9.5 Controllability and Observablity of Linear time-invariant system

- Background knowledge
- Concept of Controllability and Observability of Linear Continual System
- Criteria of Controllability and Observability of Linear Continual System
- Duality Principle(对偶原理)

➢ The background of Controllability and Observability

In 1960s, Kalman ... from state space description.

In modern control theory, we consider about a issue, that whether all the states of the system are *affected* and *reflected* by the *input* and *output* in state equation and output equation description of the system, which is the Controllability and Observability Issue.

Controllability: system input affect all the state of the system to achieve the control.

Observability: system output can reflect all the state of the system to achieve the observation.

For SS only, rather than TF representation.

Ex.9-19 System state equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analysis: the differential equations

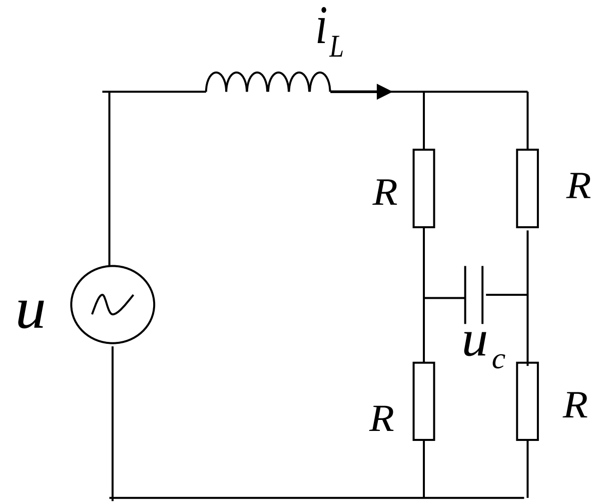
$$\dot{x}_1 = 2x_1 + u$$
$$\dot{x}_2 = -3x_2 + 2u$$
$$y = -4x_2$$

By the input u(t), the state variables x_1 and x_2 can transform from initial value to zero. Thus both states are controllable.

The output y can reflect the state x_2 only, and has not connection with x_1. Thus the system is incomplete observable.

Ex.9-20(a) in a bridge circuit, choose state variables as the current of the inductance i_L and the voltage of the capacitance u_c, the input is u and the output is $y=u_c$

Analysis:



Assume $x_1 = i_L, \quad x_2 = u_c$

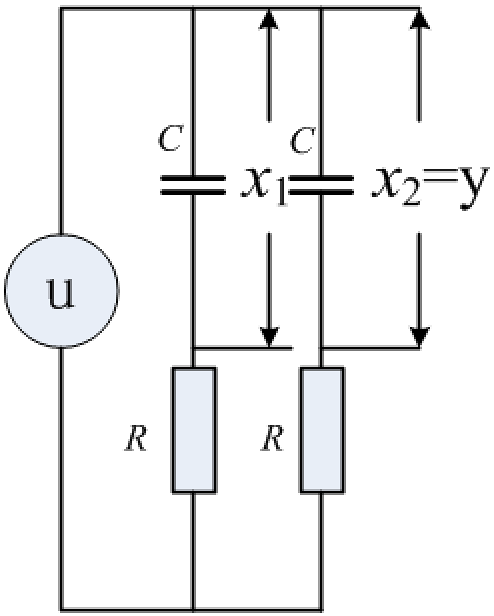
if $x_2(t_0) = 0$

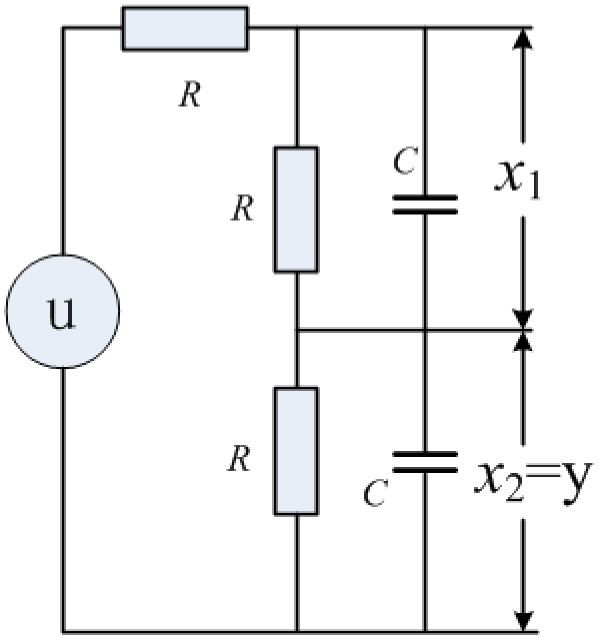
and $t \geq t_0, \quad x_2(t) \equiv 0$

Thus, input u cannot control the state x_2, x_2 is uncontrollable.

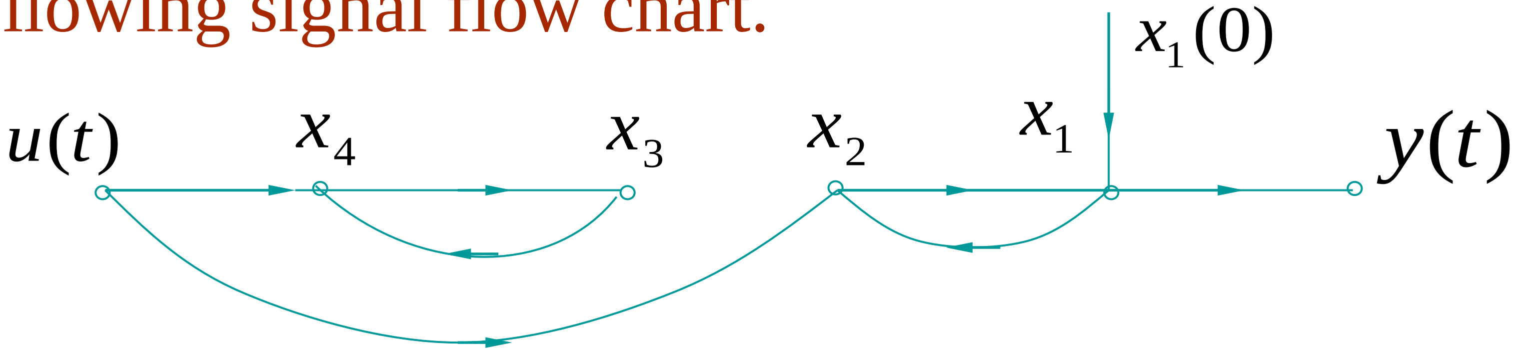
Furthermore, because $y=u_c \equiv 0$, output y cannot reflect the changes of variable x_1, thus x_1 is unobservable.

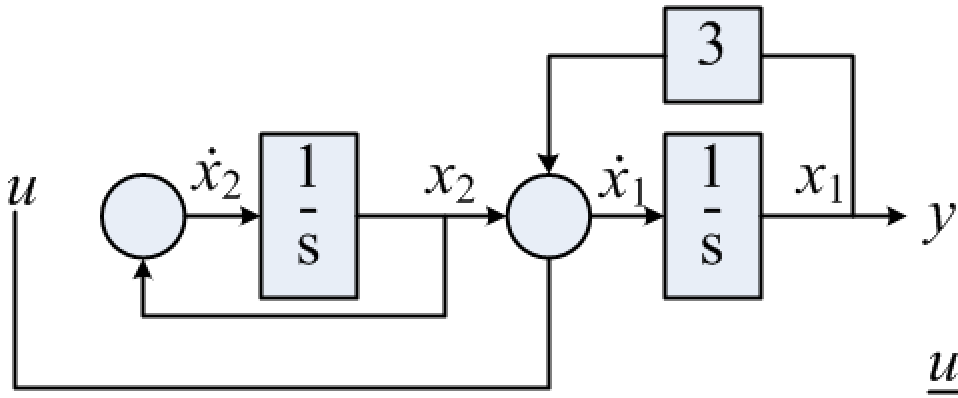
Ex.9-20(b) analysis the controllability and observability of the following circuit.

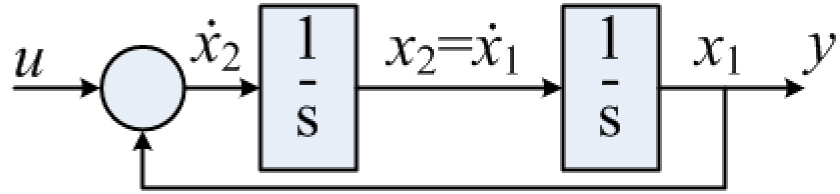


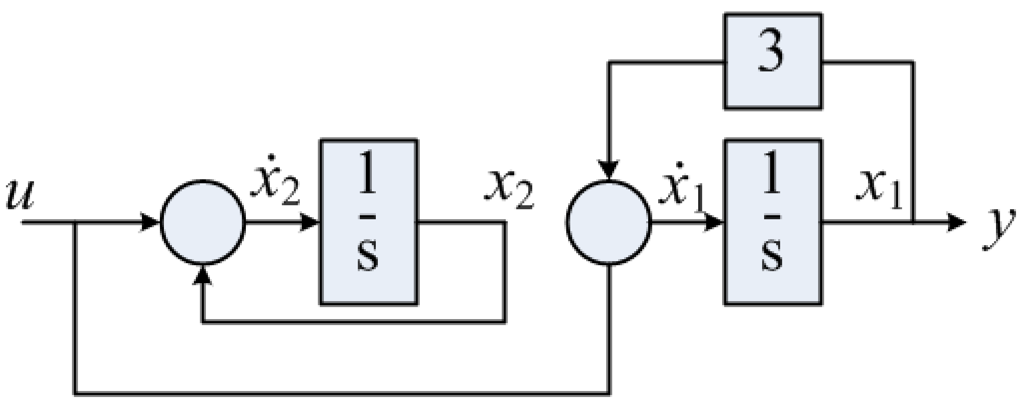


Ex.9-21(a) analysis the controllability and observability of the following signal flow chart.









9.5.1 the Controllability of the linear time-invariable continual system

1. Definition of Controllability

State-equations: $\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in T_t$

$x(t) \in R^n$, $u(t) \in R^r$, $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times r}$, T_t: time space

State controllability

It is called the state x_0 is controllable at t_0, if for a non-zero initial state $x(t_0)=x_0$ with the initial time $t_0 \in T$, exists certain time $t_1 \in T_t$, $t_1>t_0$ and an unrestricted control $u(t)$, which makes the state transfer from $x(t_0)=x_0$ to $x(t_1)=0$.

System Controllability

It is called the system is controllable at time t_0, if at time $t_0 \in T$, the non-zero initial states in the state space are all controllable.

Incomplete Controllable

It is called the system is incomplete controllable, if there are one or some non-zero state variables uncontrollable in the state space.

The controllability of linear time-invariable system has no relation to the initial time t_0.

➢ Cayley- Hamilton Theorem

companion matrix
(伴随矩阵)

Consider a n×n matrix A, whose eigenpolynomial is:

$$f(\lambda)=|\lambda I-A|=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0$$

Then the matrix A satisfy its eigenpolynomial as well:

$$A^n+a_{n-1}A^{n-1}+\cdots+a_1A+a_0I=0$$

Proof:

$$(\lambda I-A)^{-1}=\frac{adj(\lambda I-A)}{|\lambda I-A|}=\frac{adj(\lambda I-A)}{f(\lambda)}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \lambda I-A=\begin{bmatrix} \lambda-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda-a_{nn} \end{bmatrix}$$

$$adj(\lambda I - A) = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & & \vdots \\ -a_{n2} & & \lambda - a_{nn} \end{vmatrix} \cdots\cdots (-1)^{n+1}\begin{vmatrix} -a_{12} & \cdots & -a_{1n} \\ \vdots & & \vdots \\ -a_{n-1,2} & \cdots & -a_{n-1,n} \end{vmatrix} \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ (-1)^{1+n}\begin{vmatrix} -a_{21} & \cdots & -a_{2,n-1} \\ \vdots & & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} \end{vmatrix} \cdots\cdots (-1)^{n+n}\begin{vmatrix} \lambda - a_{11} & \cdots & -a_{1,n-1} \\ \vdots & & \vdots \\ -a_{n-1,1} & \cdots & \lambda - a_{n-1,n-1} \end{vmatrix} \end{bmatrix}$$

The elements in the companion matrix $adj(\lambda \boldsymbol{I}\text{-}A)$ are the multinomial with λ. The companion matrix can be decomposed into the summary of n matrix:

$$\mathrm{adj}(\lambda I - A) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0$$

In which, Bn-1,…,B0 are n-step matrices.

$$(\lambda I - A)^{-1} = \frac{adj(\lambda I - A)}{|\lambda I - A|} = \frac{adj(\lambda I - A)}{f(\lambda)}$$

right multiply (λI-A)

right multiply (λI-A)

$$f(\lambda)I = adj(\lambda I - A)(\lambda I - A)$$

$$\text{adj}(\lambda I - A) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0$$

$$\lambda^n I + a_{n-1}\lambda^{n-1} I + \cdots + a_1\lambda I + a_0 I$$

$$= \lambda^n B_{n-1} + \lambda^{n-1}(B_{n-2} - B_{n-1}A) + \lambda^{n-2}(B_{n-3} - B_{n-2}A) + \cdots + \lambda(B_0 - B_1 A) - B_0 A$$

Compare the coefficients:

$$\begin{aligned} B_{n-1} &= I \\ B_{n-2} - B_{n-1}A &= a_{n-1}I \\ &\vdots \\ B_1 - B_2 A &= a_2 I \\ B_0 - B_1 A &= a_1 I \\ -B_0 A &= a_0 I \end{aligned} \xrightarrow[A^n,\, A^{n-1},\, \cdots,\, A]{\text{right multiply}} \begin{aligned} B_{n-1}A^n &= A^n \\ B_{n-2}A^{n-1} - B_{n-1}A^n &= a_{n-1}A^{n-1} \\ &\vdots \\ B_1 A^2 - B_2 A^3 &= a_2 A^2 \\ B_0 A - B_1 A^2 &= a_1 A \\ -B_0 A &= a_0 I \end{aligned}$$

Add together the equations on the right :

$$f(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

- **Corollary 1:** Matrix A's k power (k>n) can be denoted by A's (n－1) step polynomial:

$$A^k = \sum_{m=0}^{n-1} a_m A^m, \qquad k \geq n$$

Proof: from **Cayley-Hamilton** Theorem,

$$A^n = -a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \cdots - a_1 A - a_0 I$$

thus

$$\begin{aligned}
A^{n+1} = A^n A &= -a_{n-1}A^n - a_{n-2}A^{n-1} - \cdots - a_1 A^2 - a_0 A \\
&= -a_{n-1}(-a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \cdots - a_1 A - a_0 I) \\
&\quad -a_{n-2}A^{n-1} - \cdots - a_1 A^2 - a_0 A \\
&= (a_{n-1}^2 - a_{n-2})A^{n-1} + (a_{n-1}a_{n-2} - a_{n-3})A^{n-2} + \cdots \\
&\quad +(a_{n-1}a_2 - a_1)A^2 + (a_{n-1}a_1 - a_0)A + a_{n-1}a_0 I
\end{aligned}$$

- **Corollary 2:** the matrix exponent e^{At} can be denoted by a (n-1) steps polynomial of matrix A.

$$e^{At} = \sum_{m=0}^{n-1} a_m(t) A^m$$

2. the State Controllability algebraic criteria of time-invariable system

Assume the final state is the origin of the state space, and the initial time t_0=0.

$$x(t) = e^{At}x(0) + \int_o^t e^{A(t-\tau)}Bu(\tau)d\tau$$

From the definition of controllability of the state,

we have: $$x(t_1) = 0 = e^{At_1}x(0) + \int_o^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

or $$x(0) = -\int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau$$

From corollary 2, we can rewrite the $e^{-A\tau}$ to the polynomial of matrix A:

$$e^{-A\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau) A^k$$

we have:

$$\int_0^{t_1} a_k(\tau) u(\tau) d\tau = \beta_k$$

$$x(0) = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} a_k(\tau) u(\tau) d\tau = -[B\ AB \cdots A^{n-1}B] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

Thus if system is controllable, the following equation about the initial state $x(0)$ has unique solution.

$$x(0) = -[B\ AB \cdots A^{n-1}B]\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

In which $\beta_k\,(k = 0,1,2,.....,n-1)$ is the scalar quantity set.

Which is equivalent to the situation that the nxn matrix Q:

$$Q = [B\ AB \cdots A^{n-1}B]$$

Satisfy rank(Q)=n

If rank(Q)=n is existed, solve the equation about β_k.

$$\beta_k = \int_0^{t_1} a_k(\tau) u(\tau) d\tau$$

Then the related control input u(t) is available, which could make the linear time-invariable system transfer from any initial state x(0) to the origin point (原点) in limitary period ($0 \sim t_1$).

ALL SUM UP, we have n.s. conditions of controllability for linear time-invariable system:

The algebraic criteria of State Controllability :

iff (if and only if) the n×n matrix Q is full rank:

$$rank Q = rank[B \;\; AB \cdots A^{n-1}B] = n$$

the system is controllable.

Extensive result:

For ***r*** dimension control vector ***u***, if the system state equation is:

$$\dot{x} = Ax + Bu$$

$$x(t) \in R^n, u(t) \in R^r, A \in R^{n \times n}, B \in R^{n \times r}$$

Controllable condition is for n × nr matrix Q:

$$Q = [B \; AB \cdots A^{n-1}B]$$

Controllability Matrix

rank(Q)=*n*, which is equivalent that ***n*** column vectors in Q are linearly independent.

Ex.9-22 Determine the controllability of following system:

System 1

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$\det Q=\det[B\quad AB]=\begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix}=0$$

Q is a singular matrix, so system is uncontrollable.

System 2

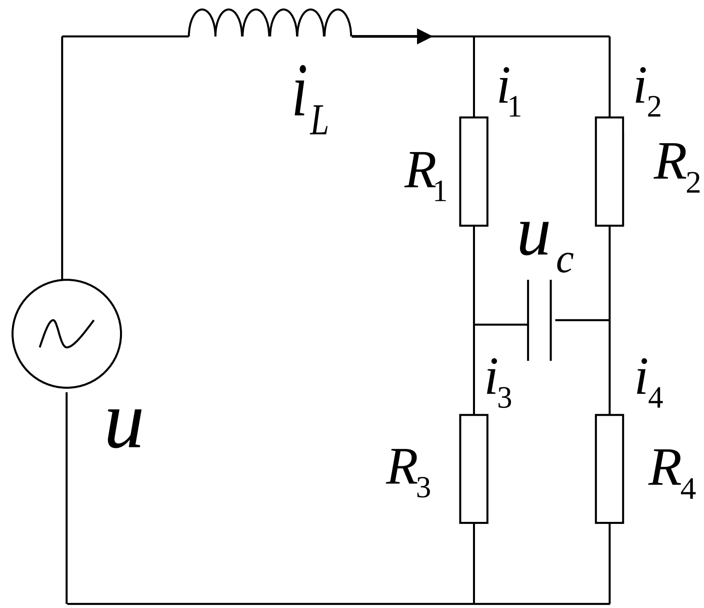
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}1 & 3 & 2\\ 0 & 2 & 0\\ 0 & 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}2 & 1\\ 1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$Q=[B\quad AB\quad A^2B]=\left[\begin{array}{cc:cc:cc}2 & 1 & 3 & 2 & 5 & 4\\ 1 & 1 & 2 & 2 & 4 & 4\\ -1 & -1 & -2 & -2 & -4 & -4\end{array}\right]$$

The 2nd and 3rd line are linearly dependent

$rank(Q)=2<3$ So the system is uncontrollable.

- Ex.9-20(a) a bridge circuit



- Solution:

  Dynamic differential equations:

$$i_L = i_1 + i_2 = i_3 + i_4$$

$$R_3 i_3 - R_4 i_4 = u_c$$

$$R_2 i_2 - R_1 i_1 = u_c$$

$$L\frac{di_L}{dt} + R_1 i_1 + R_3 i_3 = u \qquad u_c = \frac{1}{C}\int (i_1 - i_3)dt$$

select state variables: $x_1 = i_L \qquad x_2 = u_c$

eliminate the intermediate variables: i_1, i_2, i_3, i_4

the state equations:

$$\begin{cases} \dot{x}_1 = -\dfrac{1}{L}\left(\dfrac{R_1 R_2}{R_1 + R_2} + \dfrac{R_3 R_4}{R_3 + R_4}\right)x_1 + \dfrac{1}{L}\left(\dfrac{R_1}{R_1 + R_2} - \dfrac{R_3}{R_3 + R_4}\right)x_2 + \dfrac{1}{L}u \\ \dot{x}_2 = \dfrac{1}{C}\left(\dfrac{R_2}{R_1 + R_2} - \dfrac{R_4}{R_3 + R_4}\right)x_1 - \dfrac{1}{C}\left(\dfrac{1}{R_1 + R_2} + \dfrac{1}{R_3 + R_4}\right)x_2 \end{cases}$$

- Controllable matrix S_3

$$\mathrm{rank}S_3 = \mathrm{rank}[b \quad Ab] = rank\begin{bmatrix} \frac{1}{L} & -\frac{1}{L^2}\left(\frac{R_1R_2}{R_1+R_2}+\frac{R_3R_4}{R_3+R_4}\right) \\ 0 & -\frac{1}{LC}\left(\frac{R_4}{R_3+R_4}-\frac{R_2}{R_1+R_2}\right) \end{bmatrix}$$

- if $\frac{R_4}{R_3+R_4} \neq \frac{R_2}{R_1+R_2}$, $\mathrm{rank}S_3 = 2 = n$, system is controllable

- If $\frac{R_1}{R_1+R_2} = \frac{R_3}{R_3+R_4}$ and $\frac{R_2}{R_1+R_2} = \frac{R_4}{R_3+R_4}$, state equation:

$$\begin{cases} \dot{x}_1 = -\frac{1}{L}\left(\frac{R_1R_2}{R_1+R_2}+\frac{R_3R_4}{R_3+R_4}\right)x_1 + \frac{1}{L}u \\ \dot{x}_2 = -\frac{1}{C}\left(\frac{1}{R_1+R_2}+\frac{1}{R_3+R_4}\right)x_2 \end{cases}$$

$$\operatorname{rank} S_3 = \operatorname{rank}\begin{bmatrix} b & Ab \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \frac{1}{L} & -\frac{1}{L^2}\left(\frac{R_1R_2}{R_1+R_2}+\frac{R_3R_4}{R_3+R_4}\right) \\ 0 & 0 \end{bmatrix} = 1 < n$$

System is uncontrollable

•Ex.9-20(b) the controllability of the Circuit

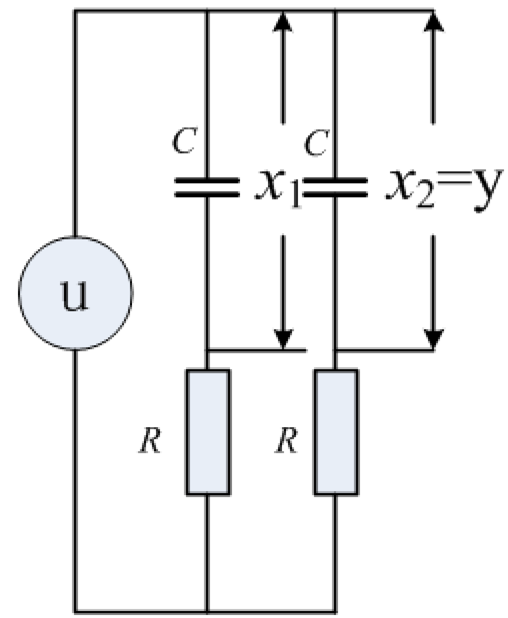
Solution:

For the circuit (*a*), the differential equation is:

$$x_1 + R_1C_1\dot{x}_1 = x_2 + R_2C_2\dot{x}_2 = u$$

in which,

$$x_1 = u_{c1} = \frac{1}{C_1}\int i_1 dt,\; x_2 = u_{c2} = \frac{1}{C_2}\int i_2 dt$$



circuit (*a*)

its state equations:

$$\begin{cases} \dot{x}_1 = -\dfrac{1}{R_1C_1}x_1 + \dfrac{1}{R_1C_1}u \\ \dot{x}_2 = -\dfrac{1}{R_2C_2}x_2 + \dfrac{1}{R_2C_2}u \end{cases}$$

$$\text{rank}\begin{bmatrix} b & Ab \end{bmatrix} = \text{rank}\begin{bmatrix} \dfrac{1}{R_1C_1} & -\dfrac{1}{R_1^2C_1^2} \\ \dfrac{1}{R_2C_2} & -\dfrac{1}{R_2^2C_2^2} \end{bmatrix}$$

If $R_1C_1 \neq R_2C_2$, system is controllable.

If $R_1=R_2$, $C_1=C_2$, thus $R_1C_1=R_2C_2$, rank[$\boldsymbol{b}$ $\boldsymbol{Ab}$]=1<n, system is uncontrollable.

For circuit (b), the differential equation is:

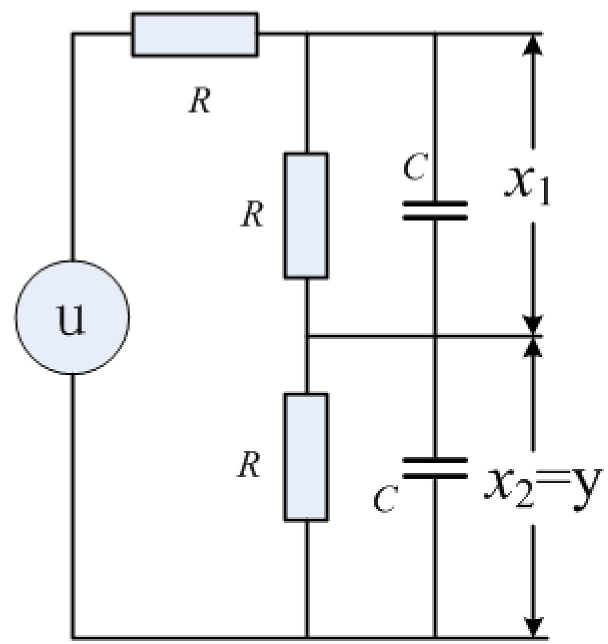
$$R_3(i_1+i_2)+x_1+x_2=u$$

$$x_1=R_1i_2, x_2=R_2i_4$$

$$i_1+i_2=i_3+i_4, \dot{x}_1=i_1/C_1, \dot{x}_2=i_3/C_2$$

in which:

$$x_1=u_{c1}=\frac{1}{C_1}\int i_1 dt, x_2=\frac{1}{C_2}\int i_3 dt$$



circuit (*b*)

Eliminate $i_1 \sim i_4$, state equations:

$$\begin{cases} \dot{x}_1 = -\left(\dfrac{1}{R_3C_1} + \dfrac{1}{R_1C_1}\right)x_1 - \dfrac{1}{R_3C_1}x_2 + \dfrac{1}{R_3C_1}u \\ \dot{x}_2 = -\dfrac{1}{R_3C_2}x_1 - \left(\dfrac{1}{R_3C_2} + \dfrac{1}{R_2C_2}\right)x_2 + \dfrac{1}{R_3C_2}u \end{cases}$$

$$\operatorname{rank}\begin{bmatrix} b & Ab \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \dfrac{1}{R_3C_1} & -\dfrac{1}{R_3C_1}\left(\dfrac{1}{R_3C_1} + \dfrac{1}{R_1C_1}\right) - \dfrac{1}{R_3^2C_1C_2} \\ \dfrac{1}{R_3C_2} & -\dfrac{1}{R_3C_2}\left(\dfrac{1}{R_3C_2} + \dfrac{1}{R_2C_2}\right) - \dfrac{1}{R_3^2C_1C_2} \end{bmatrix}$$

If $R_1 \neq R_2$, $C_1 \neq C_2$, rank[$\boldsymbol{b}$ $\boldsymbol{Ab}$]=2=n, system is controllable.

If $R_1 = R_2$, $C_1 = C_2$, rank[$\boldsymbol{b}$ $\boldsymbol{Ab}$]=1<n, system is uncontrollable.

3. Output Controllability

Consider the following state space for a linear time-invariable system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

$$\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u}$$

$$x \in R^n, u \in R^r, y \in R^m, A \in R^{n\times n}, B \in R^{n\times r}, C \in R^{m\times n}, D \in R^{m\times r}$$

If we can find an unrestricted control $u(t)$ that in limitary period $t_0 \le t \le t_1$, which could make any initial output $y(t_0)$ transfer to any final output $y(t_1)$, the system is called output controllable.

The n.s. condition of output controllable:

Iff the $m\times(n+1)r$ dimension output controllability matrix:

$$Q' = [\ CB \vdots\ CAB \vdots\ CA^2B\ \vdots \cdots\ CA^{n-1}B\ \vdots\ D\ ]$$

Satisfy rank(Q')=m, the system is output controllable.

9.5.2 Observability of Linear Continual System

In practical project, sometimes, the states $x(t)$ of the system cannot be measured totally or even are unmeasured. One possible way is to **reflect the states $x(t)$ by the output $y(t)$**, which is the **observability** of the system.

1. Definitions

Completely Observable

For initial time $t_0 \in T_t$, existing a limitary time $t_1 \in T_t$, $t_1 > t_0$, for all $t \in [t_0, t_1]$, the initial value of the states $x(t_0)$ can be determined uniquely by system output $y(t)$, therefore the system is called Completely Observable in $[t_0, t_1]$.

For the whole time field $[t_0, \infty)$, if the system is observable, the system is observable at $t > t_0$.

If every state $x(t)$ can be observed by $y(t)$ in the period $t_0 \le t \le t_1$, the system is Completely Observable.

Incompletely Observable

For initial time $t_0 \in T_t$, existing a limitary time $t_1 \in T_t$, $t_1 > t_0$, for all $t \in [t_0, t_1]$, if the initial value of all states $x_i(t_0)$, i=1,2,…,n, cannot be determined by the system output $y(t)$ totally, in other word, at least one state cannot be determined by y(t), therefore the system is called Incompletely Observable in $[t_0, t_1]$, or Unobservable.

For the following state space:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(t) = e^{At}x(0) + \int_o^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x(0) + C\int_o^t e^{A(t-\tau)}Bu(\tau)d\tau + Du$$

The matrix A, B, C, D and input $u(t)$ are known, therefore the integral part of above equation on the right side is known, which can be removed from measured value $y(t)$.

To discuss the n.s. condition of the system's observability, the zero-input system will be considered.

2. the State Observability Algebraic Criteria of Linear time-invariable system

Consider the zero-input state space equation:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \qquad x \in R^n, y \in R^m, A \in R^{n\times n}, C \in R^{m\times n}$$

Output vector is: $y(t) = Ce^{At}x(0)$

Rewrite the e^{At} into the polynomial of matrix A:

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t)A^k \qquad y(t) = \sum_{k=0}^{n-1} \alpha_k(t)CA^k x(0)$$

If system is observable, in the time period $t_0 \le t \le t_1$, give a output $y(t)$, then the state $x(0)$ can be determined by the above equation uniquely.

$$\boldsymbol{y}(t)=\alpha_0(t)C\boldsymbol{x}(0)+\alpha_1(t)CA\boldsymbol{x}(0)+\cdots+\alpha_{n-1}(t)CA^{n-1}\boldsymbol{x}(0)$$

$$=[a_0(t)I_m \quad a_1(t)I_m \quad \cdots \quad a_{n-1}(t)I_m]\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}x(0)$$

Column linear independence

n.s. condition of system Observability

For the linear time-invariable system:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

If and only if the rank of the nm×n dimension observable matrix R is n, system is observable.

namely

$$R = \begin{bmatrix} C \\ CA \\ \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad rankR = n$$

or

$$R^T = [\, C^T \;\; A^T C^T \;\; \cdots (A^T)^{n-1} C^T \,] \qquad rankR^T = n$$

Ex.9-23 try to confirm the controllability and observability of following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

The state controllable matrix: $Q = [\ B \ \ AB\ ] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$rankQ = 2 = n$ Thus, system is **state controllable**.

The output controllable matrix: $Q' = [\ CB \ \ CAB\ ] = \begin{bmatrix} 0 & 1 \end{bmatrix}$

$rankQ' = 1 = m$ Thus, system is **output controllable**.

The observable matrix: $R^T = [\ C^T \ \ A^T C^T\ ] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$rankR^T = 2 = n$ Thus, system is **observable.**

➢ Transfer function description of the State Controllability and Observability Conditions

State Controllability and Observability conditions can be described by transfer function, as well.

The N.S. condition of state controllable and observable is No Cancellation Appeared in the Transfer Function.

If there is cancellation in the transfer function, the system is uncontrollable or unobservable, or even uncontrollable and unobservable simultaneously.

$$G(s) = C(sI - A)^{-1}B = \frac{(s - z_1)(s - z_2)\cdots\cancel{(s - z_i)}\cdots}{(s - p_1)(s - p_2)\cdots\cancel{(s - p_j)}\cdots}$$

$$z_i = p_j$$

Ex.9-24 **Consider following transfer function:**

$$\frac{Y(s)}{U(s)} = \frac{s+2.5}{(s+2.5)(s-1)}$$

The reducible factors (s+2.5) are included in the numerator and denominator of the transfer function, therefore the system state is uncontrollable or unobservable.

The same result can be derived if we transform the transfer function to the state function:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \Longrightarrow \quad \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2.5 & 1 \\ 2.5 & 1 \end{bmatrix}$$

$$y = [2.5 \quad 1] x$$

Rank of the observable matrix $[C \mid \; CA]^{T}$ is 1, the system is state unobservable.

- if we transform the transfer function to another state function:

$$\dot{x} = \begin{bmatrix} 0 & 2.5 \\ 1 & -1.5 \end{bmatrix} x + \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} u \quad \Longrightarrow \quad [B \vdots AB] = \begin{bmatrix} 2.5 & 2.5 \\ 1 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Rank of the controllable matrix $[B \mid AB]^{\mathrm{T}}$ is 1, the system is state uncontrollable.

- We can also found the state space: $\begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{cases}$

- Thus: if there is Cancellation Appeared in the Transfer Function, the state space representing will appear uncontrollable or unobservable if we select different state variables to establish the state space.

Ex.9-25 **Analyze the observability of the following system**

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

In which: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$

Solution: the observable matrix:

$$R^T = [\, C^T \quad A^T C^T \quad (A^T)^2 C^T \,] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

The system is unobservable.

Attention that: there is reducible factor(可约因子) in the transfer function of the system.

The transfer function between $X_1(s)$ and $U(s)$ is:

$$\frac{X_1(s)}{U(s)} = \frac{1}{(s+1)(s+2)(s+3)}$$

The one between $Y(s)$ and $X_1(s)$ is:

$$\frac{Y(s)}{X_1(s)} = (s+1)(s+4) \qquad \text{then} \qquad \frac{Y(s)}{U(s)} = \frac{(s+1)(s+4)}{(s+1)(s+2)(s+3)}$$

The factor $(s+1)$ in the numerator and denominator polynomial can be reduced. Therefore, the system is unobservable, or some non-zero initial state x(0) cannot be measured by y(t).

If and only if, system is controllable and observable, its transfer function has no reducible factor. That is to say, the reducible transfer function doesn't have complete information to describe the dynamic system.

9.5.3 Duality Principle (对偶原理) -- R.E.Kalman

Discuss the relationship between Controllability and Observability.

Consider the following system state space S_1 and S_2:

$$S_1:\ \begin{cases}\dot{x} = Ax + Bu \\ y = Cx\end{cases} \qquad x \in R^n, u \in R^r, y \in R^m, A \in R^{n\times n}, B \in R^{n\times r}, C \in R^{m\times n}$$

$$S_2:\ \begin{cases}\dot{z} = A^T z + C^T v \\ n = B^T z\end{cases} \qquad z \in R^n, v \in R^m, n \in R^r, A^T \in R^{n\times n}, C^T \in R^{n\times m}, B^T \in R^{r\times n}$$

System S_1 and S_2 are called **Dually System(对偶系统)**.

Duality Principle:
if and only if the system S_1 is state observable / state controllable, system S_2 will be state controllable / state observable.

➢ Analysis of Duality Principle:

For system S_1:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

1. The n.s. condition of state controllable is the rank of n×nr dimension controllable matrix:

$$rank[\, B \quad AB \quad \cdots \quad A^{n-1}B \,] = n$$

2. The n.s. condition of state observable is the rank of n×nm dimension observable matrix:

$$rank[\, C^T \quad A^T C^T \quad \cdots \quad (A^T)^{n-1} C^T \,] = n$$

For system S_1:

$$\dot{z} = A^T z + C^T v$$
$$n = B^T z$$

1. The n.s. condition of state controllable is the rank of n×nm dimension controllable matrix:

$$rank[\, C^T \quad A^T C^T \quad \cdots \quad (A^T)^{n-1} C^T \,] = n$$

2. The n.s. condition of state observable is the rank of n×nr dimension observable matrix:

$$rank[\, B \quad AB \quad \cdots \quad A^{n-1} B \,] = n$$

Based on the Duality Principle, the observability of a certain system can be determined by the state controllability of its dually system.

In brief, the duality is:

$$A \Rightarrow A^T, \quad B \Rightarrow C^T, \quad C \Rightarrow B^T$$

9.5.4 the criterion of Controllability and Observability for a certain linear continual system

➢ Controllability criterion

☞Controllable Canonical form：

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix},\quad B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

if the state equation is: $\dot{x} = Ax + Bu$

$$x(t) \in R^n, u(t) \in R^r, A \in R^{n\times n}, B \in R^{n\times r}$$

Criterion 1: the n.s. condition of linear time-invariable continual system state complete controllable is that the controllable matrix Q_c should be full rank.

$$Q_C = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] \qquad rankQ_c = n$$

Ex.9-26 **The system:** $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

try to analyze the controllability of the system.

Solution: $$Q_c=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$$

$$rankQ_c=2=n$$

System is controllable

Ex.9-27 **Analyze the controllability of the following 3-order 2-input system:**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\mathbf{Q_c}=\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A^2B}\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1}\\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{1}\end{bmatrix}$$

System is uncontrollable

Criterion 2: if linear time-invariable system has unequal eigenvalues, the n.s. condition of system controllable is: the diagonal canonical equation from nonsingular transfer satisfies that there is no row of zero in the input matrix $\bar{\mathrm{B}}$

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} + \bar{\mathrm{B}} u$$

Ex.9-28 Determine the controllability of following diagonal canonical system.

1) $$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} u$$ √

2) $$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix} u$$ ×

3) $$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 4 & 0 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$ √

4) $$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$ ×

Criterion 3: For the Jordan Canonical form $\dot{\bar{x}} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix} \bar{x} + \bar{B}u$

the elements in the relative **rows** in $\bar{\mathbf{B}}$ related with the **last row** in each Jordan block $J_i(i=1,2,\ldots,k)$ are **not completely zero**.
(If two of Jordan Blocks have the same eigenvalue, the result is not exists.)

Ex.9-29 Analysis the controllability of following Jordan canonical system.

1) $$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$$ √

2) $$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$ ×

➢ Observability Criterion

☞Observable canonical form:

$$A=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \quad C=\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

Criterion 1: the n.s. condition of linear time-invariable continual system state complete observable is that the observable matrix Q_o should be **full rank**.

$$Q_0=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\left(=Q_c^T\big|_{B\to C}\right)$$

Namely, $rankQ_o=n$

Ex.9-30 from the matrix A and C, analyze system's observability.

$$A=\begin{bmatrix}-4 & 5\\ 1 & 0\end{bmatrix} \qquad C=\begin{bmatrix}1 & -1\end{bmatrix}$$

Solution:

$$CA=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}-4 & 5\\ 1 & 0\end{bmatrix}=\begin{bmatrix}-5 & 5\end{bmatrix}$$

$$Q_o=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}1 & -1\\ -5 & 5\end{bmatrix}$$

$\boxed{\times}$

Ex.9-31 try to determine the observability of the following system.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}2 & -1\\ 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}-1\\ 1\end{bmatrix}u \qquad \begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Solution:

$$Q_o=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}1 & 0\\ -1 & 0\\ 2 & -1\\ -2 & 1\end{bmatrix}$$

$\boxed{\surd}$

Criterion 2: if the linear time-invariable continual system has unequal eigenvalue, the n.s. condition of state observability is that the diagonal canonical form of the system from nonsingular transfer satisfy that there is no column of zero in the output matrix $\bar{C}$

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \bar{x} \\ y = \overline{cx} \end{cases}$$

Ex.9-32 **determine the observability of the system.**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

√

Criterion 3: In the Jordan Canonical Form:

$$\begin{cases} \dot{\bar{x}} = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix} \bar{x} \\ y = \overline{cx} \end{cases}$$

the elements in the relative **column** in $\bar{C}$ related with the **first row** in each Jordan block J_i(i=1,2,…,k) are **not completely zero**.

(If two of Jordan Blocks have the same eigenvalue, the result is not exists.)

Ex.9-33 **try to determine the observability of following system.**

1) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}2 & 1 & & 0\\ 0 & 2 & & \\ & & 3 & 1\\ 0 & & 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \qquad \begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$ ×

2) $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}u \qquad y=\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$ √

9.5.5 Controllability and Observability of Linear Time-invariant Discrete System

➢ Critierion of Controllability

State equations of discrete system: $\begin{cases} \boldsymbol{x}(k+1) = \mathrm{A}\boldsymbol{x}(k) + \mathrm{B}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k) \end{cases}$

$x \in R^{n\times 1}, \quad \mathrm{A} \in R^{n\times n}$ is nonsingular matrix

$\mathrm{B} \in R^{n\times r}, \quad C \in R^{m\times n}, \quad D \in R^{m\times r}, \quad y \in R^{m\times 1}, \quad u \in R^{r\times 1}$

If a sequence of the unrestricted control vector $u(0)$, $u(1)$,…,$u(n$-1) can be obtained to make the system transform from $x(0)$ to x(n)=0, the system is controllable.

The solution of the state equation is $x(k) = \mathrm{A}^{k} x(0) + \sum_{i=0}^{k-1} \mathrm{A}^{k-1-i} \mathrm{B} u(i)$

Let $k = n, \quad x(n) = 0$ and left multiply A^{-k}:

we have: $x(0) = -\sum_{i=0}^{n-1} A^{-1-i} B u(i) = -[A^{-1}Bu(0) + A^{-2}Bu(1) + \cdots + A^{-n}Bu(n-1)]$

$$= -[A^{-1}B \quad A^{-2}B \quad \cdots \quad A^{-n}B]\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix}$$

From the solution existence theorem, the n.s. condition of MIMO linear discrete system controllability is

$$rank[A^{-1}B \quad A^{-2}B \quad \cdots \quad A^{-n}B] = n$$

or $\quad rank[A^{n-1}B \cdots \quad AB \quad B] = n$

assume $\quad rank\, Q_d = rank\,[B \quad AB \quad \cdots \quad A^{n-1}B\,]$

then $\quad rank\, Q_d = n$

The criterion of output complete controllability

$$rank\, Q_d^o = rank\,[CB \quad CAB \quad \cdots \quad CA^{n-1}B \,\vdots\, D] = n$$

Ex.9-34 state equations of the time-invariable discrete system:

$$x(k+1)=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & 0 & 3\end{bmatrix}x(k)+\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}u(k)$$

Determine the controllability of the system.

Solution: The rank of the controllable matrix:

$$rank[B \quad AB \quad A^2B]=rank\begin{bmatrix}1 & 0 & 1 & 2 & 0 & 4\\ 0 & 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 & 4 & 2\end{bmatrix}=3$$

The system is controllable。

The controllable matrix of multi-input time-invariable discrete system is $n\times nr$ dimension. And the controllable condition is that the rank of the matrix is n. Therefore, the calculation could be enough when rank is n.

➢ Criterion of Observability

Consider the state equation of the system

$$x(k+1) = \mathrm{A}x(k)$$

$$y(k) = Cx(k)$$

If the output $y(k)$ in the finite sample period can determine the initial state vector $x(0)$, the system is observable.

If the system is observable, we have:

$$x(k) = \mathrm{A}^k x(0)$$

$$y(k) = C\mathrm{A}^k x(0)$$

Because:

$$\begin{aligned} y(0) &= Cx(0) \\ y(1) &= C\mathrm{A}x(0) \\ &\vdots \\ y(n-1) &= C\mathrm{A}^{n-1}x(0) \end{aligned}$$

Notice that $y(k)$ is an m dimension vector, thus the above n matrix equations will have nm algebraic equations, which contains $x_1(0), x_2(0), \ldots, x_n(0)$.

To obtain unique solution $x_1(0), x_2(0), \ldots, x_n(0)$ from the nm equations, the rank of the $nm \times n$ coefficient matrix should be n.

$$rank\begin{bmatrix} C \\ C\mathrm{A} \\ \vdots \\ C\mathrm{A}^{n-1} \end{bmatrix} = n$$

The ranks for the matrix and its transposition are the same, therefore, the n.s. condition of system observable is:

$$rank\ R_d^T = rank[\ C^T \quad \mathrm{A}^T C^T \quad \cdots \quad (\mathrm{A}^T)^{n-1} C^T\ ] = n$$

Ex.9-35 determine the observability of the following systems

S_1:

$$x(k+1)=\begin{bmatrix}2 & 0 & 3\\ -1 & -2 & 0\\ 0 & 1 & 2\end{bmatrix}x(k)$$

$$y(k)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}x(k)$$

Obtain the observable matrix

$$rank\begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix}=rank\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 3\\ -1 & -2 & 0\\ 4 & 3 & 12\\ 0 & 4 & -3\end{bmatrix}=3$$

or

$$rank\begin{bmatrix}C^T & A^TC^T & (A^T)^2C^T\end{bmatrix}=rank\begin{bmatrix}1 & 0 & 2 & -1 & 4 & 0\\ 0 & 1 & 0 & -2 & 3 & 4\\ 0 & 0 & 3 & 0 & 12 & 3\end{bmatrix}=3$$

System 1 is observable.

S_2

$$x(k+1)=\begin{bmatrix}1 & 0 & -1\\ 0 & -2 & 1\\ 3 & 0 & 2\end{bmatrix}x(k)+\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}x(k)$$

Obtain the observable matrix

$$rank\begin{bmatrix}C\\ CA\\ CA^2\end{bmatrix}=rank\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 3 & 0 & 2\\ 1 & 0 & -1\\ 9 & 0 & 1\\ -2 & 0 & -3\end{bmatrix}<3$$

System 2 is unobservable

9.5.6 Structure analysis of continual time linear time-invariable system

➢ Controllable Decomposition

The dynamic equation of an uncontrollable system is:

$$\dot{x} = Ax + Bu \quad y = Cx$$

The rank of the controllable matrix r<n, select r linear independent columns and any n-r columns, to construct the nonsingular transformation ***T**^{-1}* and satisfy:

$$x = T^{-1}\bar{x} \qquad \bar{x} = \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$y = \bar{C}\bar{x}$$

In which

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \quad \bar{B} = TB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \quad \bar{C} = CT^{-1} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$$

after nonsingular transformation, dynamic equations of the system should be:

$$\begin{bmatrix} \dot{x}_C \\ \dot{x}_{\bar{C}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix}$$

Thus, the dynamic equation of controllable subsystem is:

$$\dot{x}_C = \bar{A}_{11} x_C + \bar{A}_{12} x_{\bar{C}} + \bar{B}_1 u$$

$$y_1 = \bar{C}_1 x_C$$

And the dynamic equation of uncontrollable subsystem is:

$$\dot{x}_{\bar{C}} = \bar{A}_{22} x_{\bar{C}}$$

$$y_2 = \bar{C}_2 x_{\bar{C}}$$

Ex.9-36 For the system:

$$A=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 3\end{bmatrix},\quad b=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad c=\begin{bmatrix}1 & -1 & 1\end{bmatrix}$$

Try to decomposed the system by controllability

Solution:

$$rank\begin{bmatrix}b & Ab & A^2b\end{bmatrix}=rank\begin{bmatrix}0 & -1 & -4\\ 0 & 0 & 0\\ 1 & 3 & 8\end{bmatrix}=2<3$$

System is incompletely controllable,

$$T^{-1}=\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & 1\\ 1 & 3 & 0\end{bmatrix},\quad T=\begin{bmatrix}3 & 0 & 1\\ -1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}$$

$$\bar{A} = TAT^{-1} = \left[\begin{array}{cc|c} 0 & -4 & 2 \\ 1 & 4 & -2 \\ \hline 0 & 0 & 1 \end{array}\right] \quad \bar{b} = Tb = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] \quad \bar{c} = cT^{-1} = \left[\begin{array}{cc|c} 1 & 2 & -1 \end{array}\right]$$

The dynamic equation of the controllable subsystem is:

$$\dot{x}_C = \begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix} x_C + \begin{bmatrix} 2 \\ -2 \end{bmatrix} x_{\bar{C}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & 2 \end{bmatrix} x_C$$

And the dynamic equation of the uncontrollable subsystem is:

$$\dot{x}_{\bar{C}} = x_{\bar{C}}$$

$$y_2 = -x_{\bar{C}}$$

➢ Observable Decomposition

The dynamic equation of the unobservable system is

$$\dot{x} = Ax + Bu \qquad y = Cx$$

The rank of the observable matrix is $l<n$, select l linear independent rows and any n-l rows to construct a nonsingular transformation ***T***

$$x = T^{-1}\bar{x} \qquad \bar{x} = \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad \bar{B} = TB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \quad \bar{C} = CT^{-1} = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix}$$

The dynamic of observable subsystem is:

$$\dot{x}_o = \bar{A}_{11} x_o + \bar{B}_1 u$$

$$y_1 = \bar{C}_1 x_o$$

The dynamic of unobservable subsystem is:

$$\dot{x}_{\bar{o}} = \bar{A}_{21} x_o + \bar{A}_{22} x_{\bar{o}} + \bar{B}_2 u$$

$$y_2 = 0$$

- Ex9-38: try to find the observable subsystem of:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x \end{cases}$$

Solution:

$$\operatorname{rank} Q_o = \operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix} = 2 < 3$$

System is incomplete observable and the rank of observable part is 2. Choose transform matrix:

$$T = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- New system after linear transforming:

$$\bar{A} = TAT^{-1} = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ -1 & -2 & 0 \\ \hdashline 1 & 0 & -1 \end{array}\right] \qquad \bar{B} = TB = \left[\begin{array}{c} 1 \\ -1 \\ \hdashline 0 \end{array}\right]$$

$$\bar{C} = CT^{-1} = \left[\begin{array}{cc:c} 1 & 0 & 0 \end{array}\right]$$

the state system of observable subsystem is:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

System Canonical decomposition by Controllability and Observability

Assume system (A, B, C) is uncontrollble and unobservable, first of all, decompose system by controllability

$$x = T_c^{-1} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix}$$

Then decompose the controllable subsystem and uncongtrollable subsystem by observability

$$x_C = T_{O1}^{-1} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \end{bmatrix} \qquad x_{\bar{C}} = T_{O2}^{-1} \begin{bmatrix} x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix}$$

At last we have:

$$x = T_c^{-1} \begin{bmatrix} x_C \\ x_{\bar{C}} \end{bmatrix} = \begin{bmatrix} T_c^{-1}T_{O1}^{-1} & & & \\ & T_c^{-1}T_{O1}^{-1} & & \\ & & T_c^{-1}T_{O2}^{-1} & \\ & & & T_c^{-1}T_{O2}^{-1} \end{bmatrix} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \\ x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix} = T^{-1} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \\ x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix}$$

From T^{-1} transformation, system dynamic is:

$$\begin{bmatrix} \dot{x}_{CO} \\ \dot{x}_{C\bar{O}} \\ \dot{x}_{\bar{C}O} \\ \dot{x}_{\bar{C}\bar{O}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{21} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{33} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{44} \end{bmatrix} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \\ x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_1 & 0 & \bar{C}_3 & 0 \end{bmatrix} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \\ x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix}$$

the dynamic equation of controllable and observable subsystem is:

$$\dot{x}_{CO} = \bar{A}_{11} x_{CO} + \bar{A}_{13} x_{\bar{C}O} + \bar{B}_1 u$$

$$y_1 = \bar{C}_1 x_{CO}$$

The dynamic of controllable but unobservable subsystem is:

$$\dot{x}_{C\bar{O}} = \bar{A}_{21}x_{CO} + \bar{A}_{22}x_{C\bar{O}} + \bar{A}_{23}x_{\bar{C}O} + \bar{A}_{24}x_{\bar{C}\bar{O}} + \bar{B}_2 u$$

$$y_2 = 0$$

The dynamic of uncontrollable but observable subsystem is:

$$\dot{x}_{\bar{C}O} = \bar{A}_{33}x_{\bar{C}O}$$

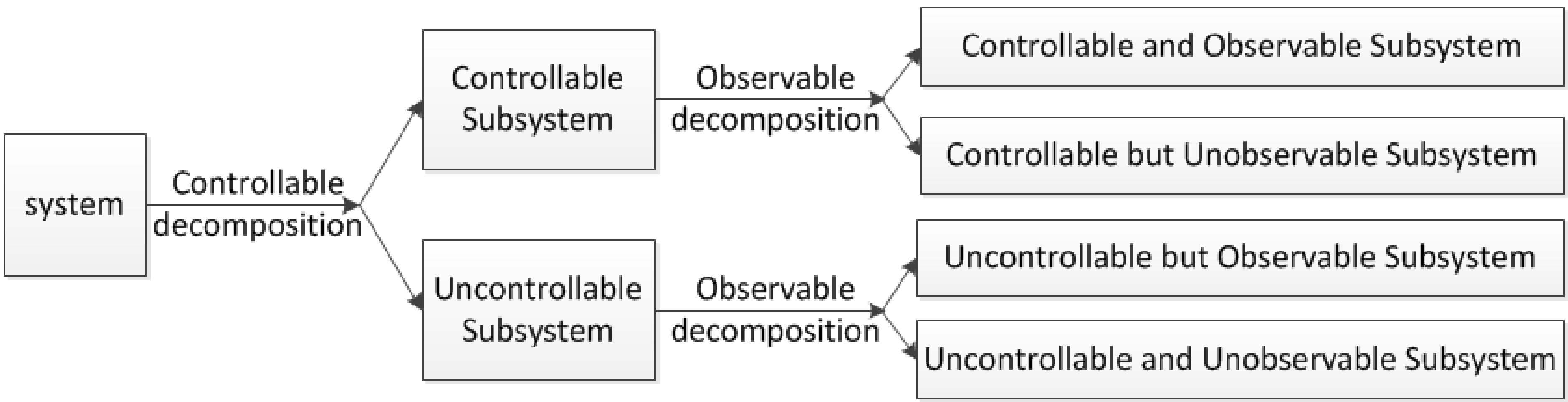
$$y_3 = \bar{C}_3 x_{\bar{C}O}$$

The dynamic of uncontrollable and unobservable sub system is:

$$\dot{x}_{\bar{C}\bar{O}} = \bar{A}_{43}x_{\bar{C}O} + \bar{A}_{44}x_{\bar{C}\bar{O}}$$

$$y_4 = 0$$

- However, it is complicated to calculate transform matrix T_{co}^{-1} directly.
- The normal method is to decompose the system gradually to controllable, observable and uncontrollable & unobservable subsystem.
- The general decomposition process is:



- Ex9-38: system dynamics:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x \end{cases}$$

which is incomplete controllable and incomplete observable, try to decompose the system by controllability and observability.

Solution:

(1) Controllability decomposition:

Form the decomposition matrix:

$$T_c^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- Decompose the system by controllability:

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} = \left[\begin{array}{cc:c} 0 & -1 & -1 \\ 1 & -2 & -2 \\ \hdashline 0 & 0 & -1 \end{array}\right] \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \left[\begin{array}{c} 1 \\ 0 \\ \hdashline 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc:c} 1 & -1 & -2 \end{array}\right] \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix}$$

- The uncontrollable subsystem is one dimension system and observable. Thus, it is the uncontrollable but observable subsystem we need.
- (2) Decompose the controllable subsystem by observability.

$$\dot{x}_c = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x_c + \begin{bmatrix} -1 \\ -2 \end{bmatrix} x_{\bar{c}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & -1 \end{bmatrix} x_c$$

- Form the observable decomposition matrix:

$$T_{co}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad T_{co} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Then, the decompose the controllable subsystem by observability:

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} x_{\bar{c}o} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$$

- (3) Combine above transforming, the controllability and observability decomposition equation is:

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \end{bmatrix}$$

- The transform matrix:

$$T = T_c \begin{bmatrix} T_{co} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$