Final Exam
Total points (100 pts)

Problem 1: (20 pts)

(a) Suppose that $\Omega = \{x \in \mathbb{R}^n | h(x) = c\}$ is convex, where $h : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$. Show that h is convex and concave over Ω.

(b) Given $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^n$, $m \leq n, \text{rank}(A) = m$, and $x_0 \in \mathbb{R}^n$. Derive an expression for the closest point to x_0 that satisfies $Ax = b$ in terms of A, b and x_0.

Problem 2: [20 pts]

Suppose that we use the golden section algorithm to find the minimizer of a function.

(a) Consider the function $f(x) = 2x^2 - 5x + 3$ over the range $[0, 2]$. Perform one iteration of Golden section algorithm and determine the uncertainty range. (Recall the Golden section number $\rho = \frac{3-\sqrt{5}}{2} \approx 0.382$)

(b) Let u_k be the uncertainty range at the $k-$th iteration. Find the order of convergence of $\{u_k\}$ and determine the iterations number that is needed to reduce the uncertainty range to 0.1 (express the result in logarithm function).

Problem 3: [20 pts] Consider the minimization of

$$f(x_1, x_2) = -\frac{3}{2}x_1 + x_1^2 + x_2^2 - x_1x_2 + 50$$

(a) Express $f(\mathbf{x})$, $\mathbf{x} = [x_1, x_2]^T$ in the form of $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x}^T - \mathbf{x}^T b + c$ and determine an optimal solution $\mathbf{x}^*$ analytically. Is the solution local or global minimizer?

(b) If we perform the <u>fixed-step-size gradient</u> method $\mathbf{x}^{(k)} := \mathbf{x}^{(k-1)} - \alpha\nabla f(\mathbf{x}^{(\mathbf{k-1})})$, find the range of the step size α for the sequence $\mathbf{x}^{(k)}$ convergent to $\mathbf{x}^*$?

(c) Perform one iteration of <u>steepest descent</u> method with the starting point $\mathbf{x}^{(0)} = \mathbf{0}$ and compute $\mathbf{x}^{(1)}$.

(d) Use the <u>conjugate gradient algorithm</u> to construct a vector $\mathbf{d}^{(1)}$ that is Q conjugate with $\mathbf{d}^{(0)} = -\nabla f(x^{(0)})$, where $\mathbf{x}^{(0)} = \mathbf{0}$.

Problem 4: [20 pts] Consider the linear program

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 \leq 5 \\ & x_1 + x_2 \leq 9 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) Solve it using the simplex method. Show all your work to get the full credit.

(b) Write down the dual of the linear programming problem, and find the solution to the dual using the complementary slackness condition.

Problem 5: (20 pts) Consider the problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & (x_1 - 1)^2 + x_2 - 2 \\ \text{subject to} \quad & x_2 - x_1 - 1 = 0 \\ & x_1 + x_2 - 2 \leq 0 \end{aligned}$$

(a) State the KKT conditions for this problem and find all the points (and KKT multipliers) that satisfy the condition. In each case, determine if the point is regular.

(b) Use the second-order conditions to specify which points are (strict) local minimizers.