Problem 4.

$\forall x, y \in M$, assume that $x = \sum_{i=1}^{l} \alpha_i x_i \ (\alpha_i \geq 0, \sum_{i=1}^{l} \alpha_i = 1)$ $\quad y = \sum_{i=1}^{l} \beta_i x_i \ (\beta_i \geq 0, \sum_{i=1}^{l} \beta_i = 1)$

$\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda) y = \sum_{i=1}^{l} [\lambda \alpha_i + (1-\lambda)\beta_i] x_i$, let $y_i = \lambda \alpha_i + (1-\lambda)\beta_i$

if $y_i \geq 0$ and $\sum_{i=1}^{l} y_i = 1$, then $\lambda x + (1-\lambda) y \in M$

because $\lambda, 1-\lambda, \alpha_i, \beta_i \geq 0$, $\therefore y_i \geq 0$. $\sum_{i=1}^{l} y_i = \sum_{i=1}^{l} \lambda \alpha_i + (1-\lambda)\beta_i = \lambda \sum_{i=1}^{l} \alpha_i + (1-\lambda) \sum_{i=1}^{l} \beta_i$

$\therefore \lambda x + (1-\lambda) y \in M \Rightarrow M$ is convex. $\qquad = \lambda + (1-\lambda) = 1$

Problem 5.

$f(x) = (a^T x)(b^T x) = (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)(b_1 x_1 + b_2 x_2 + \cdots + b_n x_n)$

$$\therefore \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix} = \begin{bmatrix} a_1(b_1 x_1 + b_2 x_2 + \cdots + b_n x_n) + b_1(a_1 x_1 + \cdots + a_n x_n) \\ \vdots \\ a_n(b_1 x_1 + b_2 x_2 + \cdots + b_n x_n) + b_n(a_1 x_1 + \cdots + a_n x_n) \end{bmatrix}$$

$$F(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix} = \begin{bmatrix} 2a_1 b_1 & a_1 b_2 + a_2 b_1 & \cdots & a_1 b_n + b_1 a_n \\ \vdots & & & \\ a_1 b_n + b_1 a_n & \cdots & \cdots & 2 a_n b_n \end{bmatrix}$$

Problem 6.

$f(x) = x_1 \cdot x_2 / 2$, $g(s,t) = \begin{bmatrix} 4s+3t \\ 2s+t \end{bmatrix}$. $\quad \frac{\partial f(g(s,t))}{\partial s} = \frac{\partial f}{\partial (g(s,t))} \cdot \frac{\partial g(s,t)}{\partial s} = \begin{bmatrix} \frac{x_2}{2} & \frac{x_1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\qquad = 2x_2 + x_1$

$\frac{\partial f(g(s,t))}{\partial t} = \frac{\partial f}{\partial (g(s,t))} \cdot \frac{\partial (g(s,t))}{\partial t} = \begin{bmatrix} \frac{x_2}{2} & \frac{x_1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{3}{2} x_2 + \frac{1}{2} x_1 = 5s + 3t$

$\qquad = 4s + 2t + 4s + 3t$

$\qquad = 8s + 5t$

Problem. 7.

$f(x) = x_1 e^{-x_2} + x_2 + 1 \qquad \nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^{-x_2} \\ -x_1 e^{-x_2} + 1 \end{bmatrix} \qquad \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & -e^{-x_2} \\ -e^{-x_2} & x_1 e^{-x_2} \end{bmatrix}$

$f(x) = f(x_0) + \nabla f(x)^T \big|_{x=x_0} (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x) \big|_{x=x_0} (x - x_0) + o(\|x - x_0\|^2)$

$= f(x_0) + x_1 - 1 + \frac{1}{2} x_2^2 + x_2 - x_1 x_2 = \frac{1}{2} x_2^2 + x_1 + x_2 - x_1 x_2 + 1$