

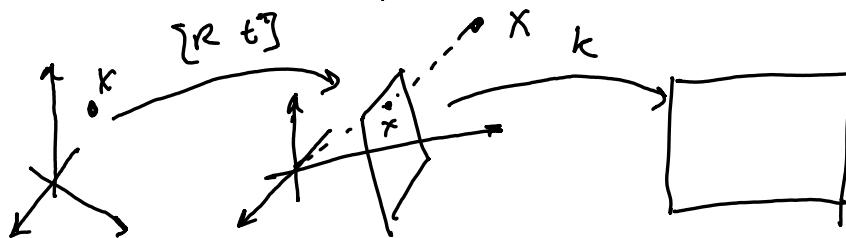
## Lecture 4:

### Reprojection

The camera equation:  $\lambda \mathbf{x} = \underbrace{k [R \ t]}_P \mathbf{X}$

$k$  - intrinsic parameters

$R, t$  - extrinsic parameters



The structure for motion problem:

Solve the cam eq:  $A_{ij} \mathbf{x}_{ij} = P_i \mathbf{X}, \forall_{ij}$

Find both camera matrices  $P_i$ , and 3D points  $\mathbf{x}_i$ !

Two versions:

- Projective Reconstruction: Nothing is known about  $P_i$
- Euclidean:  $P_i = K_i \{R_i \ t_i\}$ , where  $K_i$  is known

## Projective VS. Euclidean Reconstruction

Projective: The reconstruction is determined up to a projective transformation. If  $\lambda \mathbf{x} = P \mathbf{x}$ , then for any projective transformation

$$\tilde{\mathbf{x}} = H^{-1} \mathbf{x}$$

we have

$$\lambda \mathbf{x} = PHH^{-1} \mathbf{x} = PH \tilde{\mathbf{x}}$$

$PH$  is also valid camera.

Calibrated cameras: A camera:  $P = K[R \ t]$

where the inner parameters  $K$  are known is called calibrated. If we change coordinates in the image using

$$\tilde{\mathbf{x}} = K^{-1} \mathbf{x}$$

we get a so called normalized (calibrated) camera

$$\tilde{\mathbf{x}} \sim K^{-1} K [R \ t] \mathbf{x} = [R \ t] \mathbf{x}$$

Euclidean: The reconstruction is determined up to a similarity transformation. If  $\lambda \mathbf{x} = [R \ t] \mathbf{x}$  then for any similarity transformation

$$\tilde{\mathbf{x}} = t^{-1} \mathbf{x} = \begin{pmatrix} S Q & v \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{x}$$

We have:

$$\frac{\lambda}{S} \mathbf{x} = [R \ t] \begin{pmatrix} Q & V \\ 0 & \frac{1}{S} \end{pmatrix} \tilde{\mathbf{x}} = [RQ \ Rv + \frac{t}{S}] \tilde{\mathbf{x}}$$

Since  $RQ$  is a rotation this is an normalized camera.

The projective coord. system makes things look  
strange

The inner parameters - k:

$$k = \begin{pmatrix} f & sf & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

f - focal length  
 $s$  - aspect ratio  
 $(x_0, y_0)$  - principal points.

The focal length f:

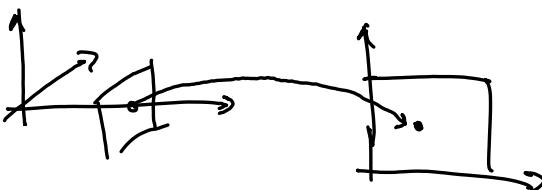
$$\begin{pmatrix} fx \\ fy \\ 1 \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Re-scales the image (e.g. meters  $\rightarrow$  pixels)

The principal point:

$$\begin{pmatrix} fx + x_0 \\ fy + y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Re-centres the image. Typically transforms the point  $(0,0,1)$  to the middle of the image.



## Aspect ratio:

$$\begin{pmatrix} \gamma f x + x_0 \\ \gamma f y + y_0 \end{pmatrix} = \begin{pmatrix} \gamma f & 0 & x_0 \\ 0 & \gamma f & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Pixels are not always square but can be rectangular  
In such cases the scaling in the x-dir should be different  
from the y-dir.

## Skew:

$$\begin{pmatrix} \gamma f & s f & x_0 \\ 0 & \gamma f & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



Corrects for tilted pixels. Typically 0.

## Finding $k$ :

- Solve the resection problem:

Find  $P$  from the camera equations:

$$d_i x_i = P_i \bar{x}_i$$

Where both  $x_i$  and  $\bar{x}_i$  are known  $\forall i$   
(Structure from motion with known 3D points)

- Use QR-factorization to extract  $k$  from  $P$

## Q Q - factorization :

Theorem: If  $A$  is an  $n \times n$  matrix then there is an orthogonal matrix  $Q$  and a right triangular matrix  $R$  such that  $A = QR$

(If  $A$  is invertible the diagonal elements are chosen to be positive, then the factorization is unique.)

Note: In our case we will use  $K$  for the triangular matrix and  $R$  for the rotation.

$$\begin{pmatrix} A_1^T \\ A_2^T \\ A_3^T \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} R_1^T \\ R_2^T \\ R_3^T \end{pmatrix} = \begin{pmatrix} aR_1^T + bR_2^T + cR_3^T \\ dR_2^T + eR_3^T \\ fR_3^T \end{pmatrix}$$

Algo: (Gram-Schmidt)

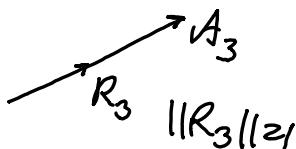
- Determine  $f$  and  $R_3$  from last eq  $A_3 = fR_3$
- Divide  $A_2$  into two perpendicular components  $eR_3$  (projection onto  $R_3$ ) and  $dR_2 = A_2 - eR_3$
- Divide  $A_1$  into three perpendicular components  $cR_3$  (proj. on to  $R_3$ ),  $bR_2$  (projection on to  $R_2$ ) and  $aR_1 = A_1 - cR_3 - bR_2$

Ex 1:

$$\text{If } P = \begin{pmatrix} 3000 & 0 & -1000 & 1 \\ 1000 & 2000\sqrt{2} & 1000 & \frac{1}{2} \\ 2 & 0 & 2 & 3 \end{pmatrix}$$

find  $f$  and  $R_3$

$$A_3^T = f R_3^T, \quad A_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

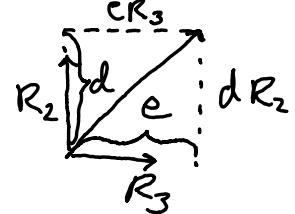


$$A_3 = f R_3^T = \underbrace{2\sqrt{2}}_{f} \cdot \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{R_3}$$

Ex 2: Determine  $e$ ,  $R_2$  and  $d$  for  $P$  in Ex 1

$$A_2^T = d R_2^T + e R_3^T, \quad 1000 \begin{pmatrix} 1 \\ 2\sqrt{2} \\ 1 \end{pmatrix} = A_2$$

$$\begin{aligned} e &= A_2^T R_3 = 1000 \cdot (1 \ 2\sqrt{2} \ 1) \cdot \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{R_2} = \\ &= \frac{1000}{\sqrt{2}} 2 = 1000\sqrt{2} \end{aligned}$$



$$\begin{aligned} dR_2 &= A_2 - eR_3 = 1000 \begin{pmatrix} 1 \\ 2\sqrt{2} \\ 1 \end{pmatrix} - 1000 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \underbrace{2000\sqrt{2}}_d \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ d &= 2000\sqrt{2} \end{aligned}$$

Ex 3:

$$A_1^T = aR_1^T + bR_2^T + cR_3^T, A_1 = 1000 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

Projection of  $A_1$  onto  $R_2$  and  $R_3$ .

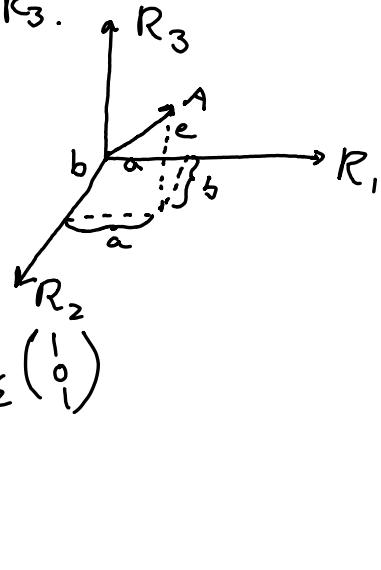
$$c = A_1^T R_3 = 1000(3 \ 0 \ -1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \frac{1000}{\sqrt{2}} 2 = 1000\sqrt{2}$$

$$b = A_1^T R_2 = 1000(3 \ 0 \ -1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$aR_1 = A_1 - CR_3 = 1000 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} - 1000\sqrt{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= 1000 \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = \underbrace{2000 \frac{1}{\sqrt{2}}}_{a} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$



The RQ-factorization of  $A$  is

$$A = \begin{pmatrix} 2000\sqrt{2} & 0 & 1000\sqrt{2} \\ 0 & 2000\sqrt{2} & 1000\sqrt{2} \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}$$

Should be 1  
i.e. divided by this number

$$f = \frac{2000\sqrt{2}}{2\sqrt{2}} = 1000$$

$$x_0 = y_0 = \frac{1000\sqrt{2}}{2\sqrt{2}} = 500$$

$$\gamma = 1, S = 0$$

Remark:  $\det(R) = |\gamma f f'| > 0$   
 if  $\det(A) < 0$  then  $\det(R) < 0$   
 $\Rightarrow R$  is orthogonal but not a rotation.  
 switch from  $P$  to  $-P$

## Direct linear Transform: (DLT)

### Finding the camera matrix: (The Resection Problem)

Use images of a known object to eliminate the projective ambiguity. If  $\mathbf{x}_i$  are 3d-points of a known object, and  $\mathbf{x}_i$  corresponding projections we have

$$\begin{aligned}\lambda_i \mathbf{x}_i &= \mathbf{P} \mathbf{x}_i \\ \lambda_2 \mathbf{x}_2 &= \mathbf{P} \mathbf{x}_2 \\ &\vdots \\ \lambda_n \mathbf{x}_n &= \mathbf{P} \mathbf{x}_n\end{aligned}$$

There are  $3N$  equations and  $11+N$  unknowns. We need  $3N \geq 11+N \Rightarrow N \geq 6$  points to solve the problem.

### Matrix Formulation:

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \mathbf{p}_3^T \end{pmatrix} \text{ where } \mathbf{p}_i \text{ are the rows of } \mathbf{P}.$$

The first equality is:

$$\mathbf{x}_i^T \mathbf{p}_1 - \lambda_i x_i = 0$$

$$\mathbf{x}_i^T \mathbf{p}_2 - \lambda_i y_i = 0$$

$$\mathbf{x}_i^T \mathbf{p}_3 - \lambda_i = 0$$

$$\begin{pmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \mathbf{p}_3^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{p}_1^T \mathbf{x} \\ \mathbf{p}_2^T \mathbf{x} \\ \mathbf{p}_3^T \mathbf{x} \end{pmatrix}$$

When  $\mathbf{x}_i = (x_i, y_i, 1)$ , in matrix form:

$$\begin{pmatrix} \mathbf{x}_i^T & 0 & 0 & -x_i \\ 0 & \mathbf{x}_i^T & 0 & -y_i \\ 0 & 0 & \mathbf{x}_i^T & -1 \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \lambda_i \end{pmatrix}$$

## Homogeneous Least Square:

Want to find a solution to:  $Mv=0$

To avoid trivial sol.  $v=0$  Use constraint  $\|v\|=1$   
No exact ...

Ex 4:  $M = USV^T$  where

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$$

find a vector of length 1 that minimizes  $\|Mv\|^2$  and the minimal value

$$M = USV^T$$

$$\|Mv\|^2 = v^T M^T M v = v^T (USV^T)^T (USV^T) v =$$

$$= v^T \underbrace{(V S^T U^T)}_W \underbrace{(U S V^T)}_I v = w^T S^T S w$$

$$1 = \|v\|^2 = v^T v = w^T \underbrace{V^T V}_I w = \|w\|^2 \quad \begin{cases} w = V^T v \quad (2) \\ V^T w = v \end{cases}$$

$$Sw = \begin{pmatrix} 3 & & \\ & 2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3w_1 \\ 2w_2 \\ w_3 \end{pmatrix}$$

$$\|Sw\|^2 = 2w_1^2 + 4w_2^2 + w_3^2$$

$$1 = \|w\|^2 = w_1^2 + w_2^2 + w_3^2$$

$$w = (0, 0, 1)$$

last column  
of  $V$

$$V = VW = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$