补：外微分与场论公式.

$\cdot\mathbb{R}^3$ 取直角坐标系 (x^1, x^2, x^3)

(每点处看) 维数 微分形式

$\binom{3}{0}=1$ 0阶 f $\longleftrightarrow$ f 标量场

$\binom{3}{1}=3$ 1阶 $\omega_A^1 = A^1 dx^1 + A^2 dx^2 + A^3 dx^3$ $\longleftrightarrow$ $A=(A^1, A^2, A^3)$ 向量场

$\binom{3}{2}=3$ 2阶 $\omega_B^2 = B^1 dx^2\wedge dx^3 + B^2 dx^3\wedge dx^1 + B^3 dx^1\wedge dx^2$ $\longleftrightarrow$ $B=(B^1, B^2, B^3)$ 向量场

$\binom{3}{3}=1$ 3阶 $\omega_f^3 = f\, dx^1\wedge dx^2\wedge dx^3$ $\longleftrightarrow$ f 标量场 (数量场)

$$f \xrightarrow{d} df = \sum_{i=1}^{3}\frac{\partial f}{\partial x^i}dx^i \ (1\text{-form}) \longleftrightarrow \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\right) = \nabla f$$

$$= \omega^1_{\mathrm{grad} f} \qquad = (\mathrm{grad} f)$$

$$d\omega_A^1 = \left(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}\right)dx^2\wedge dx^3 + \left(\frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}\right)dx^3\wedge dx^1 + \left(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right)dx^1\wedge dx^2 = \omega^2_{\mathrm{rot} A}$$

$$\longleftrightarrow \left(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}, \frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}, \frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right) = \mathrm{rot} A (= \mathrm{curl} A)$$

$$d\omega_B^2 = \left(\frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3}\right)dx^1\wedge dx^2\wedge dx^3 \longleftrightarrow \frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3} = \mathrm{div} B$$

$$= \omega^3_{\mathrm{div} B}$$

应用: $\mathrm{rot}(\mathrm{grad} f) = 0 \Longleftarrow \boxed{d^2=0}$ $\qquad f \xrightarrow{d} \omega^1_{\nabla f} \xrightarrow{d} \underbrace{\omega^2_{\mathrm{rot}(\nabla f)}}_{=0} = 0$

$\mathrm{div}(\mathrm{rot} A) = 0$ $\qquad \omega_A^1 \xrightarrow{d} \omega^2_{\mathrm{rot} A} \xrightarrow{d} \underbrace{\omega^3_{\mathrm{div}(\mathrm{rot} A)}}_{=0} = 0$

可证: $\mathrm{rot}(fA) = f\,\mathrm{rot} A + \nabla f \times A$

$\mathrm{div}(fA) = f\,\mathrm{div} A + \nabla f\cdot A$

$\mathrm{div}(A\times B) = B\cdot \mathrm{rot} A - A\cdot \mathrm{rot} B$

用"d与$\wedge$"与对应关系

其他对应 ① $\omega_A^1\wedge\omega_B^2 = (A^1B^1 + A^2B^2 + A^3B^3)\, dx^1\wedge dx^2\wedge dx^3$ (3-form)

$= \omega^3_{\langle A,B\rangle}$

$\updownarrow$

$A = (A^1, A^2, A^3)$

$B = (B^1, B^2, B^3)$ $\qquad \langle A, B\rangle_{\mathbb{R}^3}$

$\updownarrow$

② (2-form) $\omega_A^1\wedge\omega_B^1 = (A^2B^3 - A^3B^2)\,dx^2\wedge dx^3 + (A^3B^1 - A^1B^3)\,dx^3\wedge dx^1 + (A^1B^2 - B^1A^2)\,dx^1\wedge dx^2 = \omega^2_{A\times B}$

$$A\times B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{vmatrix}$$

卓里奇《数学分析》

第2卷 12章~15章

(Stokes)

· 一般的斯托克斯公式

定理：$S\subset\mathbb{R}^n$，S为k维(光滑)可定向（紧致）曲面，

$\underbrace{\partial S}_{\text{边界(为(k-1)维曲面)}}$ 诱导定向　　$S\subset G$，G为$\mathbb{R}^n$中的开区域.

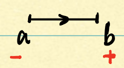
ω是G上的(光滑)$(k-1)$形式.

则

$$\boxed{\int_{\partial S}\omega=\int_S d\omega}$$

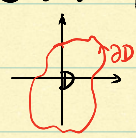
特别地，$\partial S=\phi$，则 $\int_S d\omega=0$.

例：① $I=\overset{1维}{[a,b]}$，$\partial I=\overset{0维}{\{a,b\}}$ $\Rightarrow$ $\int_a^b df=+f(b)-f(a)$



f (0-形式) 函数.　　(Newton-Leibniz)

② D为$\mathbb{R}^2$中区域，$\bar{D}$紧，∂D光滑.

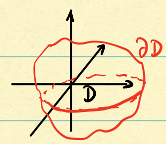


$\omega=P\,dx+Q\,dy$

$$\int_{\partial D}P\,dx+Q\,dy=\left(\int_D d\omega\right)=\int_D\left(-\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right)dx\wedge dy$$

(Green公式)

③ D为$\mathbb{R}^3$中的区域，$\bar{D}$紧，∂D光滑.

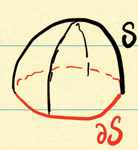


$\omega=P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy$

$$\int_{\partial D}P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy\left(=\int_D d\omega\right)$$

$$=\int_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\wedge dy\wedge dz$$　(Gauss-Ostrogradskii公式)

④ S为$\mathbb{R}^3$中(2维)曲面，∂S.



$\omega=A\,dx+B\,dy+C\,dz$

$$\int_{\partial S}A\,dx+B\,dy+C\,dz=\left(\int_S d\omega\right)=\int_S\left(\frac{\partial C}{\partial y}-\frac{\partial B}{\partial z}\right)dy\wedge dz+\left(\frac{\partial A}{\partial z}-\frac{\partial C}{\partial x}\right)dz\wedge dx+\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)dx\wedge dy$$

($\mathbb{R}^3$中的Stokes公式)