9.6 the feedback structure and state observer of linear time-invariable system

The poles of the closed loop system have significant connection with system performance, which is introduced in classic control theory. In state space analysis, the **State Feedback and Output Feedback** are often used to allocate the poles to improve system performance.

- State/Output feedback and Poles Allocation
- Poles Allocation Conditions
- Poles Allocation Algorithm
- State Observer

9.6.1 The feedback structure of linear time-invariable system

The feedback structures: state feedback and output feedback

1. State Feedback

An n-dimension linear time-invariable system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

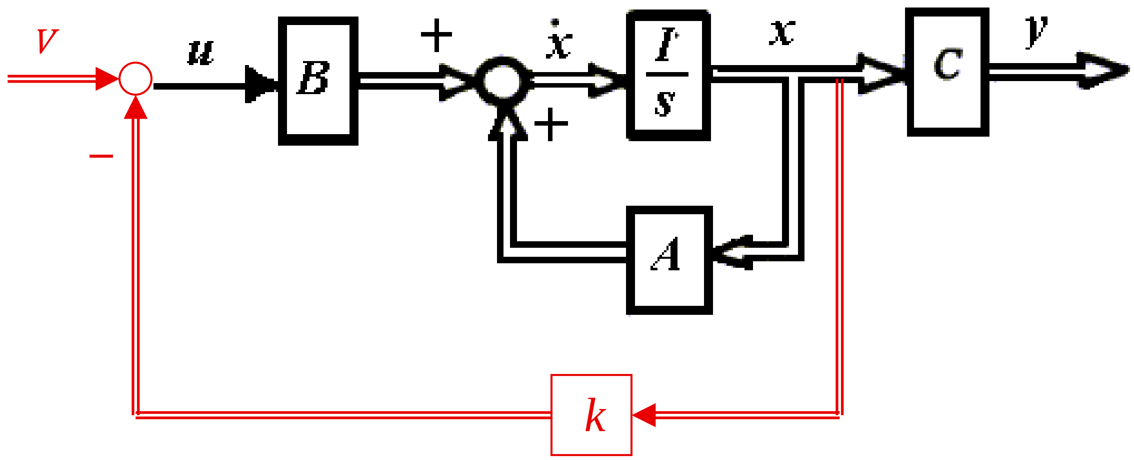
$$y = Cx$$

In which, x_n, u_p, y_q are n-dimension, p-dimension and q-dimension vectors; A,B,C are $n\times n$, $n\times p$ and $q\times n$ real matrix.

If we choose the linear function of state variable as the control value u.

$$u = v - Kx$$

it is called linear direct state feedback(线性直接状态反馈), or state feedback in short. (all states can be the feedback).



The system with state feedback

The dynamic of the state feedback system.

$$\dot{x} = (A - BK)x + Bv$$

$$y = Cx \qquad \text{(unchanged)}$$

Transfer function matrix is:

$$G_K(s) = C(sI - A + BK)^{-1}B$$

2. Output Feedback

The state the system won't always be observed, or the state will be uncontrollable, thus, the application of state feedback will be limited.

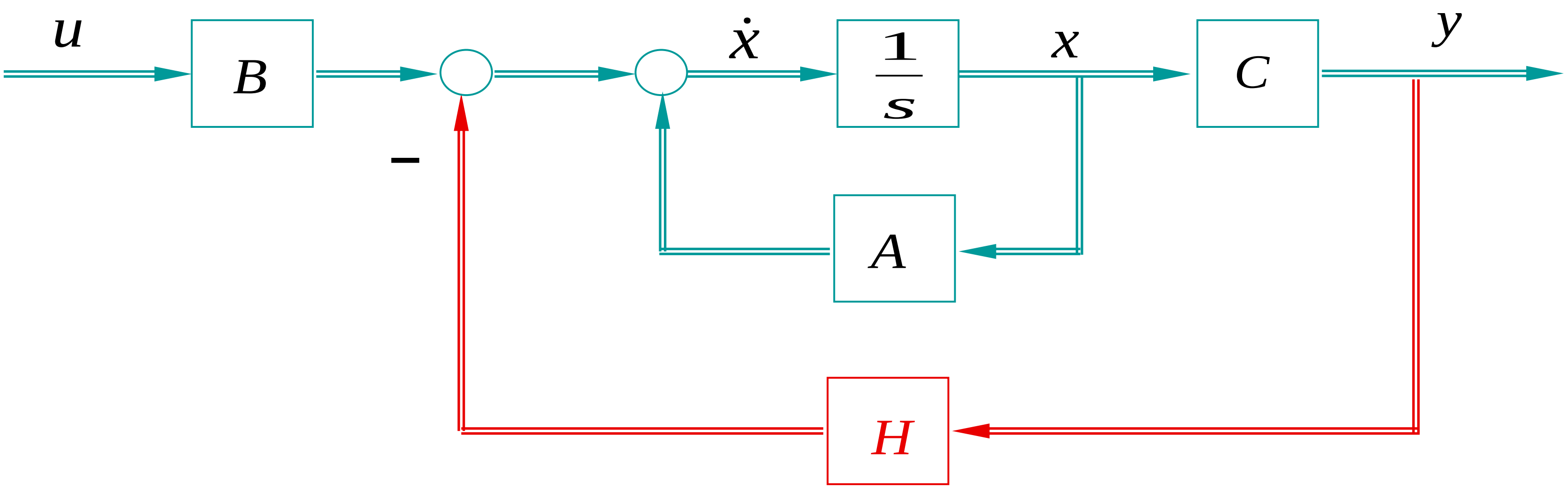
Another kind of feedback is the output feedback, whose purpose is to keep system stable at first and then improve the performance of the system.

Two types of output feedback:

1 Feedback to the **differential of the state**;

2 Feedback to the **consulting input**.

- **Feedback to the differential of the state**

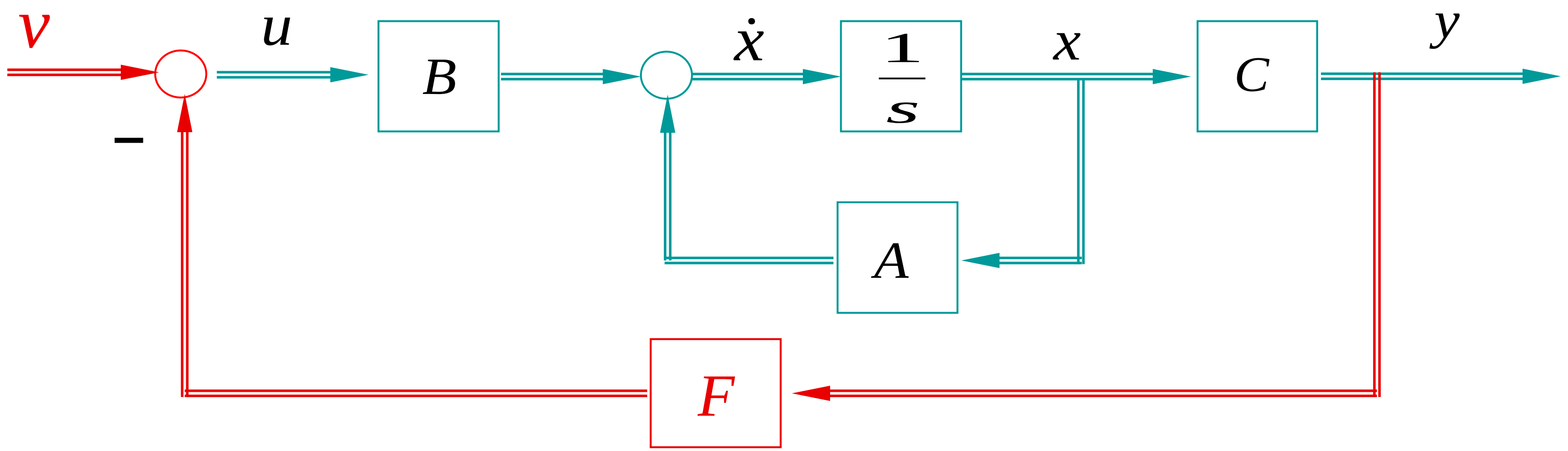


State space : $\dot{x} = Ax + Bu - Hy = (A - HC)x + Bu$

$y = Cx$

Transfer function: $G_H(s) = C(sI - A + HC)^{-1}B$

- **Feedback to the consulting input**



Choose the linear function of output y to be the control value u

$$u = v - Fy$$

State space expression:

$$\dot{x} = (A - BFC)x + Bv$$

$$y = Cx$$

Transfer function matrix is:

$$H_F = C(sI - A + BFC)^{-1}B$$

9.6.2 State feedback and poles allocation

➢ Formulation of the issues

A SISO linear time-invariable system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

$$x(t) \in R^n, u(t) \in R^1, A \in R^{n\times n}, B \in R^{n\times 1}$$

Select linear feedback controller: $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{K}\boldsymbol{x} \left(K \in R^{1\times n} \right)$

In which, $K \in R^{1\times n}$ is called the <u>state feedback gain matrix</u> or <u>linear state feedback matrix</u>.

$$\dot{x}(t) = (A - BK)x + Bv$$

The eigenvalues of the matrix *A*-*BK* are the poles of the closed-loop system. Choose the value of feedback matrix *K*, we can construct the matrix *A*-*BK* to an asymptotically stable system(渐近稳定系统), and allocate the closed poles to the expected place, which is called **Poles Allocation**.

Poles Allocation Conditions

Assume the control input u is unrestricted, if we choose following control input: $u=v-Kx$, with the linear state feedback matrix K.

Consider the allocation conditions, we have the poles allocation theorem:

Theorem (n.s. condition): if a linear time-invariable system is **completely controllable**, all poles can be allocated by the linear state feedback. (Such condition will be available in both SISO systems and MIMO systems.)

Proof: (SISO system for example)

1. Sufficiency

If system (A,b) is controllable, exists nonsingular linear transform -ation $x = P\bar{x}$, which can transform the system to the controllable canonical form: $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \bar{b} = P^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Import state feedback: $u = v - kx = v - kP\bar{x} = v - \bar{k}\bar{x}$

Then: $\bar{k} = kP = [\bar{k}_n \quad \bar{k}_{n-1} \quad \cdots \bar{k}_2 \quad \bar{k}_1]$

The coefficient matrix of the closed system is:

$$\bar{A} - \overline{bk} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ (-a_n - \bar{k}_n) & (-a_{n-1} - \bar{k}_{n-1}) & & \cdots & (-a_1 - \bar{k}_1) \end{bmatrix}$$

From the controllable canonical form, we have the system’s closed-loop eigenequation:

$$\det[sI - (\bar{A} - \overline{bk})] = s^n + (a_1 + \bar{k}_1)s^{n-1} + \cdots + \left(a_{n-1} + \bar{k}_{n-1}\right)s + a_n + \bar{k}_n$$

There are n coefficients in this n-order eigenequation, which is unrestricted and selected arbitrarily by $\bar{k}_1,\ \bar{k}_2,\cdots\bar{k}_n$, as well as the eigenvalues of $(\bar{A}-\overline{bk})$. Thus, the poles can be allocated freely.

2. Necessary

If system (A,b) is uncontrollable, it means that some states will not be controlled by u. Thus, the uncontrollable poles won't be affected by u after importing the state feedback.

Use the reduction to absurdity to prove: if the system is uncontrollable we have : $rank[B\ AB\ \ldots\ A^{n-1}B]=q<n$.

Some states x_i will be independent to the control u. Therefore, the full state feedback is unavailable, thus the eigenvalues cannot be allocated freely. Therefore, it is necessary that system should be completely controllable, for the eigenvalues allocation of matrix (A-BK).

➢ Transforming matrix P of Controllable Canonical Form

If system is completely controllable, there exists nonsingular transformation **P** satisfy $\boldsymbol{x}=\mathbf{P}\boldsymbol{x}_c$, which can transform the system to the controllable canonical form.

Define the transforming matrix: **P=QW**

$$Q=[B \quad AB \quad \cdots \quad A^{n-1}B]$$

$$W=\begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

In which, a_i are coefficients of the eigenpolynomials.

$$|sI-A|=s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n$$

➢ P of Observable Canonical Form

- Define the transforming matrix: $\mathbf{P=(WR)^{-1}}$

- R is observable matrix:

$$R=\begin{bmatrix} C \\ AC \\ \vdots \\ A^{n-1}C \end{bmatrix}$$

- W is coefficient matrix of eigen-equation:

$$W=\begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$|sI-A|=s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n$$

➢ Poles allocation algorithm for SISO system

For system (A,b) and expected eigenvalues $(\lambda_1,\lambda_2,\ldots,\lambda_n)$

We can obtain $1\times n$ dimention feedback gain vector $\boldsymbol{k}$ by following steps:

Step 1: Analyze the controllability of the system, then continue the following steps if system is states completely controllable;

Step 2: Calculate the eigenpolynomial of A;

$$\det[sI - A] = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

Step 3: Calculate the expected eigenpolynomial from closed-loop eigenvalues

$$a^*(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$
$$= s^n + a_1^* s^{n-1} + \cdots + a_{n-1}^* s + a_n^*$$

$$
\bar{A}-\overline{bk}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ (-a_n-\bar{k}_n) & (-a_{n-1}-\bar{k}_{n-1}) & & \cdots & (-a_1-\bar{k}_1) \end{bmatrix}
$$

Step 4: Calculate $\bar{\mathbf{k}}=[a_n^*-a_n \quad a_{n-1}^*-a_{n-1} \quad \cdots \quad a_1^*-a_1]$

Step 5: Calculate transforming matrix **P**=**QW**, if the given state equation is controllable canonical, the **P**=**I**.

Step 6: The state feedback matrix is: $\mathbf{K}=\bar{\mathbf{k}}\mathbf{P}^{-1}$

Ex.9-37 Consider the linear time-invariable system:

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Try to find the state feedback matrix K, by which the poles of the system can be allocated to s= $-2 \pm j4$ and s= -10.

Solution:

Step 1: check the controllable matrix of the system:

$$Q = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$rankQ = 3$$

System is controllable, thus we can allocate the poles freely.

Then obtain the State Feedback Matrix K:

Method 1:the eigenpolynomial of the system is:

$$|sI-A| = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{bmatrix} = s^3+6s^2+5s+1 = s^3+a_1s^2+a_2s+a_3=0$$

$$a_1=6, a_2=5, a_3=1$$

$$(s+2-j4)(s+2+j4)(s+10) = s^3+14s^2+60s+200 = s^3+a_1^*s^2+a_2^*s+a_3^*=0$$

$$\bar{k} = [a_n^*-a_n \quad a_{n-1}^*-a_{n-1} \quad \cdots \quad a_1^*-a_1]$$

$$K = [\ 200-1 \vdots 60-5 \vdots 14-6\ ]P^{-1}$$
$$= [199 \ \ 55 \ \ 8]$$

(here P = I)

Method 2: assume the expected state feedback gain matrix is

$$K = [k_1 \quad k_2 \quad k_3]$$

$$[sI - A + BK] = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3]$$

$$|sI - A + BK| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+k_1 & 5+k_2 & s+6+k_3 \end{vmatrix}$$

$$= s^3 + (6+k_3)s^2 + (5+k_2)s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

$$6 + k_3 = 14, \quad 5 + k_2 = 60, \quad 1 + k_1 = 200$$

$$k_1 = 199, \quad k_2 = 55, \quad k_3 = 8$$

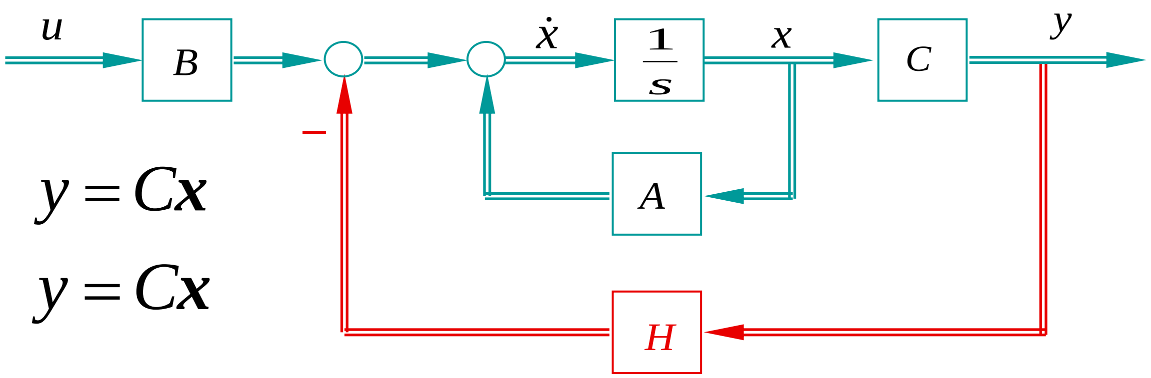
$$K = [199 \quad 55 \quad 8]$$

The same result.

9.6.3 Output feedback and poles allocation

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} - \boldsymbol{h}y \qquad y = C\boldsymbol{x}$$

$$\dot{\boldsymbol{x}} = (A - \boldsymbol{h}C)\boldsymbol{x} + B\boldsymbol{u} \qquad y = C\boldsymbol{x}$$



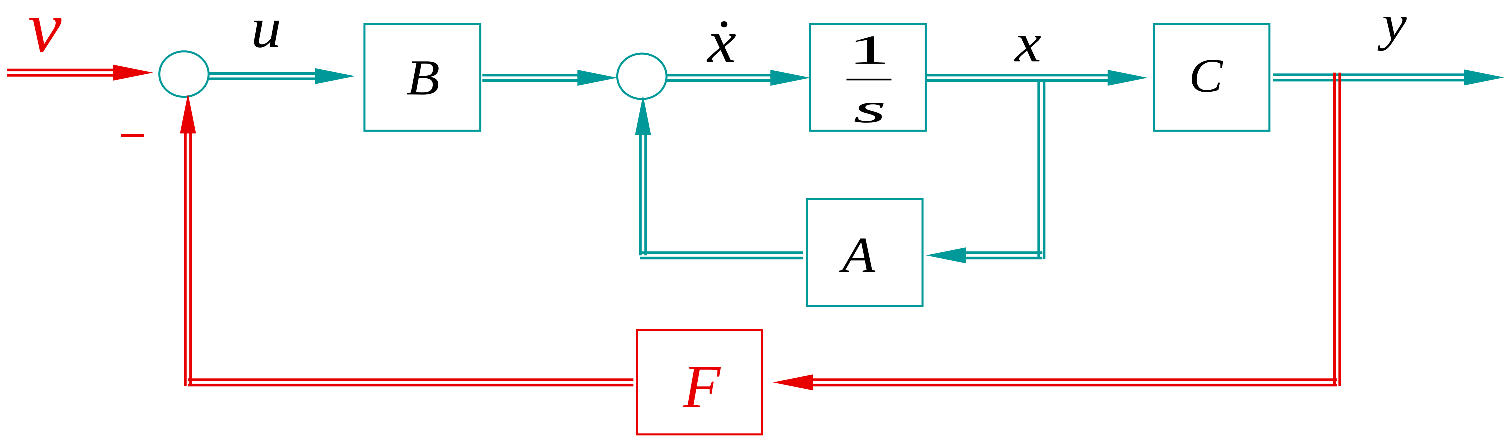
Theorem (n.s. condition): if the system is completely observable, all feedback poles can be allocated by the output to the differential of the state feedback.

Proof: using the MISO system as an example: based on the Dually Theorem, if (A, B, C) is observable, its dual system (A^T, C^T, B^T) is controllable. From the state feedback poles allocation theorem, the eigenvalue of $(A^T - C^T h^T)$ can be allocated freely.

Since the eigenvalues of $(A^T - C^T h^T)$ are equal to which of $(A^T - C^T h^T)^T = A - hC$, therefore, if and only if (A, B, C) is observable, the eigenvalue of $(A - hC)$ can be allocated freely.

To design the output feedback matrix h from expected closed-loop poles, we should compare the eigenpolynomials of expected system with the output feedback system $|\lambda I - (A - hC)|$.

If we use Output to Input feedback (in figure), using MISO system e.g., the feedback matrix $F \in R^{p\times 1}$,



$$u = v - Fy$$

$$\dot{x} = (A - BFC)x + Bv$$

Assume the output feedback FC=K is equal to the state feedback, the poles can be allocated freely by selecting the proper F. However, when state feedback transforms to the output feedback, then F is not a constant matrix, It'll be hard to realize the physics system.

The examples that we cannot allocate the poles by the output to input proportion feedback.

Ex.9-38 Double-in-Single-out system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 5 \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

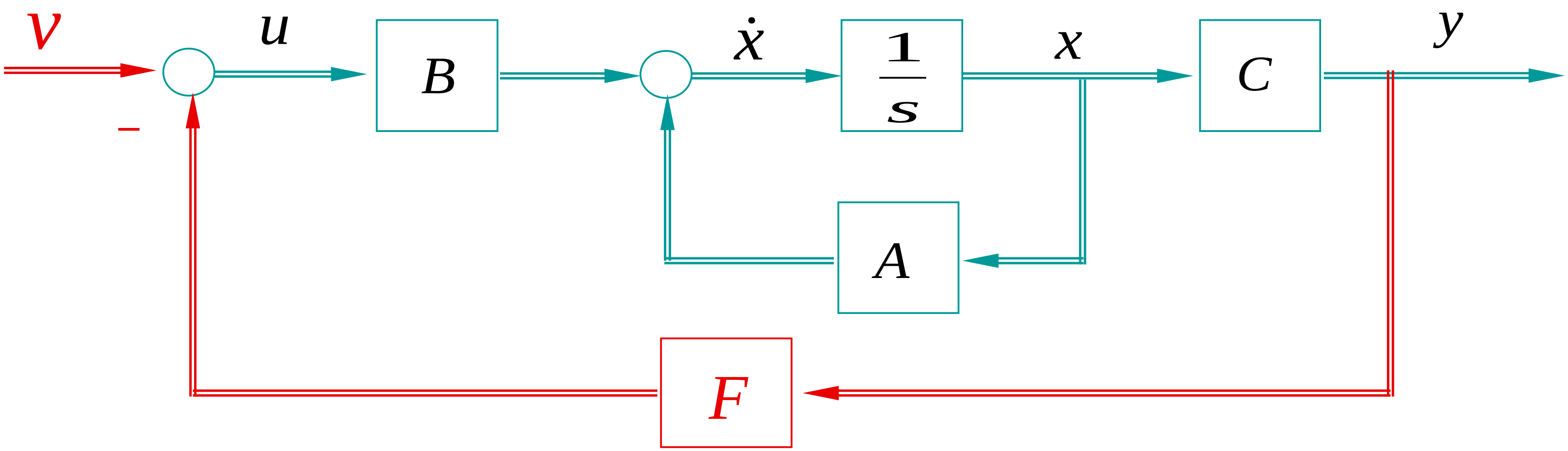
Solution: its eigenpolynomial is:

$$|sI - A| = \begin{vmatrix} s & 0 & -5 \\ -1 & s & 1 \\ 0 & -1 & s+3 \end{vmatrix} = s^3 + 3s^2 + 3s - 5$$

The system is unstable but observable.

Poles allocation by Output to Input feedback, to make system stable.

If use Output to Input feedback (in the figure): $F \in R^{2\times 1}$



$$u = v - Fy$$

$$\dot{x} = (A - BFC)x + Bv$$

Obtain the linear feedback matrix $F=[f_1,f_2]^T$.

$$A-BFC=\begin{bmatrix}0&0&5\\1&0&-1\\0&1&-3\end{bmatrix}-\begin{bmatrix}-2&0\\1&-2\\0&1\end{bmatrix}\begin{bmatrix}f_1\\f_2\end{bmatrix}\begin{bmatrix}0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&5+2f_1\\1&0&-1-f_1+2f_2\\0&1&-3-f_2\end{bmatrix}$$

The closed-loop eigenpolynomial is:

$$\left|sI-(A-BFC)\right|=\begin{bmatrix}s&0&-2f_1-5\\-1&s&f_1-2f_2+1\\0&-1&s+f_2+3\end{bmatrix}$$

$$=s^3+(f_2+3)s^2+(f_1-2f_2+1)s-(2f_1+5)$$

We can stabilize the system by choosing f_1=-3, f_2=-2 rather than achieve the poles allocation freely.

Ex.9-39 Single-in-double-out system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

The eigenpolynomial is:

$$|sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & 1 \\ 1 & 0 & s \end{vmatrix} = s^3 - 1$$

System is unstable.

$$Q = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad rankQ = 3$$

System is controlable.

$$R^T = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad rank R = 3$$

System is observable.

Using the Output to Input feedback to allocate the poles and stabilize the system.

$$u = v - Fy$$

$$\dot{x} = (A - BFc)x + Bv$$

$$A - BFC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -f_1 & 0 & -1 - f_2 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\left| sI - (A - BFc) \right| = \begin{bmatrix} s & -1 & 0 \\ f_1 & s & 1 + f_2 \\ 1 & 0 & s \end{bmatrix} = s^3 + f_1 s - (f_2 + 1)$$

In this situation, the system is always unstable with any matrix F, thus the pole allocation is failed.

9.6.4 System performance affection by the feedback

- The feedback, which modify system's parameter matrix, will influent the system performance?
- Controllability, Observability, Stability, other performance?

✓ The State feedback and Output feedback can stabilize the system

✓ The State feedback won't affect the system controllability, but might change its observability

✓ The output to differential of state feedback won't affect system observability, but might change its controllability

✓ The Output to input feedback won't affect system controllability and observability

✓ The State feedback and Output feedback won't change the zeros of the system.

9.6.5 State Observer

1 Full order state observer

(1) the Proposed Problem

The state feedback needs whole state variables, however, actually most of the state variables are hard to be measured or cannot be measured.

When the state variables cannot be measured, the state observer (or state estimator, state reconfigurer) are proposed, which reconfigures the state x by the output $y=cx$.

If the dimension of the reconfigured state vector is the same with the Plant, the observer is called Full Order State Observer.

(2) Model of the observer

The linear time-invariable system

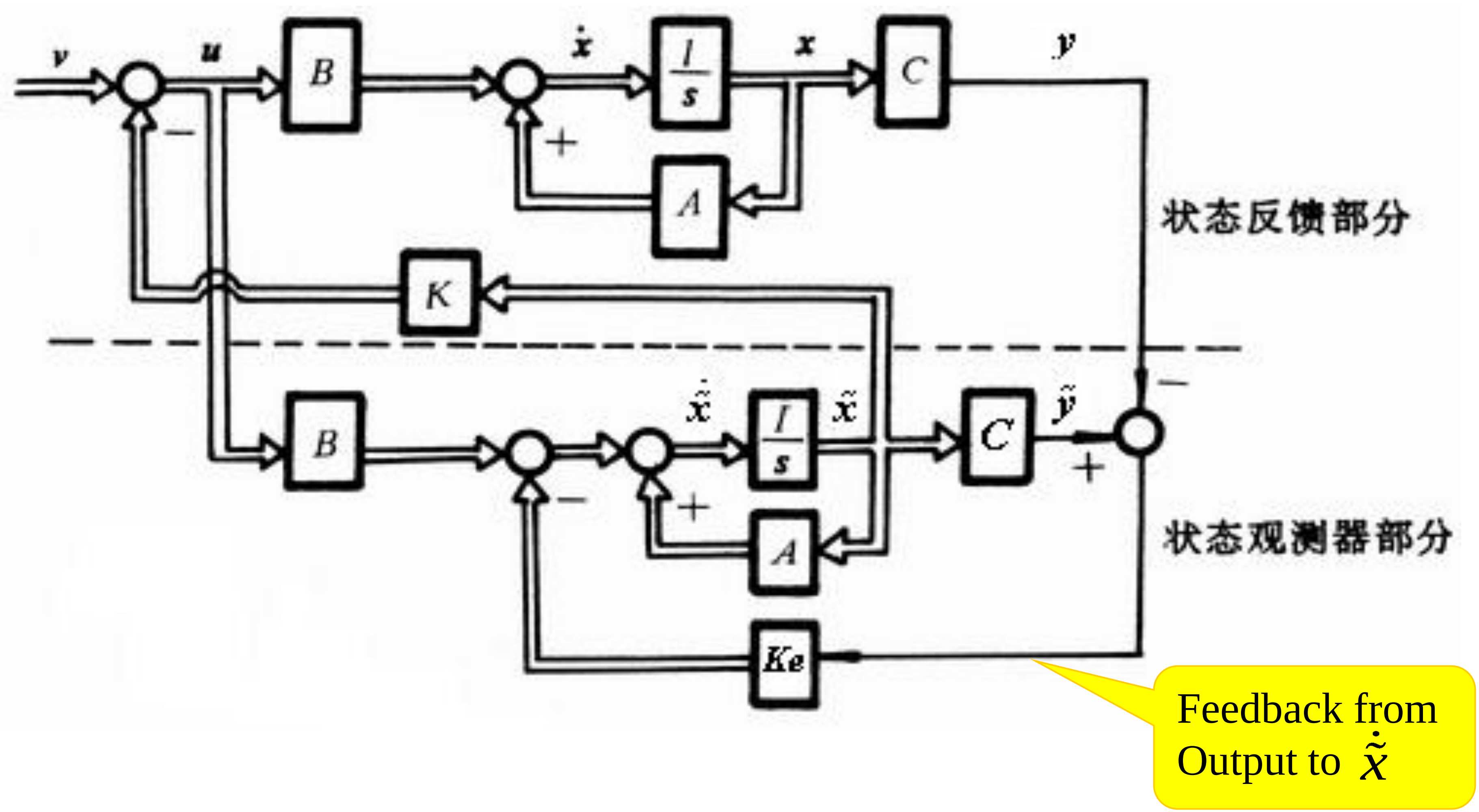
$$\dot{x} = Ax + Bu \qquad y = Cx$$

Compose a simulation system:

$$\dot{\tilde{x}} = A\tilde{x} + Bu \qquad \tilde{y} = c\tilde{x}$$

If $\dot{x}$ and $\dot{\tilde{x}}$ has the same initial value, we can use simulated state variables $\dot{\tilde{x}}$ to estimate $\dot{x}$.

When $\dot{x}$ and $\dot{\tilde{x}}$ are varying, the output y and $\tilde{y}$ will be different as well. We can use $y-\tilde{y}$ to revise the observer model $\dot{\tilde{x}}$, by the feedback from $y-\tilde{y}$ to $\dot{\tilde{x}}$, to achieve $x-\tilde{x}\rightarrow 0$



The observer model is: $\dot{\tilde{x}} = A\tilde{x} + Bu + K_e c(x - \tilde{x})$

If it is satisfied for any initial value that: $\lim\limits_{t \to \infty} [\tilde{x}(t) - x(t)] = 0$

Then, the state observer is: $\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - \tilde{y})$

(3) the selection of linear feedback matrix K_e

The condition of: $\lim\limits_{t\to\infty}[\tilde{x}(t)-x(t)]=0$

How to choose K_e ?

➢ the existent conditions of K_e

Step 1: Confirm the differential equation of $\tilde{x}-x$

Step 2: Consider the state equations of system and observer:

$$\dot{x}=Ax+Bu$$

$$\dot{\tilde{x}}=A\tilde{x}+Bu+K_e c(x-\tilde{x})$$

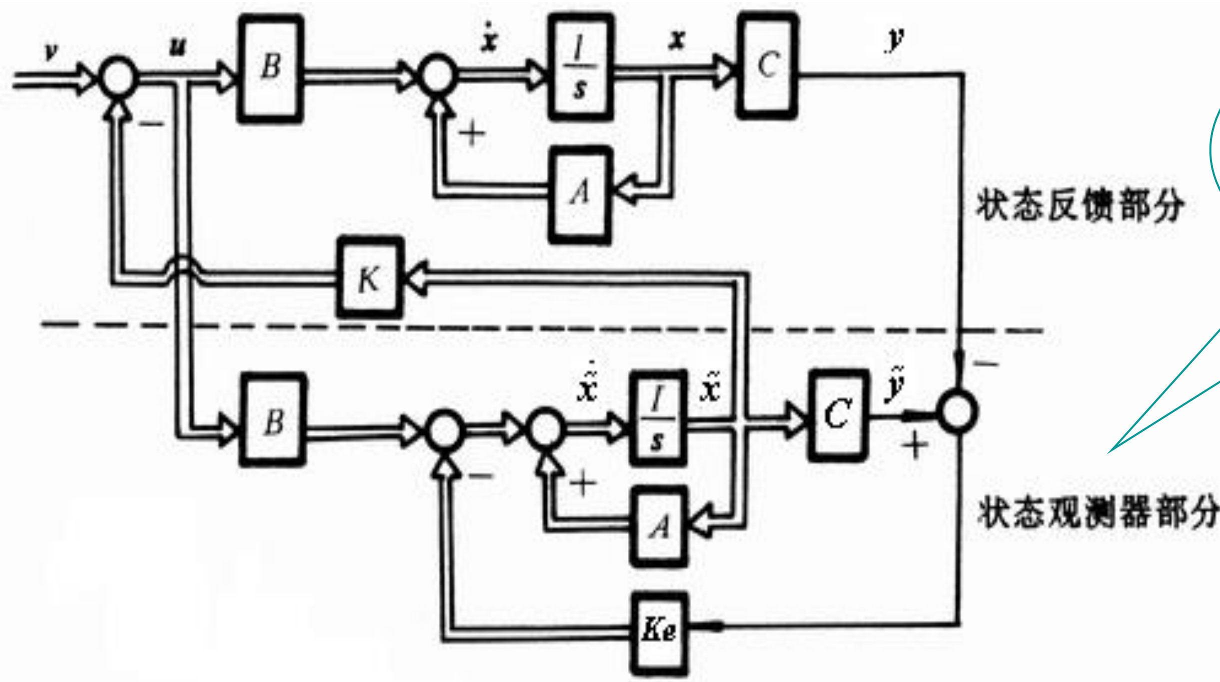
Calculate the minus of above equations: $\dot{\tilde{x}}-\dot{x}=(A-K_e c)(\tilde{x}-x)$

To satisfy: $\lim\limits_{t\to\infty}[\tilde{x}(t)-x(t)]=0$

If and only if the eigenvalue of (***A-K_e c***) are all in the left s plane, $\tilde{x}-x$ will tend to 0 by the exponential law of time *t*.

Solve the differential equation: $\dot{\tilde{x}} - \dot{x} = (A - K_e c)(\tilde{x} - x)$

We have: $x(t) - \tilde{x}(t) = e^{(A-K_eC)(t-t_0)}[x(t_0) - \tilde{x}(t_0)]$



Analyze the existence of $\mathbf{K_e}$ by the figure

If the system (A,B,C) is observable, its state can be estimated by the full order state observer:

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e c(x - \tilde{x}) = (A - K_e c)\tilde{x} + Bu + K_e y$$

In which, matrix K_e is selected as the condition of pole allocation.

➢ the selection of K_e

The decay rate of $\lim_{t\to\infty}[\tilde{x}(t) - x(t)] = 0$ is decided by poles allocation of the estimator.

To satisfy the rapidity of $\tilde{x}$ approaching to x, the poles of state estimator should be allocated to the place far away to the imaginary axis in the ***s plane*** (more minus than the poles of estimated system). However, the parameters of matrix $\mathbf{K_e}$ should not be too large.

1 the selection of $\mathbf{K_e}$ will be restricted by the real condition of the equipment (such as capability, saturation, heat, and stress, etc.)

2 Since the frequency band of observer is broadened, the noise of input u and output y will cause **larger noise** of $\tilde{x}$, which is sensitive to other kind of noise as well.

Therefore, in practical applications, the selection of $\mathbf{K_e}$ should be little faster than the plant.

➢ Calculation of the linear feedback matrix $\mathbf{K_e}$

Transform the system to the observable canonical form to design $\mathbf{K_e}$ by the following steps:

Define the transforming matrix P: $$P = (WR)^{-1}$$

Here, R is the observable matrix:

$$R^T = [C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T]$$

The symmetrical matrix **W** is defined as follow:

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

a_i is the parameters of following eigenequation (i=1,2,…,n):

$$|sI - A| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

Since the system is supposed to be completely observable, thus, the inverse of matrix WR is existed. Under the effect of the linear transformation $x = P\xi$, system can be transformed to the observable canonical form.

$$\tilde{A} = P^{-1}AP = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

$$\tilde{B} = P^{-1}B = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{bmatrix} \qquad \tilde{C} = CP = [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]$$

Follow the method to obtain state feedback matrix $\mathbf{K_e}$ in the poles allocation of state feedback.

$$K_e = P\begin{bmatrix} a_n^* - a_n \\ a_{n-1}^* - a_{n-1} \\ \vdots \\ a_1^* - a_1 \end{bmatrix} = (WR)^{-1}\begin{bmatrix} a_n^* - a_n \\ a_{n-1}^* - a_{n-1} \\ \vdots \\ a_1^* - a_1 \end{bmatrix}$$

In which, a_i and a_i^*, (i=1,2,…,n) are the eigenequations coefficients of the original system and expected state observer. Above equation achieves the desired state observer gain matrix.

If the expected eigenvalues are selected (or expected eigenequation), and the system is completely observable, the full dimensional observer can be designed.

Ex.9-40 Consider the following linear time-invariable system

$$\dot{x} = Ax + Bu \qquad A = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = [0 \;\; 1]$$

$$y = Cx$$

Design a full dimensional state observer with expected eigenvalues: $\mu_1 = -1.8 + j2.4, \quad \mu_2 = -1.8 - j2.4$

Analysis:

The observable matrix:

$$[\, C^T \vdots A^T C^T \,] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The rank of the above matrix is 2, thus, system is completely observable. Then try to derive the expected observer gain matrix in two ways.

Method 1: the state matrix A *is observable canonical form*, therefore, the transforming matrix is: $P=(WR)^{-1}=I$

The eigenequation of the given system is:

$$|sI-A|=\begin{vmatrix} s & -20.6 \\ -1 & s \end{vmatrix}=s^2-20.6=s^2+a_1s+a_2=0$$

$$a_1=0, \quad a_2=-20.6$$

The expected eigenequation of the observer is:

$$(s+1.8-j2.4)(s+1.8+j2.4)=s^2+3.6s+9=s^2+a_1^*s+a_2^*$$

$$a_1^*=3.6, \quad a_2^*=9$$

The gain matrix of the observer is:

$$K_e=(WR)^{-1}\begin{bmatrix} a_2^*-a_2 \\ a_1^*-a_1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9+20.6 \\ 3.6-0 \end{bmatrix}=\begin{bmatrix} 29.6 \\ 3.6 \end{bmatrix}$$

Method 2: Assume $\dot{e} = (A - K_e C)e = 0$

The eigenequation of the observer is:

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} s & -20.6 + k_{e1} \\ -1 & s + k_{e2} \end{vmatrix}$$

$$= s^2 + k_{e2}s - 20.6 + k_{e1} = 0$$

And the expected eigenequation is:

$$s^2 + 3.6s + 9 = 0$$

Compare the coefficient of both eigenequations:

$$k_{e1} = 29.6,\ k_{e2} = 3.6$$

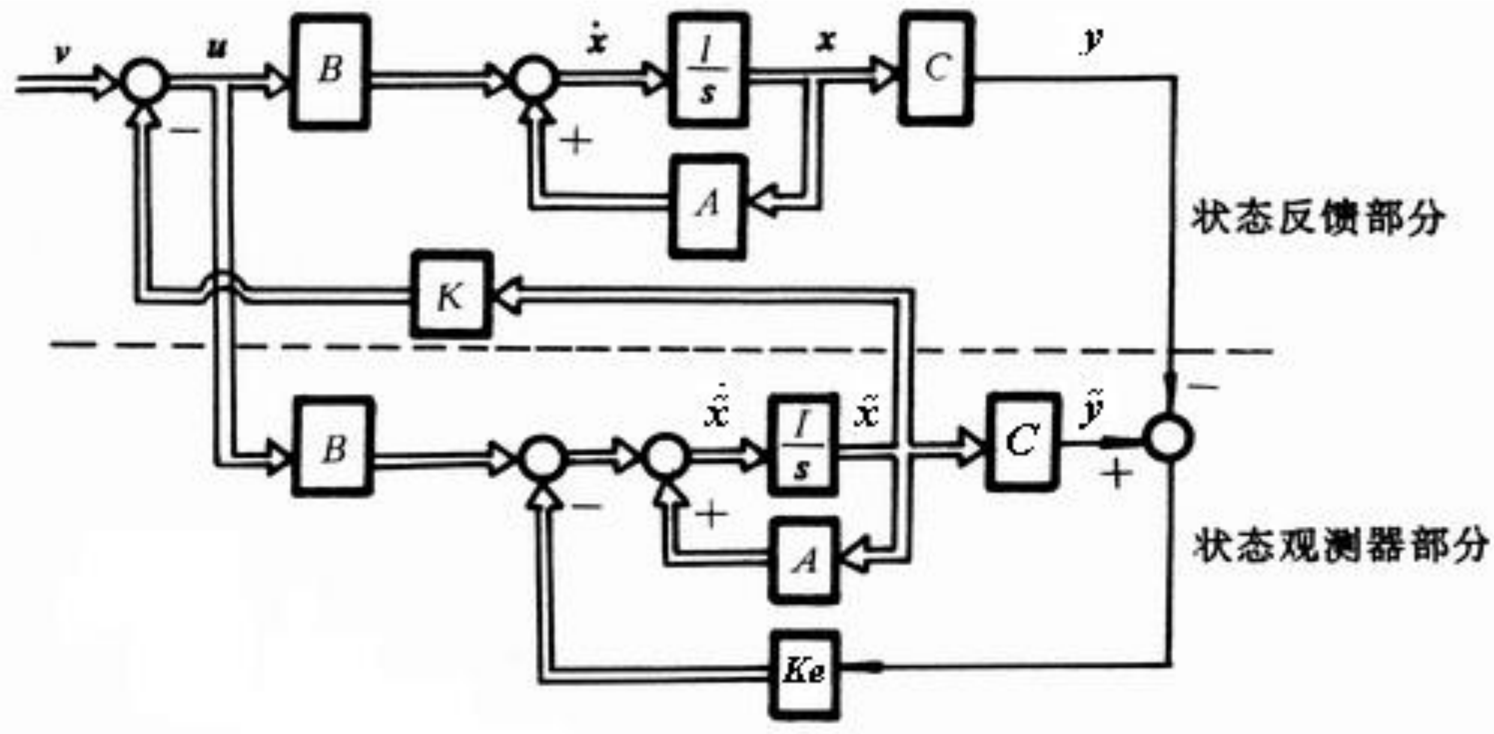
The full dimensional observer is:

$$\dot{\tilde{x}} = (A - K_e C)\tilde{x} + Bu + K_e y$$

$$= \begin{bmatrix} 0 & -9 \\ 1 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 29.6 \\ 3.6 \end{bmatrix} y$$

(4) the affect of the observer to the closed-loop system

- According to the demand of the system performance, we can allocate the poles of the system by importing the state feedback into the system. If we import the observer, does the state feedback matrix K need to be redesigned?
- After the state observer design, will observer feedback matrix K_e be influenced by the state feedback?



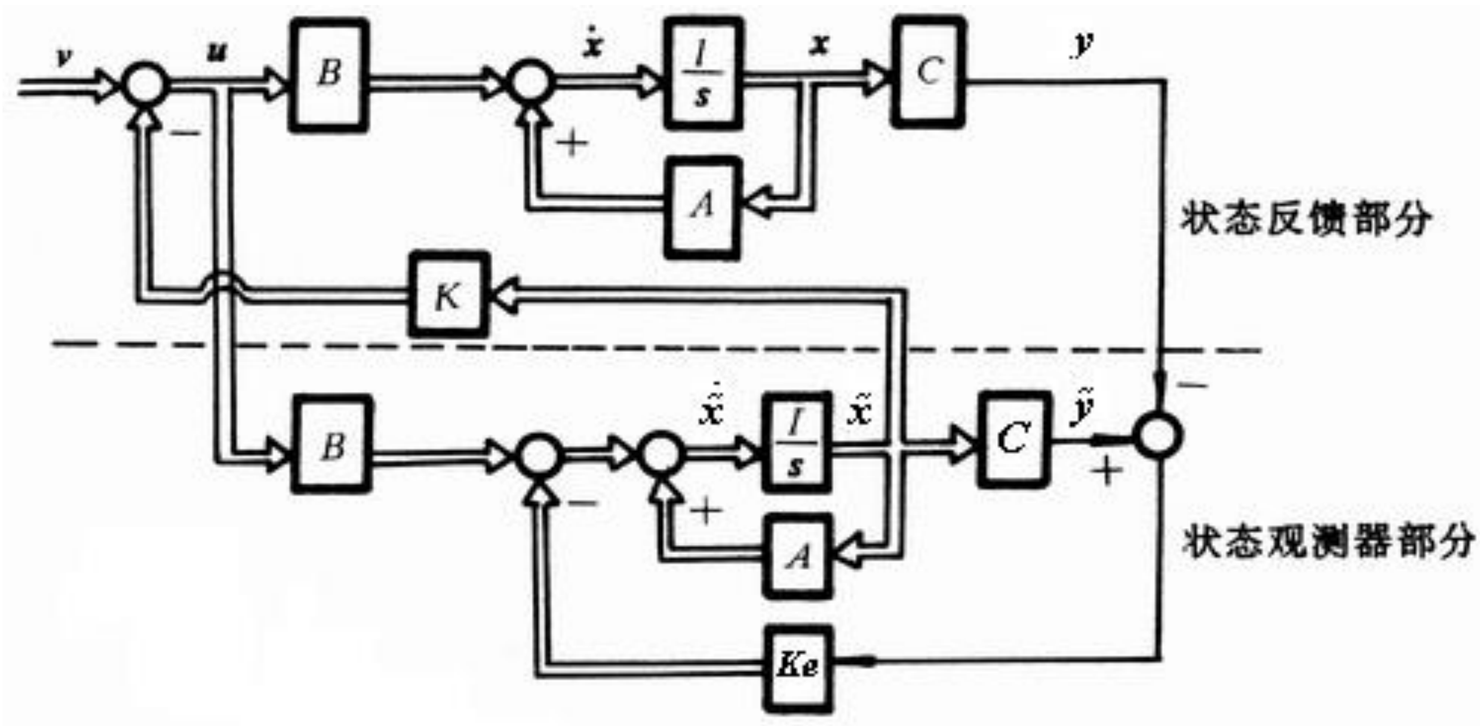
The state function of the state completely controllable and observerable linear time-invariable system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Import the state observer, achieve the feedback from observed state $\tilde{x}(t)$ to the input: $u = v - K\tilde{x}$

$$\dot{x} = Ax - BK\tilde{x} + Bv = (A - BK)x + BK(x - \tilde{x}) + Bv$$



The different between real state and estimate state of the system is:

$$\dot{\tilde{x}} - \dot{x} = (A - K_e c)(\tilde{x} - x)$$

$$\dot{x} = Ax - BK\tilde{x} + Bv = (A - BK)x + BK(x - \tilde{x}) + Bv$$

Combine 2 equations above, we have:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} - \dot{x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - K_e C \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} - x \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

which describe the dynamic character of the observe-state feedback control system, whose eigenequation is:

$$\begin{vmatrix} sI - A + BK & -BK \\ 0 & sI - A + K_e C \end{vmatrix} = 0$$

therefore $\left| sI - A + BK \right| \left| sI - A + K_e C \right| = 0$

The Poles from pure poles allocation

The Poles from pure observer design

Therefore, closed-loop poles of the observe-state feedback control system include the poles from the pure poles allocation and pure observer design.

If the system and observer are both n-order (full dimensional observer), the eigenequation of the whole closed-loop system is 2n-order.

Separation Theorem(分离原理):

If the control system (A,B,C) is controllable and observable, the pole allocation (matrix $\mathbf{K}$) and observer design (matrix $\mathbf{K_e}$) of the system can be calculated separately to derive the state feedback by the estimate value of the state observer.

2 Reduced-order state observer

Assume state vector ***x*** is *n*-dimensional vector, the output vector ***y*** is measurable *m*-dimensional vector. Since *m* output variables are the linear combination of the state variables, *m* of *n* state variables needn't be estimated. Thus, only (*n*-*m*) state variables need to be estimated, which is called (*n*-*m*) order observer, or reduced-order observer.

(1) the model of the reduced-order observer

The state equation of the observable linear system is:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Import the nonsingular transformation:

$$x = Q^{-1}\overline{x}$$

In which $Q_{n\times n} = \begin{bmatrix} D \\ C \end{bmatrix}$ $\begin{array}{l} \text{(n-m) rows} \\ \text{m rows} \end{array}$

C is (m×n) matrix, D is the arbitrary (n-m)×n matrix, which make Q nonsingular.

Separate x to x1 and x2, in which x2 is composed by m states obtained directly from output y.

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \qquad \bar{y} = \bar{C}\bar{x}$$

In which， $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$ $\qquad \bar{A} = QAQ^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$

$$\bar{B} = QB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \qquad \bar{C} = CQ^{-1} = C\begin{bmatrix} D \\ C \end{bmatrix}^{-1}$$

Since: $$C = C\begin{bmatrix} D \\ C \end{bmatrix}^{-1}\begin{bmatrix} D \\ C \end{bmatrix} = \bar{C}\begin{bmatrix} D \\ C \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} D \\ C \end{bmatrix}$$

We have $\bar{C} = \begin{bmatrix} 0 & I \end{bmatrix}$ m rows

(n-m) columns (m) columns

Thus $$\bar{y} = \bar{C}\bar{x} = \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \bar{x}_2$$

State equation is: $$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u$$

The state equation can be written as:

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \underline{\bar{A}_{12}\bar{y} + \bar{B}_1 u}$$

$$\dot{\bar{y}} = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{y} + \bar{B}_2 u$$

Assume: $v = \bar{A}_{12}\bar{y} + \bar{B}_1 u$

Then v is the input of (n-m) dimensional subsystem

Assume $z = \dot{\bar{y}} - \bar{A}_{22}\bar{y} - \bar{B}_2 u$

The dynamic of (n-m) dimensional subsystem is:

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + v$$

$$z = \bar{A}_{21}\bar{x}_1$$

The plant and subsystem are observable, the state observer of the (n-m) dimensional subsystem is:

$$\dot{\tilde{\bar{x}}}_1 = \bar{A}_{11}\tilde{\bar{x}}_1 + v + K_e(z - \tilde{z}) \qquad \dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - \tilde{y})$$

(2) Design of K_e

The state observer can be written further:

$$\begin{aligned}\dot{\tilde{\bar{x}}}_1 &= \bar{A}_{11}\tilde{\bar{x}}_1 + v + K_e(z - \tilde{z}) \\ &= \bar{A}_{11}\tilde{\bar{x}}_1 + \bar{A}_{12}\bar{y} + \bar{B}_1 u + K_e(\bar{A}_{21}\bar{x}_1 - \bar{A}_{21}\tilde{\bar{x}}_1) \\ &= (\bar{A}_{11} - K_e\bar{A}_{21})\tilde{\bar{x}}_1 + (\bar{A}_{12}\bar{y} + \bar{B}_1 u) + K_e(\dot{y} - \bar{A}_{22}\bar{y} - \bar{B}_2 u)\end{aligned}$$

Then select state variables to for the differential part $\dot{y}$ cancellation:

$$w = \tilde{\bar{x}}_1 - K_e y$$

$$\dot{w} = \dot{\tilde{\bar{x}}}_1 - K_e \dot{y}$$

$$\dot{w} = (\bar{A}_{11} - K_e\bar{A}_{21})w + (\bar{B}_1 - K_e\bar{B}_2)u + [(\bar{A}_{11} - K_e\bar{A}_{21})K_e + \bar{A}_{12} - K_e\bar{A}_{22}]\bar{y}$$

From above analysis, the state feedback vector is composed by two parts:

From observer

$$\tilde{\bar{x}} = \begin{bmatrix} \tilde{\bar{x}}_1 \\ \bar{y} \end{bmatrix} = \begin{bmatrix} w + K_e \bar{y} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix} w + \begin{bmatrix} K_e \\ I_m \end{bmatrix} \bar{y} = \begin{bmatrix} I_{n-m} & K_e \\ 0 & I_m \end{bmatrix} \begin{bmatrix} w \\ \bar{y} \end{bmatrix}$$

From output $\bar{y} = \bar{x}_2$

The error of the reduced order observer.

$$\dot{\bar{x}} - \dot{\tilde{\bar{x}}} = (\bar{A}_{11} - K_e \bar{A}_{21})(\bar{x}_1 - \tilde{\bar{x}}_1)$$

Select the proper matrix K_e to allocate poles of the state observer arbitrarily and make the error having approving decay rate.

Ex.9-41 the state equation of the system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Try to design the reduced observer with the eigenvalue: λ=3.

Solution: (1) Check the observability of the system

$$rank\begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T \end{bmatrix} = rank\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} = 3$$

System is observable.

And m=2, the reduced-order observer should be 1 dimension.

(2) Choose the matrix Q

$$Q=\begin{bmatrix} D \\ C \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},\quad Q^{-1}=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(3) The dynamic equations and matrices after the nonsingular linear transformation

$$\dot{\bar{x}}=\bar{A}\bar{x}+\bar{B}u \qquad \bar{y}=\bar{C}\bar{x}$$

$$\bar{A}=QAQ^{-1}=\left[\begin{array}{c:cc} 1 & 0 & 0 \\ \hdashline 0 & -1 & 0 \\ 1 & 0 & 1 \end{array}\right]=\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

$$\bar{B}=QB=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix},\bar{C}=CQ^{-1}=\begin{bmatrix} 0 & I_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4) Dynamic equation of reduced-order observer

$$\dot{w}=(\bar{A}_{11}-K_e\bar{A}_{21})w+(\bar{B}_1-K_e\bar{B}_2)u+[(\bar{A}_{11}-K_e\bar{A}_{21})K_e+\bar{A}_{12}-K_e\bar{A}_{22}]\bar{y}$$

import

$$\bar{y}=\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix},\qquad u=\begin{bmatrix}u_1\\ u_2\end{bmatrix},\qquad \bar{A}_{11}-K_e\bar{A}_{21}=1-k_{e2}$$

$$\bar{B}_1-K_e\bar{B}_2=\begin{bmatrix}-k_{e1} & 1-2k_{e2}\end{bmatrix},\qquad \bar{A}_{12}=\begin{bmatrix}0 & 0\end{bmatrix},\qquad K_e\bar{A}_{22}=\begin{bmatrix}-k_{e1} & k_{e2}\end{bmatrix}$$

$$\dot{w}=(1-k_{e2})w+[-k_{e1}\quad 1-2k_{e2}]\begin{bmatrix}u_1\\ u_2\end{bmatrix}+[2k_{e1}-k_{e1}k_{e2}\quad -k_{e2}^2]\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}$$

$$\tilde{\bar{x}}_1=w+K_e y=w+k_{e1}\bar{x}_2+k_{e2}\bar{x}_3$$

(5) Derive matrix K_e

Eigenequation of observer: $\left|sI-(\bar{A}_{11}-K_e\bar{A}_{21})\right|=s-(1-k_{e2})=0$

Expected eigenequation: s +3=0

Combined above equations: $k_{e2}=4$

k_{e1} has no relationship with the eigenvalue allocation, which is 0。

Observer equation is:

$$\dot{w} = -3w + \begin{bmatrix} 0 & -7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & -16 \end{bmatrix} \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

$$= -3w - 7u_2 - 16\bar{x}_3$$

$$\tilde{\bar{x}}_1 = w + 4\bar{x}_3$$

$$\tilde{\bar{x}} = \begin{bmatrix} \tilde{\bar{x}}_1 \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \tilde{\bar{x}}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

(6) Transform to the original state space by the estimated values

$$\tilde{x} = Q^{-1}\tilde{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\bar{x}}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ -\tilde{\bar{x}}_1 + \bar{x}_3 \\ \tilde{\bar{x}}_1 \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ -w - 3\bar{x}_3 \\ w + 4\bar{x}_3 \end{bmatrix}$$

Then replace the real state value x by the estimated one: $\tilde{x}$ to derive the state feedback.

From the output equation:

$$y_1 = x_1 = \tilde{x}_1 = \bar{x}_2$$
$$y_2 = x_2 + x_3 = \tilde{x}_2 + \tilde{x}_3 = (-w - 3\bar{x}_3) + (w + 4\bar{x}_3) = \bar{x}_3$$

The relationship between $\tilde{x}$ and original system output is:

$$\tilde{x} = \begin{bmatrix} y_1 \\ -w - 3y_2 \\ w + 4y_2 \end{bmatrix}$$

Ex.9-42 Consider the design of a regulator system, the given linear time-invariable system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

$$y = C\boldsymbol{x}$$

in which:

$$A = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0]$$

its closed-loop poles are: $s = u_i (i = 1, 2)$

$$\mu_1 = -1.8 + j2.4, \mu_2 = -1.8 - j2.4$$

using observe-state feedback control, the expected eigenvalue is:

$$\mu_1 = \mu_2 = -8$$

try to derive the related state feedback gain matrix K and observer gain matrix K_e.

Solution:

(1) Design state feedback matrix K

The expected eigenequation is:

$$(s+1.8+j2.4)(s+1.8-j2.4)=s^2+3.6s+9$$

The closed-loop eigenequation after the state feedback is:

$$|sI-A+BK| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \right| = \begin{vmatrix} s & -1 \\ k_1-20.6 & k_2 \end{vmatrix}$$

$$= s^2+k_2s+k_1-20.6$$

Thus $\quad K=[29.6 \quad 3.6]$

If we use such state feedback gain matrix *K*, the control input *u is:*

$$u=v-K\boldsymbol{x}=v-[29.6 \;\; 3.6]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which can be written by using the estimated state variable as the feedback, as well.

$$u=v-K\tilde{\boldsymbol{x}}=v-[29.6 \;\; 3.6]\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

(2) Derive the state observer gain matrix K_e

The eigenvalue of system matrix A is:

$$|sI - A| = \begin{vmatrix} s & -1 \\ -20.6 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1 s + a_2$$

$$a_1 = 0, \quad a_2 = -20.6$$

The expected eigenequation of such observer is:

$$(s-\mu_2)(s-\mu_2) = (s+8)(s+8) = s^2 + 16s + 64$$

$$= s^2 + a_1^* s + a_2^*$$

$$a_1^* = 16, \quad a_2^* = 64$$

Derive the observer gain matrix:

$$K_e = (WR)^{-1} \begin{bmatrix} a_2^* - a_2 \\ a_1^* - a_1 \end{bmatrix}$$

$$R^T = [C^T \vdots A^T C^T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$K_e = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 64 + 20.6 \\ 16 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 84.6 \\ 16 \end{bmatrix} = \begin{bmatrix} 16 \\ 84.6 \end{bmatrix}$$

State equation of the observer is:

$$\dot{\tilde{x}} = (A - K_e C)\tilde{x} + Bu + K_e y$$

Import $u = v - K\tilde{\mathbf{x}}$ into the observer equation:

$$\dot{\tilde{\mathbf{x}}} = (A - K_e C - BK)\tilde{\mathbf{x}} + K_e y + Bv$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} - \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} [1 \ \ 0] - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [29.6 \ \ 3.6] \right\} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$= \begin{bmatrix} -16 & 1 \\ -93.6 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$\frac{U(s)}{-Y(s)} = K(sI - A + K_e C + BK)^{-1} K_e$$

$$= [29.6 \quad 3.6] \begin{bmatrix} s+16 & -1 \\ 93.6 & s+3.6 \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 84.6 \end{bmatrix}$$

$$= \frac{778.16s + 3690.72}{s^2 + 19.6s + 151.2}$$