Lecture 2

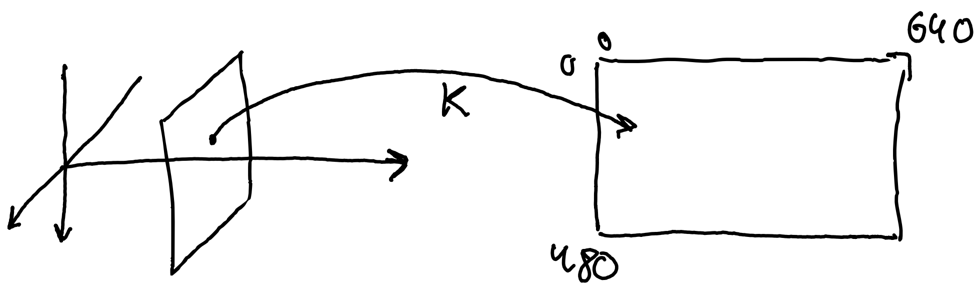
The camera equations:

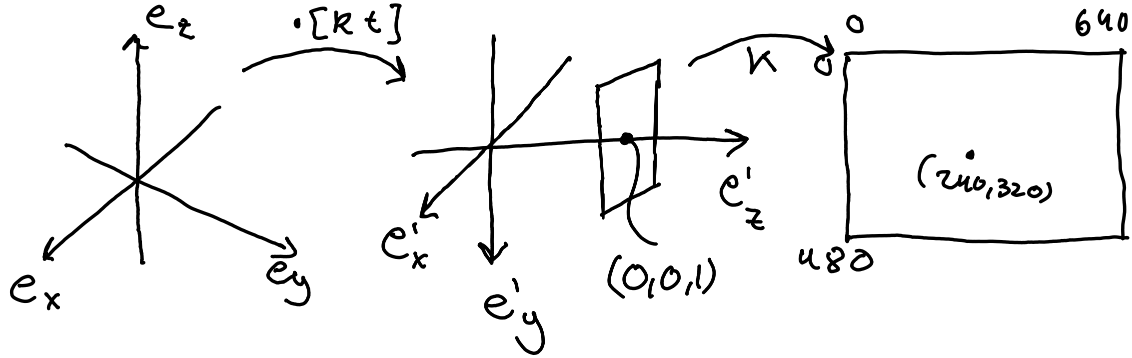
$$x_3' \begin{pmatrix} x_1'/x_3' \\ x_2'/x_3' \\ 1 \end{pmatrix} = \begin{bmatrix} R & t \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

Algorithm:

- Comp $v = \begin{bmatrix} R & t \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$
- Divide v by its third coordinate

The Intrinsic Parameters:





K: 3x3 upper triangular, invertable, $k_{33}=1$

$$K = \begin{pmatrix} \gamma f & sf & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

K assumed to have positive diagonal elements

$\Rightarrow \det(K) > 0$

- f - focal length
- (x_0, y_0) - principal point (middle of the im.)
- γ - aspect ratio (typically 1)
- s - skew (typically 0)

Projection:

camera equation.

$$\lambda \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}}_{\mathbf{x}} = \underbrace{K[R\ \ t]}_{P} \underbrace{\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{pmatrix}}_{\mathbf{X}} \Big\} X \Rightarrow \lambda \mathbf{x} = P\mathbf{X}$$

Algorithm:

1. compute $v = P\mathbb{X}$

2. Divide by its third coord.

$$K\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma f & sf & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ z \end{pmatrix}$$

Last coord. is retained.

Camera center: $0 = RC + t = \begin{bmatrix} R & t \end{bmatrix}\begin{bmatrix} C \\ 1 \end{bmatrix} \Leftrightarrow$

$$0 = K\begin{bmatrix} R & t \end{bmatrix}\underbrace{\begin{bmatrix} C \\ 1 \end{bmatrix}}_{\mathbb{C}} \Leftrightarrow 0 = P\mathbb{C}$$

$\mathbb{C}$ is in the nullspace of P.

Ex1: compute the projection of $X=(0,0,1)$ in the cameras

$$P_1 = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 1 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \\ -1 & 0 & -1 & -\sqrt{2} \end{pmatrix}$$

What is the center of the cameras?

$$P_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2}+1 \end{pmatrix} \qquad \text{Projection: } \left(\frac{-1}{1+\sqrt{2}}, 0\right)$$

add a one

same

$$P_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{\sqrt{2}}-1 \end{pmatrix} \qquad \text{Projection: } \left(\frac{-1}{1+\sqrt{2}}, 0\right)$$

Camera center:

$$0 = P_1 C \iff P_1 \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 0 \iff \begin{cases} \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} z = 0 \\ y = 0 \\ \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} z + w = 0 \end{cases}$$

$$\begin{cases} x = z \\ y = 0 \\ 2x + \sqrt{2} w = 0 \end{cases} \qquad \begin{cases} x \ \ 1/\sqrt{2} \cdot t \\ y = 0 \\ z = 1/\sqrt{2} \cdot t \\ w = t \end{cases} \Rightarrow C = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 1 \end{pmatrix}$$

center: $w=1$

Ex 1:

$P_2 = \sqrt{2}\, P_1$

$P_1 C = 0 \Rightarrow P_2 C = 0$

$$C = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 1 \end{pmatrix}$$

if $P_1 = s$ than P_1 and P_2 represent the Same camera.

$\uparrow$
scalar $\neq 0$

We write $P_1 \sim P_2$

Projective Spaces:

We say that $u \sim v$ if $u = \lambda v$, $\lambda \neq 0$

$\mathbb{P}^2$ = "All elements that can be represented with vectors in $\mathbb{R}^3 \setminus \{0\}$

A regular point in 2D is represented with a 3D vector.
Its regular coord. are comp by division w/ third coord.

$\mathbb{P}^n$ = " All elem. that can be represented w/ vectors in $\mathbb{R}^{n+1} \setminus \{0\}$."

Homogeneous Coords.

$v = (3,2,1)$ gives $(3,2)$ $\mathbb{R}^2$

$v = (6,4,2)$ gives $(3,2)$ $\mathbb{R}^2$

We say that $(\lambda x, \lambda y, \lambda) \sim (x, y, 1)$ for $\lambda \neq 0$

$\mathbb{P}^2$ = all "elements" that can be represented w/ vectors in $\mathbb{R}^3 \setminus \{\bar{0}\}$

$\mathbb{P}^3$ = —— " —— $\mathbb{R}^4 \setminus \{\bar{0}\}$

Ex: Both vectors $(3,2,1)$ and $(6,4,2)$ represents the same element.

Ex: $\underbrace{v}_{\in \mathbb{P}^2} = [R \;\; t] \underbrace{\begin{bmatrix} X \\ 1 \end{bmatrix}}_{\in \mathbb{P}^3}$

$$\begin{bmatrix} \lambda X \\ \lambda \end{bmatrix} \sim \begin{bmatrix} X \\ 1 \end{bmatrix}$$

$$[R \;\; t] \begin{bmatrix} \lambda X \\ \lambda \end{bmatrix} = \lambda [R \;\; t] \begin{bmatrix} X \\ 1 \end{bmatrix} = \lambda v \sim v$$

In $\mathbb{P}^2$ "are dual to lines

The point $x \sim (x, y, z)$ is on the line $l \sim (a, b, c)$ if

$$ax + by + cz = 0$$

In $\mathbb{P}^3$ "points are dual to planes"

<u>Lines in $\mathbb{P}^2$:</u> In $\mathbb{R}^2$ (x, y) is on the line $l : (a, b, c)$ if $ax + by + c = 0$

In $\mathbb{P}^3$ $x \sim (x, y, z)$, $x \sim (\frac{x}{z}, \frac{y}{z}, 1)$

$$0 = a\frac{x}{z} + b\frac{x}{z} + c \Leftrightarrow 0 = ax + by + cz$$

$x \sim (\lambda x, \lambda y, \lambda z)$

$$0 = a(\lambda x) + b(\lambda y) + c(\lambda z) \overset{\lambda \neq 0}{\Leftrightarrow} 0 = ax + by + cz$$

$x \sim (x, y, z)$, $l \sim (a, b, c)$

In $\mathbb{P}^2$ lines are dual to lines

$\mathbb{R}^3$ $ax + by + cz + d = 0$ is a plane

In $\mathbb{P}^3$ planes and points are dual

Ex2: compute the point of intersection $x \in \mathbb{P}^2 (x \sim (x, y, z))$ of the two lines $l_1 \sim (-1, 1, 0)$ and $l_2 \sim (0, -1, 1)$.

$$\begin{cases} l_1^T x = 0 \\ l_2^T x = 0 \end{cases} \Leftrightarrow \begin{cases} -x + z = 0 \\ -y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

$x \sim (1, 1, 1)$ $\quad ((1, 1) \text{ in } \mathbb{R}^2)$

Ex 3: Comp. the line $l \sim (a, b, c)$ passing through the points $x_1 \sim (-1, 0, 1)$ and $x_2 \sim (0, -1, 1)$. $\quad l \sim (a, b, c)$

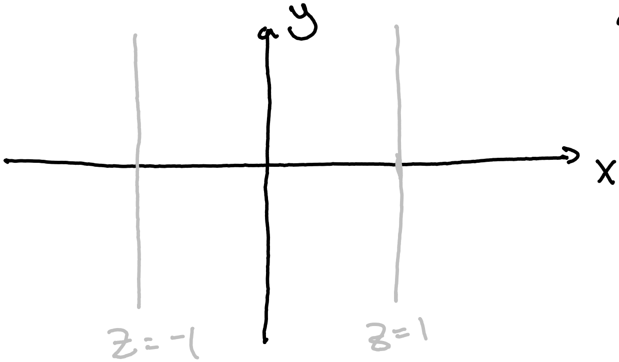
$$\begin{cases} l^T x_1 = 0 \\ l^T x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} -a + c = 0 \\ -b + c = 0 \\ c = t \end{cases} \Leftrightarrow \begin{cases} a = t \\ b = t \\ c = t \end{cases}$$

$l \sim (1, 1, 1)$

What is the intersection between the two lines $l_1 \sim (-1,0,1)$, $l_2 \sim (1,0,1)$?

$$\begin{cases} l_1^T \mathbf{x} = 0 \\ l_2^T \mathbf{x} = 0 \end{cases} \Leftrightarrow \begin{cases} -x + z \\ x + z \end{cases} \overset{+}{\Leftrightarrow} \begin{cases} -x + z = 0 \\ 2z = 0 \Leftrightarrow \\ y = t \end{cases} \begin{cases} x = 0 \\ y = t \\ z = 0 \end{cases}$$

$$(x,y,z) \sim (0,1,0)$$



$l_1 : -x + \overset{=1}{z} = 0$

$x = 1$

$l_2 : -x + \overset{=-1}{z} = 0$

$x = -1$

$l_1 \parallel l_2$, what is the intersection?

$$(0,1,\varepsilon) \sim (0, 1/\varepsilon, 1) \Longleftrightarrow \left(0, \tfrac{1}{\varepsilon}\right) \text{ in } \mathbb{R}^2$$

$\varepsilon > 0$, small number

x coord.

large y coord. (far away)

$(0,1,0)$: point infinitly far away in the $(0,1)$ direction

$\{(x,y,z) \mid z=0\}$: line at infinity

Called vanishing points / points at infinity

Conics:

Conics are curves of the form

$$x^T C x = 0$$

Where C is sym. matrix.

Ex: Circle of radius 1.

$$(x \; y \; 1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = x^2 + y^2 - 1 = 0$$

Similar objects in 3D are called quadrics.

$I = Cx$ is a tangent line to the conic at x.

The dual conic $C^* =$ is the set of all tangents. If the C is invertible.

$$0 = x^T C x = x^T C C^{-1} C x = (Cx)^T C^{-1} C x =$$

$$= I^T C^{-1} I$$