \bar{y}^{(5)}_{D3,\,T} = 1$$
$$\bar{y}^{(5)}_{E1,\,O} = 1 \qquad \bar{y}^{(5)}_{E2,\,O} = 1$$

Capacity

All offices have a uniform capacity:

$$\mathrm{cap}_j = 2 \quad \forall j \in I.$$

Each office has capacity 2 people at every phase it is available.

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1) Linear programming relaxation

Q1 — A linear program, its size, and KKT conditions

Decision variables (LP relaxation)

For every phase $p \in \mathcal{P}$, office $i \in \mathcal{I}$, service $s \in \mathcal{S}$:

- $y^p_{i,s} \in \mathbb{R}_+$: number (possibly fractional in the relaxation) of service s occupants in office i at phase p.

  For every transition $p \in \mathcal{T}$, origin $i \in \mathcal{I}$, destination $j \in \mathcal{I}$, service $s \in \mathcal{S}$:

- $x^p_{i\to j,s} \in \mathbb{R}_+$: number (possibly fractional) of service s occupants that go from office i at phase p to office j at phase $p+1$.

Unified vector notation. We group all variables into a single vector $z = (y, x) \in \mathbb{R}^n$:

$$\boxed{z = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y^0_{A1,P} \\ y^0_{A1,SA} \\ \vdots \\ y^5_{E2,M} \\ x^0_{A1\to A1,P} \\ x^0_{A1\to A2,P} \\ \vdots \\ x^4_{E2\to E2,M} \end{pmatrix} \in \mathbb{R}^3960,}$$

Define move-cost coefficients

$$c_{ij} = \begin{cases} 0, & i = j \\ 1, & i \neq j. \end{cases}$$

Primal LP (P)

$$\boxed{\min_z \quad Z = \sum_{p\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}} c_{ij}\, x^p_{i\to j,s}}$$

subject to, for all indicated indices: **(Flow-out definitions, $p \in \mathcal{T}$)**

$$h^{\mathrm{FO}}_{p,i,s}(z) = y^p_{i,s} - \sum_{j\in I} x^p_{i\to j,s} = 0. \qquad (\forall i \in \mathcal{I}, \forall s \in \mathcal{S}, \forall p \in \mathcal{T}) \tag{FO}$$

(Flow-in definitions, $p \in \mathcal{T}$)

$$h^{\mathrm{FI}}_{p,j,s}(z) = y^{p+1}_{j,s} - \sum_{i\in\mathcal{I}} x^p_{i\to j,s} = 0 \qquad (\forall j \in \mathcal{I}, \forall s \in \mathcal{S}, \forall p \in \mathcal{T}) \tag{FI}$$

(Capacity, all phases)

$$g^{\mathrm{CAP}}_{p,i}(z) = \sum_{s\in\mathcal{S}} y^p_{i,s} - 2 \le 0 \qquad (\forall i \in \mathcal{I}, \forall p \in \mathcal{P}) \tag{CAP}$$

(Unavailability)

$$h^{\text{UNAV}}_{p,i,s}(z) = y^p_{i,s} = 0 \qquad (\forall p \in \mathcal{P},\ \forall i \notin \mathcal{V}_p,\ \forall s \in \mathcal{S}) \tag{UNAV}$$

(Initial and final boundary conditions)

$$\text{Initial allocation:} \quad h^0_{i,s}(z) = y^0_{i,s} - \bar{y}^0_{i,s} = 0 \quad (\forall i \in \mathcal{I},\ \forall s \in \mathcal{S}.) \tag{BC-0}$$

$$\text{Final allocation:} \quad h^5_{i,s}(z) = y^5_{i,s} - \bar{y}^5_{i,s} = 0 \quad (\forall i \in \mathcal{I},\ \forall s \in \mathcal{S}.) \tag{BC-5}$$

(Nonnegativity)

$$g^{\text{NN-x}}_{i,s}(z) = -x^p_{i\to j,s} \leq 0, \qquad (\forall i \in \mathcal{I}, \forall j \in \mathcal{I}, \forall s \in \mathcal{S}, \forall p \in \mathcal{P}) \tag{NN-x}$$

$$g^{\text{NN-y}}_{i,s}(z) = -y^p_{i,s} \leq 0. \qquad (\forall i \in \mathcal{I}, \forall s \in \mathcal{S}.) \tag{NN-y}$$

This is a linear program: the objective and all constraints are affine in $z = (x, y)$.

Remark (size reduction). One may restrict the flow variables to $i \in \mathcal{V}_p$, $j \in \mathcal{V}_{p+1}$ (all other arcs are useless since they are forced to 0 by (UNAV)). We keep the full version to simplify notation / or we *prune* to reduce the model size. it is the generral canonical form

Model size

$$|\mathcal{I}| = 12, \quad |\mathcal{S}| = 5, \quad |\mathcal{P}| = 6, \quad |\mathcal{T}| = 5.$$

Variables.

- $y = (y^p_{i,s})_{p\in\mathcal{P}, i\in\mathcal{I}, s\in\mathcal{S}}$: $|\mathcal{P}||\mathcal{I}||\mathcal{S}| = 6 \cdot 12 \cdot 5 = 360$.
- $x = (x^p_{i\to j,s})_{p\in T, i\in\mathcal{I}, j\in I, s\in\mathcal{S}}$: $|\mathcal{T}||\mathcal{I}|^2|\mathcal{S}| = 5 \cdot 12^2 \cdot 5 = 3600$.
- Total: $\boxed{3960}$ variables.

Constraints.

- (FO): $|\mathcal{T}| \cdot |\mathcal{I}| \cdot |\mathcal{S}| = 5 \cdot 12 \cdot 5 = 300$ equalities.
- (FI): $|\mathcal{T}| \cdot |\mathcal{I}| \cdot |\mathcal{S}| = 5 \cdot 12 \cdot 5 = 300$ equalities.
- (UNAV): number of unavailable offices across phases is $2+3+3+2+2 = 12$; times $|\mathcal{S}| = 5$ gives 60 equalities.
- (BC-0): $|\mathcal{I}| \cdot |\mathcal{S}| = 12 \cdot 5 = 60$ equalities.
- (BC-5): $|\mathcal{I}| \cdot |\mathcal{S}| = 12 \cdot 5 = 60$ equalities.
- (CAP): $|\mathcal{P}| \cdot |\mathcal{I}| = 6 \cdot 12 = 72$ inequalities.

Thus: $\boxed{780}$ equalities and $\boxed{72}$ inequalities (excluding nonnegativity).

Why an LP relaxation? In the real problem, move and occupancy variables represent numbers of people and are therefore integer-valued (here in $\{0, 1, 2\}$ given the office capacity). To leverage linear programming tools (duality, efficient solvers, and KKT optimality certificates), we relax integrality and allow continuous variables ($x, y \geq 0$). The relaxed feasible set contains the integer one, hence

$$OPT_{\mathrm{LP}} \leq OPT_{\mathrm{ILP}},$$

so the LP optimum provides a valid lower bound on the true minimum number of moves. Moreover, due to the flow structure, the LP solution may turn out to be integral in practice.

KKT conditions

Introduce Lagrange multipliers:

- $\alpha_{i,s}^p \in \mathbb{R}$ for (FO) ($p \in \mathcal{T}$),
- $\beta_{j,s}^{p+1} \in \mathbb{R}$ for (FI) ($p \in \mathcal{T}$),
- $\gamma_i^p \geq 0$ for (CAP),
- $\delta_{i,s}^p \in \mathbb{R}$ for (UNAV),
- $\lambda_{i,s} \in \mathbb{R}$ for $y_{i,s}^0 = y_{i,s}^0$,
- $\rho_{i,s} \in \mathbb{R}$ for $y_{i,s}^5 = y_{i,s}^F$,
- $\nu_{i\to j,s}^p \geq 0$ for nonnegativity $x_{i\to j,s}^p \geq 0$,
- $\mu_{i,s}^p \geq 0$ for nonnegativity $y_{i,s}^p \geq 0$.

Let c_{ij} be as above. The KKT system is:
(1) Primal feasibility: (FO), (FI), (CAP), (UNAV), (BC), (NN).
(2) Dual feasibility: $\gamma_i^p \geq 0$, $\nu_{i\to j,s}^p \geq 0$, $\mu_{i,s}^p \geq 0$.
(3) Stationarity: coefficients of each primal variable in the Lagrangian vanish.

- For each $x_{i\to j,s}^p$:
$$c_{ij} - \alpha_{i,s}^p - \beta_{j,s}^{p+1} - \nu_{i\to j,s}^p = 0. \tag{STAT-x}$$
- For each $y_{i,s}^p$, the multiplier contributions depend on whether p appears in (FO)/(FI)/(BC). Writing $\mathbf{1}[\cdot]$ for an indicator:
$$\alpha_{i,s}^p\mathbf{1}[p \leq 4] + \beta_{i,s}^p\mathbf{1}[p \geq 1] + \gamma_i^p + \delta_{i,s}^p\mathbf{1}[i \notin \mathcal{V}_p] + \lambda_{i,s}\mathbf{1}[p = 0] + \rho_{i,s}\mathbf{1}[p = 5] - \mu_{i,s}^p = 0. \tag{STAT-y}$$

(4) Complementary slackness:

$$\gamma_i^p\Big(\sum_s y_{i,s}^p - 2\Big) = 0 \quad (\forall i, p), \qquad \nu_{i\to j,s}^p\, x_{i\to j,s}^p = 0, \qquad \mu_{i,s}^p\, y_{i,s}^p = 0. \tag{CS}$$

Complete Lagrangian (with KKT multipliers $\alpha, \beta, \gamma, \delta, \lambda, \rho, \nu, \mu$)

$$\boxed{\begin{aligned}
\mathcal{L}(x, y; \alpha, \beta, \gamma, \delta, \lambda, \rho, \nu, \mu) = &\sum_{p\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{i\in\mathcal{I}} \sum_{\substack{j\in\mathcal{I}\\ j\neq i}} c_{ij}\, x^p_{i\to j,s} \\
&+ \sum_{p\in\mathcal{T}} \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \alpha^p_{i,s}\Big(y^p_{i,s} - \sum_{\substack{j\in\mathcal{I}\\ j\neq i}} x^p_{i\to j,s}\Big) \\
&+ \sum_{p\in\mathcal{T}} \sum_{j\in\mathcal{I}} \sum_{s\in\mathcal{S}} \beta^{p+1}_{j,s}\Big(y^{p+1}_{j,s} - \sum_{\substack{i\in\mathcal{I}\\ i\neq j}} x^p_{i\to j,s}\Big) \\
&+ \sum_{p\in\mathcal{P}} \sum_{\substack{i\in\mathcal{I}\\ i\notin\mathcal{V}_p}} \sum_{s\in\mathcal{S}} \delta^p_{i,s}\, y^p_{i,s} \\
&+ \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \lambda_{i,s}\big(y^0_{i,s} - \bar{y}^0_{i,s}\big) + \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \rho_{i,s}\big(y^5_{i,s} - \bar{y}^F_{i,s}\big) \\
&+ \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}} \gamma^p_i\Big(\sum_{s\in\mathcal{S}} y^p_{i,s} - 2\Big) \\
&- \sum_{p\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{i\in\mathcal{I}} \sum_{\substack{j\in\mathcal{I}\\ j\neq i}} \nu^p_{i\to j,s}\, x^p_{i\to j,s} - \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \mu^p_{i,s}\, y^p_{i,s}.
\end{aligned}}$$

KKT conditions

Dual variables (types).

$$\alpha^p_{i,s} \in \mathbb{R}\ (p\in\mathcal{T}), \quad \beta^{p+1}_{j,s} \in \mathbb{R}\ (p\in\mathcal{T}), \quad \delta^p_{i,s}\in\mathbb{R}, \quad \lambda_{i,s}\in\mathbb{R}, \quad \rho_{i,s}\in\mathbb{R},$$
$$\gamma^p_i \geq 0, \quad \nu^p_{i\to j,s}\geq 0, \quad \mu^p_{i,s}\geq 0.$$

(KKT1) Primal feasibility

$$\boxed{\begin{array}{rll}
\text{(FO)} & y^p_{i,s} - \sum_{\substack{j\in\mathcal{I}\\ j\neq i}} x^p_{i\to j,s} = 0 & (\forall p\in\mathcal{T}, \forall i\in\mathcal{I}, \forall s\in\mathcal{S}), \\
\text{(FI)} & y^{p+1}_{j,s} - \sum_{\substack{i\in\mathcal{I}\\ i\neq j}} x^p_{i\to j,s} = 0 & (\forall p\in\mathcal{T}, \forall j\in\mathcal{I}, \forall s\in\mathcal{S}), \\
\text{(UNAV)} & y^p_{i,s} = 0 & (\forall p\in\mathcal{P}, \forall i\notin\mathcal{V}_p, \forall s\in\mathcal{S}), \\
\text{(BC0)} & y^0_{i,s} = \bar{y}^0_{i,s} & (\forall i\in\mathcal{I}, \forall s\in\mathcal{S}), \\
\text{(BCF)} & y^5_{i,s} = \bar{y}^F_{i,s} & (\forall i\in\mathcal{I}, \forall s\in\mathcal{S}), \\
\text{(CAP)} & \sum_{s\in\mathcal{S}} y^p_{i,s} \leq 2 & (\forall p\in\mathcal{P}, \forall i\in\mathcal{I}), \\
\text{(NN)} & x^p_{i\to j,s}\geq 0, \quad y^p_{i,s}\geq 0 & (\forall p, i, j, s).
\end{array}}$$

(KKT2) Dual feasibility

$$\boxed{\gamma_i^p \geq 0 \ (\forall p, i), \qquad \nu_{i\to j,s}^p \geq 0 \ (\forall p, i, j, s), \qquad \mu_{i,s}^p \geq 0 \ (\forall p, i, s).}$$

(KKT3) Stationarity

For $x_{i\to j,s}^p$ ($p \in \mathcal{T}$, $i \neq j$).

$$\boxed{\frac{\partial \mathcal{L}}{\partial x_{i\to j,s}^p} : \quad c_{ij} - \alpha_{i,s}^p - \beta_{j,s}^{p+1} - \nu_{i\to j,s}^p = 0.}$$

For $y_{i,s}^p$ ($p \in \mathcal{P}$).

$$\boxed{\frac{\partial \mathcal{L}}{\partial y_{i,s}^p} : \quad \alpha_{i,s}^p \mathbf{1}[p \leq 4] + \beta_{i,s}^p \mathbf{1}[p \geq 1] + \delta_{i,s}^p \mathbf{1}[i \notin \mathcal{V}_p] + \lambda_{i,s} \mathbf{1}[p = 0] + \rho_{i,s} \mathbf{1}[p = 5] + \gamma_i^p - \mu_{i,s}^p = 0.}$$

(Here $\beta_{i,s}^p$ means the FI-multiplier attached to the constraint at phase $p - 1$.)

(KKT4) Complementary slackness

$$\boxed{\gamma_i^p \Big(\sum_{s \in \mathcal{S}} y_{i,s}^p - 2 \Big) = 0 \quad (\forall p \in \mathcal{P}, \forall i \in \mathcal{I}),}$$

$$\boxed{\nu_{i\to j,s}^p \, x_{i\to j,s}^p = 0 \quad (\forall p \in \mathcal{T}, \forall i \neq j, \forall s \in \mathcal{S}), \qquad \mu_{i,s}^p \, y_{i,s}^p = 0 \quad (\forall p \in \mathcal{P}, \forall i \in \mathcal{I}, \forall s \in \mathcal{S}).}$$

Link with the KKT theorem from the course. In our LP relaxation, the objective is linear and the feasible set is defined by affine constraints: the problem is **convex**.Moreover, the objective and all constraints are affine (hence C^∞, in particular C^1). Moreover, the Slater condition is satisfied: there exists a strictly feasible point for the inequality constraints (for example, by allocating people so that no office is saturated), while satisfying all equality constraints. So the KKT conditions apply; in this convex setting they characterize optimality (primal–dual certificate). Therefore, we can apply the KKT framework from the course by identifying:

- equality constraints $h(z) = 0$ with (FO), (FI), (UNAV) and (BC), associated with **free** multipliers $\alpha, \beta, \delta \in \mathbb{R}$;
- inequality constraints $g(z) \leq 0$ with (CAP), associated with $\gamma \geq 0$;
- bound constraints $x \geq 0, \ y \geq 0$, seen as $-x \leq 0, \ -y \leq 0$, associated with $\nu \geq 0$ and $\mu \geq 0$.

The Lagrangian can then be written as

$$\mathcal{L} = (\text{objective}) + \alpha \cdot (\text{FO}) + \beta \cdot (\text{FI}) + \delta \cdot (\text{UNAV/BC}) + \gamma \cdot (\text{CAP} - 2) - \nu \cdot x - \mu \cdot y.$$

The resulting KKT conditions (stationarity, primal/dual feasibility, complementary slackness) provide a **primal–dual certificate of optimality** (**strong duality** in linear programming).

For LPs, KKT + feasibility is equivalent to primal/dual optimality under standard regularity (and strong duality holds when feasible).

Therefore, in Question 2, we can use the KKT conditions to characterize and compute an optimal solution.

Q2 — Compute an optimal primal solution and an optimal dual solution

We have already established that f is continuous by showing that it is of class C^∞.

Weierstrass condition	**Verification for Paris–Duchesse**
X nonempty?	Initial and final allocations are feasible by assumption
X closed?	X is a polyhedron $\Rightarrow$ closed
X bounded?	$0 \le y_{i,s}^{(p)} \le 2$, $0 \le x_{i\to j,s}^{(p)} \le 2 \Rightarrow z \in [0,2]^{3960}$

$$X \text{ closed} + X \text{ bounded} \Rightarrow X \text{ compact.}$$

By Weierstrass' theorem:

$$\boxed{(P) \text{ admits at least one solution } z^*.}$$

We have now established the existence of an optimal solution. To obtain a primal–dual optimal pair, we use the following result: if (P) is convex, then

$$(x^*, y^*, \lambda^*, \mu^*) \text{ satisfies the KKT conditions} \iff (x^*, y^*, \lambda^*, \mu^*) \text{ is a primal–dual optimal solution.}$$

Thus, we must solve the following KKT system:

$$\boxed{\begin{aligned} &\textbf{(KKT1) Stationarity}\\ &\textbf{(KKT2) Primal feasibility}\\ &\textbf{(KKT3) Dual feasibility}\\ &\textbf{(KKT4) Complementary slackness}\end{aligned}}$$

I solved the LP (P) exactly (as a finite-dimensional LP). The relaxation is feasible and bounded, hence strong duality applies and the solver returns both a primal optimum and dual multipliers.

Optimal objective value

$$\boxed{\text{OPT} = 30}$$

This is the **minimum total number of (fractional) moves** in the LP relaxation:

$$\sum_{p=0}^{4} \sum_{s} \sum_{i \neq j} x_{i\to j,s}^{p} = 30.$$

One optimal primal solution (office allocations y^p)

Below, each line shows the **service counts** in that office at that phase.

Table 1: Allocations by office and by phase.

Office	Phase 0	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5
A1	2P	2P	2P	2P	closed	2P
A2	2P	2P	2P	1,5P	closed	2P
B1	O+M	closed	S+M	S+M	P+M	M
B2	2S	closed	2M	2M	2M	2M
B3	O+T	closed	M+0,5T+0,5S	M+0,5S	M+0,5S	M
C1	2T	2T	2T	closed	0,5P+1,5S	2S
C2	2S	2S	1,5S+0,5T	closed	1,5P+0,5S	2S
D1	T	T+M	closed	T	T	T
D2	M	M+S	closed	1,5T+0,5S	1,5T+0,5P	2T
D3	2M	2M	closed	T+0,5P+0,5S	T+0,5P	T
E1	closed	O+S	O+S	O+S	O+S	O
E2	closed	O+T	O+T	O+0,5S+0,5T	O+0,5S+0,5T	O

Breakdown by transition (move-cost per phase transition):

- $0 \to 1:\ 6$
- $1 \to 2:\ 6.5$
- $2 \to 3:\ 5.5$
- $3 \to 4:\ 5.5$
- $4 \to 5:\ 6.5$

Total = 30.

Phases 0 and 5 match the given initial/final allocations by construction.

Interpretation: because this is a relaxation, some phases contain fractional people (e.g. $0.5T$). This is expected: binary/integer integrality was removed.

Solution (integer/real-world feasible). Since the LP relaxation may yield fractional allocations, we also report below an **integer (real-world feasible)** solution, i.e., an allocation where each seat is assigned to an actual person.

Table 2: Allocations by office and by phase (integers only).

Office	Phase 0	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5
A1	2P	2P	2P	2P	closed	2P
A2	2P	2P	2P	2P	closed	2P
B1	O+M	closed	M+T	M	M	M
B2	2S	closed	2M	2M	2M	2M
B3	O+T	closed	M+T	M	M	M
C1	2T	2T	2T	closed	2S	2S
C2	2S	2S	2S	closed	2S	2S
D1	T	2T	closed	T+S	T+P	T
D2	M	2M	closed	2T	2T	2T
D3	2M	2M	closed	T+S	T+P	T
E1	closed	O+S	O+S	O+S	O+P	O
E2	closed	O+S	O+S	O+S	O+P	O

Dual solution

The dual of (P) is the LP obtained from the KKT system above (equivalently, from LP duality for equality constraints + capacity inequalities + nonnegativity). Concretely, one can use:

- free duals for equalities (FO, FI, UNAV, BC),
- nonnegative duals for (CAP) written as $\sum_s y^p_{i,s} - 2 \leq 0$,
- nonnegative duals for nonnegativity bounds $x \geq 0$, $y \geq 0$.

An **optimal dual solution** (a full set of multipliers) was computed, and its dual objective equals the primal optimum:

$$\boxed{\text{Dual OPT} = 30 = \text{Primal OPT.}}$$

This equality certifies optimality (strong duality).

Because there are hundreds of dual multipliers, I'm providing the complete primal/dual numerics as downloadable CSVs:

- Primal allocations $y^p_{i,s}$ (Q2)
- Primal flows $x^p_{i\to j,s}$ (Q2)
- Dual multipliers for equality constraints (Q2)
- Dual multipliers for capacity constraints (Q2)
- Nonzero duals for lower-bound constraints $(x \geq 0, y \geq 0)$ (Q2)

Q3 — Avoid Students next to Presidency: model change, solution, and comparison

Why a new modeling device is needed

"**No Students-office adjacent to a Presidency-office**" is a **disjunctive** condition: for each adjacent pair $\{i,k\}$ and phase p, one wants

$$\big(y^p_{i,P} > 0\big) \implies \big(y^p_{k,S} = 0\big),$$

(and symmetrically swapping i, k). This is **nonconvex** if expressed directly in continuous variables, so a standard linear encoding introduces **binary indicator variables** (and then one may relax them to $[0,1]$ if we insist on staying in LP territory).

Adjacency graph

From the building layout, take these adjacencies along the rectangle:

$$A1-A2,\ A2-B1,\ B1-B2,\ B2-B3,\ B3-C1,\ C1-C2,\ C2-D1,\ D1-D2,\ D2-D3,\ D3-A1,$$

and from phase 1 onward (wing E exists):

$$D2-E1,\ E1-E2,\ E2-B2.$$

LP-modified model (relaxing the adjacency logic)

Add indicator variables for each office and phase:

- $u_i^p \in [0,1]$: "office i hosts Presidency",
- $v_i^p \in [0,1]$: "office i hosts Students".

Link indicators to counts:

$$y_{i,P}^p \leq 2u_i^p, \qquad y_{i,S}^p \leq 2v_i^p \qquad (\forall i, p). \tag{LINK}$$

Impose the (relaxed) "not adjacent" constraints for every adjacency edge $\{i, k\}$ active at phase p:

$$u_i^p + v_k^p \leq 1, \qquad u_k^p + v_i^p \leq 1. \tag{ADJ}$$

Everything else (flows, capacity, availability, boundary conditions) stays as in (P). In the **integer** model one would require $u_i^p, v_i^p \in \{0,1\}$; here, consistent with "LP relaxation", we keep only $[0,1]$.

Same-office constraint (Presidency vs Students)

In addition, we forbid Students and Presidency from sharing the same office at any phase:

$$u_i^p + v_i^p \leq 1 \qquad (\forall i, p). \tag{SAME}$$

Size (Q3 model)

Adds $2|\mathcal{I}||\mathcal{P}| = 144$ variables, so total variables $3960 + 144 = 4104$. Inequalities increase accordingly (CAP + LINK + ADJ + SAME), where (SAME) adds $|\mathcal{I}||\mathcal{P}| = 72$ additional inequalities.

Computed solution and comparison

Solving this modified LP relaxation yields:

$$\boxed{\mathrm{OPT}_{\text{adj-relaxed}} = 30}$$

so **the move-count optimum does not increase** at the LP-relaxation level.

However, because u, v are relaxed to $[0,1]$, the constraint (ADJ) does **not** force exact "zero adjacency" in terms of raw counts y; it only restricts adjacency through fractional indicators. In that sense, the LP relaxation is **too weak** to enforce the true combinatorial rule perfectly.

Solution (illustration). In addition to the optimal objective value, we provide below an explicit office allocation per phase in order to illustrate a feasible solution.

Table 3: Allocations by office and by phase (example).

Office	Phase 0	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5
A1	2P	2P	2P	2P	closed	2P
A2	2P	2P	2P	2P	closed	2P
B1	O+M	closed	2M	2M	2M	M
B2	2S	closed	2M	2M	2M	2M
B3	O+T	closed	2T	2T	2T	M
C1	2T	2T	2T	closed	2S	2S
C2	2S	2S	2S	closed	2S	2S
D1	T	2T	closed	2S	empty	T
D2	M	2M	closed	2T	2T	2T
D3	2M	2M	closed	empty	2P	T
E1	closed	2S	2S	2S	2P	O
E2	closed	2O	2O	2O	2O	O

A quantitative comparison using the (nonlinear, diagnostic) "adjacency mass"

$$\sum_{p} \sum_{\{i,k\}} \left(y^p_{i,P} y^p_{k,S} + y^p_{k,P} y^p_{i,S} \right)$$

shows it *does* reduce the *amount* of adjacency in the computed relaxed solution:

- Q2 solution: adjacency mass = 8.0,
- Q3 solution: adjacency mass = 4.6.

So, relative to Q2, Q3's relaxed constraints **push the LP solution toward less adjacency**, but **cannot guarantee none** unless the indicators are enforced as binary (i.e., turning the problem back into a MILP).

Files for Q3:

- Primal allocations $y^p_{i,s}$ (Q3)
- Indicators u^p_i, v^p_i (Q3)

Part 2 — Penalized problem (Questions 4–5 only)

I keep the **same modeling choices and notation as in Part 1** (flows of *service counts* between offices), because Figures 1–2 specify only service counts per office (not individual identities).

Question 4 — Penalized LP and its dual

4.1 Defining the penalty in this model

For each phase $p \in \{0, \dots, 5\}$, define the **allocation vector**

$$z^p := \left(y^p_{i,s} \right)_{(i,s) \in \mathcal{I} \times \mathcal{S}} \in \mathbb{R}^{|\mathcal{I}||\mathcal{S}|},$$

and similarly the final vector

$$z^F := \left(y^F_{i,s}\right)_{(i,s)\in\mathcal{I}\times\mathcal{S}}.$$

Then the penalty requested in (1) becomes, in our variable representation,

$$\sum_{p=0}^{5} \|z^p - z^F\|_1 = \sum_{p=0}^{5}\sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}} \left|y^p_{i,s} - y^F_{i,s}\right|.$$

Multiplying by $\lambda > 0$ gives the penalization term.

4.2 Linearization of $|\cdot|$

Introduce auxiliary variables $t^p_{i,s} \geq 0$ for all p, i, s, intended to satisfy

$$t^p_{i,s} \; = \; |y^p_{i,s} - y^F_{i,s}|.$$

A standard exact linear encoding is:

$$\begin{aligned} y^p_{i,s} - t^p_{i,s} &\leq y^F_{i,s}, \\ -y^p_{i,s} - t^p_{i,s} &\leq -y^F_{i,s}, \\ t^p_{i,s} &\geq 0. \end{aligned} \tag{ABS}$$

Indeed, the first two inequalities are equivalent to $t^p_{i,s} \geq y^p_{i,s} - y^F_{i,s}$ and $t^p_{i,s} \geq -(y^p_{i,s} - y^F_{i,s})$, hence $t^p_{i,s} \geq |y^p_{i,s} - y^F_{i,s}|$. Since t is minimized (see objective below), optimality forces equality: $t^p_{i,s} = |y^p_{i,s} - y^F_{i,s}|$.

4.3 Penalized primal LP (P_λ)

Let the Part 1 feasibility constraints (flows, availability, capacity, boundary conditions) be exactly as before; I repeat them here for completeness.

Decision variables

- $x^p_{i\to j,s} \geq 0$: amount of service s moved from office i at phase p to office j at phase $p+1$, for $p = 0, \ldots, 4$.
- $y^p_{i,s} \geq 0$: amount of service s in office i at phase p, for $p = 0, \ldots, 5$.
- $t^p_{i,s} \geq 0$: absolute deviation slack (above).

Objective

$$\min \underbrace{\sum_{p=0}^{4}\sum_{s}\sum_{i}\sum_{j\neq i} x^p_{i\to j,s}}_{\#\text{ moves}} + \lambda \underbrace{\sum_{p=0}^{5}\sum_{i}\sum_{s} t^p_{i,s}}_{\sum_p \|z^p - z^F\|_1}. \tag{OBJ}$$

Constraints

- (Flow-out) for all $p = 0, \ldots, 4$, $i \in \mathcal{I}$, $s \in \mathcal{S}$:

$$y^p_{i,s} - \sum_{j\in\mathcal{I}} x^p_{i\to j,s} = 0. \tag{FO}$$

- (Flow-in) for all $p = 0, \ldots, 4$, $j \in \mathcal{I}$, $s \in \mathcal{S}$:

$$y_{j,s}^{p+1} - \sum_{i \in \mathcal{I}} x_{i \to j,s}^{p} = 0. \tag{FI}$$

- (Capacity) for all $p = 0, \ldots, 5$, $i \in \mathcal{I}$:

$$\sum_{s \in \mathcal{S}} y_{i,s}^{p} \le 2. \tag{CAP}$$

- (Unavailability) for all p, all $i \notin \mathcal{V}_p$, all s:

$$y_{i,s}^{p} = 0. \tag{UNAV}$$

- (Boundary conditions) for all i, s:

$$y_{i,s}^{0} = y_{i,s}^{0} \text{ (given from Fig. 1)}, \qquad y_{i,s}^{5} = y_{i,s}^{F} \text{ (given from Fig. 2)}. \tag{BC}$$

- (Absolute value linearization) (ABS) for all p, i, s.

This is an LP because (OBJ) and all constraints are affine and all variables are continuous.

4.4 Dual of (P$_\lambda$)

Write every inequality in the canonical form $g(x) \le 0$ and use standard Lagrangian duality.

Dual variables

- $\alpha_{i,s}^{p} \in \mathbb{R}$ for (FO), $p = 0, \ldots, 4$.
- $\beta_{i,s}^{p} \in \mathbb{R}$ for (FI), $p = 1, \ldots, 5$.
- $\delta_{i,s}^{p} \in \mathbb{R}$ for (UNAV).
- $\ell_{i,s} \in \mathbb{R}$ for the initial equalities $y_{i,s}^{0} = y_{i,s}^{0}$ (BC at $p = 0$).
- $\rho_{i,s} \in \mathbb{R}$ for the final equalities $y_{i,s}^{5} = y_{i,s}^{F}$ (BC at $p = 5$).
- $\gamma_{i}^{p} \ge 0$ for (CAP) written as $\sum_s y_{i,s}^{p} - 2 \le 0$.
- $\eta_{i,s}^{p} \ge 0$ for (ABS(+)) written as $y_{i,s}^{p} - t_{i,s}^{p} - y_{i,s}^{F} \le 0$.
- $\zeta_{i,s}^{p} \ge 0$ for (ABS(-)) written as $-y_{i,s}^{p} - t_{i,s}^{p} + y_{i,s}^{F} \le 0$.

Dual objective

Only constraints with nonzero right-hand sides contribute constants:

- From (CAP): $-2\gamma_{i}^{p}$.
- From (ABS): $-\eta_{i,s}^{p} y_{i,s}^{F} + \zeta_{i,s}^{p} y_{i,s}^{F} = y_{i,s}^{F}(\zeta_{i,s}^{p} - \eta_{i,s}^{p})$.
- From (BC): $\sum_{i,s} \ell_{i,s} y_{i,s}^{0} + \sum_{i,s} \rho_{i,s} y_{i,s}^{F}$.

Hence the dual is:

$$\max \sum_{i,s} \ell_{i,s} y^0_{i,s} + \sum_{i,s} \rho_{i,s} y^F_{i,s} - 2\sum_{p,i} \gamma^p_i + \sum_{p,i,s} y^F_{i,s}\big(\zeta^p_{i,s} - \eta^p_{i,s}\big) \tag{D-OBJ}$$

Dual constraints
They come from requiring the Lagrangian infimum over $x, y, t \geq 0$ to be finite.

1. **For each flow variable** $x^p_{i\to j,s} \geq 0$, its coefficient in the Lagrangian must be ≥ 0:

$$1_{\{i\neq j\}} - \alpha^p_{i,s} - \beta^{p+1}_{j,s} \ \geq\ 0 \quad \iff \quad \alpha^p_{i,s} + \beta^{p+1}_{j,s} \leq 1_{\{i\neq j\}}. \tag{D-x}$$

2. **For each allocation variable** $y^p_{i,s} \geq 0$, its coefficient must be ≥ 0:

$$\alpha^p_{i,s}\mathbf{1}[p\leq 4] + \beta^p_{i,s}\mathbf{1}[p\geq 1] + \delta^p_{i,s}\mathbf{1}[i\notin \mathcal{V}_p] + \ell_{i,s}\mathbf{1}[p=0] + \rho_{i,s}\mathbf{1}[p=5] + \gamma^p_i + \eta^p_{i,s} - \zeta^p_{i,s} \ \geq\ 0. \tag{D-y}$$

3. **For each deviation variable** $t^p_{i,s} \geq 0$, its coefficient must be ≥ 0:

$$\lambda - \eta^p_{i,s} - \zeta^p_{i,s} \ \geq\ 0 \quad \iff \quad \eta^p_{i,s} + \zeta^p_{i,s} \leq \lambda. \tag{D-t}$$

Together with $\gamma, \eta, \zeta \geq 0$ and the free variables $\alpha, \beta, \delta, \ell, \rho$, this defines the dual LP (D_λ).

Question 5 — Solve (P_λ) and (D_λ) for $\lambda = 100$

I solved (P_{100}) exactly as the LP defined above (HiGHS solver). The solution is **integral** in this instance (even though the model is a relaxation).

5.1 Optimal objective value and decomposition

The optimal penalized objective is

$$\boxed{\mathrm{OPT}_{100} = 5238.}$$

Decomposing the objective:

- total move count:

$$\sum_{p=0}^{4}\sum_{s}\sum_{i\neq j} x^p_{i\to j,s} = \boxed{38},$$

- total L_1 deviation:

$$\sum_{p=0}^{5} \|z^p - z^F\|_1 = \sum_{p,i,s} |y^p_{i,s} - y^F_{i,s}| = \boxed{52},$$

so

$$5238 = 38 + 100\cdot 52.$$

Per-phase L_1 distances are:

$$\|z^0 - z^F\|_1 = 20, \quad \|z^1 - z^F\|_1 = \|z^2 - z^F\|_1 = \|z^3 - z^F\|_1 = \|z^4 - z^F\|_1 = 8, \quad \|z^5 - z^F\|_1 = 0.$$

Move counts per transition:

$$0\to 1:\ 10, \quad 1\to 2:\ 8, \quad 2\to 3:\ 8, \quad 3\to 4:\ 8, \quad 4\to 5:\ 4.$$

5.2 One optimal primal solution (allocations y^p)

I list each phase's **nonempty** offices as "service-counts".

Table 4: Allocations by office and by phase (example).

Office	Phase 0	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5
A1	2P	2P	2P	2P	closed	2P
A2	2P	2P	2P	2P	closed	2P
B1	O+M	closed	M+T	M+S	M+P	M
B2	2S	closed	2M	2M	2M	2M
B3	O+T	closed	M+T	M+S	M+P	M
C1	2T	2S	2S	closed	2S	2S
C2	2S	2S	2S	closed	2S	2S
D1	T	T+M	closed	T	T+P	T
D2	M	2T	closed	2T	2T	2T
D3	2M	T+M	closed	T+S	T	T
E1	closed	O+T	O+T	O+S	O+P	O
E2	closed	O+M	O+T	O	O	O

(Phases 0 and 5 are exactly the given initial/final allocations.)
Downloads:

- Primal allocations $y^p_{i,s}$ for $\lambda = 100$
- All flows $x^p_{i\to j,s}$ for $\lambda = 100$
- Only "true moves" $(i \neq j)$ for $\lambda = 100$
- Absolute deviations $t^p_{i,s}$ for $\lambda = 100$

5.3 Optimal dual value and a dual solution

Solving the dual (D_{100}) yields

$$\boxed{\mathrm{OPT}^{\mathrm{dual}}_{100} = 5238 = \mathrm{OPT}_{100}},$$

certifying optimality by strong duality.
Downloads (full dual multipliers):

- Dual multipliers for equality constraints (FO/FI/UNAV/BC)
- Dual multipliers for inequality constraints (CAP and ABS)

A structural feature visible in the dual (and consistent with constraint (D-t)) is:

- whenever $t^p_{i,s} > 0$, the corresponding ABS multipliers satisfy $\eta^p_{i,s} + \zeta^p_{i,s} = 100$;
- in this instance, for each mismatch entry exactly one of η, ζ equals 100 and the other 0 (sign indicates whether $y^p_{i,s} - y^F_{i,s}$ is positive or negative).

5.4 Comparison with the move-only optimum (Section 1)

For the **move-only** LP (no penalty), an optimal solution has:

- move objective = 30 (as before),
- but cumulative deviation
$$\sum_{p=0}^{5} \|z^p - z^F\|_1 = 74,$$
with per-phase distances (one optimal move-only solution):
$$(20,\ 18,\ 13,\ 10,\ 13,\ 0).$$

So, compared to Section 1:

- **moves increase** from 30 to $\boxed{38}$,
- but the renovation trajectory stays **much closer to the final layout**: total L_1 deviation drops from 74 to $\boxed{52}$, and already reaches distance 8 from phase 1 onward.

Preliminaries (notation + exact 0–1 model to be relaxed)

Sets

- Offices $\mathcal{I} = \{A1, A2, B1, B2, B3, C1, C2, D1, D2, D3, E1, E2\}$ (12 offices).
- Phases $\mathcal{P} = \{0, 1, 2, 3, 4, 5\}$, transitions $\mathcal{T} = \{0, 1, 2, 3, 4\}$.
- Services $\mathcal{S} = \{P, S, O, T, M\}$.
- People: we model **18 individuals** (4P, 4S, 2O, 4T, 4M). Let $\mathcal{A}$ be this set, partitioned as $\mathcal{A} = \bigsqcup_{s\in\mathcal{S}} \mathcal{A}_s$ with $|\mathcal{A}_s|$ equal to the total number of people of service s.

Availability (offices that exist and are not under renovation)

Let $\mathcal{V}_p \subseteq \mathcal{I}$ be:

- $\mathcal{V}_0 = \{A1, A2, B1, B2, B3, C1, C2, D1, D2, D3\}$ (wing E not yet built),
- $\mathcal{V}_1 = (\mathcal{V}_0 \cup \{E1, E2\}) \setminus \{B1, B2, B3\}$,
- $\mathcal{V}_2 = (\mathcal{V}_0 \cup \{E1, E2\}) \setminus \{D1, D2, D3\}$,
- $\mathcal{V}_3 = (\mathcal{V}_0 \cup \{E1, E2\}) \setminus \{C1, C2\}$,
- $\mathcal{V}_4 = (\mathcal{V}_0 \cup \{E1, E2\}) \setminus \{A1, A2\}$,
- $\mathcal{V}_5 = \mathcal{I}$.

Data (from Figures 1–2)

Let $y^0_{i,s}$ and $y^F_{i,s}$ be the **given service-counts** in office i at phase 0 and phase 5.

0–1 variables and objective (the "integer" problem)

Assignment variables

For each person $a \in \mathcal{A}$, office $i \in \mathcal{I}$, phase $p \in \mathcal{P}$,

$$v_{a,i,p} \in \{0,1\} \quad \text{meaning "person } a \text{ is in office } i \text{ at phase } p\text{".}$$

Move-count linearization

For each $a \in \mathcal{A}$, $i \in \mathcal{I}$, $p \in \mathcal{T}$, introduce

$$d_{a,i,p} \in \{0,1\},$$

intended to model $|v_{a,i,p} - v_{a,i,p+1}|$. Enforce with linear inequalities:

$$\begin{aligned} d_{a,i,p} &\geq v_{a,i,p} - v_{a,i,p+1}, \\ d_{a,i,p} &\geq -v_{a,i,p} + v_{a,i,p+1}. \end{aligned} \tag{ABS-move}$$

If v is binary, then these imply $d_{a,i,p} \geq |v_{a,i,p} - v_{a,i,p+1}|$, and minimizing makes them tight.

For a fixed person a and transition p, the quantity

$$\frac{1}{2} \sum_{i \in \mathcal{I}} d_{a,i,p}$$

equals 1 if the person changes office and 0 otherwise (because two coordinates flip when moving).

Feasibility constraints

1. **Exactly one office per person per phase**

$$\sum_{i \in \mathcal{I}} v_{a,i,p} = 1 \qquad (\forall a \in \mathcal{A},\ \forall p \in \mathcal{P}). \tag{ONE}$$

2. **Unavailability**

$$v_{a,i,p} = 0 \qquad (\forall a,\ \forall p,\ \forall i \notin \mathcal{V}_p). \tag{UNAV}$$

3. **Office capacity (max 2)**

$$\sum_{a \in \mathcal{A}} v_{a,i,p} \leq 2 \qquad (\forall i \in \mathcal{I},\ \forall p \in \mathcal{P}). \tag{CAP}$$

4. **Match the given initial and final service-counts** For $p \in \{0,5\}$, for every office i and service s,

$$\sum_{a \in \mathcal{A}_s} v_{a,i,p} = y^p_{i,s}, \quad \text{where } y^0_{i,s} \text{ is from Fig.1 and } y^5_{i,s} = y^F_{i,s} \text{ from Fig.2.} \tag{BC-count}$$

Integer objective (minimize total moves)

$$\min \sum_{p \in \mathcal{T}} \sum_{a \in \mathcal{A}} \frac{1}{2} \sum_{i \in \mathcal{I}} d_{a,i,p}. \tag{IP}$$

This is a 0–1 linear program. The SDP relaxation below is built from the binary vector of all its variables.

Part 3 — Semidefinite programming relaxation

Motivation for Model Change

Recall: Parts 1–2 used service-flow aggregation. In Questions 1–5, we modeled **service flows** using variables $y_{i,s}^p$ (count of service s in office i at phase p) and $x_{i \to j,s}^p$ (flow of service s from office i to j between phases). This aggregation was justified by the **interchangeability** of individuals within the same service (no individual preferences specified).

Why switch to individual tracking for Q6–Q8? Questions 6–8 require a **0–1 formulation** suitable for **semidefinite programming (SDP)** relaxation. The classical SDP framework (Goemans-Williamson style) naturally applies to **binary quadratic programs**, where variables represent individual decisions $v_{a,i,p} \in \{0, 1\}$ ("person a is in office i at phase p").

Equivalence of the two models. The individual-assignment model and the service-flow model are **equivalent in terms of feasible allocations**:

$$y_{i,s}^p = \sum_{a \in \mathcal{A}_s} v_{a,i,p} \quad \forall i, s, p,$$

where $\mathcal{A}_s$ is the set of individuals belonging to service s. Any feasible solution $(v_{a,i,p})$ induces a feasible $(y_{i,s}^p)$ via this aggregation, and conversely, any $(y_{i,s}^p)$ can be disaggregated into a $(v_{a,i,p})$ by arbitrarily assigning individuals to offices respecting the service counts (since individuals within a service are interchangeable by assumption).

Question 6 — SDP relaxation, dual, and KKT conditions

6.1 Vectorization and $\{-1, 1\}$ encoding

Stack all binary variables. Define

$$w := (v_{a,i,p}, d_{a,i,p})_{a,i,p} \in \{0, 1\}^N,$$

where $N = |\mathcal{A}||\mathcal{I}||\mathcal{P}| + |\mathcal{A}||\mathcal{I}||\mathcal{T}|$.

Convert to $\{-1, 1\}$ variables using the standard affine map (your footnote [3]):

$$u := 2w - \mathbf{1} \in \{-1, 1\}^N, \qquad \text{equivalently } w = \frac{u + \mathbf{1}}{2}.$$

All linear constraints in (v, d) can be written as

$$Aw = b, \qquad Gw \le h$$

for appropriate matrices A, G and vectors b, h. Substituting $w = (u + \mathbf{1})/2$ yields linear constraints in u:

$$Au = 2b - A\mathbf{1}, \qquad Gu \le 2h - G\mathbf{1}. \tag{Lin-u}$$

The objective is linear in w, hence also linear in u:

$$c^\top w = c^\top \frac{u + \mathbf{1}}{2} = \frac{1}{2} c^\top u + \text{constant}.$$

The constant can be dropped without affecting the minimizer.

6.2 SDP relaxation based on (2)

Introduce $U \in \mathbb{S}^N$ and impose:

$$U \succeq uu^\top, \qquad \mathrm{diag}(U) = \mathbf{1}. \tag{SDP-core}$$

As standard, it is convenient to package (u, U) into one matrix

$$X := \begin{pmatrix} 1 & u^\top \\ u & U \end{pmatrix} \in \mathbb{S}^{N+1}.$$

Then $X \succeq 0$ and $X_{00} = 1$ is equivalent (by the Schur complement) to $U \succeq uu^\top$. Also, $\mathrm{diag}(U) = \mathbf{1}$ is the same as $X_{ii} = 1$ for all $i = 1, \dots, N$.

Primal SDP (standard form)

Choose $C \in \mathbb{S}^{N+1}$ so that $\langle C, X\rangle = \frac{1}{2}c^\top u$ (e.g., set $C_{0i} = C_{i0} = c_i/4$, others 0). Define matrices $\{A_k\}$, $\{G_\ell\}$ so that the linear constraints (Lin-u) become linear in X via the identification $u_i = X_{0i}$. Then:

$$\begin{aligned}
\min_{X \in \mathbb{S}^{N+1}} \quad & \langle C, X\rangle \\
\text{s.t.} \quad & \langle A_k, X\rangle = \tilde{b}_k \quad (k = 1, \dots, m), \\
& \langle G_\ell, X\rangle \le \tilde{h}_\ell \quad (\ell = 1, \dots, r), \\
& X_{00} = 1, \quad X_{ii} = 1 \ (i = 1, \dots, N), \\
& X \succeq 0,
\end{aligned} \tag{P-SDP}$$

where $\tilde{b}, \tilde{h}$ are the right-hand sides in (Lin-u).

This is exactly the relaxation idea (2): we keep the linear constraints, and replace "$u \in \{-1, 1\}^N$" by the semidefinite lift.

6.3 Dual SDP

Introduce dual variables:

- $y_k \in \mathbb{R}$ for equalities $\langle A_k, X\rangle = \tilde{b}_k$,
- $z_\ell \ge 0$ for inequalities $\langle G_\ell, X\rangle \le \tilde{h}_\ell$,
- $\pi_0 \in \mathbb{R}$ for $X_{00} = 1$,
- $\pi_i \in \mathbb{R}$ for $X_{ii} = 1$, $i = 1, \dots, N$.

Let E_{ii} be the matrix with a 1 at (i, i) and 0 elsewhere. The dual is:

$$\begin{aligned}
\max \quad & \sum_{k=1}^m \tilde{b}_k y_k + \sum_{\ell=1}^r \tilde{h}_\ell z_\ell + \pi_0 + \sum_{i=1}^N \pi_i \\
\text{s.t.} \quad & S := C - \sum_{k=1}^m y_k A_k - \sum_{\ell=1}^r z_\ell G_\ell - \pi_0 E_{00} - \sum_{i=1}^N \pi_i E_{ii} \ \succeq \ 0, \\
& z_\ell \ge 0 \ \ (\ell = 1, \dots, r).
\end{aligned} \tag{D-SDP}$$

6.4 KKT conditions

Let $X^\star$ be primal-feasible and $(y^\star, z^\star, \pi^\star, S^\star)$ dual-feasible. KKT for SDP is:

1. **Primal feasibility:** $X^\star$ satisfies all constraints of (P-SDP).
2. **Dual feasibility:** $z^\star \geq 0$, $S^\star \succeq 0$, and the defining relation for $S^\star$ holds.
3. **Complementary slackness (inequalities):**
$$z_\ell^\star \big(\langle G_\ell, X^\star \rangle - \tilde{h}_\ell \big) = 0 \quad (\forall \ell).$$
4. **Complementary slackness (PSD cone):**
$$\langle S^\star, X^\star \rangle = 0 \quad \text{(equivalently, since both are PSD, } S^\star X^\star = 0\text{)}.$$
5. **Stationarity:** already encoded by the construction of $S^\star$ (the dual constraint is precisely the stationarity condition for the Lagrangian).

Question 7 — Compute the SDP solution

Key fact (and why the SDP is easy to "solve" here)

In this model, the objective and constraints depend on X **only through the first moments** $u_i = X_{0i}$ and the diagonal constraints $X_{ii} = 1$. The lift U appears only in the PSD constraint.

Claim. If $u \in [-1, 1]^N$, then there exists U such that $\operatorname{diag}(U) = \mathbf{1}$ and $U \succeq uu^\top$.

Proof. Let
$$U := uu^\top + \operatorname{diag}(1 - u_1^2, \dots, 1 - u_N^2).$$
Then $\operatorname{diag}(U) = u_i^2 + (1 - u_i^2) = 1$. Moreover
$$U - uu^\top = \operatorname{diag}(1 - u_1^2, \dots, 1 - u_N^2) \succeq 0$$
because each diagonal entry is nonnegative when $|u_i| \leq 1$. □

So the SDP relaxation does **not** further restrict u beyond $|u_i| \leq 1$. Therefore the SDP optimum equals the optimum of the **LP box relaxation** of the 0–1 model.

Computation (numerical result)

Solving that LP relaxation (with HiGHS) yields an **integral** optimum with objective value:
$$\boxed{\mathrm{OPT}_{\mathrm{SDP}} = 30.}$$

Because the LP optimum is integral, it corresponds to a feasible integer schedule, hence the SDP (being a relaxation) cannot do better than 30, and we have equality.

An explicit optimal SDP primal point

Let $w^\star \in \{0,1\}^N$ be the optimal (integral) vector of variables (v, d) from the LP solution, and set

$$u^\star = 2w^\star - \mathbf{1} \in \{-1, 1\}^N.$$

Then the rank-one matrix

$$X^\star := \begin{pmatrix} 1 \\ u^\star \end{pmatrix} \begin{pmatrix} 1 \\ u^\star \end{pmatrix}^\top = \begin{pmatrix} 1 & (u^\star)^\top \\ u^\star & u^\star (u^\star)^\top \end{pmatrix}$$

satisfies $X^\star \succeq 0$, $X^\star_{00} = 1$, $X^\star_{ii} = 1$, and the linear constraints (because $w^\star$ satisfies the original constraints). Its objective value is exactly 30, hence it is SDP-optimal.

The corresponding (feasible) allocation schedule

Aggregating the computed integer solution into **service counts per office** :

Table 5: Allocations by office and by phase.

Office	Phase 0	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5
A1	2P	2P	2P	2P	closed	2P
A2	2P	2P	2P	2P	closed	2P
B1	O+M	closed	S+M	S+M	S+M	M
B2	2S	closed	2M	2M	2M	2M
B3	O+T	closed	T+M	M	M	M
C1	2T	2T	2T	closed	2P	2S
C2	2S	S+T	S+T	closed	2P	2S
D1	T	T+M	closed	T	T	T
D2	M	S+M	closed	2T	2T	2T
D3	2M	2M	closed	S+T	S+T	T
E1	closed	S+O	S+O	S+O	S+O	O
E2	closed	S+O	S+O	S+O	S+O	O

(Phases 0 and 5 are exactly Figures 1 and 2.)
Move counts per transition for this schedule:

$$0 \to 1 : 7, \quad 1 \to 2 : 6, \quad 2 \to 3 : 5, \quad 3 \to 4 : 4, \quad 4 \to 5 : 8,$$

total = 30.

Question 8 — Randomized rounding and feasibility/optimality

8.1 Randomized rounding (classical SDP hyperplane method)

Given an SDP solution $X^\star \succeq 0$, factorize $X^\star = VV^\top$ (e.g., Cholesky or eigen-factorization). Draw $r \sim \mathcal{N}(0, I)$ and define rounded signs by

$$\hat{s}_i = \operatorname{sign}(V_i^\top r) \quad (i = 0, 1, \dots, N),$$

where $V_i^\top$ is the i-th row of V. To remove the global sign ambiguity and enforce the constant coordinate $u_0 = 1$, set

$$\hat{u}_i := \hat{s}_i \, \hat{s}_0 \quad (i = 1, \dots, N).$$

Then $\hat{u} \in \{-1, 1\}^N$, hence $\hat{w} = (\hat{u} + \mathbf{1})/2 \in \{0, 1\}^N$.

8.2 What happens here (rank-one optimal SDP)

For our computed SDP optimum,

$$X^\star = \begin{pmatrix} 1 \\ u^\star \end{pmatrix} \begin{pmatrix} 1 \\ u^\star \end{pmatrix}^\top$$

is **rank one**, so we can take $V = \begin{pmatrix} 1 \\ u^\star \end{pmatrix}$ (a single column). Then $V_i^\top r$ is just a scalar multiple of V_i, and after the sign-correction by $\hat{s}_0$ one obtains

$$\hat{u} = u^\star \quad \text{with probability 1 (except the measure-zero event } r = 0).$$

Therefore the rounding returns exactly the same integer vector $w^\star$ and hence the same schedule.

8.3 Feasibility and (apparent) optimality

- **Feasibility:** yes — because $\hat{w} = w^\star$ satisfies (ONE), (UNAV), (CAP), (BC-count), and the move-linearization constraints.
- **Optimality:** yes — because the rounded schedule has objective value 30, and we proved in Q7 that 30 is the SDP optimum (hence a lower bound on any integer feasible solution). Achieving the lower bound certifies global optimality.