作业2

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Problem 1.

$J = \int_{x_0}^{x_1} F(x, y, y')\,dx$

$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\delta F}{\partial y'}\partial y' \right) dx$

$J[y(x)] = \int_{x_0}^{x_1} (x^4 y' + x^3 y + 3)\,dx$

$\delta J(y, \delta y) = \frac{d}{d\alpha} J(y + \alpha\delta y)\Big|_{\alpha=0}$

$= \frac{d}{d\alpha}\int_{x_0}^{x_1} \left[x^4 (y+\alpha\delta y)' + x^3 (y + \alpha\delta y) + 3\right] dx \Big|_{\alpha=0}$

$= \int_{x_0}^{x_1} \frac{d}{d\alpha}\left[x^4 y' + \alpha x^4 \delta y' + x^3 y + \alpha x^3 \delta y + 3\right] dx \Big|_{\alpha=0}$

$= \int_{x_0}^{x_1} \left[x^4 \delta y' + x^3 \delta y\right] dx \Big|_{\alpha=0}$

$= \int_{x_0}^{x_1} x^4 \delta y' + x^3 \delta y \; dx$

Problem 2.

$\int_0^{\frac{\pi}{2}} (x^2 + x' \sin 2t)\,dt$

$J(y) = \int_0^{\pi/2} (y^2 + y' \sin 2x)\,dx$

$g = y^2 + y' \sin 2x$，不显含t

必要条件：

$\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = C_1$

$\Rightarrow 2y - \frac{d}{dx}(\sin 2x) = 0$

$\Rightarrow y = \cos 2x$

带入 $y(\frac{\pi}{2}) = -1 \Rightarrow y = \cos 2x,\ y(0) = 1,\ K = 1$

存在 $y = \cos 2x,\ K = 1$

Problem 3.

$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} \Rightarrow \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{matrix}$

$\bar{J} = \int_{t_0}^{t_f} \frac{1}{2}u^2 + p_1(x_2 - \dot{x}_1) + p_2(u - \dot{x}_2)\,dt$

$H = \frac{1}{2}u^2 + p_1 x_2 + p_2 u$

$\bar{J} = \int_{t_0}^{t_f} (H - p_1\dot{x}_1 - p_2\dot{x}_2)\,dt$

极值条件： $0 = \frac{\partial H}{\partial u} \Rightarrow u = -p_2 \Rightarrow u = C_1 t - C_2$

$\dot{x} = \frac{\partial H}{\partial p} \Rightarrow \begin{cases} \dot{x}_1 = x_2 & \Rightarrow x_1 = \frac{1}{6}C_1 t^3 - \frac{1}{2}C_2 t^2 + C_3 t + C_4 \\ \dot{x}_2 = u & \Rightarrow x_2 = \frac{1}{2}C_1 t^2 - C_2 t + C_3 \end{cases}$

$\dot{p} = -\frac{\partial H}{\partial x} \Rightarrow \begin{matrix} \dot{p}_1 = 0 \Rightarrow p_1 = C_1 \\ \dot{p}_2 = -p_1 \Rightarrow p_2 = -C_1 t + C_2 \end{matrix}$

边界条件： $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ x(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 代入可得

故 $u(t) = 3t - \frac{7}{2}$

$x_1(t) = \frac{1}{2}t^3 - \frac{7}{4}t^2 + t + 1$

$x_2(t) = \frac{3}{2}t^2 - \frac{7}{2}t + 1$

Problem 4.

$T_m\dot{w} + w = \frac{1}{K_e}u \Rightarrow u = K_e T_m \dot{w} + K_e w$

$J = \int_0^{t_f} \frac{1}{2}u^2\,dt = \int_0^{t_f} \frac{1}{2}\left[K_e^2 T_m^2 (w')^2 + 2K_e^2 T_m w w' + K_e^2 w^2\right] dt$

$\bar{J} = \lambda x_{t_f} + \int_0^{t_f} \frac{1}{2}\left[K_e^2 T_m^2 (w')^2 + 2K_e T_m w w' + K_e w^2\right] dt$

必要条件：

$0 = \frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial x'} \Rightarrow 0 = K_e^2 T_m w' + K_e^2 w - (K_e^2 T_m^2 w'' + K_e^2 T_m w')$

$w = T_m^2 w''$

$\Rightarrow w = C_1 e^{-\frac{1}{T_m}t} + C_2 e^{\frac{1}{T_m}t}$

$0 = \frac{1}{2}\left[K_e^2 T_m^2 (w')^2 + 2K_e T_m w w' + K_e w^2\right]\Big|_{t_f} - \left[K_e^2 T_m (w')^2 + K_e T_m w w'\right]\Big|_{t_f}$

$\Rightarrow \frac{1}{2}K_e^2 T_m^2 (w')^2\Big|_{t_f} + \frac{1}{2}K_e w^2\Big|_{t_f} = K_e^2 T_m (w')^2\Big|_{t_f}$

$(K_e^2 T_m - \frac{1}{2}K_e^2 T_m^2)(w')^2\Big|_{t_f} = \frac{1}{2}K_e w^2\Big|_{t_f} \Rightarrow w|_{t_f} = 0 \Rightarrow w'|_{t_f} = 0$

$w_{t_f} = C_1 e^{-\frac{1}{T_m}t_f} + C_2 e^{\frac{1}{T_m}t_f} = 0,\quad w_{t=0} = C_1 + C_2 = 1$

$\therefore w'\Big|_{t_f} = 0 = -\frac{C_1}{T_m}e^{-\frac{1}{T_m}t_f} + \frac{C_2}{T_m}e^{\frac{1}{T_m}t_f} = 0$

记 $T = \frac{1}{T_m}$ $\begin{cases} -C_1 e^{-Tt_f} + C_2 e^{Tt_f} = 0 \\ C_1 + C_2 = 1 \\ C_1 e^{-Tt_f} + C_2 e^{Tt_f} = 0 \end{cases}$

解得

$C_1 e^{-Tt_f} = C_2 e^{Tt_f}$

$2C_2 e^{Tt_f} = 0$

$C_2 = 0$

$C_1 = 1$

$t_f = \infty$

$\therefore w = e^{-\frac{1}{T_m}t}$

$w' = -\frac{1}{T_m}e^{-\frac{1}{T_m}t}$

$u(t) = 0$