7.6.3 Steady-state error of discrete systems

(1) General method: obtain system response

(2) Final value theorem $\begin{cases} G(z) \to \Phi_e(z) \\ D(z) \to \text{ Stability} \\ e(\infty) = \lim\limits_{z \to 1}(z-1)R(z)\Phi_e(z) \end{cases}$

(3) Static error constant $\begin{cases} G(z) \to v,\ K_p,\ K_v,\ K_a \\ \text{Obtain } e(\infty) \end{cases}$

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

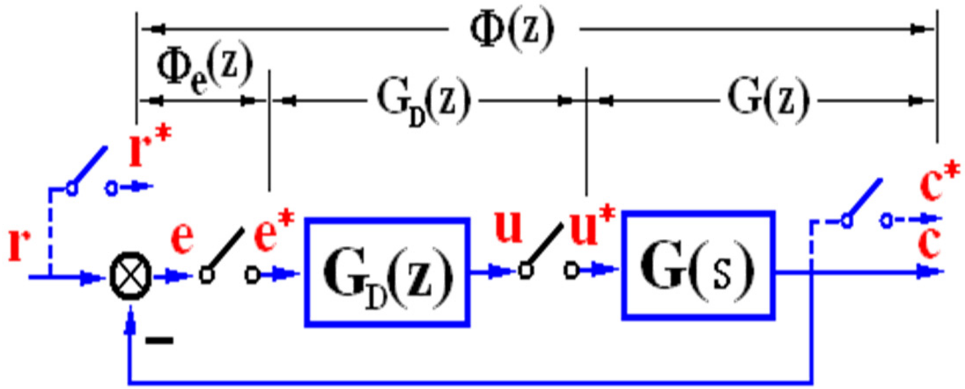
Design for discrete-time systems can be done in s-domain, z-domain and w-domain, respectively.

7.7.1 The Impulse Transfer Function for the Digital Controller

$$\Phi(z)=\frac{G_D(z)\cdot G(z)}{1+G_D(z)\cdot G(z)}$$

$$\Phi_e(z)=\frac{1}{1+G_D(z)\cdot G(z)}=1-\Phi(z)$$

$$G_D(z)\cdot G(z)=\frac{\Phi(z)}{1-\Phi(z)}=\frac{\Phi(z)}{\Phi_e(z)}$$



$$\boxed{G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}}$$

$$\Phi_e(z)=1-\Phi(z),\ \mathrm{E}(z)=\Phi_e(z)R(z)$$

7.7.2 Deadbeat Control Design 最少拍控制

Deadbeat Control Systems: Matching a particular test input within a number of steps. —— No steady-state error on the sampling point.

(典型输入作用下，能在有限拍内结束响应过程且在采样点上无稳态误差的系统。)

1. A unified description of typical test inputs

$$r(t)=\begin{cases}1(t)\\ t\\ t^2/2\end{cases}\qquad R(z)=\left\{\begin{matrix}\dfrac{z}{z-1}=\dfrac{1}{1-z^{-1}}\\ \dfrac{Tz}{(z-1)^2}=\dfrac{Tz^{-1}}{(1-z^{-1})^2}\\ \dfrac{T^2z(z+1)}{2(z-1)^3}=\dfrac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}\end{matrix}\right\}\ \frac{A(z)}{(1-z^{-1})^{\nu}}$$

ν	$A(z)$
1	1
2	Tz^{-1}
3	$\dfrac{T^2z^{-1}(1+z^{-1})}{2}$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z) \cdot G(z)}$$

Design Idea: **Obtain $G_D(z)$ by constructing $\Phi(z)$ so that the output can match the typical test signal within the minimum steps.**

No $\begin{cases} \text{Zeros} \\ \text{Poles} \end{cases}$ of G(z) on or beyond the unit circle, except for (1, j0)

$$\mathrm{R}(z) = \frac{A(z)}{(1 - z^{-1})^{\nu}}$$

$$\mathrm{E(z)} = \Phi_e(z) R(z), \quad \Phi_e(z) = 1 - \Phi(z)$$

$$e(\infty) = \lim_{z \to 1} (z-1)\, \Phi_e(z)\, R(z)$$

From the design idea, we know that $e(\infty T) = 0$

$$E(z) = \Phi_e(z) \cdot R(z) = \frac{A(z)}{(1 - z^{-1})^{\nu}} \Phi_e(z)$$

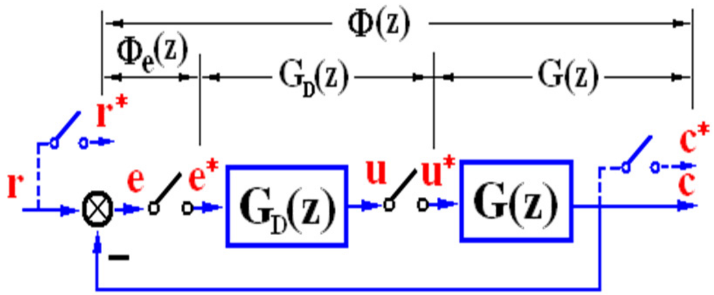
$$e(\infty T) = \lim_{z\to 1}(1-z^{-1})\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z) = 0$$

$$\Rightarrow \Phi_e(z) = (1-z^{-1})^{\nu} F(z^{-1})$$

To make the D(z) simplest and of the lowest-order, we can choose $F(z^{-1})$ as 1.

$$\Phi_e(z) = (1-z^{-1})^{\nu} F(z) \overset{F(z)=1}{=} (1-z^{-1})^{\nu}$$

$$\Phi(z) = 1-\Phi_e(z) = 1-(1-z^{-1})^{\nu}$$



Hence:

$$\Phi(z) = 1-\Phi_e(z) = 1-(1-z^{-1})^{\nu} = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{\nu} z^{-\nu}$$

$$= \frac{b_1 z^{\nu-1} + b_2 z^{\nu-2} + \cdots + b_{\nu}}{z^{\nu}}$$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

The rule to construct Φ(z): All poles of Φ(z) are located on the origin of z-plane.

2. Φ(z) for typical test inputs

（1） for $r(t) = 1(t)$

- The C.L. impulse transfer function:

$$\nu = 1 \qquad \Phi(z) = z^{-1}$$

$$E(z) = \sum_{n=0}^{\infty} e(nT) z^{-n}$$

$$= e(0) + e(T) z^{-1} + \ldots = 1$$

The system can track the input by 1 step only。

（2） for $r(t) = t \cdot 1(t)$

- The C.L. impulse transfer function:

$$\nu = 2 \quad \Phi(z) = 2z^{-1} - z^{-2}$$

$$E(z) = \sum_{n=0}^{\infty} e(nT) z^{-n}$$

$$= e(0) + e(T) z^{-1} + \ldots = Tz^{-1}$$

The system can track the input by 2 steps。

（3） for $r(t)=\frac{1}{2}t^2\cdot 1(t)$

- **The C.L.impulse transfer function:**

$$v=3 \qquad \Phi(z)=3z^{-1}-3z^{-2}+z^{-3}$$

$$E(z)=\sum_{n=0}^{\infty}e(nT)z^{-n}$$

$$=e(0)+e(T)z^{-1}+\ldots=\frac{1}{2}T^2z^{-1}+\frac{1}{2}T^2z^{-2}$$

The system can track the input by 3 steps。

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z) \cdot G(z)}$$

Deadbeat Control Design Table

$r(t)$	$R(z)$	$\Phi_e(z) = (1-z^{-1})^v$	$\Phi(z) = 1-\Phi_e(z)$	$G_D(z)$	t_s
$1(t)$	$\dfrac{1}{1-z^{-1}}$	$1-z^{-1}$	z^{-1}	$\dfrac{z^{-1}}{(1-z^{-1}) \cdot G(z)}$	T
t	$\dfrac{Tz^{-1}}{(1-z^{-1})^2}$	$(1-z^{-1})^2$	$2z^{-1}-z^{-2}$	$\dfrac{z^{-1}(2-z^{-1})}{(1-z^{-1})^2 G(z)}$	$2T$
$\dfrac{t^2}{2}$	$\dfrac{T^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$	$(1-z^{-1})^3$	$3z^{-1}-3z^{-2}+z^{-3}$	$\dfrac{z^{-1}(3-3z^{-1}+z^{-2})}{(1-z^{-1})^3 G(z)}$	$3T$

3. Algorithm for Deadbeat Control Design

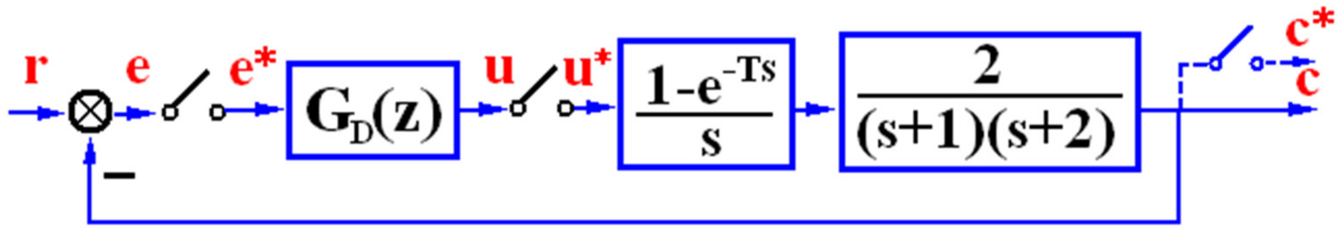
① **Obtain G(z) — Suppose there are no poles and zeros of G(z) on or beyond the unit circle.**

② **Determine $\Phi_e(z)$ for the particular test input by v**

$$\boldsymbol{r(t)} \quad \Rightarrow \quad R(z)=\frac{A(z)}{(1-z^{-1})^{\nu}} \quad \Rightarrow \quad \Phi_e(z)=(1-z^{-1})^{\nu}$$

③ **Obtain** $\boldsymbol{\Phi(z)=1-\Phi_e(z)}$

④ **Achieve** $\boldsymbol{G_D(z)=\dfrac{\Phi(z)}{\Phi_e(z)\cdot G(z)}}$



Example 1. **Consider the system shown in the above figure (T=1). Design deadbeat controllers $G_D(z)$ for r(t)=1(t)、t .**

Solution.

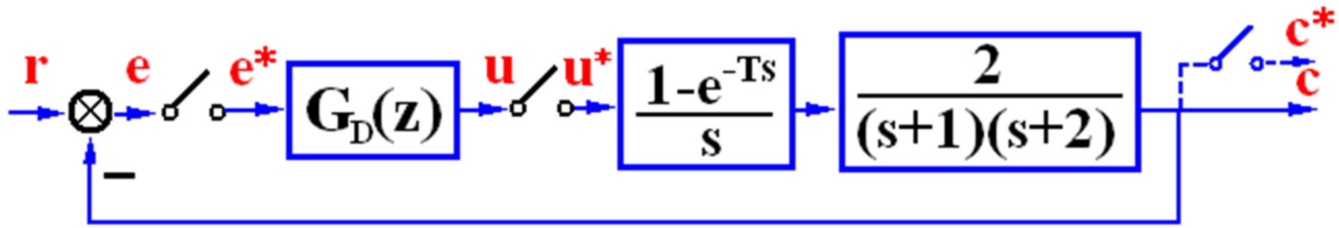
$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{2}{(s+1)(s+2)}\right] = 2(1-z^{-1})\cdot Z\left[\frac{C_0}{s} - \frac{C_1}{s+1} + \frac{C_2}{s+2}\right]$$

$$= 2\cdot\frac{z-1}{z}\cdot Z\left[\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}\right]$$

$$= \frac{z-1}{z}\left[\frac{z}{z-1} - \frac{2z}{z-e^{-T}} + \frac{z}{z-e^{-2T}}\right] = 1 - \frac{2(z-1)}{z-e^{-T}} + \frac{z-1}{z-e^{-2T}}$$

$$= \frac{(1+e^{-2T}-2e^{-T})z + (e^{-3T}+e^{-T}-2e^{-2T})}{(z-e^{-T})(z-e^{-2T})}$$

$$\overset{T=1}{=} \frac{0.4(z+0.365)}{(z-0.368)(z-0.136)}$$



Referring to the result for $r(t) = 1(t)$ in the Design Table

$$R(z) = \frac{z}{z-1} \qquad \text{Choose} \begin{cases} \Phi_e(z) = 1 - z^{-1} \\ \Phi(z) = 1 - \Phi_e(z) = z^{-1} \end{cases}$$

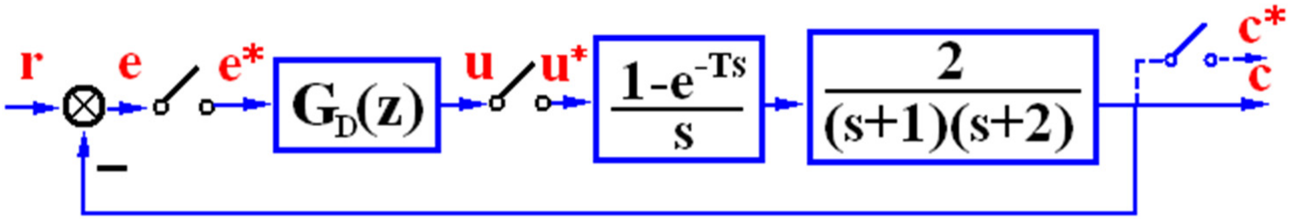
$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)} = \frac{z^{-1}}{1-z^{-1}} \cdot \frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$= \frac{2.5(z-0.368)(z-0.136)}{(z-1)(z+0.365)}$$

$$C(z) = \Phi(z)R(z) = z^{-1} \cdot \frac{1}{1-z^{-1}}$$

$$= z^{-1}[1 + z^{-1} + z^{-2} + \cdots] = z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$E(z) = \Phi_e(z)R(z) = (1-z^{-1}) \cdot \frac{1}{1-z^{-1}} = 1$$



For $r(t) = t$

$$R(z) = \frac{Tz^{-1}}{(1-z^{-1})^2} \qquad \text{Choose} \begin{cases} \Phi_e(z) = (1-z^{-1})^2 \\ \Phi(z) = 1 - \Phi_e(z) = 2z^{-1} - z^{-2} \end{cases}$$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)} = \frac{2z^{-1} - z^{-2}}{(1-z^{-1})^2} \cdot \frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$= \frac{5(z-0.5)(z-0.368)(z-0.136)}{(z-1)^2(z+0.365)}$$

$$E(z) = \Phi_e(z)\cdot R(z) = Tz^{-1}$$

$$C(z) = \Phi(z)R(z) = (2z^{-1} - z^{-2})\cdot\frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$= R(z) - E(z) = 2Tz^{-2} + 3Tz^{-3} + 4Tz^{-4} + \cdots$$

- **Attention:**
- The setting time (the minimum steps) of the system is decided by designed Φ(z), rather than the typical input signal: 1(t), t, $t^2/2$.
- Such as: if we use the Deadbeat system of velocity input r(t)=t.
- We have $\Phi(z) = 2z^{-1} - z^{-2}$
- For r(t)=1(t)

$$R(z) = \frac{1}{1-z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$C(z) = \frac{2z^{-1} - z^{-2}}{1-z^{-1}} = 0 + 2z^{-1} + z^{-2} + z^{-3} + \ldots$$

- For r(t)=t

$$R(z)=\frac{Tz^{-1}}{(1-z^{-1})^2}=0+Tz^{-1}+2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\dots$$

$$C(z)=\frac{Tz^{-1}(2z^{-1}-z^{-2})}{(1-z^{-1})^2}=0+0+2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\dots$$

- For r(t)=t^2/2

$$R(z)=\frac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}=0+0.5T^2z^{-1}+2T^2z^{-2}+4.5T^2z^{-3}+8T^2z^{-4}+\dots$$

$$C(z)=\frac{T^2z^{-1}(1+z^{-1})(2z^{-1}-z^{-2})}{2(1-z^{-1})^3}=0+0+T^2z^{-2}+3.5T^2z^{-3}+7T^2z^{-4}+\dots$$

4. G(z) has poles or zeros on or beyond the unit circle

suppose

$$G(z) = \frac{z^{-v}\prod_{i=1}^{L}(1 - z_i z^{-1})}{\prod_{i=1}^{n}(1 - p_i z^{-1})}$$

where z_i **is the zero of G(z);**

p_i **is the pole of G(z).**

v **is delay.**

Then

$$\mathrm{G}_D(z) = \frac{\Phi(z)}{\Phi_e(z)G(z)} = \frac{z^{v}\prod_{i=1}^{n}(1 - p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1 - z_i z^{-1})\Phi_e(z)}$$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi_e(z)}$$

①If there is z^{v} in $G_D(z)$， $G_D(z)$ is un-realizable.

Thus, we have to ensure that there exists z^{-v} in $\Phi(z)$, which promises $G_D(z)$ is realizable.

② If there is z_i on or beyond the unit circle, $G_D(z)$ is unstable.

Then, those z_i will be designed as the zeros of $\Phi(z)$.

③**Note that** $\Phi(z)=G_D(z)G(z)\Phi_e(z)$

$$=\frac{z^{-v}\prod_{i=1}^{L}(1-z_i z^{-1})}{\prod_{i=1}^{n}(1-p_i z^{-1})}G_D(z)\Phi_e(z)$$

If there are p_i on or beyond the unit circle,

$\Phi(z)$ will be unstable，

Then **those p_i will be designed as the zeros of $\Phi_e(z)$.**

④ Determine the relative parameters by

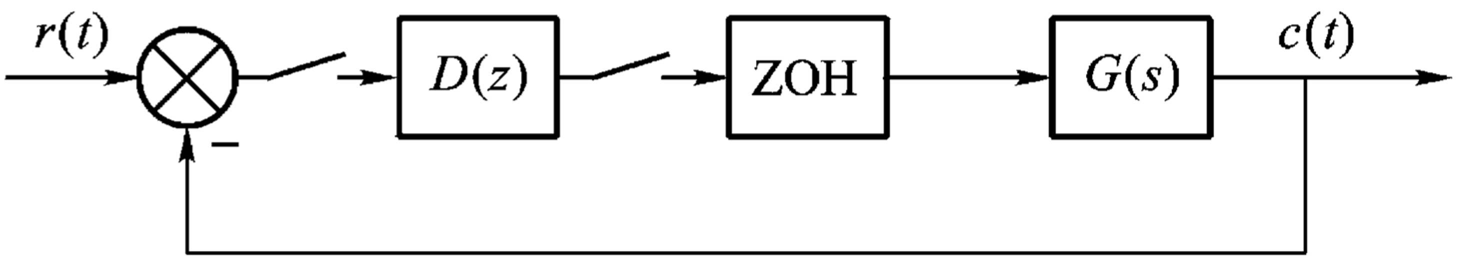
$$\Phi(z)=1-\Phi_e(z)$$

Example Given the discrete system described as in the Following figure, where

$$G(s)=\frac{10}{s(0.1s+1)(0.05s+1)}, \quad G_{zoh}(s)=\frac{1-e^{-Ts}}{s}$$

with $T = 0.2s$

Design a deadbeat controller for $r(t)=1(t)$



Solution: the O. L. impulse transfer function is

$$G(z)=Z[G_{zoh}(s)G(s)]=\frac{0.76z^{-1}(1+0.05z^{-1})(1+1.065z^{-1})}{(1-z^{-1})(1-0.135z^{-1})(1-0.0185z^{-1})}$$

For $r(t)=1(t)$ **, we can design**

$$\varphi_e(z)=1-z^{-1} \qquad (1)$$

$$\varphi(z)=z^{-1} \qquad (2)$$

Because there exists z **= −1.065 (beyond the unit circle),**

Thus, z should also be the zero of Φ(z)

There exist z^{-1} ***in G(z)*** **,** z^{-1} **should be in Φ(z), thus**

$$\varphi(z)=\quad z^{-1}(1+1.065z^{-1}) \qquad (3)$$

Because that

$$\varphi(z) = 1 - \varphi_e(z) \qquad (4)$$

from (3), $\varphi(z)$ **is now a polynomial on** z^{-1} **of order 2,**
To satisfy（4）， $\varphi_e(z)$ **must be a polynomial on** z^{-1} **of order 2， thus based on (1), we redesign：**

$$\varphi_e(z) = (1 - z^{-1})\,(1 + a_1 z^{-1}) \qquad (5)$$

Where a_1 **is a constant to be chosen later。**

Thus multiplied by a constant b_1 to be designed later, we get

$$\varphi(z) = b_1 z^{-1}(1 + 1.065 z^{-1}) \qquad (6)$$

From（5）and（6），we get：

$$a_1 = 0.516 \qquad b_1 = 0.484$$

Thus,

$$\varphi_e(z) = (1 - z^{-1})\ (1 + 0.516z^{-1}) \qquad \textbf{（7）}$$

$$\varphi(z) = 0.484z^{-1}(1 + 1.065z^{-1}) \qquad \textbf{（8）}$$

Then the deadbeat controller is

$$D(z) = \frac{1 - \varphi_e(z)}{G(z)\varphi_e(z)}$$

$$= \frac{1 - (1 - z^{-1})\ (1 + 0.516z^{-1})}{\dfrac{0.\ 76z^{-1}(1 + 0.05z^{-1})\ (1 + 0.065z^{-1})}{(1 - z^{-1})\ (1 - 0.135z^{-1})\ (1 - 0.0185z^{-1})}(1 - z^{-1})\ (1 + 0.516z^{-1})}$$

$$D(z) = \frac{0.637(1-0.0185z^{-1})\,(1-0.135z^{-1})}{(1+0.05z^{-1})\,(1+0.516z^{-1})}$$

Then the *Z*-transform is

$$C(z) = \varphi(z)R(z) = 0.484z^{-1}(1+1.085z^{-1})\frac{1}{1-z^{-1}}$$
$$= 0.484z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-4} + \cdots$$

System can follow the input at the 2nd step, which is one step later.

Although the deadbeat control system tracks a particular test input accurately within a number of steps, it has the following disadvantages:

(1) It is designed only for a particular input.

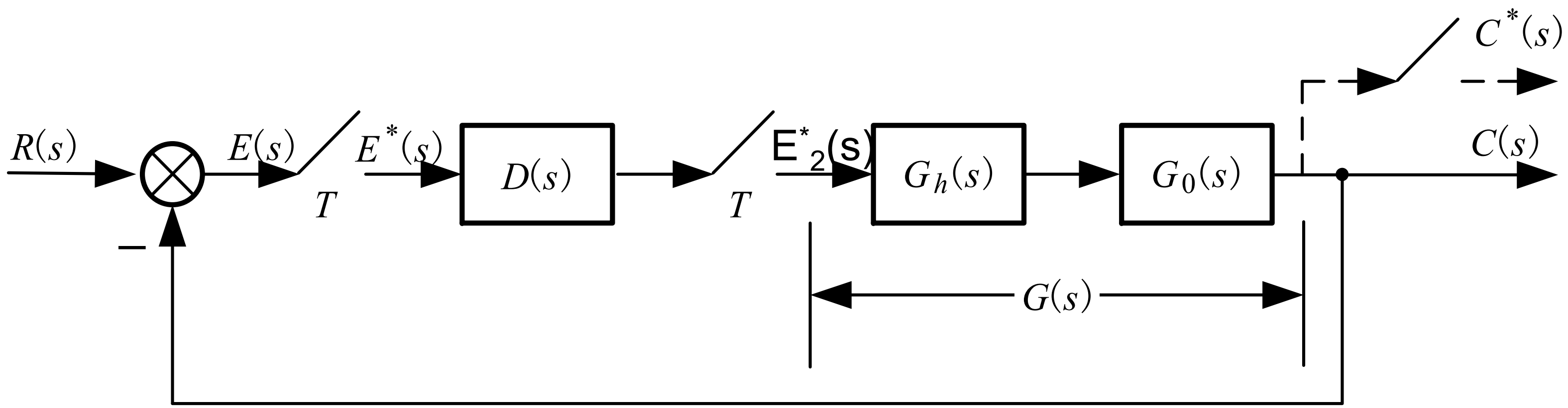
(2) Although there are no errors on the sampling points, the output has ripples between the sampling points.

(3) The high-order controller will make the control process (the output of controller) changes drastically.

5. Ripple-free deadbeat control design

Ripple：though the system outputs are stable at the sampling time, they are varying between two sampling time。

Objective: Not only tracking the input at the sampling time, but also the one between two sampling point. Then, the outputs are ripple-free.



$$E_2(z) = D(z)E(z)$$

solution： ensure $E_2(z)$ being a polynomial on z^{-1} of a finite order.

Condition: $E_2(z)$ is a polynomial on $z^{-1}$of finite order。

$$E_2(z) = D(z)E(z) = D(z)\varphi_e(z)R(z), \quad D(z)\varphi_e(z) = \frac{\varphi(z)}{G(z)}$$

➔ $D(z)\Phi_e(z)$=（*）$/z^r$, that is the zero of $G(z)$ must be a zero of $\Phi(z)$

最少拍设计中，$\Phi(z)$和$\Phi_e(z)$选取时应遵循的原则：

1。D(z)零点的数目不能大于极点的数目；

2。$\Phi_e(z)$应把G(z)在单位圆上及单位圆外的极点作为自己的零点；

3。$\Phi(z)$应把G(z)在单位圆上及单位圆外的零点作为自己的零点；

4。当G(z)含有z^{-1}因子时，要求$\Phi(z)$也含有z^{-1}的因子；

5。因为$\Phi(z)=1-\Phi_e(z)$，他们应该是关于z^{-1}同样阶次的多项式，而且$\Phi_e(z)$还应包含常数项1。

6。当最小拍系统还有无纹波要求时，闭环脉冲传函$\Phi(z)$的零点应抵消G(z)的全部零点（因为最少拍系统设计中G(z)单位圆上及单位圆外的零极点已经被补偿，因此在无纹波的设计中只需抵消G(z)单位圆内的零点）。

Homework:

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7-15. Consider the system as shown in Fig 7-69, T=1s, design deadbeat controller D(z) for r(t)=t. Draw $r^*(t)$, $e_1^*(t)$, $e_2^*(t)$, x(t), y(t) and $y^*(t)$.

7-16. Furthermore, design a ripple-free deadbeat controller for the system in 7-15.