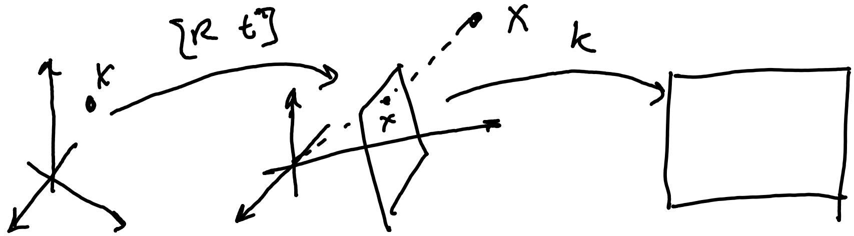
Lecture 4:

Repitition

The camera equation: $\lambda \mathbf{x} = \underbrace{K[R \;\; t]}_{P} \mathbf{X}$

K - intrinsic parameters

R, t - extrinsic parameters



The structure for motion problem:

solve the cam eq: $\lambda_{ij} \mathbf{x}_{ij} = P_i \mathbf{X}, \forall_{ij}$

Find both camera matricies P_i, and 3D points X_j!

Two versions:

- Projective Reconstruction: Nothing is known about P_i
- Euclidean —"— : $P_i = K_i[R_i \;\; t_i]$, when K_i is known

Projective VS. Euclidian Reconstruction.

Projective: The instruction is determined up to a projective transformation, If $\lambda \mathbb{x} = P \mathbb{X}$, then for any projective transformation

$$\tilde{\mathbb{X}} = H^{-1} \mathbb{X}$$

H is a projective transformation

we have

$$\lambda \mathbb{x} = P \underbrace{H H^{-1}}_{=I} \overbrace{\mathbb{X}}^{\text{new 3D-points.}} = PH \tilde{\mathbb{X}}$$

By multiplying with an abritrary transformation H Inf. many sol. can be created

PH is also valid camera.

Calibrated cameras: A camera : $P = K[R \;\; t]$

where the inner parameters K are known is called calibrated. If we change coordinates in the image using.

$$\tilde{x} = K^{-1} x$$

we get a so called normalized (calibrated) camera

$$\tilde{x} \sim K^{-1} K [R \;\; t] X = [R \;\; t] X$$

Euclidian: The reconstruction is determined up to a similarity transformation. If $\lambda \mathbb{x} = [R \; t] \mathbb{X}$ then for any similarity transformation

$$\tilde{X} = H^{-1} \mathbb{X} = \begin{pmatrix} sQ & v \\ 0 & 1 \end{pmatrix}^{-1} \mathbb{X}$$

We have:

$$\frac{\lambda}{s} x = [R \;\; t] \begin{bmatrix} Q & v \\ 0 & \frac{1}{s} \end{bmatrix} \tilde{X} = [RQ \;\; Rv + \frac{t}{s}] \tilde{X}$$

Since RQ is a rotation this is an normalized camera.

The projective coord. system makes thing look strange

The inner parameters - K:

$$K = \begin{pmatrix} \gamma f & sf & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

f - focal length
γ - aspect ratio
s - skew
(x_0, y_0) - principal points.

The focal length f:

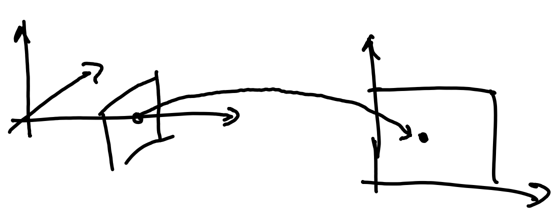
$$\begin{pmatrix} fx \\ fy \\ 1 \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Re scales the image (e.g. meters $\rightarrow$ pixels)

The principal point:

$$\begin{pmatrix} fx + x_0 \\ fy + y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Recenters the image. Typically transforms the point (0,0,1) to the middle of the image.



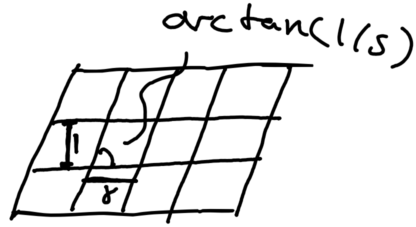
Aspect ratio:

$$\begin{pmatrix} \gamma f x + x_0 \\ f y + y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Pixels are not always square but can be rectangular. In such cases the scaling in the x-dir should be different from the y-dir.

Skew:

$$\begin{pmatrix} \gamma f & s f & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



Corrects for tilted pixels. Typically 0.

Finding K:

1. Solve the resection problem:

   Find P from the camera equations:

   $$\lambda_i \mathbf{x}_i = P_i \mathbf{X}_i$$

   where both $\mathbf{x}_i$ and $\mathbf{X}_i$ are known $\forall i$
   (structure from motion with known 3D points)

2. Use RQ-factorization to extract K from P

RQ-factorization:

<u>theorem:</u> If A is an $n \times n$ matrix then there is an orthogonal Matrix Q and a right triangular matrix R such that $A = RQ$

(If A is invertible the diagonal elements are chosen the (to?) be positive, then the factorization is unique.)

<u>Note:</u> In our case we will use K for the triangular Matrix and R for the rotation.

$$\begin{pmatrix} A_1^T \\ A_2^T \\ A_3^T \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} R_1^T \\ R_2^T \\ R_3^T \end{pmatrix} = \begin{pmatrix} aR_1^T + bR_2^T + cR_3^T \\ dR_2^T + eR_3^T \\ fR_3^T \end{pmatrix}$$

<u>Algo:</u> (Gram-Schmidt)

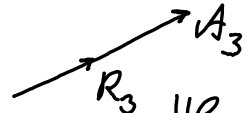
- Determine f and R_3 from last eq $A_3 = fR_3$
- Divide A_2 into two perpendicular components eR_3 (projection onto R_3) and $dR_2 = A_2 - eR_3$
- Divide A_1 into three perpendicular components cR_3 (proj. onto R_3), bR_2 (projection onto R_2) and $aR_1 = A_1 - cR_3 - bR_2$

Ex 1:

If $P = \begin{pmatrix} 3000 & 0 & -1000 & 1 \\ 1000 & 2000\sqrt{2} & 1000 & 2 \\ 2 & 0 & 2 & 3 \end{pmatrix}$

find f and R_3

$A_3^T = f R_3^T$, $A_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$



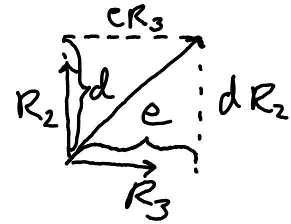
$\|R_3\| = 1$

$\|A_3\| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$$A_3 = f R_3^T = \underbrace{2\sqrt{2}}_{f} \cdot \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{R_3}$$

Ex 2: Determine e, R_2 and d for P in Ex 1

$A_2^T = d R_2^T + e R_3^T$, $1000 \begin{pmatrix} 1 \\ 2\sqrt{2} \\ 1 \end{pmatrix} = A_2$



$$e = A_2^T R_3 = 1000 \cdot (1 \;\; 2\cdot\sqrt{2} \;\; 1) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \frac{1000}{\sqrt{2}} 2 = 1000\sqrt{2}$$

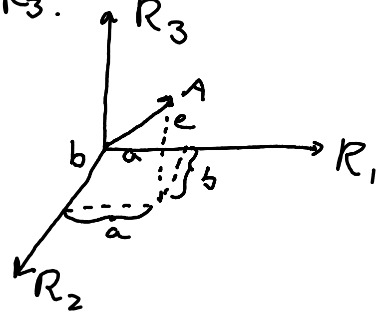
$$dR_2 = A_2 - eR_3 = 1000 \begin{pmatrix} 1 \\ 2\sqrt{2} \\ 1 \end{pmatrix} - 1000 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \underbrace{2000\sqrt{2}}_{d} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{R_2}$$

$d = 2000\sqrt{2}$

Ex 3:

$$A_1^T = aR_1^T + bR_2^T + cR_3^T \quad , \quad A_1 = 1000\begin{pmatrix}3\\0\\-1\end{pmatrix}$$

Projection of A_1 onto R_2 and R_3.



$$c = A_1^T R_3 = 1000(3 \;\; 0 \;\; -1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} =$$

$$= \frac{1000}{\sqrt{2}}\, 2 = 1000\sqrt{2}$$

$$b = A_1^T R_2 = 1000(3 \;\; 0 \;\; -1)\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$$

$$aR_1 = A_1 - cR_3 = 1000\begin{pmatrix}3\\0\\-1\end{pmatrix} - 1000\cancel{\sqrt{2}}\frac{1}{\cancel{\sqrt{2}}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$= 1000\begin{pmatrix}2\\0\\-2\end{pmatrix} = \underbrace{2000}_{a}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}}_{R_1}$$

The RQ-factorization of A is

$$A = \begin{pmatrix} 2000\sqrt{2} & 0 & 1000\sqrt{2} \\ 0 & 2000\sqrt{2} & 1000\sqrt{2} \\ 0 & 0 & 2\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

should be 1
i.e. divided by this number

$$f = \frac{2000\sqrt{2}}{2\sqrt{2}} = 1000$$

$$x_0 = y_0 = \frac{1000\sqrt{2}}{2\sqrt{2}} = 500$$

$$\gamma = 1 \;,\quad s = 0$$

Remark: $\det(K) = \gamma f f 1 > 0$
if $\det(A) < 0$ then $\det(R) < 0$
$\Rightarrow R$ is orthogonal but not a rotation.
Switch from P to $-P$

Direct linear Transform: (DLT)

Finding the camera matrix: (The Resection Problem)

Use images of a known object to eliminate the projective ambiguity. If $\mathbb{X}_i$ are 3d-points of a known Object, and x_i corresponding projections we have

$$\begin{aligned} \lambda_1 \mathbb{x}_1 &= P \mathbb{X}_1 \\ \lambda_2 \mathbb{x}_2 &= P \mathbb{X}_2 \\ &\vdots \\ \lambda \mathbb{x}_n &= P \mathbb{X}_n \end{aligned}$$

There are $3N$ equations and $11+N$ unknowns. We need $3N \geq 11+N \Rightarrow N \geq 6$ points to solve the problem.

Matrix Formulation:

$$P = \begin{pmatrix} p_1^T \\ p_2^T \\ p_3^T \end{pmatrix}$$ where p_i are the rows of P.

$$\begin{pmatrix} p_1^T \\ p_2^T \\ p_3^T \end{pmatrix} \mathbb{X} = \begin{pmatrix} p_1^T \mathbb{X} \\ p_2^T \mathbb{X} \\ p_3^T \mathbb{X} \end{pmatrix}$$

The first equality is:

$$\begin{aligned} \mathbb{X}_1^T p_1 - \lambda_1 x_1 &= 0 \\ \mathbb{X}_1^T p_2 - \lambda_1 y_1 &= 0 \\ \mathbb{X}_1^T p_3 - \lambda_1 &= 0 \end{aligned}$$

When $\mathbb{x}_1 = (x_1, y_1; 1)$, in matrix form:

$$\begin{pmatrix} \mathbb{X}_1^T & 0 & 0 & -x_1 \\ 0 & \mathbb{X}^T & 0 & -y_1 \\ 0 & 0 & \mathbb{X}^T & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \lambda_1 \end{pmatrix}$$

Homogeneous Least square:

Want to find a solution to : $Mv = 0$

To avoid trivial sol. $v = 0$ Use constraint $\|v\| = 1$

No exact ...

Ex 4: $M = USV^T$ where

$$S = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$$

find a vector of length 1 that minimizes $\|Mv\|$ and the minimal value

$$M = USV^T$$

$$\|Mv\|^2 = v^T M^T M v = v^T (USV^T)^T (USV^T) v =$$

$$= \underbrace{v^T (V}_{w} S^T \underbrace{U^T U}_{I} S \underbrace{V^T v}_{w}) = w^T S^T S w$$

$$1 = \|v\|^2 = v^T v = w^T \underbrace{V^T V}_{=I} w = \|w\|^2 \quad \begin{cases} w = V^T v \quad (2) \\ Vw = v \end{cases}$$

$$Sw = \begin{pmatrix} 3 & & \\ & 2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3w_1 & & \\ & 2w_2 & \\ & & w_3 \end{pmatrix}$$

$$\|Sw\|^2 = 2w_1^2 + 4w_2^2 + w_3^2$$

$$1 = \|w\|^2 = w_1^2 + w_2^2 + w_3^2$$

$$w = (0, 0, 1)$$

last column of V

$$v = Vw = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$