9.3 State-space Establishing of Linear System

General Methodology:

1. From Physics Mechanism of System

2. From Differential Equations of System

3. From Transfer Functions of System

4. From State-variable Diagram of System

5. Linear Transformation of State space
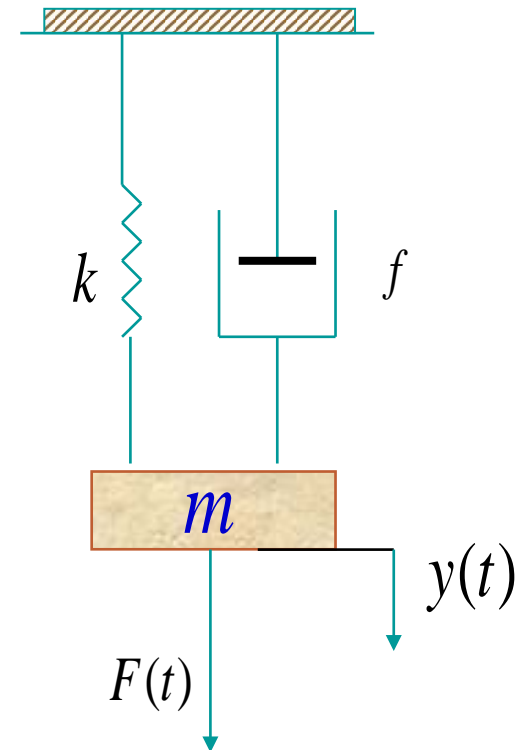
9.3.1 From Physics Mechanism of System

Ex.9-2 Mechanism system composed by force, spring(弹簧) and damper(阻尼器) without gravity(重力).

from Newton's law

$$m\frac{d^2y}{dt^2} + f\frac{dy}{dt} + ky = F(t)$$

in which, F(t) is Input, y(t) is Output.

Then, if the original displacement and velocity are available, the system's solution of the certain input is available as well.

Select the displacement and velocity as the state variables

$$x_1 = y, \quad x_2 = v = \dot{y}$$

Input is:

$$u(t) = F(t)$$

State-equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{m}x_1 - \dfrac{f}{m}x_2 + \dfrac{1}{m}u \end{cases}$$

State space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{f}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ex.9-3 The State-space representation of Mechanism without gravity, the input is the pull F, the outputs are the mass: m_1 and m_2, and displacement y_1 and y_2.

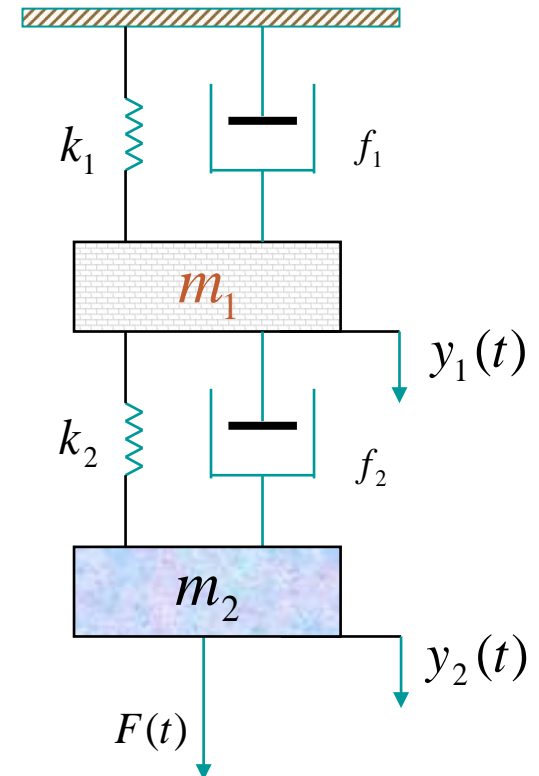From Newton's first Law, we have physics relationship of m_1 and m_2:

$$m_2\ddot{y}_2 = F(t) - k_2(y_2 - y_1) - f_2(\dot{y}_2 - \dot{y}_1)$$

$$m_1\ddot{y}_1 = k_2(y_2 - y_1) + f_2(\dot{y}_2 - \dot{y}_1) - k_1y_1 - f_1\dot{y}_1$$

select 4 independent state variables:

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \dot{y}_1, \quad x_4 = \dot{y}_2$$

$$\begin{cases} \dot{x}_1 = x_3 \\[2mm] \dot{x}_2 = x_4 \\[2mm] \dot{x}_3 = \dfrac{k_2}{m_1}(x_2 - x_1) + \dfrac{f_2}{m_1}(x_4 - x_3) - \dfrac{k_1}{m_1}x_1 - \dfrac{f_1}{m_1}x_3 \\[4mm] \dot{x}_4 = \dfrac{1}{m_2}F(t) - \dfrac{k_2}{m_2}(x_2 - x_1) - \dfrac{f_2}{m_2}(x_4 - x_3) \end{cases}$$

k_1

f_1

m_1

$y_1(t)$

k_2

f_2

m_2

$y_2(t)$

$F(t)$

4

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{f_1+f_2}{m_1} & \dfrac{f_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & \dfrac{f_2}{m_2} & -\dfrac{f_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix} F$$
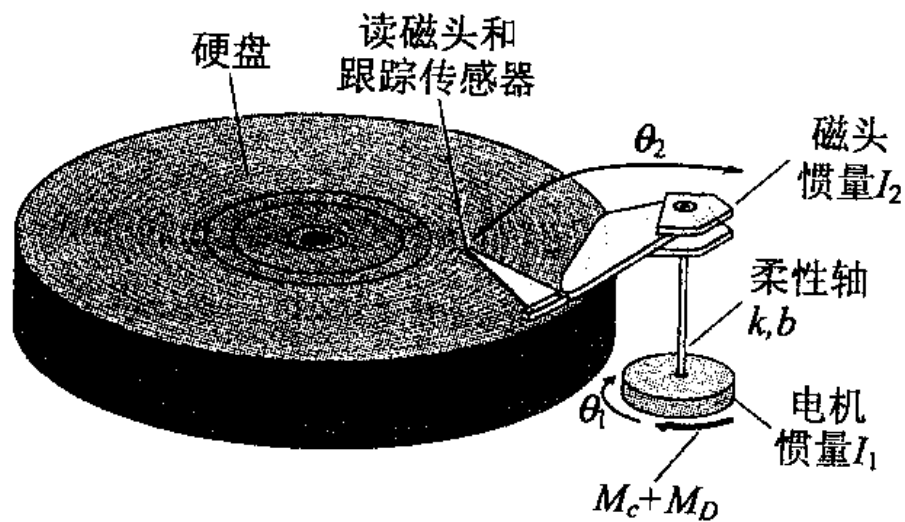
A

B

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

C

5

产品标签

安装螺丝

透气孔

空气过滤片

主轴（马达
电机与轴承
在其下方）

磁盘

磁头

音圈马达

磁头臂

永磁铁

6

读磁头和
硬盘　跟踪传感器

θ_2　磁头
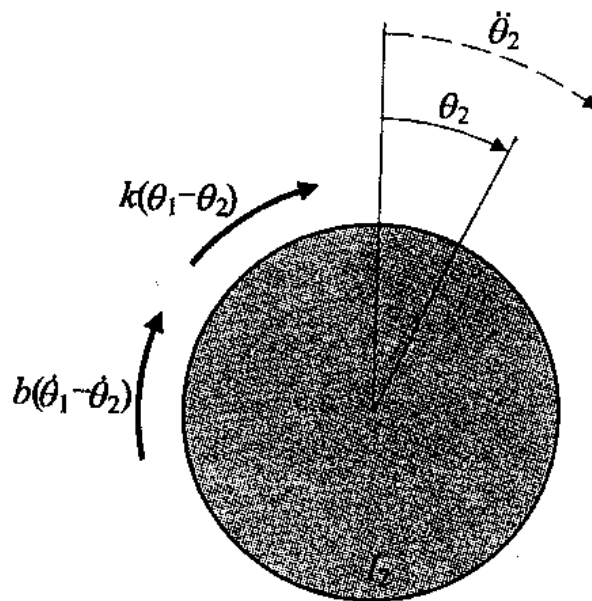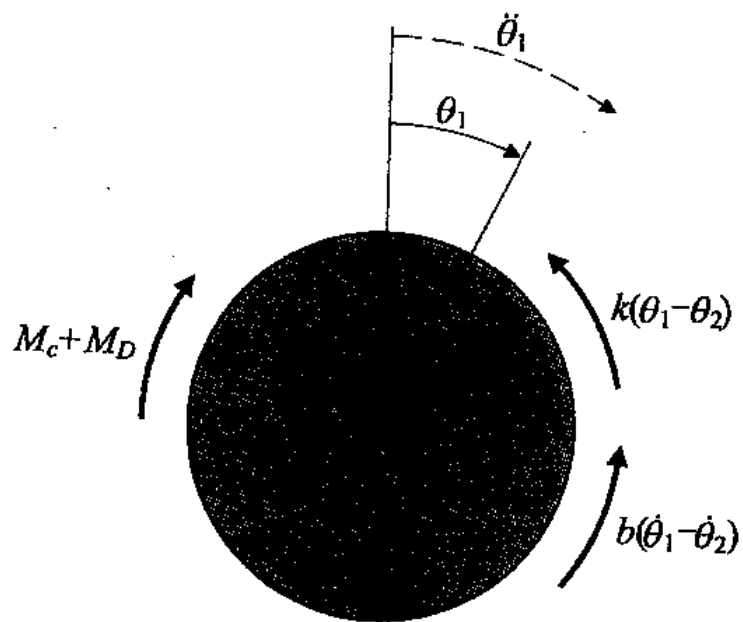惯量I_2

柔性轴
k, b

θ_1　电机
惯量I_1

$M_c + M_D$

$$J_1\ddot{\theta}_1 = M_c + M_D - k(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2)$$

$$J_2\ddot{\theta}_2 = k(\theta_1 - \theta_2) + b(\dot{\theta}_1 - \dot{\theta}_2)$$

$\ddot{\theta}_1$

θ_1

$M_c + M_D$

$k(\theta_1 - \theta_2)$

$b(\dot{\theta}_1 - \dot{\theta}_2)$

$\ddot{\theta}_2$

θ_2

$k(\theta_1 - \theta_2)$

$b(\dot{\theta}_1 - \dot{\theta}_2)$

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9.3.2 From Differential Equations of System

➢ Methodology:

- Establish the differential/difference equations by the physics mechanism of system;

- Establish the state equation focusing on the equations and a group of state-variables;

- Establish the output function based on the relationship between system's outputs and states.

➢ State-variable Selection

✓ Section of state variable is not unique.

✓ Methodology:

☞1. Select variable in the initial conditions or related.

☞2. Select characteristic variable of independent storage components (energy or information) with certain physical meaning, such as the electric current i of inductance, the voltage u_c of capacitor, mass m and velocity v, etc.

Scenario (1): No derivatives(微分) of input u contained in n-order linear differential equations

Assume the dynamic process of the SISO control system is described as follow:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

$y^{(n)}, y^{(n-1)}, \cdots, \dot{y}, y$ —— Derivatives of output

$\boldsymbol{u}$ —— Input

If initial conditions $y(0), y'(0), \ldots, y^{(n-1)}(0)$ of output and the input u(t) of t≥0, the behavior of system at any time can be confirmed.

Select state variables:
$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ \vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} \end{cases}$$

then:
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots a_1 x_n + bu \end{cases}$$

therefore:
$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

State space:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B u$$

$$y = C\boldsymbol{x}$$

in which, the matrixes:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ b \end{bmatrix}$$

$$C = [1 \ \ 0 \ \ 0 \ \ \ldots \ \ 0]$$

➢ Then, draw the following block diagram (state variable diagram) among the state variables.

✓ The output of each integrator corresponds to each state variable.

✓ The state equations are decided by the relationship of I/O.

✓ The output equation is on output part.
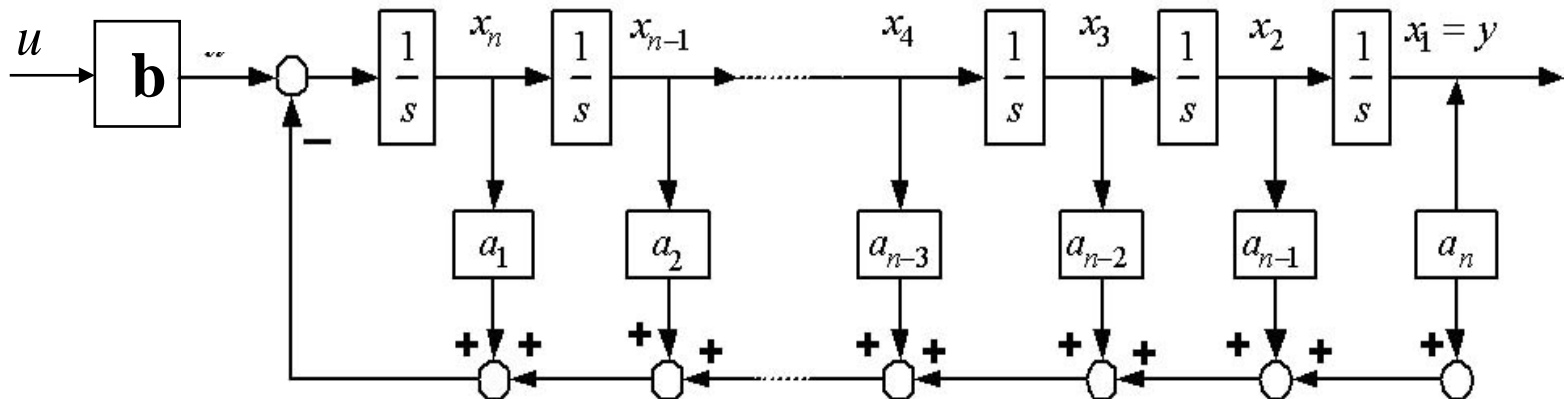
$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

Ex.9-4 assume the differential equation of the system's dynamic

process is
$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

in which, u and y are input and output.

Try to find the state space representation of the system.

Select state-variables: $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y},$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + 6u \end{cases}$$

Standard Form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

A

B

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

C

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Scenario(2): Derivatives of input u contained in n-order linear differential equation system

n-order linear differential equation representation:

$$y^{(n)} + a_1 \, y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_o \, u^{(n)} + b_1 \, u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

Reference scenario(1):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n y - a_{n-1}\dot{y} - \cdots - a_1 y^{(n-1)} + b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u \\ \quad = -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + b_o u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u \end{cases}$$

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \\ \vdots \\ x_{n-1} = y^{(n-2)} \\ x_n = y^{(n-1)} \end{cases}$$

However, the derivative of input u is still contained in

state equation, which is INCONSEQUENCE(不合理).

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Analysis: if input u is a limitary Step signal: Step Function, u'
will be the Impulsive Function δ, $u^{(i)}(i=2,3,\ldots)$ will be the higher-
order impulse function, and state trajectory will have infinite
jump at t_0.

Hence, we cannot choose the output y and its derivatives to be
state variables of the system. Such group of state variables cannot
decide the future state of the system based on the known system
input and original state conditions.

The principle of state variable selection:
No derivative of the input/operation function could be included in
any differential function in the system state equations represented
by one-order differential equation sets.

The equations of a 2input/2output 2nd-order system are:

$$\ddot{y}_1 + a_1 \dot{y}_1 + a_2 y_2 = b_1 \dot{u}_1 + b_2 u_1 + b_3 u_2$$

$$\dot{y}_2 + a_3 y_2 + a_4 y_1 = b_4 u_2$$

try to give its state space description.

Solution: find derivative of y_1, y_2 with highest-order

$$\ddot{y}_1 = -a_1 \dot{y}_1 + b_1 \dot{u}_1 - a_2 y_2 + b_2 u_1 + b_3 u_2$$

$$\dot{y}_2 = -a_3 y_2 - a_4 y_1 + b_4 u_2$$

calculate their integration

$$y_1 = \iint \left[(-a_1 \dot{y}_1 + b_1 \dot{u}_1) + (-a_2 y_2 + b_2 u_1 + b_3 u_2) \right] dt^2$$

$$= \int (-a_1 y_1 + b_1 u_1) dt + \iint (-a_2 y_2 + b_2 u_1 + b_3 u_2) dt^2$$

$$\int [(-a_1 y_1 + b_1 u_1) + \int (-a_2 y_2 + b_2 u_1 + b_3 u_2) dt] dt$$

$$y_2 = \int [(-a_3 y_2 - a_4 y_1 + b_4 u_2) dt$$

Select state variables: $x_1 = y_1$

$$x_2 = y_2$$

From the equation of y_1:

$$\dot{x}_1 = -a_1 y_1 + b_1 u_1 + \int (-a_2 y_2 + b_2 u_1 + b_3 u_2) dt$$

$$= -a_1 x_1 + b_1 u_1 + \int (-a_2 x_2 + b_2 u_1 + b_3 u_2) dt$$

Select another state variable:

$$x_3 = \int (-a_2 x_2 + b_2 u_1 + b_3 u_2) dt$$

and

$$\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$$

From the equation of y_2:

$$\dot{x}_2 = -a_3 x_2 - a_4 x_1 + b_4 u_2$$

The equation set: $\dot{x}_1 = -a x_1 + x_3 + b_1 u_1$

$$\dot{x}_2 = -a_4 x_1 - a_3 x_2 + b_4 u_2$$

$$\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$$

$$\dot{x}_1 = -ax_1 + x_3 + b_1 u_1$$

$$\dot{x}_2 = -a_4 x_1 - a_3 x_2 + b_4 u_2$$

$$\dot{x}_3 = -a_2 x_2 + b_2 u_1 + b_3 u_2$$

Rewrite the equations by the matrixes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 0 & 1 \\ -a_4 & -a_3 & 0 \\ 0 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \\ b_2 & b_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The output matrix equation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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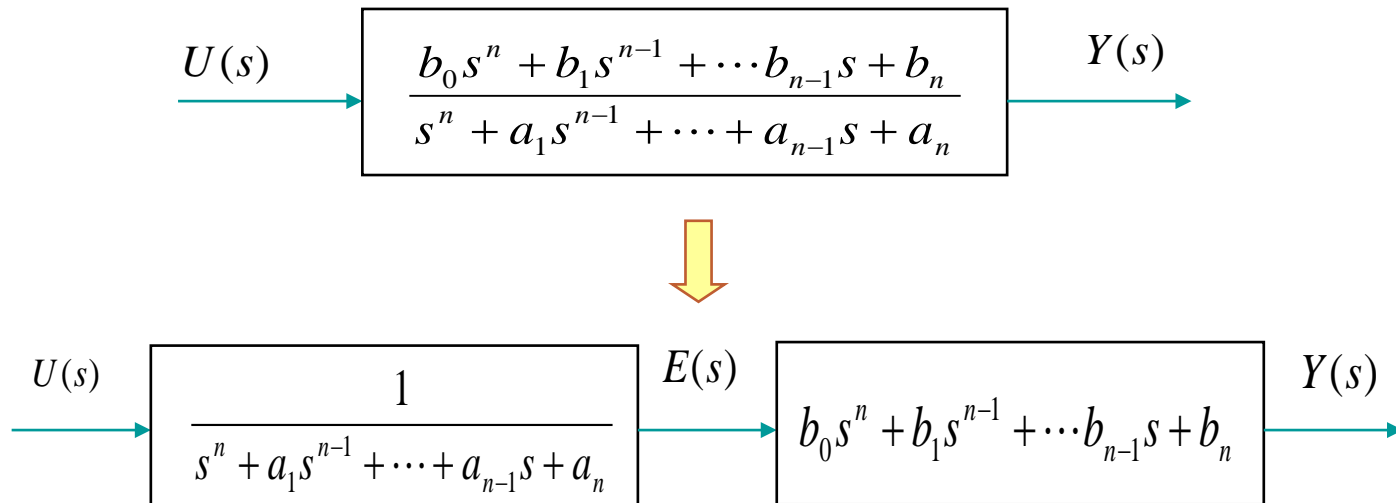
5. Linear Transformation of State space

9.3.3 From Transfer Functions of System

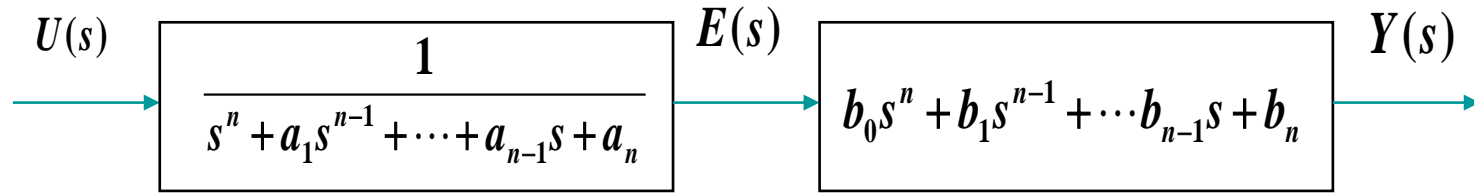- **Transfer Functions** ⟷ **State Space**

1. Transfer Functions to State Space

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$U(s)$ → $\boxed{\dfrac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}$ → $Y(s)$

$U(s)$ → $\boxed{\dfrac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}$ → $E(s)$ → $\boxed{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}$ → $Y(s)$

$$U(s) \rightarrow \boxed{\dfrac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}} \rightarrow E(s) \rightarrow \boxed{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1}s + b_n} \rightarrow Y(s)$$

$$U(s) = (s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n)E(s)$$

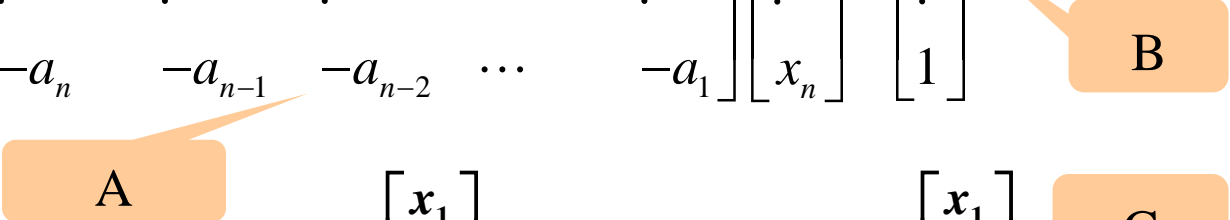$$Y(s) = (b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1}s + b_n)E(s)$$

Select state variables:

$$\begin{cases} x_1 = e(t) \\ x_2 = \dot{e}(t) \\ \vdots \\ x_n = e^{(n-1)}(t) \end{cases} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \end{cases}$$
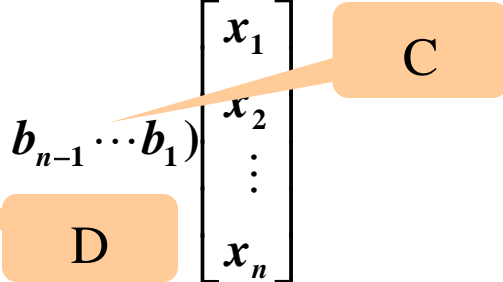
$$u = \dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-1} x_2 + a_n x_1$$

$$y = b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \cdots + b_{n-1} x_2 + b_n x_1$$

$$u = \dot{x}_n + a_1 x_n + a_2 x_{n-1} + \cdots + a_{n-1} x_2 + a_n x_1$$

$$y = b_0 \dot{x}_n + b_1 x_n + b_2 x_{n-1} + \cdots + b_{n-1} x_2 + b_n x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

B

A

C

$$y = b_0(-a_n \quad -a_{n-1} \quad \cdots \quad -a_1) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u + (b_n \quad b_{n-1} \cdots b_1) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

D

If $b_0 = 0$, the output equation will be simplified.

For the situation that the derivative of input u is included in the differential equations.

Homework: Comparing with the method before.

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Ex.9-7 Transfer function of a control system is:

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 4s + 1}{s^3 + 9s^2 + 8s}$$

transform it to the state space representation.

Solution:

$$a_1 = 9, \quad a_2 = 8, \quad a_3 = 0, \quad b_0 = 0, \quad b_1 = 1, \quad b_2 = 4, \quad b_3 = 1$$

The state equation is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

The output equations is:

$$y = \begin{bmatrix} 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

➢ **Parallel of the Transfer Function:**

$$G(s) = \frac{Num(s)}{Den(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad m \leq n$$

The Denominator:

$$Den(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

is the Characteristic Equations of the system.

Assume there are n characteristics roots: $p_i, i = 1, 2, \cdots, n$

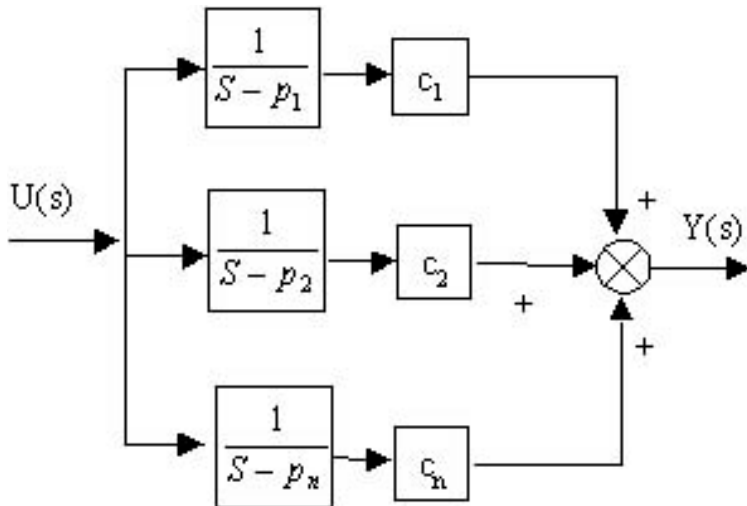G(s) can be decomposed by the summation of n fractions:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n}$$

$$= \sum_{i=1}^{n} \frac{c_i}{s - p_i}$$

In which, $c_i = \lim_{s \to p_i} (s - p_i) G(s)$, is called Residue (留数) of pole p_i.
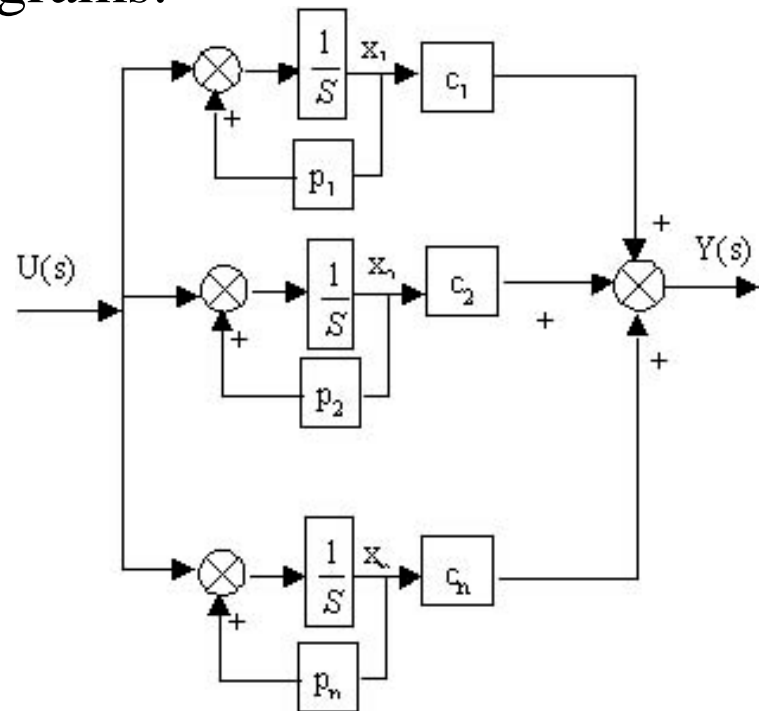
$$Y(s) = \frac{c_1}{s - p_1} U(s) + \frac{c_2}{s - p_2} U(s) + \cdots + \frac{c_n}{s - p_n} U(s)$$

$$= \sum_{i=1}^{n} \frac{c_i}{s - p_i} U(s)$$

The parallel connection diagrams:

(a)

(b) Parallel connection
(No repeated roots)

The state-equations of figure (b):

$$\begin{cases} \dot{x}_1 = p_1 x_1 + u \\ \dot{x}_2 = p_2 x_2 + u \\ \quad\vdots \\ \dot{x}_n = p_n x_n + u \end{cases}$$

The output equation:

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Matrix representation:

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ \\ Y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} X \end{cases}$$

PS: the system matrix A is a diagonal matrix.

If the denominator of G(s) Den(s)=0 has repeated roots:

$$Den(s) = (s - p_1)^q (s - p_{q+1}) \cdots (s - p_n)$$

$s = p_1$ is the only q times repeated root, we have G(s):

$$G(s) = \frac{Num(s)}{Den(s)}$$

$$= \frac{c_{11}}{s - p_1} + \frac{c_{12}}{(s - p_1)^2} + \cdots + \frac{c_{1q}}{(s - p_1)^q} + \frac{c_{q+1}}{s - p_{q+1}} + \cdots + \frac{c_n}{s - p_n}$$

In which：

$$c_{1i} = \frac{1}{(q-i)!} \bullet \lim_{s \to p_1} \frac{d^{q-i}}{ds^{q-i}} [(s - p_1)^q G(s)] \quad i = 1, 2, \ldots, q$$

$$c_j = \lim_{s \to p_j} [(s - p_j) G(s)] \qquad j = q+1, q+2, \ldots, n$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{c_{11}}{s - p_1} + \frac{c_{12}}{(s - p_1)^2} + \cdots + \frac{c_{1q}}{(s - p_1)^q} + \frac{c_{q+1}}{s - p_{q+1}} + \cdots + \frac{c_n}{s - p_n}$$
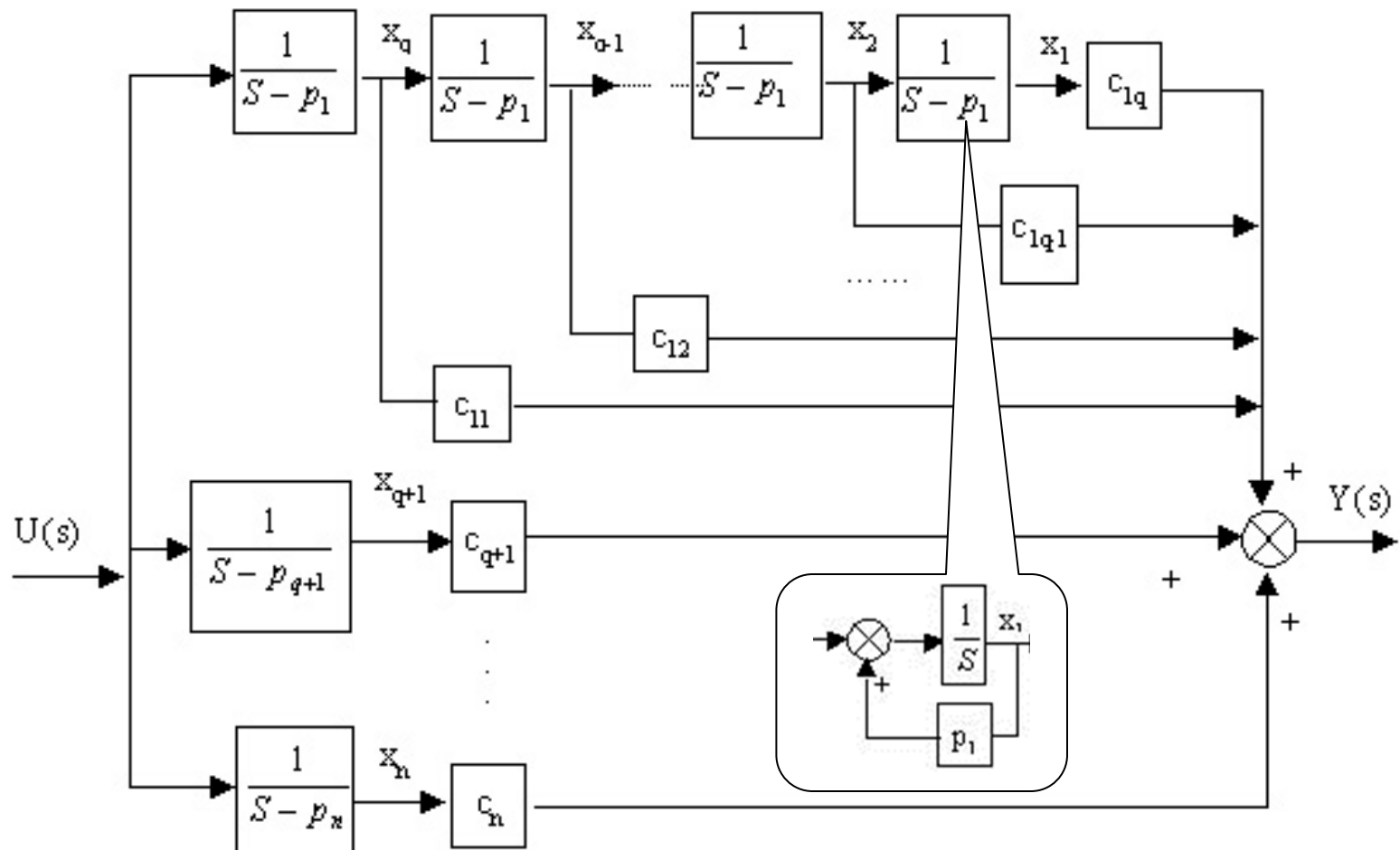
$$Y(s) = \frac{c_{11}}{s - p_1} U(s) + \frac{c_{12}}{(s - p_1)^2} U(s) + \cdots + \frac{c_{1q}}{(s - p_1)^q} U(s) + \frac{c_{q+1}}{s - p_{q+1}} U(s) + \cdots + \frac{c_n}{s - p_n} U(s)$$

Parallel Connection (Repeated roots)

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$$Y(s) = \frac{c_{11}}{s - p_1} U(s) + \frac{c_{12}}{(s - p_1)^2} U(s) + \cdots + \frac{c_{1q}}{(s - p_1)^q} U(s) + \frac{c_{q+1}}{s - p_{q+1}} U(s) + \cdots + \frac{c_n}{s - p_n} U(s)$$

Select the state-variables as the output of the integral items in the diagram:

$$
\begin{cases}
\dot{x}_1 = p_1 x_1 + x_2 \\
\dot{x}_2 = p_1 x_2 + x_3 \\
\vdots \\
\dot{x}_{q-1} = p_1 x_{q-1} + x_q \\
\dot{x}_q = p_1 x_q + u \\
\vdots \\
\dot{x}_n = p_n x_n + u
\end{cases}
$$

$$y = c_{1q} x_1 + c_{1q-1} x_2 + \cdots + c_{11} x_q + c_{q+1} x_{q+1} + \cdots + c_n x_n$$

$$\begin{cases} \dot{X} = \begin{bmatrix} p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & p_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & p_{q+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & p_n \end{bmatrix} X + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} U \\ \\ Y = \begin{bmatrix} c_{1q} & c_{1q-1} & \cdots & c_{11} & c_{q+1} & \cdots & c_n \end{bmatrix} X \end{cases}$$

System matrix A is Jordan Standard Form(约当标准型).

Ex.9-8 find the parallel connection of the follow system:

$$G(s) = \frac{4s^2 + 10s + 5}{s^3 + 5s^2 + 8s + 4}$$

Solution：the Denominator: $(s+2)^2(s+1)$

$$G(s) = \frac{4s^2 + 10s + 5}{(s+2)^2(s+1)} = \frac{c_{11}}{s+2} + \frac{c_{12}}{(s+2)^2} + \frac{c_3}{s+1}$$

$$c_{12} = \lim_{s \to -2}(s+2)^2 G(s) = -1$$

$$c_{11} = \frac{1}{(2-1)!}\lim_{s \to -2}\frac{d^{(2-1)}}{ds^{(2-1)}}[(s+2)^2 G(s)] = \lim_{s \to -2}\frac{d}{dS}[\frac{4s^2 + 10s + 5}{(s+1)}] = 5$$

$$c_3 = \lim_{s \to -1}(s+1)G(s) = \lim_{s \to -1}\frac{4s^2 + 10s + 5}{(s+2)^2} = -1$$

The state equation of the system by parallel connection:

$$\begin{cases} \dot{X} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \\ \\ y = \begin{bmatrix} -1 & 5 & -1 \end{bmatrix} X \end{cases}$$

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2. State Space to Transfer Functions

State Space (Dynamic Equations) VS Transfer Functions

➢ State Space denote both the input/output relationship and

the internal state variables of the system;

Transfer functions present the input/output relationship

only.

➢ From Transfer Functions to State Space: system realization

process, which is complicated and non-unique.

From State Space to Transfer Functions: simple and unique

process.

❖ For SISO system: State-space to TF

The State-space representation of a SISO system:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ \boldsymbol{y} = C\boldsymbol{x} + Du \end{cases}$$

$x, \dot{x} \in R^{n\times 1}; A \in R^{n\times n}; B \in R^{n\times 1}; C \in R^{1\times n}; D$ is a scalar quantity.

Assume the initial condition zero-input, and use Laplace Transform:

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$

The Transfer Function is:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

<u>Ex.9-8</u> the state-space of the system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find its transfer function:

Solution: write the related matrices [A ,B ,C] at first:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
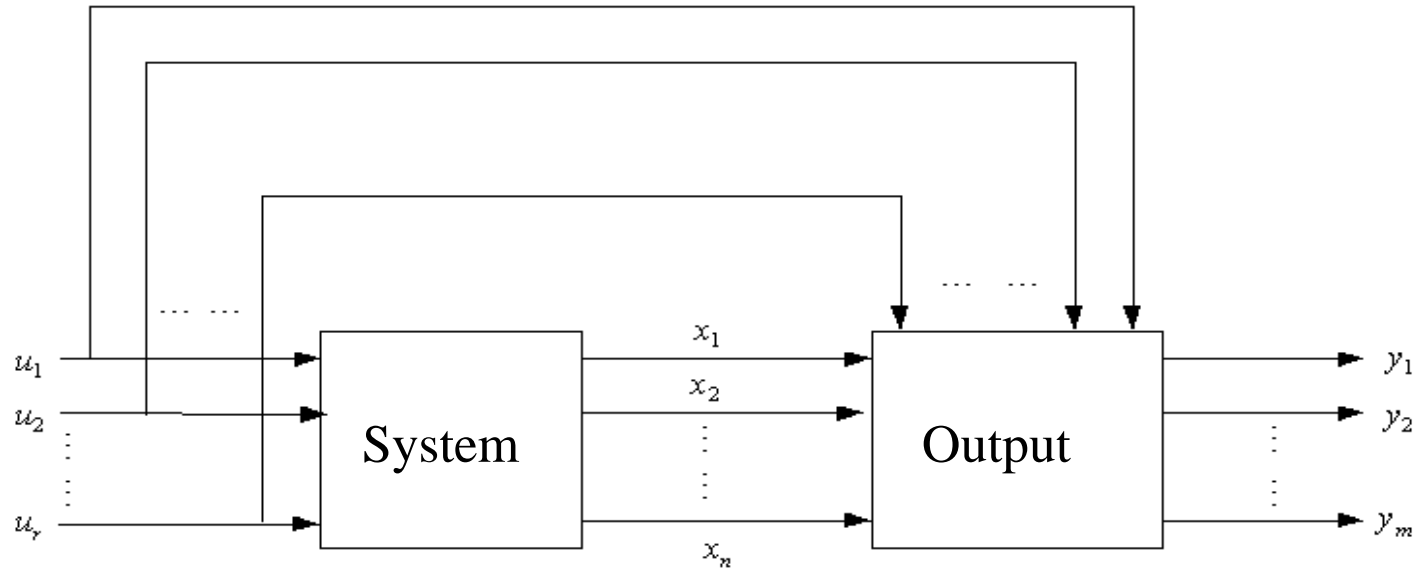
$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+3)} \begin{bmatrix} s+3 & 1 \\ -1 & s \end{bmatrix}$$

Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{s+3}{s(s+3)+1} & \dfrac{1}{s(s+3)+1} \\[2ex] \dfrac{-1}{s(s+3)+1} & \dfrac{s}{s(s+3)+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s+3)+1} = \frac{1}{s^2 + 3s + 1}$$

❖ **For MIMO system: State-space to TF**

$$u_1, u_2, \cdots, u_r \quad \text{———} \quad \text{input signals of the system}$$

$$y_1, y_2, \cdots, y_m \quad \text{———} \quad \text{output signals of the system}$$

$$x_1, x_2, \cdots, x_n \quad \text{———} \quad \text{state variables of the system}$$

The dynamic equations:
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Equations have the same formation with SISO system, however, the matrices *B*, *C* ,*D* have different dimension.

$$x, \dot{x} \in R^{n \times 1}, \quad U \in R^{r \times 1}, \quad Y \in R^{m \times 1},$$

$$A \in R^{n \times n}, \quad B \in R^{n \times r}, \quad C \in R^{m \times n}, \quad D \in R^{m \times r}$$

Laplace transformation:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$G \in R^{m \times r} \ ----\text{Transfer Function Matrix}$$

Ex.9-9 **The dynamic equations are:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the Transfer function of the system.

Solution: $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = 0$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}$$

➢ the Eigen-equation and Eigenvalue of the system matrix A

The Eigen-equation of the system is:
$$| \lambda I - A | = 0$$

The Eigenvalue is one of its solutions.

Expand the equation: $| \lambda I - A | = 0$

we have the polynomial: $\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots a_n = 0$

Then solve the equation we have its n eigenvalues.

e.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ then $| \lambda I - A | = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix}$
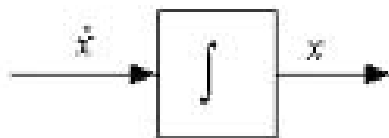
$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$
$$= (\lambda+1)(\lambda+2)(\lambda+3) = 0$$

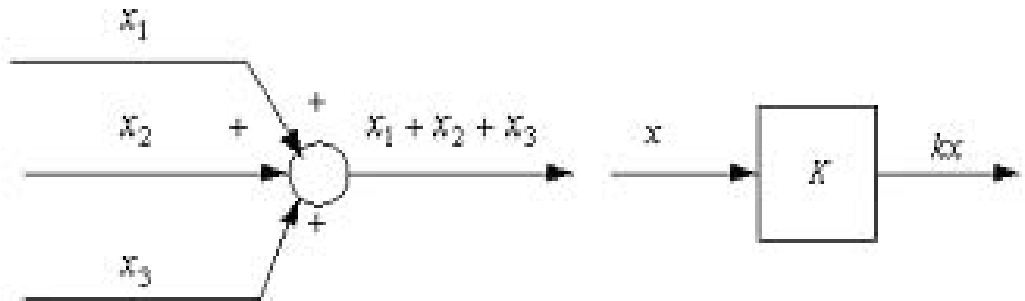3 eigenvalues are -1、-2 and -3.

9.3.4 From State variable diagram of the System

➢ **State variable diagram:** the diagram description of the relationship of state variables, which is composed by the integral items, proportion items and sum symbols.
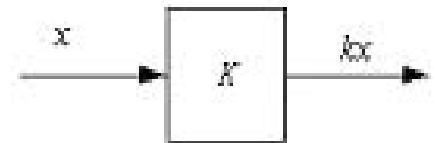
◆ The output of each integral item is one of the state variables of the system.

(a)   (b)   (c)

Ex.9-10   The Close-loop TF of the system is:

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 3s + 2}{s(s^2 + 7s + 12)}$$

Draw the State variable diagram and find the state space representation of the system.

Solution: rewrite the close-loop TF of the system:
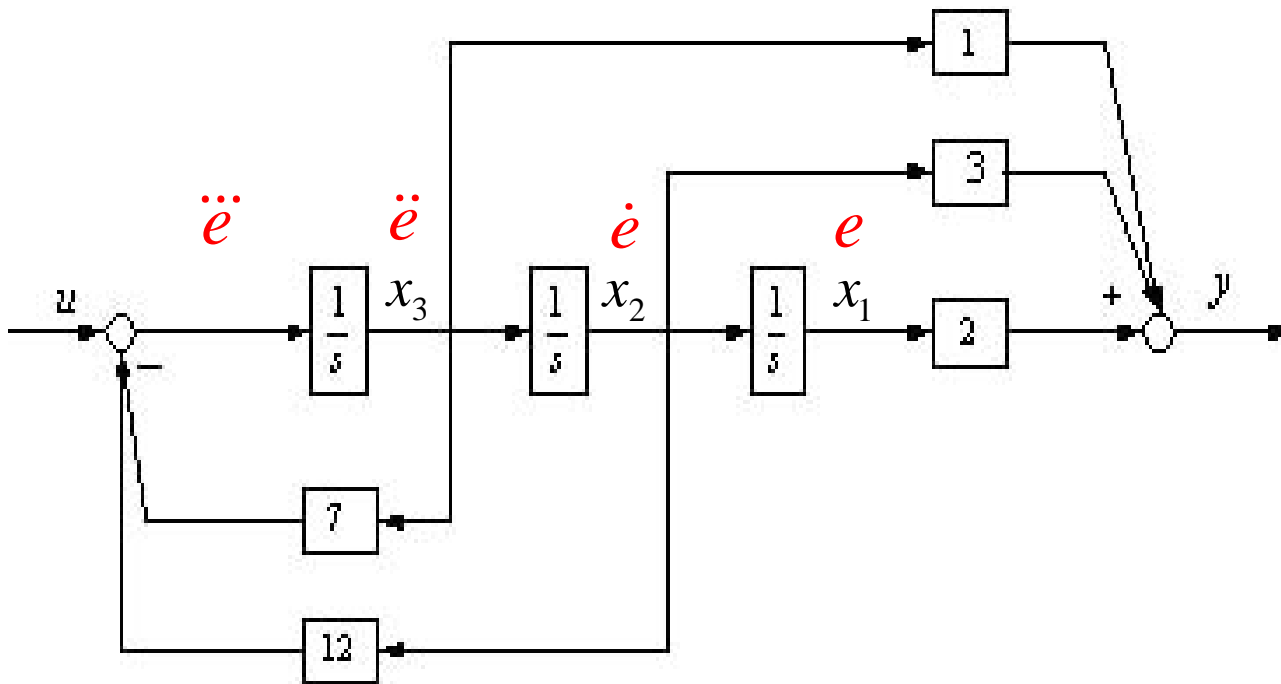
$$\frac{Y(s)}{U(s)} = \frac{s^2 + 3s + 2}{s^3 + 7s^2 + 12s}$$

Assume:

$$Y(s) = (s^2 + 3s + 2)E(s) \quad \Longrightarrow \quad Y(s) = s^2 E(s) + 3sE(s) + 2E(s)$$

$$\boxed{y(t) = \ddot{e}(t) + 3\dot{e}(t) + 2e(t)}$$

$$U(s) = (S^3 + 7s^2 + 12s)E(s) \quad \Longrightarrow \quad S^3 E(s) = U(s) - 7s^2 E(s) - 12sE(s)$$

$$\boxed{\dddot{e}(t) = u(t) - 7\ddot{e}(t) - 12\dot{e}(t)}$$

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$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -12 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ \\ y = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

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9.3.5 Linear Transformation of State space

- State-space equation establishing method review:
Physics Mechanism / Differential Equations/ Transfer Functions / State-variable Diagram

- State-variables selection: Non-unique

- State-space equation: Non-unique

- The amount of the independent state variables in different state-space equation for a certain physics system: Uniform

- The connection of different state-space representations:
Linear Transformation

- **Non-uniqueness of State-space Variables**

Assume a State-space Equation:

$$\dot{x} = Ax + Bu$$

Linear Transform the state-variables: $(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)$ into another group: $x_1, x_2, \ldots, x_n$

We have

$$\begin{cases} x_1 = p_{11}\overline{x}_1 + p_{12}\overline{x}_2 + \cdots + p_{1n} \\ x_2 = p_{21}\overline{x}_1 + p_{22}\overline{x}_2 + \cdots + p_{2n} \\ \cdots\cdots \\ x_n = p_{n1}\overline{x}_1 + p_{n2}\overline{x}_2 + \cdots + p_{nn} \end{cases}$$

and also

$$x = P\overline{x}$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

If P is a non-singular(非奇异) constant matrix: $|P| \neq 0$

The vector $\overline{x}$ is the state-variable vector of the system: $\dot{x} = Ax + Bu$

Proof: $\dot{\overline{x}} = P^{-1}AP\overline{x} + P^{-1}Bu$

Symbolize: $A_1 = P^{-1}AP, \quad B_1 = P^{-1}B$

then $\dot{\overline{x}} = A_1\overline{x} + B_1u$

A_1 and A are similar matrices, the characteristic polynomial:

$$|sI - A_1| = |sI - A|$$

The state-equations from x and $\overline{x}$ have same eigenvalues

Result: for any control system, selection of state variables is non-unique. Any state variables from linear transformation, which satisfy the **non-singular condition** of the transform matrix P, are the appropriate state variables of the system.

Invariability(不变性) of linear transformation

System: $\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ \mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{cases}$ If $\mathbf{x} = P\overline{\mathbf{x}}$ $\begin{cases} \dot{\overline{\mathbf{x}}} = P^{-1}AP\overline{\mathbf{x}} + P^{-1}B\mathbf{u} \\ \mathbf{y} = CP\overline{\mathbf{x}} + D\mathbf{u} \end{cases}$

(1) Invariability of eigenequations and eigenvalues

Eigenvalues after transformation:

Analysis:

$$\left| \lambda I - P^{-1}AP \right| = \left| \lambda P^{-1}P - P^{-1}AP \right| = \left| P^{-1}\lambda P - P^{-1}AP \right|$$

$$= \left| P^{-1}(\lambda I - A)P \right| = \left| P^{-1} \right|\left| \lambda I - A \right|\left| P \right|$$

$$= \left| P^{-1} \right|\left| P \right|\left| \lambda I - A \right| = \left| P^{-1}P \right|\left| \lambda I - A \right|$$

$$= \left| I \right|\left| \lambda I - A \right| = \left| \lambda I - A \right|$$

Obviously, the eigenvalues of the system are same before and after the non-singular linear transformation.

(2) Invariant of system transfer function matrix

The transfer matrix after transformation:

$$G'(s) = CP(sI - P^{-1}AP)^{-1}P^{-1}B + D$$

$$= CP(P^{-1}sIP - P^{-1}AP)^{-1}P^{-1}B + D$$

$$= CP\left[P^{-1}(sI - A)P\right]^{-1}P^{-1}B + D$$

$$= CPP^{-1}(sI - A)^{-1}PP^{-1}B + D$$

$$= C(sI - A)^{-1}B + D = G(s)$$

The transfer matrix of the system is invariant before and after the non-singular linear transformation.

✓ **Why need Linear Transformation?**

Although there are infinite forms of state-space equations according to a certain system, which satisfy the non-singular linear transformation, only several kind of canonical forms are benefit for us:

☐ **Controllability Canonical Form**

☐ **Observability Canonical Form**

☐ **Diagonal Canonical Form**

☐ **Jordan Canonical Form.**

❑ **Controllability Canonical Form**(能控标准型)

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_n & c_{n-1} & \cdots & c_1 \end{bmatrix} x \end{cases}$$

Matrix A is also called companion matrix(友矩阵) .

☐ **Observability Canonical Form(能观标准型)**

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} x + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} x \end{cases}$$

☐ Diagonal Canonical Form (I)

$$\begin{cases} \dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} x \end{cases}$$

❑ Diagonal Canonical Form (II)

Select state variables:

$$X_i(s) = \frac{c_i}{s - \lambda_i} U(s)$$

System Output is:

$$Y(s) = \sum_{i=1}^{n} X_i(s)$$

Inverse Laplace Transformation:

$$
\begin{cases}
\dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} x
\end{cases}
$$

☐ **Jordan Canonical Form**

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & & & & \\ & \lambda_1 & 1 & & & 0 & \\ & & \lambda_1 & & & & \\ & & & \lambda_4 & & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_4 & \cdots & c_n \end{bmatrix} x$$

The Block of repeated poles $\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$ is called Jordan Block.

(1) Outlines：

$$\left.\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}\right\} \xrightarrow[\text{Equivalent transforming}]{x = P\bar{x}} \left.\begin{array}{l} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ \bar{y} = \bar{C}\bar{x} + \bar{D}u = y \end{array}\right\}$$

Non-canonical form   Canonical form

(2) Relationship between the coefficient matrices

$$\because \quad x = P\bar{x}, \quad \dot{x} = P\dot{\bar{x}},$$

P is NxN non-singular constant matrix

Substitute into the equations:

$$\left.\begin{array}{l} P\dot{\bar{x}} = AP\bar{x} + Bu \\ y = CP\bar{x} + Du \end{array}\right\} \quad \Rightarrow \quad \left.\begin{array}{l} \dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu \\ y = CP\bar{x} + Du \end{array}\right\}$$

$$\Downarrow$$

$$\bar{A} = P^{-1}AP, \quad \bar{B} = P^{-1}B, \quad \bar{C} = CP, \quad \bar{D} = D$$

- The constraint satisfied systems {A,B,C,D} and { $\bar{A},\bar{B},\bar{C},\bar{D}$ } are called Similar Systems;

- The related dynamic equations are called Equivalent Dynamic Equations;

- The linear transformation is called Equivalent Transformation.

Ex.9-12 consider the follow transfer function of a certain system:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2 + 3s + 2}$$

try to find its controllability canonical form, observability canonical form, and diagonal form.

Solution:

Its controllability canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Observability canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal canonical form is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [2 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$