# Computer Vision
# Assignment 3

**Joel Bäcker, jo4383ba-s**

Spring 2022

## 1 The Fundamental Matrix

***Exercise 1***

The fundamental matrix is calculated by $[t]_x A$ where $A$ and $t$ is from $P_2 = [A \;\; t]$.

$$F = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

The epipolar line in the second images going from $x = (1, 1)$, in the first image, is found by

$$l = F\mathtt{x} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}.$$

The epipolar constraint states that $\bar{\mathtt{x}} F \mathtt{x} = 0$, where $\mathtt{x}$ is the projection in the first image and $\bar{\mathtt{x}}$ is the projection in the second image. If $\bar{\mathtt{x}} = (\bar{\mathtt{x}}_1, \bar{\mathtt{x}}_2, \bar{\mathtt{x}}_3)$ we have

$$\begin{pmatrix} \bar{\mathtt{x}}_1 & \bar{\mathtt{x}}_2 & \bar{\mathtt{x}}_3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = 2\bar{\mathtt{x}}_1 - 4\bar{\mathtt{x}}_3 \iff \bar{\mathtt{x}}_1 = 2\bar{\mathtt{x}}_3.$$

We can now determine which points that could be the same projection into $P_2$, a point in $P_2$ should satisfy the equation $\bar{\mathtt{x}}_1 = 2\bar{\mathtt{x}}_3$.

1) $(2, 0) : 2 \cdot 2 - 4 \cdot 1 = 0$ Possible projection

2) $(2, 1) : 2 \cdot 2 - 4 \cdot 1 = 0$ Possible projection

3) $(4, 2) : 2 \cdot 4 - 4 \cdot 1 = 4$

***Exercise 2***

If $P_1 = [I \quad 0]$ and

$$P_2 = [A \quad t] = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

the epipoles can be found by $e_2 \sim P_2C_1$ and $e_1 \sim P_1C_2$, $C_n$ is the camera centers for the cameras. To calculate the centers, remember $P_nC_n = 0$, the only solution with the first camera matrix is $C_1 = (0 \quad 0 \quad 0 \quad 1)^T$ (in homogeneous coordinates). The second center can be found by

$$C_2 = \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \iff P_2C_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \iff$$

$$\begin{cases} X + Y + Z + 2W & = 0 \\ 2Y + 2W & = 0 \\ Z & = 0 \end{cases} \iff \begin{cases} X = -t \\ Y = -t \\ Z = 0 \\ W = t \end{cases}$$

thus $C_2$ is, in homogeneous coordinates, $C_2 = (-1 \quad -1 \quad 0 \quad 1)^T$. We can now calculate $e_1$ and $e_2$

$$e_1 \sim P_1C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$e_2 \sim P_2C_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

From the second epipole, the fundamental matrix can be computed by $F = [e_2]_\times A$, where $[e_2]_\times$ is a skew symmetric matrix and $A$ is the first $3 \times 3$ matrix in $P_2$.

$$F = [e_2]_\times A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{pmatrix}$$

The determinant of $F$ is

$$\det(F) = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} = 2(0 \cdot 2 - 0 \cdot (-2)) = 0$$

and

$$e_2^T F = \begin{pmatrix} 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$F e_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

***Exercise 3***

The matrix $F$ can be expressed as $F = N_2^T \tilde{F} N_1$.

***Computer Exercise 1***

The fundamental matrix is

$$F = \begin{pmatrix} -3.39011 \cdot 10^{-8} & -3.72006 \cdot 10^{-6} & 0.00577 \\ 4.66737 \cdot 10^{-6} & 2.89361 \cdot 10^{-7} & -0.02668 \\ -0.00719 & 0.02630 & 1 \end{pmatrix}.$$

The mean epipolar distances obtained

- With normalization: 0.1535
- Without normalization: 0.2421

See figure 1–3.

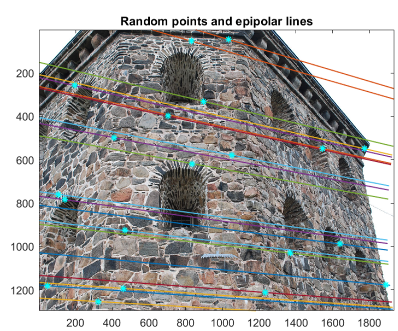


Figure 1: Random lines and points in second image

***Exercise 4***

We have a matrix

$$F = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

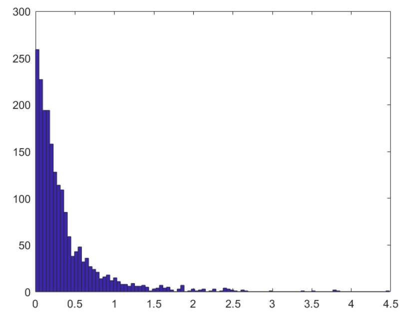


Figure 2: Histogram of distances between points and lines

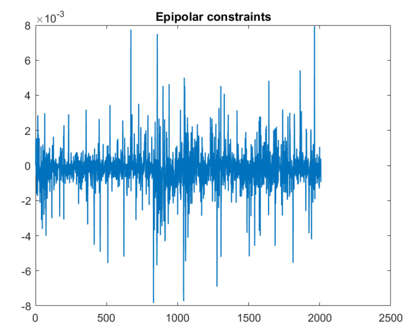


Figure 3: Errors of epipolar constraint

and want to verify that the projection of the points $x_1 = (1, 2, 3)$ and $x_2 = (3, 2, 1)$ in $P_1 = [I \quad 0]$ respectively $P_2 = [[e_2]_\times F \quad e_2]$ satisfies the epipolar constraint. We also want to find the center of $P_2$. To start we remember that

$$\begin{cases} e_1 \sim P_2 C_1 \\ e_2 \sim P_1 C_2 \end{cases} \quad \text{and} \quad \begin{cases} e_2^T F = 0 \\ F e_1 = 0 \end{cases} \quad .$$

So, if $e_2$ is found then we could calculate $P_2$ and then its center. If $e_2 = (x \ \ y \ \ z)^T$ then

$$e_2^T F = (x \quad y \quad z) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 0 \iff \begin{cases} y = 0 \\ x + z = 0 \\ x + z = 0 \end{cases} \iff \begin{cases} x = -t \\ y = 0 \\ z = t \end{cases}$$

which means that $e_2 \sim (-1 \;\; 0 \;\; 1)^T$. Calculating $P_2$ then becomes

$$
\begin{aligned}
[e_2]_\times F &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ -1 & 0 & 0 \end{pmatrix} \iff \\
P_2 &= \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Then $x_1$ and $x_2$ are projected using the camera matrices.

$$
\begin{aligned}
x_{1,1} = P_1 x_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} \\
x_{2,1} = P_1 x_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \\
x_{1,2} = P_2 x_1 &= \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 0 \end{pmatrix} \\
x_{2,2} = P_2 x_2 &= \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}
\end{aligned}
$$

and to verify that the epipolar constraint holds

$$
\begin{aligned}
x_{1,2}^T F x_{1,1} = 0 \\
x_{2,2}^T F x_{2,1} = 0
\end{aligned}
$$

Lastly, the center of $P_2$ is found by calculating the null space of it. We say that $C_2 = (x \;\; y \;\; z \;\; w)^T$ then

$$
P_2 C_2 = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \iff \begin{cases} -x - w = 0 \\ 2y + 2z = 0 \\ -x + w = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = -t \\ z = t \\ w = 0 \end{cases}
$$

which means that $C_2 \sim (0 \;\; -1 \;\; 1 \;\; 0)$.

***Computer Exercise 2***

The camera matrices were

$$P1 = \begin{pmatrix} 420.2724 & 0 & 0 & 851.2594 \\ 0 & 333.0697 & 0 & 667.6759 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$P2 = \begin{pmatrix} -29.9348 & -67.6594 & 497.7678 & 509.0913 \\ -90.7747 & 228.7685 & 110.2791 & 458.8298 \\ -0.0588 & -0.0121 & 0.0692 & 0.6974 \end{pmatrix}.$$

See figure 4–5.

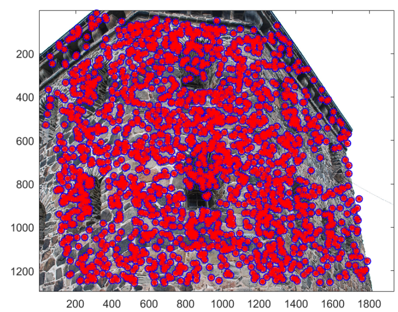


Figure 4: First image, red are projection and blue are correct.

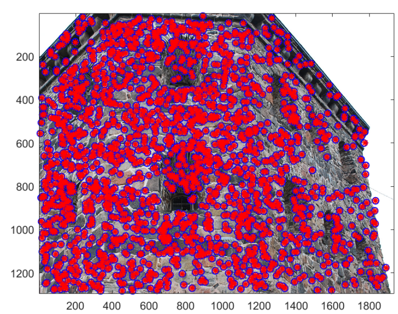


Figure 5: Second image, red are projection and blue are correct.

### ***Computer Exercise 3***

The essential matrix is

$$E = \begin{pmatrix} -8.8885 & -1005.8067 & 377.07825 \\ 1252.5231 & 78.3677 & -2448.1743 \\ -472.7888 & 2550.1917 & 1 \end{pmatrix}.$$

See figure 6–7.

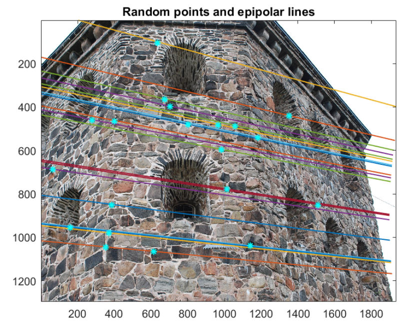


Figure 6: Random lines and points in second image

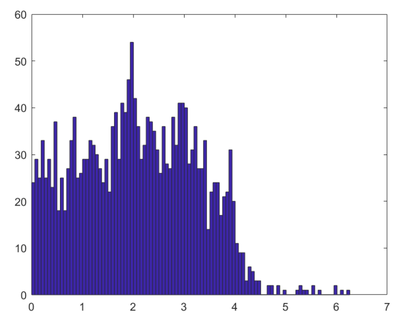


Figure 7: Histogram of distances between points and lines

### ***Exercise 6***

We have $E = U\text{diag}([1\ 1\ 0]V^T$ where

$$U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We start by verifying the determinant of $UV^T$ equals 1.

$$UV^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \implies$$

$$\det(UV^T) = \begin{vmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -1/\sqrt{2} \\ 1 & 0 \end{vmatrix} + \frac{1}{\sqrt{2}} \begin{vmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \left(0 - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} - 0\right) = \frac{1}{2} + \frac{1}{2} = 1$$

Compute the essential matrix and verify that $x_1 = (0,0)$ (in camera 1) and $x_2 = (1,1)$ (in camera 2) is plausible correspondence.

$$E = U\text{diag}([1\ 1\ 0]V^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

To verify that the two points are a plausible correspondance we use the formula $(x')^T Ex = 0$, where $x'$ is the point projected in the second camera and $x$ in the first.

$$x_2^T Ex_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $x_1$ is the projection of $\mathbf{X}$ in $P_1$, show that $\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 & s \end{pmatrix}^T$. Lets say that we do not know what $\mathbf{X}$ is, i.e. $\mathbf{X}(s) = \begin{pmatrix} x & y & z & w \end{pmatrix}^T$, then we solve for the values using $x_1 = P_1\mathbf{X}$.

$$x_1 = P_1\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \iff \begin{cases} x = 0 \\ y = 0 \\ z = 1 \end{cases} .$$

Note, we can not find $w$ because there arises no equation to solve for it, thus we put it to $s$. This this shows that $x_1$ is the projection of $\mathbf{X}(s)$.

For each of

$$P_2 = [UWV^T\ u_3] \text{ or } [UWV^T\ -u_3] \text{ or } [UW^TV^T\ u_3] \text{ or } [UW^TV^T\ -u_3]$$

where

$$W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $u_3$ the third column of $U$, compute $\mathbf{X}(s)$ such that it projects to $x_2$.

1)

$$[UWV^T \;\; u_3]X(s) = \begin{pmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ s \end{pmatrix} \implies$$

$$s = -1/\sqrt{2}$$

2)

$$[UWV^T \;\; -u_3]X(s) = \begin{pmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ -s \end{pmatrix} \implies$$

$$s = 1/\sqrt{2}$$

3)

$$[UW^TV^T \;\; u_3]X(s) = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ s \end{pmatrix} \implies$$

$$s = 1/\sqrt{2}$$

4)

$$[UW^TV^T \;\; -u_3]X(s) = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -s \end{pmatrix} \implies$$

$$s = -1/\sqrt{2}$$

To determine which 3D points is in front of the camera the depth is calculated, if the depth is larger than 0, it is in front of the camera. The depth can be found by

$$\text{depth}(P, X) = \frac{\text{sign}(\det(A))}{||A_3||}[A_3^T \;\; a_3]X,$$

where $P = [A \;\; a]$. Doing this for the matrices above we get

1) $\text{depth}([UWV^T \;\; u_3], X(-1/\sqrt{2})) = 1$
2) $\text{depth}([UWV^T \;\; -u_3], X(1/\sqrt{2})) = -1$
3) $\text{depth}([UW^TV^T \;\; u_3], X(1/\sqrt{2})) = 1$
4) $\text{depth}([UW^TV^T \;\; -u_3], X(-1/\sqrt{2})) = -1$

and for $P_1$

1) $\text{depth}(P_1, X(-1/\sqrt{2})) = -\sqrt{2}$
2) $\text{depth}(P_1, X(1/\sqrt{2})) = \sqrt{2}$
3) $\text{depth}(P_1, X(1/\sqrt{2})) = \sqrt{2}$
4) $\text{depth}(P_1, X(-1/\sqrt{2})) = -\sqrt{2}$

Pairs where $\mathbf{X}(s)$ is in front of both cameras both depth needs to be positive, this is only satisfied by pair 3).

### ***Computer Exercise 4***

See figure 8–10.

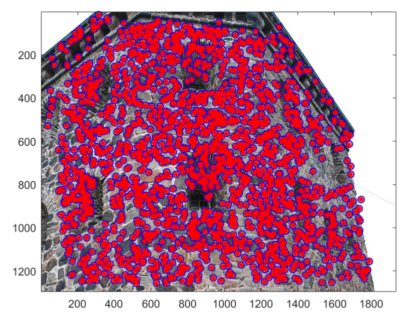

Figure 8: First image, red are projection and blue are correct.

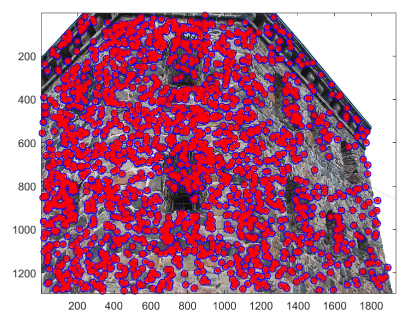

Figure 9: Second image, red are projection and blue are correct.

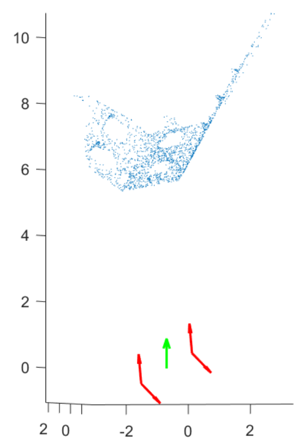


Figure 10: 3D points and camera matrices, red are different $P_2$ and green is $P_1$.