Chapter 8 Nonlinear Systems Theory

8.1 Overview of Nonlinear Systems (1h)

8.2 Typical Nonlinear Characteristics and Mathematical Description (1h)

8.3 Describing Function Approach (4h)

8.4 Phase Plane Analysis (6h)

Nonlinear Models and Nonlinear Phenomena

Examples:

Pendulum Equation:

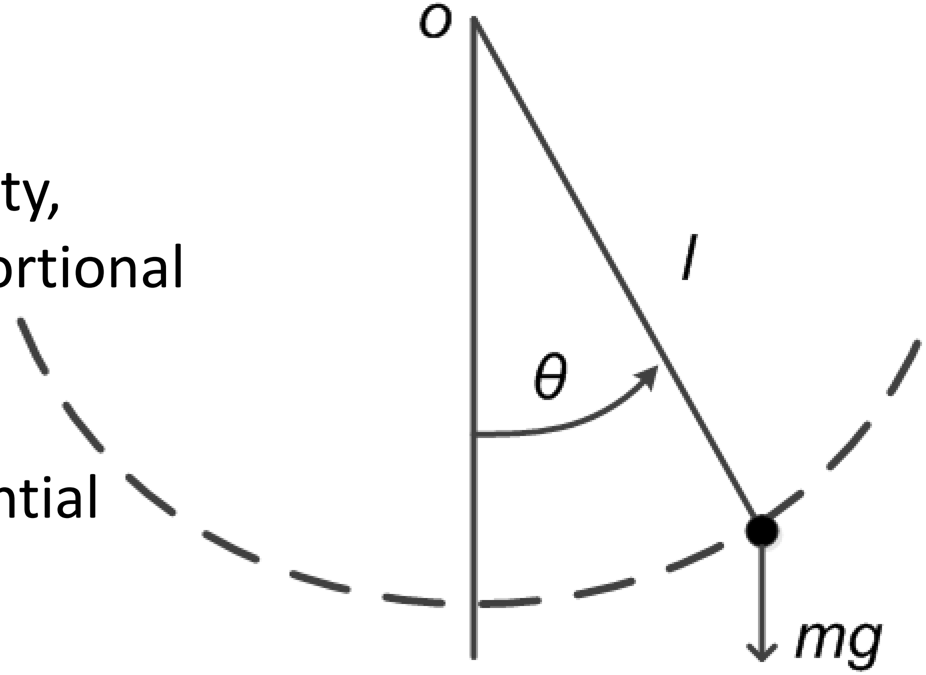
l denotes the length of the rigid rod with zero mass,

m denotes the mass of the bot,

g is the acceleration due to gravity,

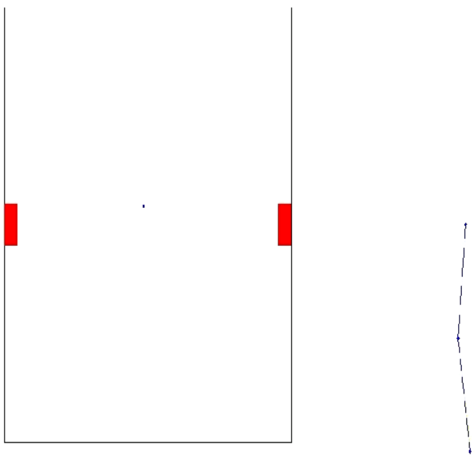
k is a coefficient of friction proportional to the speed of the bob.

From **Newton's second law**, the equation of motion in the tangential direction is:



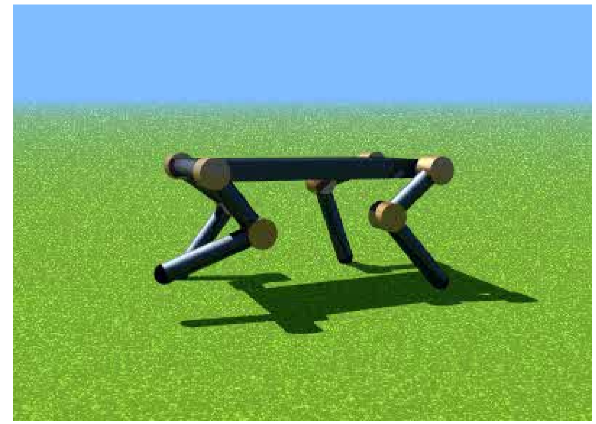
$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

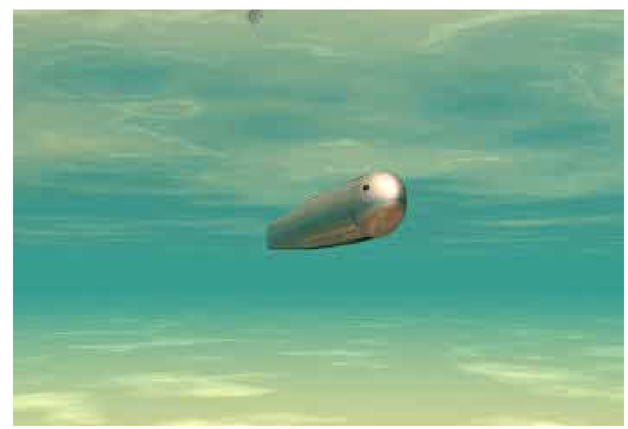
Interesting Examples of Nonlinearity



Other pendulums: Acrobot Robots

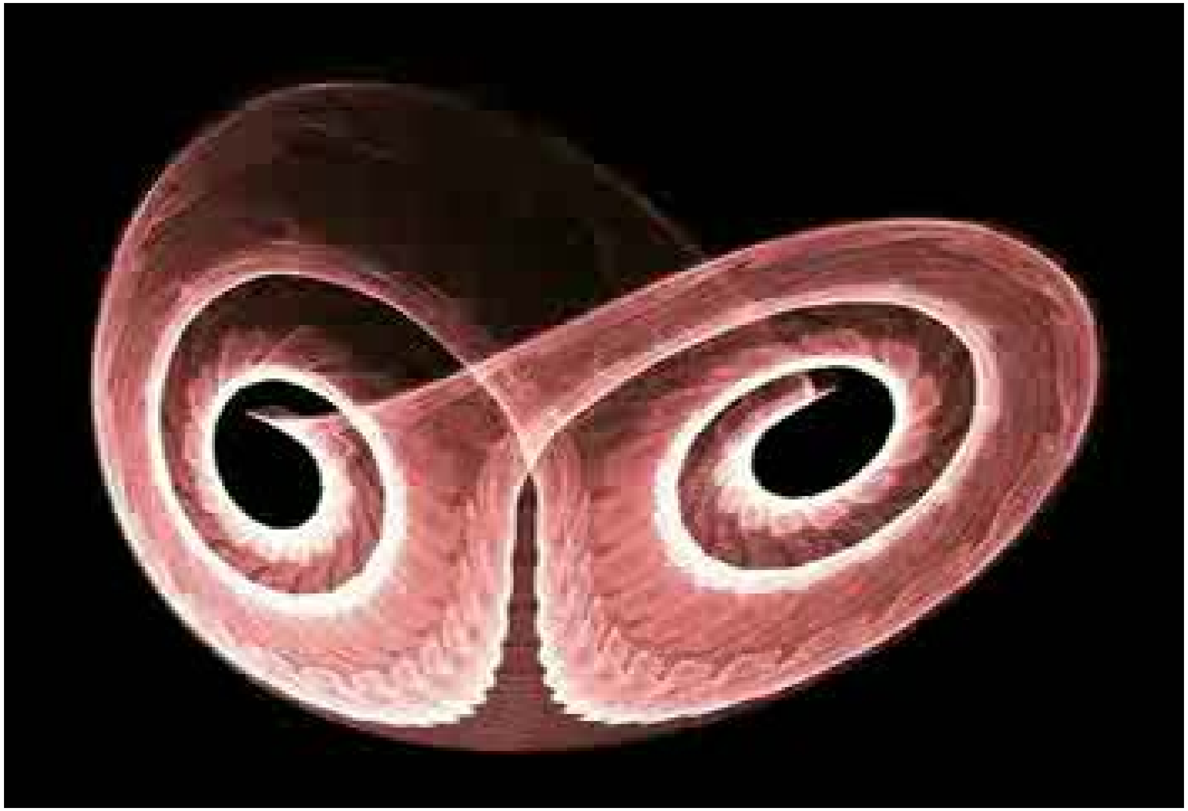
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = F$$
$$q = (x, \theta_1, \theta_2)$$





Biomimetic robots

Interesting Examples of Nonlinearity



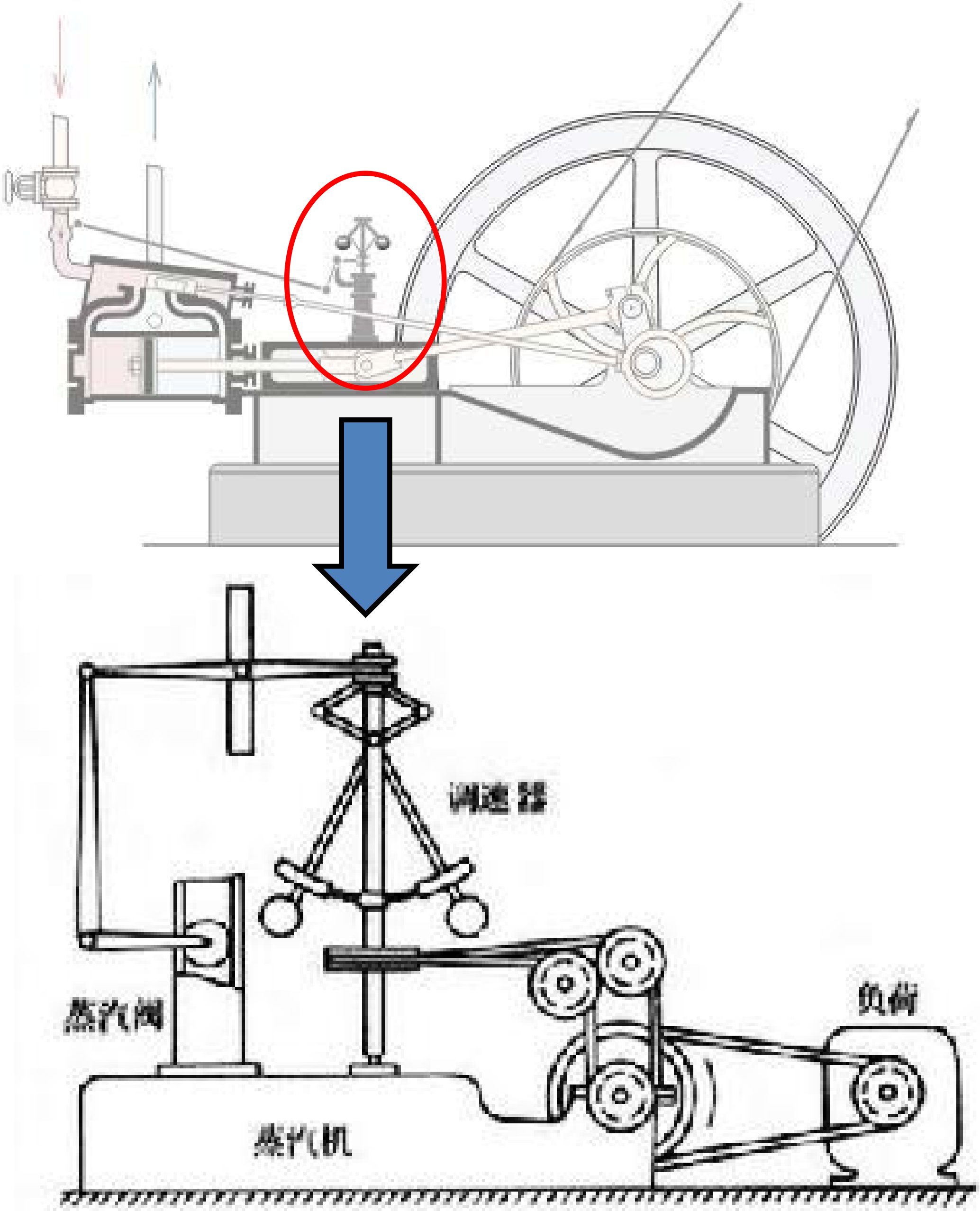
$$\frac{dx}{dt} = -c(x - y)$$

$$\frac{dy}{dt} = ax - y - xz$$

$$\frac{dz}{dt} = b(xy - z)$$

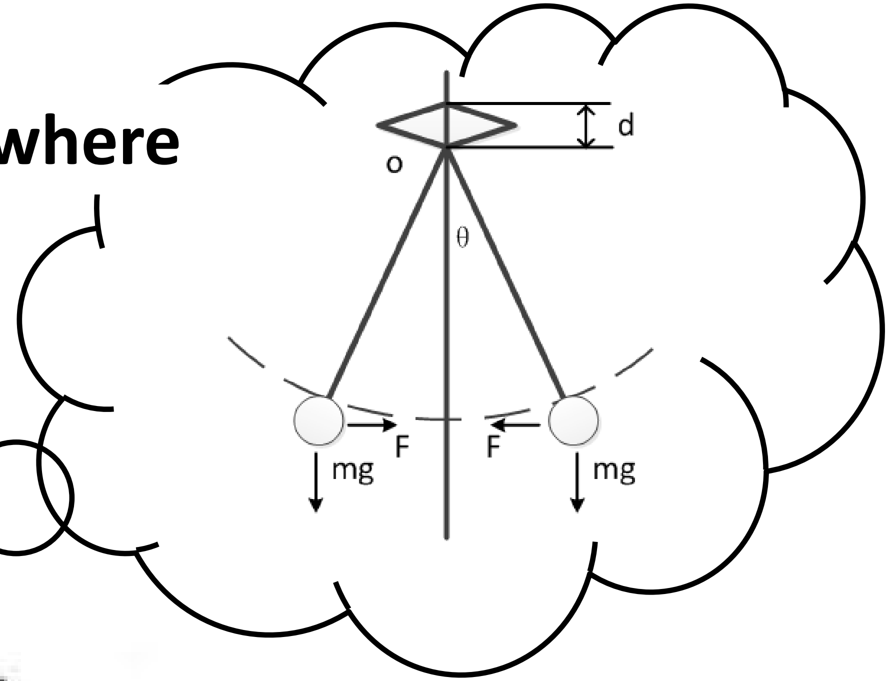
Lorenz Chaotic Attractor

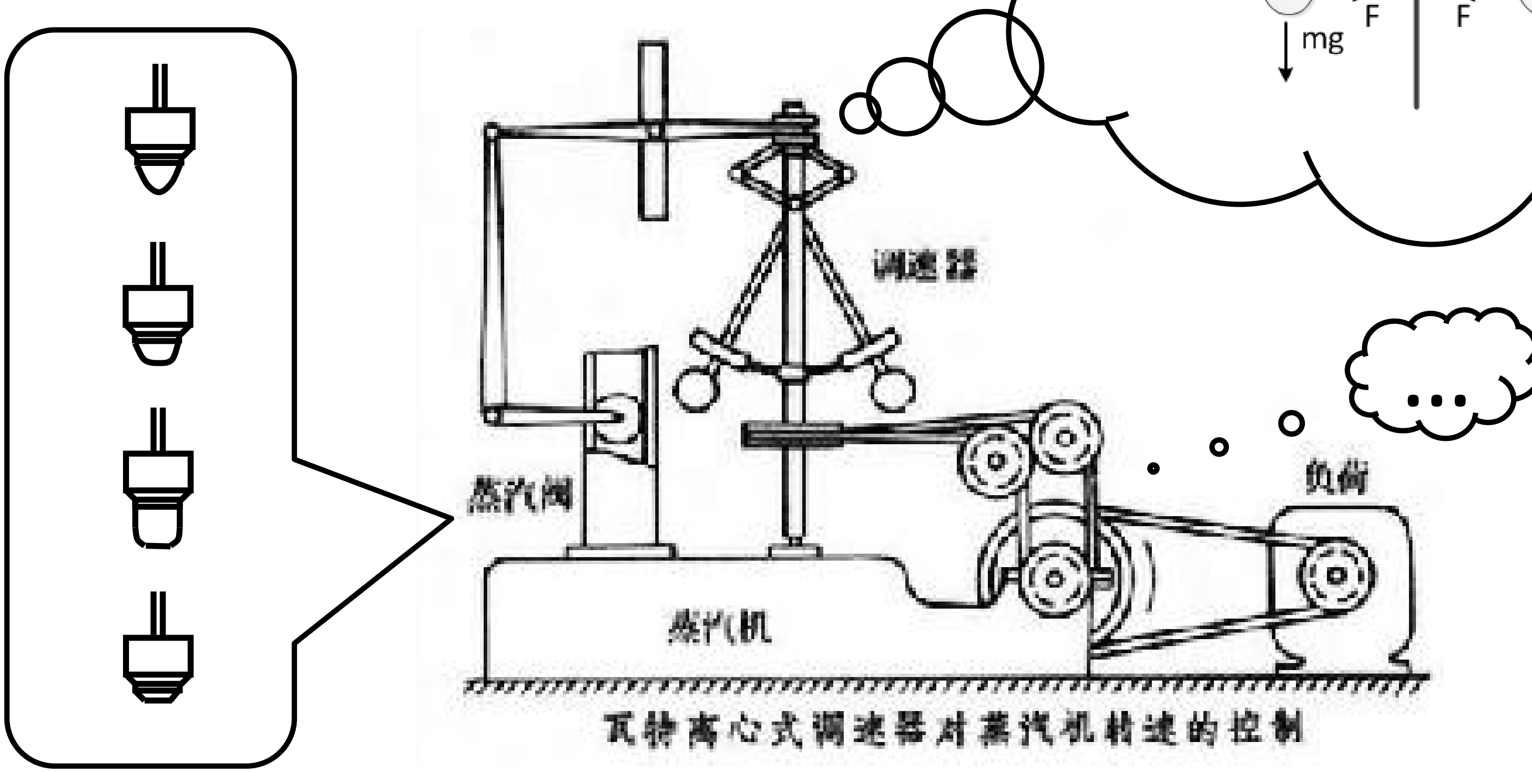
Mr. J. Watt



瓦特离心式调速器对蒸汽机转速的控制

Nonlinear can be found everywhere





§ 8.1 Overview of Nonlinear Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y + \varepsilon \cdot f(y, y', \cdots y^{(n)}) = x$$

If ε→0, it is a linear System;

If ε cannot be ignored, it is a Nonlinear System.

Such as： $\ddot{y} + \dot{y}y + y = \sin \omega t$

$$(\ddot{y})^2 + 3\dot{y} + y = e^t$$

$$\ddot{y} + 3\dot{y} + y^2 = x$$

Thinking: How to distinguish nonlinearity？

1. Significance of Studying Non-linear Systems

1) There are **no** systems without nonlinearity.

- ***Dead-zone*** characteristics of *measurement element*;
- ***Saturation*** characteristics of *amplification element*;
- ***Dead-zone*** and ***saturation*** characteristics of *actuator*;
- ***Gap*** Characteristics of actuating unit and so on…

2) The ***inherent nonlinearities*** make the linear system theory cannot be applied in analyzing the actual systems. The influences of nonlinear factors can not be explained by linear system theory.

3) The nonlinear characteristics do not always have negative impacts on systems. ***Optimal control laws are often nonlinear laws***. The relay(继电器) and waveform generator(波形发生器) are also widely used.

2. Features of Nonlinear Systems

Comparing with linear control systems，non-linear systems have many *new features*:

1. **A linear systems satisfies the *principle of superposition(叠加原理)*, while a non-linear system does not, always.**

(1) Additivity(叠加性):

$$y = f(x) \Rightarrow f(x_1 + x_2) = f(x_1) + f(x_2)$$

$f(x) = ax$ Obviously, a linear function satisfies the principle of Superposition.

Consider: $f(x) = ax + b$

(2) Multiplicativity (均匀性,可乘性):

$$f(ax) = af(x)$$

- Nonlinear systems may be additive (rarely), but it is completely not multiplicative.

Fig. 8—1 Nonlinear system with filters

$$X_1 \rightarrow Y_1, \quad X_2 \rightarrow Y_2$$

Additivity:

$$X_1 + X_2 \rightarrow Y_1 + Y_2$$

Multiplicativity :

$$nX_1 \not\rightarrow nY_1$$

2. **The stabilities of non-linear systems depend on not only the inherent structure and parameters of control systems, but also the *initial conditions* and the *inputs*.**

Example: An nonlinear systems described by the non-linear differential equation: $\dot{x} = -x(1-x)$

which has two equilibrium points, obviously, $x_1=0$ and $x_2=1$.

The equation equals to $$\frac{dx}{x(1-x)} = -dt$$

Integrating both sides: $$\ln\frac{cx}{1-x} = -t \Rightarrow \frac{cx}{1-x} = e^{-t}$$

- Assume the initial state of the system be x_0,
- if t = 0, then: $$c = \frac{1-x_0}{x_0}$$

$$\therefore \quad x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

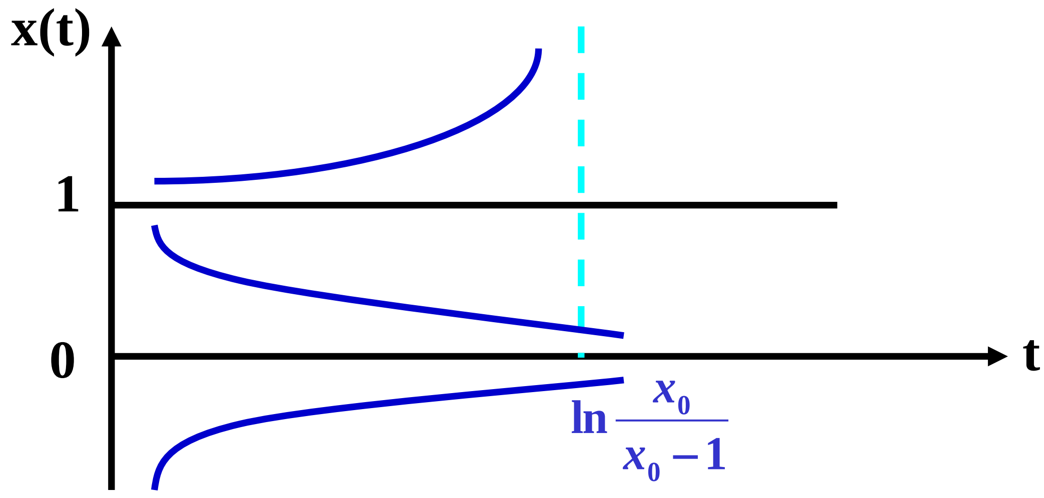


Fig. 8-2 First-order non-linear systems

If $x_0 < 1,\ t \to \infty$, then $x \to 0$

Initial conditions will affect stability of the system!

If $x_0 > 1$, when $t = \ln \dfrac{x_0}{x_0 - 1}$, we have $x \to \infty$

- The initial conditions can be even used as a key to the ***encryption*** of transmission signals in Chaotic systems

Useful signals

Chaotic signals with certain initial conditions

$+$ $\xrightarrow{\text{Transmission}}$ ***Decryption*** by the known initial conditions

3. *Periodic oscillation* **does not exist in an actual physical linear systems , while it may occur in a nonlinear system.**

4. **A stable linear system under a *periodic input* → output with the same frequency;**
   A nonlinear system under a periodic input → many complex cases of the outputs
   Distortion

(1) *Jump resonance* and *Multi-valued response*

Input signals with constant amplitude，then the *amplitude frequency characteristics* of the output is：

$$\omega\uparrow:\ 1\rightarrow 2\rightarrow 3 \vdots \overset{\updownarrow}{\rightarrow} 4 \vdots \rightarrow 5$$

$$\omega\downarrow:\ 5\rightarrow 4\rightarrow 4' \vdots \overset{\updownarrow}{\rightarrow} 2' \vdots \rightarrow 1$$

Hysteresis Loop Characteristics

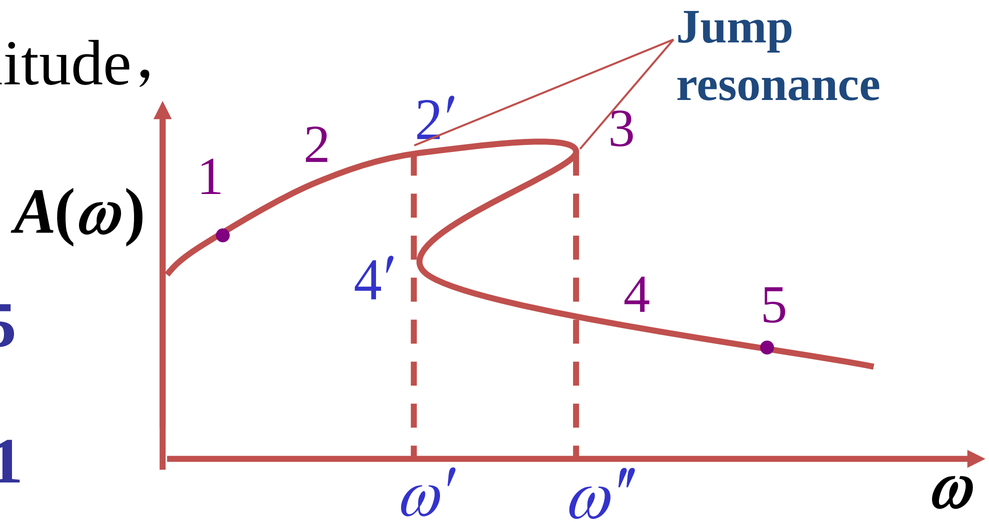


Fig. 8-3 Amplitude frequency characteristic of output of non-linear spring

(2) ***Harmonic Oscillation:*** *double frequency and dividing frequency Oscillation*

Steady-state outputs of non-linear systems can be divided into *double frequency* ***oscillation*** **and** *dividing frequency* ***oscillation*****. When the input signal is sinusoidal, showed in Fig. 8-4**

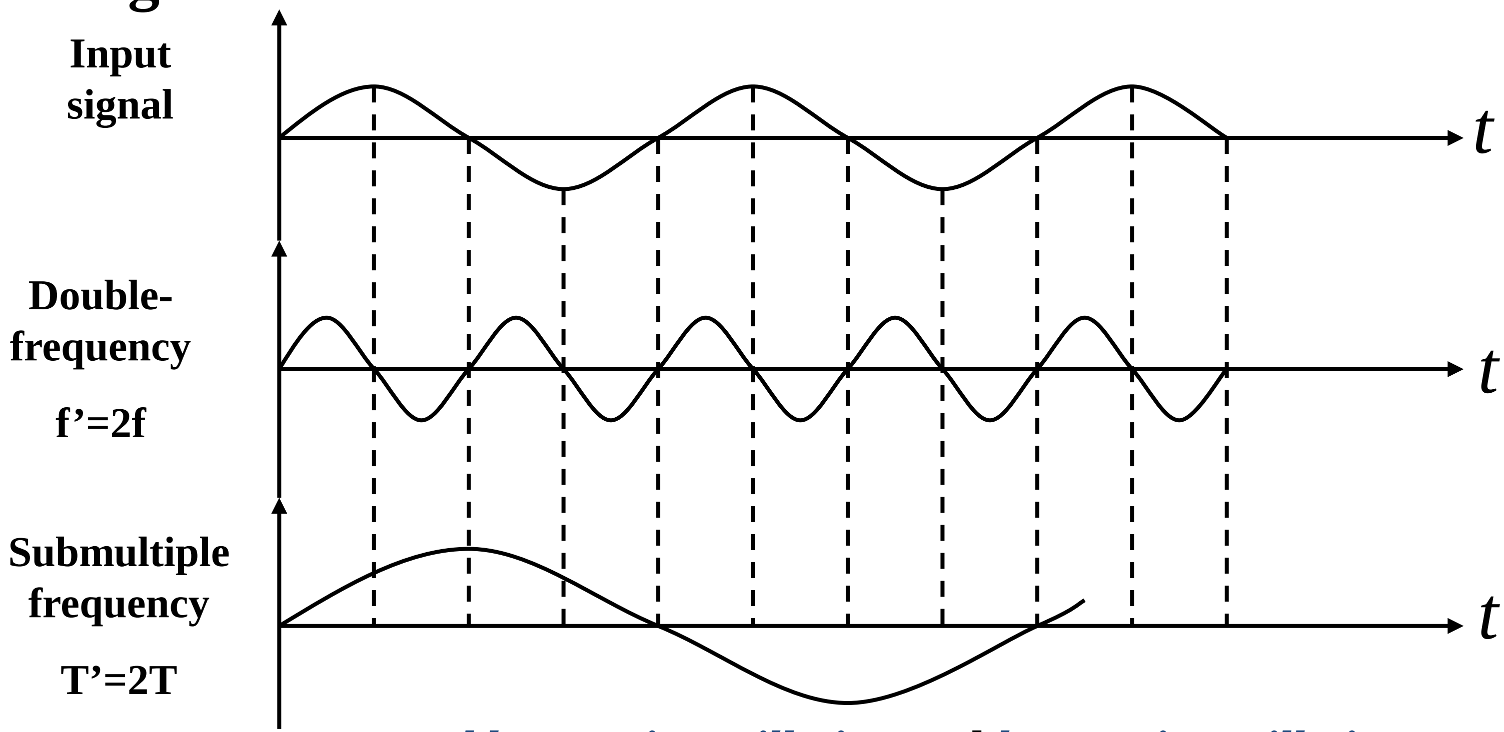


Fig. 8-4 ***subharmonic oscillation*** **and** ***harmonic oscillation***

3. Methods of Studying Nonlinear Systems

1） Phase-Plane Analysis is the graphical method used to analyze first-order and second-order Nonlinear systems. It analyzes the features of Nonlinear systems through drawing *phase portrait* to find all the solutions of the differential equations in any initial condition. It is the generalization and application of time-domain analyzing method in non-linear systems. *It can only be used in the first- and second-order nonlinear systems.*

2） Describing Function Approach is a kind of method for analyzing nonlinear systems inspired by frequency method of linear systems. It is the generalization of frequency method in nonlinear systems, and *is not restricted by the system order*.

3） Numerical Solution is a kind of numerical methods to solve the nonlinear differential equation using high-speed computers. It is almost the only effective method for analyzing and designing *complex nonlinear systems*.

Note:

- **It should be pointed out that: the above methods aim at solving the "*analysis*" problems of nonlinear systems based on analyzing the system stability.**

- The achievement of "*synthesis*" methods in nonlinear systems is much less than stability problem. There are **NO** general approaches can be used to design arbitrary nonlinear systems so far.

§ 8.2 Typical Nonlinear characteristics and Their Mathematical Description

8.2.1 Saturation characteristics

8.2.2 Dead-zone characteristics

8.2.3 Gap characteristics

8.2.4 Relay characteristics

1. Saturation

A common nonlinearity in electronic amplifiers

Mathematical description of saturation features :

$$x(t)=\begin{cases} ke(t), & |e(t)|<e_0 \\ ke_0 sign[e(t)], & |e(t)|\ge e_0 \end{cases}$$

$sign[e(t)]$ **is the *sign* function**

$$sign[e(t)]=\begin{cases} 1, & e(t)\ge 0 \\ -1, & e(t)<0 \end{cases}$$

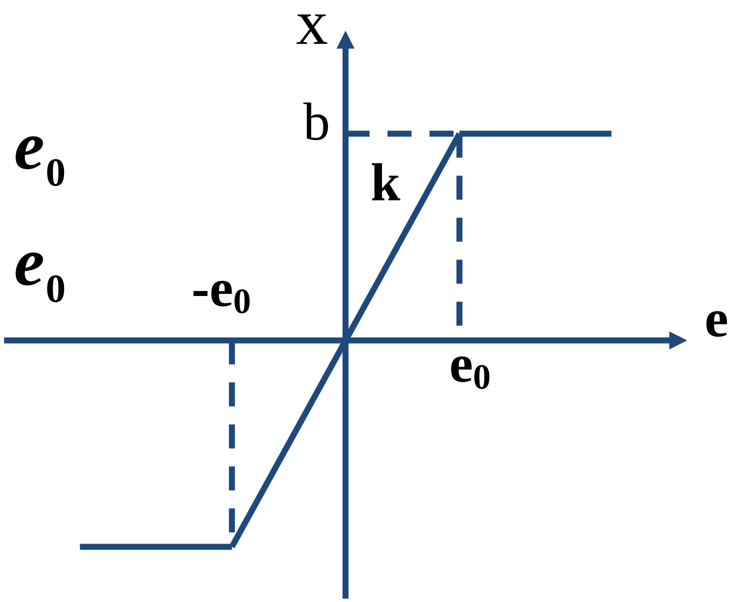


Fig. 8-5 Saturation characteristics

2. Dead-zone

Dead-zone can be also called neutral zone , its mathematical description is:

$$x(t)=\begin{cases}0, & |e(t)|\le e_0\\ k\left[e(t)-e_0 sign\left[e(t)\right]\right], & |e(t)|>e_0\end{cases}$$

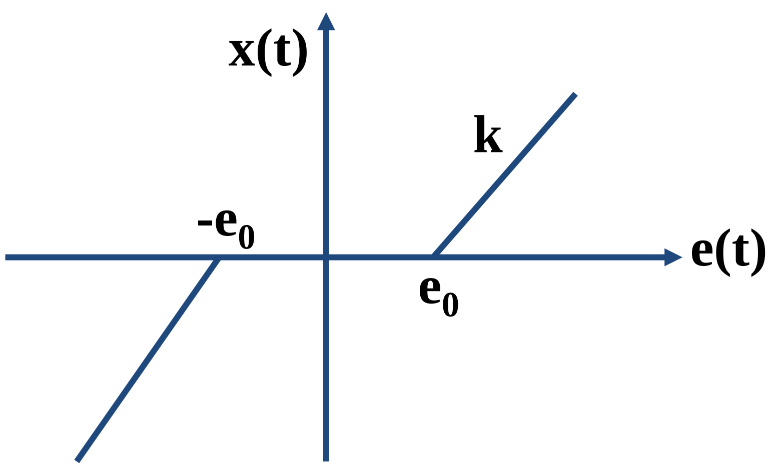


Fig. 8-6 Dead-zone characteristics

3. Gap

Mechanical transmission devices are based on gears, there must exist some gaps for sliding and reversing transmission, that means the gears have to pass a few distances when reversing transmission is needed.

Its mathematical description is :

$$x(t)=\begin{cases} k\left[e(t)-e_0\right], & \dot{x}(t)>0 \\ k\left[e(t)+e_0\right], & \dot{x}(t)<0 \\ b\,sign\left[e(t)\right], & \dot{x}(t)=0 \end{cases}$$

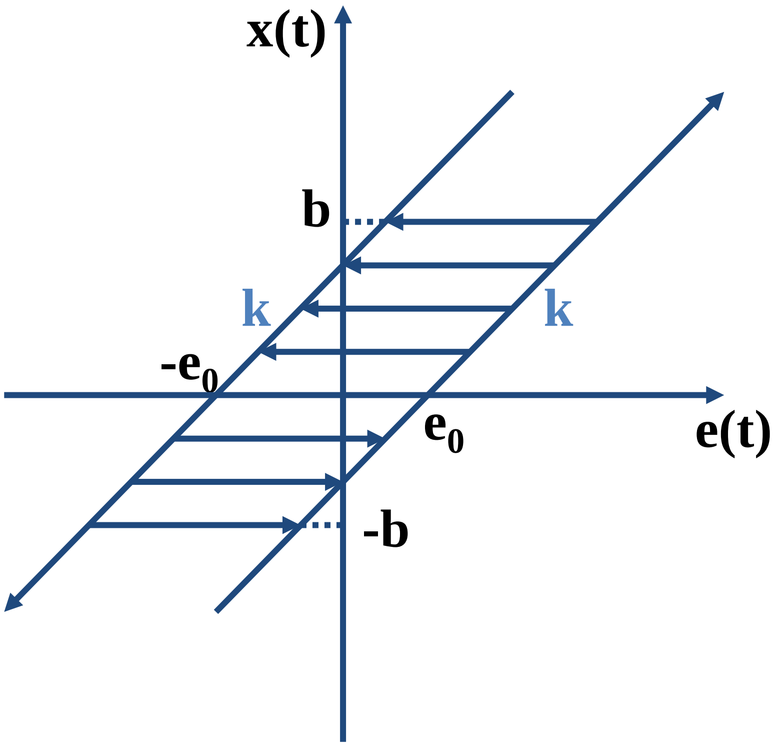
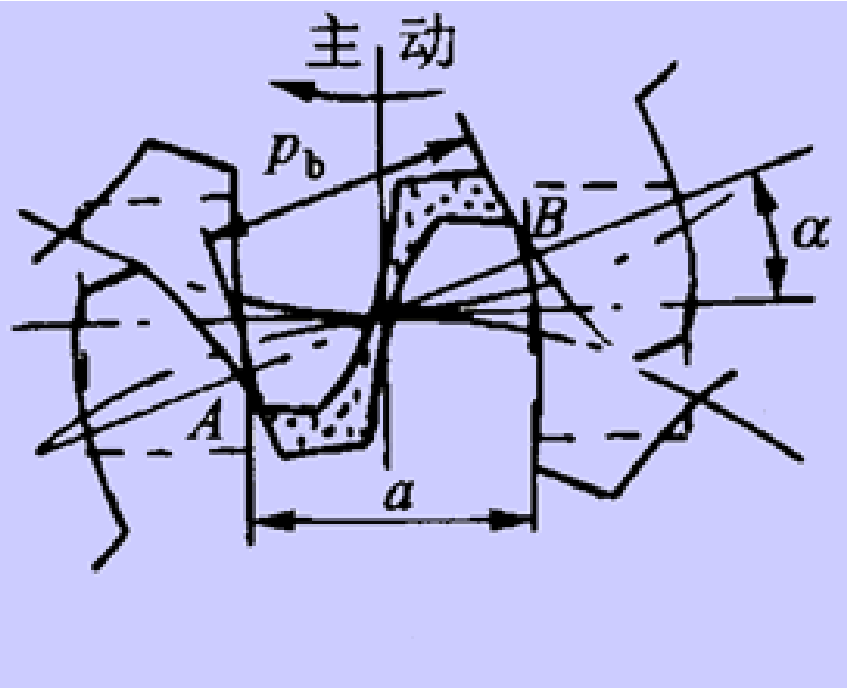
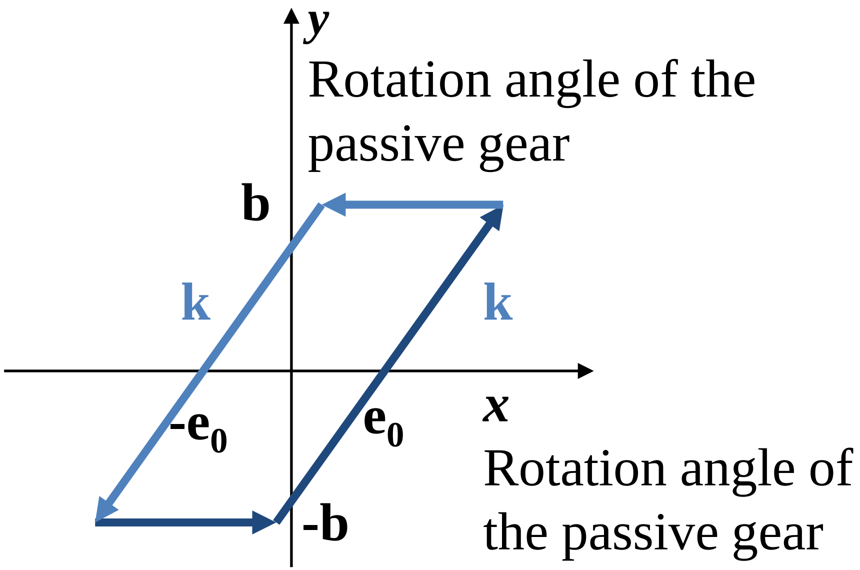


Fig. 8-7 Gap characteristics





Input x is the rotation angle of the driven gear
Output y is the rotation angle of the passive gear

4. Relay(继电特性)

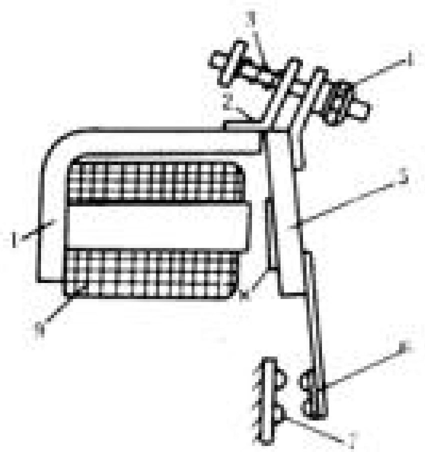


图1 电磁式继电器原理图

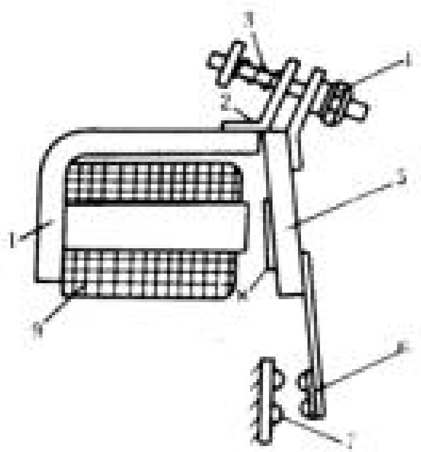
1- 铁心 2-旋转棱角 3-释放弹簧 4-调节螺母 5--衔铁

6-动触点 7-静触点 8-非磁性垫片 9-线圈

***Principle of relay*:**

Input voltage → Current in coil → generates the electromagnetic force → Close the relay contact

If the input voltage is e_0, the electromagnetic force generated by current in coil is enough to make the switch to be closed, then e_0 is called ***Operation Voltage*.**



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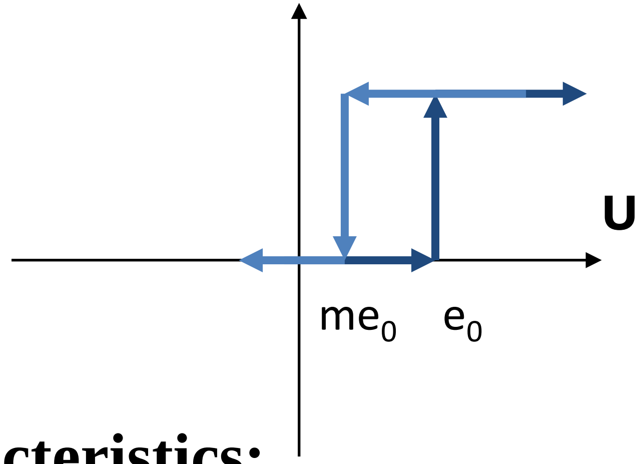
The relay contact will not release when the input voltage is reduced to e_0 because of the influence of Hysteresis.

When it is further reduced to me_0 ($m<1$), the relay contact will be released.

Then me_0 is called Release Voltage.



There are four forms of relay characteristics:

1. Ideal relay characteristics

$$x(t)=\begin{cases} M, & e>0 \\ -M, & e<0 \end{cases}$$

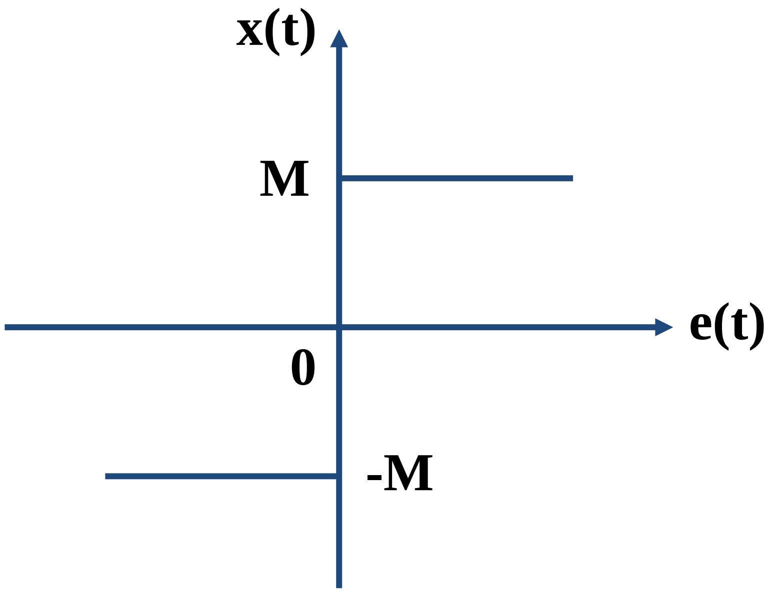


Fig. 8-8(a) Perfect relay characteristics

2. Relay characteristics with Dead-zone

$$x(t)=\begin{cases} M, & e(t)>e_0 \\ 0, & -e_0\le e(t)\le e_0 \\ -M, & e(t)<-e_0 \end{cases}$$

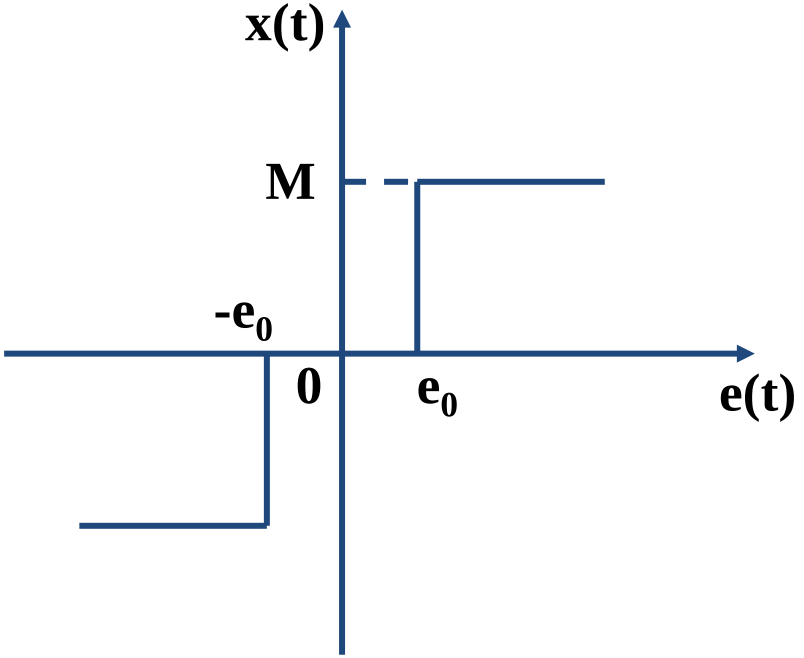


Fig. 8-8(b) Relay characteristics with Dead-zone

3. Relay characteristics with Hysteresis loop

$$x(t)=\begin{cases} M, & \dot{e}>0, e>e_0 \\ & \dot{e}<0, e>-e_0 \\ -M, & \dot{e}>0, e<e_0 \\ & \dot{e}<0, e<-e_0 \end{cases}$$

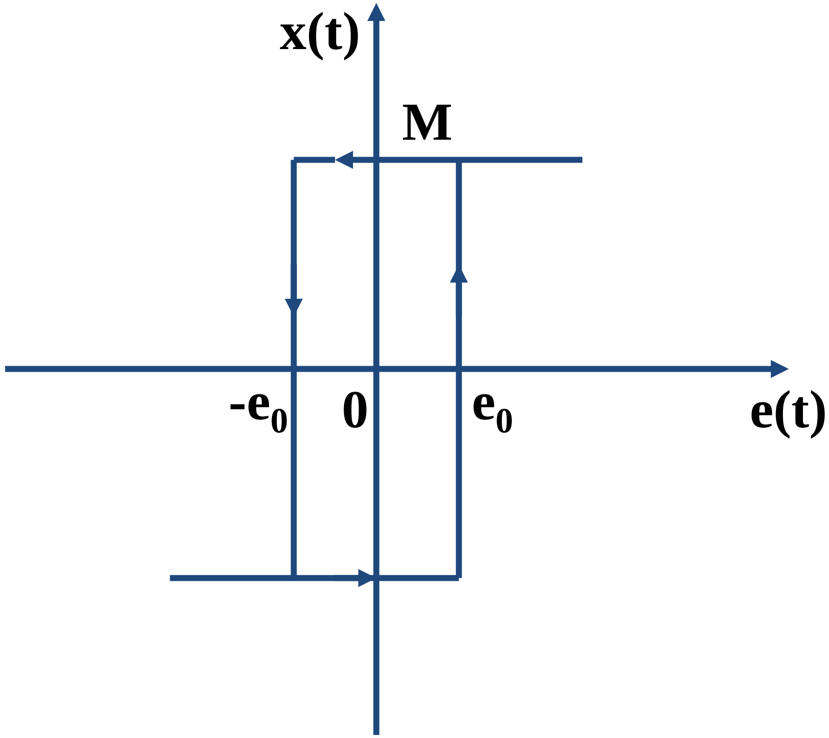


Fig. 8-8(c) Relay characteristics with Hysteresis loop

4. Relay characteristics with Dead-zone and Hysteresis loop

$$x(t)=\begin{cases} M, & \dot{e}>0, e\geq e_0 \\ & \dot{e}<0, e>me_0 \\ 0, & \dot{e}>0, -me_0<e<e_0 \\ & \dot{e}<0, -e_0<e<me_0 \\ -M, & \dot{e}>0, e<-me_0 \\ & \dot{e}<0, e\leq -e_0 \end{cases}$$

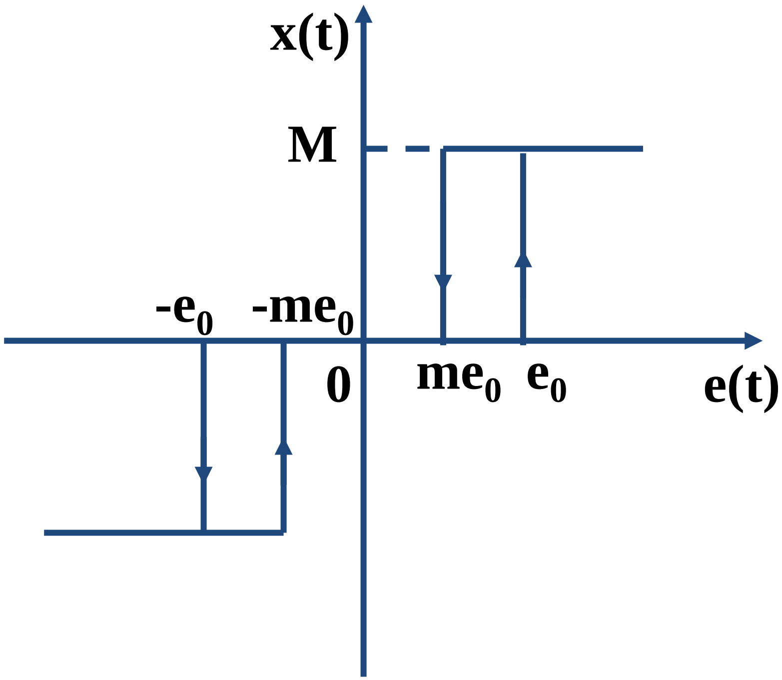
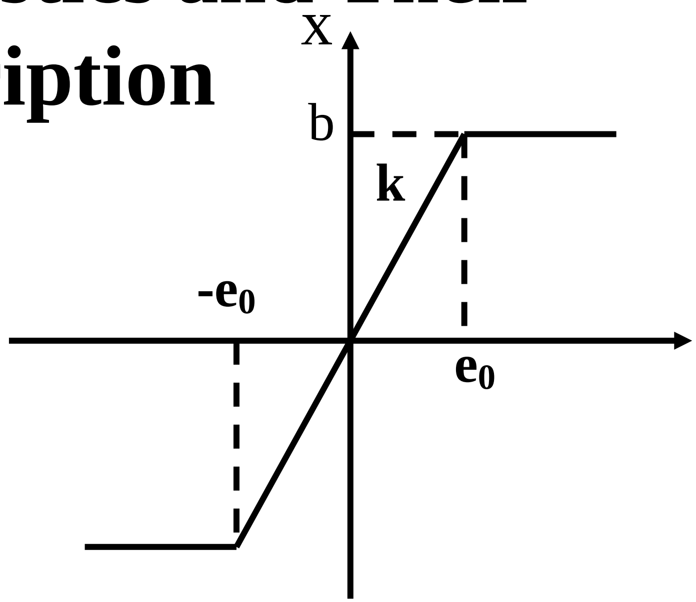
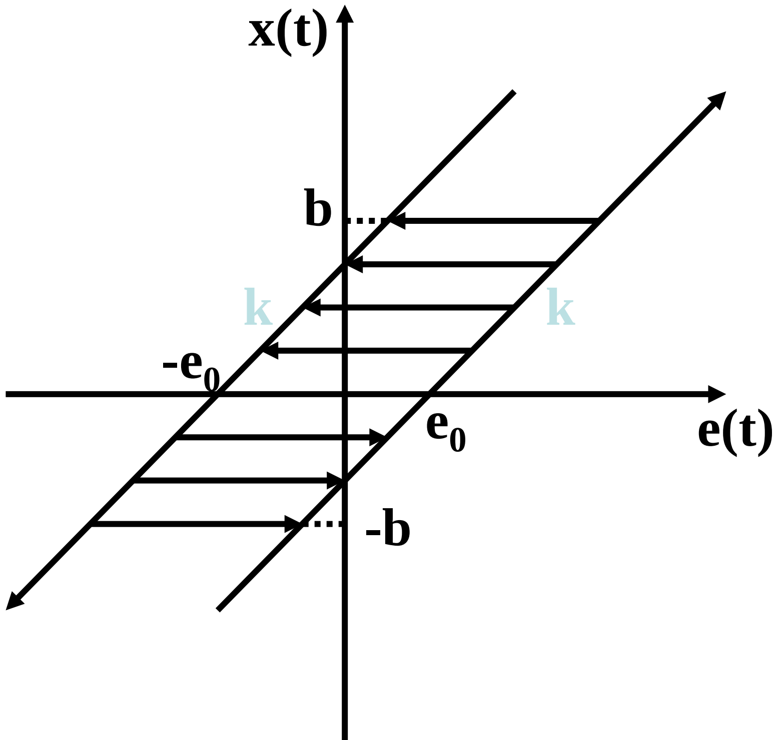


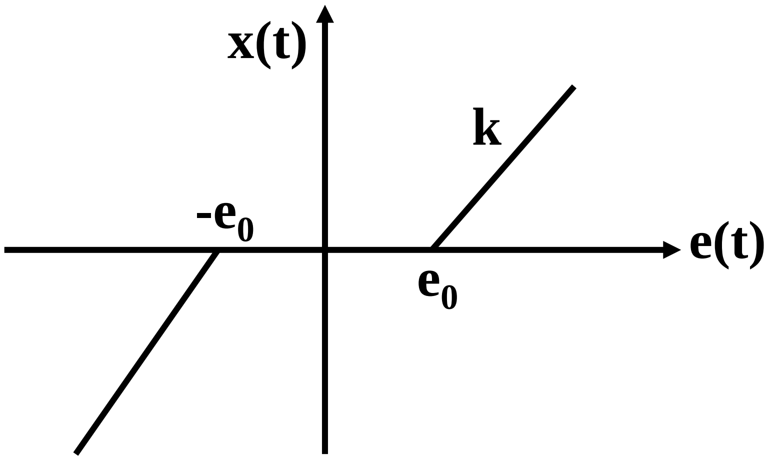
Fig. 8-8(d) Relay characteristics with Dead-zone and Hysteresis loop

Typical Nonlinear characteristics and Their Mathematical Description

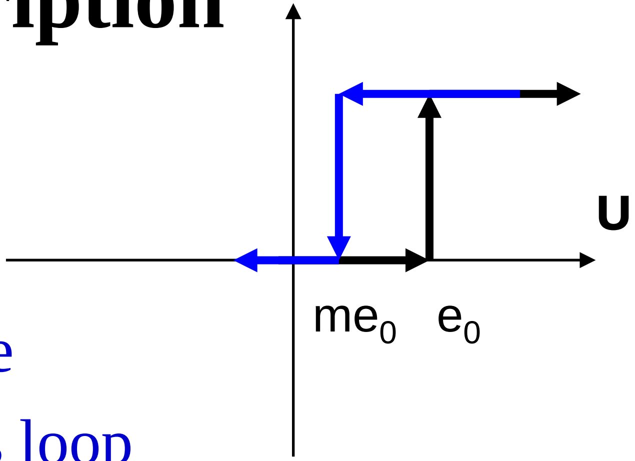
- Saturation characteristics
- Dead-zone characteristics
- Gap characteristics



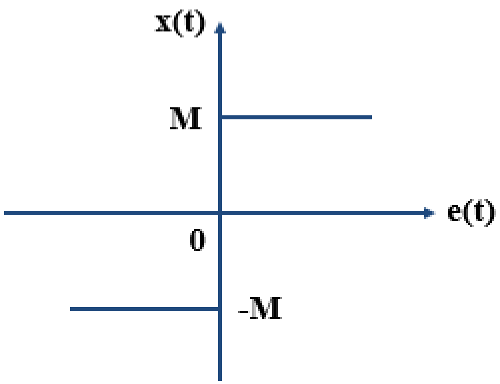


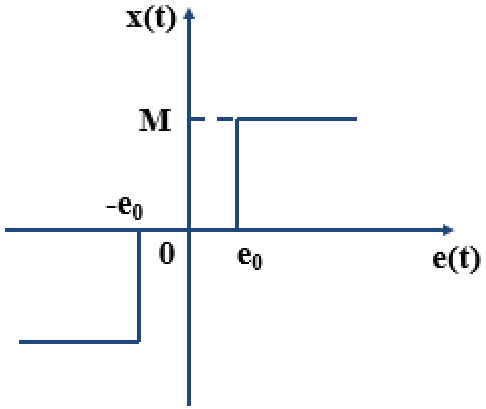


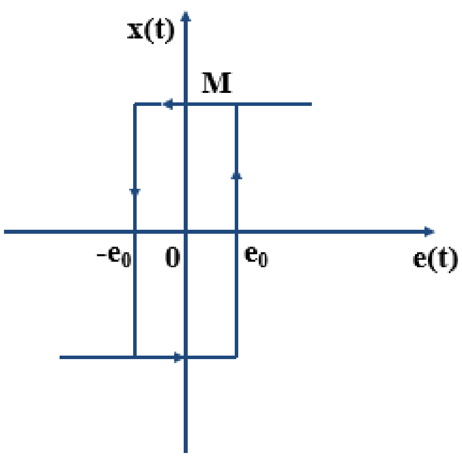
Typical Nonlinear characteristics and Their Mathematical Description

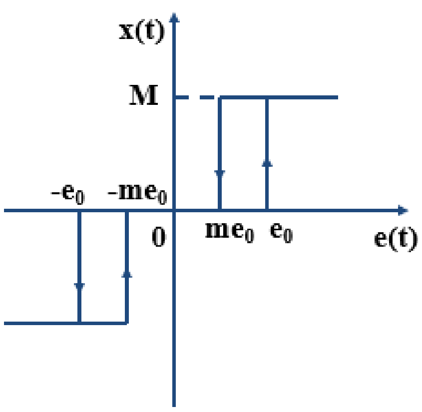


- Relay characteristics
  - Ideal relay characteristics
  - Relay characteristics with Dead-zone
  - Relay characteristics with Hysteresis loop
  - Relay characteristics with Dead-zone and Hysteresis loop





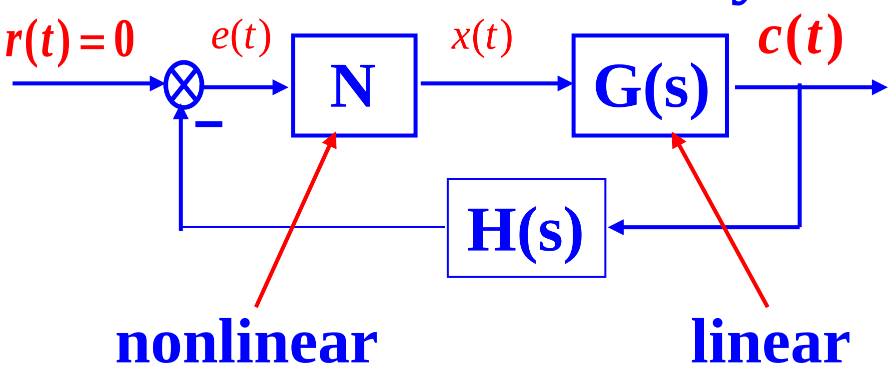




§ 8.3 Describing function method (Harmonic Linearizing method)

Basic idea

For the nonlinear system



Typical structure of the nonlinear systems

$If \ : e(t) = A\sin\omega t$ ⟹ **a sinusoidal input,**

$x(t)$, maybe it is not a sinusoidal but a periodic function, can be expressed as a Fourier series.

Assumption:

The harmonic of $x(t)$ could be neglected, then:

$$x(t) \approx x_1 \sin(\omega t + \phi_1) \Longrightarrow$$

output frequency is equal to input frequency approximately.

Similar to the *frequency analysis* of linear system, we can perform frequency analysis for the nonlinear system based on the assumption.

$\Longrightarrow$ **Describing function**

- The describing function method is mainly used to analyze the ***stability*** and ***self-oscillation*** of the nonlinear system without external excitation.

- Advantage:
  - It is not confined by ***the order of the system***.

- Disadvantages:
  - It is an ***approximate analysis method***.
  - It can only be used to study the system ***frequency characteristics***.

8.3.1 Concept of describing function

The describing function method can be applied to nonlinear systems with the following features:

1. The linear part and the nonlinear part can be separated.

Shown in Fig. 8-9， NL is a *nonlinear part*， G is the transfer function of the *linear part*.

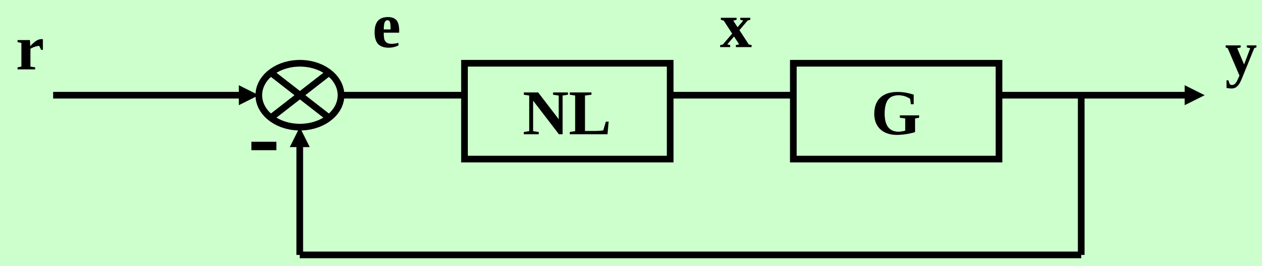


Fig. 8-9 Typical structure of nonlinear system

2. The system has an *odd-symmetric nonlinearity*, and the input-output relationship of the nonlinear part is static (without energy storage elements).

- **If so, sinusoidal input → periodic output**
- **The output can be expanded into Fourier series with a *zero D. C. component*.**

$$f(x)=-f(-x)$$

3. The linear part is a good low-pass filter

We can suppose the *higher-oder harmonic* is filtered out.

⇩

There is only a *fundamental component* in the the output .

If all the conditions above are satisfied, we can describe the nonlinear components by the frequency response like as that we did in the linear systems.

So we have:

Definition of the describing function

The describing function *N*(*A*) of the nonlinear element is: the *complex ratio* of the fundamental component of the output *x*(*t*) and the sinusoidal input *e*(*t*), that is:

$$For\ e(t) = A\sin\omega t,$$

$$x(t) \approx A_1\cos\omega t + B_1\sin\omega t$$

$$= x_1\sin(\omega t + \phi_1) \Longrightarrow N(A) = \frac{x_1 e^{j\phi_1}}{A}$$

Assume the input of nonlinear is sinusoidal $e(t) = A\sin\omega t$

Normally, the output is periodic, which can be expressed as a *Fourier series*:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\omega t + B_n \sin n\omega t\right)$$

The nonlinearity is *odd-symmetric*(奇对称).

$$\Rightarrow \quad A_0 = 0$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos n\omega t \, d(\omega t)$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin n\omega t \, d(\omega t)$$

For the fundamental component, we have

$$\boxed{A_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos\omega t \; d(\omega t)}$$

$$\boxed{B_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin\omega t \; d(\omega t)}$$

Thus, the fundamental component is

$$\boxed{x_1(t) = A_1\cos\omega t + B_1\sin\omega t = x_1\sin(\omega t + \varphi_1)}$$

where $\boxed{x_1 = \sqrt{A_1^2 + B_1^2}}$ $\boxed{\varphi_1 = arctg\,\frac{A_1}{B_1}}$

The describing function is then given by

$$N(A)=\frac{X_1}{A}e^{j\varphi_1}$$

Obviously, the describing function is a function of the input amplitude *A*. So we can regard it as a *variable gain amplifier*.

$$\boxed{N(A)=\frac{\sqrt{A_1^2+B_1^2}}{A}e^{j\,arctg\frac{A_1}{B_1}}=\frac{B_1}{A}+j\frac{A_1}{A}}$$

Replacing the nonlinear part by N(A), we can extend the *frequency response method* of linear system to the nonlinear system so as to analyze the *frequency characteristics* of nonlinear system.

Remarks:

Normally, the describing function N is *a function of the amplitude and the frequency of input signal*, it should be expressed as $N(A,\omega)$.

In most of the nonlinear components, there are no energy storage elements. The frequencies of output and input are then independent. So the describing function N of common nonlinear components is *only a function of the amplitude* of input, which can be expressed as $N(A)$。

Remarks: (cont.)

If the nonlinearity is *single-valued odd-symmetric* ⇨

The output $x(t)$ is *an odd function*.

$$A_1 = 0 \qquad N(A) = {B_1}/{A}$$

The describing function is a *real function* of input amplitude A.

If the nonlinearity is not *single-valued odd-symmetric* ⇨

The output $x(t)$ is *neither an odd nor even*.

$$A_1 \neq 0, B_1 \neq 0$$

The describing function is a *complex function* of input amplitude A.

8.3.2 Describing function of typical nonlinear characteristic

1. Saturation

Assume the input is $e(t) = A\sin\omega t$

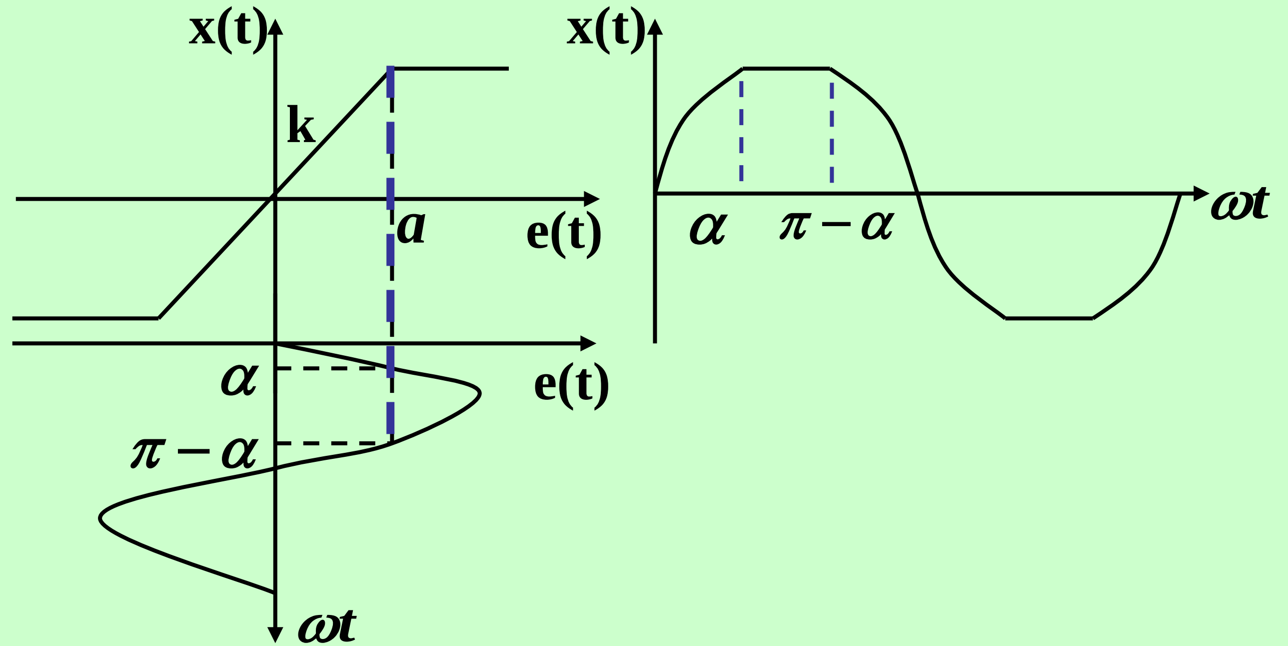


Fig 8-10 Input and output waveform of a saturation nonlinearity

when $A > a$, the output $x(t)$ is

$$x(t)=\begin{cases} KA\sin\omega t, & 0\le \omega t\le \alpha \\ Ka, & \alpha<\omega t\le \pi-\alpha \\ KA\sin\omega t, & \pi-\alpha<\omega t\le \pi \end{cases}$$

where,

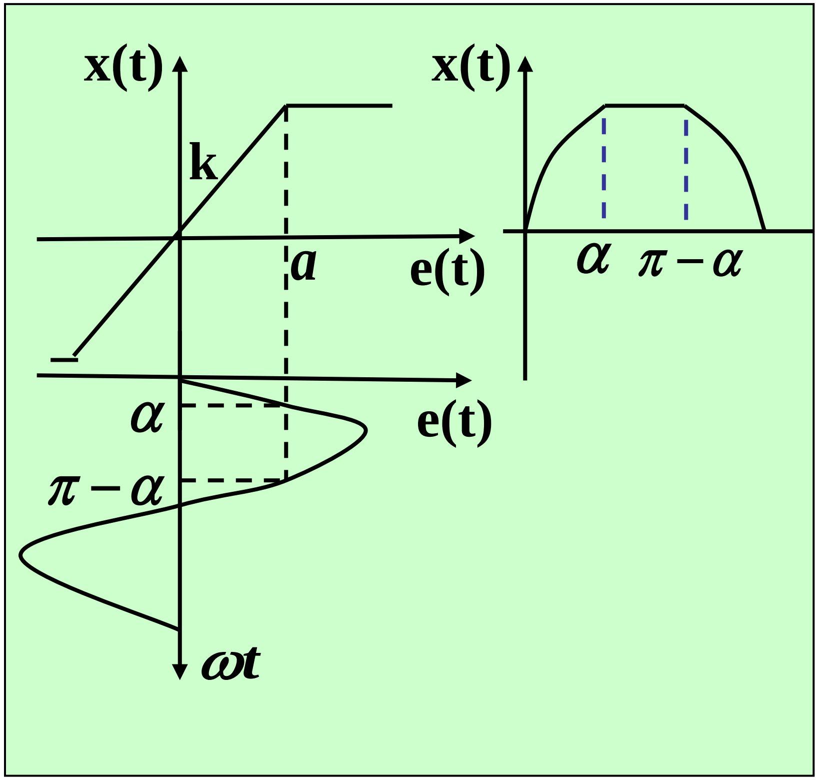
$$A\sin\alpha = a, \therefore \alpha = \sin^{-1}\frac{a}{A}$$

Because the output is odd,

$A_1 = 0$ (*single-valued odd-symmetric*)

$$\phi_1 = tg^{-1}\frac{A_1}{B_1} = 0$$

$$\boxed{N(A)=\frac{B_1}{A}+j\frac{A_1}{A}=\frac{B_1}{A}}$$



$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t \; d(\omega t)$$

$$= \frac{2}{\pi}\left[\int_0^{\alpha} KA\sin^2\omega t d(\omega t) + \int_{\alpha}^{\pi-\alpha} Ka\sin\omega t d(\omega t) + \int_{\pi-\alpha}^{\pi} KA\sin^2\omega t d(\omega t)\right]$$

$$= \frac{2}{\pi}KA\left[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function of saturation nonlinearity is:

$$N(A) = \frac{B_1}{A} = \frac{2}{\pi}K\left[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

N*(*A*) is a** ***nonlinear real function **of input amplitude** ***A*****.**
It can be regarded as a ***variable gain amplifier*****.**

2. Dead-zone

Assume the input is $e(t) = A\sin\omega t$,

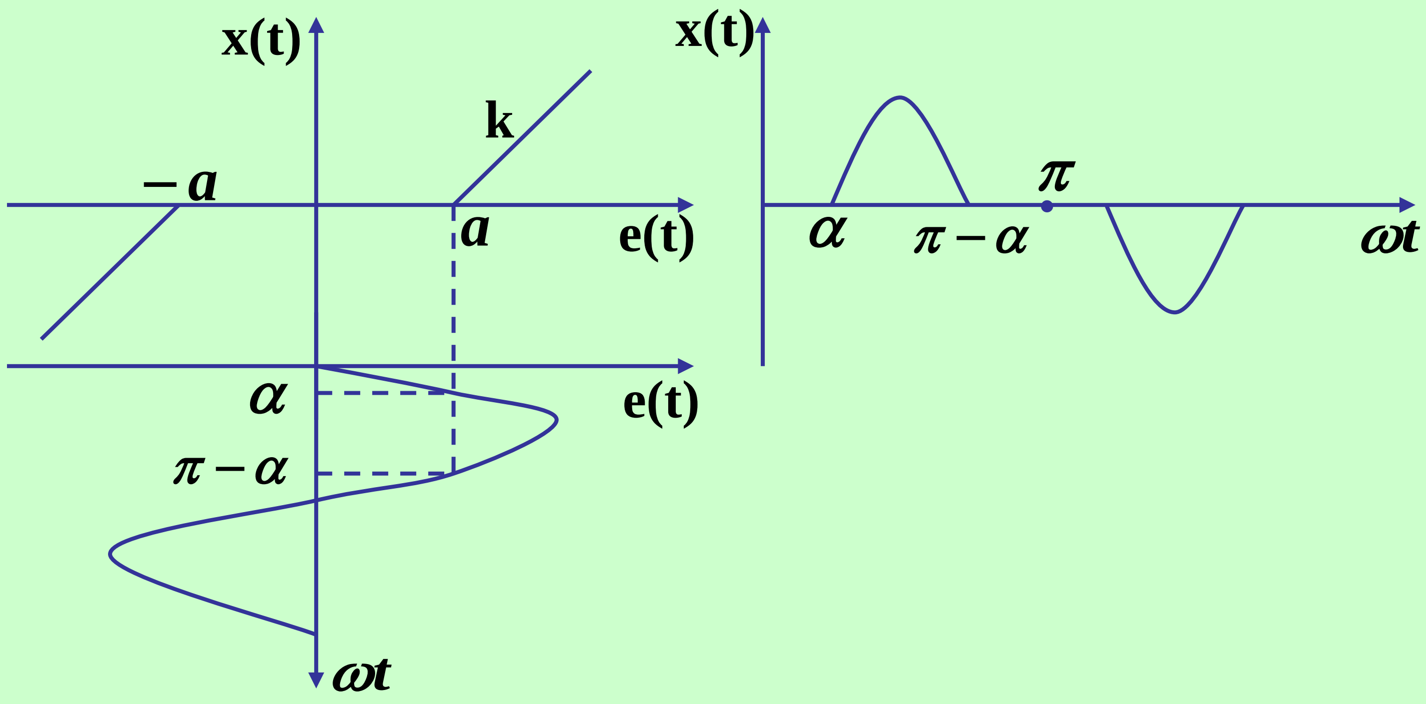


Fig8-11 input and output waveforms of dead zone nonlinearity

When $A > a$, **the output of dead zone nonlinearity is**

$$x(t) = \begin{cases} 0, & 0 \le \omega t \le \alpha \\ K(A\sin\omega t - a), & \alpha < \omega t \le \pi - \alpha \\ 0, & \pi - \alpha < \omega t \le \pi \end{cases}$$

where, $A\sin\alpha = a, \therefore \alpha = \sin^{-1}\frac{a}{A}$

The output is odd→ $A_1 = 0$, $\phi_1 = 0$

$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t\, d(\omega t)$$

$$= \frac{2}{\pi}\int_{\alpha}^{\pi-\alpha} K\left(A\sin\omega t - a\right)\sin\omega t\, d(\omega t)$$

$$B_1 = \frac{2}{\pi} KA \left[\frac{\pi}{2} - \sin^{-1}\frac{a}{A} - \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2} \right]$$

The describing function of dead zone nonlinearity is

$$N(A) = \frac{B_1}{A} = \frac{2}{\pi} K \left[\frac{\pi}{2} - \sin^{-1}\frac{a}{A} - \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2} \right]$$

Note:

(1) When a/A is very small, i.e., the non-sensible zone is small, $N(A)$ is approximately equal to K;

(2) $N(A)$ decreases as a/A becomes larger

(3) $a/A = 1 \qquad \Rightarrow \qquad N(A) = 0$

3. Gap

Assume the input is $e(t) = A\sin\omega t$,

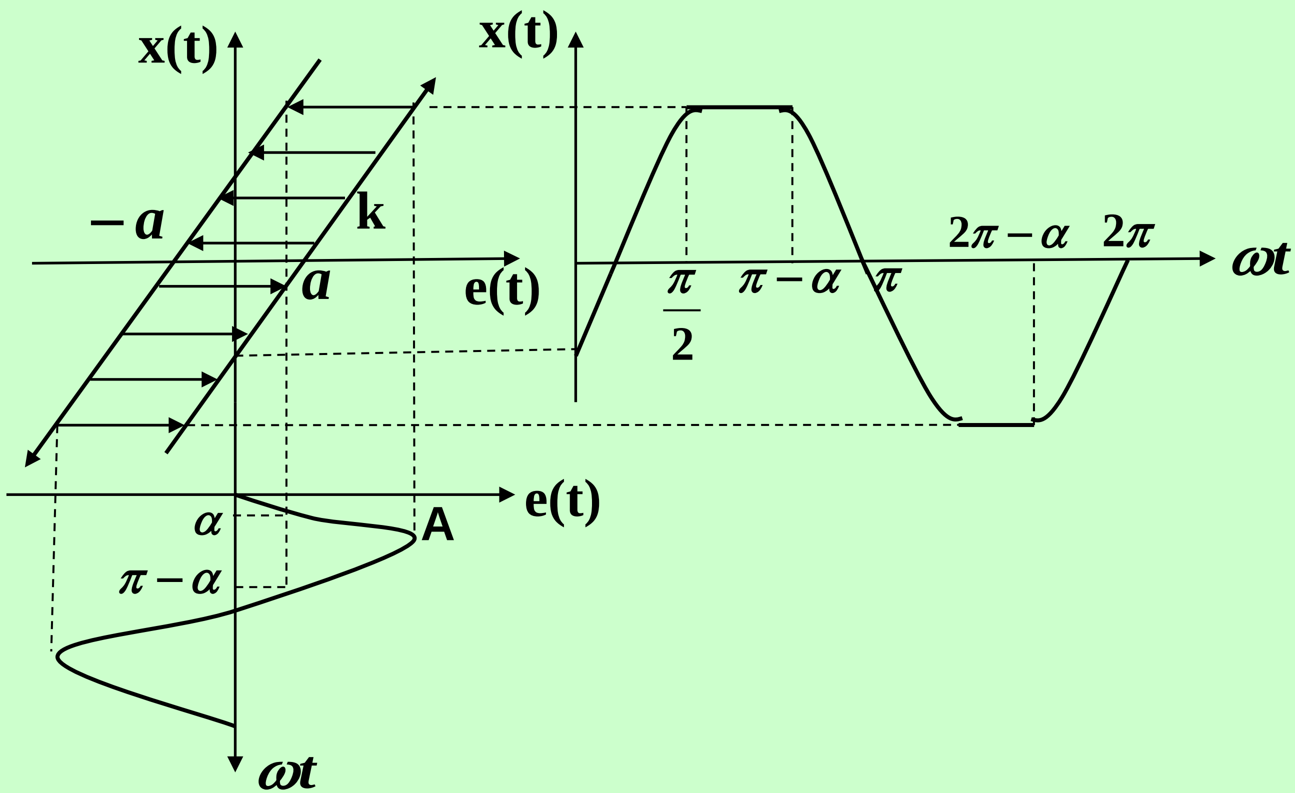


Fig 8-12 input and output waveforms of gap nonlinearity

From the mathematical description of gap nonlinearity, the *x*(*t*) is given by

$$x(t) = \begin{cases} K(A\sin\omega t - a), & 0 \le \omega t < \dfrac{\pi}{2} \\ K(A - a), & \dfrac{\pi}{2} \le \omega t < \pi - \alpha \\ K(A\sin\omega t + a), & \pi - \alpha \le \omega t \le \pi \end{cases}$$

where， $A\sin(\pi - \alpha) = A - 2a, \ \therefore \alpha = \sin^{-1}\dfrac{A - 2a}{A}$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\cos\omega t \; d(\omega t)$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} K(A\sin\omega t - a)\cos\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\alpha} K(A-a)\cos\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\pi-\alpha}^{\pi} K(A\sin\omega t + a)\cos\omega t d(\omega t) \quad = \frac{4KA}{\pi}\left[\left(\frac{a}{A}\right)^2 - \frac{a}{A}\right]$$

$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t \; d(\omega t) \quad = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} K(A\sin\omega t - a)\sin\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\alpha} K(A-a)\sin\omega t d(\omega t) + \frac{2}{\pi}\int_{\pi-\alpha}^{\pi} K(A\sin\omega t + a)\sin\omega t d(\omega t)$$

$$= \frac{KA}{\pi}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{A-2a}{A}\right) + \frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right]$$

So, we can obtain the describing function N(A) of gap nonlinearity as follows:

$$
\begin{aligned}
N(A) &= \frac{B_1}{A} + j\frac{A_1}{A} \\
&= \frac{K}{\pi}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{A-2a}{A}\right) + \frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right] + j\frac{4K}{\pi}\left[\frac{a(a-A)}{A^2}\right] \\
&= \left|N(A)\right| e^{j\varphi_1}
\end{aligned}
$$

$$
\left|N(A)\right| = \sqrt{\left[\frac{4K}{\pi}\left(\frac{a(a-A)}{A^2}\right)\right]^2 + \left[\frac{K}{\pi}\left(\frac{\pi}{2} + \sin^{-1}\frac{A-2a}{A} + \frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right)\right]^2}
$$

$$
\varphi_1 = tg^{-1}\frac{4\dfrac{a(a-A)}{A^2}}{\left[\dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{A-2a}{A}\right) + \dfrac{A-2a}{A}\sqrt{1-\left(\dfrac{A-2a}{A}\right)^2}\right]}
$$

4. Relay

Assume the input is $e(t) = A\sin\omega t$,

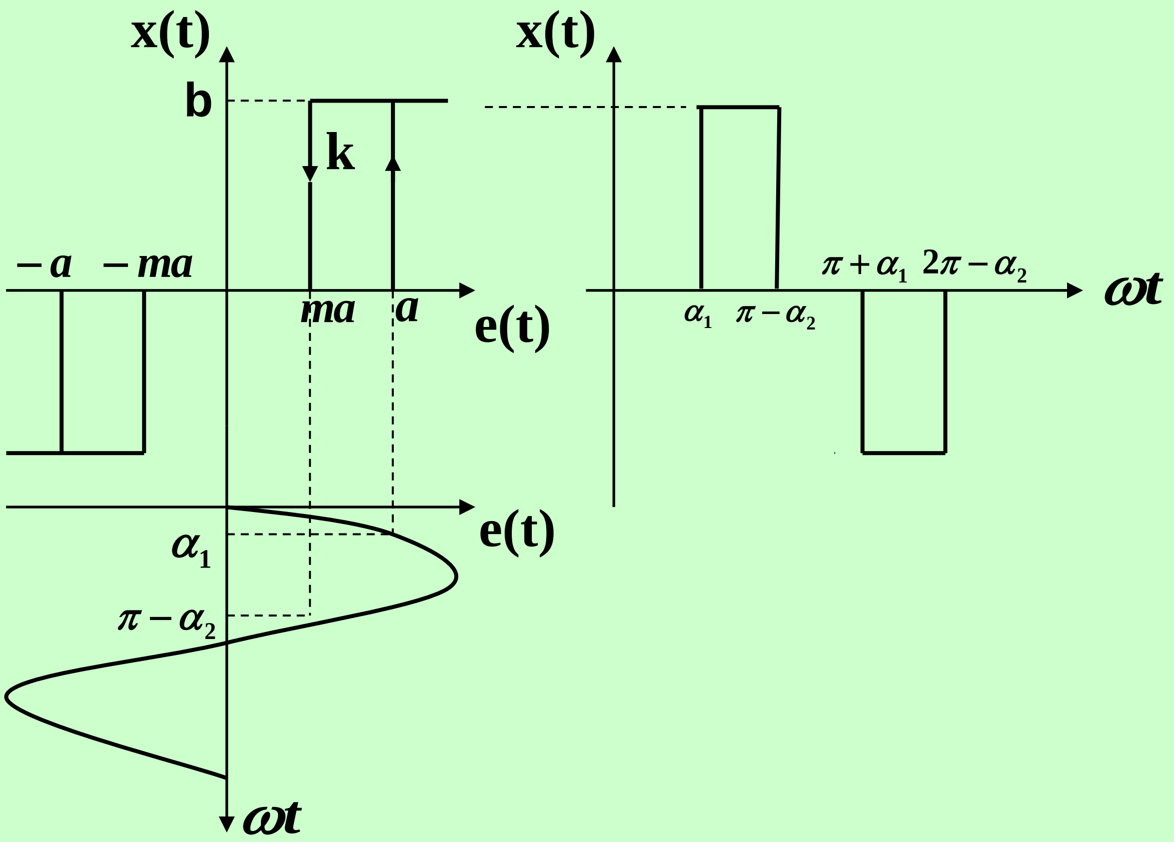


Fig 8-13 input and output waveforms of relay nonlinearity with dead zone and hysteresis ring

The output of the relay characteristic is :

$$x(t)=\begin{cases}0, & 0\le \omega t<\alpha_1\\ b, & \alpha_1\le \omega t<\pi-\alpha_2\\ 0, & \pi-\alpha_2\le \omega t\le \pi\end{cases}$$

where,

$$A\sin\alpha_1=a,\ \therefore \alpha_1=\sin^{-1}\frac{a}{A}$$

$$A\sin(\pi-\alpha_2)=ma,\ \therefore \alpha_2=\sin^{-1}\frac{ma}{A}$$

$$A_1 = \frac{2}{\pi}\int_{\alpha_1}^{\pi-\alpha_2} b\cos\omega t d(\omega t)$$

$$= \frac{2b}{\pi}(\sin\alpha_2 - \sin\alpha_1) = \frac{2ab(m-1)}{\pi A}$$

$$B_1 = \frac{2}{\pi}\int_{\alpha_1}^{\pi-\alpha_2} b\sin\omega t d(\omega t)$$

$$= \frac{2b}{\pi}(\cos\alpha_2 + \cos\alpha_1) = \frac{2b}{\pi}\left[\sqrt{1-\left(\frac{ma}{A}\right)^2} + \sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function N(A) of relay nonlinearity with dead zone and hysteresis ring is

$$N(A) = |N(A)| e^{j\phi_1} = \sqrt{\left(\frac{A_1}{A}\right)^2 + \left(\frac{B_1}{A}\right)^2} e^{jtg^{-1}\frac{A_1}{B_1}}$$

$$|N(A)| = \frac{2b}{\pi A}\sqrt{2\left[1 - m\left(\frac{a}{A}\right)^2 + \sqrt{1 + m^2\left(\frac{a}{A}\right)^4 - (m^2+1)\left(\frac{a}{A}\right)^2}\right]}$$

$$\phi_1 = tg^{-1}\frac{(m-1)\left(\frac{a}{A}\right)}{\sqrt{1 - m^2\left(\frac{a}{A}\right)^2} + \sqrt{1 - \left(\frac{a}{A}\right)^2}}$$

Corollary:

When a = 0, we can obtain the describing function of a *ideal relay nonlinearity*

$$N(A) = \frac{4b}{\pi A}$$

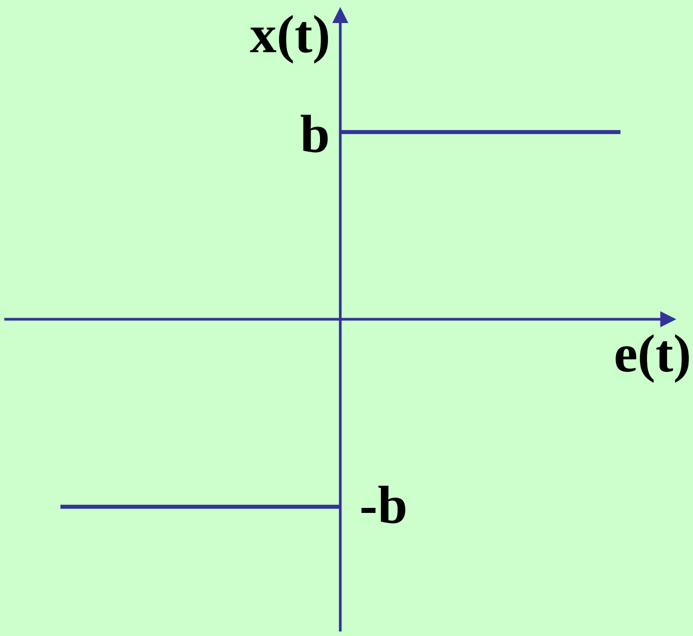


Fig 8-14 Ideal relay nonlinearity

When m = 1 and a≠0, we can obtain the describing function of a relay with dead zone

$$N(A) = \frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$

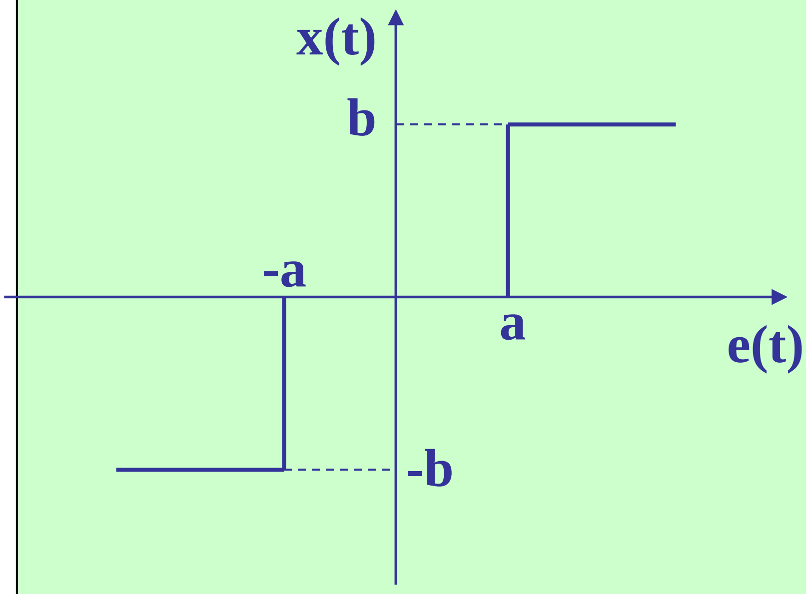


Fig 8-15 A relay with dead zone

When m = − 1, we can obtain the describing function of a relay with hysterics

$$N(A)=\frac{4b}{\pi A}e^{\,jtg^{-1}\frac{-\left(\frac{a}{A}\right)}{\sqrt{1-\left(\frac{a}{A}\right)^{2}}}}$$

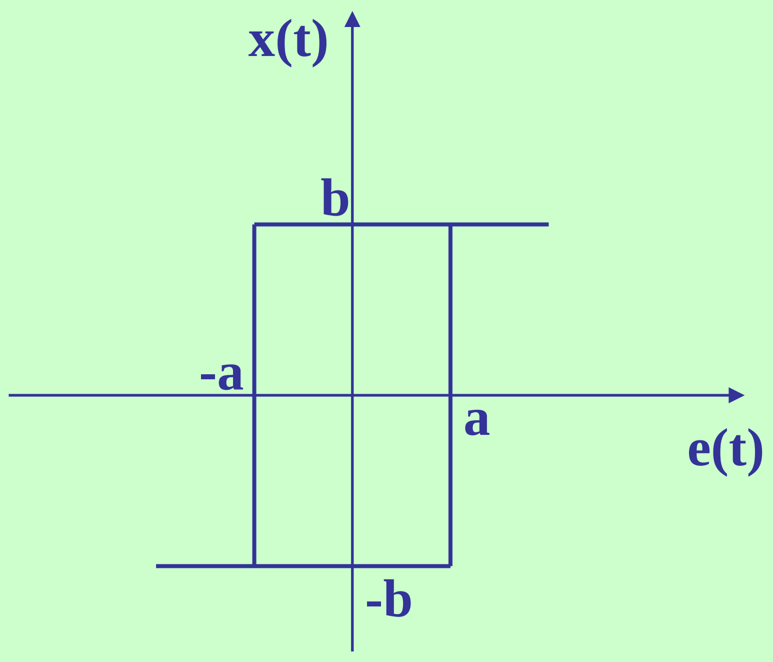


Fig 8-16 A relay with hysteresis

- Summary:
  - Nonlinear system analysis by the Describing Function Method:
    1. Draft of e-x, e-t, x-t;
    2. Decide the odd and Single-value property of x(e);
    3. Decide the symmetry property of x(t);
    4. Calculate A1, B1 - by integration;
    5. Calculate N(A).

8.3.3 Describing function of multiple nonlinearities

1. Cascaded nonlinear system

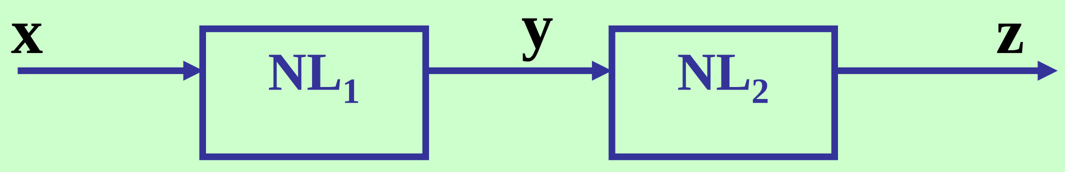


Fig 8-17 cascaded nonlinear system

The describing function of a cascaded nonlinear system *is not equal to* the product of the two describing functions of each nonlinear elements.

$$N(A) \neq N_1(A) \cdot N_2(A)$$

assume NL_1 is a dead zone nonlinearity, NL_2 is a saturation nonlinearity, the composite nonlinearity of the cascaded nonlinear system is shown in Fig.8-18.

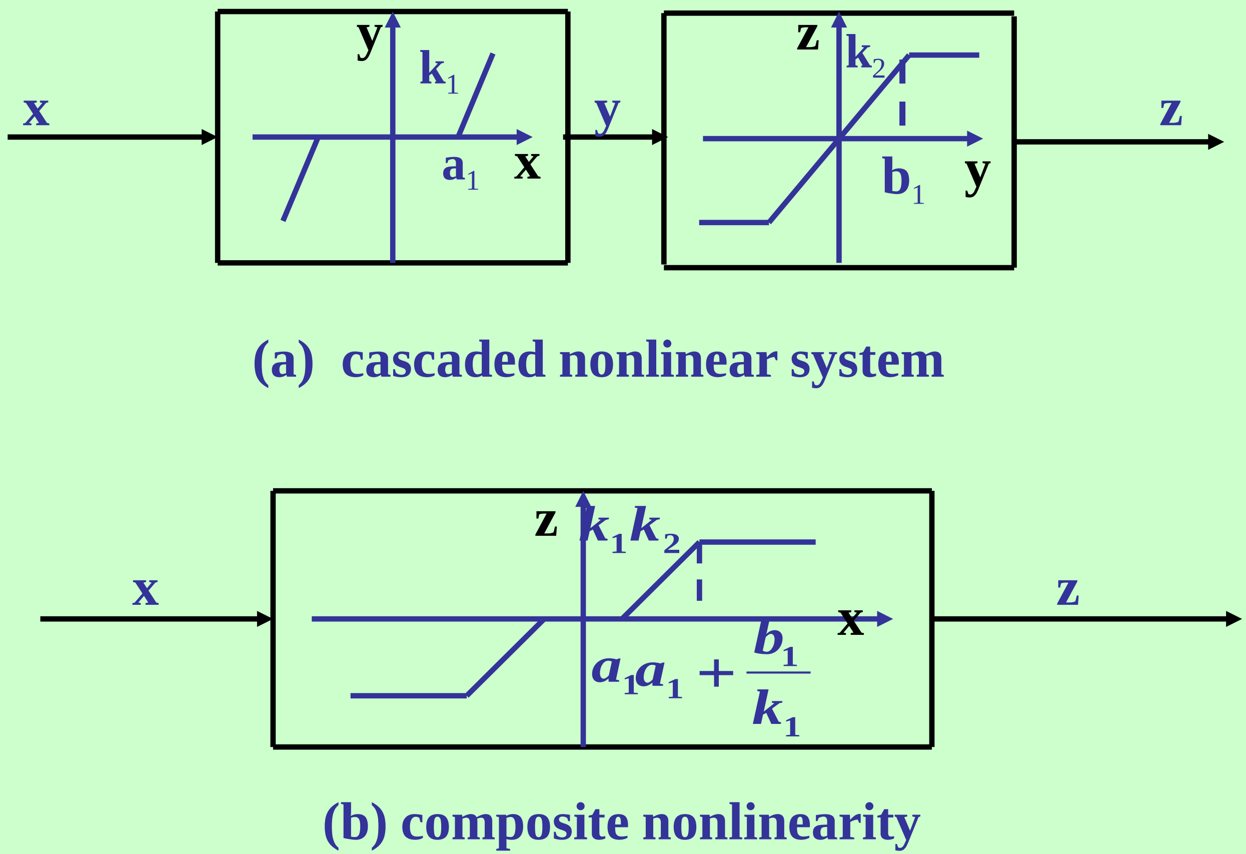


Fig 8-18 the cascaded nonlinear system and its composite nonlinearity

2. Parallel nonlinear system

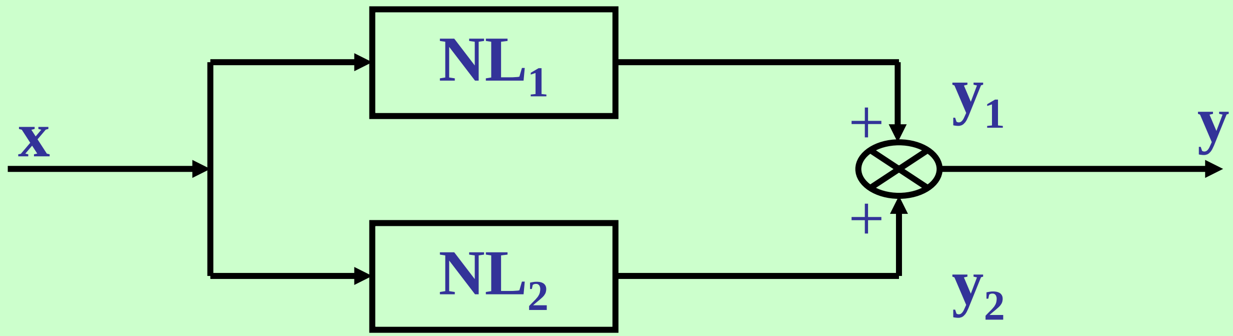


Fig 8-19 Parallel nonlinear system

According to the definition of describing function, the describing function N(A) of the output y and the input x is equal to the sum of two describing functions:

$$N(A) = N_1(A) + N_2(A)$$

[Example 1] **Obtain the describing function of the nonlinear system shown in the following figure.**

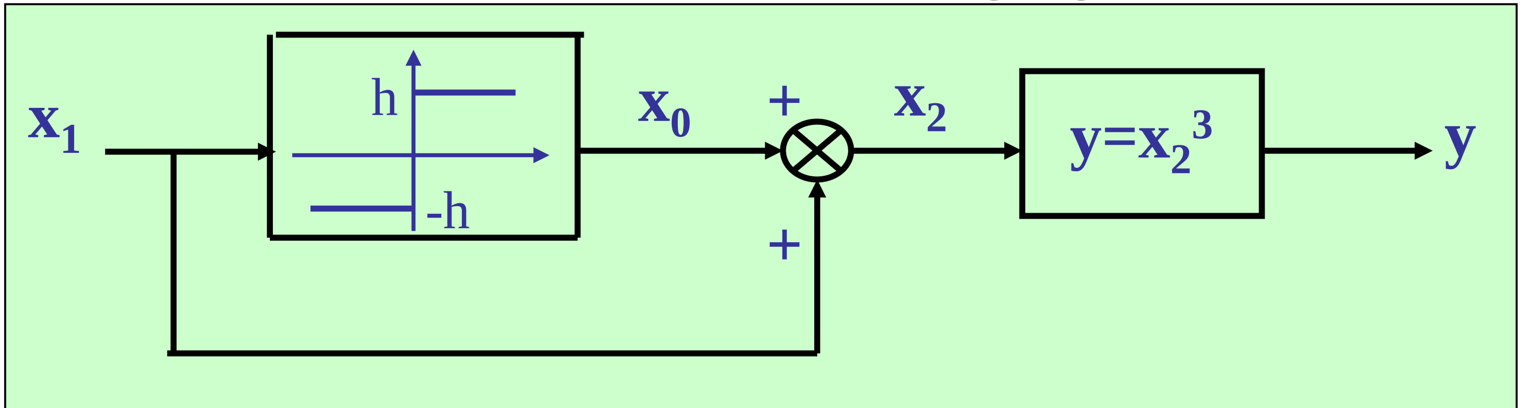


Fig 8-20 multiple nonlinearities

Solution: $y = x_2^3 = (x_0 + x_1)^3 = x_0^3 + 3x_0^2 x_1 + 3x_0 x_1^2 + x_1^3$

then $N(A) = N_1(A) + N_2(A) + N_3(A) + N_4(A)$

$assume\ x_1 = A\sin\omega t$

NL$_1$ is an ideal relay nonlinearity when a=0,

$\therefore\ A_1 = 0$

Obtain $N_1(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} h^3 \sin\omega t d(\omega t) = \frac{4h^3}{\pi}$$

$$\therefore \quad N_1(A) = \frac{B_1}{A} = \frac{4h^3}{\pi A}$$

Obtain $N_2(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} 3h^2 A \sin\omega t \cdot \sin\omega t d(\omega t) = 3h^2 A$$

$$\therefore \quad N_2(A) = 3h^2$$

Obtain $N_3(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} 3hA^2 \sin^2 \omega t \cdot \sin \omega t d(\omega t) \; = \frac{8hA^2}{\pi}$$

$$\therefore \quad N_3(A) = \frac{8hA}{\pi}$$

Obtain $N_4(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} A^3 \sin^3 \omega t \cdot \sin \omega t d(\omega t) \qquad \text{suppose} \;\; \theta = \omega t$$

$$= \frac{2}{\pi}\int_0^{\pi} - A^3 \sin^3 \theta \cdot d(\cos\theta)$$

$$= \frac{2A^3}{\pi}\left[\left(-\sin^3\theta\cos\theta\right)_0^{\pi} + \int_0^{\pi} 3\sin^2\theta\cos^2\theta d\theta\right] = \frac{3}{4}A^3$$

$$\therefore \quad N_4(A) = \frac{3}{4} A^2$$

Then, the describing function of the multiple nonlinearity is

$$N(A) = \frac{4h^3}{\pi A} + 3h^2 + \frac{8hA}{\pi} + \frac{3}{4} A^2$$

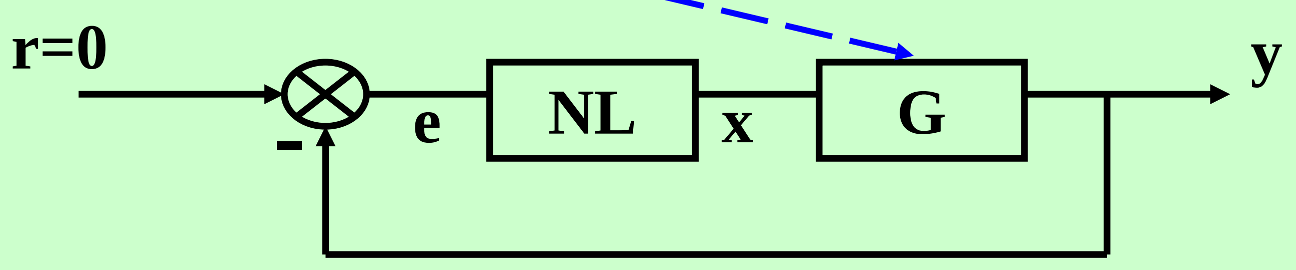
8.3.4 Analyze nonlinear system with describing function method

$$N(A) = \frac{\textit{Fundamental component} \text{ of the output } x(t)}{\text{sinusoidal input } e(t)}$$

It can only reflect *partial dynamic characteristics* of the system.

If a *self-excited oscillation* occurs in the system, any input $x(t)$ can be regarded as a *sinusoidal* signal because the linear part is a *low-pass filter*.

The describing function method is then *applicable*.



Typical structure of nonlinear system

Assume the input of a nonlinear system is zero, N(A) is the describing function of a nonlinear element, analyze the *condition for self-exited oscillation*.

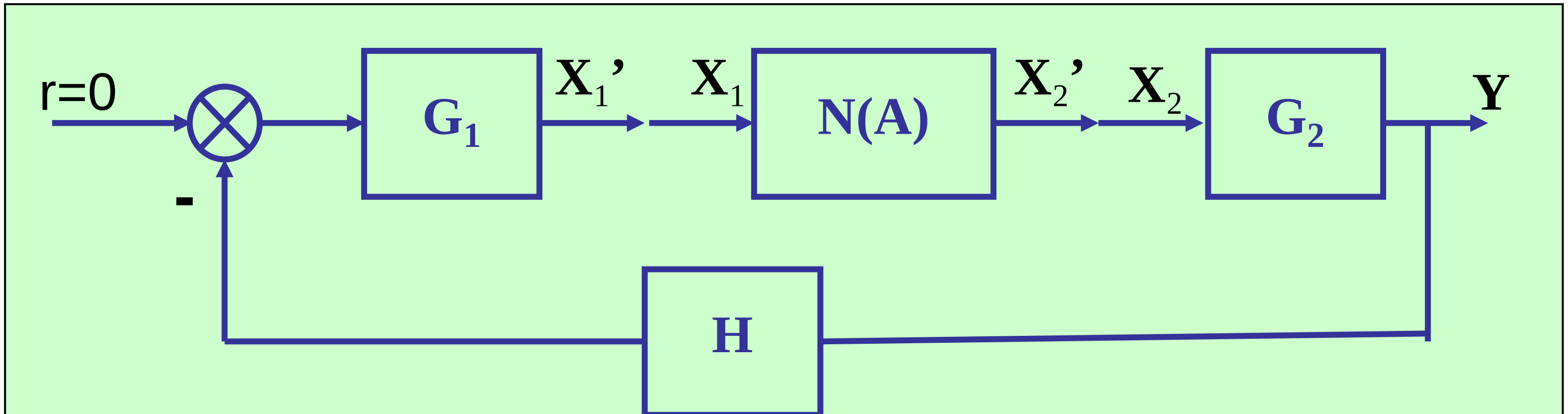


Fig. 8-21 nonlinear system

Assume $X_2 = A_2 \sin \omega t$

Then $X_1' = -\left|G_1(j\omega)G_2(j\omega)H(j\omega)\right| A_2 \sin(\omega t + \theta)$

where: $\theta = \angle G_1(j\omega) + \angle G_2(j\omega) + \angle H(j\omega)$

Assume: $N(A) = |N(A)|e^{j\phi}$

then $x_2'(t) = -|N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)|A_2 \sin(\omega t + \theta + \phi)$

Note: $x_2'(t) = x_2(t)$ $\Longleftrightarrow$ **The self-oscillation occurs.**

Considering we have $x_2 = A_2 \sin \omega t$ **,**

the *condition of self-oscillation* is

$$\begin{cases} |N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)| = 1 \\ \theta + \phi = (2n+1)\pi \end{cases}$$

$$\begin{cases} |N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)| = 1 \\ \theta + \phi = (2n+1)\pi \end{cases}$$

Suppose the transfer function of linear part satisfies

$$G(s) = G_1(s)G_2(s)H(s)$$

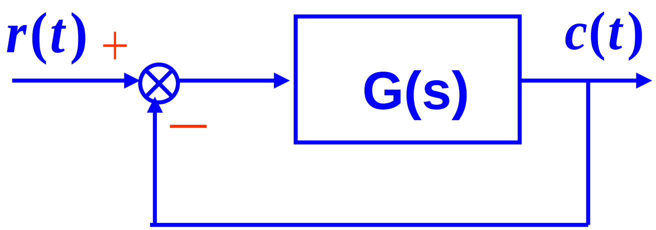
We can conclude that the ***condition of self-excited oscillation*** **is**

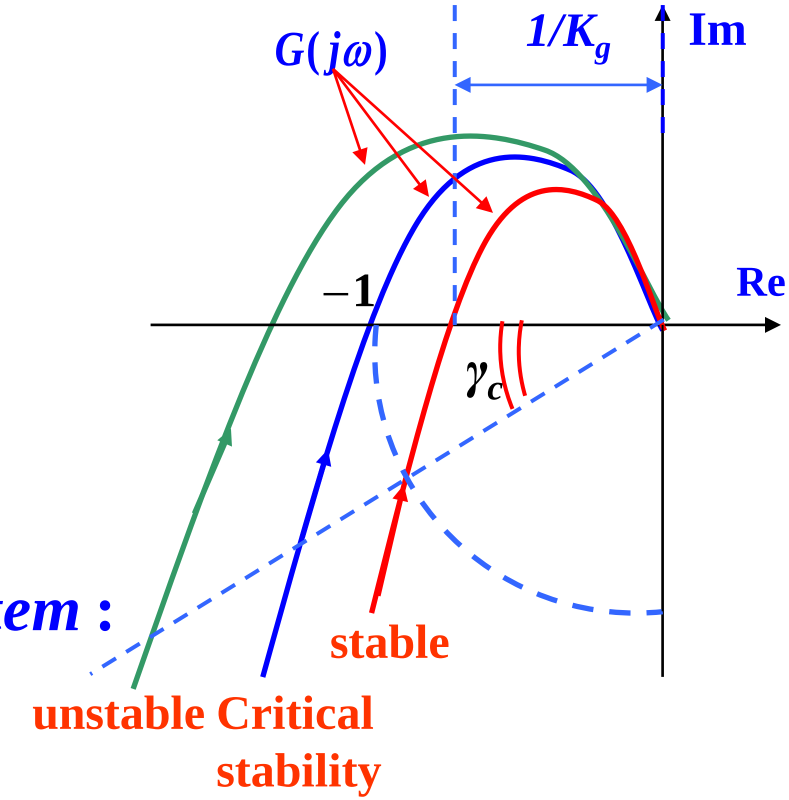
$$G(j\omega) = -\frac{1}{N(A)}$$

$$\text{or} \quad 1 + N(A)G(j\omega) = 0$$

Review of Nyquist criterion

For the linear system:





The characteristic equation of the system :

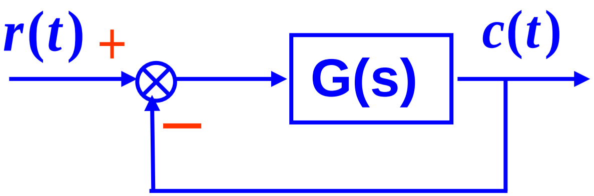
$$1+G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1+j0$$

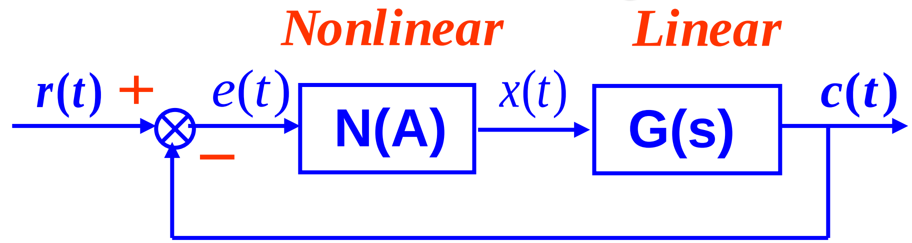
If $G(s)$ is a minimum phase transfer function, the necessary and sufficient condition of the stable system is:

$G(j\omega)$ *does not circle the point* $(-1, j0)$

Compare the nonlinear system with the linear system



Linear system



nonlinear system

Transfer function of the system:

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}$$

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{N(A)G(j\omega)}{1+N(A)G(j\omega)}$$

Characteristic equation:

$$1+G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1$$

In the $G(j\omega)$ plane

A point

$$1+N(A)G(j\omega)=0$$

$$\Rightarrow \ G(j\omega)=-\frac{1}{N(A)}$$

A curve

Because the describing function N(A) actually is a linearized frequency response, we can expand the Nyquist criterion to the nonliear system:

Stability analysis of the nonlinear system

(For example the minimum phase system)

compare with linear system

(1) $G(j\omega)$ don't circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear system is stable;

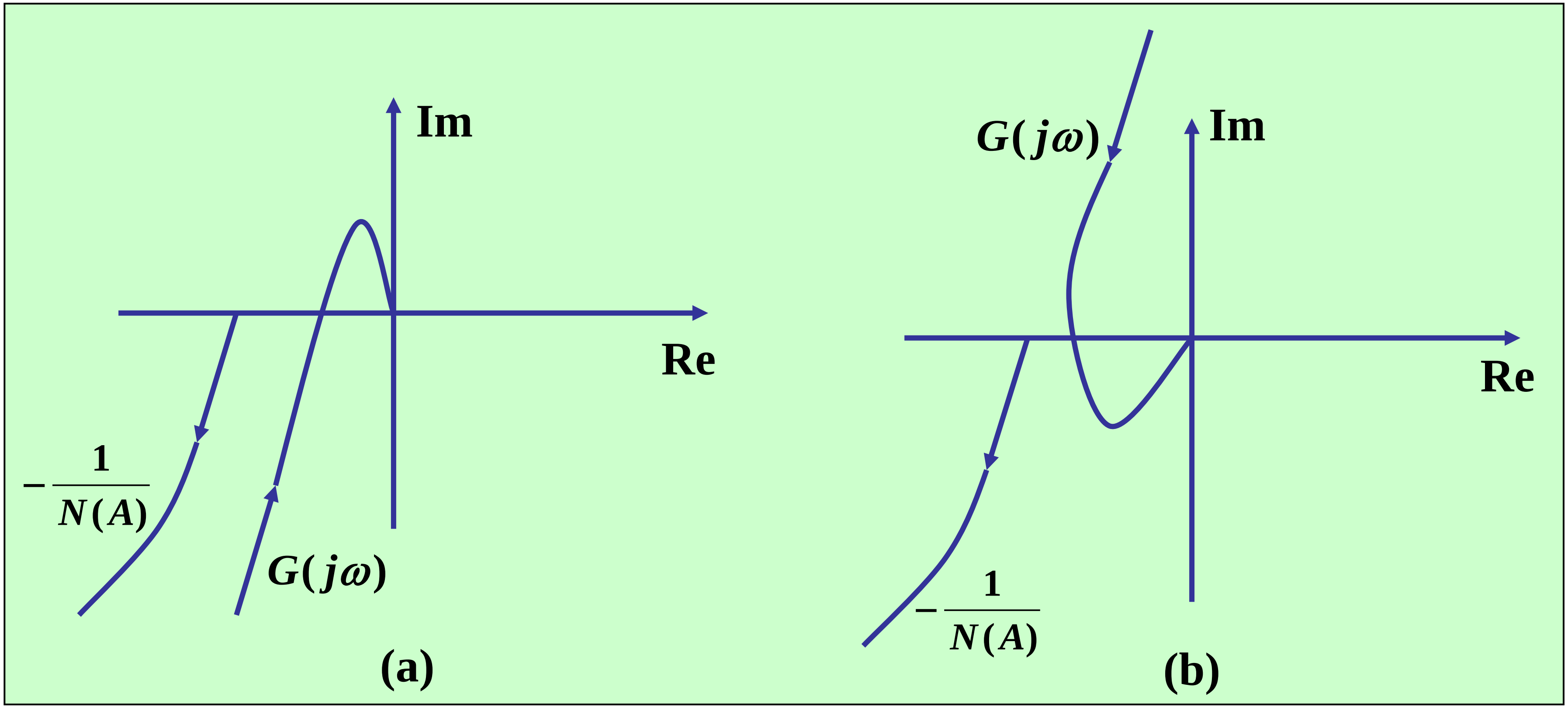
(2) $G(j\omega)$ circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear system is unstable;

(3) $G(j\omega)$ intersect with the $-\dfrac{1}{N(A)}$ curve, there will be a self-oscillation in the nonlinear system .

(1) $G(j\omega)$ don't circle the point $(-1, j0)$, the system is stable;

(2) $G(j\omega)$ circle the point $(-1, j0)$, the system is unstable;

(3) $G(j\omega)$ intersect with the point $(-1, j0)$, the system is critically stable.

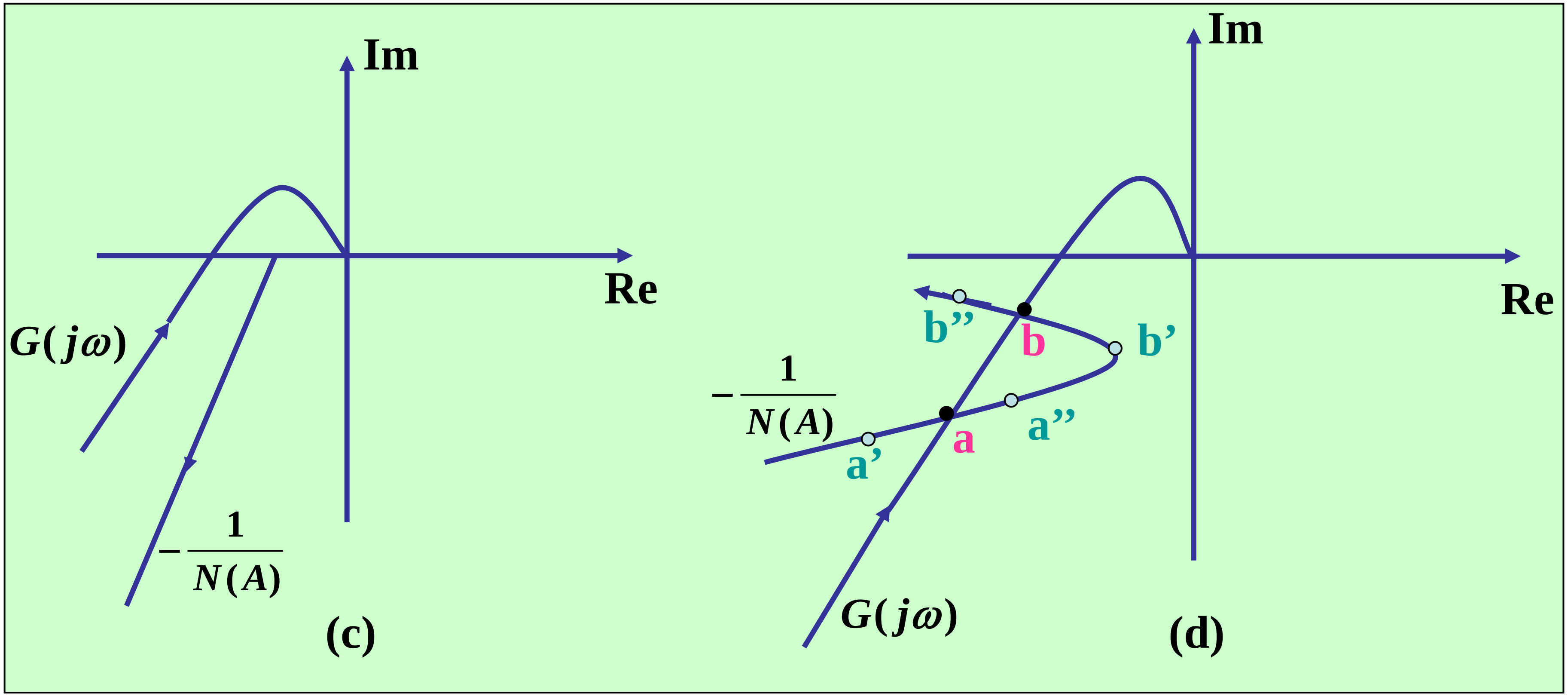


$G(j\omega)$ **do not circle —1/N(A)**

(Stable)

$G(j\omega)$ **circle —1/N(A)**

(Unstable)



Unstable

$G(j\omega)$ **intersect with** $-1/N(A)$

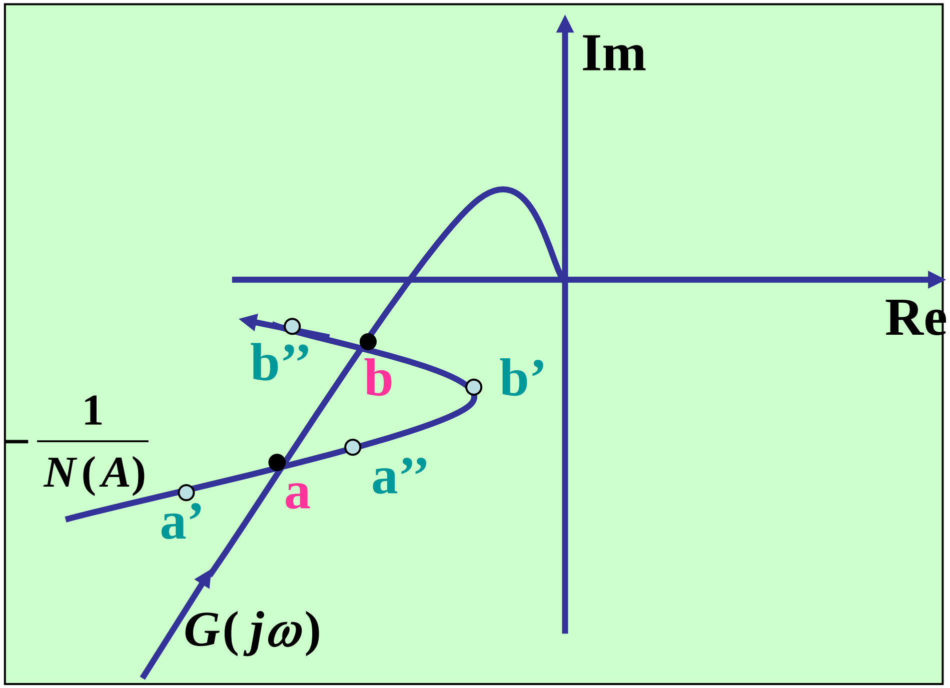
(Self-oscillation)

Analysis:

(1) Self-oscillation occurs at point a and point b.

(2) The self-oscillation at point a is unstable while the self-oscillation at point b is stable.

(3) There is only one stable self-oscillation in an actual physical system.



The ***amplitude*** and ***frequency*** of the self-oscillation can be obtained by solving

$$\left|G(j\omega)N(A)\right| = 1$$

$$\theta + \phi = -\pi$$

[Example 2] A relay control system structure is shown in Fig. 8-23. Suppose a = 1, b = 3.

(1) Is there a self-oscillation in the system? If there is, obtain the amplitude and frequency of the oscillation.

(2) How to adjust the parameters if you want to eliminate the self-oscillation?

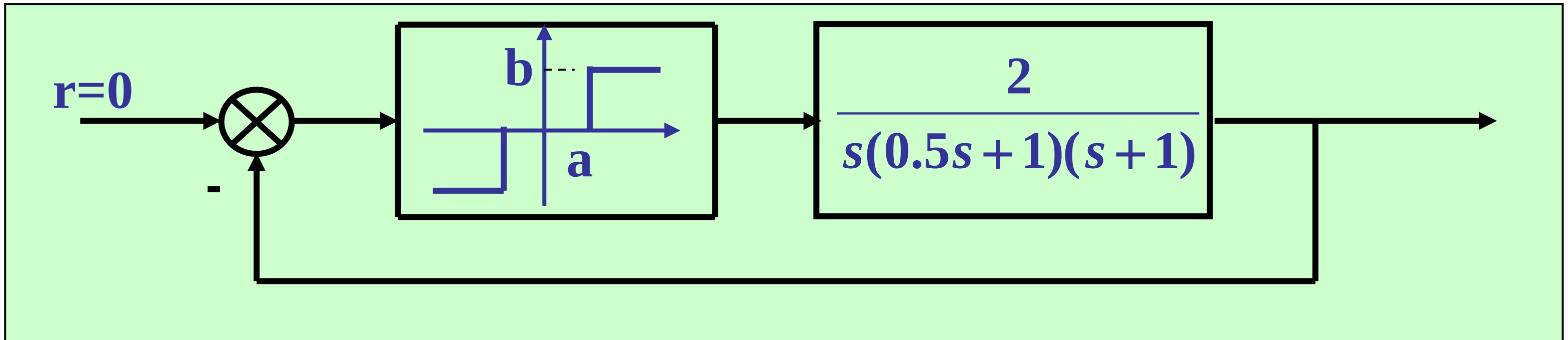


Fig.8-23 relay control sytem

Solution: the describing function of a relay nonlinearity with dead zone is

$$N(A) = \frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$

$$\therefore\ -\frac{1}{N(A)} = -\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}$$

$$when \qquad A = a, -\frac{1}{N(A)} \to -\infty$$

$$when \qquad A \to \infty, -\frac{1}{N(A)} \to -\infty$$

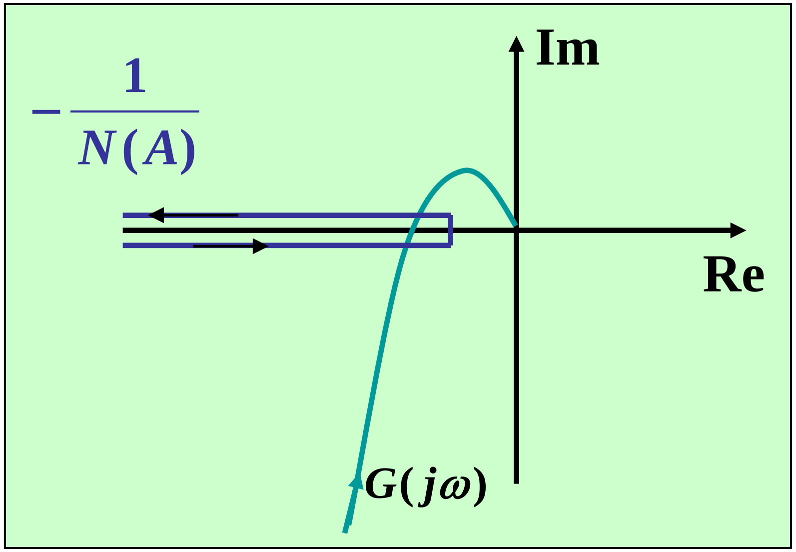
There is an *extreme value* of function $-\frac{1}{N(A)}$ **on the real axis.**

$$\frac{d}{dA}\left(-\frac{1}{N(A)}\right) = 0 \Rightarrow 1-2\left(\frac{a}{A}\right)^2 = 0 \qquad \therefore\ A = \sqrt{2}a$$

Substitute a = 1, b = 3 into above equations, we have

$$A = \sqrt{2}$$

$$-\left.\frac{1}{N(A)}\right|_{A=\sqrt{2}} = -\frac{\pi}{6} \approx -0.52$$



$$G(s) = \frac{2}{s(0.5s+1)(s+1)}$$

$$G(j\omega) = -\frac{3\omega}{\omega(0.25\omega^4+1.25\omega^2+1)} - j\frac{2(1-0.5\omega^2)}{\omega(0.25\omega^4+1.25\omega^2+1)}$$

Set the imaginary part to zero, we have $\omega = \sqrt{2}$

Substituting $\omega = \sqrt{2}$ **into the real part, we have**

$$\mathrm{Re}\,G(j\omega)\Big|_{\omega=\sqrt{2}} = -\frac{1}{1.5} \approx -0.66$$

Let $$-\frac{1}{N(A)}=\frac{-\pi A}{12\sqrt{1-\left(\frac{1}{A}\right)^{2}}}=-\frac{1}{1.5}$$

We can obtain two amplitude: $A_1 = 1.11$, $A_2 = 2.3$

(not exist in the reality)

There is a self-oscillation in the system with the amplitude 2.3 and the frequency $\sqrt{2}$.

(2) To eliminate the self-oscillation, let

$$-\frac{1}{N(A)} = -\left.\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}\right|_{A=\sqrt{2}a} \leq -\frac{1}{1.5}$$

We can get the ratio of relay parameters

$$\frac{b}{a} \leq \frac{1.5\pi}{2} \approx 2.36$$

Adjust the ratio of a and b to b = 2a, we can eliminate the self-oscillation.

[Example 3] **A multi-loop control system is shown in Fig. 8-25. If $G_1(s)=1$, the natural oscillation frequency and damping ratio of system is $\omega_n = 2$ and $\varsigma = 1$ when working on the linear area of the saturation. If we suppose $G_1(s) = 1 + \dfrac{1}{8s}$, try to find the minimal ratio T_1/T_2 when the system is stable.**

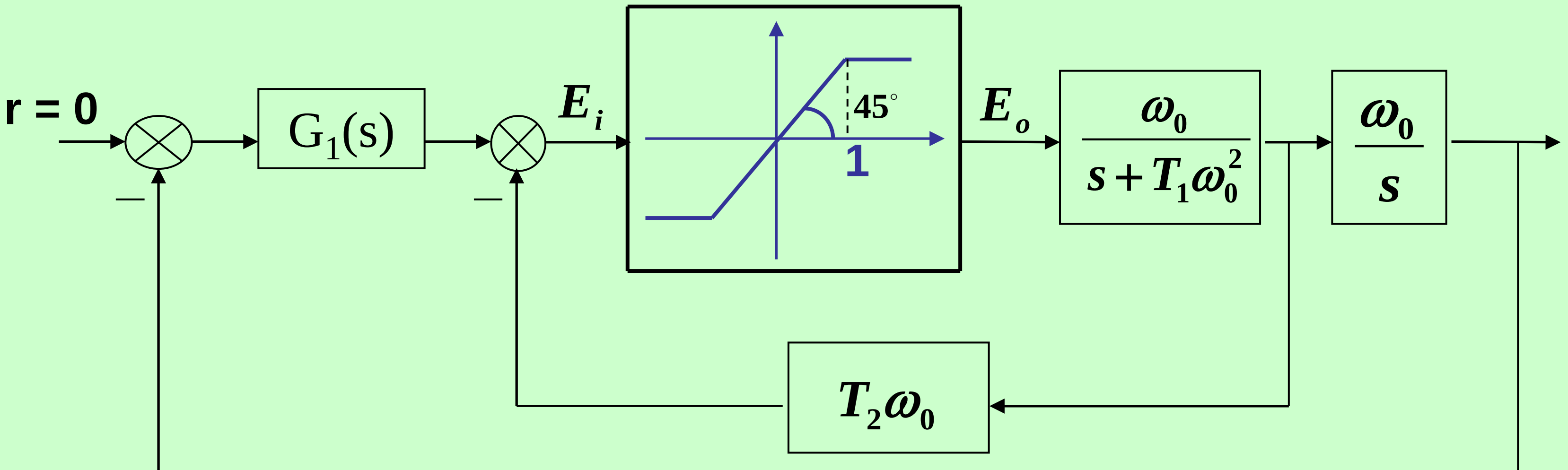


Fig.8-25 multi-loop control system

Solution: When $G_1(s) = 1$, the closed-loop transfer function of the inner loop is:

$$G_{内}(s)=\frac{\dfrac{\omega_0}{s+T_1\omega_0^2}}{1+\dfrac{T_2\omega_0^2}{s+T_1\omega_0^2}}=\frac{\omega_0}{s+(T_1+T_2)\omega_0^2}$$

The closed-loop transfer function of the whole system is

$$G_B(s)=\frac{\dfrac{\omega_0^2}{s[s+(T_1+T_2)\omega_0^2]}}{1+\dfrac{\omega_0^2}{s[s+(T_1+T_2)\omega_0^2]}}=\frac{\omega_0^2}{s^2+(T_1+T_2)\omega_0^2 s+\omega_0^2}$$

$$\therefore\ \begin{cases}\omega_0=\omega_n=2\\ T_1+T_2=\varsigma=1\end{cases}$$

when $G_1(s) = 1 + \dfrac{1}{8s}$, the transfer function of inner loop is

$$G_{内}(s) = \frac{\dfrac{\omega_0}{s + T_1\omega_0^2} N(A)}{1 + \dfrac{T_2\omega_0^2}{s + T_1\omega_0^2} N(A)} = \frac{\omega_0 N(A)}{s + T_1\omega_0^2 + T_2\omega_0^2 N(A)}$$

The open-loop transfer function of the whole system

$$G(s) = G_1(s) G_{内}(s) \frac{\omega_0}{s} = \frac{\omega_0^2 (1 + \dfrac{1}{8s}) N(A)}{s[s + T_1\omega_0^2 + T_2\omega_0^2 N(A)]}$$

From the characteristic equation of closed-loop system $1 + G(s) = 0$, we have

$$s^2 + T_1\omega_0^2 s + T_2\omega_0^2 s N(A) + \omega_0^2 (1 + \frac{1}{8s}) N(A) = 0$$

Substituting $\omega_0 = 2$ to the above equation, we can obtain

$$8s^3 + 32T_1 s^2 + (32T_2 s^2 + 32s + 4) N(A) = 0$$

$$\therefore \quad -\frac{1}{N(A)} = \frac{8T_2 s^2 + 8s + 1}{2s^3 + 8T_1 s^2} = \frac{8T_2 s^2 + 8s + 1}{s^2(2s + 8T_1)}$$

Note that the describing function of saturation nonlinearity is

$$N(A) = \frac{2}{\pi}\left[\sin^{-1}\frac{1}{A} + \frac{1}{A}\sqrt{1 - \left(\frac{1}{A}\right)^2}\,\right]$$

So the function of $-\dfrac{1}{N(A)}$ is given by

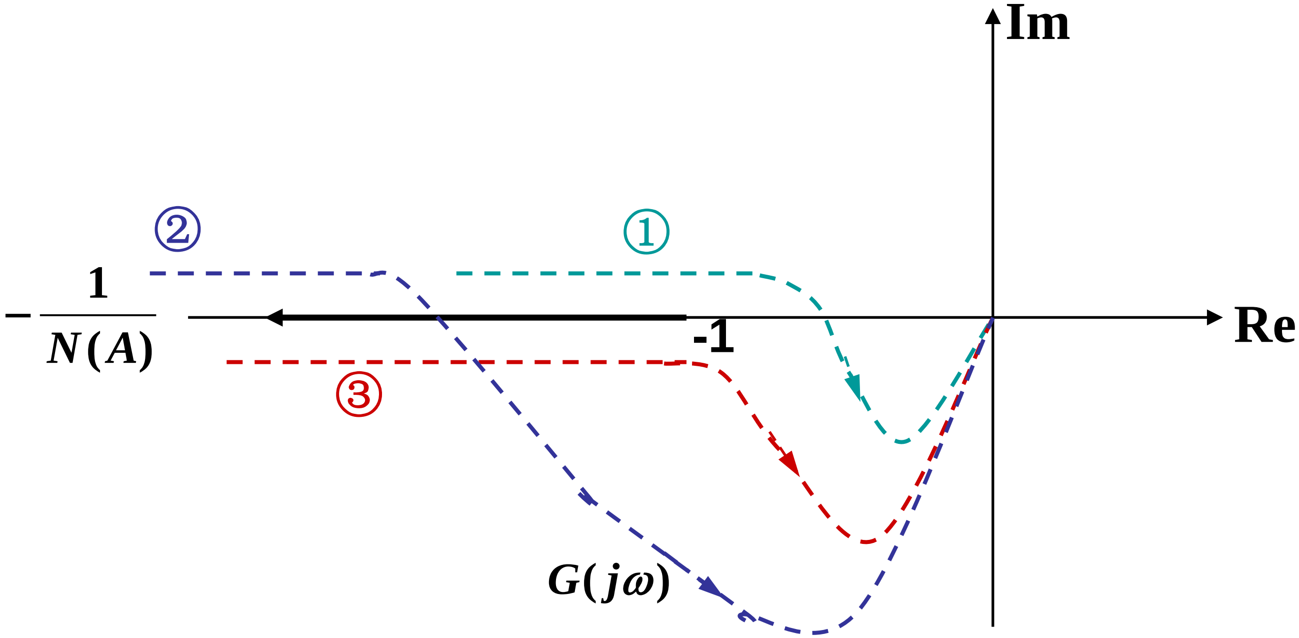
$$-\frac{1}{N(A)} = -\frac{\pi}{2}\frac{1}{\left[\sin^{-1}\frac{1}{A}+\frac{1}{A}\sqrt{1-\left(\frac{1}{A}\right)^2}\right]}$$

The open-loop transfer function of the whole system is

$$G(s) = -\frac{1}{N(A)} = \frac{8T_2s^2+8s+1}{s^2(2s+8T_1)}$$

$$\therefore \ G(j\omega) = \frac{-8T_2\omega^2+j8\omega+1}{-\omega^2(j2\omega+8T_1)} = u(\omega)+jv(\omega)$$

$$G(j\omega) = -\frac{4T_1 - 32\omega^2 T_1 T_2 + 8\omega^2}{2\omega^2(\omega^2 + 16T_1^2)} + j\frac{32T_1 - 1 + 8T_2\omega^2}{2\omega(\omega^2 + 16T_1^2)}$$



To guarantee the system to be stable, we should choose parameters T_1 and T_2 which make $G(j\omega)$ has no intersection with the negative real axis. So we choose curve ③

Let $v(\omega) = \dfrac{32T_1 - 1 + 8T_2\omega^2}{2\omega(\omega^2 + 16T_1^2)} = 0$

Then we have $\omega = \sqrt{\dfrac{1 - 32T_1}{8T_2}}$

If $T_1 > \dfrac{1}{32}$ **holds,** $G(j\omega)$ **has no intersection with the negative real axis.**

On the other hand, $T_1 + T_2 = 1$

$$\therefore \; T_2 < \frac{31}{32} \quad \Rightarrow \quad \therefore \; \frac{T_1}{T_2} \geq \frac{1}{31}$$

So, the minimal ratio of $\dfrac{T_1}{T_2}$ **to guarantee the system to be stable is** $\dfrac{T_1}{T_2} = \dfrac{1}{31}$

§ **8.4** Phase Plane Method

- Phase plane method was first proposed in 1885 by Poincare . It is a graphical method for studying first-order, second-order systems.
- *The essence of this method is visually transforming the motion process of the system into the motion of a point in the phase plane.*
- We can obtain all information regarding the motion patterns of system by studying the motion trajectory of this point. Now this method is widely used, because it can intuitively, accurately and comprehensively character the motion states of the system.

The Role of The Phase Plane Method

- The phase plane method can be used for analyzing the ***stability, equilibrium position*** and ***steady-state*** accuracy of the first-order, second-order linear systems or nonlinear systems.

- It also can be used for analyzing the impact on the system motion of ***initial conditions*** and ***parameters*** of this system.

- When the nonlinearity of system is serious or we can not use the describing function method while there are some ***non-periodic*** inputs, the phase plane method is still available for these problems.

Basic Concepts of The Phase Plane Method

(1) Phase plane and Phase portrait

Phase plane:

The x_1-x_2 plane is called Phase Plane, Where x_1, x_2 are the system state and its derivative (c, $\dot{c}$).

Phase portrait (相图):

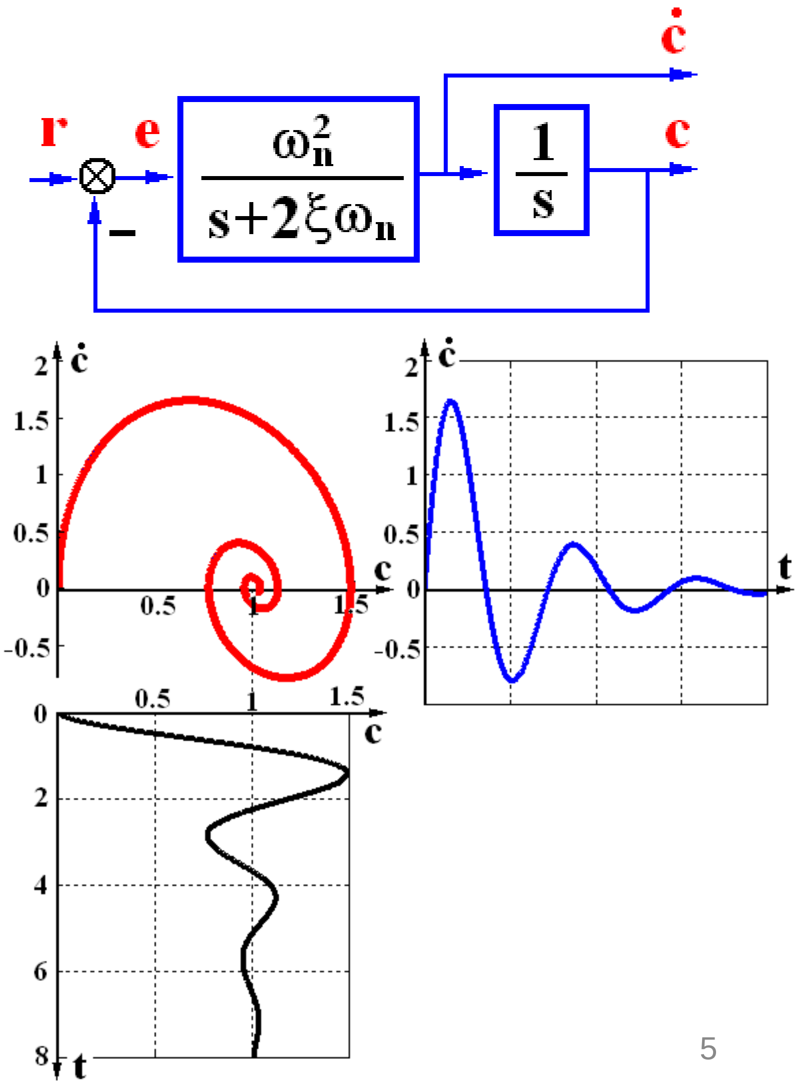
The locus in the x_1-x_2 plane of the solution x(t) for all t>= 0 is a curve named ***trajectory*** or ***orbit*** that passes through the point x0.

The family of phase plane trajectories corresponding to various initial conditions is called ***Phase Portrait*** of the system.

Example 1: A unit feedback system

$$G(s) = \frac{5}{s(s+1)} \quad \begin{cases} \omega_n = 2.236 \\ \xi = 0.2236 \end{cases}$$

$$r(t) = 1(t)$$



Basic Concepts of The Phase Plane Method

(2) The features of phase trajectory

System equation: $\ddot{x} + f(x, \dot{x}) = 0$

Motion Direction

$$\left\{\begin{array}{l}\text{Upper half-plane } \dot{x} > 0 \text{ — move right} \\ \text{lower half-plane } \dot{x} < 0 \text{ — move left}\end{array}\right\}$$

clockwise motion

Passing the X-axis $(\dot{x} = 0)$ **perpendicularly.**

Singular Points (Equilibrium Points)

$$\alpha = \frac{d\dot{x}}{dx} = \frac{d\dot{x}/dt}{dx/dt} = \frac{-f(x,\dot{x})}{\dot{x}} = \frac{0}{0} \Rightarrow \begin{cases}\ddot{x} = 0 \\ \dot{x} = 0\end{cases}$$

For the linear time-invariant system, there is only one equilibrium point.

The equilibrium point is the original point? How about Nonlinear system?

Basic Concepts of The Phase Plane Method

(2) The features of phase trajectory

Except the equilibrium points, there is only one phase trajectory passing through any point in the phase plane.

It is determined by the existence and uniqueness of solutions of differential equations.

Methods of Constructing Phase Plane Trajectories

- Analytical Method
- Isocline Method
- Experimental Method

Analytical Method

For an arbitrary second-order nonlinear differential equations $\ddot{x} + f(x, \dot{x}) = 0$

Or $\ddot{x} + a_1(x, \dot{x})\dot{x} + a_0(x, \dot{x})x = 0$

Let $x_1 = x$

$x_2 = \dot{x}_1 = \dot{x}$

Then: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} = -a_1(x_1, x_2)x_2 - a_0(x_1, x_2)x_1 \end{cases}$

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{-a_1(x_1, x_2)x_2 - a_0(x_1, x_2)x_1}{x_2}$$

Rewrite the system equations in a general form:

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) \\ \dot{x}_2 = Q(x_1, x_2) \end{cases}$$

$$\frac{dx_2}{dx_1} = \frac{Q(x_1, x_2)}{P(x_1, x_2)}$$

→ the slope of the trajectory at point (x_1, x_2)

If $P(x_1, x_2)$, $Q(x_1, x_2)$ is analytic, the differential equation can then be solved. Given a initial condition, the solution can be plotted in the phase plane. This curve is named Phase trajectory. The family of phase plane trajectories is called Phase portrait.

Assume

$$\begin{cases} P(x_1, x_2) = 0 \\ Q(x_1, x_2) = 0 \end{cases}$$

The solutions (x_{10}, x_{20}) is called the equilibrium points of the system.

Note:
The "***time***" is ***eliminated*** here → The responses $\boldsymbol{x_1(t)}$ and $\boldsymbol{x_2(t)}$ cannot be obtained directly.

Only ***qualitative behavior*** can be concluded, such as ***stability*** or ***oscillatory response***.

The purpose of plotting the phase trajectory is to analyze the dynamic characteristics.

Because there are infinite phase trajectories leaving or arriving at the ***equilibrium point*** the phase trajectories near the ***equilibrium point*** reflect the dynamic characteristics of the system.

Equilibrium points is also called ***singular points***.

Limit cycle is another phase trajectory which can reflect the dynamic characteristics of the system.

Limit cycle is an ***Isolated and Closed*** phase trajectory, which describes the harmonic oscillation of a system. It divides the infinite phase plane into two parts.

Singular Point and Limit Cycle

1. Singular Point

Singular Points are the equilibrium points (x_{10}, x_{20}) ,

which are obtained by solving the following equations.

$$\begin{cases} \dot{x}_1 = P(x_1, x_2) = 0 \\ \dot{x}_2 = Q(x_1, x_2) = 0 \end{cases}$$

The singular point can only appear on the X-axis.

To study the shape and dynamic characteristics of the phase trajectories near the equilibrium (x_{10}, x_{20}) , we expand the function $P(x_1, x_2)$, $Q(x_1, x_2)$ into Taylor series around it.

Ignoring the higher-order terms, without loss of generality we assume that

$$x_{10} = x_{20} = 0$$

then

$$P(x_1, x_2) = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

$$Q(x_1, x_2) = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} x_2$$

Assume $a = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)}$ $\quad b = \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$

$$c = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} \qquad d = \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)}$$

then
$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases}$$

the characteristic equation of system is given by

$$|\lambda I - A| = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

the roots of the above equation is

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}$$

According to the property of these roots, the singular points can be divided into the following classes:

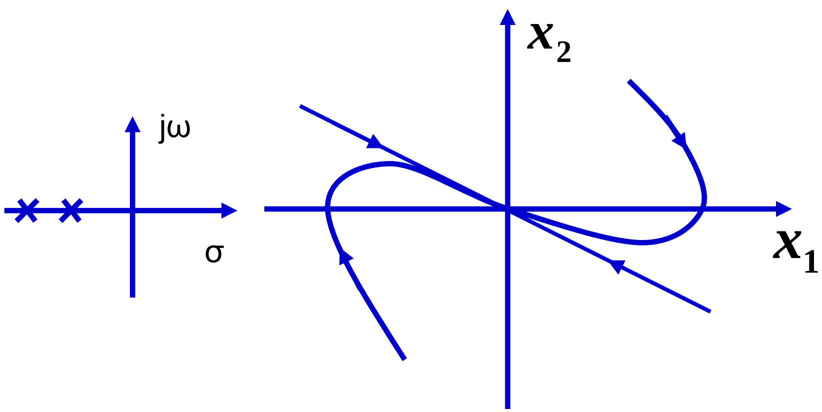
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

1) different real roots with the same sign

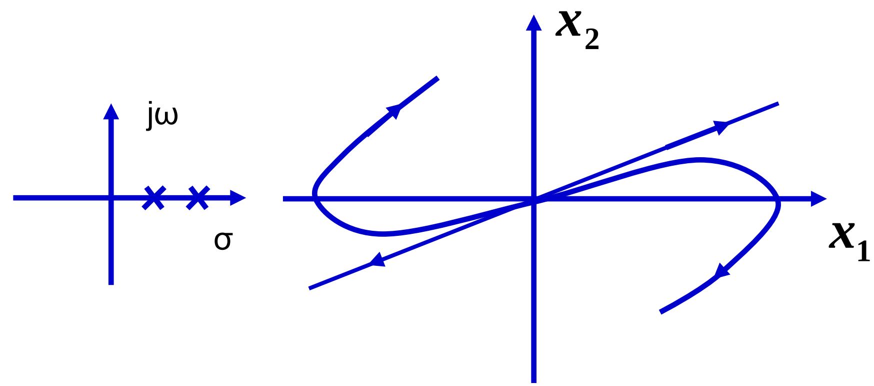
$$(a+d)^2 > 4(ad-bc)$$

If $a+d<0$, two roots are all negative, singular point is called stable node.

If $a+d>0$, two roots are all positive, singular point is called unstable node.



(a) stable node



（b） unstable node

Fig 8-28 Phase trajectory in this case

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

2) Real roots with different signs

$$ad - bc < 0$$

Singular point is called saddle point (鞍点)

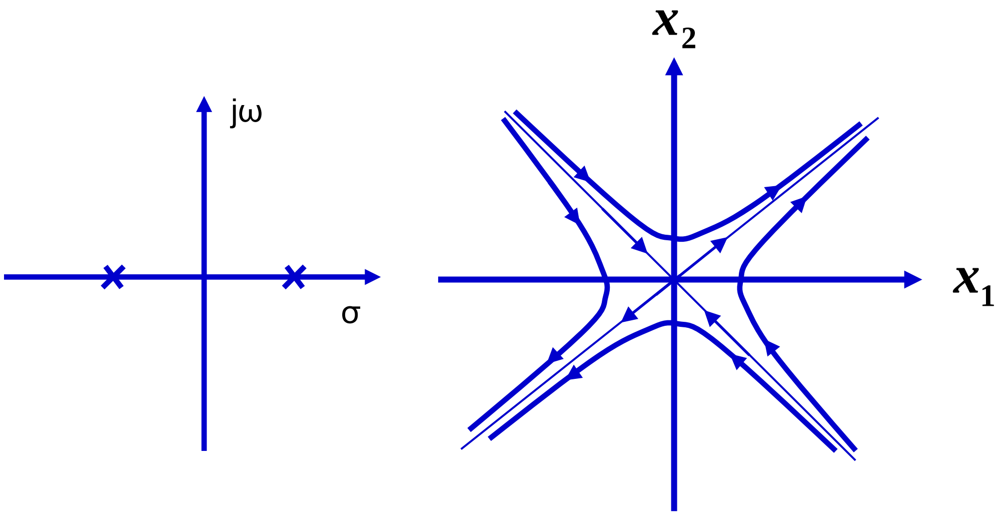


Fig 8-29 Phase trajectories that are corresponding to a saddle point

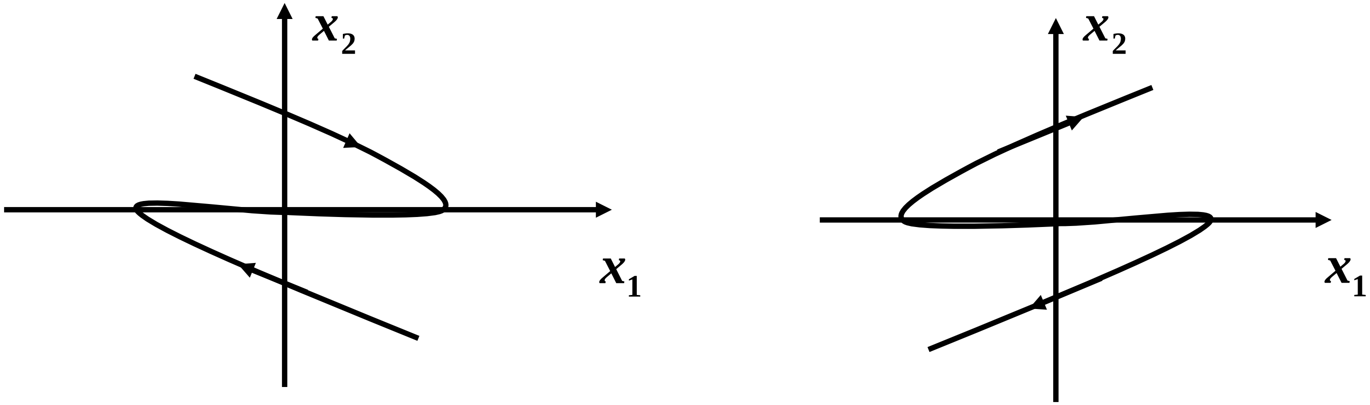
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

3) Double Root

$$(a+d)^2 = 4(ad-bc)$$

If $a+d<0$, there are two equal negative real roots . Singular point is called degraded stable node;

If $a+d>0$, there are two equal positive real roots . Singular point is called degraded unstable node;



(a) double negative roots **(b) double positive roots**

Fig 8-30 Phase trajectories that are corresponding to double point

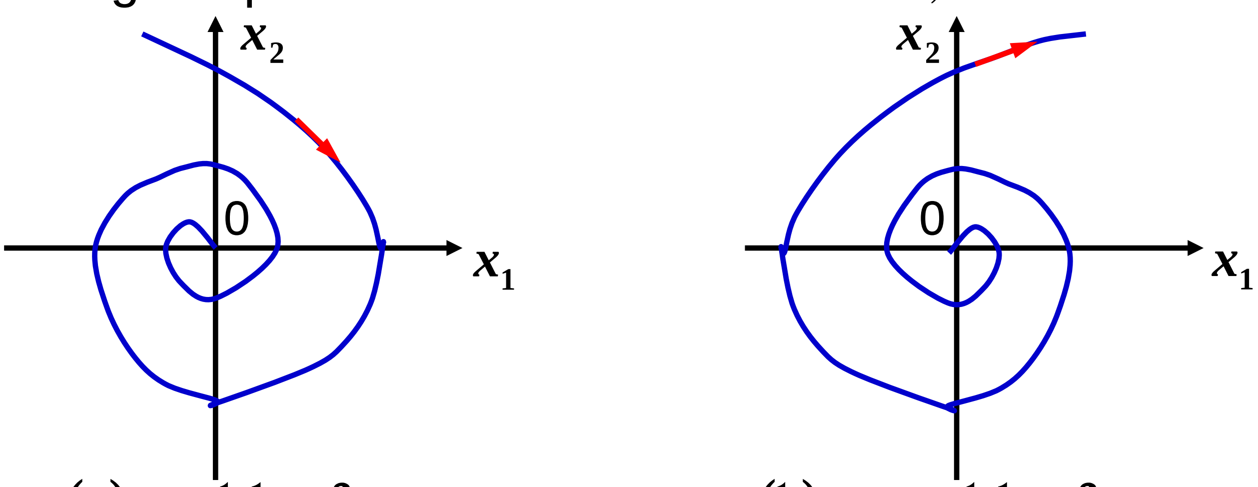
$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

4) Complex Conjugate Root

$$(a+d)^2 < 4(ad-bc)$$

If $a+d<0$, there are complex conjugate roots with negative real component. Singular point is called stable focus;

If $a+d>0$, there are complex conjugate roots with positive real component. Singular point is called unstable focus;



(a) stable focus (b) unstable focus

Fig 8-31 Phase trajectories that are corresponding to complex conjugate roots

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

5) Purely imaginary roots

$$(a+d) = 0, \quad ad - bc > 0$$

Singular point is called center.

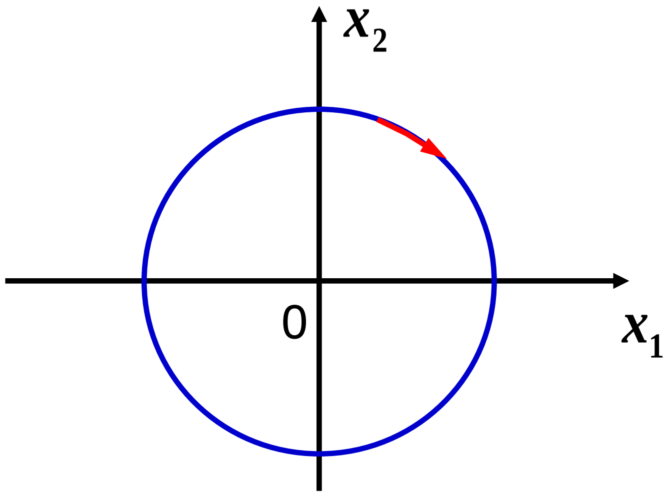


Fig 8-32 Phase trajectories that are corresponding to Purely imaginary roots

2. Limit Cycle

A closed and isolated phase trajectory in the phase plane is called a limit cycle. It is corresponding to the harmonic oscillation state of a system. Limit cycle divides the phase-plane into two parts: the part inside the limit cycle and the part outside the limit cycle. Any phase trajectory can not enter one part from the other.

Limit cycle can be easily found in actual physical systems. For example , the response of an unstable linear control system is theoretically a divergent oscillation. Whereas, in reality the amplitude of response may tends to a constant value due to the non-linear characteristics like saturation.

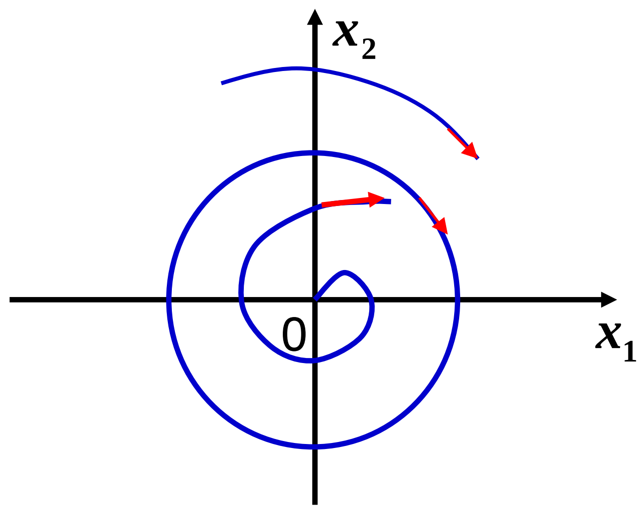
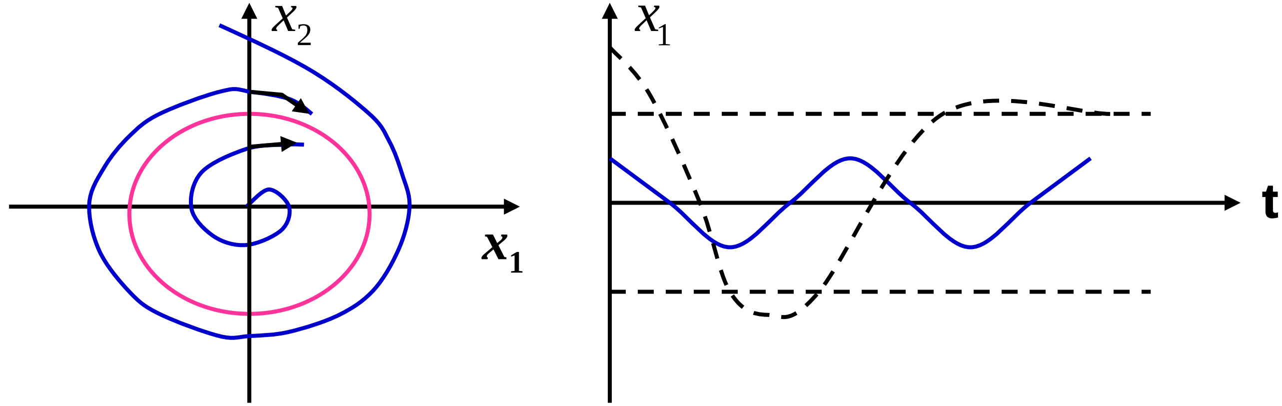
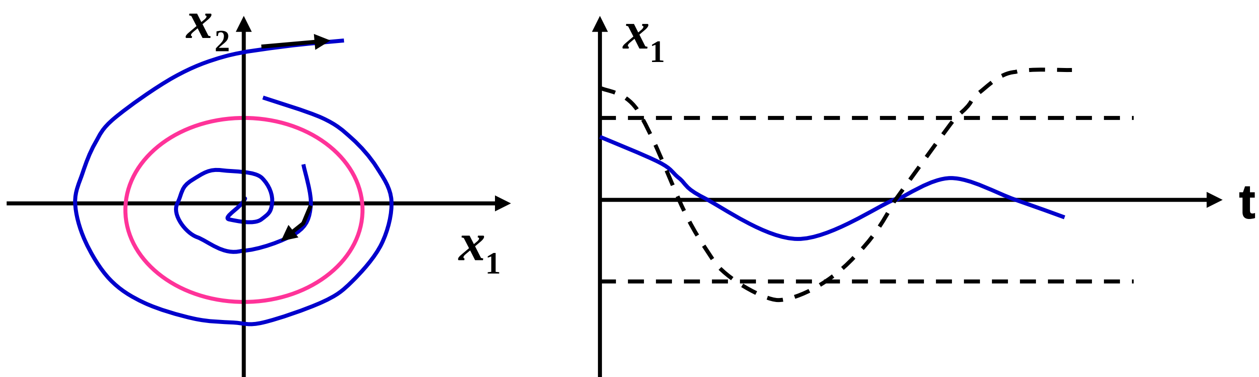


Fig 8-33 limit cycle

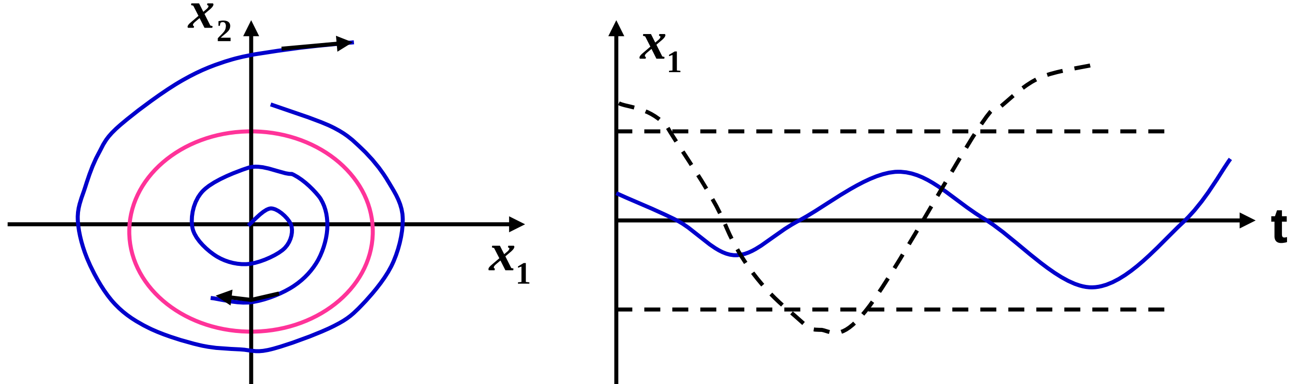
- Something should be pointed out, not all the closed curves in phase-plane are limit cycles. (Think about the trajectories corresponding to a center.) This kind of curves are not limit cycle, because they are not isolated.
- Limit cycle is a special phenomena which only exists in non-conservation systems. It is caused by nonlinearity of systems, not the non-damping feature of linear systems.



(a) Stable limit cycle



(b) Unstable limit cycle



(c) Semi-stable limit cycle

[Example 1] **The equation of a non-linear system is given by**

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

analyze the stability of this system.

Solution:

The Cartesian coordinate is transformed to the polar one as following:

Assume $x_1 = r\cos\theta, \quad x_2 = r\sin\theta$

Then $$\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta}$$

$$\dot{x}_2 = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}$$

Substituting the above equations into the system equations, we have

$$\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta} = r\sin\theta + r\cos\theta(1-r^2) \qquad \cdots(1)$$

$$\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta} = -r\cos\theta + r\sin\theta(1-r^2) \qquad \cdots(2)$$

It follows from (2) that

$$\dot{\theta} = \frac{-r\cos\theta + r\sin\theta(1-r^2) - \dot{r}\sin\theta}{r\cos\theta} \qquad \cdots(3)$$

Substituting (3) into (1), we have $\dot{r} = r(1-r^2)$

It follows from (1) that

$$\dot{r} = \frac{r\sin\theta + r\cos\theta(1-r^2) + r\sin\theta\cdot\dot{\theta}}{\cos\theta} \qquad \cdots(4)$$

Substituting (4) into (2), we have $\dot{\theta} = -1$

$$\therefore \quad \begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = -1 \end{cases}$$

There are two cases: $r = 0$ *and* $1 - r^2 = 0$

(1) $r = 0, \quad x_1 = 0, \quad x_2 = 0$ **is the singular point**

$$a = \left.\frac{\partial P(x_1, x_2)}{\partial x_1}\right|_{(0,0)} = 1 \qquad b = \left.\frac{\partial P(x_1, x_2)}{\partial x_2}\right|_{(0,0)} = 1$$

$$c = \left.\frac{\partial Q(x_1, x_2)}{\partial x_1}\right|_{(0,0)} = -1 \qquad d = \left.\frac{\partial Q(x_1, x_2)}{\partial x_2}\right|_{(0,0)} = 1$$

The roots of characteristic equation are:

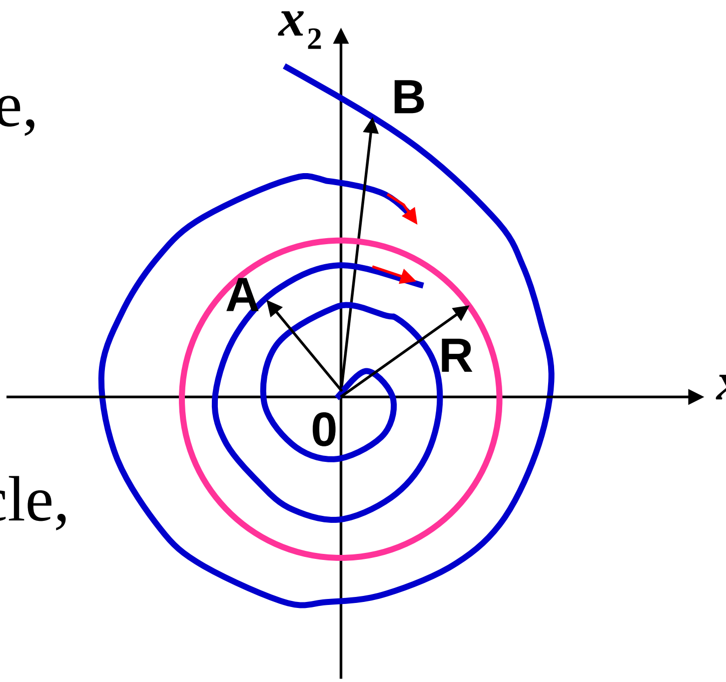
$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} = 1 \pm j$$

Singular point (0,0) is an ***unstable focus***, corresponding phase trajectories nearby are all divergent oscillations.

(2) $r = 1, x_1^2 + x_2^2 = 1$ **the unit cycle is limit cycle of systems.**

For an arbitrary point ***A*** inside the unit circle, inequality $\dot{r} = r(1 - r^2) > 0$ holds since we have ***OA***<***r***=1.

Then the phase trajectory crossing point ***A*** will finally approach to the unit circle.



For an arbitrary point ***B*** outside the unit circle, inequality $\dot{r} = r(1 - r^2) < 0$ holds since we have ***OB***>***r***=1.

Then the phase trajectory crossing point ***B*** will also finally approach to the unit circle.

$\therefore x_1^2 + x_2^2 = 1$ **is a stable limit cycle. Equilibrium (0, 0) is an unstable focus.**

3. Time domain analysis of typical nonlinear systems using Phase Plane Analysis

Algorithm of phase plane analysis:

1. Divide the phase plane into ***several areas*** according to nonlinear characteristics. Establish ***linear differential equations*** for each area.

2. Select ***appropriate coordinate axis*** during the analysis.

3. Establishing equations for the ***switching lines*** in the phase plane according to different nonlinear characteristics.

4. ***Solve the differential equations*** of each area and then draw phase trajectory.

5. The phase trajectory of the whole system can be obtained by ***connecting all the partial trajectories*** in different areas.

[Example 2] The following nonlinear system is excited with a step input signal of amplitude 6. If the initial state of the system is $e(0)=6,\ \dot{e}(0)=0$, how many seconds will it take for the system state to reach the origin.

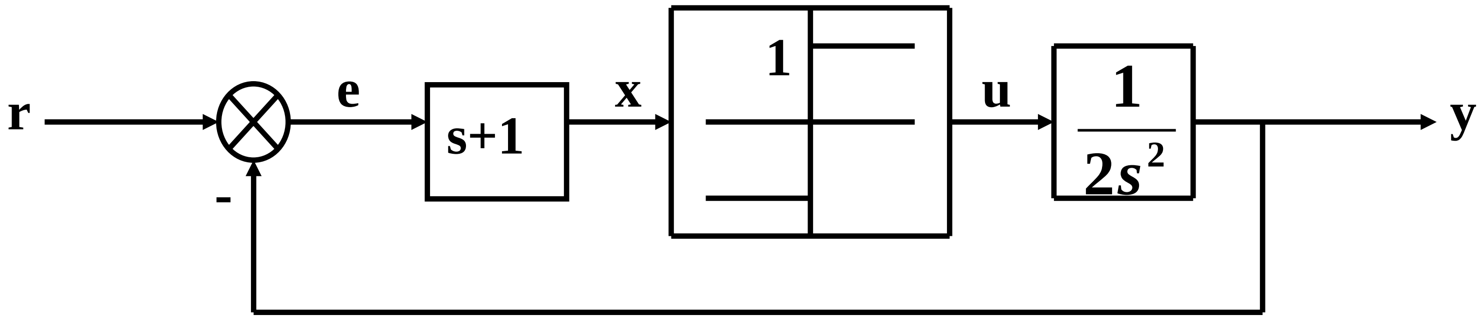


Fig 8-36 control system with a relay module

Solution: The dynamic equation is: $2\ddot{y}=u$

$$u=\begin{cases}1 & x>0 \\ -1 & x<0\end{cases}$$

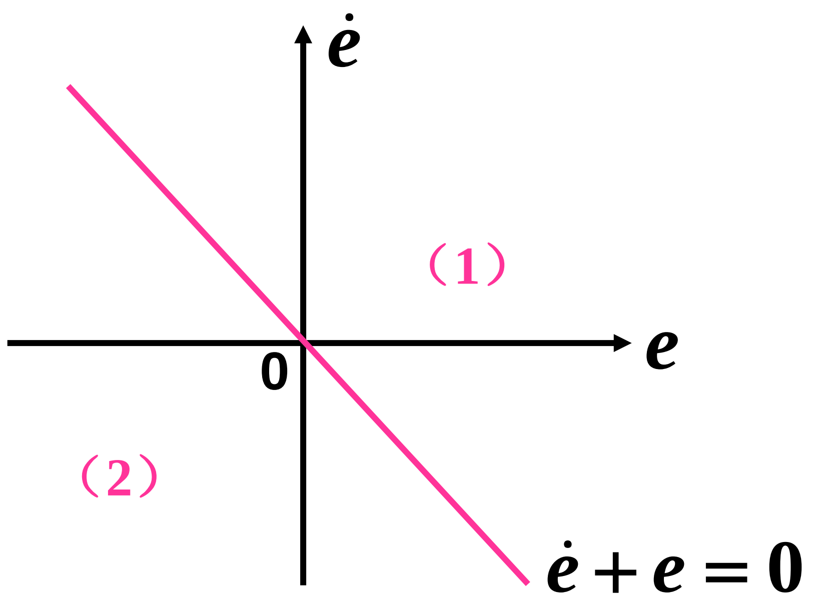
$$x = \dot{e} + e$$

and $e = r - y \qquad \therefore \ \dot{e} = -\dot{y}$

$$\therefore \ \ddot{e} = -\ddot{y} = \begin{cases} -0.5 & \dot{e} + e > 0 \\ 0.5 & \dot{e} + e < 0 \end{cases}$$

area (1)

area (2)



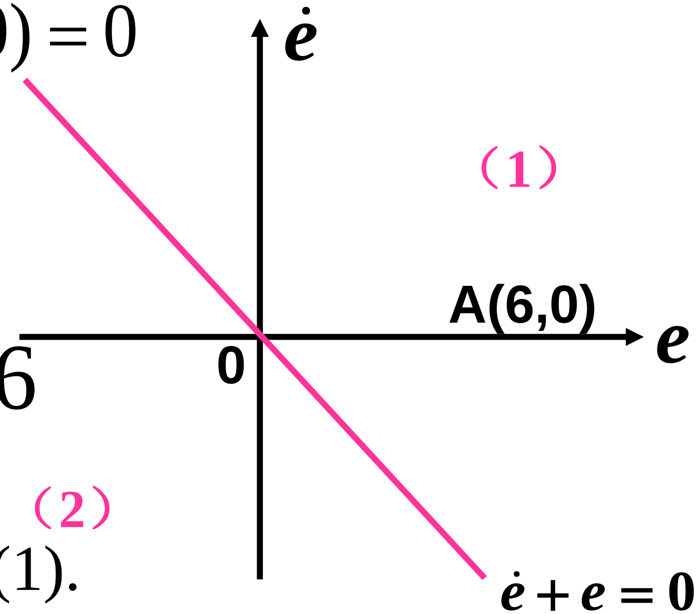
Area (1):

$$\begin{cases} \ddot{e} = -0.5 \\ \dot{e} = -0.5t + c_1 \\ e = -0.25t^2 + c_1 t + c_2 \end{cases}$$

The initial conditions is $e(0) = 6, \quad \dot{e}(0) = 0$

then $c_1 = 0, c_2 = 6$

$$\therefore \begin{cases} \dot{e} = -0.5t \\ e = -0.25t^2 + 6 \end{cases} \rightarrow e = -\dot{e}^2 + 6$$



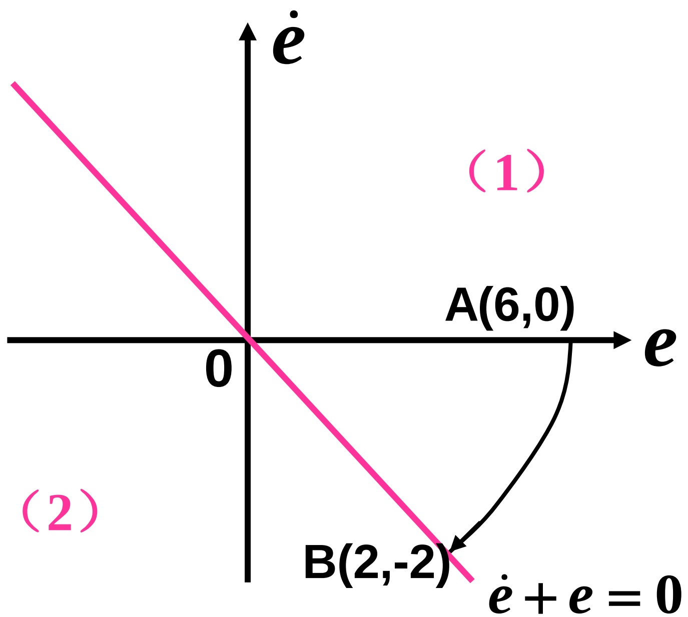
——This is the phase trajectory in area (1).

The phase trajectory is a ***parabola***. From point A , the system state reaches point B and enter area (2).

$$\begin{cases} e_B = -\dot{e}_B^2 + 6 \\ \dot{e}_B + e_B = 0 \end{cases}$$

Solution: $\boldsymbol{e_B = 2, \dot{e}_B = -2}$

area (2): $$\begin{cases} \ddot{e} = 0.5 \\ \dot{e} = 0.5t + c_3 \\ e = 0.25t^2 + c_3 t + c_4 \end{cases}$$



Consider the initial condition

$$e_B = 2, \dot{e}_B = -2$$

We obtain $c_3 = -2, c_4 = 2$

$$\therefore \begin{cases} \dot{e} = 0.5t - 2 \\ e = 0.25t^2 - 2t + 2 \end{cases}$$

Eliminating *t* we have

$$e = \dot{e}^2 - 2$$ ——**This is the trajectory in area (2).**

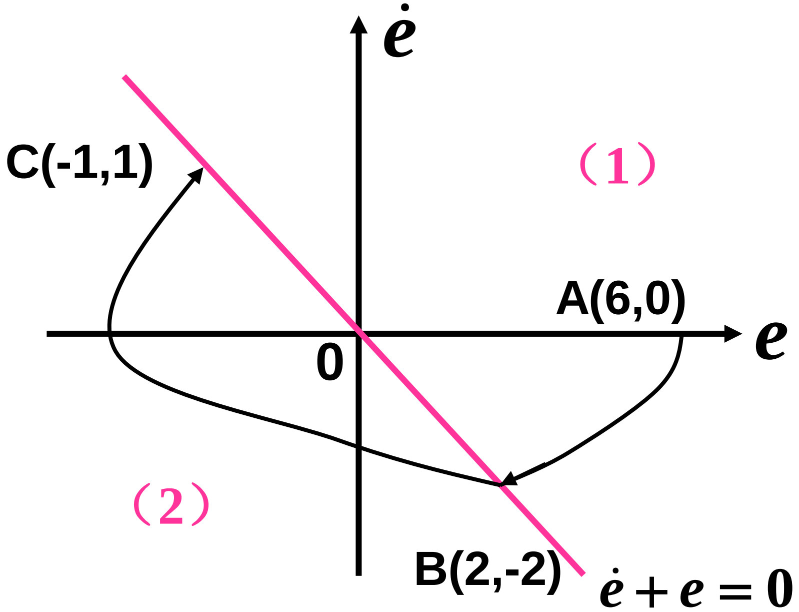
The system state moves along the parabola from point B to point C, then enter area (1).

The coordinate of point C satisfies

$$\begin{cases} e_C = \dot{e}_C^2 - 2 \\ \dot{e}_C + e_C = 0 \end{cases}$$

We can obtain

$$e_C = -1, \dot{e}_C = 1$$



Area (1):

$$\begin{cases} \ddot{e} = -0.5 \\ \dot{e} = -0.5t + c_5 \\ e = -0.25t^2 + c_5 t + c_6 \end{cases}$$

In terms of the initial condition C(-1,1), we have

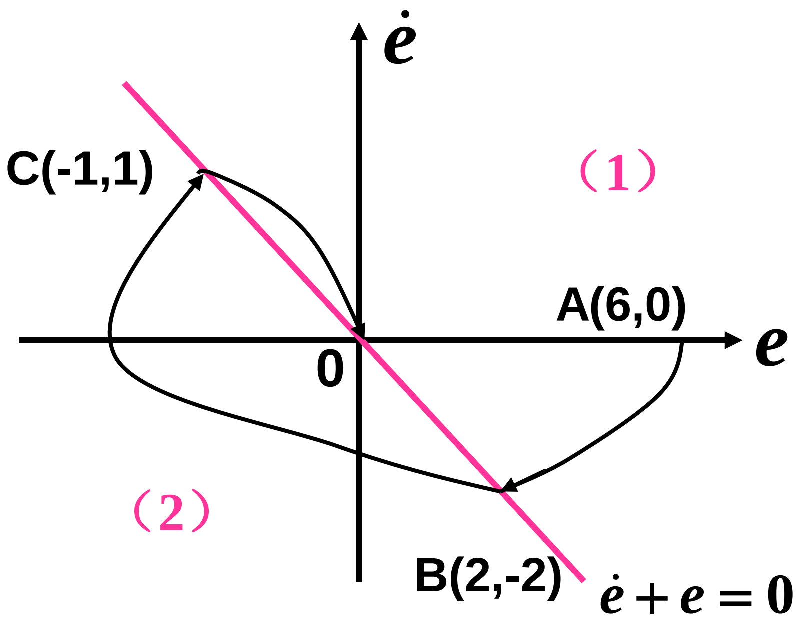
$$\boldsymbol{c_5 = 1, c_6 = -1}$$

$$\therefore \begin{cases} \dot{e} = -0.5t + 1 \\ e = -0.25t^2 + t - 1 \end{cases}$$

Eliminating *t* we have

$$\boldsymbol{e = -\dot{e}^2}$$

In area (1), the system state moves along the parabola from point C to the origin.



The time t_{AO} of moving to the origin from point A can be obtained by the following equation:

$$\boldsymbol{t_{AO} = t_{AB} + t_{BC} + t_{CO}}$$

Based on the dynamic equations in different areas ,

$t_{AB}: \ \dot{e} = -0.5t \qquad \boldsymbol{-2 = -0.5t_{AB}} \qquad \boldsymbol{\therefore \ t_{AB} = 4}$

$t_{BC}: \ \dot{e} = 0.5t - 2 \qquad \boldsymbol{3 = 0.5t_{BC}} \qquad \boldsymbol{\therefore \ t_{BC} = 6}$

$t_{CO}: \ \dot{e} = -0.5t + 1 \qquad \boldsymbol{-1 = -0.5t_{CO}} \qquad \boldsymbol{\therefore \ t_{CO} = 2}$

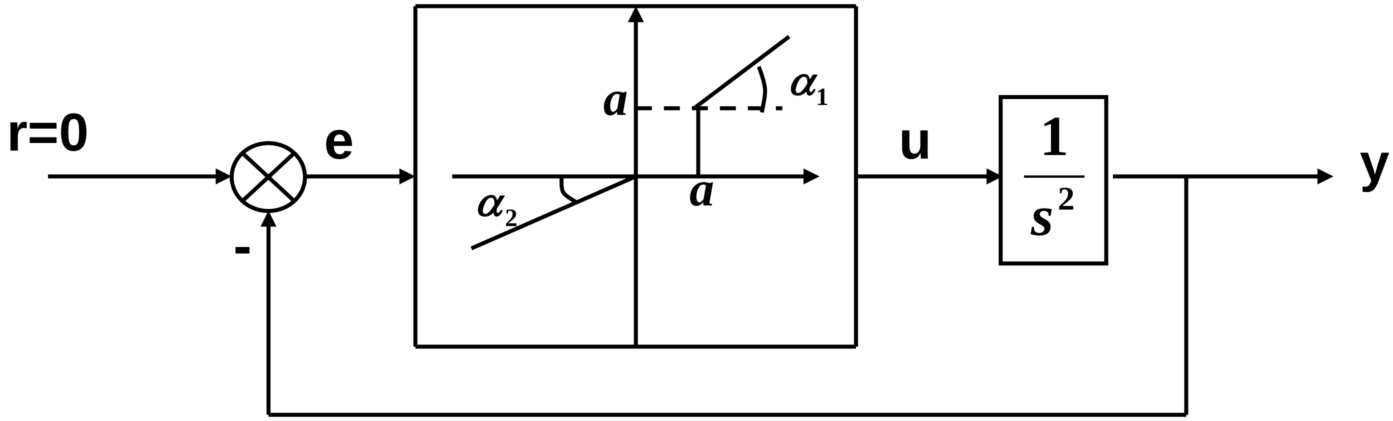
$$\therefore \ t_{AO} = t_{AB} + t_{BC} + t_{CO} = 12 \ \text{Seconds}$$

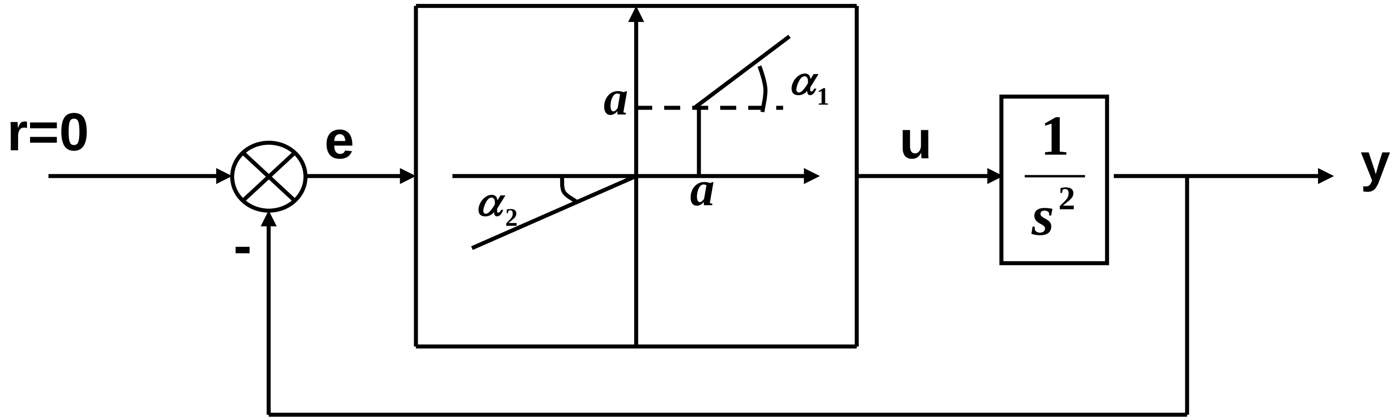
[Example3] **The structure of a nonlinear system as shown in the following figure , where a = 1,** $tg\alpha_1 = 1,\ tg\alpha_2 = 1/2.$

(1) Plot the phase trajectory of this system with the initial state $y(0) = -1, \dot{y}(0) = -1.$

(2) Draw briefly the corresponding curve $y(t)$. Try to Obtain the values of t when $y(t)$= 0.

(3) If $y(t)$ is periodic, obtain the value of this period.





Solution: Dynamic equations $\ddot{y} = u$

$$u = \begin{cases} a + (e-a)tg\alpha_1 & e > a \\ 0 & 0 \le e \le a \\ etg\alpha_2 & e < 0 \end{cases}$$

When r=0, e=-y. Substituting the known conditions into the above equation, we obtain

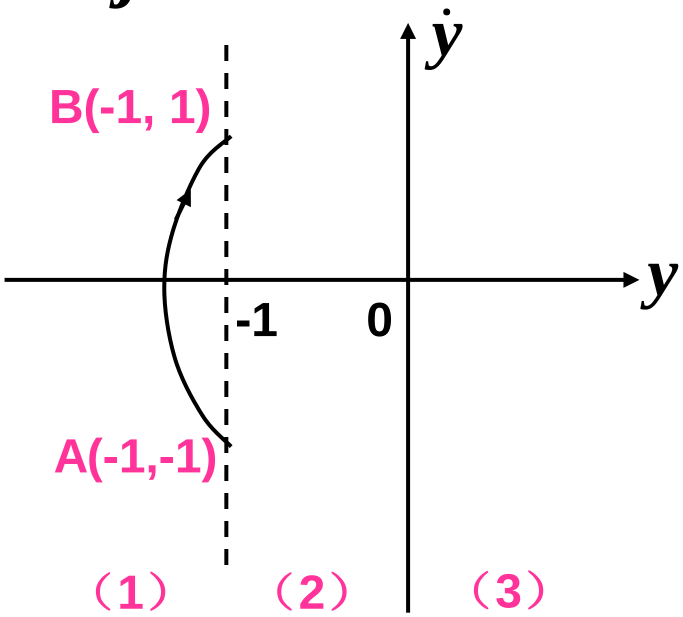
$$\ddot{y} = u = \begin{cases} -y & y < -1 \\ 0 & -1 \le y \le 0 \\ -0.5y & y > 0 \end{cases}$$

Area (1): $\ddot{y} = -y$

$$\ddot{y} + y = 0 \;\rightarrow \lambda^2 + 1 = 0,\;\; \lambda = \pm j$$

$$\therefore \;\; y = c_1 \cos t + c_2 \sin t$$

$$\dot{y} = -c_1 \sin t + c_2 \cos t$$



Substitute the initial conditions A(-1,-1) into the above equations. We obtain:

$$c_1 = -1, c_2 = -1$$

$$\therefore \;\; y = -\cos t - \sin t = -\sqrt{2}\sin(t + \frac{\pi}{4})$$

$$\dot{y} = \sin t - \cos t$$

Eliminating t we have

$$\dot{y}^2 + y^2 = 2$$

The system state moves in area(1) along the arc from point A to point B, and then enter area（2）.

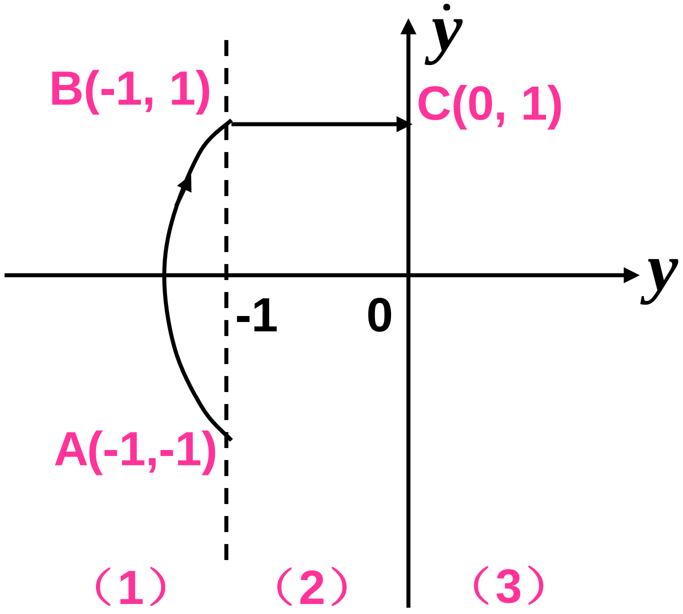
Area (2): $\ddot{y} = 0$

$$\dot{y} = c_3, \quad y = c_3 t + c_4$$

Substituting the initial conditions B(-1,1) into above equations, we obtain

$$c_3 = 1, c_4 = -1$$

$$\therefore \quad y = t - 1, \quad \dot{y} = 1$$



The system state moves in area (2) along the line $\dot{y} = 1$ from point A to point B, and then enter area (3).

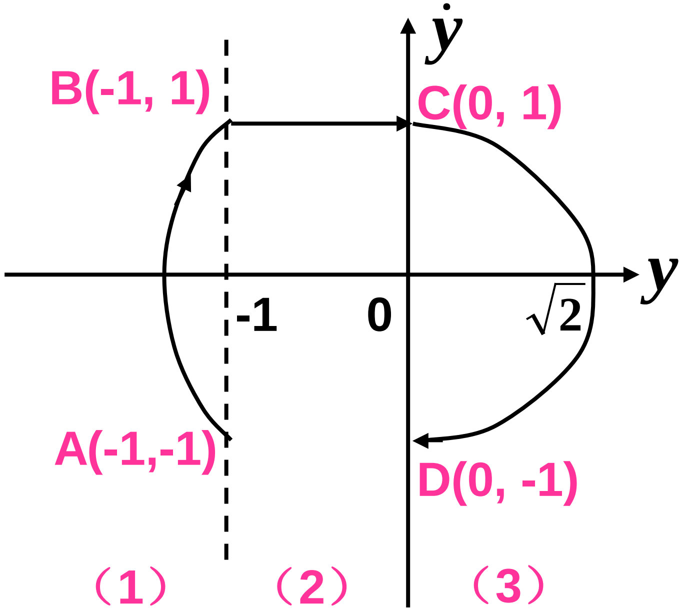
Area (3): $\ddot{y} = -0.5y$

$$\ddot{y} + \frac{1}{2}y = 0 \;\rightarrow \lambda^2 + \frac{1}{2} = 0, \;\; \lambda = \pm\frac{\sqrt{2}}{2}j$$

$$\therefore \quad y = c_5 \cos\frac{\sqrt{2}}{2}t + c_6 \sin\frac{\sqrt{2}}{2}t$$

$$\dot{y} = -\frac{\sqrt{2}}{2}c_5 \sin\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}c_6 \cos\frac{\sqrt{2}}{2}t$$

Substituting the initial conditions C(0, 1) into above equations, we obtain



$$c_5 = 0, c_6 = \sqrt{2}$$

$$\therefore \begin{cases} y = \sqrt{2}\sin\frac{\sqrt{2}}{2}t \\ \dot{y} = \cos\frac{\sqrt{2}}{2}t \end{cases}$$

then $\left(\frac{y}{\sqrt{2}}\right)^2 + \dot{y}^2 = 1$

The system state moves in area (3) along the ellipse from point C to point D, and then enter area (2).

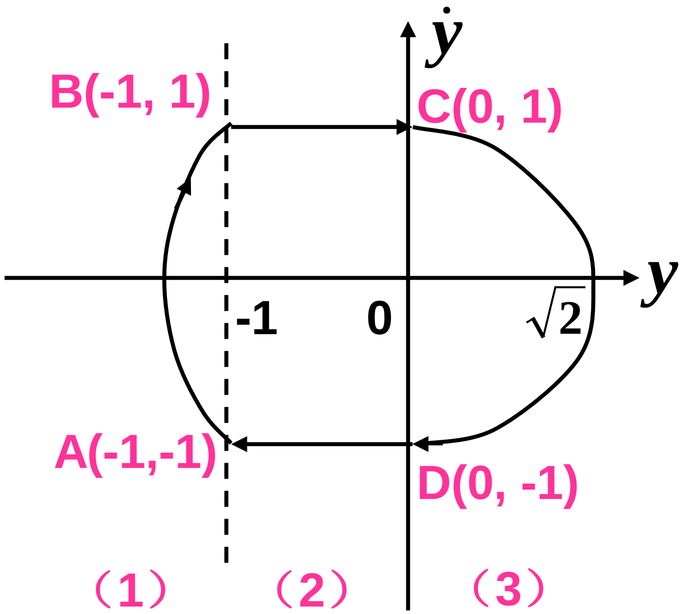
Area (2): $\ddot{y} = 0$

$$\dot{y} = c_7, \quad y = c_7 t + c_8$$

Substituting the initial conditions D(0,-1) into above equations, we obtain

$$c_7 = -1, c_8 = 0$$

$$\therefore \begin{cases} y = -t \\ \dot{y} = -1 \end{cases}$$



The system state moves in area (2) along the line $\dot{y} = -1$ from point D to point A, and then enter area (1). Because it is a closed phase trajectory, the motion of system is periodic。

Based on the dynamic equations in different areas ,

$$t_{AB}: \quad y = -\sqrt{2}\sin(t + \frac{\pi}{4}) \qquad \mathbf{\sin(t_1 + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}} \qquad \therefore\ \mathbf{t_1 = \frac{\pi}{2}}$$

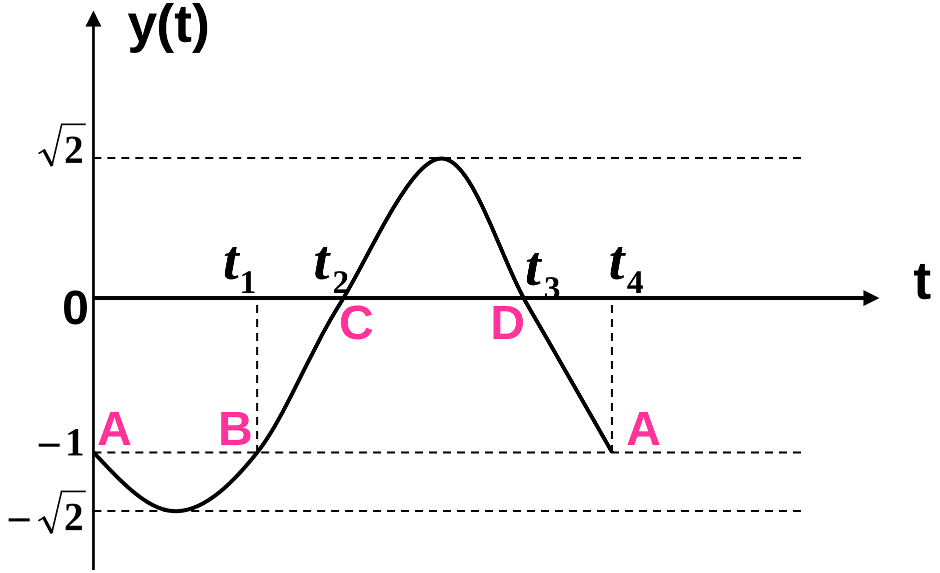
$$t_{BC}: \quad y = t - 1 \qquad 0 = t_2 - 1 \qquad \therefore t_2 = 1$$

$$t_{CD}: \quad y = \sqrt{2}\sin\frac{\sqrt{2}}{2}t \qquad 0 = \sqrt{2}\sin\frac{\sqrt{2}}{2}t_3 \qquad \therefore\ t_3 = \sqrt{2}\pi$$

$$t_{DA}: \quad y = -t \qquad -1 = -t_4 \qquad \therefore\ t_4 = 1$$

The period of system motion is

$$T = t_1 + t_2 + t_3 + t_4 = 2 + \frac{\pi}{2} + \sqrt{2}\pi$$



where,

$$t^*_1 = \frac{\pi}{2} \qquad t^*_2 = 1 + \frac{\pi}{2} \qquad t^*_3 = 1 + \frac{\pi}{2} + \sqrt{2}\pi \qquad t^*_4 = 2 + \frac{\pi}{2} + \sqrt{2}\pi$$