

# Riemannian Multinomial Logistics Regression for SPD Neural Networks

– CVPR 2024 –

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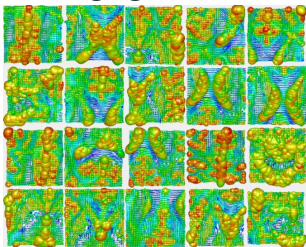


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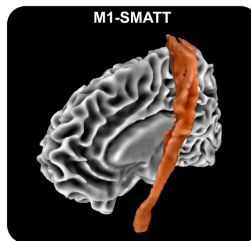
# Applications of SPD Manifolds

Image generation



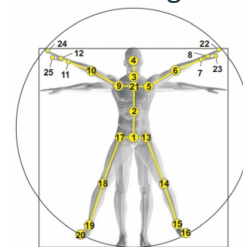
Huang et al., 2019

Medical Imaging



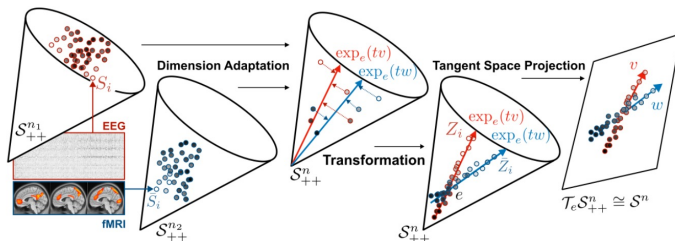
Chakraborty et al., 2020

Action Recognition



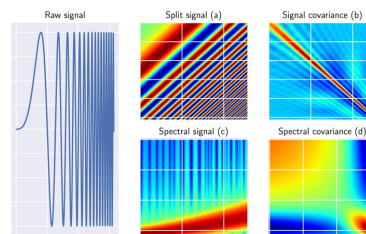
Nguyen, Xuan Son 2021

Brain-Computer Interfaces



Ju et al., 2024

Radar Classification



Brooks et al., 2020

Huang, Zhiwu, Jiqing Wu, and Luc Van Gool. "Manifold-valued image generation with wasserstein generative adversarial nets." AAAI, 2019.

Chakraborty, Rudrasis, et al. "Manifoldnet: A deep neural network for manifold-valued data with applications." IEEE-TPAMI, 2020.

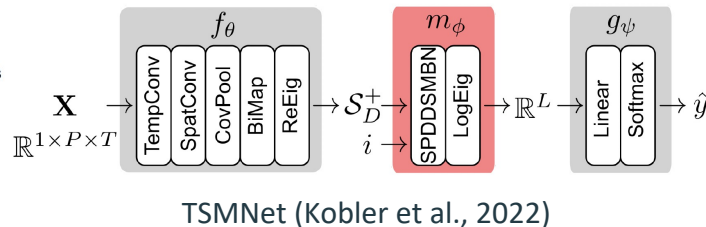
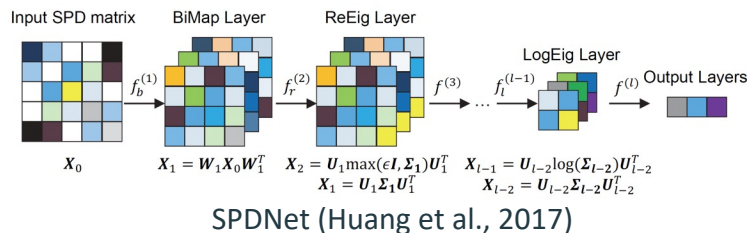
Ju, Ce, et al. "Deep geodesic canonical correlation analysis for covariance-based neuroimaging data." ICLR, 2024.

Nguyen, Xuan Son. "Geomnet: A neural network based on riemannian geometries of SPD matrix space and cholesky space for 3D skeleton-based interaction recognition." ICCV, 2021.

Brooks, Daniel, et al. "Deep learning and information geometry for drone micro-Doppler radar classification." RadarConf, 2020.

# Classification on SPD Neural Networks

Tangent Space:



Parameterization:

$$\begin{aligned}
 Y_t &= \text{FM} \left( \left\{ M_{t-1}^{(\alpha)} \right\}, \left\{ w^{(y, \alpha)} \right\} \right), & R_t &= \mathbb{T} \left( Y_t, g^{(r)} \right) \\
 T_t &= \text{FM} \left( \left\{ R_t, X_t \right\}, w^{(t)} \right), & \Phi_t &= \mathbb{T} \left( T_t, g^{(p)} \right) \\
 \forall \alpha \in J, & & M_t^{(\alpha)} &= \text{FM} \left( \left\{ M_{t-1}^{(\alpha)}, \Phi_t \right\}, \alpha \right) \\
 S_t &= \text{FM} \left( \left\{ M_t^{(\alpha)} \right\}, \left\{ w^{(s, \alpha)} \right\} \right), & O_t &= \text{Chol} \left( \text{ReLU} \left( \text{Chol} \left( \mathbb{T} \left( S_t, g^{(y)} \right) \right) \right) \right)
 \end{aligned}$$

SPD-SRU (Chakraborty et al., 2018)

They rely on approximated spaces



How to build intrinsic classification layers on manifolds?

Huang, Zhiwu, and Luc Van Gool. "A riemannian network for spd matrix learning." AAAI, 2017.

Kobler, Reinmar, et al. "SPD domain-specific batch normalization to crack interpretable unsupervised domain adaptation in EEG." Neurips, 2022.

Chakraborty, Rudrasis, et al. "A statistical recurrent model on the manifold of symmetric positive definite matrices." Neurips, 2018.

## SPDMLR:

- A general framework for SPD Multinomial Logistics Regression (MLR) under PEMs
- Specific SPD MLRs under parameterized LCM and LEM
- An intrinsic theoretical explanation of the most popular LogEig classifier

$$\forall k \in \{1, \dots, C\}, p(y = k \mid x) \propto \exp((\langle a_k, x \rangle - b_k))$$

## Euclidean MLR:

Reformulation into margin distance  
to hyperplane



$$p(y = k \mid x) \propto \exp(\text{sign}(\langle a_k, x - p_k \rangle) \|a_k\| d(x, H_{a_k, p_k}))$$

$$H_{a_k, p_k} = \{x \in \mathbb{R}^n : \langle a_k, x - p_k \rangle = 0\}$$

## Gyro SPD MLR:

- Requires gyro vector structures
- Relies on gyro distance, instead of geodesic distance
- Solves formulation case by case

## Our SPD MLR:

- Focus on Pullback Euclidean Metrics (PEMs)
- Only needs Riemannian geometry
- Relies on geodesic distance
- Proposes a general formulation for PEMs

## From Euclidean to SPD

$$p(y = k \mid x) \propto \exp(\text{sign}(\langle a_k, x - p_k \rangle) \|a_k\| d(x, H_{a_k, p_k})) \quad H_{a_k, p_k} = \{x \in \mathbb{R}^n : \langle a_k, x - p_k \rangle = 0\}$$

Riemannian reformulation



**Definition 3.1** (SPD hyperplanes). Given  $P \in \mathcal{S}_{++}^n$ ,  $A \in T_P \mathcal{S}_{++}^n \setminus \{0\}$ , we define the SPD hyperplane as

$$\tilde{H}_{A,P} = \{S \in \mathcal{S}_{++}^n : g_P(\text{Log}_P S, A) = \langle \text{Log}_P S, A \rangle_P = 0\}, \quad (12)$$

where  $P$  and  $A$  are referred to as shift and normal matrices, respectively.

**Definition 3.2** (SPD MLR). SPD MLR is defined as

$$p(y = k \mid S) \propto \exp(\text{sign}(\langle A_k, \text{Log}_{P_k}(S) \rangle_{P_k}) \|A_k\|_{P_k} d(S, \tilde{H}_{A_k, P_k})), \quad (13)$$

where  $P_k \in \mathcal{S}_{++}^n$ ,  $A_k \in T_{P_k} \mathcal{S}_{++}^n \setminus \{0\}$ ,  $\langle \cdot, \cdot \rangle_{P_k} = g_{P_k}$ , and  $\|\cdot\|_{P_k}$  is the norm on  $T_{P_k} \mathcal{S}_{++}^n$  induced by  $g$  at  $P_k$ , and  $\tilde{H}_{A_k, P_k}$  is a margin hyperplane in  $\mathcal{S}_{++}^n$  as defined in Eq. (12).  $d(S, \tilde{H}_{A_k, P_k})$  denotes the margin distance between  $S$  and SPD hyperplane  $\tilde{H}_{A_k, P_k}$ , which is formulated as:

$$d(S, \tilde{H}_{A_k, P_k}) = \inf_{Q \in \tilde{H}_{A_k, P_k}} d(S, Q), \quad (14)$$

where  $d(S, Q)$  is the geodesic distance induced by  $g$ .

**Proposition 3.3** (Submanifolds). *The SPD hyperplane (as defined in Eq. (12)) under any geometrically complete Riemannian metric  $g$  is a regular submanifold of SPD manifolds.*

**Submanifolds are natural generalizations of the Euclidean hyperplanes.**

## Margin distance and MLR

Margin distance:

**Lemma 3.5.** *Given a PEM  $g$ , the margin distance defined in Eq. (14) has a closed-form solution:*

$$d(S, \tilde{H}_{A_k, P_k}) = d(\phi(S), H_{\phi_*, P_k}(A_k), \phi(P_k)), \quad (15)$$

$$= \frac{|\langle \phi(S) - \phi(P_k), \phi_{*, P_k}(A_k) \rangle|}{\|A_k\|_{P_k}}, \quad (16)$$

where  $|\cdot|$  is the absolute value.

Optimization:

**Lemma 3.6.** *Given a PEM, any parallel transportation is equivalent to the differential map of a left translation and vice versa.*

**Lemma 3.7.** *Given two fixed SPD matrices  $Q_1, Q_2 \in \mathcal{S}_{++}^n$ , we have the following equivalence for parallel transportations under a PEM,*

$$\begin{aligned} \forall \tilde{A}_{1,k} \in T_{Q_1} \mathcal{S}_{++}^n, \exists! \tilde{A}_{2,k} \in T_{Q_2} \mathcal{S}_{++}^n, \\ s.t. \Gamma_{Q_1 \rightarrow P_k}(\tilde{A}_{1,k}) = \Gamma_{Q_2 \rightarrow P_k}(\tilde{A}_{2,k}). \end{aligned} \quad (18)$$

General formulation:

**Theorem 3.8** (SPD MLR under a PEM). *Under any PEM, SPD MLR and SPD hyperplane is*

$$p(y = k | S) \propto \exp(\langle \phi(S) - \phi(P_k), \phi_{*, I}(\tilde{A}_k) \rangle), \quad (19)$$

$$\tilde{H}_{\tilde{A}_k, P_k} = \{S \in \mathcal{S}_{++}^n : \langle \phi(S) - \phi(P_k), \phi_{*, I}(\tilde{A}_k) \rangle = 0\}, \quad (20)$$

where  $\tilde{A}_k \in T_I \mathcal{S}_{++}^n / \{0\} \cong \mathcal{S}^n / \{0\}$  is a symmetric matrix, and  $P_k \in \mathcal{S}_{++}^n$  is an SPD matrix.

# MLR on the parameterized LEM and LCM

**Corollary 4.1** (SPD MLRs under the deformed LEM and LCM). *The SPD MLRs under  $(\alpha, \beta)$ -LEM is*

$$p(y = k | S) \propto \exp \left[ \langle \text{mlog}(S) - \text{mlog}(P_k), \tilde{A}_k \rangle^{(\alpha, \beta)} \right], \quad (21)$$

where  $\tilde{A}_k \in T_I \mathcal{S}_{++}^n \cong \mathcal{S}^n$  and  $P_k \in \mathcal{S}_{++}^n$ . *The SPD MLRs under  $(\theta)$ -LCM is*

$$p(y = k | S) \propto \exp \left[ \frac{1}{\theta} \langle \lfloor \tilde{K} \rfloor - \lfloor \tilde{L}_k \rfloor + \left[ \mathbb{D} \log(\mathbb{D}(\tilde{K})) - \mathbb{D} \log(\mathbb{D}(\tilde{L}_k)) \right], \lfloor \tilde{A}_k \rfloor + \frac{1}{2} \mathbb{D}(\tilde{A}_k) \rangle \right], \quad (22)$$

where  $\tilde{K} = \text{Chol}(S^\theta)$ ,  $\tilde{L}_k = \text{Chol}(P_k^\theta)$ , and  $\mathbb{D}(\tilde{A}_k)$  denotes a diagonal matrix with diagonal elements of  $\tilde{A}_k$ .

## Visualization of SPD hyperplane

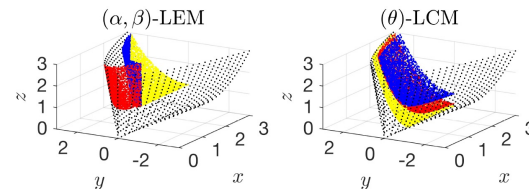


Figure 1. Conceptual illustration of SPD hyperplanes induced by  $(\alpha, \beta)$ -LEM and  $(\theta)$ -LCM. In each subfigure, the black dots are symmetric positive semi-definite (SPSD) matrices, denoting the boundary of  $\mathcal{S}_{++}^2$ , while the blue, red, and yellow dots denote three SPD hyperplanes.



# Experiments



Backbone	Classifier	[20,16,8]	[20,16,14,12,10,8]
SPDNet	LogEig MLR	92.88±1.05	93.47±0.45
	Gyro-AIM	94.53±0.95	94.32±0.94
	(1,0)-LEM	93.55±1.21	94.60±0.70
	(1,1)-LEM	<b>95.64±0.83</b>	<b>95.87±0.58</b>
	(1)-LCM	93.49±1.25	93.93±0.98
	(0.5)-LCM	<b>94.59±0.82</b>	<b>95.16±0.67</b>

Table 3. Results of SPDNet with different classifiers on the Radar dataset.

Backbone	Classifier	Inter-session	Inter-subject
SPDDSMBN	LogEig MLR	53.83±9.77	49.68±7.88
	Gyro-AIM	53.36±9.92	50.65±8.13
	(1,0)-LEM	53.16±9.73	51.41±7.98
	(1)-LCM	55.71±8.57	51.60±8.43
	(1.5)-LCM	<b>56.43±8.79</b>	<b>51.65±5.90</b>

Table 5. Results of SPDDSMBN with different classifiers on the Hinss2021 dataset under inter-subject and inter-session scenarios. The presented results are the ones of balanced accuracy under the leaving 5% out cross-validation scenario.

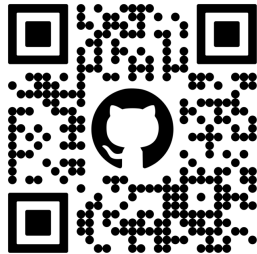
Backbone	Classifier	[93,30]	[93,70,30]	[93,70,50,30]
SPDNet	LogEig MLR	57.42±1.31	60.69±0.66	60.76±0.80
	Gyro-AIM	58.07±0.64	60.72±0.62	61.14±0.94
	(1,0)-LEM	57.02±0.75	61.34±0.62	60.78±0.86
	(1)-LCM	62.04±1.05	62.11±2.11	62.89±2.09
	(0.5)-LCM	<b>65.66±0.73</b>	<b>65.79±0.63</b>	<b>65.71±0.75</b>

Table 4. Results of SPDNet with different classifiers on the HDM05 dataset.

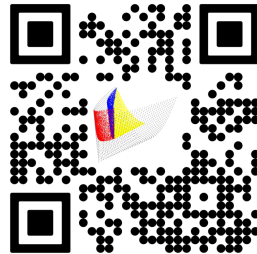
Methods	Radar	HDM05	Hinss2021	
			inter-session	inter-subject
Baseline	1.36	1.95	0.18	8.31
MLR-Gyro-AIM	1.75	31.64	0.38	13.3
MLR-LEM	1.5	4.7	0.24	10.13
MLR-LCM	<b>1.35</b>	<b>3.29</b>	<b>0.18</b>	<b>8.35</b>

Table 6. Comparison of training efficiency (s/epoch) of SPDNet (SPDDSMBN) under different classifiers. The most efficient MLR is highlighted in **bold**.

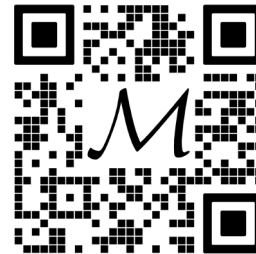
## Thanks you Q & A



Code



Paper



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