Homework 2

Problem 1: Consider the problem:

$$\min f(x)$$
$$x \in \Omega$$

where $f \in \mathcal{C}^2$(twice continuously differentiable). For each of the following specifications for Ω, x^* and f, determine if the given point x^* is : (i) definitively a local minimizer; (ii) definitively not a local minimizer; (iii) possibly a local minimizer.

1. $f : \mathbb{R}^2 \to \mathbb{R}, \Omega = \{x = [x_1, x_2]^T : x_2 \geq 1\}, x^* = [1, 2]^T$, and gradient $\nabla f(x^*) = [1, 1]^T$.
2. $f : \mathbb{R}^2 \to \mathbb{R}, \Omega = \{x = [x_1, x_2]^T : x_1 \geq 1, x_2 \geq 2\}, x^* = [1, 2]^T$, and gradient $\nabla f(x^*) = [1, 0]^T$.
3. $f : \mathbb{R}^2 \to \mathbb{R}, \Omega = \{x = [x_1, x_2]^T : x_1 \geq 0, x_2 \geq 0\}, x^* = [1, 2]^T$, and gradient $\nabla f(x^*) = [0, 0]^T$ and Hessian $F(x^*) = I$(identity matrix).
4. $f : \mathbb{R}^2 \to \mathbb{R}, \Omega = \{x = [x_1, x_2]^T : x_1 \geq 1, x_2 \geq 2\}, x^* = [1, 2]^T$, and gradient $\nabla f(x^*) = [1, 0]^T$, and Hessian

$$F(x^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2: Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = x^T \begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix} x + x^T \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 7$$

1. Find the directional derivative of f at $[0, 1]^T$ in the direction $[1, 0]^T$.
2. Find all points that satisfy the first-order necessary condition for f. Does f have a minimizer? If it does, then find all minimizer(s); otherwise, explain why it does not.

Problem 3: Consider the problem

$$\begin{aligned} &\min f(x) \\ &x \in \Omega, \end{aligned}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = -3x_1$ with $x = [x_1, x_2]^T$ and $\Omega = \{x = [x_1, x_2]^T : x_1 + x_2^2 \leq 2\}$. Answer each of the following questions, showing complete justification.

1. Does the point $x^* = [2, 0]^T$ satisfy the first-order necessary condition?
2. Does the point $x^* = [2, 0]^T$ satisfy the second-order necessary condition?
3. Is the point $x^* = [2, 0]^T$ a local minimizer?

Problem 4: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = (x_1 - x_2)^4 + x_1^2 - x_2^2 - 2x_1 + 2x_2 + 1$$

where $x = [x_1, x_2]^T$. Suppose that we wish to minimize f over $\mathbb{R}^2$. Find all points satisfying the FONC. Do these points satisfy the SONC?

Problem 5: Suppose that we are given a set of vectors $\{x^{(1)}, \cdots, x^{(p)}\}$, $x^{(i)} \in \mathbb{R}^n$, $i = 1, \cdots, p$. Find the vector $\tilde{x} \in \mathbb{R}^n$ such that the average squared distance (norm) between $\tilde{x}$ and $x^{(1)}, \cdots, x^{(p)}$,

$$\frac{1}{p} \sum_{i=1}^{p} \|\tilde{x} - x^{(i)}\|^2$$

is minimized. Use the SOSC to prove that the vector $\tilde{x}$ found above is a strict local minimizer. How is $\tilde{x}$ related to the centroid (or center of gravity) of the given set of points $\{x^{(1)}, \cdots, x^{(p)}\}$?

Problem 6: Consider the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \frac{1}{2} x^T Q x - x^T b,$$

where $Q = Q^T > 0$ (Q is positive definite). Show that x^* minimizes f if and only if x^* satisfies the FONC.

Problem 7: (Computational assignments) Let $f(x) = x^2 + 4\cos(x), x \in \mathbb{R}$ and we wish to find the minimizer x^* of f over $[1, 2]$.

(a) Use Matlab (or other graphic tool) to plot $f(x)$ versus x over the interval $[1, 2]$, and verify if f is unimodal over $[1, 2]$.

(b) Write a program to implement the golden section method that locates the minimizer x^* within an uncertainty of 0.2. Display all intermediate steps using a table as

Iteration k	a_k	b_k	$f(a_k)$	$f(b_k)$	New uncertainty interval
1	?	?	?	?	[?,?]
...	...	...	...	...	...

(c) Apply Newton's method, using the same number of iterations as in part (b) with $x^{(0)} = 1$.

Problem 8: (Computational assignment)

(a) Write a program to implement the secant method to locate the root of the equation $g(x) = 0$. For the stopping criterion, use the condition $|x^{(k+1)} - x^{(k)}| < \epsilon|x^{(k)}|$, where $\epsilon > 0$ is a given constant.

(b) Let $g(x) = (2x - 1)^2 + 4(4 - 1024x)^4$. Find the root of $g(x) = 0$ using the secant method with $x^{(0)} = 0, x^{(1)} = 1$, and $\epsilon = 10^{-5}$. Also determine the value of g at the solution obtained.