Problem 2:

(a) $f(x)=\frac{1}{2}x^T\begin{bmatrix}1&0\\0&2\end{bmatrix}x+\begin{bmatrix}-1\\-2\end{bmatrix}^T x+\begin{bmatrix}1&1\end{bmatrix}x$

$\Leftrightarrow$ $\nabla f(x^{(k)})=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\x_2^{(k)}\end{bmatrix}+\begin{bmatrix}-1\\-2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}$

$f(x^{(k)})=0$ $\Leftrightarrow$

$x^{(k)}=x^{(k)}+\alpha\left(\begin{bmatrix}x_1^{(k)}\\x_2^{(k)}\end{bmatrix}+\begin{bmatrix}-1\\-2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}x_1^{(k)}\\x_2^{(k)}\end{bmatrix}$

$f(x^{(k)})=0$ $\Leftrightarrow$

$x^{(k)}=x^{(k)}+\alpha\left(\begin{bmatrix}x_1^{(k)}\\x_2^{(k)}\end{bmatrix}+\begin{bmatrix}-1\\-2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}x_1^{(k)}\\x_2^{(k)}\end{bmatrix}$

$f(x^{(k)})=0.035$

$x^{(k)}=x^{(k)}+\alpha\left(\begin{bmatrix}x_1^{(k)}\\x_2^{(k)}\end{bmatrix}+\begin{bmatrix}-1\\-2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}x_1^{(k)}\\x_2^{(k)}\end{bmatrix}$

$f(x^{(k)})=0.035$

$x^{(k)}\to x^{(k)}$ & $x^{(k)}\to x^{(k)}$

So the minimizer is in $\{x^{(k)},x^{(k)}\}=\left\{\begin{bmatrix}-0.9996\\-0.9996\end{bmatrix},\begin{bmatrix}-0.9996\\-0.9996\end{bmatrix}\right\}$, $\alpha\in\left\{\begin{bmatrix}-0.5\\-0.5\end{bmatrix}\right\}$

(b) Golden section

$[a_0,b_0]=\left[\begin{bmatrix}-0.9996\\-0.9996\end{bmatrix},\begin{bmatrix}-0.9996\\-0.9996\end{bmatrix}\right]$

$f(a_0)=0.0448$, $f(b_0)=0.035$

using code in HW2, result:

```
>> C:\Users\pc\Desktop\golden - Windows 7 SP1 x
Iteration k |        ak                |        bk                |        f(ak)             |        f(bk)             |  New uncertainty interval
            |                          |                          |                          |                          |          ak,bk]
            |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-1.0000000 -0.9996455]
            |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
1           |  -1.0000000 -0.9996455   |  -0.9996455 -0.9996455   |  0.0000000               |  0.0447425               |  [-1.0000000 -0.9996455]
2           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
3           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
4           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
5           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
6           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
7           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
8           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
9           |  -0.9996455 -0.9996455   |  -0.9996455 -0.9996455   |  0.0447425               |  0.0447425               |  [-0.9996455 -0.9996455]
Optimal ak = -0.9996455
```

Problem 2:

$f=\frac{1}{2}x^T\begin{bmatrix}1&0\\0&4\end{bmatrix}x-x^T\begin{bmatrix}-1\\-1\end{bmatrix}+3=\frac{1}{2}x^TQx-x^Tb+c$

$x^{(k)}=x^{(k)}-\alpha\nabla f(x^{(k)})$, $g^{(k)}=Qx^{(k)}-b=\begin{bmatrix}1\\1/4\end{bmatrix}$

$\alpha_0=\frac{g^{(0)T}g^{(0)}}{g^{(0)T}Qg^{(0)}}=\frac{5}{6}$

$x^{(1)}=\begin{bmatrix}1/6\\-1/6\end{bmatrix}-\frac{5}{6}\begin{bmatrix}1\\1/4\end{bmatrix}=-\begin{bmatrix}1/6\\1/24\end{bmatrix}$

$x^{(1)}=x^{(0)}-\alpha_1\nabla f(x^{(1)})$

$\nabla f(x^{(1)})=g^{(1)}=Qx^{(1)}-b=\begin{bmatrix}1/6\\-1/24\end{bmatrix}$

$\alpha_1=\frac{g^{(1)T}g^{(1)}}{g^{(1)T}Qg^{(1)}}=\frac{5}{9}$

$x^{(2)}=x^{(1)}-\alpha_1 g^{(1)}=\begin{bmatrix}-\frac{25}{54}\\-\frac{25}{108}\end{bmatrix}$

Optimal solution is $Qx^*=b\Rightarrow x^*=\begin{bmatrix}-1\\-1/4\end{bmatrix}$

Problem 3:

$\lim_{k\to\infty}x^{(k)}=\lim_{k\to\infty}2^{-2^k}=\lim_{k\to\infty}\left(\frac{1}{2}\right)^{2^k}=0$

$\frac{|x^{(k+1)}|}{|x^{(k)}|^p}=\frac{2^{-2^{k+1}}}{2^{-2^kp}}=2^{-2^{k+1}+p2^k}=2^{-2^k(2^{k+1}-p)}=\left(\frac{1}{2}\right)^{2^k(2^{k+1}-p)}$

$k\to\infty$, $-2^k(2^{k+1}-p)\to 0$. So for any p. $\lim_{k\to\infty}\frac{|x^{(k+1)}|}{|x^{(k)}|^p}=0$. the convergence is ∞.

Problem 4:

Gold section method $U_{k+1}=(1-p)U_k$

$\lim_{k\to\infty}U_k=0$. $\lim_{k\to\infty}\frac{|U_{k+1}|}{|U_k|}=1-p>0$, so the convergence is 1.

Problem 5:

$Q=\begin{bmatrix}6&4\\4&6\end{bmatrix}$, $|Q-\lambda I|=0\Rightarrow\lambda_1=2,\lambda_2=10$

So $0<\alpha<\frac{2}{\lambda_{max}(Q)}=\frac{2}{10}$

Problem 6:

(a) $x^*=\frac{f'(x^{(k)})}{f''(x^{(k)})}=x^{(k)}-\frac{4(x^{(k)}-x_0)^3}{12(x^{(k)}-x_0)^2}=\frac{2x^{(k)}+x_0}{3}$

(b) $y^{(k+1)}=|x^{(k+1)}-x^{(k)}|=\left|\frac{2x^{(k)}+x_0}{3}-x^{(k)}\right|=\frac{2x^{(k)}-2x^{(k)}}{3}=\frac{4(x^{(k)}-x_0)}{3}=\frac{4y^{(k)}}{3}=\frac{4}{9}(x^{(k-1)}-x^{(k-2)})$

$y^{(k)}=|x^{(k)}-x^{(k-1)}|=\frac{2(x^{(k-1)}-x^{(k-2)})}{3}=\frac{2}{3}(x^{(k-1)}-x^{(k-2)})$

$\therefore y^{(k+1)}=\frac{2}{3}y^{(k)}$

(c) $y^{(k+1)}=\left(\frac{2}{3}\right)^k y^{(1)}=\left(\frac{2}{3}\right)^k(x^{(1)}-x^{(0)})$

$k\to\infty$. $y^{(k+1)}\to 0\Rightarrow x^{(k+1)}\to x^{(0)}$

$f''(x)=12(x-x_0)^2$

(d) $\lim_{k\to\infty}\frac{|y^{(k+1)}|}{|y^{(k)}|}=\frac{2}{3}>0$. So the convergence is 1.

(e) because in this problem, $f''(x)=12(x-x_0)^2$, when $x=x_0$, $f''(x)=0$, do not satisfy the condition $f''(x)$ is invertible.

Problem 7.

(a) $f(x)\geq 0$. so $x^*=[1,1]^T$, $f(x^*)=0$. it is a global minimizer

(b) $\nabla f(x)=\begin{bmatrix}400x_1^3-400x_1x_2+2x_1-2\\200(x_2-x_1^2)\end{bmatrix}$, $F(x)=\begin{bmatrix}1200x_1^2-400x_2+2&-400x_1\\-400x_1&200\end{bmatrix}$

$F(x)^{-1}=\frac{1}{800x_1^2-400x_2+200}\begin{bmatrix}200&400x_1\\400x_1&1200x_1^2-400x_2+2\end{bmatrix}$

$x^{(1)}=x^{(0)}-F(x^{(0)})\nabla f(x^{(0)})=[1,0]^T$

$x^{(2)}=x^{(1)}-F^{-1}(x^{(1)})\nabla f(x^{(1)})=[1,1]^T$

(c) $x^{(1)}=x^{(0)}-\alpha\nabla f(x^{(0)})=[0.1,0]^T$

$x^{(2)}=x^{(1)}-\alpha\nabla f(x^{(1)})=[0.07,0.03]^T$

Problem 8.

assume $d^{(j)}\neq 0$, $j=1,2,\cdots k$, $d^{(i)T}Qd^{(j)}\neq 0$

① For $n=0$, it is obvious true.

② assume $n=k$: $\{d^{(0)},d^{(1)},\cdots d^{(k)}\}$ is Q-conjugate.

③ prove than $n=k+1$, $\{d^{(0)},d^{(1)},\cdots d^{(k)},d^{(k+1)}\}$ is Q-conjugate

for each $j=0,1,\cdots k$,

$d^{(k+1)T}Qd^{(j)}=\left(p^{(k+1)T}-\sum_{i=0}^{k}\frac{p^{(k+1)T}Qd^{(i)}}{d^{(i)T}Qd^{(i)}}\right)Qd^{(j)}$

$=p^{(k+1)T}Qd^{(j)}-\sum_{i=0}^{k}\frac{p^{(k+1)T}Qd^{(i)}}{d^{(i)T}Qd^{(i)}}Qd^{(i)T}Qd^{(j)}$

$\because d^{(i)T}Qd^{(j)}=0$. for $i\neq j$,

$\therefore d^{(k+1)T}Qd^{(j)}=p^{(k+1)T}Qd^{(j)}-\frac{p^{(k+1)T}Qd^{(j)}}{d^{(j)T}Qd^{(j)}}d^{(j)T}Qd^{(j)}=0$

④ prove $d^{(k)}\neq 0$

$\Rightarrow d^{(k)}=\alpha_0p^{(0)}+\alpha_1p^{(1)}+\cdots+\alpha_kp^{(k)}$ $\alpha_0,\alpha_1,\cdots\alpha_k$ not all zero

1. For $n=0$. $d^{(0)}=p^{(0)}$

2. Assume $n=k$, $d^{(k)}=\alpha_0p^{(0)}+\alpha_1p^{(1)}+\cdots+\alpha_kp^{(k)}$ $\alpha_0,\alpha_1,\cdots,\alpha_k$ not all zero

3. when $n=k+1$, $d^{(k+1)}=p^{(k+1)}-\sum_{j=0}^{k}\beta_jd^{(j)}$

$=p^{(k+1)}-\sum_{j=0}^{k}\beta_j\sum_{i=0}^{j}\alpha_ip^{(i)}$

$=p^{(k+1)}-\sum_{i=0}^{k}\sum_{j=i}^{k}\beta_j\alpha_ip^{(i)}$

So $d^{(k+1)}$ is a nonzero linear combination of $p^{(0)},p^{(1)}\cdots p^{(k+1)}$

Problem 9.

because Q is symmetric, exists $\{d^{(1)},d^{(2)}\cdots d^{(n)}\}$, such that $Qd^{(i)}=\lambda_id^{(i)}$, $i=1,2,\cdots n$

$d^{(i)T}d^{(j)}=0$, $\forall i\neq j$, λ_i are eigenvalues of Q. Therefore $i\neq j$, $d^{(i)T}Qd^{(j)}=d^{(i)T}(\lambda_jd^{(j)})=\lambda_jd^{(i)T}d^{(j)}=0$

hence that $\{d^{(1)},d^{(2)}\cdots d^{(n)}\}$ is Q-conjugate

we define $\lambda_i=\frac{d^{(i)T}Qd^{(i)}}{d^{(i)T}d^{(i)}}$

since Q is positive define and $\{d^{(1)},d^{(2)}\cdots d^{(n)}\}$ Q-conjugate, the set is linearly independent

so $D=\begin{bmatrix}d^{(1)}\\\vdots\\d^{(n)}\end{bmatrix}$ is nonsingular.

because $d^{(i)T}Qd^{(j)}=d^{(i)T}\lambda_jd^{(j)}$

so $DQD^{T}=D(\lambda_jd^{(j)})\Rightarrow QD^{T}=\lambda_jD^{T}$.

Problem 10.

命令行窗口

```
>> options(2) = 10^(-4);
>> options(3) = 10^(-4);
>> steep_desc('g',[-4;5;1],options)
Terminating: Norm of difference between iterates :
    1.0000e-04
Final point =
    4.0023e+00    3.0000e+00   -4.9859e+00
Number of iterations =
        7
```

Rosenbrock's function:

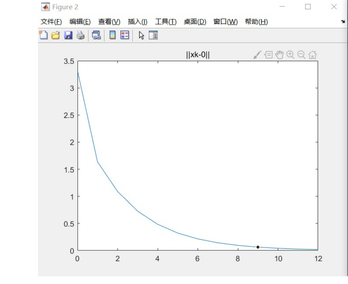
命令行窗口

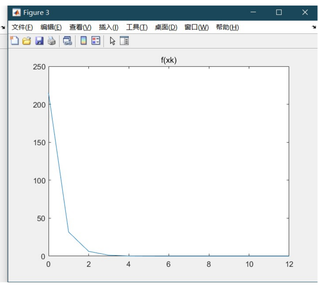
```
>> options(2) = 10^(-4);
>> options(3) = 10^(-4);
>> Untitled2
>>steep_desc('g',[-2;2],options)
Terminating: Norm of difference between iterates le
    1.0000e-04
Final point =
    9.9992e-01    9.9993e-01
Number of iterations =
       6031
```

Problem 11:

命令行窗口

```
>> test
x_optimization =
    0.0184
   -0.0018
    0.0029
    0.0029
f_optimization =
    2.8761e-06
```





Problem 12.

see the attached notable $f(x)$.