Distance Regularized Level Sets Evolution on Active Contour

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Abstract

This paper explores active contour segmentation using level set methods. Instead of explicitly representing contours, the method uses implicit functions governed by partial differential equations to evolve contours toward object boundaries. We examine the evolution of level set methods from classical approaches through reinitialization techniques, and then focus on the Distance Regularized Level Set Evolution (DRLSE) innovation, which removes the need for reinitialization while maintaining numerical stability and accuracy.

1 Introduction

Active contours aim to detect object boundaries in images. Level set methods introduce an implicit representation of contours via a level set function $\phi(x, y)$ whose zero-level set approximates object boundaries. Unlike explicit active contours, this approach naturally handles topological changes, allowing contours to split or merge during evolution.

The evolution of the level set function is typically governed by partial differential equations (PDEs) that move the zero-level set toward desired features in the image. This paper traces the development of level set methods for segmentation, from fundamental approaches through the introduction of reinitialization techniques, to the more recent and efficient Distance Regularized Level Set Evolution (DRLSE) method.

2 Fundamental Level Sets (1980s-1990s)

2.1 Introduction

Instead of representing active contours explicitly by a set of ordered points, level set methods define the contour implicitly as the zero-level set of a function $\phi(x, y)$. By evolving this function in time, we can approach our desired contour.

During evolution:

- ϕ is updated over time using a partial differential equation (PDE)
- The function ϕ deforms under a speed function so that the zero-level set (the contour) moves toward object boundaries in the image

2.2 Mathematical Formulation

The contour C is defined where $\phi(x, y, t) = 0$. The evolution is governed by:

$$\frac{\partial \phi}{\partial t} + F|\nabla \phi| = 0 \tag{1}$$

where F is the speed function and $\nabla \phi$ is the spatial gradient.

2.3 Basic Algorithm Components

2.3.1 Level Set Function Initialization

ϕ is initialized as a signed distance function:

$$\phi(x, y, 0) = \pm d$$

where d is the distance to the initial contour.

2.3.2 Speed Function Design

Typical form:

$$F = g \cdot \kappa + \nabla g \cdot \nabla \phi \tag{2}$$

where $g = \frac{1}{1+|\nabla I|^2}$ is an edge detector, I is the image, and κ is the curvature.

2.3.3 Level Set Evolution

At each iteration, we update ϕ using:

$$\phi^{n+1} = \phi^n - \Delta t \cdot F|\nabla \phi^n| \tag{3}$$

2.4 Complete Algorithm

Algorithm 1 Basic Level Set Method

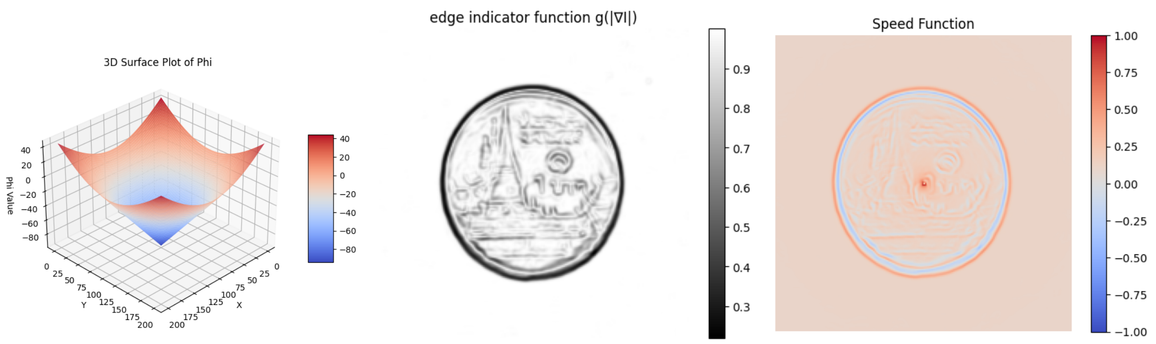
Initialize $\phi(x, y, 0)$ as a signed distance function
Calculate the edge indicator function $g(|\nabla I|)$
for each iteration n **do**
    Calculate $\nabla \phi^n$ using finite differences
    Compute curvature κ
    Evaluate speed function F
    Update $\phi^{n+1} = \phi^n - \Delta t \cdot F|\nabla \phi^n|$
end for
Extract final contour from zero level set $\phi = 0$

2.5 Implementation Notes

In our implementation:

- We initialize the LSF as a signed distance function ($|\nabla \phi| = 1$)
- We calculate the edge indicator function: $g(|\nabla I|) = \frac{1}{1+|\nabla I|^2}$ using Sobel filters
- The speed function is defined as: speed $= g \cdot (\text{curvature} + \alpha) + \beta \cdot \text{edge\_attraction}$
- At each iteration: $\phi^{n+1} = \phi^n + \Delta t \cdot \text{speed\_function} \cdot |\nabla \phi^n|$

2.6 Implementation on python



(a) LSF Initialization (b) Edge Indicator g (c) Speed Function

Figure 1: Key steps in level set evolution implementation.

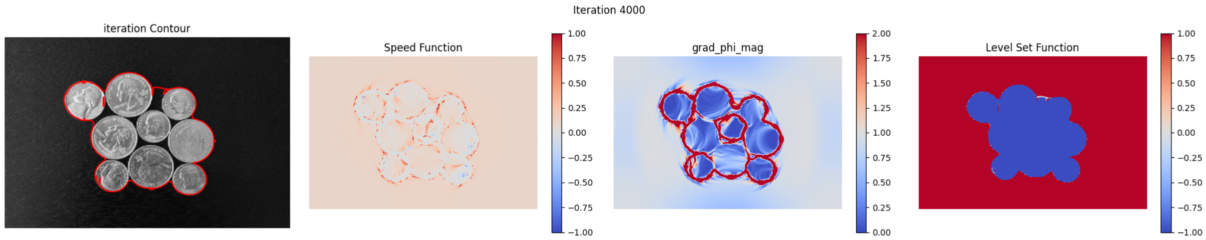


Figure 2: Fundamental level set evolution example.

2.7 Fundamental Level Set Challenges

The basic level set method faces several challenges:

- To prevent evolution from diverging into artifacts, the time step must be kept very small, resulting in significant computational cost
- Parameter selection for α (balloon force) and β (edge attraction) is highly image-dependent
- As iterations progress, $|\nabla\phi|$ becomes increasingly steep, causing instabilities, These instabilities result in anomalies within the LSF and irregular, unwanted contours near some inner edges

3 Level Set Methods with Reinitialization (1990s)

3.1 The Need for Reinitialization

As the level set evolves over iterations, it gradually loses its signed distance property ($|\nabla\phi| = 1$) due to numerical errors and the nature of the evolution equation. The level set function develops regions where $|\nabla\phi|$ deviates from 1, causing:

- Regions where $|\nabla\phi| \gg 1$ (very steep gradient): Numerical instability and excessive contour movement
- Regions where $|\nabla\phi| \ll 1$: Slow or stalled contour evolution

The signed distance property ($|\nabla\phi| = 1$) is important because:

- **Numerical stability**: Gradients remain well-behaved during calculations
- **Improved accuracy**: Evolution equations assume ϕ is a signed distance function
- **Consistent evolution speed**: Prevents uneven contour movement

3.2 Reinitialization Process

To address these issues, we periodically *reinitialize* ϕ to restore the signed distance property while preserving the zero-level set (contour). This reshapes the level set function without altering the current contour position.

3.2.1 Mathematical Formulation

Reinitialization solves the following PDE to steady state:

$$\frac{\partial\phi}{\partial\tau} = \text{sign}(\phi_0)(1 - |\nabla\phi|) \tag{4}$$

This equation acts as follows:

- When $|\nabla\phi| > 1$: ϕ decreases, When $|\nabla\phi| < 1$: ϕ increases, When $|\nabla\phi| = 1$: ϕ remains unchanged

3.3 Implementation on python

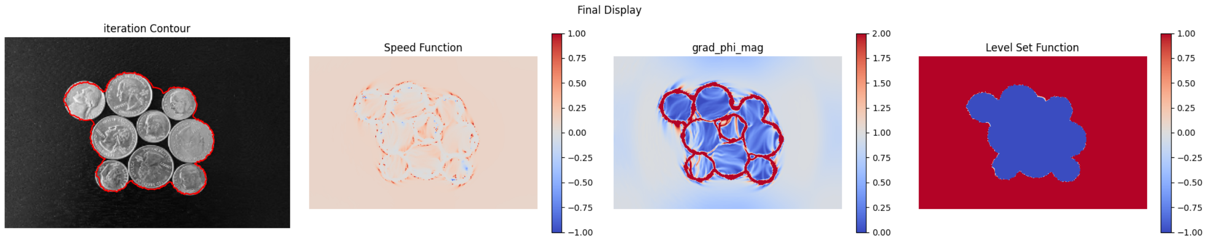


Figure 3: Fundamental level set with Reinitialisation evolution example.

3.4 Limitations of Reinitialization

While reinitialization improves stability, it introduces new issues:

- The irregular unwanted contours disappear, but the LSF struggles to transform into an accurate distance function
- The reinitialization process is computationally expensive
- It can cause the zero level set to shift unintentionally, affecting segmentation accuracy

4 Variational Level Set Methods (Early 2000s)

A major advancement came with variational formulations where level set evolution was derived by minimizing energy functionals:

$$E(\phi) = \iint [\text{internal energy terms} + \text{external energy terms}]\, dxdy \tag{5}$$

Where:

- E_{internal}: Controls regularity of the contour (smoothness, length)
- E_{external}: Drives the contour toward desired features (edges, regions)

The evolution equation became:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial E}{\partial \phi} \tag{6}$$

4.1 Common Energy Terms

4.1.1 Length Regularization

$$E_{\text{length}}(\phi) = \iint \delta(\phi)|\nabla \phi|\, dxdy \tag{7}$$

This term penalizes the length of the contour, encouraging smoothness. The Dirac delta function $\delta(\phi)$ focuses the integral on the zero level set.

4.1.2 Area Constraint

$$E_{\text{area}}(\phi) = \iint H(-\phi)\, dxdy \tag{8}$$

This term controls the area enclosed by the contour, preventing collapse or excessive expansion. The Heaviside function $H(-\phi)$ equals 1 inside the contour and 0 outside.

4.2 Variational Level Set Algorithm

4.2.1 Step 1: Formulate the Energy Functional

Define $E(\phi)$ with appropriate terms for the application.

4.2.2 Step 2: Derive the Evolution Equation

Using calculus of variations, compute the Euler-Lagrange equation:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial E}{\partial \phi} \tag{9}$$

4.2.3 Step 3: Discretize the Evolution Equation

Convert the continuous equation to a discrete update scheme:

$$\phi^{n+1} = \phi^n + \Delta t \cdot \left(-\frac{\partial E}{\partial \phi}\right) \tag{10}$$

5 Distance Regularized Level Set Evolution (DRLSE)

Li et al. (2010) recognized the drawbacks of reinitialization

Their key insight was to incorporate distance regularization directly into the energy functional:

$$E(\phi) = \mu R_p(\phi) + \lambda L_g(\phi) + \alpha A_g(\phi) \tag{11}$$

Where:

- $R_p(\phi)$ is the distance regularization term
- $L_g(\phi)$ is the length term weighted by edge indicator g
- $A_g(\phi)$ is the area term weighted by edge indicator g
- μ, λ, and α are weighting parameters

5.1 The Distance Regularization Term

The critical innovation was the distance regularization term:

$$R_p(\phi) = \iint p(|\nabla \phi|)\, dxdy \tag{12}$$

With potential function p designed so that $|\nabla \phi| = 1$ is a minimum of $p(s)$, ensuring that the gradient flow of R_p automatically maintains the signed distance property.

5.1.1 Potential Function Design

Double-Well Potential:

$$p_2(s) = \begin{cases} \frac{\sin(2\pi s)}{2\pi} & \text{if } s \le 1 \\ (s-1) & \text{if } s > 1 \end{cases} \tag{13}$$

This has two minima at $|\nabla \phi| = 0$ and $|\nabla \phi| = 1$, maintaining the signed distance property near the zero level set while keeping the LSF constant farther away. It works for both edge-based and region-based models.

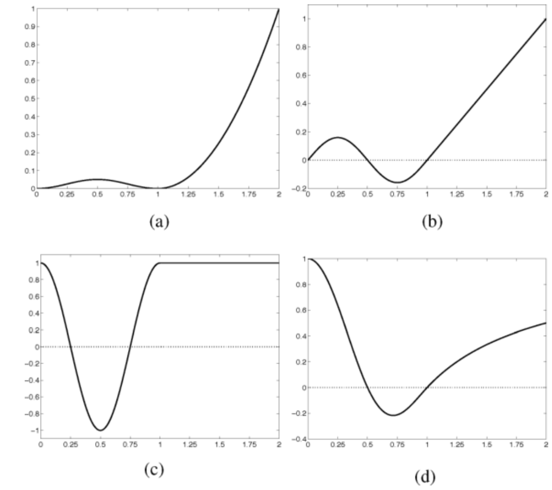


Fig. 1. Double-well potential $p = p_2(s)$ and its first derivative p_2' and second derivative p_2'', and corresponding function $d_p(s)$ are shown in (a), (b), (c), and (d), respectively.

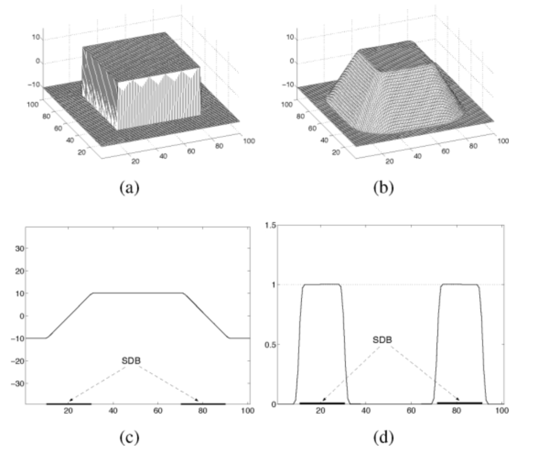


Fig. 2. Distance regularization effect on binary step function with the double-well potential $p = p_2(s)$. (a) Initial LSF ϕ_0. (b) Final LSF ϕ after the evolution. (c) and (d) Show a cross section of ϕ and $|\nabla\phi|$ for the final function ϕ, respectively.

5.2 Implementation

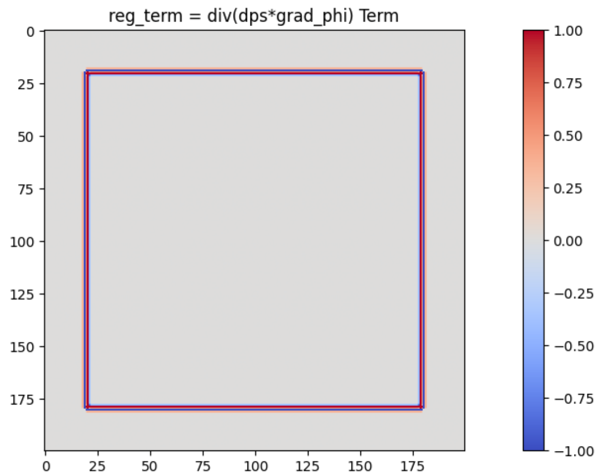


Figure 4: Distance regularization term applied to the initial binary LSF.

Around edges, we observe high positive values at the inner edge (where the LSF is negative), making the LSF increase, and negative values at the outer edge (where the LSF is positive), making the LSF decrease. Together, these smooth the abrupt jump from negative to positive values at edges into a gradual ramp.

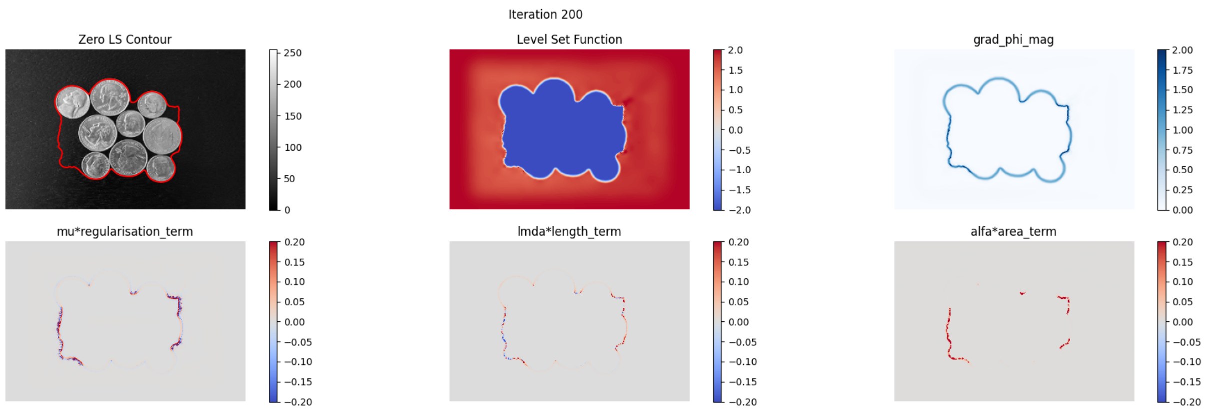


Figure 5: DRLSE level set evolution after 200 iterations.

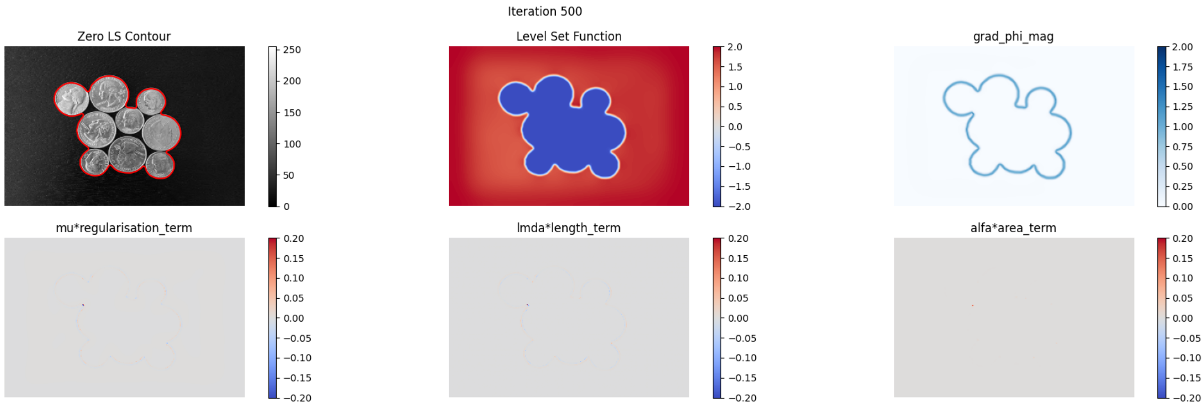


Figure 6: DRLSE level set evolution after 500 iterations.

6 Conclusion

The development of level set methods represents a significant advancement in image segmentation techniques. Our analysis has traced this evolution from:

1. **Basic level set methods**: Effective but requiring small time steps and high computational cost
2. **Level sets with reinitialization**: Improved stability but with additional computational overhead and potential contour shifts
3. **Variational formulations**: Providing better theoretical foundations
4. **Distance regularized level set evolution (DRLSE)**: Eliminating the need for reinitialization by intrinsically maintaining the signed distance property

The DRLSE method represents a significant improvement over previous approaches, providing a more elegant and efficient solution with:

- No artifacts due to regularization
- Reduced computational cost by eliminating reinitialization
- Better numerical stability through the double-well potential function
- More accurate contour evolution with improved handling of the level set function near and far from contours

These advantages make DRLSE particularly suitable for real-time or resource-constrained image segmentation applications.

References

[1] C. Li, C. Xu, C. Gui, and M. D. Fox, "Distance regularized level set evolution and its application to image segmentation," IEEE Transactions on Image Processing, vol. 19, no. 12, pp. 3243-3254, 2010.

[2] Ramesh-X's DRLSE implementation on GitHub.