Review

- Features of nonlinear systems
  - principle of superposition is not available
  - The stability of a nonlinear system depends on not only the inherent structure and parameters of control systems, but also the initial conditions and the inputs.
  - Periodic oscillation
  - Jump resonance and Multi-valued response
  - subharmonic oscillation and harmonic oscillation

Review

- Typical Nonlinear characteristics
  - Saturation
  - Dead-zone
  - Gap
  - Relay

§ **8.4** Phase Plane Method

- Phase plane method was first proposed in 1885 by Poincare . It is a graphical method for studying first-order, second-order systems.
- *The essence of this method is visually transforming the motion process of the system into the motion of a point in the phase plane.*
- We can obtain all information regarding the motion patterns of system by studying the motion trajectory of this point. Now this method is widely used, because it can intuitively, accurately and comprehensively character the motion states of the system.

The Role of The Phase Plane Method

- The phase plane method can be used for analyzing the ***stability, equilibrium position*** and ***steady-state*** accuracy of the first-order, second-order linear systems or nonlinear systems.

- It also can be used for analyzing the impact on the system motion of ***initial conditions*** and ***parameters*** of this system.

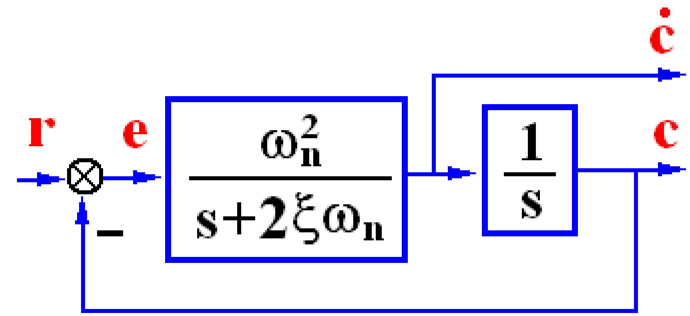
- When the nonlinearity of system is serious or we can not use the describing function method while there are some ***non-periodic*** inputs, the phase plane method is still available for these problems.

Basic Concepts of The Phase Plane Method

(1) Phase plane and Phase portrait

Phase plane:

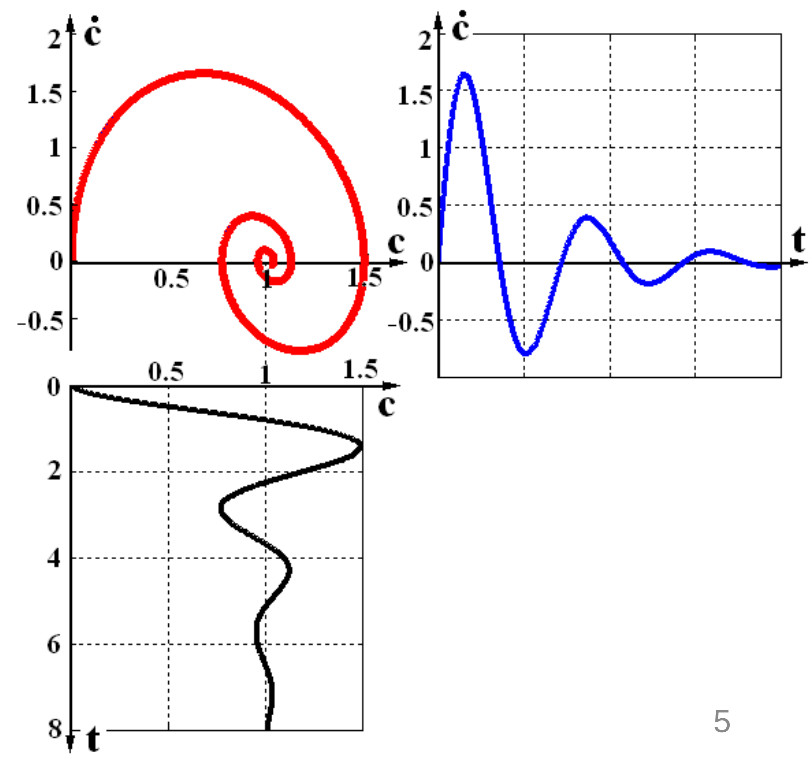
The x_1-x_2 plane is called Phase Plane,
Where x_1, x_2 are the system state and its derivative (c, $\dot{c}$).



Phase portrait (相图):

The locus in the x_1-x_2 plane of the solution x(t) for all t>= 0 is a curve named ***trajectory*** or ***orbit*** that passes through the point x0.

The family of phase plane trajectories corresponding to various initial conditions is called ***Phase Portrait*** of the system.



Example 1: A unit feedback system

$$G(s)=\frac{5}{s(s+1)} \qquad \begin{cases}\omega_n = 2.236 \\ \xi = 0.2236\end{cases}$$

$$r(t)=1(t)$$

Basic Concepts of The Phase Plane Method

(2) The features of phase trajectory

System equation: $\ddot{x} + f(x, \dot{x}) = 0$

Motion Direction

$$\left\{\begin{array}{l} \text{Upper half-plane } \dot{x} > 0 \text{ — move right} \\ \text{lower half-plane } \dot{x} < 0 \text{ — move left} \end{array}\right\}$$

clockwise motion

Passing the X-axis $(\dot{x} = 0)$ **perpendicularly.**

Singular Points (Equilibrium Points)

$$\alpha = \frac{d\dot{x}}{dx} = \frac{d\dot{x}/dt}{dx/dt} = \frac{-f(x, \dot{x})}{\dot{x}} = \frac{0}{0} \Rightarrow \begin{cases} \ddot{x} = 0 \\ \dot{x} = 0 \end{cases}$$

For the linear time-invariant system, the origin is the only equilibrium point.

Basic Concepts of The Phase Plane Method

(2) The features of phase trajectory

Except the equilibrium points , there is only one phase trajectory passing through any point in the phase plane.

It is determined by the existence and uniqueness of solutions of differential equations.

Methods of Constructing Phase Plane Trajectories

- Analytical Method
- Isocline Method
- Experimental Method

Analytical Method

For an arbitrary second-order nonlinear differential equations

$$\ddot{x} + f(x, \dot{x}) = 0$$

Or

$$\ddot{x} + a_1(x, \dot{x})\dot{x} + a_0(x, \dot{x})x = 0$$

Let

$$x_1 = x$$

$$x_2 = \dot{x}_1 = \dot{x}$$

Then:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} = -a_1(x_1, x_2)x_2 - a_0(x_1, x_2)x_1 \end{cases}$$

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{-a_1(x_1, x_2)x_2 - a_0(x_1, x_2)x_1}{x_2}$$

Rewrite the system equations in a general form:

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) \\ \dot{x}_2 = Q(x_1, x_2) \end{cases}$$

$$\frac{dx_2}{dx_1} = \frac{Q(x_1, x_2)}{P(x_1, x_2)} \longrightarrow$$

the slope of the trajectory at point (x_1, x_2)

If $P(x_1, x_2)$, $Q(x_1, x_2)$ is analytic, the differential equation can then be solved. Given a initial condition, the solution can be plotted in the phase plane. This curve is named Phase trajectory. The family of phase plane trajectories is called Phase portrait.

Assume

$$\begin{cases} P(x_1, x_2) = 0 \\ Q(x_1, x_2) = 0 \end{cases}$$

The solutions (x_{10}, x_{20}) is called the equilibrium points of the system.

Note:

The "***time***" is ***eliminated*** here → The responses $\boldsymbol{x_1(t)}$ and $\boldsymbol{x_2(t)}$ cannot be obtained directly.

Only ***qualitative behavior*** can be concluded, such as ***stability*** or ***oscillatory response***.

The Conventional Representation of Phase Trajectory
—— Isocline Method(等倾线方法)

Assume:

The term isocline derives from the Greek words for "***same slope***."

$$S(x) = \frac{dx_2}{dx_1} = \frac{-a_1(x_1, x_2)x_2 - a_0(x_1, x_2)x_1}{x_2} = \frac{Q(x_1, x_2)}{P(x_1, x_2)} = \alpha$$

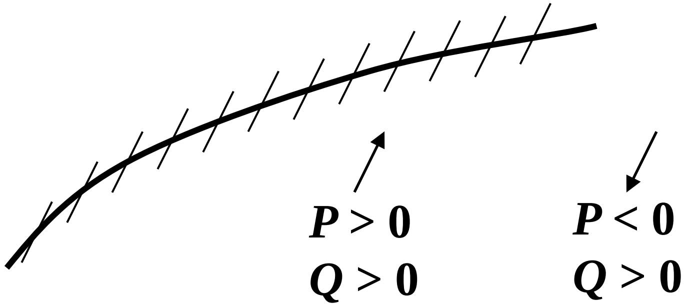
An isocline with slope is defined as $S(x) = \alpha$. α is a constant.

All the points on the curve $Q(x_1, x_2) = \alpha P(x_1, x_2)$ have the same tangent slope α.

Isocline Method

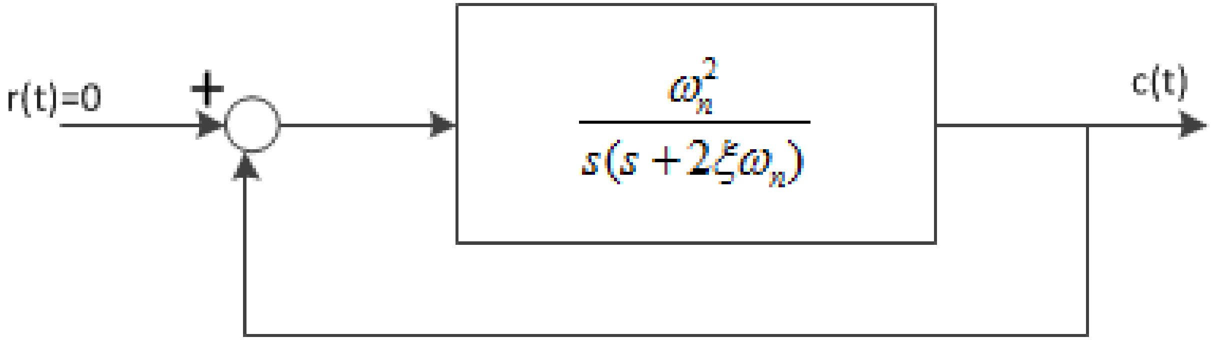
- The ***algorithm*** of constructing the phase portrait by isocline method:
  - Plot the curve $S(x) = \alpha$ in the phase plane.

□ Draw small line with slope. Note that the direction of the line depends on the sign of ***P*** and ***Q*** at that point.



□ Repeat the process for sufficient number of α until subject to the phase plane is full of isoclines

- For a linear 2nd order system:



- Dynamic equation is: $\ddot{c} + 2\zeta\omega_n\dot{c} + \omega_n^2 c = 0$
- Roots of characteristic equation are: $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
- And ***slope function*** of the phase trajectory is:

$$\frac{d\dot{c}}{dc} = \frac{-2\zeta\omega_n\dot{c} - \omega_n^2 c}{\dot{c}}$$

- Assuming that $\dfrac{d\dot{c}}{dc} = \alpha$, then $\dfrac{-2\zeta\omega_n\dot{c} - \omega_n^2 c}{\dot{c}} = \alpha$
- and $\dot{c} = \dfrac{-\omega_n^2 c}{2\zeta\omega_n + \alpha} = \beta c$, slopes of isoclines are: $\beta = \dfrac{-\omega_n^2}{2\zeta\omega_n + \alpha}$
- The isoclines are straight lines crossing the origin point.

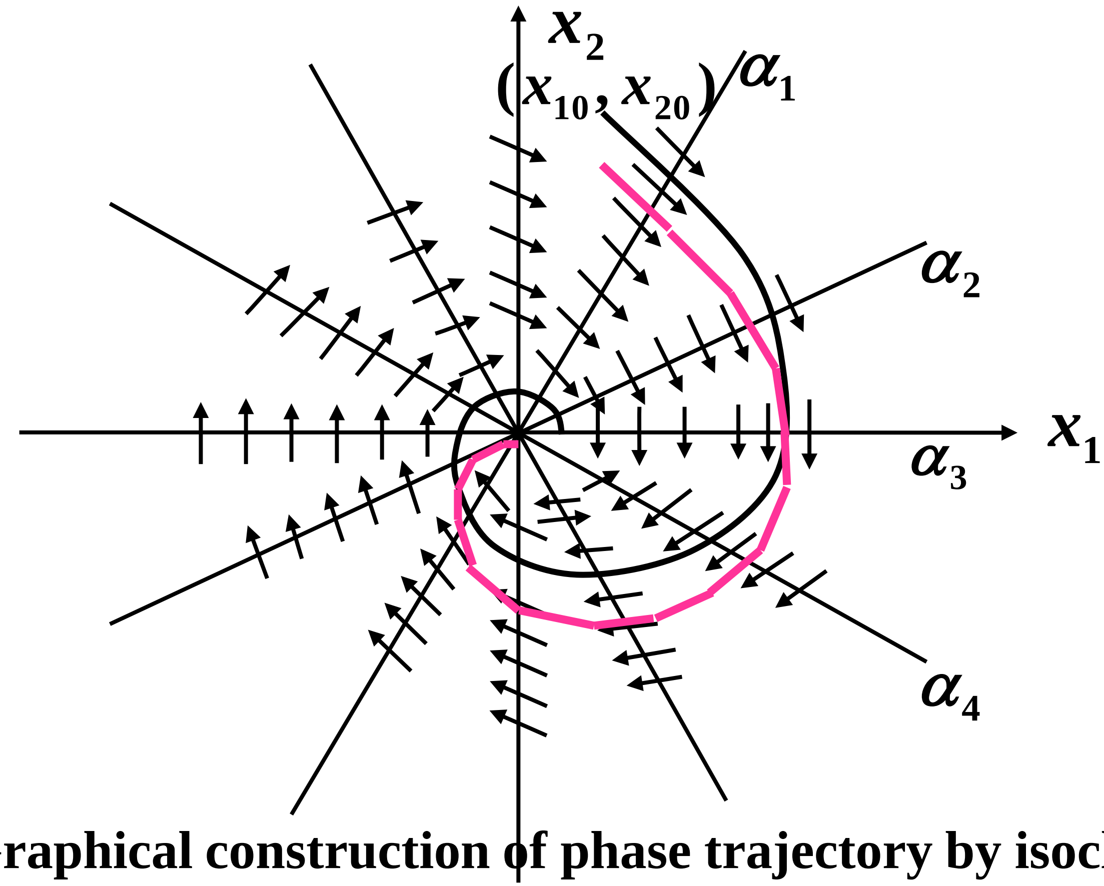


Fig.8-27 Graphical construction of phase trajectory by isocline method

α_1, α_2, α_3,… are slopes of the ***tangent of phase trajectories*** crossing the isoclines. All small lines on the constitute the ***tangential field*** of the phase trajectory.

Starting from (x_{10},x_{20}), connecting some typical points, such as ones on the coordinates, and the small lines on the adjacent isoclines smoothly, the trajectory can be constructed.

The purpose of plotting the phase trajectory is to analyze the dynamic characteristics.

Because there are infinite phase trajectories leaving or arriving at the ***equilibrium point*** the phase trajectories near the ***equilibrium point*** reflect the dynamic characteristics of the system.

Equilibrium points is also called ***singular points***.

Limit cycle is another phase trajectory which can reflect the dynamic characteristics of the system.

Limit cycle is an ***Isolated and Closed*** phase trajectory, which describes the harmonic oscillation of a system. It divides the infinite phase plane into two parts.

Singular Point and Limit Cycle

1. Singular Point

Singular Points are the equilibrium points (x_{10}, x_{20}) ,

which are obtained by solving the following equations.

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) = 0 \\ \dot{x}_2 = Q(x_1, x_2) = 0 \end{cases}$$

The singular point can only appear on the X-axis.

To study the shape and dynamic characteristics of the phase trajectories near the equilibrium (x_{10}, x_{20}) , we expand the function $P(x_1, x_2)$, $Q(x_1, x_2)$ into Taylor series around it.

Ignoring the higher-order terms, without loss of generality we assume that

$$x_{10} = x_{20} = 0$$

then

$$P(x_1, x_2) = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$Q(x_1, x_2) = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

Assume $a = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)}$ $\quad b = \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$

$$c = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} \qquad d = \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$$

then
$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases}$$

the characteristic equation of system is given by

$$\left|\lambda I - A\right| = \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

the roots of the above equation is

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

According to the property of these roots, the singular points can be divided into the following classes:

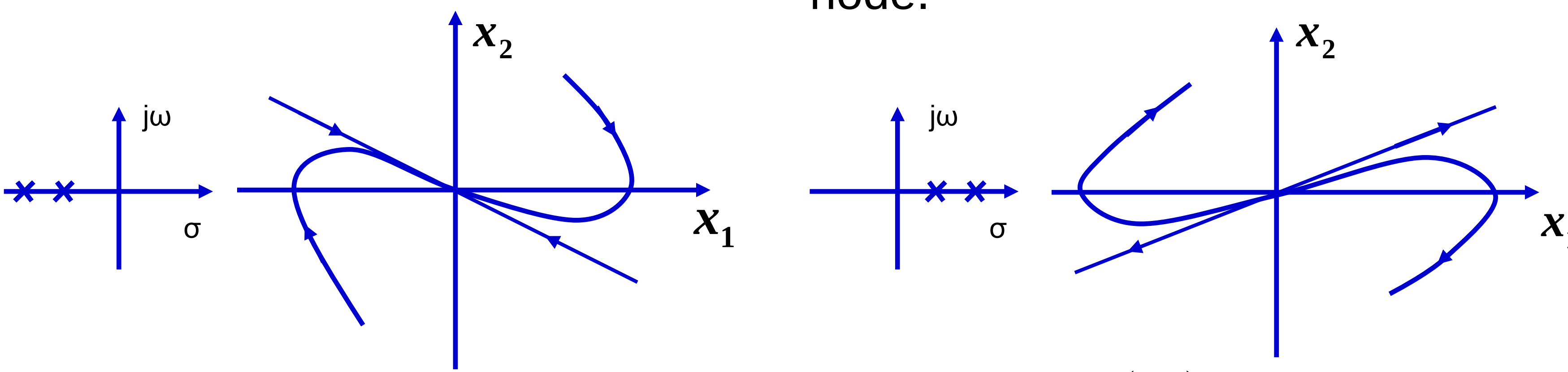
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

1) different real roots with the same sign

$$(a+d)^2 > 4(ad-bc)$$

If $a+d<0$, two roots are all negative, singular point is called stable node.

If $a+d>0$, two roots are all positive, singular point is called unstable node.



(a) stable node

（b）unstable node

Fig 8-28 Phase trajectory in this case

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

2) Real roots with different signs

$$ad - bc < 0$$

Singular point is called saddle point (鞍点)

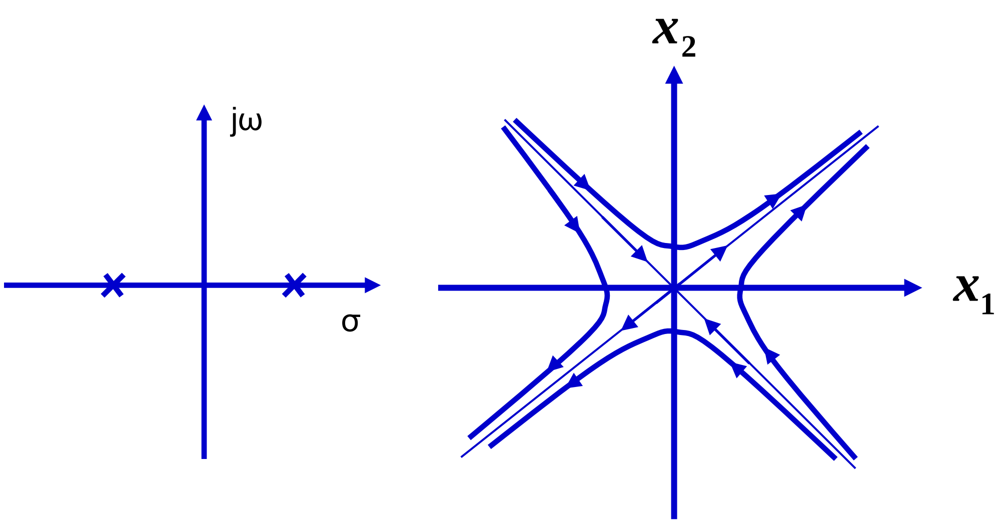


Fig 8-29 Phase trajectories that are corresponding to a saddle point

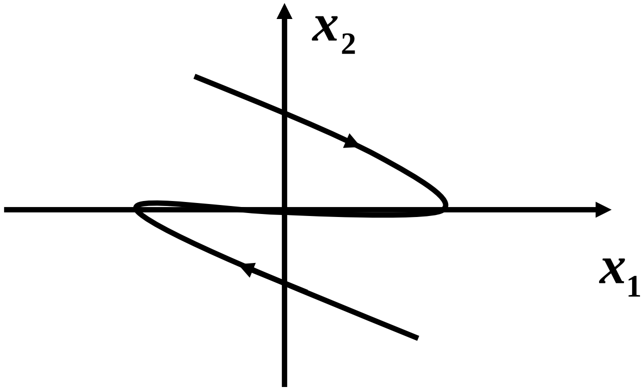
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

3) Double Root

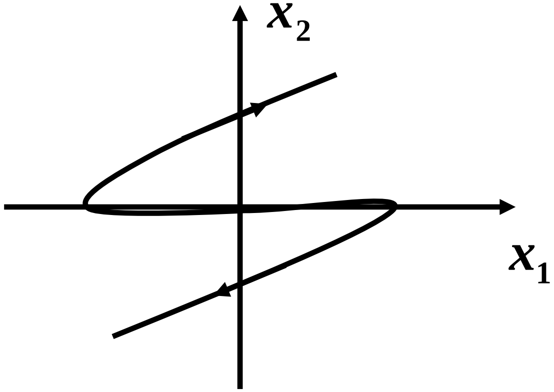
$$(a+d)^2 = 4(ad-bc)$$

If $a+d<0$， there are two equal negative real roots . Singular point is called degraded stable node;

If $a+d>0$， there are two equal positive real roots . Singular point is called degraded unstable node;



(a) double negative roots



(b) double positive roots

Fig 8-30 Phase trajectories that are corresponding to double point

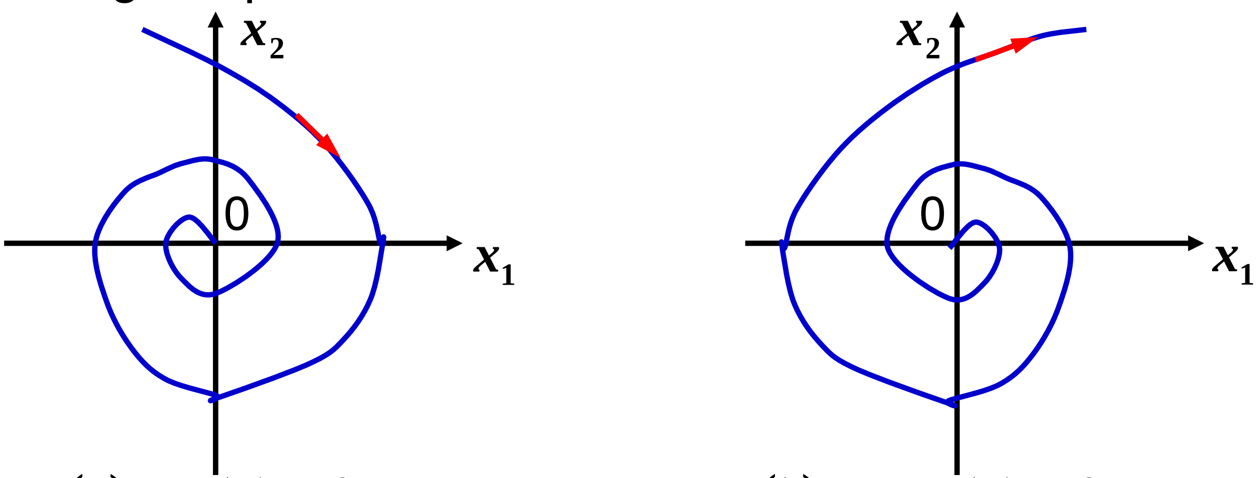
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

4) Complex Conjugate Root

$$(a+d)^2 < 4(ad-bc)$$

If $a+d<0$, there are complex conjugate roots with negative real component. Singular point is called stable focus;

If $a+d>0$, there are complex conjugate roots with positive real component. Singular point is called unstable focus;



(a) stable focus (b) unstable focus

Fig 8-31 Phase trajectories that are corresponding to complex conjugate roots

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

5) Purely imaginary roots

$$(a+d)=0, \quad ad-bc>0$$

Singular point is called center.

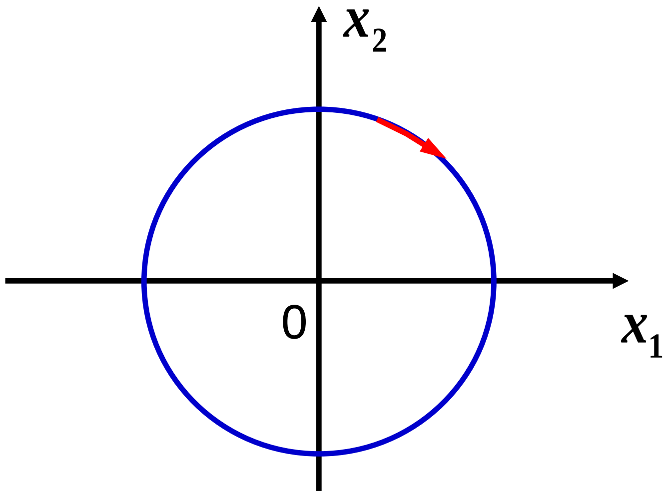


Fig 8-32 Phase trajectories that are corresponding to Purely imaginary roots

2. Limit Cycle

A closed and isolated phase trajectory in the phase plane is called a limit cycle. It is corresponding to the harmonic oscillation state of a system. Limit cycle divides the phase-plane into two parts: the part inside the limit cycle and the part outside the limit cycle. Any phase trajectory can not enter one part from the other.

Limit cycle can be easily found in actual physical systems. For example , the response of an unstable linear control system is theoretically a divergent oscillation. Whereas, in reality the amplitude of response may tends to a constant value due to the non-linear characteristics like saturation.

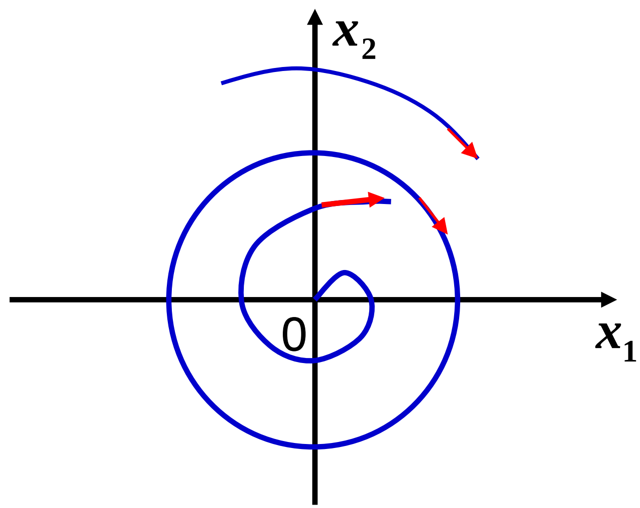
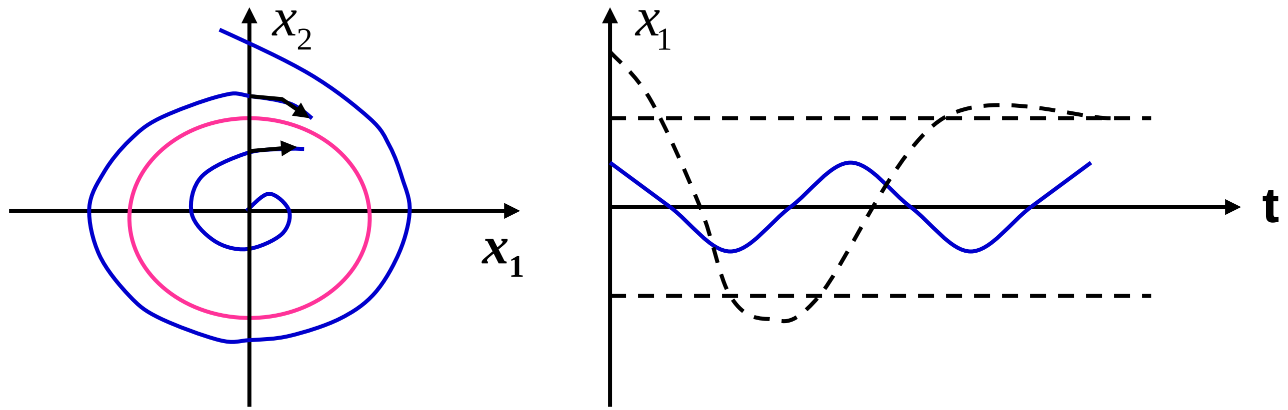
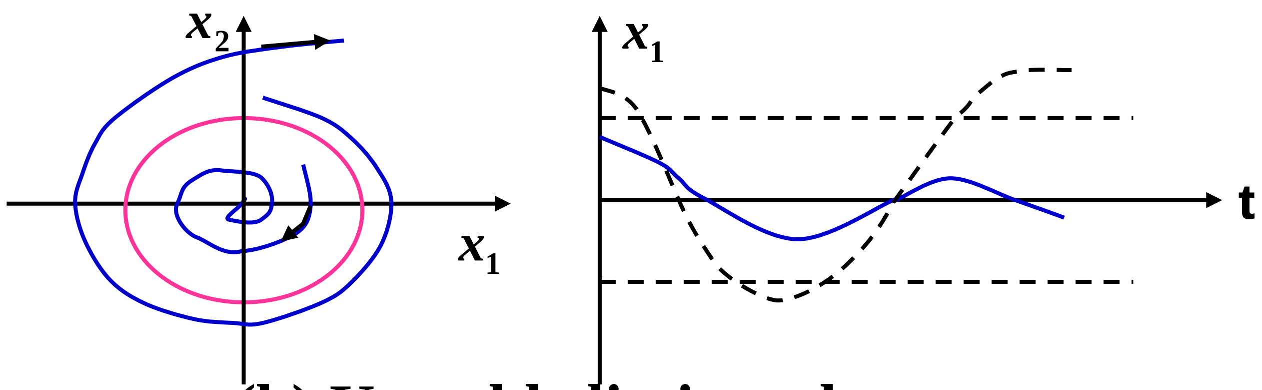


Fig 8-33 limit cycle

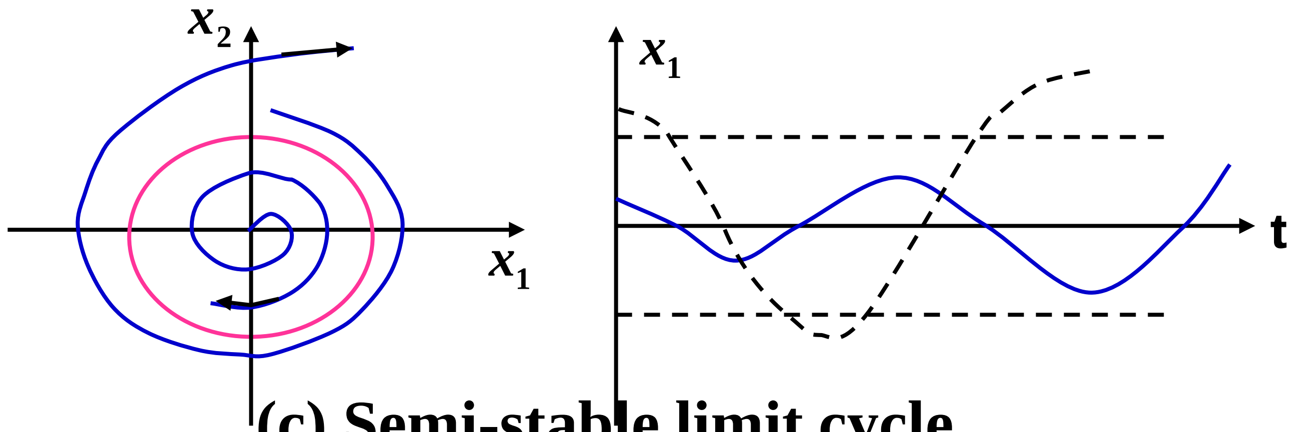
- Something should be pointed out, not all the closed curves in phase-plane are limit cycles. (Think about the trajectories corresponding to a center.) This kind of curves are not limit cycle, because they are not isolated.
- Limit cycle is a special phenomena which only exists in non-conservation systems. It is caused by nonlinearity of systems, not the non-damping feature of linear systems.



(a) Stable limit cycle



(b) Unstable limit cycle



(c) Semi-stable limit cycle

[Example 1] The equation of a non-linear system is given by

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

analyze the stability of this system.

Solution:

The Cartesian coordinate is transformed to the polar one as following:

Assume $x_1 = r\cos\theta,\quad x_2 = r\sin\theta$

Then $\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta}$

$$\dot{x}_2 = \dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta}$$

Substituting the above equations into the system equations, we have

$$\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta} = r\sin\theta + r\cos\theta(1-r^2) \qquad \cdots(1)$$

$$\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta} = -r\cos\theta + r\sin\theta(1-r^2) \qquad \cdots(2)$$

It follows from (2) that

$$\dot{\theta} = \frac{-r\cos\theta + r\sin\theta(1-r^2) - \dot{r}\sin\theta}{r\cos\theta} \qquad \cdots(3)$$

Substituting (3) into (1), we have $\dot{r} = r(1-r^2)$

It follows from (1) that

$$\dot{r} = \frac{r\sin\theta + r\cos\theta(1-r^2) + r\sin\theta\cdot\dot{\theta}}{\cos\theta} \qquad \cdots(4)$$

Substituting (4) into (2), we have $\dot{\theta} = -1$

$$\therefore \begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = -1 \end{cases}$$

There are two cases: $r = 0$ *and* $1 - r^2 = 0$

(1) $r = 0, \quad x_1 = 0, \quad x_2 = 0$ **is the singular point**

$$a = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} = 1 \qquad b = \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} = 1$$

$$c = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} = -1 \qquad d = \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} = 1$$

The roots of characteristic equation are:

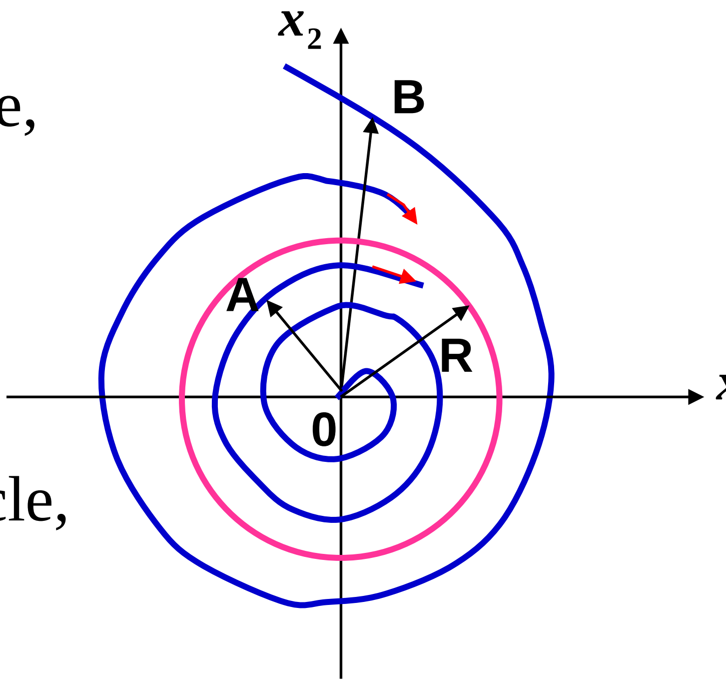
$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} = 1 \pm j$$

Singular point (0,0) is an ***unstable focus***, corresponding phase trajectories nearby are all divergent oscillations.

(2) $r = 1, x_1^2 + x_2^2 = 1$ **the unit cycle is limit cycle of systems.**

For an arbitrary point ***A*** inside the unit circle, inequality $\dot{r} = r(1-r^2) > 0$ holds since we have ***OA***<***r***=1.

Then the phase trajectory crossing point ***A*** will finally approach to the unit circle.



For an arbitrary point ***B*** outside the unit circle, inequality $\dot{r} = r(1-r^2) < 0$ holds since we have ***OB***>***r***=1.

Then the phase trajectory crossing point ***B*** will also finally approach to the unit circle.

$\therefore x_1^2 + x_2^2 = 1$ **is a stable limit cycle. Equilibrium (0, 0) is an unstable focus.**

3. Time domain analysis of typical nonlinear systems using Phase Plane Analysis

Algorithm of phase plane analysis:

1. Divide the phase plane into ***several areas*** according to nonlinear characteristics. Establish ***linear differential equations*** for each area.

2. Select ***appropriate coordinate axis*** during the analysis.

3. Establishing equations for the ***switching lines*** in the phase plane according to different nonlinear characteristics.

4. ***Solve the differential equations*** of each area and then draw phase trajectory.

5. The phase trajectory of the whole system can be obtained by ***connecting all the partial trajectories*** in different areas.

[Example 2] The following nonlinear system is excited with a step input signal of amplitude 6. If the initial state of the system is $e(0)=6,\ \dot{e}(0)=0$, how many seconds will it take for the system state to reach the origin.

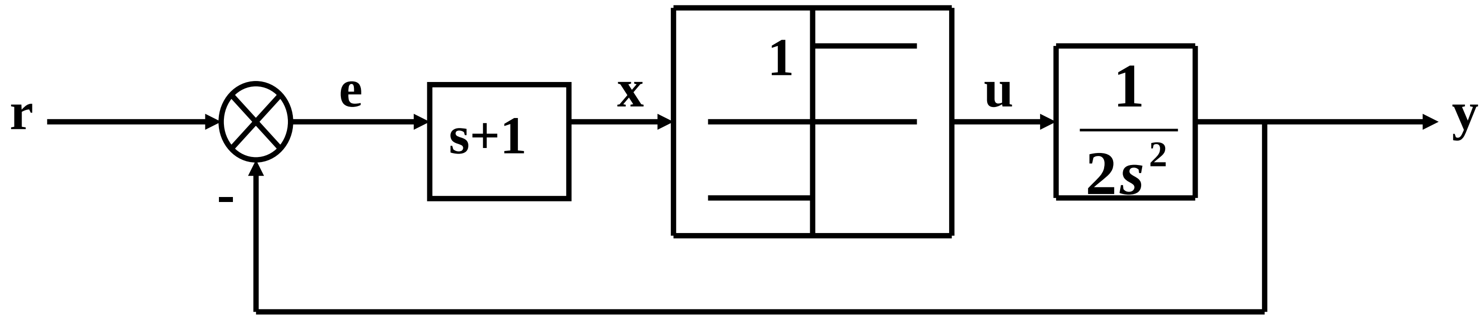


Fig 8-36 control system with a relay module

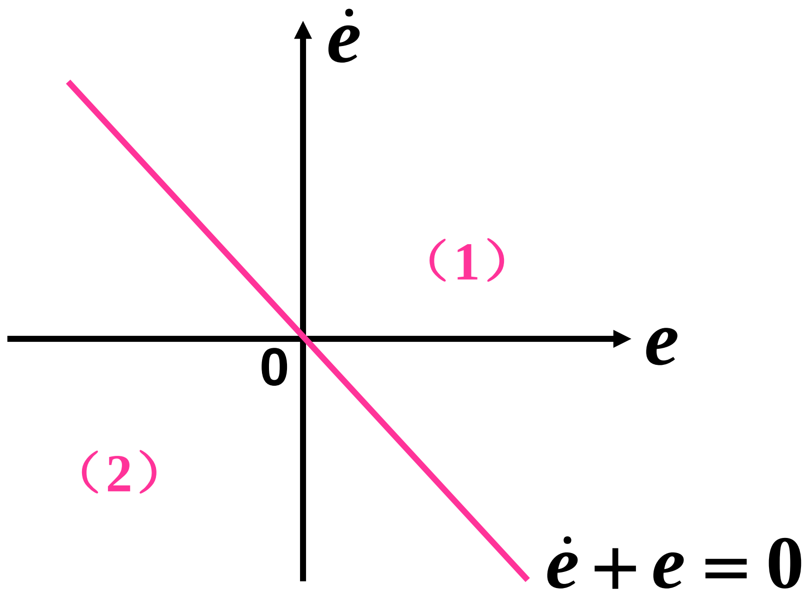
Solution: The dynamic equation is: $2\ddot{y}=u$

$$u=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$$

$$x = \dot{e} + e$$

and $e = r - y \qquad \therefore \ \dot{e} = -\dot{y}$

$$\therefore \ \ddot{e} = -\ddot{y} = \begin{cases} -0.5 & \dot{e} + e > 0 \qquad \text{area (1)} \\ 0.5 & \dot{e} + e < 0 \qquad \text{area (2)} \end{cases}$$



Area (1):

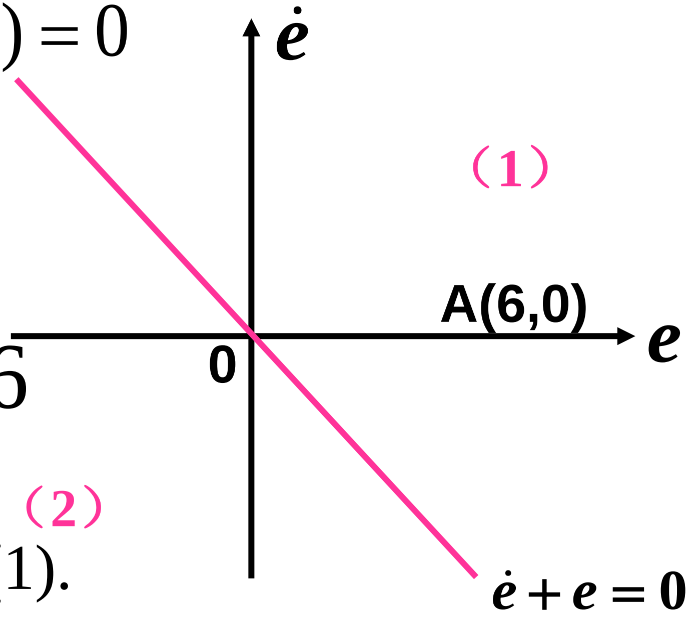
$$\begin{cases} \ddot{e} = -0.5 \\ \dot{e} = -0.5t + c_1 \\ e = -0.25t^2 + c_1 t + c_2 \end{cases}$$

The initial conditions is $e(0) = 6, \quad \dot{e}(0) = 0$

then $c_1 = 0, c_2 = 6$

$$\therefore \begin{cases} \dot{e} = -0.5t \\ e = -0.25t^2 + 6 \end{cases} \rightarrow e = -\dot{e}^2 + 6$$

——This is the phase trajectory in area (1).

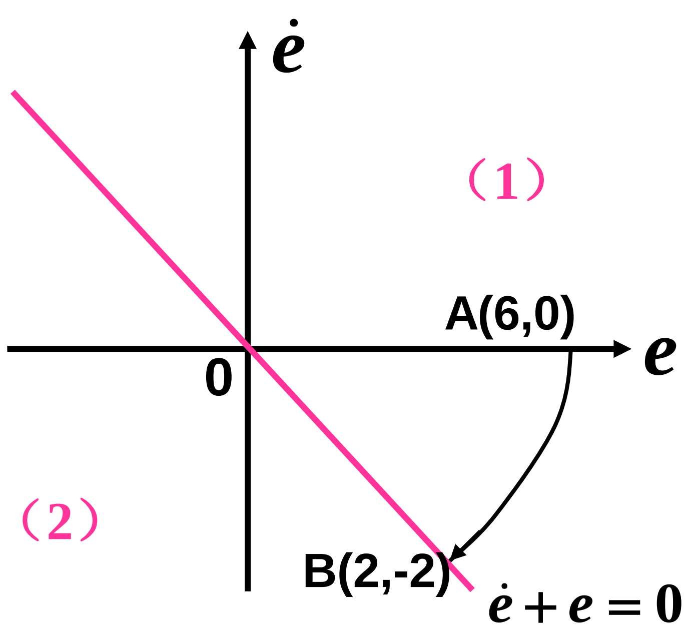


The phase trajectory is a ***parabola***. From point A , the system state reaches point B and enter area (2).

$$\begin{cases} e_B = -\dot{e}_B^2 + 6 \\ \dot{e}_B + e_B = 0 \end{cases}$$

Solution: $\boldsymbol{e_B = 2, \dot{e}_B = -2}$

area (2): $\begin{cases} \ddot{e} = 0.5 \\ \dot{e} = 0.5t + c_3 \\ e = 0.25t^2 + c_3 t + c_4 \end{cases}$



Consider the initial condition

$$e_B = 2, \dot{e}_B = -2$$

We obtain $c_3 = -2, c_4 = 2$

$$\therefore \begin{cases} \dot{e} = 0.5t - 2 \\ e = 0.25t^2 - 2t + 2 \end{cases}$$

Eliminating *t* we have

$$\boldsymbol{e = \dot{e}^2 - 2}$$ **——This is the trajectory in area (2).**

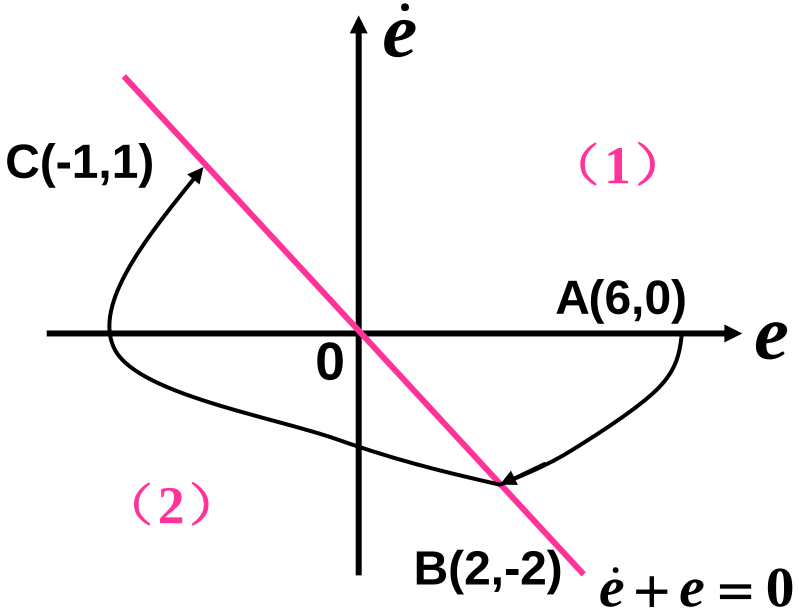
The system state moves along the parabola from point B to point C, then enter area (1).

The coordinate of point C satisfies

$$\begin{cases} e_C = \dot{e}_C^2 - 2 \\ \dot{e}_C + e_C = 0 \end{cases}$$

We can obtain

$$e_C = -1, \dot{e}_C = 1$$



Area (1):

$$\begin{cases} \ddot{e} = -0.5 \\ \dot{e} = -0.5t + c_5 \\ e = -0.25t^2 + c_5 t + c_6 \end{cases}$$

In terms of the initial condition C(-1,1), we have

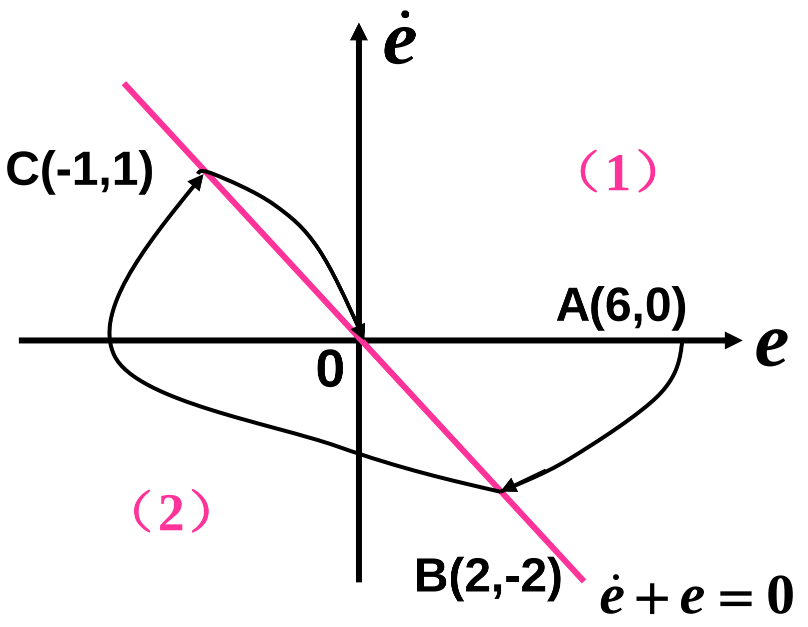
$$\boldsymbol{c_5 = 1, c_6 = -1}$$

$$\therefore \begin{cases} \dot{e} = -0.5t + 1 \\ e = -0.25t^2 + t - 1 \end{cases}$$

Eliminating *t* we have

$$e = -\dot{e}^2$$

In area (1), the system state moves along the parabola from point C to the origin.



The time t_{AO} of moving to the origin from point A can be obtained by the following equation:

$$\boldsymbol{t_{AO} = t_{AB} + t_{BC} + t_{CO}}$$

Based on the dynamic equations in different areas ,

$t_{AB}: \ \dot{e} = -0.5t \qquad \boldsymbol{-2 = -0.5t_{AB}} \qquad \boldsymbol{\therefore \ t_{AB} = 4}$

$t_{BC}: \ \dot{e} = 0.5t - 2 \qquad \boldsymbol{3 = 0.5t_{BC}} \qquad \boldsymbol{\therefore \ t_{BC} = 6}$

$t_{CO}: \ \dot{e} = -0.5t + 1 \qquad \boldsymbol{-1 = -0.5t_{CO}} \qquad \boldsymbol{\therefore \ t_{CO} = 2}$

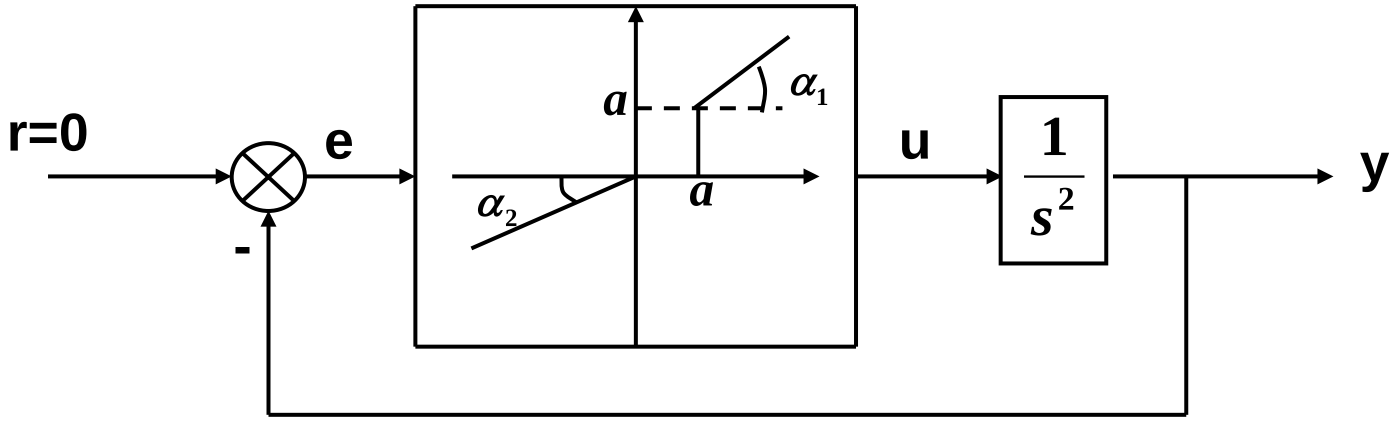
$$\therefore \ t_{AO} = t_{AB} + t_{BC} + t_{CO} = 12 \ \text{Seconds}$$

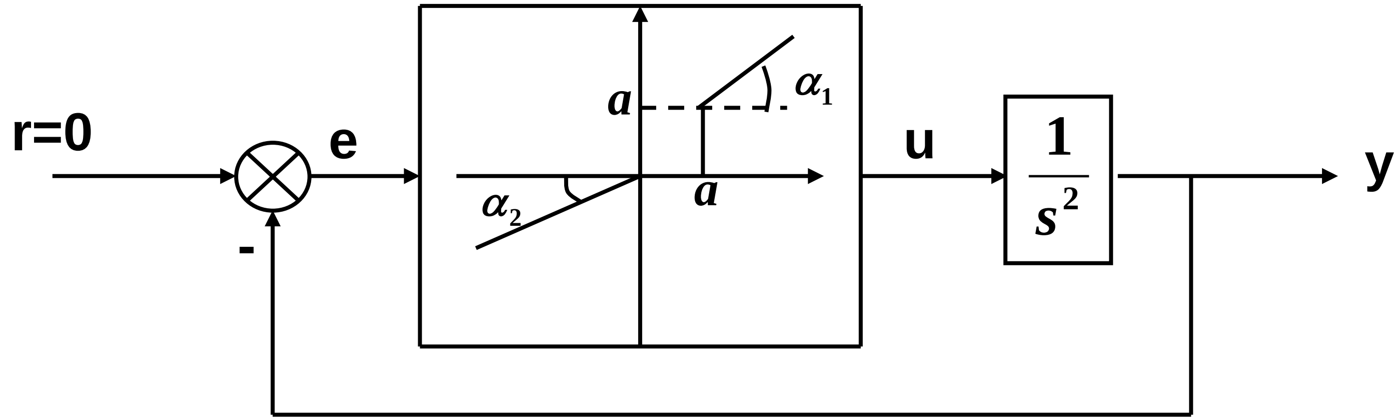
[Example3] The structure of a nonlinear system as shown in the following figure , where a = 1, $tg\alpha_1 = 1,\ tg\alpha_2 = 1/2.$

(1) Plot the phase trajectory of this system with the initial state $y(0) = -1, \dot{y}(0) = -1.$

(2) Draw briefly the corresponding curve $y(t)$. Try to Obtain the values of t when $y(t)$= 0.

(3) If $y(t)$ is periodic, obtain the value of this period.





Solution: Dynamic equations $\ddot{y} = u$

$$u = \begin{cases} a + (e-a)tg\alpha_1 & e > a \\ 0 & 0 \le e \le a \\ etg\alpha_2 & e < 0 \end{cases}$$

When r=0, e=-y. Substituting the known conditions into the above equation, we obtain

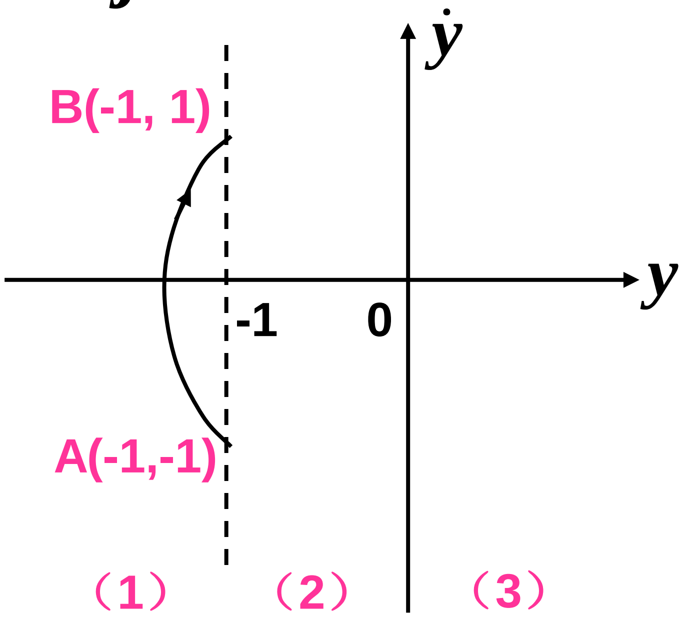
$$\ddot{y} = u = \begin{cases} -y & y < -1 \\ 0 & -1 \le y \le 0 \\ -0.5y & y > 0 \end{cases}$$

Area (1): $\ddot{y} = -y$

$$\ddot{y} + y = 0 \rightarrow \lambda^2 + 1 = 0, \ \lambda = \pm j$$

$$\therefore \quad y = c_1 \cos t + c_2 \sin t$$

$$\dot{y} = -c_1 \sin t + c_2 \cos t$$



Substitute the initial conditions A(-1,-1) into the above equations. We obtain:

$$c_1 = -1, c_2 = -1$$

$$\therefore \quad y = -\cos t - \sin t = -\sqrt{2} \sin(t + \frac{\pi}{4})$$

$$\dot{y} = \sin t - \cos t$$

Eliminating t we have

$$\dot{y}^2 + y^2 = 2$$

The system state moves in area(1) along the arc from point A to point B, and then enter area（2）.

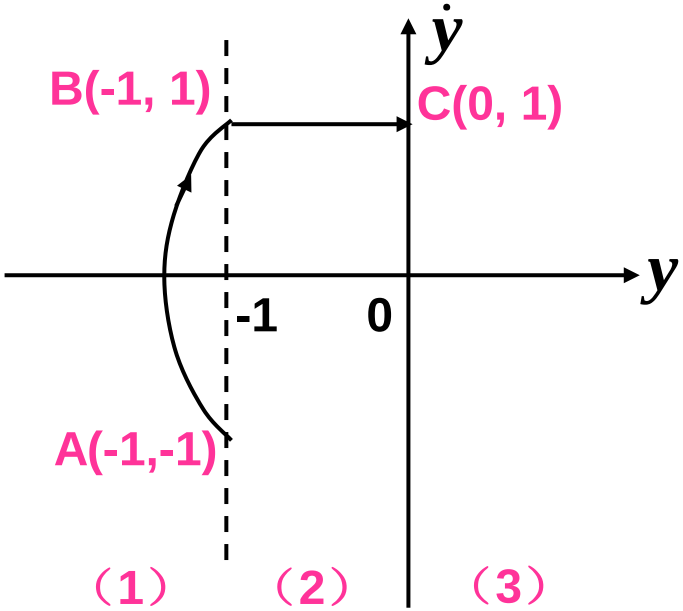
Area (2): $\ddot{y} = 0$

$$\dot{y} = c_3, \quad y = c_3 t + c_4$$

Substituting the initial conditions B(-1,1) into above equations, we obtain

$$c_3 = 1, c_4 = -1$$

$$\therefore \quad y = t - 1, \quad \dot{y} = 1$$



The system state moves in area (2) along the line $\dot{y} = 1$ from point A to point B, and then enter area (3).

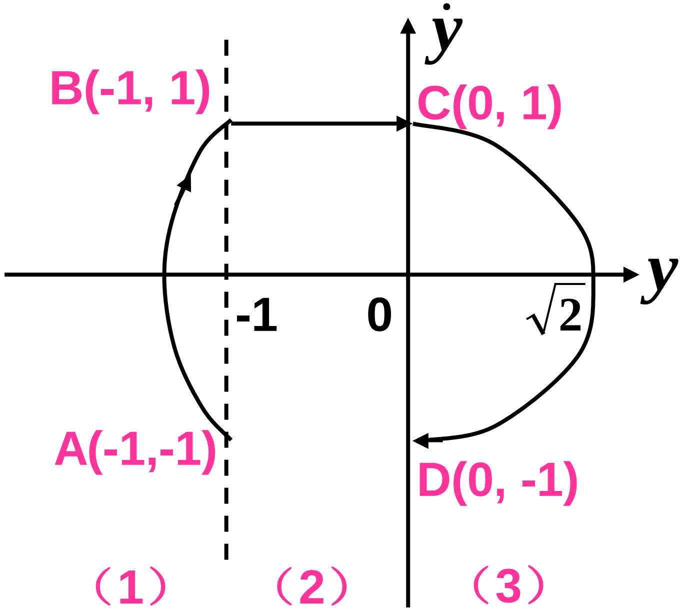
Area (3): $\ddot{y} = -0.5y$

$$\ddot{y} + \frac{1}{2}y = 0 \;\rightarrow \lambda^2 + \frac{1}{2} = 0, \;\; \lambda = \pm\frac{\sqrt{2}}{2}j$$

$$\therefore \quad y = c_5 \cos\frac{\sqrt{2}}{2}t + c_6 \sin\frac{\sqrt{2}}{2}t$$

$$\dot{y} = -\frac{\sqrt{2}}{2}c_5 \sin\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}c_6 \cos\frac{\sqrt{2}}{2}t$$

Substituting the initial conditions C(0, 1) into above equations, we obtain



$$c_5 = 0, c_6 = \sqrt{2}$$

$$\therefore \begin{cases} y = \sqrt{2}\sin\dfrac{\sqrt{2}}{2}t \\ \dot{y} = \cos\dfrac{\sqrt{2}}{2}t \end{cases}$$

then $\left(\dfrac{y}{\sqrt{2}}\right)^2 + \dot{y}^2 = 1$

The system state moves in area (3) along the ellipse from point C to point D, and then enter area (2).

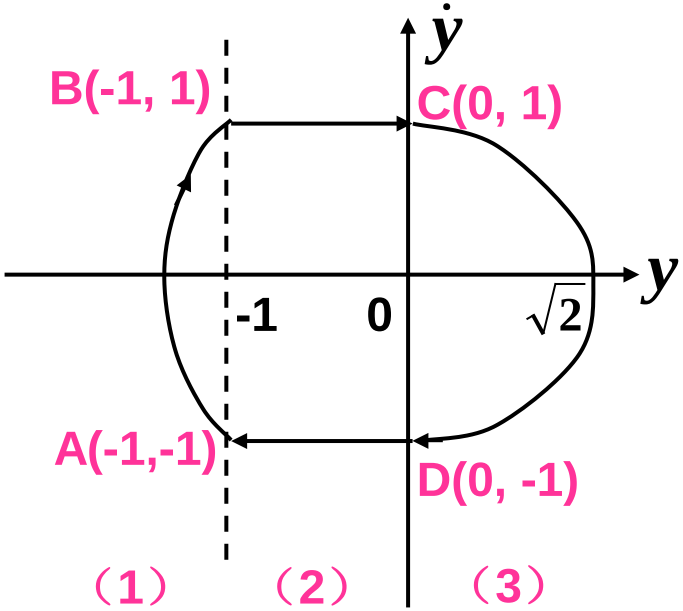
Area (2): $\ddot{y} = 0$

$$\dot{y} = c_7, \quad y = c_7 t + c_8$$

Substituting the initial conditions D(0,-1) into above equations, we obtain

$$c_7 = -1, c_8 = 0$$

$$\therefore \begin{cases} y = -t \\ \dot{y} = -1 \end{cases}$$



The system state moves in area (2) along the line $\dot{y} = -1$ from point D to point A, and then enter area (1). Because it is a closed phase trajectory, the motion of system is periodic 。

Based on the dynamic equations in different areas ,

$$t_{AB}: \quad y = -\sqrt{2}\sin(t+\frac{\pi}{4}) \qquad \sin(t_1+\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \qquad \therefore\ t_1 = \frac{\pi}{2}$$

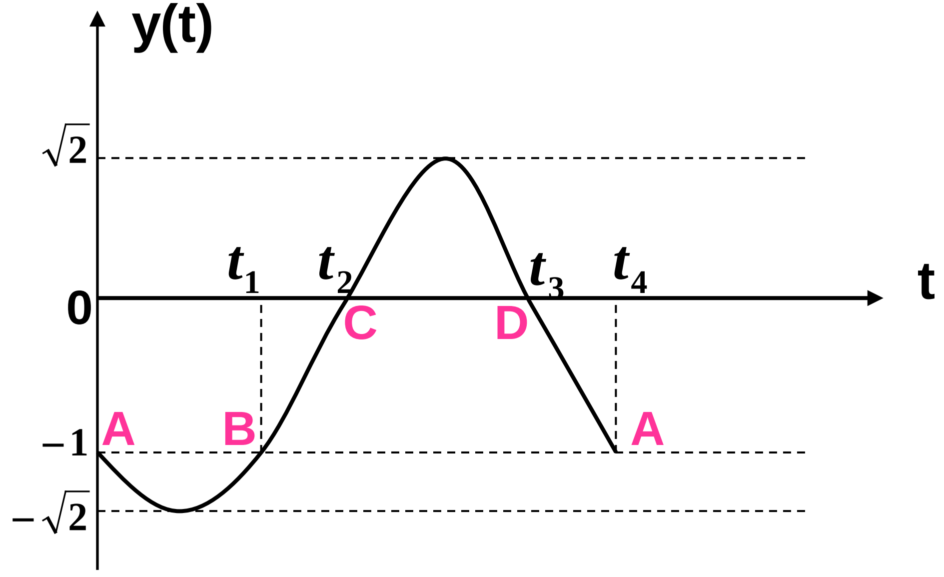
$$t_{BC}: \quad y = t-1 \qquad 0 = t_2 - 1 \qquad \therefore t_2 = 1$$

$$t_{CD}: \quad y = \sqrt{2}\sin\frac{\sqrt{2}}{2}t \qquad 0 = \sqrt{2}\sin\frac{\sqrt{2}}{2}t_3 \qquad \therefore\ t_3 = \sqrt{2}\pi$$

$$t_{DA}: \quad y = -t \qquad -1 = -t_4 \qquad \therefore\ t_4 = 1$$

The period of system motion is

$$T = t_1 + t_2 + t_3 + t_4 = 2 + \frac{\pi}{2} + \sqrt{2}\pi$$



where,

$$t^*_1 = \frac{\pi}{2} \qquad t^*_2 = 1 + \frac{\pi}{2} \qquad t^*_3 = 1 + \frac{\pi}{2} + \sqrt{2}\pi \qquad t^*_4 = 2 + \frac{\pi}{2} + \sqrt{2}\pi$$