Lecture 3:

Repetition:

We say $u \sim v$ if $u = \lambda u$, $\lambda \neq 0$

$\mathbb{P}^2 =$ "All elements that can be represented with vectors in $\mathbb{R}^3 \setminus \{0\}$"

$u = (u_1, u_2, 1)$ interpreted as point in $\mathbb{R}^2$ w\ coords. (u_1, u_2)

$u = (u_1, u_2, 0)$ interpreted as point infinitely far away in the direction (u_1, u_2)

Projective Transformations:

A P.T. is an invertible mapping $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ defined by:

$$x \sim Hy$$

Where $x \in \mathbb{R}^{n+1}$ and $y \in \mathbb{R}^{n+1}$ are homogen. coords. representing elems. of $\mathbb{P}^n$ and H is an invertible $(n+1) \times (n+1)$ matrix.

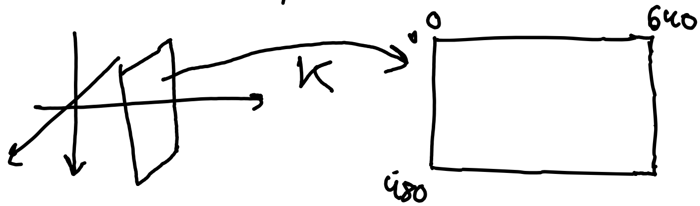
Also called homography

Ex 1: Show that it does not matter what representative we choose, the result will be the same (Hint: y and λy are two representatives of the same point

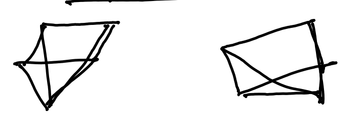
$$H(\lambda \mathbf{x}) = \lambda H \mathbf{x} \sim \lambda \mathbf{y} \sim \mathbf{y}$$

Examples

- K matrix, $x \sim K[R\ t]\,X$



- Projections of the same plane.
- Projections of a 3D plane
  $\Rightarrow$ There is a homography between the images



- Cameras with the same camera center.

Remark: Vanishing points can be mapped to regular points

$$\mathbb{y} \sim H\mathbb{x} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ h_{31}x + h_{32}y + h_{33}z \end{pmatrix}$$

$\mathbb{y}$ is a vanishing point iff:

$$h_{31}x + h_{32}y + h_{33}z = 0 \quad \text{(a line in } \mathbb{P}^2\text{)}$$

that is, iff. x is on the line

$$l \sim (h_{31}x + h_{32}y + h_{33}z)$$

Ex 2: Assume that x lies on the line l, that is, $l^T x = 0$, and that $y \sim Hx$. Show that y lies on the line $\hat{l} = (H^{-1})^T l$

$$0 = l^T x = \underbrace{l^T H^{-1}}_{\text{Target line}} \underbrace{H x}_{y} \sim \underbrace{(H^{-T} l)^T}_{(AB)^T = A^T B^T} y = \hat{l}^T y$$

Projective transformations: Special cases

Affine Transformation ($\mathbb{P}^n \to \mathbb{P}^n$):

$$H = \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} A - n \times n \text{ (invertible) and} \\ t - n \times 1 \end{cases}$$

- Parallel lines are mapped to parallel lines
- Preserves the line at infinity (points at infinity are mapped to points at infinity, and regular points are mapped to regular points)
- Can be written $y = Ax + t$ for points in $\mathbb{R}^n$

Similarity Transformations ($\mathbb{P}^n \to \mathbb{P}^n$):

$$H = \begin{pmatrix} sR & t \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} R - n\times n \text{ rotation} \\ t - n\times 1 \\ s - \text{positive number} \end{cases}$$

- Special case of affine transformation
- Preserves angles between lines.

Euclidian Transformation (Rigid body motion $\mathbb{P}^n \to \mathbb{P}^n$):

$$H = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}, \quad \begin{matrix} R - \text{rotation},\ n\times n \\ t - n\times 1 \end{matrix}$$

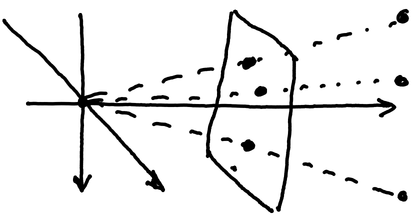
- Special case of similarity
- Peserves distances

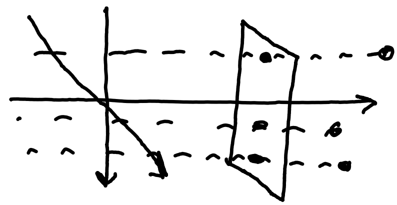
Ex 3: Assume that y lies on the conic C, that is, $y^T C y = 0$ and that $x \sim Hy$. Show that x lies on the conic

$$\hat{C} = (H^{-1})^T C H^{-1}$$

$$0 = y^T C y \sim (H^{-1} x)^T C H x = x^T \underbrace{H^{-T} C H^{-1}}_{\tilde{C}} x$$

Affine Cameras and Parallel Projection:

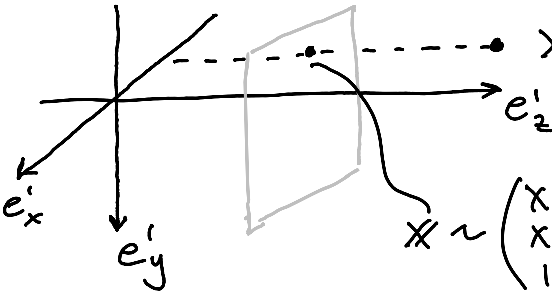




Parallel projection

Viewing rays do not intersect a common point
Incomming rays are orthogonal to the im. plane $z=1$. (z-coord is replaced by 1).

Camera Coord. sys:



$$X = \begin{pmatrix} X'_1 \\ X'_2 \\ X'_3 \end{pmatrix}$$

$$\mathbb{X} \sim \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

Projection along z-axis: $x_1 = X'_1$, $x_2 = X'_2$, $x_3 = 1$

In matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} X'_1 \\ X'_2 \\ X'_3 \\ 1 \end{pmatrix}}_{\mathbb{X}'} = \underbrace{\begin{pmatrix} I_{2\times 3} & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{3\times 4 \\ \text{(projection)}}} \mathbb{X}'$$

From world coord -> Image plane

$$\mathbb{X}' \begin{pmatrix} I_{2\times 3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \mathbb{X} = \begin{pmatrix} R_{2\times 3} & t_{2\times 1} \\ 0 & 1 \end{pmatrix}$$

From world coord -> real image (pixels)

$$\mathbb{X} \sim k \begin{pmatrix} R_{2\times 3} & t_{2\times 1} \\ 0 & 1 \end{pmatrix} \mathbb{X}$$

If $k = I$ then:

$$P = \begin{pmatrix} R_{2\times 3} & t_{2\times 1} \\ 0 & 1 \end{pmatrix}$$

orthographic projection

if $K = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $P = \begin{pmatrix} sR_{2\times3} & t_{2\times1} \\ 0 & 1 \end{pmatrix}$

scaled orthographic (weak perspective)

If $K = \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} A_{2\times3} & t_{2\times1} \\ 0 & 1 \end{pmatrix}$ general affine camera

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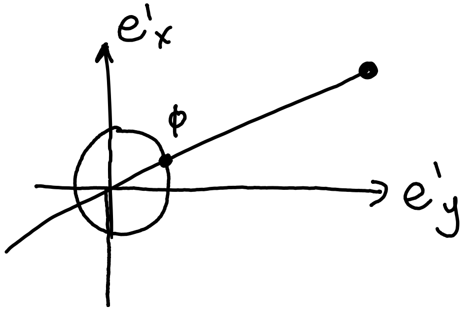
<u>EX</u> 4: Compute the camera center (null space) of

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$0 = P\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \begin{cases} X = 0 \\ 2Y = 0 \\ W = 0 \\ Z = t \end{cases} \Rightarrow C \sim \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

1D camera:



Can be seen as projection $\mathbb{P}^2 \to \mathbb{P}^1$

$$\lambda x = \underbrace{K_{2\times 2} \begin{pmatrix} R_{2\times 2} & t_{2\times 1} \end{pmatrix}}_{P_{2\times 3}} X$$

We typically don't divide by last coord, but compute ϕ:

$$\lambda \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} = P_{2\times 3} X$$

<u>Ex 5</u>: The camera $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are observing the point X at angles $\phi_1 = \pi/4$ and $\phi_2 = 0$ resp. What is the position of the point X?

$$x_1 = \begin{pmatrix} \sin \pi/4 \\ \cos \pi/4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} \sin 0 \\ \cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{cases} \Leftrightarrow \begin{cases} x = \lambda_1 \\ y = \lambda_1 \\ x - y = 0 \\ y = \lambda_2 \end{cases}$$