Linear Algebra

Riya Jana

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Overview

Matrices

Definition

A matrix is a rectangular arrangement of numbers in rows and columns. Matrices are fundamental objects in linear algebra and are used to represent and manipulate data.

Syntax is:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad (m \times n)$$

Suppose, elements are 1,3,5,7 in matrix form they will look like:

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \quad (2 \times 2)$$

Types of Matrices

In a matrix m is the number of rows and n is the number of column.

- Square matrix is a matrix, where m=n.

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \quad (2 \times 2)$$

- Rectangle matrix is a matrix, where $m \neq n$

$$\begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 0 \end{bmatrix} \quad (2 \times 3)$$

- Diagonal matrix is a matrix, where only diagonal elements are non zero and m=n.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad (3 \times 3)$$

Types of Matrices

- Scalar matrix is a matrix, where all diagonal elements are same, other elements are zero and m=n.

$$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad (3 \times 3)$$

- Null matrix is a matrix, where all elements are zero and there is no restriction for m and n.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2 \times 3)$$

- Unit/Identity matrix is a matrix, where all diagonal elements are 1, other elements are zero and m=n.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2 \times 2)$$

- Upper Triangular matrix is a matrix, where diagonal and upside the diagonal values are non zero.

$$\begin{bmatrix} 1 & 6 & 7 \\ 0 & 7 & 8 \\ 0 & 0 & 9 \end{bmatrix} \quad (3 \times 3)$$

- Lower Triangular matrix is a matrix, where diagonal and bellow the diagonal values are non zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 7 & 0 \\ 6 & 3 & 9 \end{bmatrix} \quad (3 \times 3)$$

- In Idempotent matrix, matrix A multiply with itself and produce A.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix Operations

Matrix Addition

- Two matrix A, B of same order perform addition A[i][j]+B[i][j]

$$\begin{array}{ccccc} \textbf{A} & & \textbf{B} & & \textbf{A+B} \\ \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} & + & \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} & & \begin{bmatrix} 7 & 10 \\ 11 & 14 \end{bmatrix} \end{array}$$

1. **Commutative:** A+B=B+A.
2. **Associative:** A+(B+C)=(A+B)+C.
3. **Additive Identity:** A+0=A.
4. **Additive Inverse:** A+(-A)=0.

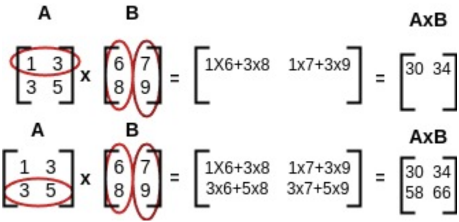
Matrix Subtraction

- Two matrix A, B of same order perform subtraction A[i][j]+(-B[i][j])

$$\begin{array}{ccccc} \textbf{A} & & \textbf{B} & & \textbf{A-B} \\ \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} & - & \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} & & \begin{bmatrix} -5 & -4 \\ -5 & -4 \end{bmatrix} \end{array}$$

Matrix Multiplication

- Two matrix A of order $m1 \times n1$, B of order $m2 \times n2$ and $n1 = m2$ then perform multiplication where resultant matrix is $m1 \times n2$:



1. **Associative:** $(\text{A}\times B) \times C = A \times (B \times C)$
2. **Distributive:** A(B + C) = AB + AC and (B + C)A = BA + CA
3. **Not Commutative:** Generally $A \times B \neq B \times A$.
4. **Multiplicative Identity:** Any $n \times n$ matrix A has its identity matrix "I" of order n.
5. **Multiplicative Inverse**: A with its inverse A^{-1}, the product $A \times A^{-1}$ results in the identity matrix I.

$$A \times A^{-1} = A^{-1} \times A = I$$

Trace of Matrix

Let's consider the matrix:

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -3 & 2 \\ -1 & 0 & 5 \end{bmatrix}$$

The trace of matrix A, denoted as $\text{tr}(A)$, is the sum of its diagonal elements:

$$\text{tr}(A) = 2 + (-3) + 5 = 4$$

So, the trace of matrix A is 4.

Properties

- $\text{Tr}(\text{A}\lambda) = \lambda \cdot \text{Tr}A$
- $\text{Tr}(\text{AB}) = \text{Tr}(\text{BA})$
- $\text{Tr}(\text{A+B}) = \text{Tr}A + \text{Tr}A$

Transpose Matrix

Let's consider the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} ==> A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$$

- Transpose of a matrix swaps its rows and columns.

properties

A is a matrix.

- $(A^T)^T$=A
- $(bA)^T$=bA^T
- $(A+B)^T$=A^T+B^T
- $(AB)^T$=B^TA^T

Inverse of Matrix

Intoduction:

Given a square matrix A, is multiplied by A^{-1}, or vice versa, the result is the identity matrix, typically denoted as I. Then $A^{-1}$is said to be inverse of A. We can state that $(A \cdot A^{-1}) = (A^{-1} \cdot A) = I$

Conditions for Existence of an Inverse

- Matrix should be a square matrix.
- Its determinant $det(A)$ must be non-zero.

Formula

- $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$

Inverse Matrix

Properties

- If A has an inverse, then A^{-1} is unique.
- If A and B are both invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.
- The inverse of the inverse of a matrix is the matrix itself: $(A^{-1})^{-1} = A$.

To calculate inverse of a matrix, we need to know Determinant and Adjoint first.

Determinant

Matrix Determinant

- The determinant of a square matrix is a scalar value that can be computed from its elements.

Let's take an example:

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 3) = -2$$

Determinant Example II

For 3×3 matrix, Let's say we have the matrix:

$$A = \begin{vmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ -2 & 0 & 1 \end{vmatrix}$$

$det(A) =$
$2 \times ((-1 \times 1) - (4 \times 0)) - 1 \times ((0 \times 1) - (-2 \times 4)) + 3 \times ((0 \times 0) - (-2 \times -1))$
$\text{det(A)} = 2\times(-1 - 0) - 1 \times (0 - (-8)) + 3 \times (0 - 2)$
$\text{det(A)} = 2\times(-1) - 1 \times 8 + 3 \times (-2)$
$\text{det(A)} =$ -2 $+$ 8 - 6
$\text{det(A)} = 0$
So,$\text{det(A)} = 0$.

- **Note:** If any matrix A, whose determinant is 0, it implies that there is no inverse matrix A^{-1}.

Properties of Determinants

- Values if det remains unchanged if it's rows are changed into column.
- If two rows or columns of a determinant are interchanged, the sign of det is changed.
- If any two rows or columns are identical then det=0.
- $|bA| = b^n|A|$ where n is oder of matrix A.
- If corresponding elements of any two rows or column are proportional then det=0.

Adjoint of a Matrix

- The adjoint of a matrix A, denoted as $\text{adj}(A)$ or A^*, is the transpose of the cofactor matrix of A.
- If A is a square matrix of order n, then the adjoint of A, denoted as $\text{adj}(A)$, is given by:

$$\text{adj}(A) = \text{Cofactor}(A)^T$$

Where the cofactor matrix of A is obtained by finding the cofactor of each element of A and arranging them in a matrix form.

Consider the following 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Adjoint of a Matrix Example

For our example matrix A, the cofactors are:

$$\text{Cofactor}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

So, the adjoint of A is the transpose of the cofactor matrix:

$$\text{adj}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus, for a 2×2 matrix, the adjoint is obtained by swapping the positions of the elements on the diagonal and changing the signs of the elements off the diagonal.

Adjoint Example

Let's consider a 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

now, instead of following same rule, we can perform shortcut for this, Arranging the matrix:

$$\begin{bmatrix} 5 & 2 & 2 & 5 \\ 1 & 1 & 3 & 1 \\ 4 & 1 & 1 & 4 \\ 5 & 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -13 \\ -3 & -2 & 11 \\ 3 & 0 & -3 \end{bmatrix} = adj(A)$$

Properties of Adjoint

- $A(adjA) = adjA.A = |A|.I$ of order n.
- $|adjA| = |A|^{n-1}$
- $|AadjA| = |A|^n$
- $adj(adjA) = |A|^{n-2}A$
- $(adjA)^T = adj(A^T)$
- $adjAB = adjB.adjA$
- $adj(kA) = k^{n-1}(adjA)$

Example of Matrix Inverse

Let's consider the same 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Now, $|A| = 6$, adj(A)=

$$\begin{bmatrix} 3 & 4 & -13 \\ -3 & -2 & 11 \\ 3 & 0 & -3 \end{bmatrix}$$

now,

$$A^{-1} = \frac{1}{|A|} \times adj(A) = \frac{1}{6} \times \begin{bmatrix} 3 & 4 & -13 \\ -3 & -2 & 11 \\ 3 & 0 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{3} & \frac{11}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

- A system of linear equations consists of multiple linear equations involving the same set of variables.
- There are two types of System of linear equations based on there representations:

Homogeneous System:

$$
\begin{cases}
a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\
a_{21}x_1 + \ldots + a_{2n}x_n = 0 \\
\vdots \\
a_{m1}x_1 + \ldots + a_{mn}x_n = 0
\end{cases}
$$

where $x_1, x_2, \ldots, x_n$ are variables, and a_{ij} are coefficients.

Non-Homogeneous System: Any of the constant value should be non-zero.

$$
\begin{cases}
a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + \ldots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + \ldots + a_{mn}x_n = b_m
\end{cases}
$$

where $x_1, x_2, \ldots, x_n$ are variables, a_{ij} are coefficients, and $b_1, b_2, \ldots, b_m$ are constants.

Homogeneous Linear Equation

A homogeneous linear equation is an equation of the form:

$$Ax = 0 \tag{1}$$

where A is a coefficient matrix, x is a variables, and 0 is the zero constant. Consider the following homogeneous linear equation:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -2 & 3 & 5 \end{bmatrix} x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Since the right-hand side vector is the zero vector, this equation is homogeneous.

Types of Solution

Homogeneous linear equations have two types of solutions.

- If $det(A) \neq 0$ then there is unique solution available.
- If $det(A) = 0$ then there is infinite solution available.

Example

Let's consider the same example of homogeneous linear equation and check what type of solution it can have:

A= $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -2 & 3 & 5 \end{bmatrix}$

$det(A) = 2\times((2\times5)-(3\times1))+1\times((4\times5)-(-2\times1))+3\times((4\times3)-(-2\times2))$

$det(A) = 2 \times (10 - 3) + 1 \times (20 + 2)) + 3 \times (12 + 4)$

$det(A) = 14 + 22 + 48$

$det(A) = 84 \neq 0$

Then we can say that, there was a solution for these linear equation.

Non-Homogeneous Linear Equation

In linear algebra, a non-homogeneous linear equation is an equation of the form:

$$A\mathbf{x} = \mathbf{B} \tag{3}$$

where $\mathbf{A}$ is a coefficient matrix, $\mathbf{x}$ is a variables, and $\mathbf{B}$ is a constant representing the right-hand side of the equation. If $\mathbf{B} \neq \mathbf{0}$, then the equation is non-homogeneous.
Consider the following system of non-homogeneous linear equations:

$$2x + 3y - z = 5 \tag{4}$$

$$-x + 4y + 2z = 10 \tag{5}$$

$$3x - y + z = 2 \tag{6}$$

This system can be represented in matrix form as:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 3 & -1 & 1 \end{bmatrix} x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix} \tag{7}$$

Here, the right-hand side vector $\mathbf{B}$ is non-zero.

Types of solutions

Non-Homogeneous linear equations have three types of solutions.

- If $det(A) \neq 0$ then unique solution available.
- If $det(A) = 0$ then calculate $(adjA)B$.
- If $(adjA)B = 0$ then infinite solutions are available.
- If $(adjA)B \neq 0$ then no solutions are available.

Examples

Consider the following system of equations in three real variables xl, x2 and x3

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 - 2x_2 + 5x_3 = 2$$
$$-x_1 + 4x_2 + 3x_3 = 3$$

This system of equations has

(A) no solution
(B) a unique solution
(C) more than one but a finite number of solutions
(D) an infinite number of solutions

Conclusion

Linear algebra is a powerful tool with a wide range of applications in mathematics, science, and engineering. It provides a framework for solving systems of equations, analyzing data, and understanding transformations in space.