HW5

2020年12月21日 星期一 09:52

Problem 1: (a)

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{1}{4} & -8 & -1 & 9 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 \end{array}\right]$$

$q=4$. $r_q=-\frac{3}{4}$

$p=\min_i \frac{y_{i0}}{y_{iq}}=1$

$$\rightarrow \left[\begin{array}{ccccccc|c} 4 & 0 & 0 & 1 & -32 & -4 & 36 & 0 \\ -2 & 1 & 0 & 0 & 4 & \frac{3}{2} & -15 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 & -4 & -\frac{7}{2} & 33 & 0 \end{array}\right]$$

$q=5$ $r_q=-4$

$p=\min_i \frac{y_{i0}}{y_{iq}}=2$ $(y_{iq}>0)$

$$\rightarrow \left[\begin{array}{ccccccc|c} -12 & 8 & 0 & 1 & 0 & 8 & -84 & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 1 & \frac{3}{8} & -\frac{15}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 18 & 0 \end{array}\right]$$

$q=6$. $r_q=-2$

$p=\min_i \frac{y_{i0}}{y_{iq}}=1$

$$\rightarrow \left[\begin{array}{ccccccc|c} -\frac{3}{2} & 1 & 0 & \frac{1}{8} & 0 & 1 & \frac{-21}{2} & 0 \\ 1/16 & -1/8 & 0 & -3/64 & 1 & 0 & \frac{3}{16} & 0 \\ \frac{3}{2} & -1 & 1 & -1/8 & 0 & 0 & 21/2 & 1 \\ -2 & 3 & 0 & 1/4 & 0 & 0 & -3 & 0 \end{array}\right]$$

$q=7$ $r_q=-3$

$p=\min_i \frac{y_{i0}}{y_{iq}}=2$

$$\rightarrow \left[\begin{array}{ccccccc|c} 2 & -6 & 0 & -5/2 & 56 & 1 & 0 & 0 \\ 1/3 & -2/3 & 0 & -1/4 & 16/3 & 0 & 1 & 0 \\ -2 & 6 & 1 & 5/2 & -56 & 0 & 0 & 1 \\ -1 & 1 & 0 & -1/2 & 16 & 0 & 0 & 0 \end{array}\right]$$

$q=1$ $r_q=-1$

$p=1$

$$\rightarrow \left[\begin{array}{ccccccc|c} 1 & -3 & 0 & -5/4 & 28 & 1/2 & 0 & 0 \\ 0 & 1/3 & 0 & 1/6 & -4 & -1/6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & -7/4 & 44 & 1/2 & 0 & 0 \end{array}\right]$$

$q=2$ $r_q=-2$

$p=2$

So the cycling occurs.

(b)

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{1}{4} & -8 & -1 & 9 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 \end{array}\right]$$

$q=4$. $r_q=-\frac{3}{4}$

$p=\min_i \frac{y_{i0}}{y_{iq}}=1$

$$\rightarrow \left[\begin{array}{ccccccc|c} 4 & 0 & 0 & 1 & -32 & -4 & 36 & 0 \\ -2 & 1 & 0 & 0 & 4 & \frac{3}{2} & -15 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 & -4 & -\frac{7}{2} & 33 & 0 \end{array}\right]$$

$q=5$ $r_q=-4$

$p=\min_i \frac{y_{i0}}{y_{iq}}=2$ $(y_{iq}>0)$

$$\rightarrow \left[\begin{array}{ccccccc|c} -12 & 8 & 0 & 1 & 0 & 8 & -84 & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 1 & \frac{3}{8} & -\frac{15}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 18 & 0 \end{array}\right]$$

$q=6$. $r_q=-2$

$p=\min_i \frac{y_{i0}}{y_{iq}}=1$

$$\rightarrow \left[\begin{array}{ccccccc|c} -\frac{3}{2} & 1 & 0 & \frac{1}{8} & 0 & 1 & \frac{-21}{2} & 0 \\ 1/16 & -1/8 & 0 & -3/64 & 1 & 0 & \frac{3}{16} & 0 \\ \frac{3}{2} & -1 & 1 & -1/8 & 0 & 0 & 21/2 & 1 \\ -2 & 3 & 0 & 1/4 & 0 & 0 & -3 & 0 \end{array}\right]$$

$q=1$. $r_q=-2$

$p=2$

$$\rightarrow \left[\begin{array}{ccccccc|c} 0 & -2 & 0 & -1 & 24 & 1 & -6 & 0 \\ 1 & -2 & 0 & -3/4 & 16 & 0 & 3 & 0 \\ 0 & 2 & 1 & 1 & -24 & 0 & 6 & 1 \\ 0 & -1 & 0 & -5/4 & 32 & 0 & 3 & 0 \end{array}\right]$$

$q=2$ $p=3$

$$\rightarrow \left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1/4 & -8 & 0 & 9 & 1 \\ 0 & 1 & 1/2 & 1/2 & -12 & 0 & 3 & 1/2 \\ 0 & 0 & 1/2 & -3/4 & 20 & 0 & 6 & 1/2 \end{array}\right]$$

$q=4$ $p=3$

$$\rightarrow \left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1/2 & 3/4 & 0 & -2 & 0 & 15/2 & 3/4 \\ 0 & 2 & 1 & 1 & -24 & 0 & 6 & 1 \\ 0 & 3/2 & 5/4 & 0 & 2 & 0 & 21/2 & 5/4 \end{array}\right]$$

So the $x^* = [3/4, 0, 0, 1, 0, 1, 0]^T$

$f(x^*) = -5/4$

Problem 2.

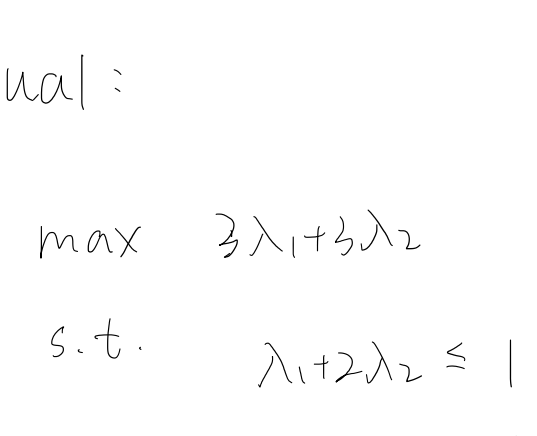
Dual:

max $3\lambda_1+3\lambda_2$

s.t. $\lambda_1+2\lambda_2 \le 1$

$2\lambda_1+\lambda_2 \le 1$

$\lambda_1, \lambda_2 \ge 0$



So $\lambda_1=\frac{1}{3}$, $\lambda_2=\frac{1}{3}$

$f(\lambda^*)=2$

Problem 3. [Programming]

See the attatch file

Problem 4

① minimize $-2x_1-3x_2$

s.t. $x_1+2x_2+x_3=4$

$2x_1+x_2+x_4=5$

$x_1, x_2, x_3, x_4 \ge 0$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ -2 & -3 & 0 & 0 & 0 \end{array}\right]$$

$q=2$. $p=1$

$$\rightarrow \left[\begin{array}{cccc|c} 1/2 & 1 & 1/2 & 0 & 2 \\ 3/2 & 0 & -1/2 & 1 & 3 \\ -1/2 & 0 & 3/2 & 0 & 6 \end{array}\right]$$

$q=1$ $p=2$

$$\rightarrow \left[\begin{array}{cccc|c} 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 \\ 1 & 0 & -\frac{1}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 4/3 & \frac{1}{3} & 7 \end{array}\right]$$

$\therefore x^* = [2, 1, 0, 0]$. $f(x^*) = -7$

② dual:

maxmize $4\lambda_1+5\lambda_2$

s.t. $\lambda_1+2\lambda_2 \le -2$

$2\lambda_1+\lambda_2 \le -3$

$\lambda_1 \le 0$

$\lambda_2 \le 0$

$\lambda^T b = -7$ and $\lambda_1+2\lambda_2=-2$

$\lambda^* = [-\frac{4}{3}, -\frac{1}{3}]$

Problem 5.

(a) dual: maxmize λ

s.t. $\lambda \le -c_1$

$\lambda \le -c_2$

$\vdots$

$\lambda \le -c_n$

$\rightarrow$ minimize λ

s.t. $\lambda \ge c_1$

$\lambda \ge c_2$

$\vdots$

$\lambda \ge c_n$

(b) $\lambda^* = c_4$

(c) $x^* = [0, 0, 0, 1, \cdots 0]^T$

Problem 6.

$\exists x$, $Ax \ge b$, $x \ge 0$ $\overset{b^Ty\le 0}{\Longleftrightarrow}$ $\forall y$. $A^Ty \le 0$, $y \ge 0$

minimize 0^Tx

s.t. $Ax \ge b$

$x \ge 0$

$\overset{\text{dual}}{\longleftrightarrow}$

maxmize y^Tb

s.t. $A^Ty \le 0$

$y \ge 0$

$\Rightarrow$: there exits a feasible solution to primal problem

so the $f(x^*)=0$, by Weaky Duality Lemma, $b^Ty \le 0$

$\Leftarrow$: any y satisfy $b^Ty \le b^T0 = 0$. So the primal has $x^*=0$.

Problem 7.

$L(x, \lambda) = x_1^2+2x_1x_2+3x_2^2+4x_1+5x_2+6x_3+\lambda_1(x_1+2x_2-3)+\lambda_2(4x_1+5x_3-6)$

$\frac{\partial L}{\partial x_1} = 2x_1+2x_2+4+\lambda_1+4\lambda_2=0$

$\frac{\partial L}{\partial x_2} = 2x_1+6x_2+5+2\lambda_1=0$

$\frac{\partial L}{\partial x_3} = 6+5\lambda_2=0$

$\frac{\partial L}{\partial \lambda_1} = x_1+2x_2-3=0$

$\frac{\partial L}{\partial \lambda_2} = 4x_1+5x_3-6=0$

$\Rightarrow$

$\lambda_1 = -\frac{27}{5}$

$\lambda_2 = -\frac{6}{5}$

$x_1 = \frac{16}{5}$

$x_2 = -\frac{1}{10}$

$x_3 = -\frac{34}{25}$

$x^* = [16/5, -\frac{1}{10}, -34/25]^T$ is a regular point.

SOSC:

$L(x^*, \lambda^*) = F(x^*) + \lambda^* H(x^*)$

$$= \begin{bmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$T(x^*) = \left\{ y \in \mathbb{R}^3 : \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \end{bmatrix} y = 0 \right\}$

$= \{a[-\frac{5}{4}, 5/8, 1]^T, a \in \mathbb{R}\}$

$\forall y \in T(x^*)$, $a \ne 0$

$y L(x^*, \lambda^*) y = \frac{75}{32}a^2 > 0$

so x^* is a strict local minimizer.

Problem 8.

(a) minimize $(x_1-1)^2+(x_2-\sqrt{3})^2$

s.t. $x_1^2+x_2^2=9$

$L(x, \lambda) = (x_1-1)^2+(x_2-\sqrt{3})^2+\lambda(x_1^2+x_2^2-9)$

$\frac{\partial L}{\partial x_1} = 2(x_1-1)+2\lambda x_1 = 0$

$\frac{\partial L}{\partial x_2} = 2(x_2-\sqrt{3})+2\lambda x_2 = 0$

$\frac{\partial L}{\partial \lambda} = x_1^2+x_2^2-9=0$

$\therefore x^* = \frac{3}{2}[1, \sqrt{3}]$ or $x^* = -\frac{3}{2}[1, \sqrt{3}]$

(b) x^* is regular point.

$L(x^*, \lambda^*) = (1+\lambda_1^*) I$

for second x^*, doesn't satisfy SONC ($L<0$)

the first x^* satisfies SONC. ($L \ge 0$)

so $x^* = \frac{3}{2}[1, \sqrt{3}]$ is a strict local minimizer

Problem 9

(a) $L(x, \lambda) = 2x_1+3x_2-6+\lambda(x_1x_2-6)$

$\frac{\partial L}{\partial x_1} = 2+\lambda x_2=0$ $\frac{\partial L}{\partial x_2} = 3+\lambda x_1=0$ $\frac{\partial L}{\partial \lambda} = x_1x_2-6=0$

$\therefore x^* = [3, 2]$ or $x^* = -[3, 2]$

(b) $L(x^*, \lambda^*) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$T(x^*) = \{a[-3, 2], a \in \mathbb{R}\}$

SOSC: For $x^* = [3, 2]$

$y^T L(x^*, \lambda^*) y = 12a^2 > 0$, so $x^* = [3, 2]$ is a strict local minimizer

For $x^* = [-3, -2]$

$y^T L(x^*, \lambda^*) y = -12a^2 < 0$. so $x^* = [-3, -2]$ is a strict local maximizer

(c) No. because local minimizer > local maximizer

$f([3, 2]) = 8$, $> f([-3, -2]) = -16$

Problem 10.

$L(x, \lambda) = \frac{1}{2}\|Ax-b\|^2 + \lambda^T(Cx-d)$

$\frac{\partial L}{\partial x} = (Ax-b)^TA + \lambda^TC = 0^T$

$\frac{\partial L}{\partial \lambda} = Cx-d = 0^T$

$\therefore x = Q^{-1}A^Tb - Q^{-1}C^T(CQ^{-1}C^T)^{-1}(CQ^{-1}A^Tb-d)$

Problem 11.

$L(x, \lambda) = \frac{1}{2}x^TQx - C^Tx + d + \lambda^T(b-Ax)$

$\frac{\partial L}{\partial x} = x^TQ - C^T - \lambda^TA = 0^T \Rightarrow x = Q^{-1}(A^T\lambda + C)$

$\frac{\partial L}{\partial \lambda} = Ax - b = 0$

$\Rightarrow AQ^{-1}(A^T\lambda + C) = b$

$\therefore \lambda = (AQ^{-1}A^T)^{-1}(b - AQ^{-1}C)$

$x^* = Q^{-1}C + Q^{-1}A^T(AQ^{-1}A^T)^{-1}(b - AQ^{-1}C)$

Problem 12.

1. $L(x, \mu) = x_1^2+4x_2^2+\mu(4-x_1^2-2x_2^2)$

KKT:

$\frac{\partial L}{\partial x_1} = 2x_1 - 2\mu x_1 = 0$

$\frac{\partial L}{\partial x_2} = 8x_2 - 4\mu x_2 = 0$

$\mu(4-x_1^2-2x_2^2) = 0$

$\mu \ge 0$

$4-x_1^2-2x_2^2 \le 0$

if $\mu=0$. doesn't satisfy

so $\mu > 0$.

if $\mu=1$, $x_1 = \pm 2$ $x_2=0$

if $\mu=2$. $x_1=0$. $x_2 = \pm\sqrt{2}$

2. SOSC

$L(x^*, \mu^*) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} + \mu \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$

if $\mu=1$. $L(x^*, \mu^*) = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$

$\tilde{T}(x^*) = \{y : [\pm 4, 0]y = 0\} = \{y = a[0, 1]^T : a \in \mathbb{R}\}$

$y^T L y = 4a^2 > 0$

so $x_1 = \pm 2$. $x_2 = 0$ satisfy SOSC to be strict local minimizer

if $\mu=2$, $L(x^*, \mu^*) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

$\tilde{T}(x^*) = \{y = a[1, 0] : a \in \mathbb{R}\}$

$y^T L y = -2a^2 < 0$

so $x_1 = 0$ $x_2 = \pm\sqrt{2}$ don't satisfy SOSC to be minimizer

Problem 13.

(a) $L(x, \lambda, \mu) = x_1x_2^2 + \lambda(x_1 - x_2) + \mu(-x_1)$

$\frac{\partial L}{\partial x_1} = x_2^2 + \lambda - \mu = 0$

$\frac{\partial L}{\partial x_2} = 2x_1x_2 - \lambda = 0$

$\frac{\partial L}{\partial \lambda} = x_1 - x_2 = 0$

$\mu(-x_1) = 0$

$x_1 \ge 0$

$\mu \ge 0$

if $\mu > 0$ doesn't satisfy

so $\mu = 0$. $x_1 = 0$. $x_2 = 0$. $\lambda = 0$

(b) SONC

$L(x^*, \lambda^*, \mu^*) = \begin{bmatrix} 0 & 2x_2 \\ 2x_2 & 2x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T(x^*) = y: \{[1, -1]y = 0, [-1, 0]y = 0\} = \{0\}$

so SONC always satisfy.

(c) SOSC.

$y^T L y = 0$, so SOSC doesn't satisfy.