LMMSE channel estimation in OFDM context: a review

Article *in* IET Signal Processing · September 2016

DOI: 10.1049/iet-spr.2016.0185

2 authors:

Vincent Savaux
IRT B-com

Yves Louet
CentraleSupélec

DOI: 10.1049/iet-spr.2016.0185

LMMSE Channel Estimation in OFDM Context: A Review

Vincent Savaux* , Yves Louët

Abstract

Linear minimum mean square error (LMMSE) is by definition the optimal channel estimator in the sense of MSE criterion, but its practical application is limited by its high complexity. Furthermore, the LMMSE estimation method requires the knowledge of both the channel and the noise statistics, which are a priori unknown at the receiver. A wide range of techniques are proposed in the literature in order to overcome these two drawbacks. In this paper, we give an overview of the LMMSE-based channel estimation in an orthogonal frequency division multiplexing (OFDM) context. A didactic reminder concerning the basics of LMMSE estimation and its performance is provided, and a survey of techniques of the literature, which enable the practical application of LMMSE and the reduction of its complexity, is presented in both single-input single-output (SISO) and multiple-input multiple-output (MIMO) contexts. Finally, some perspectives are provided, in particular the application of the LMMSE estimator to flexible waveforms beyond OFDM.

Keywords- OFDM, Channel estimation, Mean square error, LMMSE.

Manuscript submitted in April 2016

* Vincent Savaux is with IRT b<>com, Rennes, FR(corresponding author: e-mail: vincent.savaux@b-com.com Phone: +33 256358216.)

Yves Louët is with IETR-CentraleSupélec, Rennes, FR (e-mail: yves.louet@centralesupelec.fr)

LMMSE Channel Estimation in OFDM Context: A Review

I. Introduction

Cyclic prefix-orthogonal frequency division multiplexing (CP-OFDM) modulation is very robust against frequency selective channels and allows a simple per-subcarrier equalization. However, the performance of the receiver mainly depends on the channel estimation accuracy. To achieve this operation, it is very usual to multiplex pilot symbols in the data stream, from which the channel estimation is performed. In this article, we focus on the optimal estimation technique, namely the linear minimum mean square error (LMMSE) method, and we propose an overview of LMMSE-based channel estimators.

The literature is very extensive with regard to channel estimation techniques in OFDM. Detailed overviews on this topic are provided in [1]–[3]. Among the estimation methods, the most used is indisputably the least square (LS) estimator [1], [4]. This is mainly due to its low complexity, although it suffers from the noise distortion. In order to overcome this drawback, the authors of [4], [5] proposed two alternatives to LS called scaled least square (SLS) and shifted-SLS (SSLS). Alternatively, the maximum likelihood (ML) estimator allows the receiver to achieve both channel estimation and data detection [6], [7]. Nevertheless, when the number of pilot tones is not as large as the size of the OFDM symbol vector to be estimated, the ML estimator requires a prohibitive computation cost. In order to reduce the complexity of the ML method, the expectation-maximization (EM) algorithm has been developed in 1977 by A. P. Dempster *et al* in [8]. The EM algorithm has been largely studied subsequently, in particular in OFDM context [9], [10]. An other usually used estimation technique is called interpolated fast Fourier transform (iFFT) [11], [12], in which the channel is estimated in the time domain. It has a high performance, but the leakage due to the windowing may generate out-of-band interferences.

Unlike review papers such as [1]–[3], which cover a large panel of channel estimation techniques in OFDM, the present paper focuses on the LMMSE estimator. This method belongs to the class of the optimal filters called *Wiener filter* [13], which are widely used in the field of (tele)communications, as well as in signal processing, and in image processing. Despite its optimal nature, LMMSE suffers from two main drawbacks which limit its practical implementation. Firstly, it is very complex, since it requires a matrix inversion and multiplications namely $\mathcal{O}(M^3)$ scalar multiplications, where M is the discrete Fourier transform (DFT) size. Secondly, LMMSE requires the prior knowledge of the second-order moments of both the channel and the noise at the receiver side, i.e. the channel covariance matrix and the noise variance. In a wide range of systems, these statistical parameters are unknown at the receiver. For these reasons, a large number of studies investigated the way to implement the LMMSE channel estimator with a reduced complexity, and without prior knowledge of both the channel and the noise statistics. This paper presents the theoretical aspects of the LMMSE method and the main results of the literature dealing with the complexity reduction of the LMMSE estimator in both single input-single output (SISO) and multi-input multi-output (MIMO) systems. Furthermore, some recent results with regard to LMMSE in flexible waveforms beyond OFDM (in particular in filter bank multicarrier) are provided in this paper.

Notations: the normal font x is used for scalar variables, the bold face $\mathbf{x}$ is used for vectors and the underlined bold face $\underline{\mathbf{x}}$ for matrices. Moreover, small letters x refer to the variables in the time domain and capital letters X to the variables in the frequency domain.

The rest of the paper is organized as follows: the system model and basics of LMMSE in SISO OFDM are presented in Section II. Section III provides a state of the art of the techniques for the reduction of the complexity of the LMMSE estimator. Section IV deals with the LMMSE in MIMO OFDM systems. We finally give some perspectives and we conclude this paper in Section V.

II. Basics of LMMSE in SISO OFDM Systems

A. System Model

In this paper, we consider the transmission of an OFDM signal over a multipath channel, with perfect time and frequency synchronization at the receiver. The signal is transmitted in a bandwidth B and each OFDM symbol has a duration T_s. It is assumed that the CP length is larger than the channel length. After the cyclic prefix (CP) removal and the discrete Fourier transform (DFT) of size M, the n-th OFDM symbol can be written in the frequency domain as

$$\mathbf{Y}_n = \underline{\mathbf{X}}_n \mathbf{H}_n + \mathbf{W}_n, \tag{1}$$

where $\mathbf{Y}_n = [Y_{0,n}, ..., Y_{M-1,n}]^T$ and $\mathbf{W}_n = [W_{0,n}, ..., W_{M-1,n}]^T$ denote the $M \times 1$ complex vectors of the received signal and the additive white Gaussian noise (AWGN) of variance σ^2. The $M \times M$ diagonal matrix $\underline{\mathbf{X}}_n$ contains the vector $[X_{0,n}, ..., X_{M-1,n}]$ on its diagonal. Each element $X_{m,n}$ can be either a data symbol from a constellation or a pilot (also called training symbol). We denote p and n_p the indexes of the pilot tones along the frequency and time axes respectively, and we define $\mathcal{P} = E\{|X_{p,n_p}|^2\}$ the pilots power. In a given finite block of N consecutive OFDM symbols (e.g. an OFDM frame), we point out Ω the set of the positions (p, n_p). The vector $\mathbf{H}_n$ contains the channel frequency response components $H_{m,n}$ given by

$$H_{m,n} = \sum_{l=0}^{L-1} h_{l,n} \exp\left(-2j\pi \frac{m}{M} l\right), \tag{2}$$

where $0 \leq m \leq M-1$ denotes the subcarrier subscript, L the length of the impulse response and $h_{l,n}$ the zero-mean complex gain of the l^{th} path of the channel. The variable l is the discrete expression of the delay, and the L paths are assumed to be uncorrelated. Moreover, according to the channel model described by Bello in [14], the coefficients $h_{l,n}$ are supposed to be wide sense stationary.

On the pilot tones, the channel is estimated by means of the least square (LS) criterion, such as

$$\hat{H}^{LS}_{p,n_p} = \frac{Y_{p,n_p}}{X_{p,n_p}} = H_{p,n_p} + \frac{W_{p,n_p}}{X_{p,n_p}}. \tag{3}$$

IT can be noticed that the noise samples appear in (3). Therefore, the LS estimator is very sensitive to the noise distortion. In order to mitigate this noise effect, the channel estimation can be smoothed by means of the Wiener filtering.

B. 2-D Wiener Filtering

From the Wiener filter described in [13], a generalized two-dimensions smoothing filter has been proposed in [15] for noisy images processing. Its application in the time-frequency dimensions for OFDM systems has been proposed by Hoeher *et al.* in [16]. Fig. 1 illustrates the principle of the Wiener filtering: from the estimates $\hat{H}^{LS}_{p,n_p}$ on pilot positions $(p, n_p) \in \Omega$, the Wiener filter performs the channel estimation on the data subcarriers (m, n)

$$\hat{H}_{m,n} = \sum_{(p,n_p)\in\Omega} G(m, n, p, n_p)\hat{H}^{LS}_{p,n_p}, \tag{4}$$

where the optimal filter minimizes the mean square error as

$$G(m, n) = \min_{\hat{H}_{m,n}} \left(E\{|H_{m,n} - \hat{H}_{m,n}|^2\}\right), \tag{5}$$

with $E\{.\}$ the mathematical expectation. Since the Wiener filtering in (4) is performed by means of the interpolator filter $G(m, n)$ in the time and the frequency directions, it is called 2-D Wiener filter. It is shown in [16] that the minimization in (5) requires $\mathcal{O}(M^3)$ multiplications, and it must be carried out $M \times N$ times. As a consequence, although the Wiener filter (4) is optimal in the sense of the mean square error, it requires a prohibitive calculation cost.

In order to reduce the complexity, Hoeher proposed in [17] (prior to [16]) to split the 2-D filter into two separate filters, one along the frequency axis and the other one along the time axis. It must be emphasized that at the same time in the mid-1990s, the authors of [18], [19] called this kind of estimator *LMMSE*, and this naming still remains.

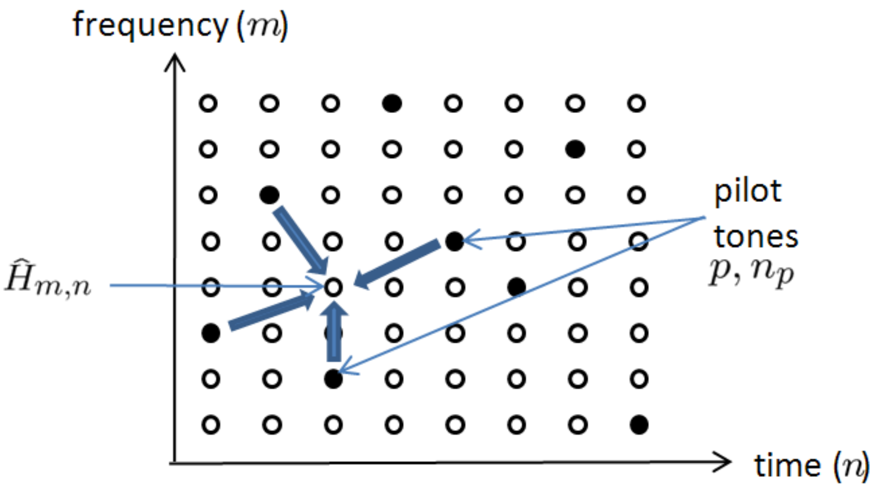


Fig. 1. Illustration of the Wiener filtering.

C. LMMSE Channel Estimation

The pilot distribution is chosen in order to track the best the time and frequency variations of the channel. In the literature, it is very usual to consider a block-type pilot arrangement (for quasi-static channels), a comb-type or an hexagonal arrangement (for frequency and time selective channel), as described in [2], [20]. In this section, a block-type pilot distribution (frequency preamble) is considered. However, for both block- and comb-type arrangement, the LMMSE channel estimation can be independently performed along the frequency and the time axis, as shown in [21], [22]. Originally proposed in [18], the LMMSE estimation is performed along the frequency axis with a pilot preamble $\underline{\mathbf{X}}_n$ (see Fig. 2) and is derived from the minimization of the cost function $J_{MMSE} = E\{||\mathbf{H}_n - \underline{\mathbf{D}}\mathbf{Y}_n||^2\}$ where $\underline{\mathbf{D}}$ is a matrix whose coefficients have to be optimized. The estimated channel frequency response vector $\hat{\mathbf{H}}_n^{LMMSE}$ is expressed as

$$\hat{\mathbf{H}}_n^{LMMSE} = \underline{\mathbf{D}}_{opt}\mathbf{Y}_n, \tag{6}$$

where $\underline{\mathbf{D}}_{opt} = \underline{\mathbf{R}}_{H,f}\underline{\mathbf{X}}_n^H(\underline{\mathbf{X}}_n\underline{\mathbf{R}}_{H,f}\underline{\mathbf{X}}_n^H + \sigma^2\underline{\mathbf{I}})^{-1}$, and $\underline{\mathbf{R}}_{H,f}$ is the channel covariance matrix along the frequency axis whose expression is detailed in Section II-D. $\underline{\mathbf{I}}$ is the $M \times M$ identity matrix, and $(.)^H$ is the Hermitian transpose. More details concerning the developments leading to (6) are provided in

[23], [24]. If the pilot vector $\underline{\mathbf{X}}_n$ is invertible (i.e. does not contain zero values), from (6) and according to [19], the usual expression of the LMMSE estimation is given by

$$\hat{\mathbf{H}}_n^{LMMSE} = \underline{\mathbf{R}}_{H,f}(\underline{\mathbf{R}}_{H,f} + (\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H)^{-1}\sigma^2\underline{\mathbf{I}})^{-1}\hat{\mathbf{H}}_n^{LS}, \tag{7}$$

with $\hat{\mathbf{H}}_n^{LS}$ the vector containing the LS estimated samples of the channel frequency response given in (3). Most of the time, the noise variance defined by $\sigma^2 = E\{|W_{m,n}|^2\}$ is supposed to be known or accurately estimated. Note that a very usual complexity reduction for LMMSE, originally proposed by Edfors *et al.* in [19] consists in using the average pilots power $E\{(\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H)^{-1}\}$ instead of $(\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H)^{-1}$ in (7). This is used in most of the papers dealing with the LMMSE estimation, as in [20], [25], [26] for instance. It has been shown in [19] that the previous approximation has a negligible effect on the performance of the LMMSE estimator. Another simple solution consists in considering pilots that are equi-powered. In that case, the invert matrix $(\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H)^{-1}$ is similar to the identity matrix and can be a priori calculated off-line.

As expressed in [22], the LMMSE channel estimation expression (7) remains the same when it is achieved along the time axis in the case of a comb-type pilot arrangement or along the frequency axis as illustrated in Fig. 2. However, note that in this scenario, the channel covariance matrix $\underline{\mathbf{R}}_{H,t}$ (along the time axis) is used in (7) instead of $\underline{\mathbf{R}}_{H,f}$. In the following, for a readability purpose, we will denote by $\underline{\mathbf{B}} = \underline{\mathbf{R}}_H(\underline{\mathbf{R}}_H + (\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H)^{-1}\sigma^2\underline{\mathbf{I}})^{-1}$ the matrix used for LMMSE in (7).

From (7), one can notice that the LMMSE estimation requires the calculation of $(\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H)^{-1}$ and $\underline{\mathbf{B}}$. The latter in particular has a very expensive computation cost, since a matrix inversion and multiplication is necessary. For this reason, a large number of methods in the literature attempts to reduce the complexity of LMMSE, such as subsequently presented.

D. Expression of the Channel Covariance Matrix

The channel covariance matrix is defined as $\underline{\mathbf{R}}_H = E\{\mathbf{H}\mathbf{H}^H\}$. When a WSS channel is assumed, $\underline{\mathbf{R}}_{H,f}$ can be computed independently from $\underline{\mathbf{R}}_{H,t}$ [21], [27]. In fact, their samples are simply obtained

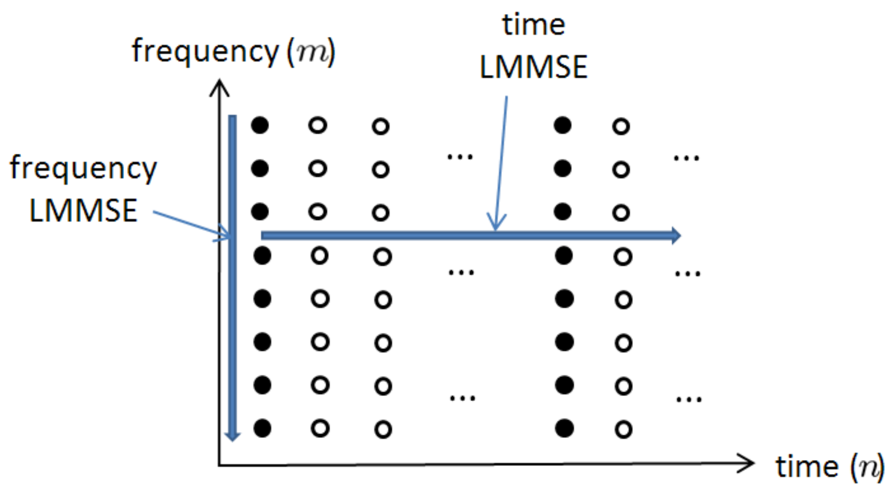


Fig. 2. Illustration of the 2×1-dimensions LMMSE filtering.

by Fourier transform (FT) of the channel intensity profile denoted by $\Gamma(\tau)$ and inverse Fourier transform (FT^{-1}) of the Doppler power spectral density (PSD) denoted by $S_H(\nu)$, respectively. Note that it is usual to model the channel intensity profile $\Gamma(\tau)$ as a decreasing exponential [28]–[30] over $\tau \in [0, \tau_{max}]$, where τ_{max} is the maximum delay of the channel. Thus, the samples $\left(\underline{\mathbf{R}}_{H,f}\right)_{u,v}$ (u-th row and v-th column of $\underline{\mathbf{R}}_{H,f}$) can be expressed as (see [28] for details)

$$\begin{aligned}\left(\underline{\mathbf{R}}_{H,f}\right)_{u,v} &= FT_{\tau}(Ce^{-\tau/\tau_{max}}) \\ &= LC.\frac{1 - e^{-2j\pi\frac{(u-v)}{M}\tau_{max}} e^{-1}}{1 + 2j\pi\frac{(u-v)}{M}\tau_{max}},\end{aligned} \tag{8}$$

where C is a normalization coefficient. A different intensity shape logically leads to a different channel covariance matrix such as in [22], [31], [32], where a constant intensity profile is assumed.

When isotropic antennas and an homogeneous transmission environment are assumed, it is also usual to consider that the Doppler PSD $S_H(\nu)$ follows Jakes' model [33] also called "U" spectrum. The corresponding channel covariance matrix $\underline{\mathbf{R}}_{H,t}$ is given by [21], [31]

$$\left(\underline{\mathbf{R}}_{H,t}\right)_{u,v} = FT^{-1}(U(f))$$
$$= \sigma_h^2 J_0(2\pi\nu_{D_{max}}(u-v)), \tag{9}$$

where $U(f)$ is the Jakes spectrum, J_0 is the Bessel function of the first kind with degree zero, and $\nu_{D_{max}}$ is the maximum Doppler frequency. In the same way as for the computation of $\underline{\mathbf{R}}_{H,f}$, a different Doppler spectrum leads to another expression of $\underline{\mathbf{R}}_{H,t}$. For instance, if far echoes are considered, as in high frequency (HF) transmissions, the Doppler PSD has a Gaussian shape [34], [35].

It can be deduced from (8) and (9) that the channel covariance matrix entirely depends on channel characteristics such as the intensity profile or the PSD and parameters such as the maximum delay or Doppler frequency. However, most of the time, these parameters are unknown at the receiver side. This is, in addition to its complexity, the main drawback of LMMSE. As a consequence, it appears that LMMSE is very difficult to implement in practice.

Thereafter, we assume that the LMMSE estimator is performed along the frequency axis. In fact, due to the structure of the OFDM and the usual pilot arrangements (block-type and comb-type arrangements), almost all LMMSE-based techniques, which are described in the literature, are carried out along the frequency axis. Moreover, from (7), the results remain the same whatever the considered dimension along which LMMSE is performed. Thus, in order to improve the readability of the paper, the subscript f is removed from $\underline{\mathbf{R}}_{H,f}$ in (7), as we only consider LMMSE estimator performed along the frequency axis.

E. Characteristics of the LMMSE Channel Estimator

In this section, we characterize the LMMSE estimator in terms of its bias $\mathcal{B}$, mean square error (MSE) $\varepsilon_{\text{LMMSE}}$ and complexity. The noise $\mathbf{W}_n$ and the channel $\mathbf{H}_n$ being zero-mean independent Gaussian processes, the LS channel estimation $\hat{\mathbf{H}}_n^{LS}$ is also a zero-mean Gaussian process. Therefore, from (7) the bias of LMMSE is zero:

$$
\begin{aligned}
\mathcal{B} &= E\{\hat{\mathbf{H}}_n^{LMMSE} - \mathbf{H}_n\} \\
&= \underline{\mathbf{B}} E\{\hat{\mathbf{H}}_n^{LS}\} - E\{\mathbf{H}_n\} = 0.
\end{aligned} \tag{10}
$$

The minimum MSE of the LMMSE estimator is calculated by minimizing the cost function $J_{\hat{\mathbf{H}}^{LMMSE}} = \frac{1}{M} E\{||\mathbf{H} - \hat{\mathbf{H}}^{LMMSE}||_F^2\}$. In SISO systems, the rank of the channel covariance matrix $\underline{\mathbf{R}}_H$ in (8) is such as $L < M$, hence $\underline{\mathbf{R}}_H$ is not invertible. However, an analytical expression of $\varepsilon_{\text{LMMSE}}$ has been derived in [36]. It is shown that $J_{\hat{\mathbf{H}}^{LMMSE}}$ can be rewritten using scalar notation as:

$$
J_{\hat{\mathbf{H}}^{LMMSE}} = \frac{1}{M} \sum_{m=0}^{L-1} \frac{\lambda_m \sigma^2}{\lambda_m X_m X_m^* + \sigma^2}, \tag{11}
$$

where λ_m are the eigenvalues of the channel covariance matrix $\underline{\mathbf{R}}_H$. The minimization of $J_{\hat{\mathbf{H}}^{LMMSE}}$ in (11) according to the variables X_m^* under the energy constraint $\mathcal{P} = \frac{1}{L} \sum_{m=0}^{L-1} X_m X_m^*$ yields

$$
\varepsilon_{\text{LMMSE}} = \frac{1}{M} . \frac{L^2 \sigma^2}{L\mathcal{P} + \sum_{m=0}^{L-1} \frac{\sigma^2}{\lambda_m}}, \tag{12}
$$

Since the MSE of the LS estimator is $\varepsilon_{\text{LS}} = \sigma^2/\mathcal{P}$ (see [4], [37] for more details), it can be noticed that $\varepsilon_{\text{LS}} > \varepsilon_{\text{LMMSE}}$ for any $\mathcal{P}/\sigma^2$ value. This result analytically proves that LMMSE is more accurate than LS. Physically, this is due to the fact that LMMSE acts like a smoothing filter, which mitigates the noise distortion.

From (7), we deduce that the complexity of LMMSE mainly lies in the computation of $\underline{\mathbf{B}}$. In fact, the matrix multiplication and inversion requires $\mathcal{O}(M^3)$ scalar operations. Moreover, $\underline{\mathbf{R}}_H$ may have to be updated, according to the channel variations or the variations of its statistics (in the case of non-WSS channels). Logically, this is the reason why a large number of methods of the literature focuses on its simplification, as shown in the following sections.

III. How to Reduce the Complexity of LMMSE in SISO-OFDM?

In this section, we provide a state of the art of the different ways to perform the LMMSE estimation in SISO system. The numerous methods proposed in the literature and reported hereafter aim at overcoming one or both drawbacks of LMMSE, i.e. the lack of knowledge of the channel and noise statistics, and the complexity of the LMMSE implementation.

A. Estimating the Channel Covariance Matrix $\underline{\mathbf{R}}_H$ and the Noise Variance σ^2

It has been aforementioned that one of the restricting factor for the practical implementation of LMMSE is the lack of knowledge of $\underline{\mathbf{R}}_H$ and σ^2 at the receiver. Therefore, a solution consists in estimating the missing parameters, independently or jointly. Thus, numerous noise variance estimators for OFDM systems are presented in [38]–[40], and in the references therein. The channel covariance matrix, in turn, can be estimated thanks to the LS estimate $\hat{\mathbf{H}}_n^{LS}$ by $\hat{\underline{\mathbf{R}}}_H^{LS} = E\{(\hat{\mathbf{H}}_n^{LS})(\hat{\mathbf{H}}_n^{LS})^H\}$, such as proposed in [22], [41]. However, this estimation suffers from the noise distortion since in practice, the expectation is approximated by an average over K pilot symbols.

In order to improve the channel covariance matrix estimation, the authors of [42] used the most significant tap (MST) technique, which has been originally presented in [43]. The MST algorithm is suitable for channels which have a sparse impulse response. In fact, this time-domain algorithm selects the J taps of highest gains among the L paths of the channel. The remaining samples of the channel impulse response (CIR) are set to zero, therefore reducing the noise distortion. The corresponding CIR is denoted $\hat{\mathbf{h}}^{MST}$. It is shown in [43] that, although the MST method misses some channel energy since J may differ from L (see the 4-th path in Fig. 3), the noise power reduction improves the LS estimator. In [42], the estimated channel covariance matrix $\hat{\underline{\mathbf{R}}}_H$ is obtained from the frequency response $\hat{\mathbf{H}}^{MST}$, which is the FFT of $\hat{\mathbf{h}}^{MST}$. Besides, the noise variance is also estimated by means of the $M-J$ last taps of $\hat{\mathbf{h}}^{LS}$, which are only noise samples. The principle of the method is depicted in Fig. 3.

Another method for the joint estimation of the channel response and the noise variance, which is based on the MMSE criterion, has been proposed in [44]. Originally proposed in a theoretical context [45], [46], in which the channel covariance matrix is supposed to be known at the receiver, the method has been extended in [44] for a practical application in which $\underline{\mathbf{R}}_H$ is estimated. To be done, an iterative algorithm is proposed: first, an LMMSE estimate $\hat{\mathbf{H}}_n^{LMMSE}(0)$ is obtained by substituting the matrix

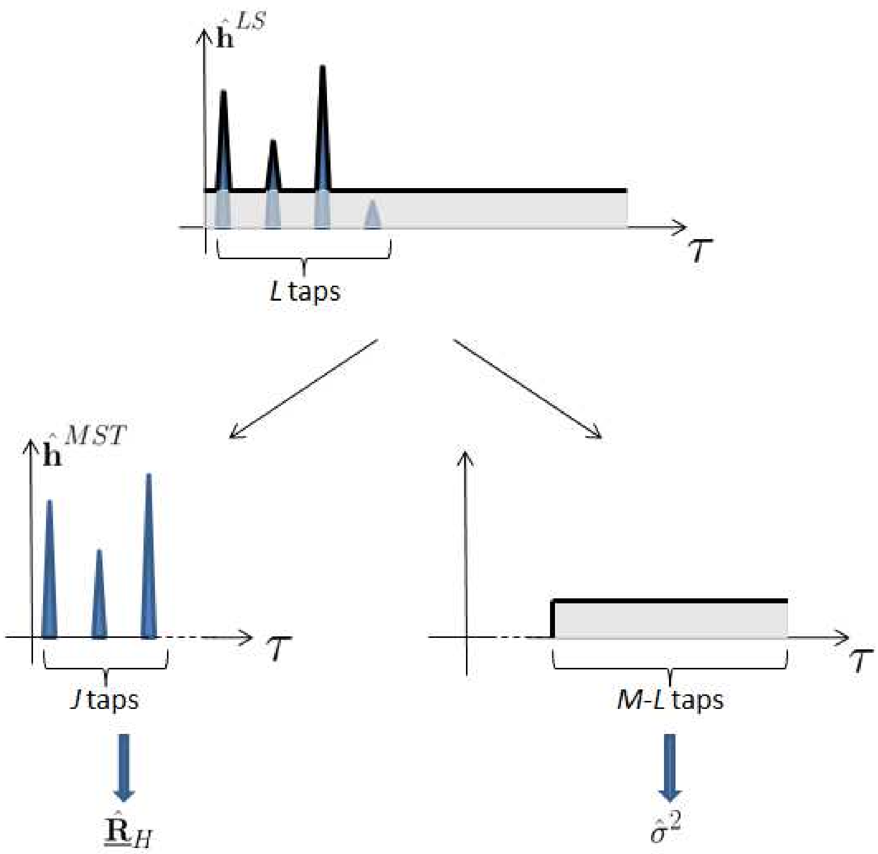


Fig. 3. Principle of Zhou's method [42] for the estimation of both the channel and the noise statistics.

$\hat{\underline{\mathbf{R}}}_H^{LS}$ in (7); second, the covariance matrix is updated as $\hat{\underline{\mathbf{R}}}_H = \hat{\mathbf{H}}_n^{LMMSE}(0)(\hat{\mathbf{H}}_n^{LMMSE}(0))^H$. The third part of the algorithm consists in alternatively estimating the noise thanks to the MMSE criterion:

$$\hat{\sigma}^2(i) = E\{||\hat{\mathbf{H}}_n^{LS} - \hat{\mathbf{H}}_n^{LMMSE}(i)||^2\}, \tag{13}$$

where i is the index of the iteration, and performing the LMMSE estimation in (7) by using the noise estimate $\hat{\sigma}^2(i)$. The principle of the algorithm is illustrated in Fig. 4. This method allows to perform the LMMSE channel estimation without any prior knowledge of $\underline{\mathbf{R}}_H$ and σ^2 at the receiver, with a performance that almost reaches the optimal LMMSE's, but without any reduction of the complexity.

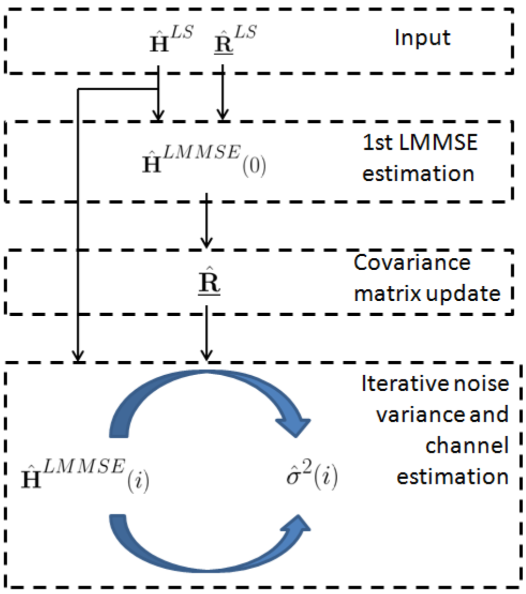


Fig. 4. Illustration of the iterative algorithm presented in [44].

B. LMMSE with Channel Covariance Matrix Substitution

An alternative to the estimation of $\underline{\mathbf{R}}_H$ consists in substituting the latter with a fixed covariance matrix denoted by $\tilde{\underline{\mathbf{R}}}_H$. This method called low-rank approximation (LRA) has been proposed by Edfors *et al.* in [19], [28], and has been used in a large number of LMMSE-based techniques and studies [43], [47]–[50]. The principle of the method is to use in (7) the matrix $\tilde{\underline{\mathbf{R}}}_H$ whose eigenvalues $\tilde{\lambda}_0, ..., \tilde{\lambda}_{M-1}$ are defined in advance, irrespective of the exact (but unknown) eigenvalues $\lambda_0, .., \lambda_{M-1}$. It is very usual to consider a constant delay profile for $\tilde{\lambda}_m$ instead of the unknown λ_m values, as illustrated in Fig. 5. The elements $(\tilde{\underline{\mathbf{R}}}_H)_{u,v}$ (at the u-th row, v-th column) of the matrix $\tilde{\underline{\mathbf{R}}}_H$ are expressed by

$$(\tilde{\underline{\mathbf{R}}}_H)_{u,v} = \frac{1 - e^{-2j\pi\tilde{L}(u-v)/M}}{2j\pi\tilde{L}(u-v)/M}, \tag{14}$$

where $\tilde{L}$ is the length of the constant delay profile. Note that, in order to have an accurate channel estimation, $\tilde{L}$ must be equal or larger than the real channel length L [28]. Some papers, like [51], deal with the different methods that can be used to evaluate the channel length.

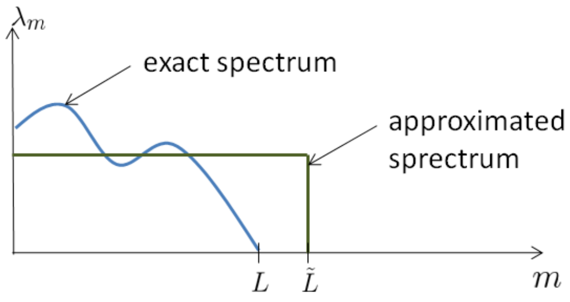


Fig. 5. Difference between the real and the approximated intensity profile.

The main advantage of using $\underline{\tilde{\mathbf{R}}}_H$ instead of $\underline{\mathbf{R}}_H$ in (7) lies in the fact that the matrix $\underline{\tilde{\mathbf{B}}} = \underline{\tilde{\mathbf{R}}}_H\big(\underline{\tilde{\mathbf{R}}}_H + (\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H)^{-1}\sigma^2\underline{\mathbf{I}}\big)^{-1}$ can be calculated with a reduced computational cost, by means of the singular value decomposition (SVD) [19], [28]. In fact, since $\underline{\tilde{\mathbf{R}}}_H$ is an Hermitian matrix, it is diagonalizable in an orthonormal basis so that $\underline{\tilde{\mathbf{R}}}_H = \underline{\tilde{\mathbf{U}}}\underline{\tilde{\mathbf{\Delta}}}_R\underline{\tilde{\mathbf{U}}}^H$. Here $\underline{\tilde{\mathbf{\Delta}}}_R$ is the diagonal matrix featuring the diagonal elements $\tilde{\lambda}_m$ (which are equal to $1/\tilde{L}$ for any $0 \leq m \leq \tilde{L}$, and zero else), and $\underline{\tilde{\mathbf{U}}}$ contains the eigenvectors of $\underline{\tilde{\mathbf{R}}}_H$. If the pilots are equi-powered (with power $\mathcal{P}$), then $\underline{\mathbf{X}}_n\underline{\mathbf{X}}_n^H = \mathcal{P}\underline{\mathbf{I}}$, $\underline{\tilde{\mathbf{B}}}$ is Hermitian as well, and can be diagonalized in the same basis as $\underline{\tilde{\mathbf{R}}}_H$. Hence, the LMMSE estimate in (7) can be rewritten as

$$\hat{\mathbf{H}}_n^{LMMSE} = \underline{\tilde{\mathbf{U}}}\underline{\tilde{\mathbf{\Delta}}}_B\underline{\tilde{\mathbf{U}}}^H\hat{\mathbf{H}}_n^{LS}, \tag{15}$$

where $\underline{\tilde{\mathbf{\Delta}}}_B$ is the eigen decomposition of $\underline{\tilde{\mathbf{B}}}$ featuring diagonal elements $\tilde{\mu}_m$ as

$$\tilde{\mu}_m = \begin{cases} \frac{\tilde{\lambda}_m}{\tilde{\lambda}_m + \frac{\sigma^2}{\mathcal{P}}} & \text{if } m = 0, 1, .., \tilde{L} - 1 \\ \\ 0 & \text{if } m = \tilde{L}, \tilde{L} + 1, .., M - 1 \end{cases} . \tag{16}$$

The LMMSE using LRA and SVD has two advantages: first, LMMSE can be performed without prior knowledge of the matrix $\underline{\mathbf{R}}_H$, and second, the SVD reduces the complexity of LMMSE to $\mathcal{O}(M^2)$ instead of $\mathcal{O}(M^3)$. Furthermore, the authors of [28] propose to use a fixed value of the noise variance $\tilde{\sigma}^2$ in (16), instead of the exact σ^2 value, in such a way that the matrix $\underline{\tilde{\mathbf{U}}}\underline{\tilde{\mathbf{\Delta}}}_B\underline{\tilde{\mathbf{U}}}^H$ can be computed off-line only once. However, $\tilde{\sigma}^2$ must have a low value in order to limit the error due to the noise variance mismatch.

In [52], [53], a method similar to the LRA-LMMSE, which is called artificial channel aided-LMMSE (ACA-LMMSE), has been proposed. The principle is to use a filter called artificial channel in order to *mask* the channel. An LMMSE estimation of the sum of the channel and the filter is then performed by using the covariance matrix of the artificial channel only. Finally, the channel frequency response is obtained by subtracting the artificial channel coefficients from the estimation LMMSE estimation. Similarly to the LRA-LMMSE estimator, this method reduces the complexity of the receiver, and does not require the prior knowledge of channel length to be performed.

In order to further reduce the complexity of the LMMSE implementation, the authors of [54] replaced the matrix $\underline{\tilde{\mathbf{B}}}$ in (15) by the matrix $\underline{\bar{\mathbf{B}}}$ defined as

$$\underline{\bar{\mathbf{B}}} = \underline{\mathbf{F}}\underline{\tilde{\mathbf{\Delta}}}_B\underline{\mathbf{F}}^H, \tag{17}$$

where $\underline{\tilde{\mathbf{\Delta}}}_B$ remains the same as defined in (16), and $\underline{\mathbf{F}}$ is the discrete Fourier transform (DFT) matrix, whose expression is given as follows:

$$\underline{\mathbf{F}} = \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(M-1)} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(M-1)} & \omega^{2(M-1)} & \cdots & \omega^{(M-1)^2} \end{pmatrix}, \tag{18}$$

with $\omega = e^{-\frac{2j\pi}{M}}$. This approximation allows to reduce the computation cost of LMMSE to $\mathcal{O}(Mlog(M))$ scalar operations. Similarly to the method in [54], the authors of [55], [56] developed the iterative dual diagonal- (DD-) LMMSE estimator, which can be expressed as

$$\hat{\mathbf{H}}_n^{DD} = \underline{\mathbf{F}}\underline{\mathbf{A}}_2\underline{\mathbf{F}}^H\underline{\mathbf{A}}_1\mathbf{Y}_n, \tag{19}$$

where $\underline{\mathbf{A}}_1$ and $\underline{\mathbf{A}}_2$ are diagonal matrices whose samples must be optimized. This operation is performed in two steps. Firstly, the optimal choice of $\underline{\mathbf{A}}_1$ is made by minimizing the cost function $J = \frac{1}{M}E\{||\underline{\mathbf{A}}_1\mathbf{Y}_n - \mathbf{H}_n||^2\}$. Secondly, $\underline{\mathbf{A}}_2$ is obtained by following a similar derivation. Since the considered matrices are diagonal, the complexity of DD-LMMSE is of order $\mathcal{O}(M\log(M))$, and the simulations shown in [55], [56] revealed that the performance of this technique is close to the one of LMMSE.

It has been demonstrated in [42], [57] that LMMSE can be performed with the same low complexity of order $\mathcal{O}(M\log(M))$ without using the DFT matrix. To do so, the techniques proposed in [42], [57] are based on fast methods for Toeplitz matrix inversion, such as the Kumar's algorithm [58]. Since $\underline{\mathbf{B}}$ is an Hermitian matrix, it is a Toeplitz matrix, and therefore such algorithms can be applied in the context of the LMMSE channel estimation.

Figs. 6 and 7 shows the performance of the LS and some aforementioned LMMSE-based estimators versus SNR (in dB). The simulations parameters are summarized as follows:

- 4-QAM and 16-QAM constellations have been used, and the OFDM symbols are composed of $M = 256$ subcarriers.
- We consider a four-paths channel with delays (given in number of samples) [0, 7, 18, 24] and gains [1, 0.4, 0.3, 0.2] (the taps are complex Gaussian variables). This channel is described in the digital radio mondiale (DRM) standard [59].
- The CP is composed of 25 samples of the OFDM symbol.
- A preamble composed of only one OFDM pilot symbol has been used for channel estimation.
- A zero-forcing (ZF) equalizer is used to invert the channel, and the BER performance has been obtained without channel coding.
- The plots have been obtained thanks to 1000 simulations runs.

Fig. 6compares the MSE behaviors of the exact LMMSE using (7), and the simplified LMMSE methods described in [28] (LRA-LMMSE), [54] (LMMSE using the FFT such as in (17)), and [41] (LMMSE using $\hat{\underline{\mathbf{R}}}_H^{LS}$). Moreover, the performance of the LS is depicted as a reference. It can be observed that the theoretical LMMSE outperforms all the other simplified LMMSE-based techniques. When the LMMSE estimation is performed by using $\hat{\underline{\mathbf{R}}}_H^{LS}$, it can be seen that $K = 8$ achieves better performance that $K = 2$, where K is the number of pilot symbols. This reflects the fact that the larger K, the better the estimation of the statistics of the channel. However, this result is obtained at the cost of a reduction of the spectral efficiency of the OFDM signal. It can also be noted that the use of an approximated channel covariance matrix $\tilde{\underline{\mathbf{R}}}_H$ greatly reduces the MSE performance.

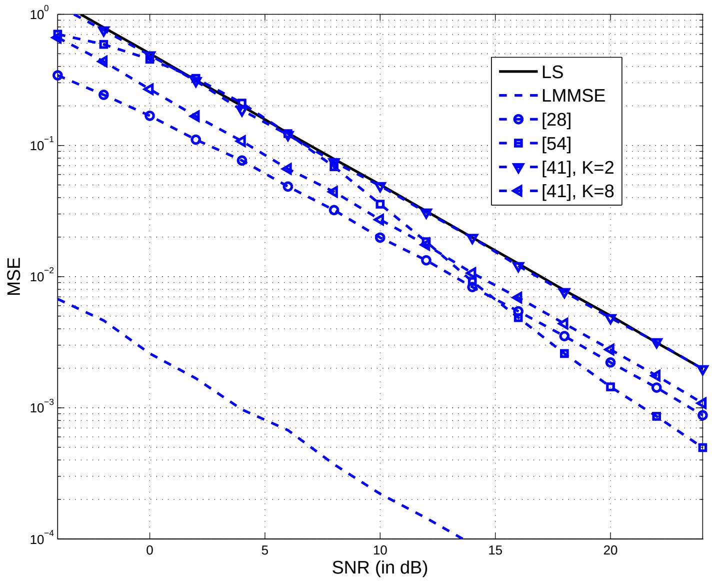
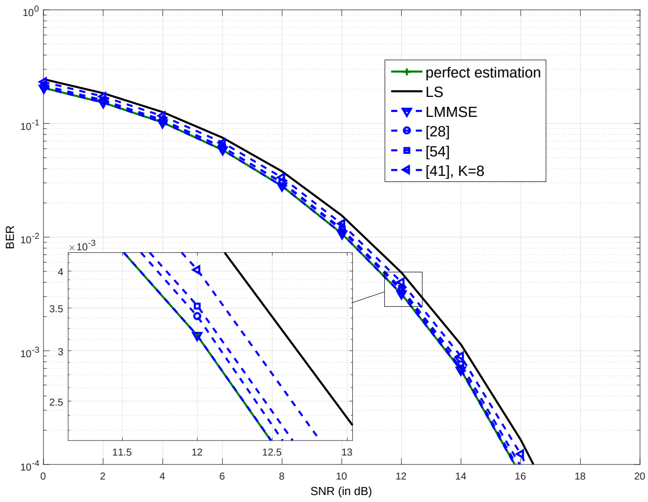
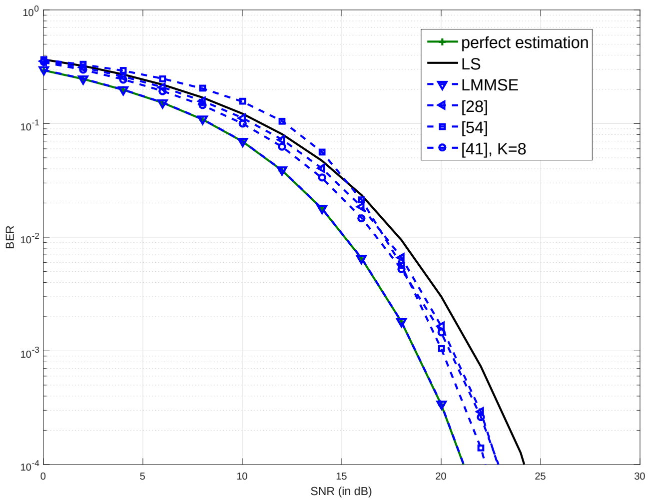


Fig. 6. MSE performance versus SNR (in dB) for different LMMSE-based estimators. The references [28], [54], and [41] correspond to LRA-LMMSE, LMMSE using (17), and LMMSE using $\hat{\underline{\mathbf{R}}}_H^{LS}$, respectively..

In addition to the MSE, Fig. 7 shows the BER performance of the same methods using (a) a 4-QAM, and (b) a 16-QAM constellation. It can be observed in Fig. 7-(a) that all the LMMSE-based techniques have behaviors close to the optimal LMMSE. In fact, a BER loss less than 0.5 dB is achieved by the different methods compared with LMMSE. This small distance between LMMSE and other techniques (in comparison with the very large MSE gain achieved by LMMSE in Fig. 6) is mainly due to the small size of the used constellation. In fact, Fig. 7 shows that the gain achieved by LMMSE is about 3 dB at BER=10^{-4} compared with LS estimator. Moreover, LMMSE estimator almost reach the perfect estimation performance. On the other hand, the other LMMSE-based methods achieve gains up to 2 dB compared with LS. These results show the capability of LMMSE to outperform LS when large-sized constellations are used.



(a) BER versus SNR, 4-QAM.



(b) BER versus SNR, 16-QAM.

Fig. 7. BER performance versus SNR (in dB) for different LMMSE-based estimators and for (a) 4-QAM and (b) 16-QAM constellations. The references [28], [54], and [41] correspond to LRA-LMMSE, LMMSE using (17), and LMMSE using $\hat{\underline{\mathbf{R}}}_H^{LS}$, respectively.

Table I summarizes the main LMMSE-based techniques proposed in the literature (for a SISO system). The methods have been classified into four approaches. The table also provides the cor-

TABLE I

COMPARISON BETWEEN CHANNEL ESTIMATION METHODS IN OFDM.

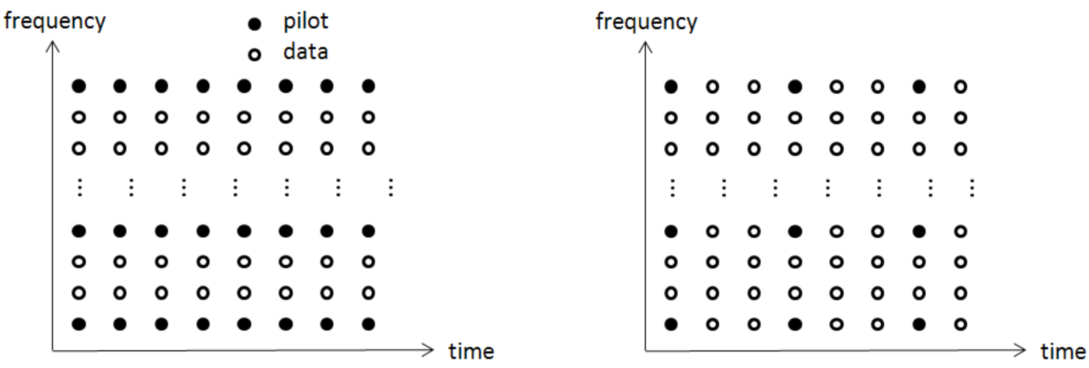
methods	complexity	performance	comments
LMMSE [16], [18]	$\mathcal{O}(M^3)$	optimal	
LRA and SVD [19], [28], [47]	$\mathcal{O}(M^2)$	sub-optimal	complex. reduction and $\underline{\mathbf{R}}_H$ substitution
Estimation of $\underline{\mathbf{R}}_H$ [41], [42], [44]	$\mathcal{O}(M^3)$	sub-optimal	$\underline{\mathbf{R}}_H$ substitution
Use of the FFT [54]–[56]	$\mathcal{O}(Mlog(M))$	sub-optimal	complex. reduction and $\underline{\mathbf{R}}_H$ substitution
Toeplitz matrix inversion algorithm [42], [57]	$\mathcal{O}(Mlog(M))$	optimal	complex. reduction Requires algo. such as [58]

responding complexity, and indicates when the methods are optimal (i.e. when they achieve the performance of the theoretical LMMSE) or sub-optimal. In the "comments" column, it is noted when the methods allow to reduce the complexity of LMMSE (it is referred as "complex. reduction"), or when the methods do not require the prior knowledge of the channel covariance matrix $\underline{\mathbf{R}}_H$ (it is referred as "$\underline{\mathbf{R}}_H$ substitution").

C. LMMSE and Interpolator Filter

The LMMSE-based techniques which have been previously presented are performed thanks to a preamble, i.e. one (or several) OFDM symbol is dedicated to the channel estimation. However, pilot symbols are usually spread in the OFDM frame in order to track both the frequency and the time variations of the channel. Fig. 8 depicts two of the possible pilot arrangements in the OFDM frame: the comb-type arrangement on Fig. 8-(a), and the rectangular pattern on Fig. 8-(b). It should also be noted that a staggered pattern is usually used in most of the telecommunication standards as the long term evolution (LTE) or the digital video broadcasting (DVB) [60]. Besides, some other pilot arrangements have been proposed in the literature, such as the hexagonal pattern or the irregular arrangement [61]. In the following, we denote by N_p (with $N_p < M$) the number of pilots in the

OFDM symbols.



(a) Comb-type arrangement. (b) Rectangular pattern.

Fig. 8. Two of the possible pilot arrangements in the OFDM frame.

When the pilots are sparse in the frequency domain[1], e.g. a frequency distance δ_d separates the pilots, the LMMSE channel estimation can be performed according to two approaches: i) using the matrix $\underline{\mathbf{B}}$ as an interpolator filter; ii) obtaining an LMMSE estimate of size N_p, and then using an interpolation filter.

The first approach is described in [22], [62]. For a readability purpose, the size of the matrices and vectors are indicated in the subscript of the variable, in the hereafter paragraph. The LMMSE estimate $\hat{\mathbf{H}}^{LMMSE}_{n,M\times 1}$ is obtained from the LS estimate $\hat{\mathbf{H}}^{LS}_{n,N_p\times 1}$ by means of a expression similar to that in (7):

$$\hat{\mathbf{H}}^{LMMSE}_{n,M\times 1} = \underline{\mathbf{R}}_{H,M\times N_p}(\underline{\mathbf{R}}_{H,N_p\times N_p} + (\underline{\mathbf{X}}_{n,N_p\times 1}\underline{\mathbf{X}}^{H}_{n,N_p\times 1})^{-1}\sigma^2\underline{\mathbf{I}})^{-1}\hat{\mathbf{H}}^{LS}_{n,N_p\times 1}, \tag{20}$$

where $\underline{\mathbf{R}}_{H,M\times N_p}$ and $\underline{\mathbf{R}}_{H,N_p\times N_p}$ are two matrices extracted from the original channel covariance matrix $\underline{\mathbf{R}}_{H,M\times M}$, the rows and columns of which correspond to the pilot positions in the OFDM symbol. Fig. 9 illustrates the principle of LMMSE used as an interpolation filter. The columns and

[1]It is assumed, nevertheless, that the number of pilots N_p is sufficient to track the channel variations, i.e. the frequency distance δ_p between pilots respects the Nyquist theorem: $\delta_p \leq M/(\tau_{max}B)$.

rows extracted from $\underline{\mathbf{R}}_{H,M\times M}$ in order to build $\underline{\mathbf{R}}_{H,M\times N_p}$ and $\underline{\mathbf{R}}_{H,N_p\times N_p}$ are highlighted in gray. It is deduced from (20) that the computation of LMMSE when pilots are sparse in the OFDM symbols requires $\mathcal{O}(N_p^3)$ scalar multiplications, which greatly reduces the complexity of the estimator. Moreover, it has been noted in [22], [28] that LMMSE used as an interpolator in (20) remains the optimal estimator in the sense of the mean square error, as soon as the condition $\delta_p \leq M/(\tau_{max}B)$ holds. Besides, the SVD can still be used in order to further reduce its implementation.

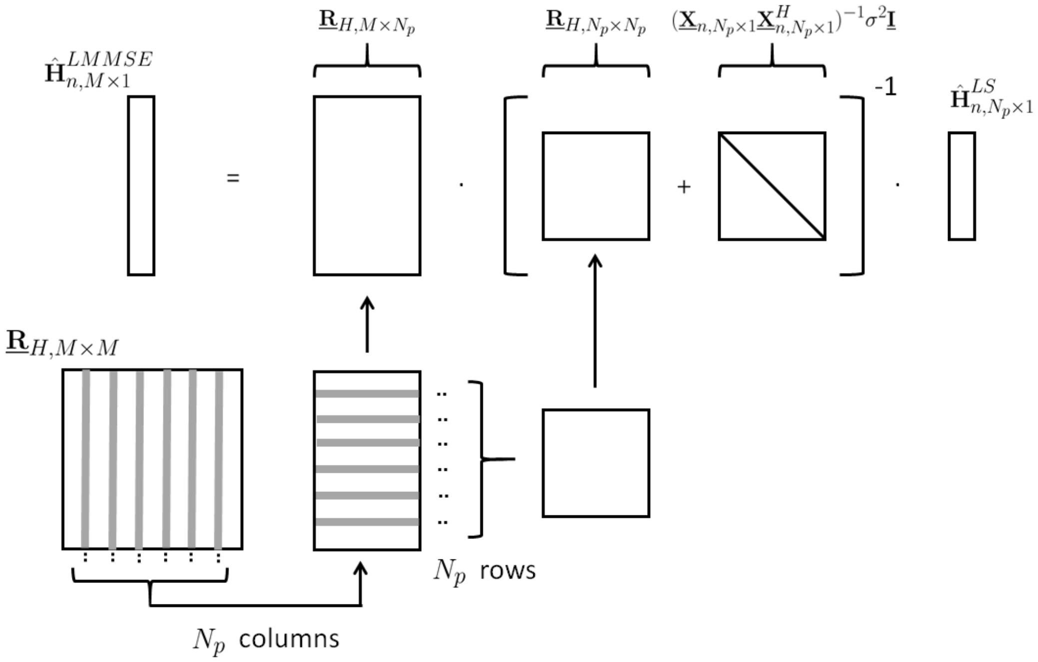


Fig. 9. Principle of the LMMSE estimator used as an interpolation filter.

The second approach consists in separating the LMMSE estimation step and the interpolation step. First, the estimate $\hat{\mathbf{H}}^{LMMSE}_{n,N_p\times 1}$ is obtained from the estimate $\hat{\mathbf{H}}^{LS}_{n,N_p\times 1}$ by using (7) where the covariance matrix $\underline{\mathbf{R}}_{H,M\times M}$ is replaced by $\underline{\mathbf{R}}_{H,N_p\times N_p}$, and second, a low-pass interpolation filtering [63] or a polynomial interpolation (linear, spline) [62] is carried out in order to obtained an estimation of the frequency response over the whole bandwidth.

Fig. 10 depicts the BER performance versus SNR (in dB) for a 16-QAM constellation. The same parameters as in Figs. 6 and 7 have been used, except for the pilot distribution (a 16-QAM constellation is considered). In fact, pilot tones are sparsely distributed in the frequency domain in order to estimate the channel. The frequency distance between pilots has been set equal to $\delta_p = 4$. The two LMMSE-based estimation methods in [22], [62] are compared with the LS estimator, which is combined with either a linear or a cubic interpolation. Note that a cubic interpolation follows the smoothing step in [62]. Fig. 10 shows that LS estimator with interpolation as well as the technique in [62] seems to tends toward non-zero lower bounds. This is due to the residual errors generated by polynomial interpolations, such as analyzed in [64], [65]. However, it can be observed that the LMMSE combined with cubic interpolation in [62] achieves a gain of 2.5 dB compared with LS with cubic interpolation. This is due to the smoothing effect of the LMMSE estimator, which reduce the noise distortion on the pilot subcarriers. It is worth noticing that the LMMSE estimator used as an interpolation filter [22] almost reaches the performance of the perfect estimation. Furthermore, it achieves a gain of 5 dB at BER=10^{-3} compared with LS using interpolation. These results shows the high performance of LMMSE compared with LS-based methods when interpolation are required[2].

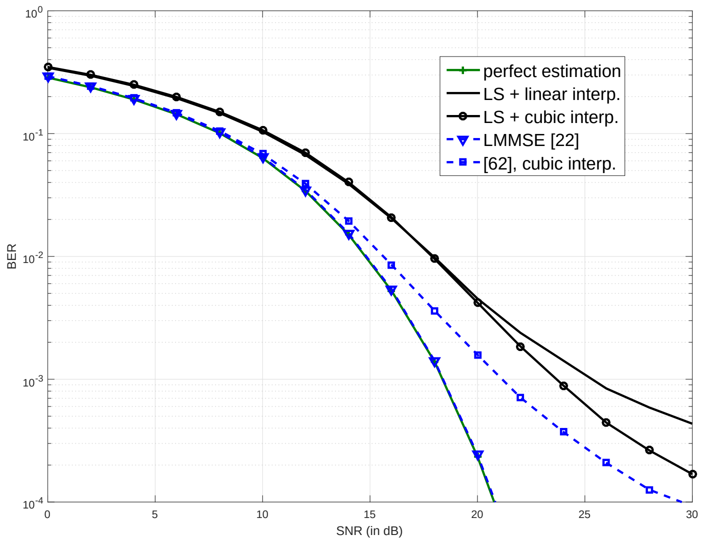


Fig. 10. BER performance versus SNR (in dB) for 16-QAM, and a distance between pilots equal to 4. LMMSE techniques in [22] and [62] are compared with the LS estimator using linear and cubic interpolations.

In this section, we have presented some techniques for the reduction of the complexity of the

[2]To be fair, it must be noted that other methods such as maximum likelihood outperform LS using interpolation as well.

LMMSE channel estimation in SISO system. Although most of the principles presented in this section remain valid in MIMO systems, the use of multiple antennas allows to design specific LMMSE-based estimation methods, such as presented hereafter.

IV. LMMSE Estimation in MIMO OFDM

A. System Model

A MIMO system with N_t transmit and N_r receive antennas is considered in this section, such as depicted in Fig. 11. In order to properly estimate the MIMO channel, a preamble composed of $K_p \geq N_p$ pilots must be transmitted [37]. The $N_r \times K_p$ matrix $\underline{\mathbf{Y}}_m$ of the received signal at the m-th subcarrier can be expressed as [4], [22], [37], [66]

$$\underline{\mathbf{Y}}_m = \underline{\mathbf{H}}_m \underline{\mathbf{X}}_m + \underline{\mathbf{W}}_m, \tag{21}$$

where $\underline{\mathbf{H}}_m$ is the $N_r \times N_t$ matrix of the channel, $\underline{\mathbf{X}}_m$ the $N_t \times K_p$ matrix of the transmitted symbols, and $\underline{\mathbf{W}}_m$ the $N_r \times K_p$ matrix of AWGN. Note that it is possible to highlight the frequency dimension in MIMO-OFDM scheme by concatenating the M received symbols (21), such as described in [67]–[69]. However, we will use the simple expression of the received signal in MIMO systems in (21), which can be used to describe both multicarrier and single carrier signals (where $M = 1$). In MIMO systems, it can be useful to rewrite (21) in the input-output notation as [70]

$$\mathrm{vec}(\underline{\mathbf{Y}}_m) = \underline{\bar{\mathbf{X}}}_m \mathrm{vec}(\underline{\mathbf{H}}_m) + \mathrm{vec}(\underline{\mathbf{W}}_m), \tag{22}$$

where the operator $\mathrm{vec}(.)$ returns a column vector obtained by stacking the columns of a matrix one below the other. Therefore, $\mathrm{vec}(\underline{\mathbf{Y}}_m)$ and $\mathrm{vec}(\underline{\mathbf{H}}_m)$ are $N_r K_p \times 1$ and $N_r N_t \times 1$ vectors, respectively. The $N_r K_p \times N_r N_t$ matrix of the transmitted signal $\underline{\bar{\mathbf{X}}}_m$ can be written as

$$\underline{\bar{\mathbf{X}}}_m = \begin{pmatrix} \underline{\tilde{\mathbf{X}}}_{m,0} \\ \underline{\tilde{\mathbf{X}}}_{m,1} \\ \vdots \\ \underline{\tilde{\mathbf{X}}}_{m,K_p-1} \end{pmatrix}, \tag{23}$$

where $\underline{\tilde{\mathbf{X}}}_{m,n}$, with $0 \leq n \leq K_p - 1$, is a $N_r \times N_r N_t$ Toeplitz matrix whose first row and first column are $[X_{m,1,n}, \overbrace{0,..0}^{N_r-1}, X_{m,2,n}, 0,..,0,..,X_{m,N_t,n}, 0,..,0]$, and $[X_{m,1,n}, \underbrace{0,..0}_{N_r-1}]^T$, respectively.

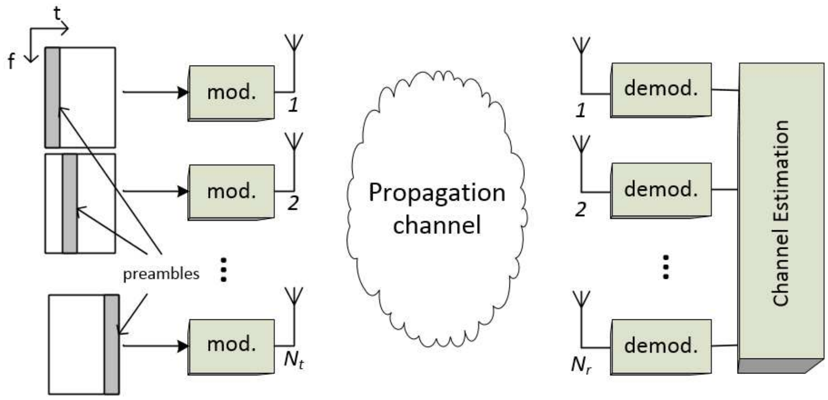


Fig. 11. Simplified representation of a MIMO-OFDM system with N_t transmit and N_r receive antennas.

Similarly to (7), the LMMSE channel estimation in MIMO-OFDM systems is obtained from (22) as follows

$$\text{vec}(\underline{\hat{\mathbf{H}}}_m)^{LMMSE} = \underline{\mathbf{R}}_H \underline{\bar{\mathbf{X}}}_m^H \left(\underline{\bar{\mathbf{X}}}_m \underline{\mathbf{R}}_H \underline{\bar{\mathbf{X}}}_m^H + \sigma^2 \underline{\mathbf{I}}\right)^{-1} \text{vec}(\underline{\mathbf{Y}}_m), \tag{24}$$

where the $N_r N_t \times N_r N_t$ channel covariance matrix $\underline{\mathbf{R}}_H$ can be expressed as [71]

$$\underline{\mathbf{R}}_H = \underline{\mathbf{R}}_{H,t} \otimes \underline{\mathbf{R}}_{H,r}, \tag{25}$$

where $\otimes$ represents the Kronecker product, $\underline{\mathbf{R}}_{H,t}$ and $\underline{\mathbf{R}}_{H,r}$ are the $N_t \times N_t$ and $N_r \times N_r$ spatial correlation matrices at the transmitter and the receiver, respectively. Note, however, that the LMMSE estimator in MIMO-OFDM can be expressed in the frequency dimension instead of the spatial dimension, such as in [68], [69].

The issue of the choice of an optimal training sequence in MIMO systems has been widely dealt with in the literature. In [67], [72], the minimization of the MSE of the LS estimator leads to optimal pilot symbols values. More recently in [69], the MSE of the LMMSE channel estimator has been chosen as the cost function to be minimized for the optimal choice of the training sequence. In both case, it has been pointed out that the optimal pilot symbols must be: equi-powered, equi-spaced in the OFDM symbol, and orthogonal between antennas. The latter feature can be simply achieved by transmitting a pilot $\underline{\mathbf{X}}_m$ at a given antenna when a null symbols are transmitted at the other antennas [69], [72]. This solution, which is illustrated in Fig. 11, has been used in practice in the long term evolution (LTE) standard (see Fig. 2.8 in [73] for instance), and in LTE-Advanced [74]. In the following, we will also assume that the pilots are equi-powered, hence $\bar{\underline{\mathbf{X}}}_m\bar{\underline{\mathbf{X}}}_m^H = \mathcal{P}\underline{\mathbf{I}}$.

B. Performance of LMMSE in MIMO-OFDM

The MSE of LMMSE in MIMO system has been analytically derived in [4], [37], [70], [75]. Unlike the SISO scheme, the spatial correlation matrix $\underline{\mathbf{R}}_H$ in MIMO may be full-rank, and then it is invertible (if the matrix is not full-rank, the problem is similar to SISO). Therefore, the MSE denoted by ε can be expressed as

$$\begin{aligned} \varepsilon =& tr\Big((\underline{\mathbf{R}}_H^{-1} + \frac{\mathcal{P}}{\sigma^2}\underline{\mathbf{I}})\Big) \\ =& \sum_{i}^{N_r N_t} \frac{\lambda_i}{1+\frac{\mathcal{P}}{\sigma^2}\lambda_i}, \end{aligned} \tag{26}$$

where λ_i is the i-th eigenvalue of $\underline{\mathbf{R}}_H$. In addition to the MSE, an estimation of the BER performance of the LMMSE estimator in space-time block-coded MIMO-OFDM systems has been proposed in [66]. Furthermore, the authors of [70], [75] have investigated the effect of the covariance mismatch on the estimator. Thus, if an estimated channel covariance matrix $\hat{\underline{\mathbf{R}}}_H$ is used instead of $\underline{\mathbf{R}}_H$ in (24), then the MSE of the estimator, which is denoted by $\tilde{\varepsilon}$, can be expressed as:

$$\tilde{\varepsilon} = \frac{\mathcal{P}}{\sigma^2} tr\Big((\underline{\hat{\mathbf{R}}}_H^{-1} + \frac{\mathcal{P}}{\sigma^2}\underline{\mathbf{I}})^{-2} \Big) + tr\Big(\big(\frac{\mathcal{P}}{\sigma^2}(\underline{\hat{\mathbf{R}}}_H^{-1} + \frac{\mathcal{P}}{\sigma^2}\underline{\mathbf{I}})^{-1} - \underline{\mathbf{I}}\big) \underline{\mathbf{R}}_H \big(\frac{\mathcal{P}}{\sigma^2}(\underline{\hat{\mathbf{R}}}_H^{-1} + \frac{\mathcal{P}}{\sigma^2}\underline{\mathbf{I}})^{-1} - \underline{\mathbf{I}}\big) \Big) \tag{27}$$

Since $\underline{\mathbf{R}}_H$ and $\underline{\hat{\mathbf{R}}}_H$ have two different eigenbases, no closed-form of the MSE can be deduced from (27). However, it is possible to use the Von Neumann's trace inequality for Hermitian matrix products [76], in order to derive a closed-form of the upper and lower bounds of the MSE:

Theorem 1. *Let* $\underline{\boldsymbol{A}}$ *and* $\underline{\boldsymbol{B}}$ *two Hermitian nonnegative definite* $M \times M$ *matrices. If the eigenvalues of* $\underline{\boldsymbol{A}}$ *and* $\underline{\boldsymbol{B}}$*, which are denoted by* $\lambda_{A,i}$ *and* $\lambda_{B,i}$ *respectively, are arranged in the decreasing order, then*

$$\sum_{i=1}^{M} \lambda_{A,M-i+1} \lambda_{B,i} \leq tr(\underline{\boldsymbol{AB}}) \leq \sum_{i=1}^{M-1} \lambda_{A,i} \lambda_{B,i}. \tag{28}$$

Proof. see [76], [77].

This theorem has been applied in [70], [75] in such a way that $\tilde{\varepsilon}$ can be upper and lower bounded as:

$$\tilde{\varepsilon}^l = \sum_{i=1}^{N_r N_t} \frac{\frac{\mathcal{P}}{\sigma^2}\tilde{\lambda}_i^2 + \lambda_i}{(1 + \frac{\mathcal{P}}{\sigma^2}\tilde{\lambda}_i)^2} \leq \tilde{\varepsilon} \leq \sum_{i=1}^{N_r N_t} \frac{\frac{\mathcal{P}}{\sigma^2}\tilde{\lambda}_i^2 + \lambda_{N_r N_t - i + 1}}{(1 + \frac{\mathcal{P}}{\sigma^2}\tilde{\lambda}_i)^2} = \tilde{\varepsilon}^u, \tag{29}$$

where $\tilde{\lambda}_i$ is the i-th eigenvalue of $\underline{\hat{\mathbf{R}}}_H$. It must be emphasizes that the same result could be applied in SISO system, where the channel covariance matrix is usually approximated by a fixed matrix (see Section III). Fig. 12 depicts the exact MSE $\tilde{\varepsilon}$, as well as the lower and upper bounds $\tilde{\varepsilon}^l$ and $\tilde{\varepsilon}^u$ versus $\frac{\mathcal{P}}{\sigma^2}$ (in dB), using a MIMO systems with $N_t = N_r = 4$. The transmit and received spatial correlation matrices are defined as $(\underline{\mathbf{R}}_{H,t})_{u,v} = (\underline{\mathbf{R}}_{H,r})_{u,v} = \rho^{|u-v|}$, where u and v are the indexes of the rows and the columns, and $\rho = 0.3$, when the estimated correlation matrix $\underline{\hat{\mathbf{R}}}_H$ is defined by using $\rho = 0.8$. It can be observed that the lower bound $\tilde{\varepsilon}^l$ almost matches the exact MSE for any $\frac{\mathcal{P}}{\sigma^2}$ values. Therefore $\tilde{\varepsilon}^l$ can be used as a good estimator of the MSE in MIMO systems.

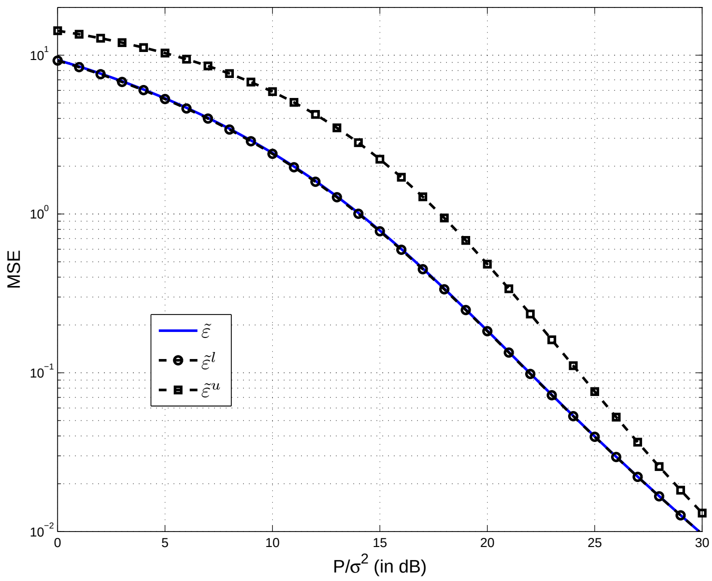


Fig. 12. Exact MSE $\tilde{\varepsilon}$, lower and upper bounds $\tilde{\varepsilon}^l$ and $\tilde{\varepsilon}^u$ versus $\frac{\mathcal{P}}{\sigma^2}$ (in dB), using $N_t = N_r = 4$.

C. Simplified LMMSE in MIMO-OFDM

As previously mentioned, most of the techniques allowing to simplify the implementation of the LMMSE estimator in SISO can be adapted to MIMO systems. Some of the LMMSE-based methods applied to MIMO systems are listed hereafter:

- Similarly to [41], the channel covariance matrix is estimated in [78], which allows to perform LMMSE without any prior knowledge of the channel covariance matrix.
- In [79], the authors propose to separate the LMMSE estimation and the interpolation stages, such as in [63] for instance.
- In [68], the low rank approximation of the channel covariance matrix is used.
- In [37], the channel covariance matrix is replaced by a diagonal matrix $\breve{\underline{\mathbf{R}}}_H = \alpha\underline{\mathbf{I}}$. The coefficient α is chosen so as to minimize the cost function

$$J = \frac{1}{M} E\{||\alpha\underline{\mathbf{I}}\left(\alpha\underline{\mathbf{I}} + \sigma^2(\underline{\mathbf{CC}}^H)^H\right)^{-1}\hat{\mathbf{H}}_n^{LS} - \mathbf{H}_n||^2\}. \quad (30)$$

This solution called relaxed MMSE (RMMSE) fits the MIMO systems with uncorrelated transmit and receive antennas, but is limited when the correlation cannot be assumed to be null.

V. Perspectives and Conclusions

A. Perspectives

Although numerous papers deal with the LMMSE channel estimation, pending issues and challenges remain. Among them, it has been previously stated that no closed-form of the MSE of the channel estimation has been derived when LMMSE is performed by using an approximated channel covariance matrix, such as in [28] in a SISO and [75] in a MIMO context. Here, the problem is to assess the trace of the product of two Hermitian matrices, which have different eigenbases.

Another issue to be dealt with is the application of LMMSE to flexible waveforms beyond OFDM, such the filter bank-based modulation. In fact these modulations are good alternatives to OFDM for fifth generation (5G) communications systems, since they are more flexible, more spectrally efficient, and generate less out-of-band interferences [80]. Among the existing candidates, the most studied solutions are the frequency multitone (FMT), the generalized frequency division multiplexing (GFDM), and the filter bank multicarrier/offset-QAM (FBMC/OQAM).

The general principle of these modulations is to relax the constraint of orthogonality among symbols and subcarriers, hence prototype filters can be designed in order to minimize the out-of-band interferences. However, the used pulse shapes generate inherent interferences among symbols and subcarriers, such as illustrated in Fig. 13. Thus, similarly to (1), the received symbol at the m-th subcarrier and n-th symbol can be expressed as

$$Y_{m,n} = H_{m,n}X_{m,n} + \underbrace{\sum_{\substack{p\neq m\\ q\neq n}} H_{p,q}X_{p,q}\Gamma_{p,q}}_{I_{m,n}} + W_{m,n}, \tag{31}$$

where $I_{m,n}$ is the interference term, and $\Gamma_{p,q}$ is a non-zero value generated by the prototype filter. More details on filter bank based modulations are provided in [80], [81].

The channel estimation is still a challenge in such modulation schemes, due to the presence of intrinsic interferences. In particular, very few papers deal with the LMMSE estimator in filter bank-

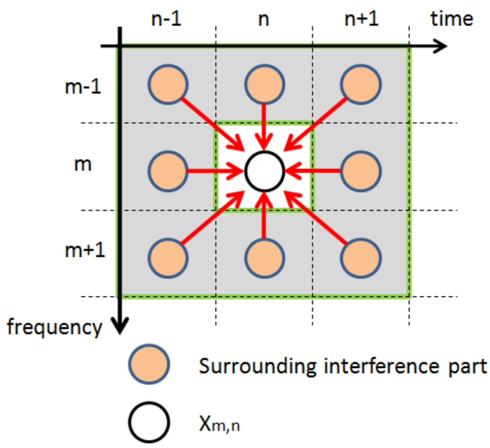


Fig. 13. Interferences generated by the subcarriers and symbols around $X_{m,n}$.

based modulations. Actually, to the best of our knowledge, there is a lack of studies on LMMSE applied to FMT and GFDM[3]. In FBMC/OQAM, recent research undertaken in [82]–[84] have shown that the LMMSE channel estimation can be performed in an equivalent manner as in OFDM (7). To do so, it is assumed that the channel is locally flat in the neighboring of the symbol $X_{m,n}$. In [85], the LMMSE estimator is used as an interpolation filter. Finally, it has also been shown in [86] that LMMSE can be used in MIMO-FBMC/OQAM systems, if the pilot arrangement avoids any interference between the antennas.

B. Summary and Conclusion

In this paper, we proposed a didactic presentation of the LMMSE channel estimation in OFDM modulation scheme, and we provided a state-of-the art of the different ways to perform such an estimator in both SISO and MIMO systems. It emerges that the issue of the practical application of LMMSE can be tackle from different approach: i) finding methods that avoid the prior knowledge of both the channel and the noise statistics; ii) reducing the complexity of implementation. It is worth noticing that some techniques allow to estimate or approximate the channel covariance matrix $\underline{\mathbf{R}}_H$ while reducing the computational cost of LMMSE. It has been pointed out that the complexity of

[3]The authors of [81] stated that they proposed the first channel estimator in GFDM.

LMMSE can be reduced from $\mathcal{O}(M^3)$ to $\mathcal{O}(Mlog(M))$ operations. Finally, it has been mentioned that the application of the LMMSE channel estimator in flexible waveforms beyond OFDM, as filter bank multicarrier, is still a pending challenge.

REFERENCES

[1] S. Coleri, M. Ergen, A. Puri, and A. Bahai, "A study of channel estimation in OFDM systems," in *Vehicular Technology Conference Fall*, vol. 2, pp. 894–898, September 2002.

[2] M. K. Ozdemir and H. Arslan, "Channel Estimation for Wireless OFDM Systems," *IEEE Communications Surveys and Tutorials*, vol. 9, pp. 18 – 48, 2nd Quarter 2007.

[3] Y. Liu, Z. Tan, H. Hu, L. J. Cimini, and G. Y. Li, "Channel Estimation for OFDM," *IEEE Communication Surveys and Tutorials*, vol. 16, pp. 1891 – 1908, Fourth quarter 2014.

[4] M. Biguesh and A. B. Gershman, "Downlink Channel Estimation in Cellular Systems with Antenna Arrays at Base Stations Using Channel Probing with Feedback," *EURASIP Journal on Applied Signal Processing*, vol. 9, pp. 1330–1339, September 2004.

[5] H. Nooralizadeh and S. S. Moghaddam, "A New Shifted Scaled LS Channel Estimator for Rician Flat Fading MIMO Channel," in *ISIEA*, vol. 1, (Kuala Lumpur, Malaysia), pp. 243 – 247, October 2009.

[6] S. M. Kay, *Fundamentals of Statistical Signal Processing : Estimation Theory*, ch. Maximum Likelihood Estimation, pp. 157–214. Prentice Hall, 2003.

[7] M. Abuthinien, S. Chen, and L. Hanzo, "Semi-blind Joint Maximum Likelihood Channel Estimation and Data Detection for MIMO Systems," *IEEE Signal Processing Letters*, vol. 15, pp. 202–205, 2008.

[8] A. P. Dempster, N. M. Laird, and D. B. Rubin, "Maximum Likelihood from Incomplete Data via the EM Algorithm," *Journal of the Royal Statistical Society. Series B*, vol. 39, no. 1, pp. 1–38, 1977.

[9] Y. Xie and C. Georghiades, "An EM-based Channel Estimation Algorithm For OFDM With Transmitter Diversity," in *GLOBECOM'01*, vol. 2, (San Antonio, TX), pp. 871 – 875, November 2001.

[10] X. Ma, H. Kobayashi, and S. C. Schwartz, "EM-Based Channel EstimationAlgorithms for OFDM," *EURASIP Journal on Applied Signal Processing*, vol. 10, pp. 1460 – 1477, 2004.

[11] J. Schoukens, R. Pintelon, and H. V. Hamme, "The Interpolated Fast Fourier Transform: A Comparative Study," *IEEE Transactions on Instrumentation and Measurement*, vol. 41, pp. 226 – 232, April 1992.

[12] G. Auer, S. Sand, and A. Dammann, "Comparison of Low Complexity OFDM Channel Estimation Techniques," in *8th International OFDM-Workshop*, (Hamburg, Germany), September 2003.

[13] S. Haykin, *Adaptive Filter Theory*, ch. 2, pp. 94 – 128. Upper Saddle River, New Jersey: Prentice Hall, fourth ed., 2002.

[14] P. Bello, "Characterization of randomly time-variant linear channels," *IEEE Trans. on Communications Systems*, vol. 11, pp. 360–393, December 1963.

[15] W. K. Pratt, "Generalized Wiener Filtering Computation Techniques," *IEEE Transactions on Computers*, vol. c-21, pp. 636 – 641, July 1972.

[16] P. Hoeher, S. Kaiser, and P. Robertson, "Two-Dimensional Pilot-Symbol-Aided Channel Estimation by Wiener Filtering," in *ICASSP*, vol. 3, (Munich), pp. 1845 – 1848, April 1997.

[17] P. Hoeher, "TCM on Frequency-Selective Land-Mobile Fading Channels," in *Proc. 5th Tirrenia International Workshop on Digital Communications*, pp. 317 – 328, 1991.

[18] J. J. van de Beek, O. Edfors, M. Sandell, S. K. Wilson, and P. O. Börjesson, "On Channel Estimation In OFDM Systems," in *IEEE conference on vehicular technology*, vol. 2, (Chicago, IL), pp. 815 – 819, September 1995.

[19] O. Edfors, M. Sandell, J.-J. van de Beek, S. K. Wilson, and P. O. Börjesson, "OFDM Channel Estimation by Singular Value Decomposition," in *VTC*, vol. 2, (Atlanta, GA), pp. 923 – 927, May 1996.

[20] Y. Shen and E. Martinez, "Channel Estimation in OFDM Systems," tech. rep., Freescale Semiconductor, 2006.

[21] Y. G. Li, L. J. Cimini, and N. R. Sollenberger, "Robust Channel Estimation for OFDM Systems with Rapid Dispersive Fading Channels," *IEEE Transactions on Communications*, vol. 46, pp. 902 – 915, July 1998.

[22] B. Le Saux, *Estimation de canal pour système multi-antennes multi-porteuses*. PhD thesis, INSA Rennes, 2007.

[23] S. Kay, *Fundamentals of statistical signal processing : estimation theory*, ch. 12, pp. 388–391. Prentice Hall, 2003.

[24] V. Savaux, *Contribution to multipath channel estimation in OFDM context*. PhD thesis, Supélec, Rennes, France, November 2013.

[25] M. Morelli and U. Mengali, "A Comparisaon of Pilot-Aided Channel Estimation Methods for OFDM Systems," *IEEE Transactions on Signal Processing*, vol. 49, pp. 3065–3073, December 2001.

[26] A. Khlifi and R. Bouallegue, "Performance Analysis of LS and LMMSE Channel Estimation Techniques for LTE Downlink Systems," *International Journal of Wireless & Mobile Networks*, vol. 3, pp. 141 – 149, October 2011.

[27] J. G. Proakis and M. Salehi, *Digital Communications*, ch. Fading Channels I : Characterization and Signaling, pp. 830–898. Mc Graw-Hill, 2008.

[28] O. Edfors, M. Sandell, J.-J. van de Beek, S. K. Wilson, and P. O. Börjesson, "OFDM Channel Estimation by Singular Value Decomposition," *IEEE Trans. on Communications*, vol. 46, pp. 931 – 939, July 1998.

[29] H. Steendam and M. Moeneclaey, "Analysis and Optimization of the Performance of OFDM on Frequency-Selective Time-Selective Fading Channels," *IEEE Trans. on Communications*, vol. 47, pp. 1811 – 1819, December 1999.

[30] J. R. Foerster, "The Effects of Multipath Interference on the Performance of UWB Systems in an Indoor Wireless Channel," in *Vehicular Technology Conference Spring*, vol. 2, (Rhodes), pp. 1176 – 1180, May 2001.

[31] G. Auer, S. Sand, A. Dammann, and S. Kaiser, "Analysis of cellular interference for MC-CDMA and its impact on channel estimation," *European Transactions on Telecommunications*, vol. 15, no. 3, pp. 173 – 184, 2004.

[32] V. Srivastava, C. K. Ho, P. H. W. Fung, and S. Sun, “Robust MMSE Channel Estimation in OFDM Systems with Practical Timing Synchronization,” in *WCNC*, vol. 2, (Altlanta, GA), pp. 711 – 716, March 2004.

[33] W. C. Jakes, *Microwave Mobile Communications*. Wiley-IEEE Press, 1994.

[34] C. C. Watterson, J. R. Juroshek, and W. D. Bensema, “Experimental Confirmation of an HF Channel Model,” *IEEE Tansactions on Communication Technology*, vol. com-18, pp. 792 – 803, December 1970.

[35] L. Ehrman, L. B. Bates, J. F. Eschle, and J. M. Kates, “Real-Time Software Simulation of the HF Radio Channel,” *IEEE Trans. on Communications*, vol. com-30, pp. 1809 – 1817, August 1982.

[36] V. Savaux, Y. Louët, M. Djoko-Kouam, and A. Skrzypczak, “Minimum mean-square-error expression of LMMSE channel estimation in SISO OFDM systems,” *IET Electronics Letters*, vol. 49, pp. 1152 – 1154, August 2013.

[37] M. Biguesh and A. B. Gershman, “Training-Based MIMO Channel Estimation: A Study of Estimator Tradeoffs and Optimal Training Signals,” *IEEE Transactions on Signal Processing*, vol. 54, pp. 884–893, March 2006.

[38] G. Ren, H. Zhang, and Y. Chang, “SNR Estimation Algorithm Based on the Preamble for OFDM Systems in Frequency Selective Channels,” *IEEE Transactions on Communications*, vol. 57, pp. 2230 – 2234, August 2009.

[39] H. Xu, G. Wei, and J. Zhu, “A Novel SNR Estimation Algorithm for OFDM,” in *IEEE Vehicular Technology Conference Spring*, vol. 5, pp. 3068 – 3071, May 2005.

[40] X. Xu, Y. Jing, and X. Yu, “Subspace-Based Noise Variance and SNR Estimation for OFDM Systems,” in *IEEE Mobile Radio Applications Wireless Communication Networking Conference*, vol. 1, pp. 23 –26, March 2005.

[41] P. Marques, A. Pereira, and A. Gameiro, “Pilot and data aided channel estimation for uplink MC-CDMA mobile systems,” in *proc. of the 14-th IST Summit*, (Dresden, Germany), pp. 1 – 5, June 2005.

[42] W. Zhou and W. H. Lam, “A Fast LMMSE Channel EstimationMethod for OFDM Systems,” *EURASIP Journal on Wireless Communications and Networking*, vol. 2009, March 2009.

[43] H. Minn and V. K. Bhargava, “An Investigation into Time-Domain Approach for OFDM Channel Estimation,” *IEEE Transactions on Broadcasting*, vol. 46, pp. 240 – 248, December 2000.

[44] V. Savaux, Y. Louët, M. Djoko-Kouam, and A. Skrzypczak, “Application of a Joint and Iterative MMSE-based Estimation of SNR and Frequency Selective Channel for OFDM Systems,” *EURASIP Journal on Advances in Signal Processing*, vol. 2013, pp. 1 – 11, July 2013.

[45] V. Savaux, Y. Louët, M. Djoko-Kouam, and A. Skrzypczak, “An Iterative and Joint Estimation of SNR and Frequency Selective Channel for OFDM Systems,” in *European Wireless*, (Poznan, Poland), pp. 1 – 7, April 2012.

[46] V. Savaux, M. Djoko-Kouam, Y. Louët, and A. Skrzypczak, “Convergence analysis of a joint signal-to-noise ratio and channel estimator for frequency selective channels in orthogonal frequency division multiplexing context,” *IET Signal Processing*, vol. 8, pp. 693 – 701, August 2014.

[47] M.-H. Hsieh and C.-H. Wei, "Channel Estimation for OFDM Systems Based on Comb-Type Pilot Arrangement in Frequency Selective Fading Channels," *IEEE Transations on Consumer Electronics*, vol. 44, pp. 217 – 225, February 1998.

[48] Y. G. Li, "Pilot-Symbol-Aided Channel Estimation for OFDM in Wireless Systems," *IEEE Transactions on Vehicular Technology*, vol. 49, pp. 1207 – 1215, July 2000.

[49] K. Yan, S. Ding, Y. Kiu, Y. Wang, and H. Liu, "A Low-Complexity LMMSE Channel EstimationMethod for OFDM-Based Cooperative Diversity Systems with Multiple Amplify-and-Forward Relays," *EURASIP Journal onWireless Communications and Networking*, vol. 2008, pp. 1 – 9, May 2008.

[50] A. Zaier and R. R. Bouallegue, "Channel Estimation Study for Block -Pilot Insertion in OFDM Systems under Slowly Time Varying Conditions," *International Journal of Computer Networks & Communications (IJCNC)*, vol. 3, pp. 39 – 54, November 2011.

[51] B. Yang, K. B. Letaief, R. S. Cheng, and Z. Cao, "Channel Estimation for OFDM Transmission in Multipath Fading Channels Based on Parametric Channel Modeling," *IEEE Trans. on Communications*, vol. 49, pp. 467 – 479, March 2001.

[52] V. Savaux, A. Skrzypczak, Y. Louët, and M. Djoko-Kouam, "Near LMMSE Channel Estimation Performance with Artificial Channel at Receiver for OFDM Systems," in *SPAWC 2012*, (Cesme, Turkey), pp. 545 – 549, June 2012.

[53] V. Savaux, Y. Louët, M. Djoko-Kouam, and A. Skrzypczak, "Artificial channel aided LMMSE estimation for time-frequency selective channels in OFDM context," *Signal Processing, Elsevier*, vol. 93, pp. 2369 – 2380, September 2013.

[54] S. Ohno, S. Munesada, and E. Manasseh, "Low-complexity approximate LMMSE channel estimation for OFDM systems," in *APSIPA ASC*, (Hollywood, CA), pp. 1 – 4, December 2012.

[55] N. Geng, L. Ping, X. Yuan, and L. F. Yeung, "Iterative Dual Diagonal LMMSE Channel Estimation in OFDM Systems," in *VTC fall*, (Ottawa, ON), pp. 1 – 4, September 2010.

[56] N. Geng, X. Yuan, and L. Ping, "Dual-Diagonal LMMSE Channel Estimation for OFDM Systems," *IEEE Transactions on Signal Processing*, vol. 60, pp. 4734 – 4746, September 2012.

[57] A. Khlifi and R. Bouallegue, "A Very Low Complexity LMMSE Channel Estimation Technique for OFDM Systems," in *proceedings of VTC'15*, (Glasgow, UK), 2015.

[58] R. Kumar, "A Fast Algorithm for Solving a Toeplitz System of Equations," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 33, pp. 254–267, February 1985.

[59] ETSI, "Digital Radio Mondiale (DRM);System Specification," Tech. Rep. ETSI ES 201 980 V 3.1.1, ETSI, August 2009.

[60] ETSI, "Digital video broadcasting (DVB) framing structure, channel coding and modulation for digital terrestrial television," Tech. Rep. ETSI EN 300 744 V1.5.1, ETSI, 2004.

[61] P. Fertl and G. Matz, "Channel Estimation in Wireless OFDM Systems With Irregular Pilot Distribution," *IEEE Transactions on Signal Processing*, vol. 58, pp. 3180 – 3194, June 2010.

[62] L. U. Khan, Z. Sabir, and M. I. Babar, "Robust Modified MMSE Estimator for Comb-Type Channel Estimation in OFDM Systems," in *ICACT'13*, (PyeongChang, Korea), pp. 924 – 928, January 2013.

[63] G. Auer and E. Karipidis, "Pilot Aided Channel Estimation for OFDM: A Separated Approach for Smoothing and Interpolation," in *ICC'05*, vol. 4, (Seoul, Korea), pp. 2173 – 2178, May 2005.

[64] V. Savaux, M. Djoko-Kouam, Y. Louët, and A. Skrzypczak, "Effect of Polynomial Interpolations on the Estimation Performance of a Frequency-Selective Rayleigh Channel in OFDM Systems," *IET Signal Processing*, vol. 9, pp. 97 – 109, February 2015.

[65] V. Savaux, A. Skrzypczak, and Y. Louët, "Theoretical Bit Error Floor Analysis of 16-QAM OFDM Signal with Channel Estimation Using Polynomial Interpolation," *IET Signal Processing*, p. 12, January 2016.

[66] L. Jacobs and M. Moeneclaey, "Effect of MMSE Channel Estimation on BER Performance of Orthogonal Space-Time Block Codes in Rayleigh Fading Channels," *IEEE Trans. on Communications*, vol. 57, pp. 1242 – 1245, May 2009.

[67] I. Barhumi, G. Leus, and M. Moonen, "Optimal training design for MIMO OFDM systems in mobile wireless channels," *IEEE Transaction on signal processing*, vol. 51, pp. 1615 – 1624, June 2003.

[68] D. Shen, Z. Diao, K.-K. Wong, and V. O. K. Li, "Simplified Channel Estimation for OFDM Systems With Multiple Transmit Antennas," *IEEE Transactions on Broadcasting*, vol. 52, pp. 193 – 202, February 2006.

[69] J. Jo and I. Sohn, "On the optimality of training signals for MMSE channel estimation in MIMO-OFDM systems," *EURASIP Journal on Wireless Communications and Networking*, vol. 2015, pp. 1 – 9, December 2015.

[70] A. Assalini, E. Dall'Anese, and S. Pupolin, "Linear MMSE MIMO Channel Estimation with Imperfect Channel Covariance Information," in *proceedings of ICC'09*, (Dresden, Germany), pp. 1 – 5, June 2009.

[71] J.-P. Kermoal, L. Schumacher, K. I. Pedersen, P. E. Mogensen, and F. Frederiksen, "A Stochastic MIMO Radio Channel Model With Experimental Validation," *IEEE Journal on Selected Areas in Communications*, vol. 20, pp. 1211 – 1226, August 2002.

[72] H. Minn and N. Al-Dhahir, "Optimal Training Signals for MIMO OFDM Channel Estimation," *IEEE Transactions on Wireless Communications*, vol. 5, pp. 1158 –1168, May 2006.

[73] M. Simko, *Pilot Pattern Optimization for Doubly-Selective MIMO OFDM Transmissions*. PhD thesis, University of Vienna, May 2013.

[74] C.-Y. Hsieh, D. W. Lin, and M. Chingwo, "LMMSE-Based Channel Estimation for LTE-Advanced MIMO Downlink Employing UE-Specific Reference Signals," in *proc. of VTC-Spring'15*, (Glasgow, UK), pp. 1 – 5, May 2015.

[75] E. Dall'Anese, A. Assalini, and S. Pupolin, "Effect of Channel Covariance Estimation Error on the MIMO Linear MMSE Channel Estimator," in *proceedings of The 12th International Symposium on Wireless Personal Multimedia Communications'09*, (Sendai, Japan), September 2009.

[76] B. Wang and F. Zhang, “Some inequalities for the Eigenvalues of the Product of Positive Semidefinite Hermitian Matrices,” *Elsevier, Linear Algebra and its Applications*, vol. 160, pp. 113 – 118, January 1992.

[77] F. Zhang and Q. Zhang, “Eigenvalue Inequalities for Matrix Product,” *IEEE Transactions on Automatic Control*, vol. 51, pp. 1506 – 1509, September 2006.

[78] P. Marques and A. Gameiro, “Uplink MIMO channel estimation for beyond 3G systems,” in *Fifth IEE International Conference on 3G Mobile Communication Technologies*, (London, England), pp. 203 – 207, October 2004.

[79] G. Auer, “Channel Estimation for OFDM Systems with Multiple Transmit Antennas by Filtering in Time and Frequency,” in *VTC Fall*, vol. 2, (Orlando, FL), pp. 1204 – 1208, October 2003.

[80] B. Farhang-Boroujeny, “OFDM Versus Filter Bank Multicarrier,” *IEEE Signal Processing Magazine*, vol. 28, pp. 92 – 112, May 2011.

[81] U. Vilaipornsawai and M. Jia, “Scattered-Pilot Channel Estimation for GFDM,” in *proc. of WCNC’14*, (Istanbul, Turkey), pp. 1053 – 1058, April 2014.

[82] L. Caro, V. Savaux, D. Boiteau, M. Djoko-Kouam, and Y. Louët, “ Preamble-Based LMMSE Channel Estimation in FBMC Modulations,” in *VTC-Spring’15*, (Glasgow, UK), pp. 1 – 5, May 2015.

[83] V. Savaux and F. Bader, “Mean square error analysis and linear minimum mean square error application for preamble-based channel estimation in orthogonal frequency division multiplexing/ offset quadrature amplitude modulation systems,” *IET Communications*, vol. 9, pp. 1763 – 1773, October 2015.

[84] V. Savaux, F. Bader, and Y. Louët, “A Joint MMSE Channel and Noise Variance Estimation for OFDM/OQAM Modulation,” *IEEE Transactions on Communications*, vol. 63, pp. 4254 – 4266, November 2015.

[85] J.-B. Doré, V. Berg, and D. Kténas, “Channel estimation techniques for 5G cellular networks: FBMC and multiuser asynchronous fragmented spectrum scenario,” *Transactions on Emerging Telecommunications Technologies*, vol. 26, pp. 15 – 30, January 2015.

[86] F. Rottenberg, Y. Medjahdi, E. Kofidis, , and J. Louveaux, “Preamble-based Channel Estimation in Asynchronous FBMC-OQAM Distributed MIMO Systems,” in *proc. of ISWCS’15*, (Brussels, Belgium), pp. 1 – 5, 2015.