Ex 2. $l_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad l_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{cases} l_1^T x = 0 \\ l_2^T x = 0 \end{cases} \Leftrightarrow \begin{cases} x + y + z = 0 \\ 3x + 2y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x + y + z = 0 \\ 2x + y = 0 \\ z = t \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2x + z = 0 \\ y = -2x \\ z = t \end{cases} \Leftrightarrow \begin{cases} x = z \\ y = -2x \\ z = t \end{cases} \Leftrightarrow \begin{cases} x = t \\ y = -2t \\ z = t \end{cases}$$

Intersection $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Corresponding point in $\mathbb{R}^2$ (by dividing with 3rd coord) is $(1, -2)$

$l_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad l_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{cases} l_3^T x = 0 \\ l_4^T x = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y + 3z = 0 \\ x + 2y + z = 0 \end{cases} \Leftrightarrow \begin{cases} 2z = 0 \\ x + 2y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -2t \\ y = t \\ z = 0 \end{cases}$$

Intersection $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

The Geometric interpretation is that l_3 and l_4 does not intersect in $\mathbb{R}^2$

$$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

l_1 and l_2 can be seen as homogenous coordinates to X_1 and X_2 due to duality.

The line l will hence be the intersection of l_1 and l_3.

From earlier calculations: $l = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Ex. 3 When solving for the intersection point we look for a point that satisfies the equation of both lines.

A line in $\mathbb{P}^2$ has the equation $ax + by + cz = 0$

$$\Rightarrow \begin{cases} l_1: 3x + 2y + z = 0 \\ l_2: x + y + z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Therfore the solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the null space to the matrix. The intersection is then in the null space.

Null space: $t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is a line of points in $\mathbb{P}^2$. So there is an infinitely amount of points.