Principle of Automatic Control II
（自动控制原理II）

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学时：48学时

考试：英文 闭卷

References

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Chapter 1. Introduction

Chapter 2. Mathematical Models of Linear Systems

Chapter 3. Analyses in Time Domain for Linear Systems

Chapter 4. Root Locus

Chapter 5. Analyses in Frequency Domain for Linear Systems

Chapter 6. Controller Design for Linear Systems

Classic Control Theory

- Classify:
  - Classic Control Theory
    - Routh-Hurwitz Stability Criterion, Bode Diagram, Nyquist plot/curve, Nichols plot/curve, Root Locus…
  - Modern Control Theory
    - State Space, Controllability, Observability …
  - Postmodern control theory
    - Large Scale System, Robust Control, Adaptive Control, Nonlinear Control, Intelligent Control, …

Modern Control Theory

Chapter 7. Linear Discrete-Time System

Chapter 8. Nonlinear System

Chapter 9. State Space Method

Chapter 7 Linear Discrete-Time System: Analysis and Design（Sampled-data System）

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

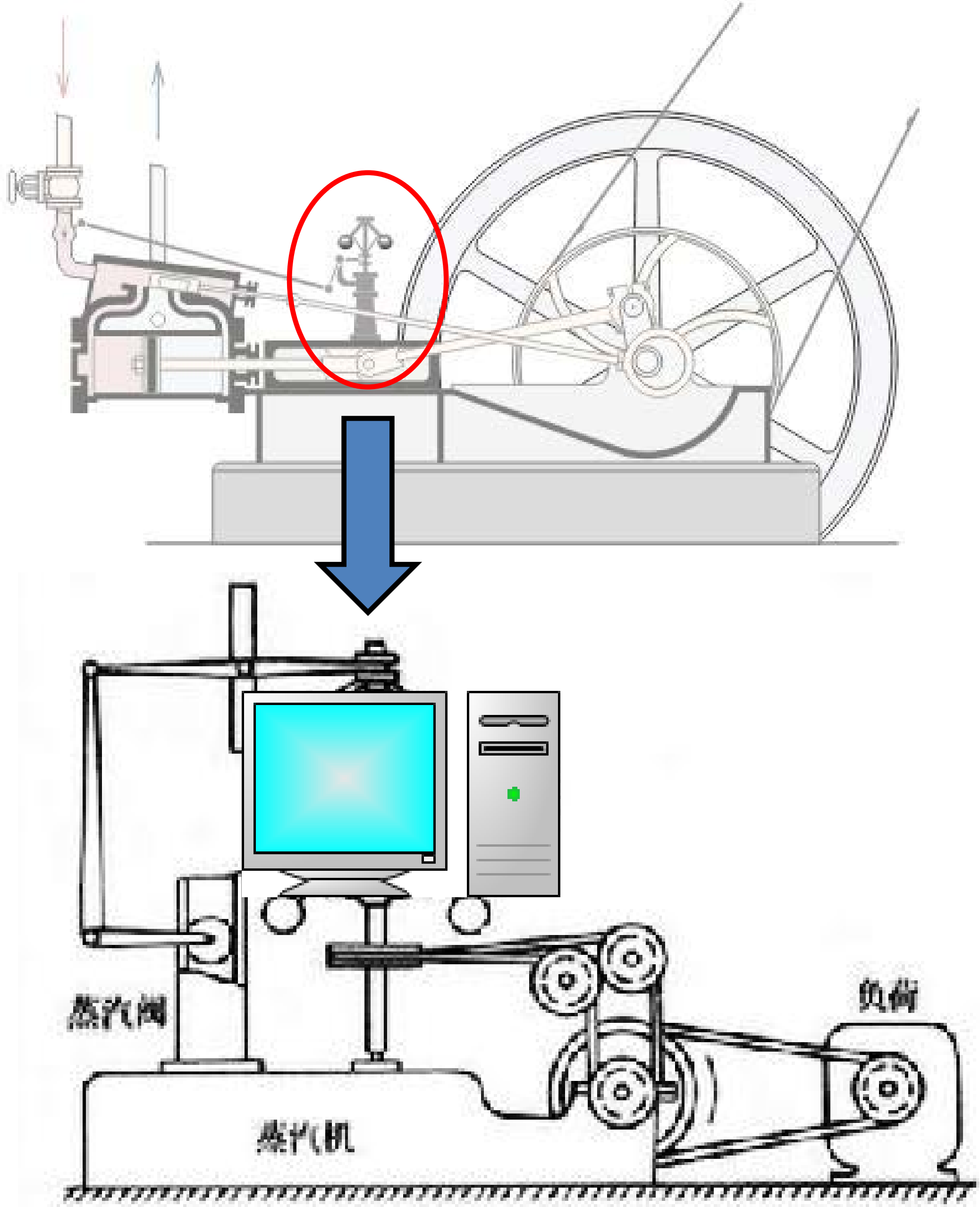
7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

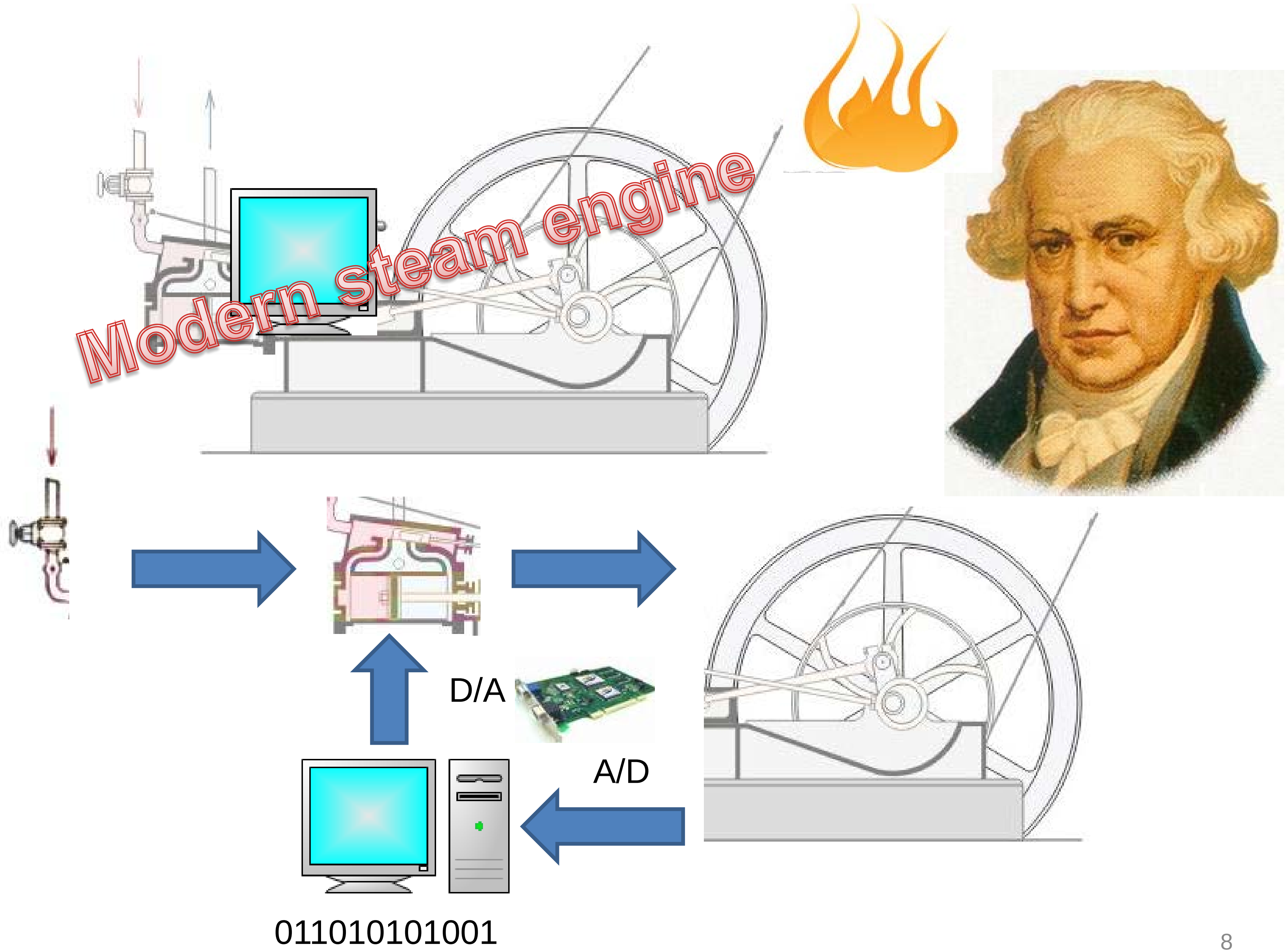
7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

Mr. J. Watt



瓦特离心式调速器对蒸汽机转速的控制


Modern steam engine
D/A
A/D
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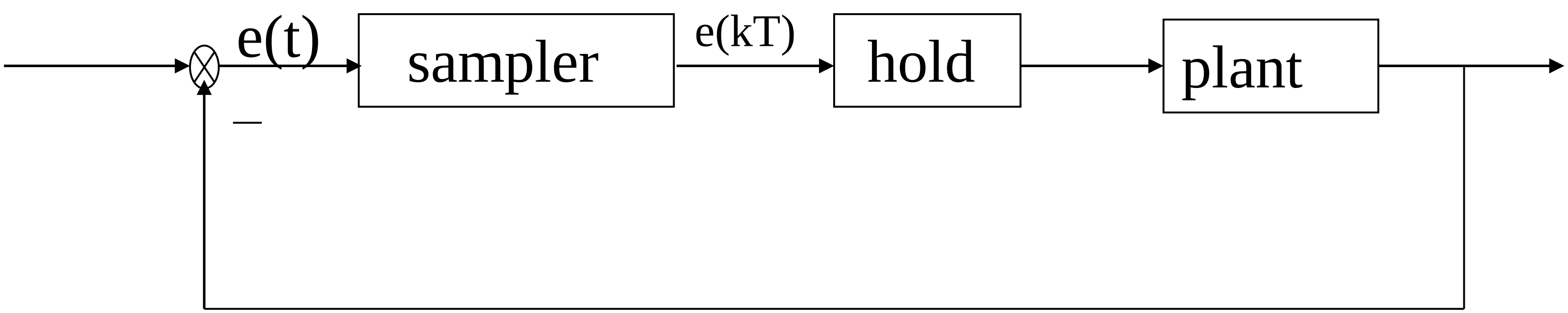
7.1 Introduction

Discrete-Time Systems:

Types:
- Sampling systems: Discrete Time, Continuous Value
- Digital systems: Discrete Time, Quantized Value

Sampled-Data System: a system that is continuous except for one or more sampling operations.

Digital System: There is one or more impulse series or digital signals in the system.



Sampled-data control system

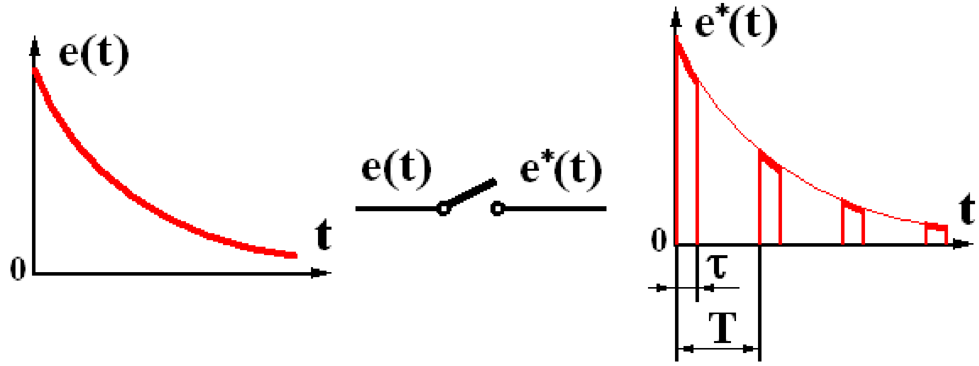
e(kT) is obtained by sampling a continuous signal e(t).

Sampler: continuous to discrete signal / AD

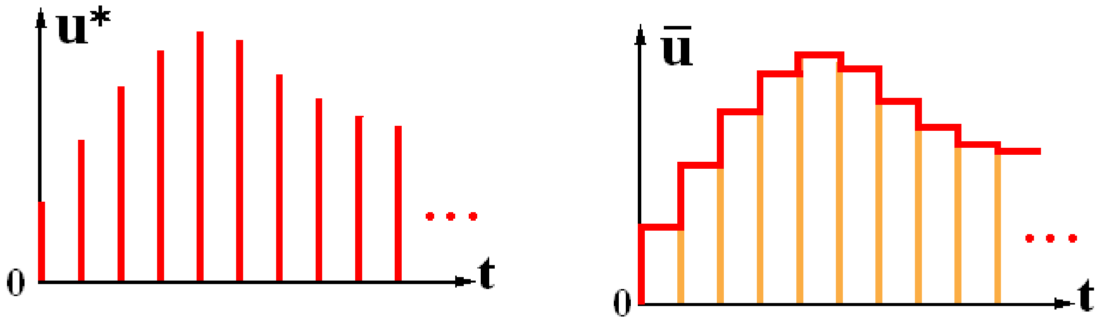
Holder : discrete to continuous signal / DA

Sampling process /AD

- Sampling — Time sampled
- Quantization — Value quantized



Holding process / DA



Advantages and Disadvantages

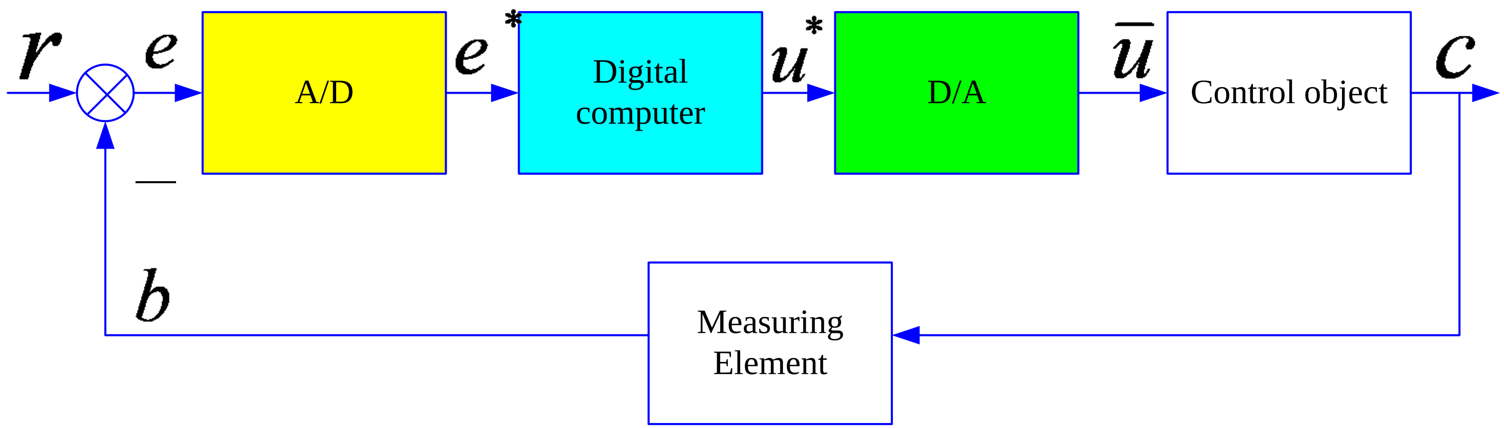
Computer Control System

- (1) Calculations are performed in the software. Easy for modification.
  (2) Complex control laws easily realized;
  (3) Reduced sensitivity to noise;
  (4) One computer for multi-tasks, high utilization；
  (5) Network for process automation, macro-management and remote control.

- (1) Information between samples is lost. Compared with continuous system in the similar condition, the performance is reduced;
  (2) Needs A/D and D/A conversion devices.

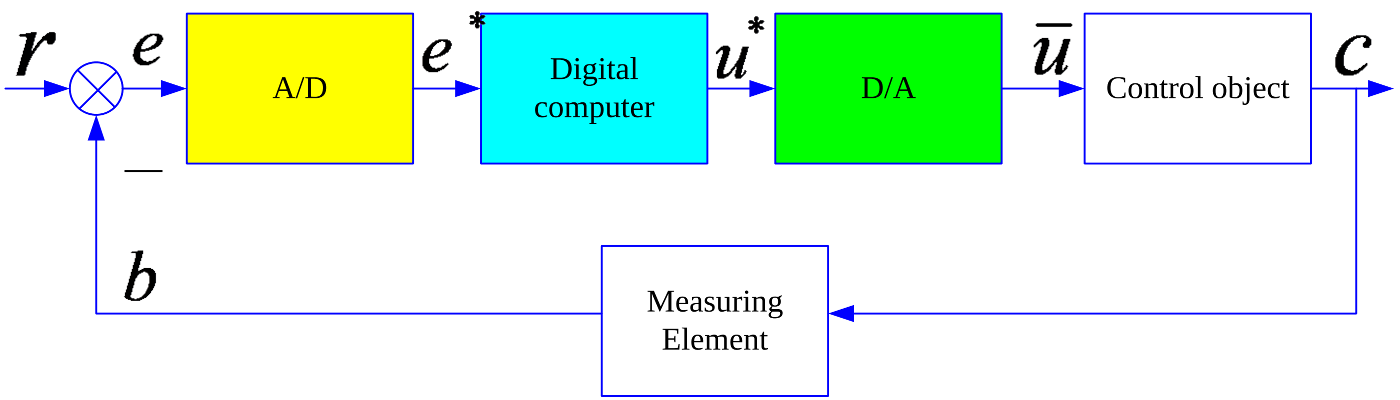
7.2 The Sampling Process and Sampling Theorem

7.2.1 The Sampling Process

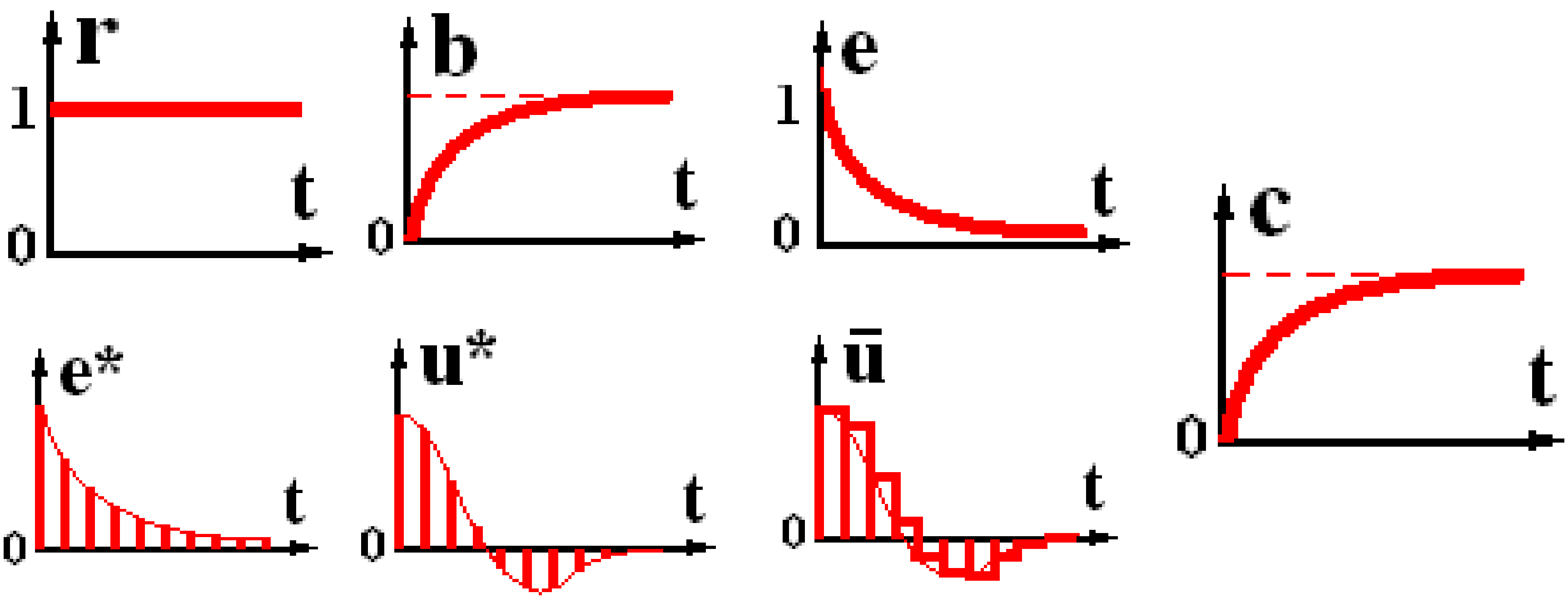


Computer Control System

Question: In the above computer control system, which signals are discrete, which signals are continuous?

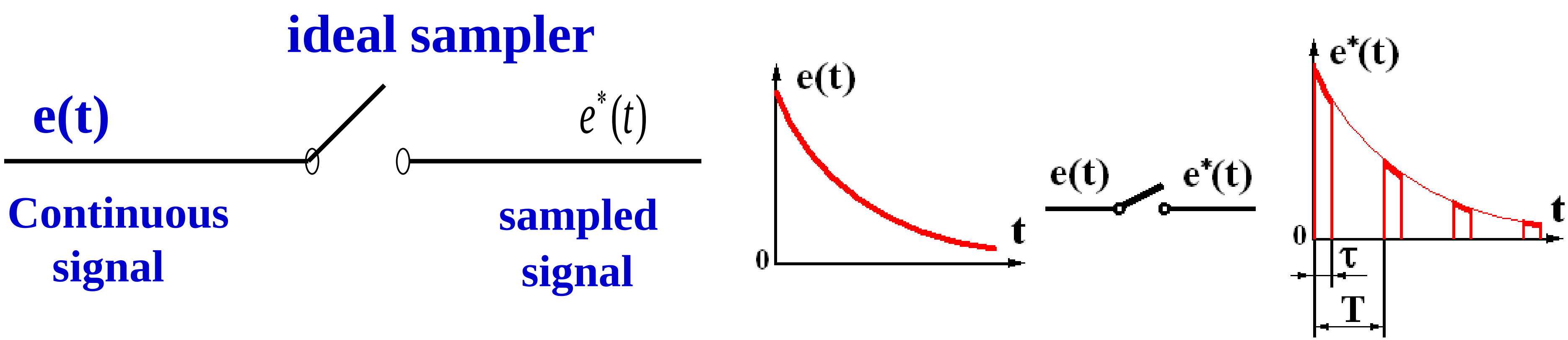


Computer Controlled Systems



- **Sampling Process: Continuous signal → Discrete Signal**
- **Holding Process: Discrete Signal → Continuous Signal.**
- **The two are inverse process to each other.**

Sampler: A switch which closes every T seconds for one instant of time.



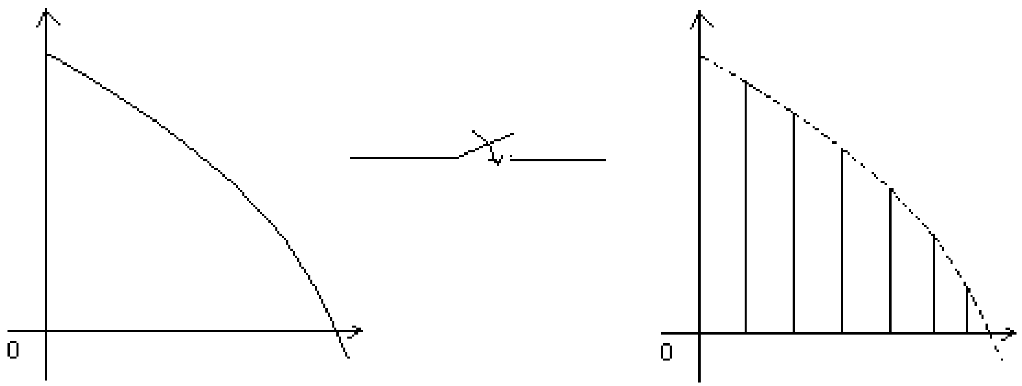
Where T is called the sampling period.

Ideal sampling process:

(1) τ <<T. The sampling process is completed instantaneously

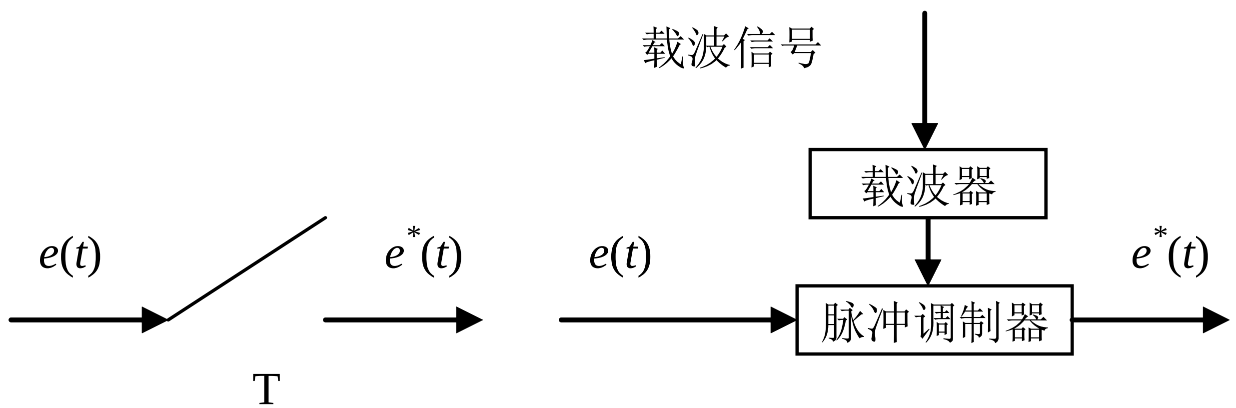
(2) Word Length is enough, for quantization, thus e*(KT)=e(KT)

Types of Samplers: ideal, periodical, random,…



(a) (b) (c)

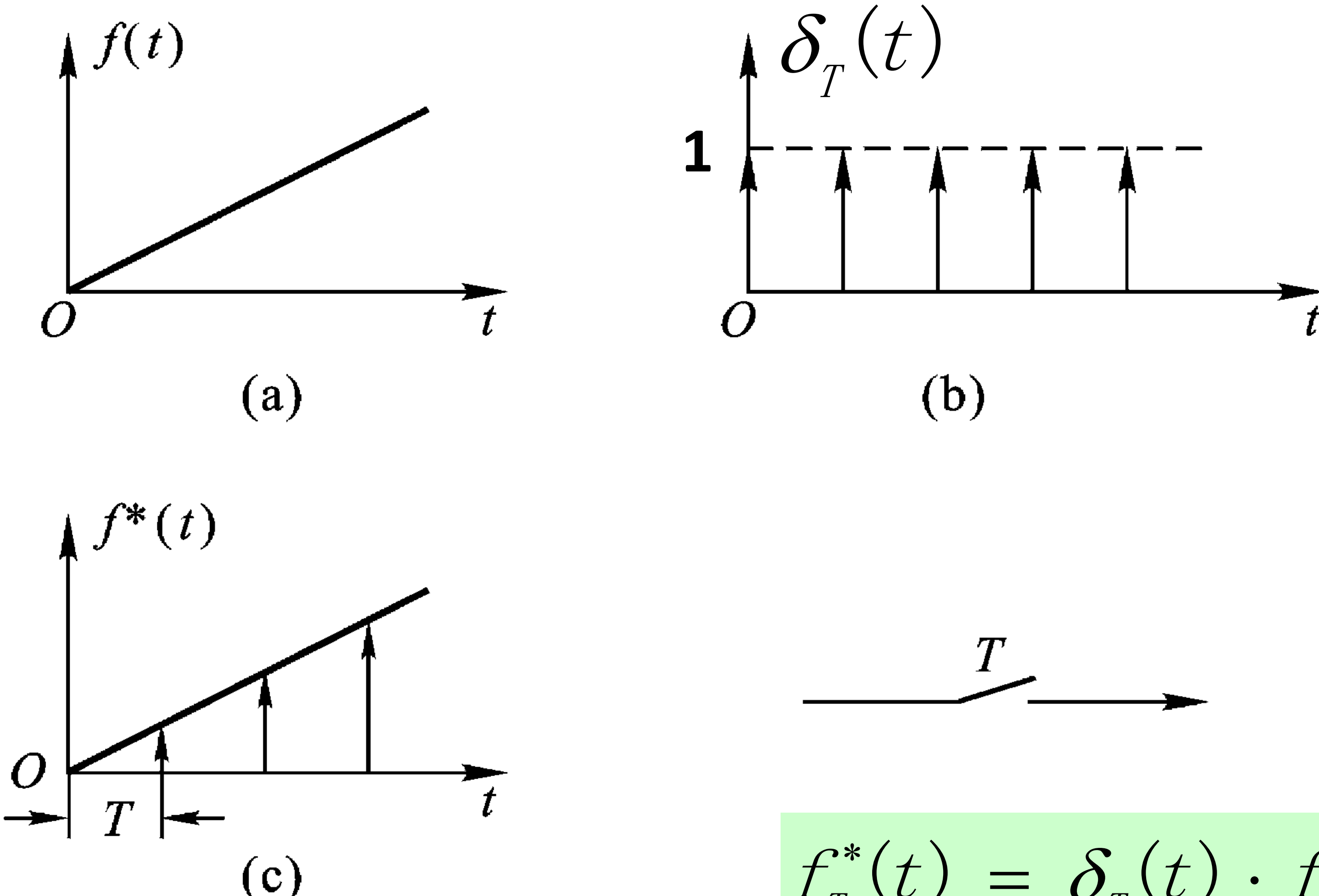
Sampling Process



7.2.2 Mathematical Model for sampling Signals

1、 Some ideal assumptions

- **The sampling process is completed instantaneously；**
- $\tau \ll T$, **that is** $\tau \to 0$;
- **The signals in and out the sampler have no difference;**
- **Output of the sampler is constant when it shuts down;**
- **Sample Period T is a constant。**



$$f_T^*(t) = \delta_T(t) \cdot f(t)$$

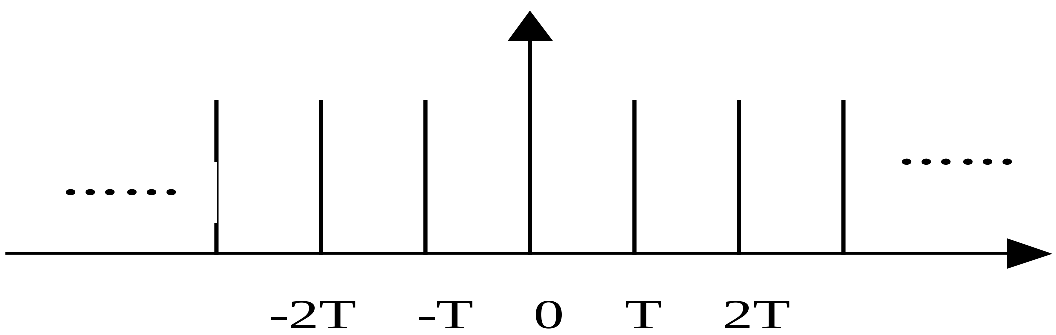
Fig 7－3 Sampling Process

2、Unit Impulsive Signal $\delta(t)$

$$\delta(t)=\begin{cases}1 & t=0\\ 0 & t\neq 0\end{cases}$$

3、Unit Impulse Sequence (Unit Impulse Train)

$$\delta_T(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)=\cdots+\delta(t+T)+\delta(t)+\delta(t-T)+\cdots+\delta(t-kT)+\cdots$$



Unit Impulse sequence

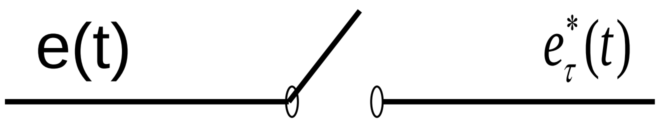
4、 Sampling Signal

$$e^{*}(t)=\sum_{k=-\infty}^{\infty} e(t)\delta(t-kT)$$

or

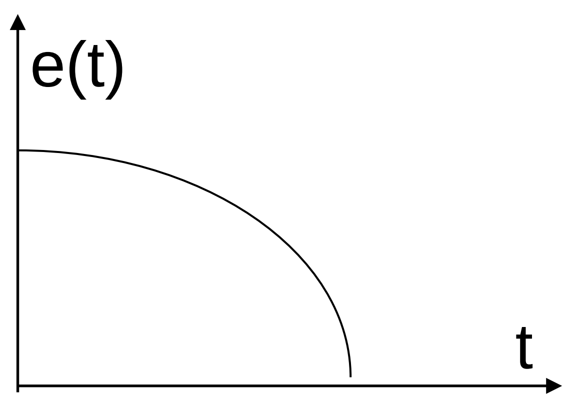
$$e^{*}(t)=\sum_{k=-\infty}^{\infty} e(kT)\delta(t-kT)$$

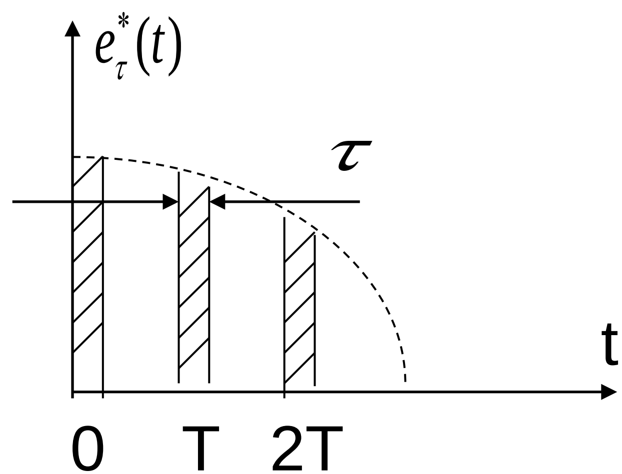
for real sampler

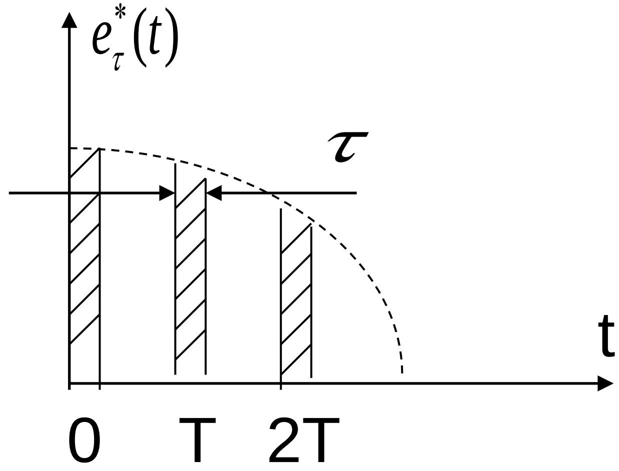


Initial working time:

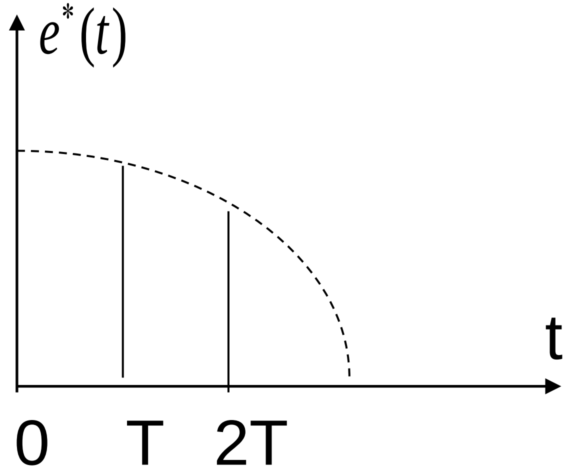
t ≥ 0







$\tau \to 0$



So the sampling operation can be expressed as

$$e^*(t) = \sum_{k=0}^{+\infty} e(kT) \cdot \delta(t - kT)$$

or

$$e^*(t) = e(t) \cdot \sum_{k=0}^{\infty} \delta(t - kT)$$

or

$$e^*(t) = e(t) \cdot \delta_T(t)$$

where

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$

Laplace Transformation

(6) 位移定理(Transposition Property):

a. 实域中的位移定理，若原函数在时间上延迟 τ ，则其象函数应乘以$e^{-\tau \cdot s}$

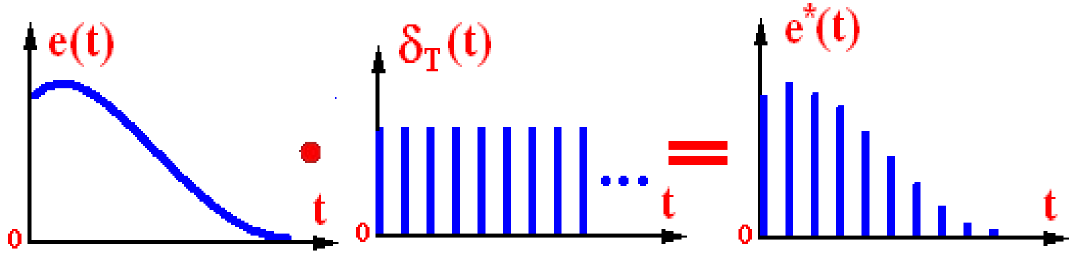
$$L[f(t-\tau)] = e^{-\tau \cdot s} F(s)$$

b. 复域中的位移定理，象函数的自变量延迟a，原函数应乘以 e^{at} ，即

$$L[e^{at} f(t)] = F(s-a)$$

Ideal sampling sequence

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$



$$e^*(t) = e(t) \cdot \delta_T(t)$$

$$= e(t) \cdot \sum_{k=0}^{\infty} \delta(t - kT) \qquad = \sum_{k=0}^{\infty} e(kT) \cdot \delta(t - kT)$$

(2) *L* :

$$E^*(s) = L\left[e^*(t)\right]$$

$$= L\left[\sum_{k=0}^{\infty} e(kT) \cdot \delta(t - kT)\right] \qquad = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

Example 7-1 $e(t)=1(t)$ to find $E^*(s)$

$$E^*(s)=\sum_{k=0}^{\infty} e(kT)\cdot e^{-kTs}$$

Solution $E^*(s)=\sum_{k=0}^{\infty} 1\cdot e^{-kTs}$

$$=1+e^{-Ts}+e^{-2Ts}+\cdots=\frac{1}{1-e^{-Ts}}=\frac{e^{Ts}}{e^{Ts}-1}$$

Example 7-2 $e(t)=e^{-at}$ to find $E^*(s)$

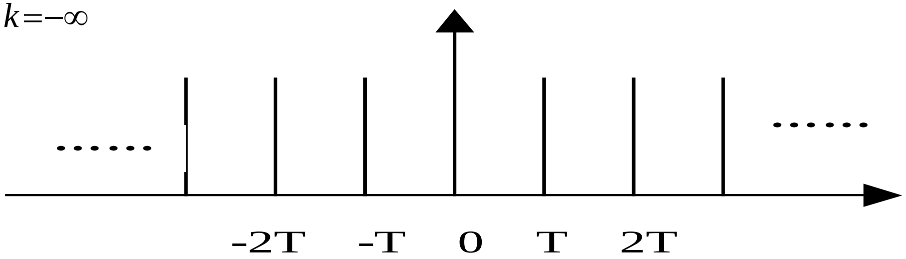
Solution $E^*(s)=\sum_{k=0}^{\infty} e^{-akT}\cdot e^{-kTs}=\sum_{n=0}^{\infty} e^{-(s+a)kT}$

$$=\frac{1}{1-e^{-(s+a)T}}=\frac{e^{Ts}}{e^{Ts}-e^{-aT}}$$

For review:

***Unit Impulse Sequence (Unit Impulse Train)

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = \delta(t) + \delta(t-T) + \cdots + \delta(t-kT) + \cdots$$



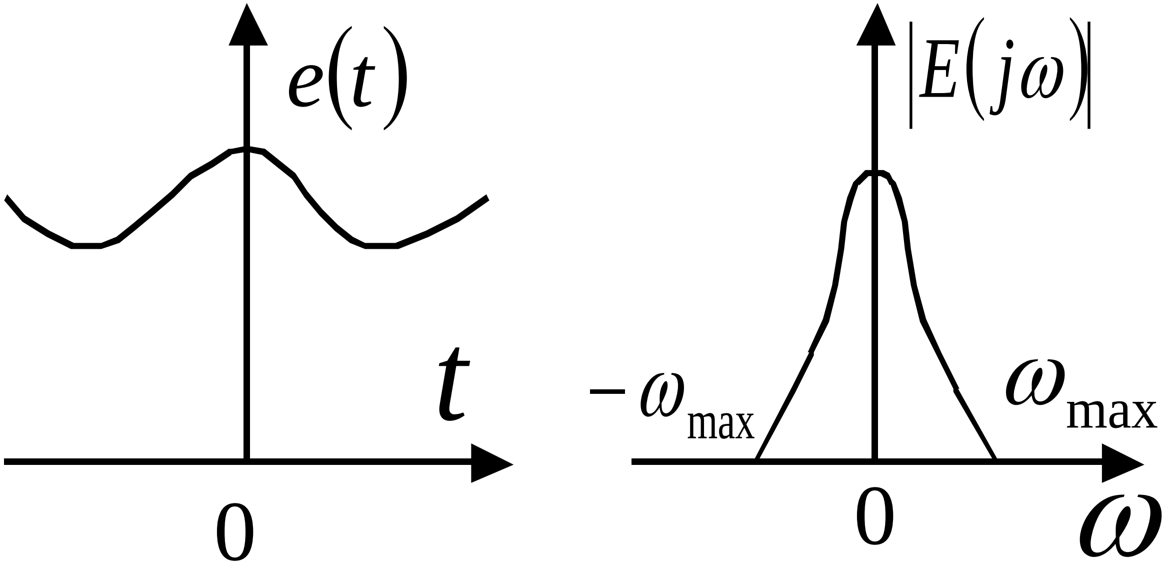
Unit Impulse sequence

***Sampling Signal $$e^*(t) = \sum_{k=-\infty}^{\infty} e(t)\delta(t-kT)$$

$$E^*(s) = L\left[e^*(t)\right] = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

7.2.3 Frequency Spectrum Analysis of Sampled Signal

Consider a continuous signal and its amplitude spectrum are:



The Fourier-series expansion of $\delta_T(t)$ is

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

So the sampled signal is

$$e^*(t) = \frac{1}{T}\cdot\sum_{k=-\infty}^{\infty} e(t)\cdot e^{jk\omega_s t}$$

which Laplace transform is $E^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s + jk\omega_s)$

$$E^*(\mathrm{j}\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} E\left[j\left(\omega + k\omega_s\right)\right]$$

where the operator s is replaced by $j\omega$

$$E^*(s) = \sum_{n=0}^{\infty} e(nT) \cdot \mathrm{e}^{-nTs}$$

① shows the value relation of E*(s) and e(t) on the sampling point;

② can be written into the closed form;

③ can be used to obtain the time response and the Z transform of e*(t)

$$E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E(s + jn\omega_s)$$

① shows relationship of E*(s) and E(s);

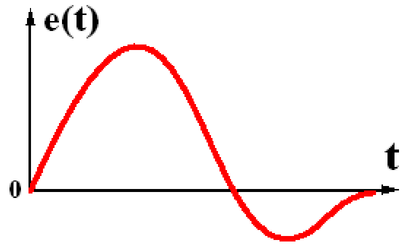
② can not be written as close form;

③ can be used for the frequency spectrum analysis of e*(t).

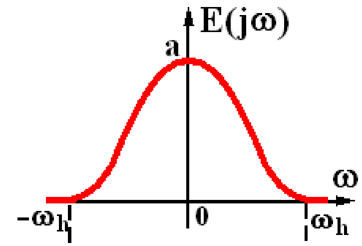
The frequency spectrum analysis of continuous signal e(t) and discrete signal e*(t)

Frequency spectrum — Frequency expansion of the signal

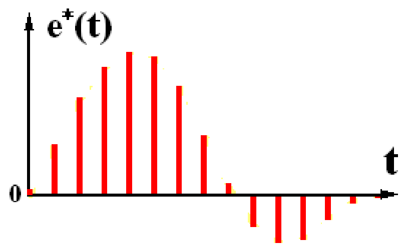
$e(t)$
continuous signal



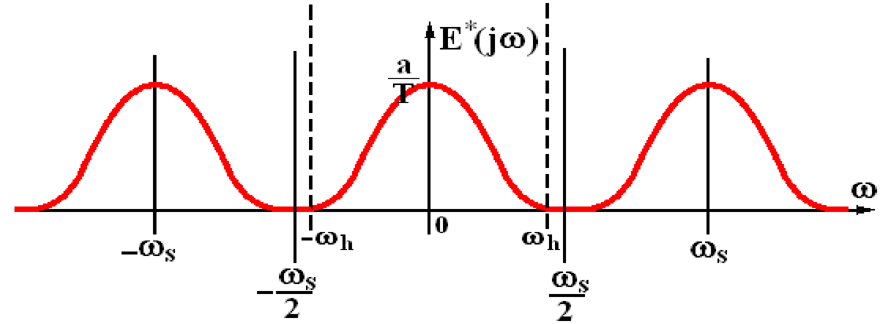
$\xrightarrow{\mathcal{F}}$



$e^*(t)$
discrete signal



$\xrightarrow{\mathcal{F}}$



$$E^*(s) = \sum_{n=0}^{\infty} e(nT)\cdot e^{-nTs}$$

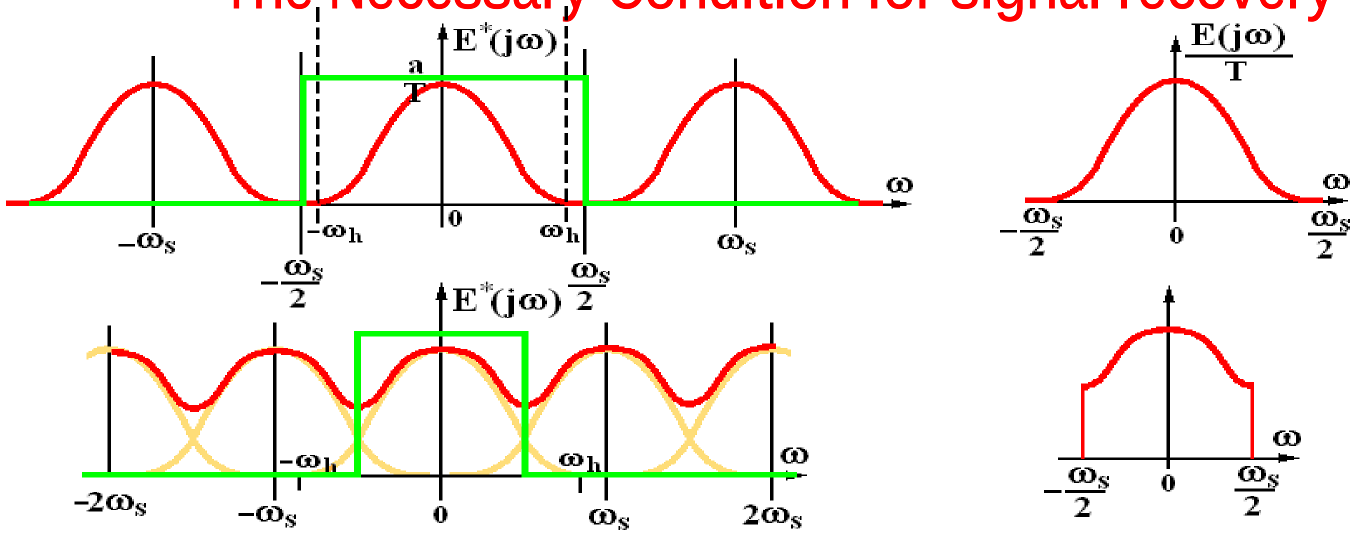
$$E^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s + jn\omega_s)$$

Shannon Sampling Theorem—

The Necessary Condition for signal recovery

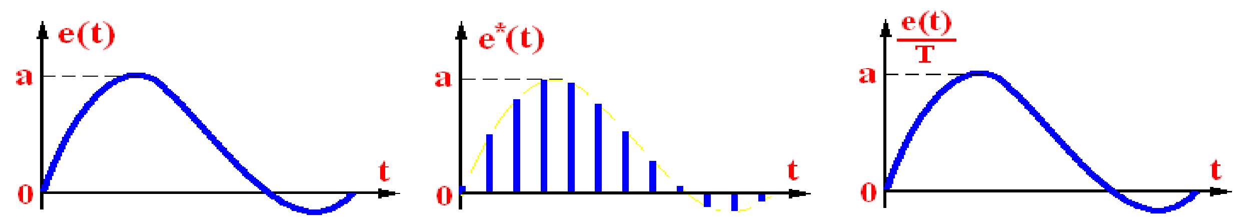
$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$

$$\omega_s = \frac{2\pi}{T} < 2\omega_h$$

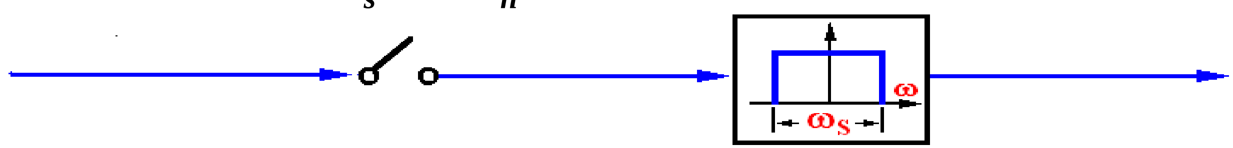


$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$

$$T < \frac{\pi}{\omega_h}$$

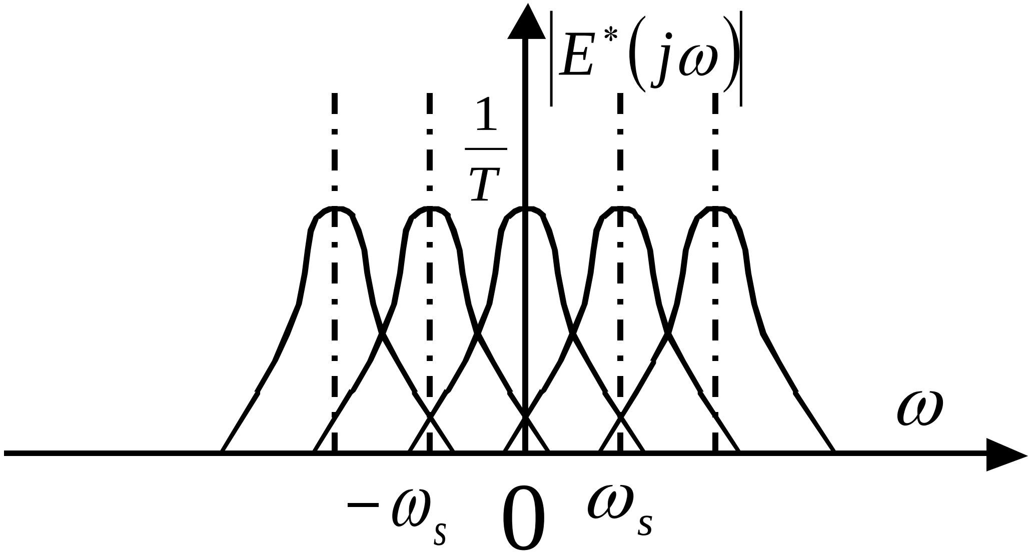


$$\omega_s > 2\omega_h$$



Sampling switch **Ideal filter**

there are no overlap of each component, so the input signal can be recovered approximately. The is called sampling theorem or Shannon's Theorem



In the figure the input signal can't be recovered.

Example Consider $e(t)=e^{-t}$，determine the sampling frequency ω_s according to Shannon Sampling Theorem.

Solution: L-Transform of e(t) is: $$E(s)=\frac{1}{s+1}$$

Frequency characteristic $$E(j\omega)=\frac{1}{j\omega+1}$$

thus

$$|E(j\omega)|=\frac{1}{\sqrt{\omega^2+1}}$$

Take $|E(j\omega)|=0.05|E(0)|$,

$$\frac{1}{\sqrt{\omega_h^2+1}}=0.05, \quad \omega_{\mathrm{h}}=20rad/s=\omega_{\max}$$

$$\omega_s \geq 2\omega_{\max}=40rad/s$$

Instant work in 10 min:

If $e(t) = \sin at$ **and** $e(t) = te^{-at}$

Try to find $E^*(s)$ **respectively.**

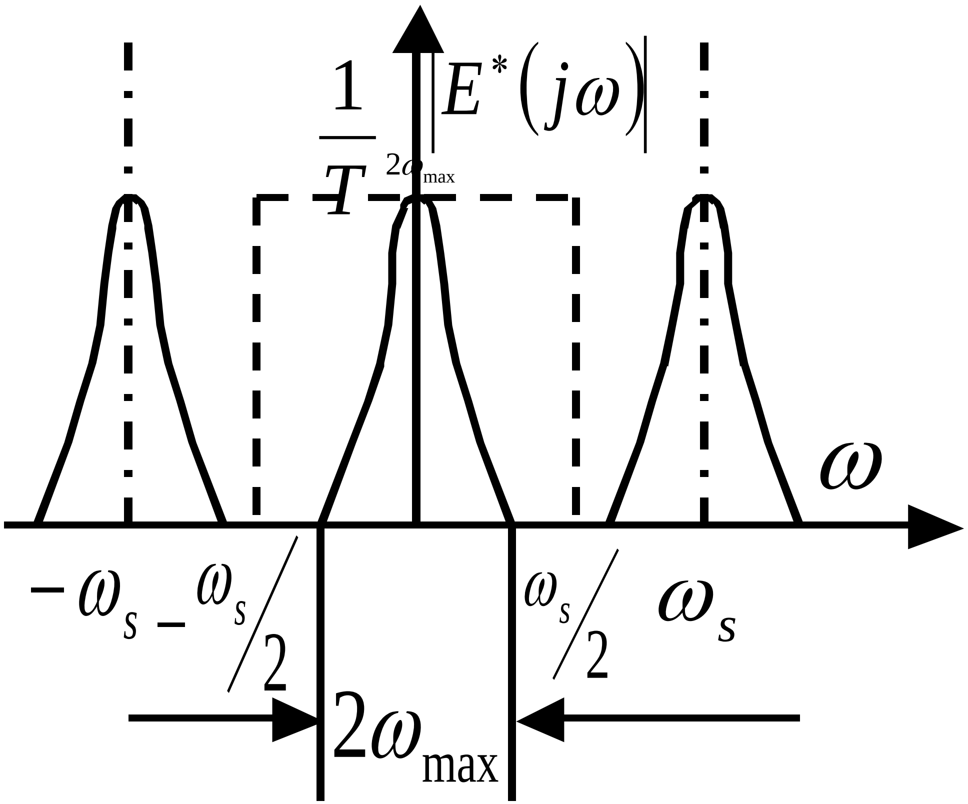
Consulting examples 7-1, 7-2

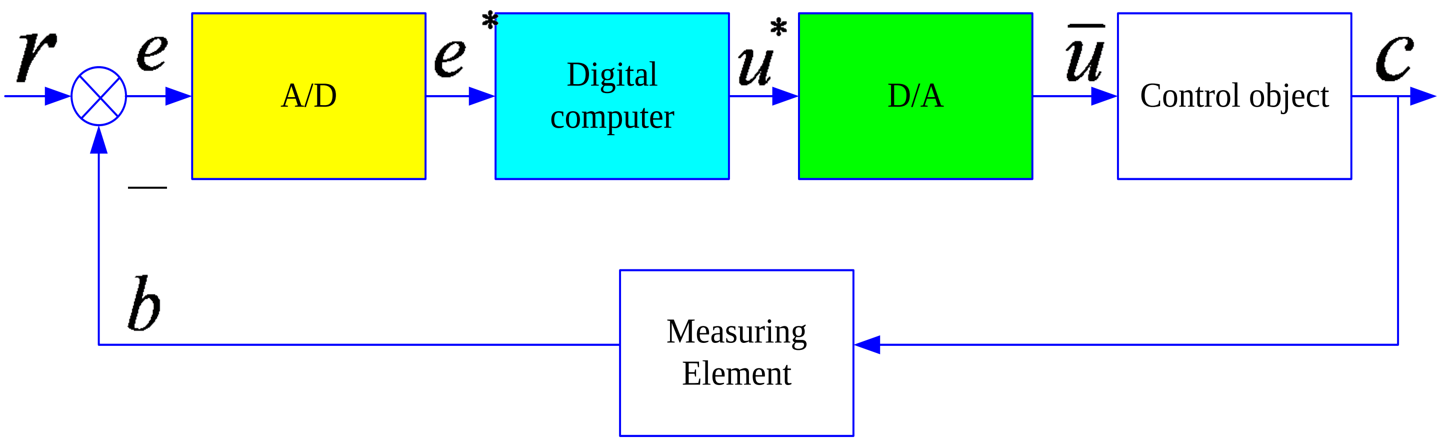
7.3 Signal Recovery and Zero-Order Hold

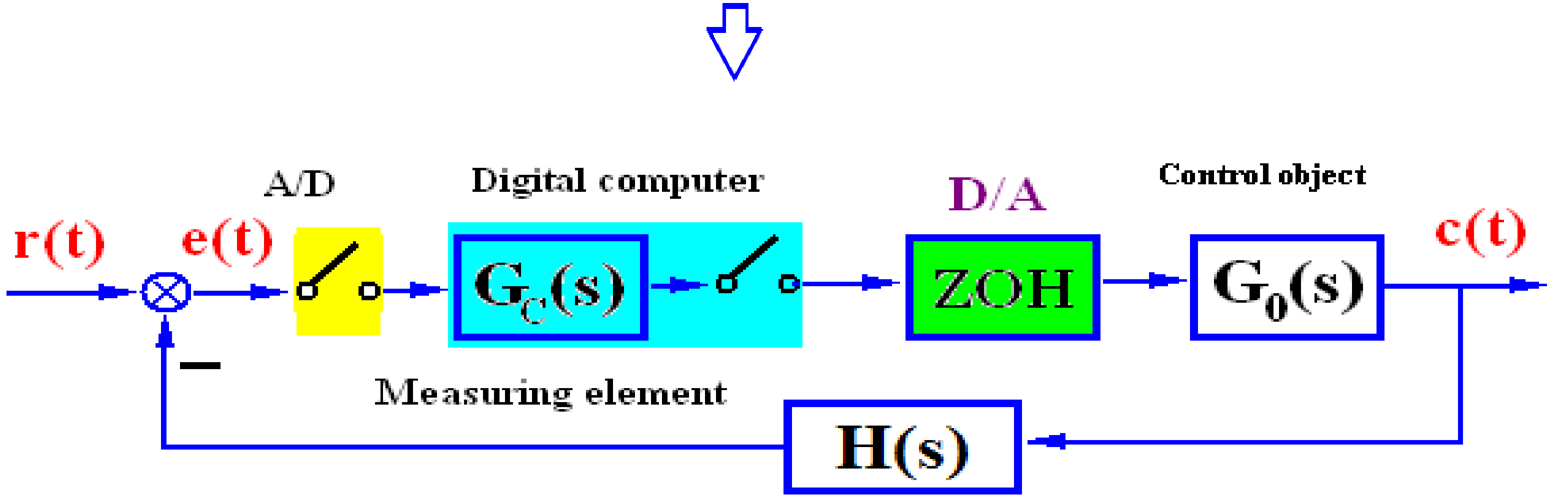
7.3.1 Signal recovery

The ideal filter is illustrated as the dotted line in the figure.

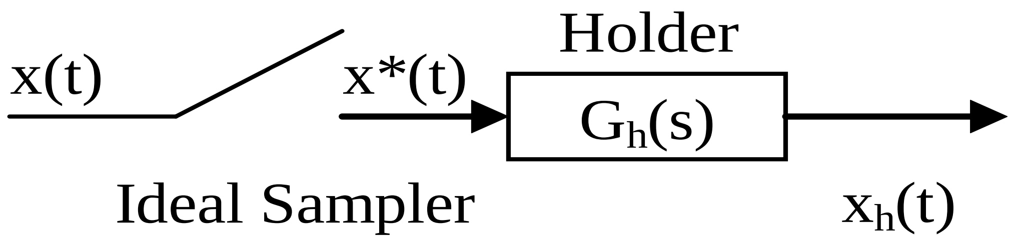
0

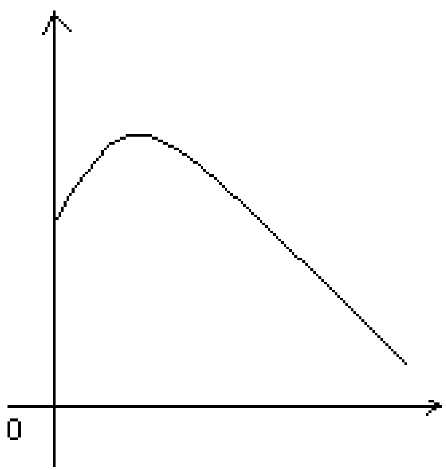




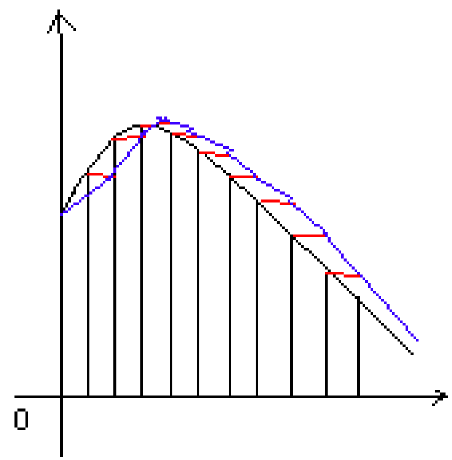


Computer Control System





Continuous Signal



Recovered Signal after ZOH

Review

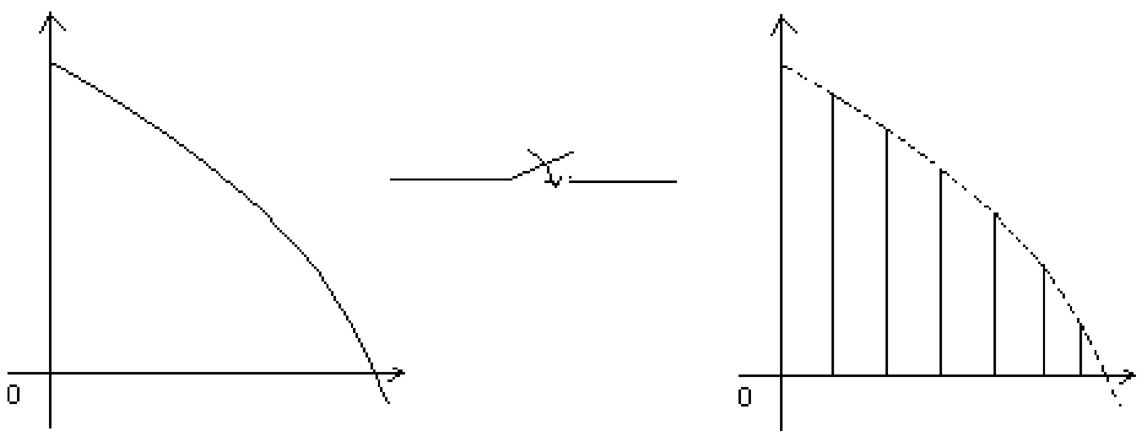
- **Unit Impulse Sequence**
- **Sampling Signal**
- **Laplace Transformation**
- **Shannon Sampling Theorem**
- **Zero-Order Hold**

7.1 Discrete-Time Control Systems

Discrete-systems: There is one or more impulse series or digital signals in the system.

Sampled-Data System: a system that is continuous except for one or more sampling operations.

Review



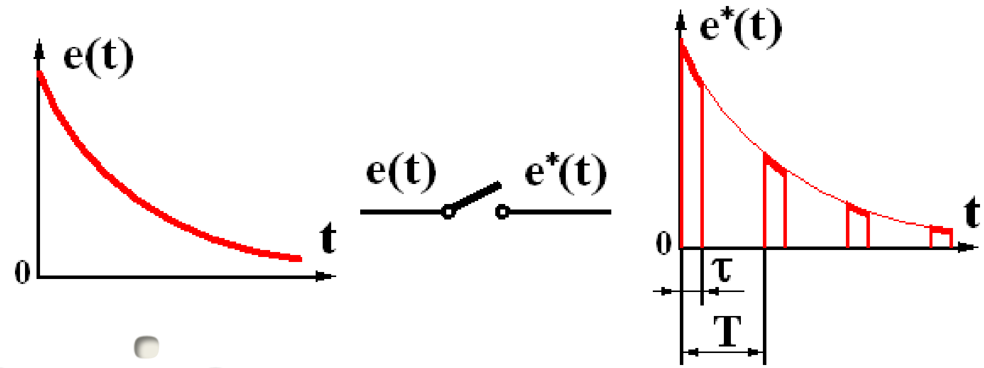
(a) (b) (c)

A/D : analog to digital converter

D/A : digital to analog converter

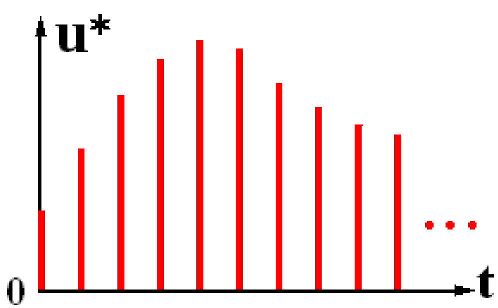
A/D process

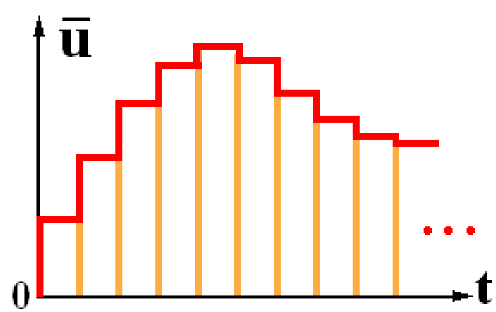
- Sampling — Time sampled
- Quantization — Value quantized

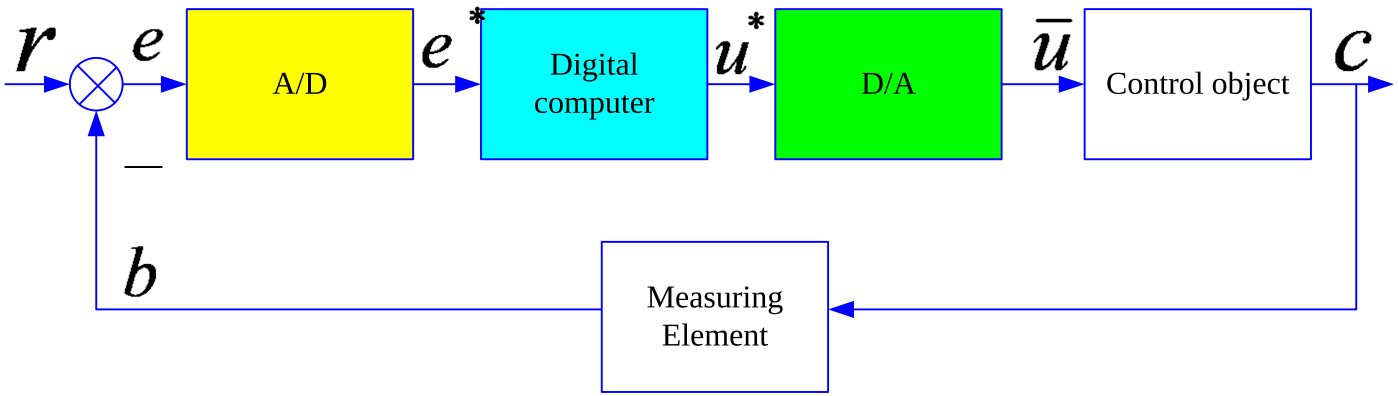


Review

D/A process

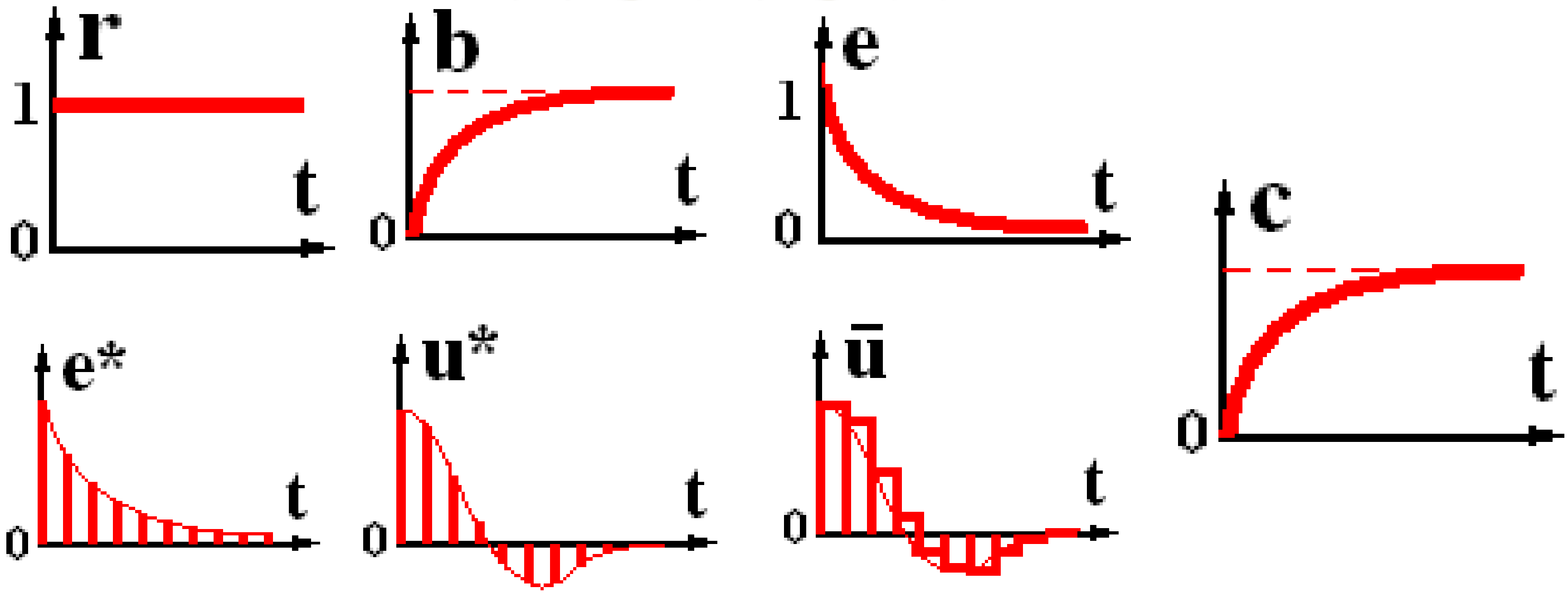






Computer Controlled Systems

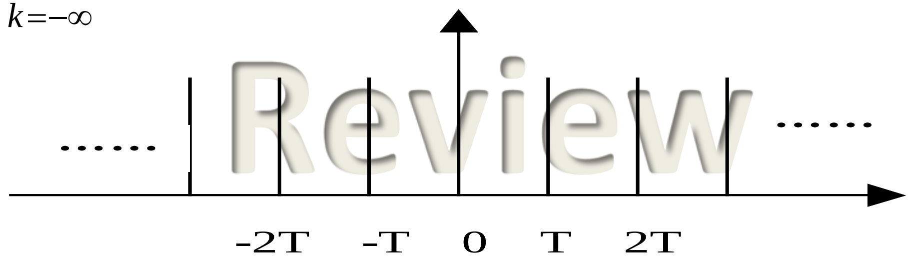
Review



7.2 Signal Sampling and Shannon Theorem

3、Unit Impulse Sequence (Unit Impulse Train)

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = \delta(t) + \delta(t-T) + \cdots + \delta(t-kT) + \cdots$$



Unit Impulse sequence

4、Sampling Signal

$$e^*(t) = \sum_{k=-\infty}^{\infty} e(t)\delta(t-kT)$$

$$E^*(s) = L\left[e^*(t)\right] = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

Shannon Sampling Theorem—

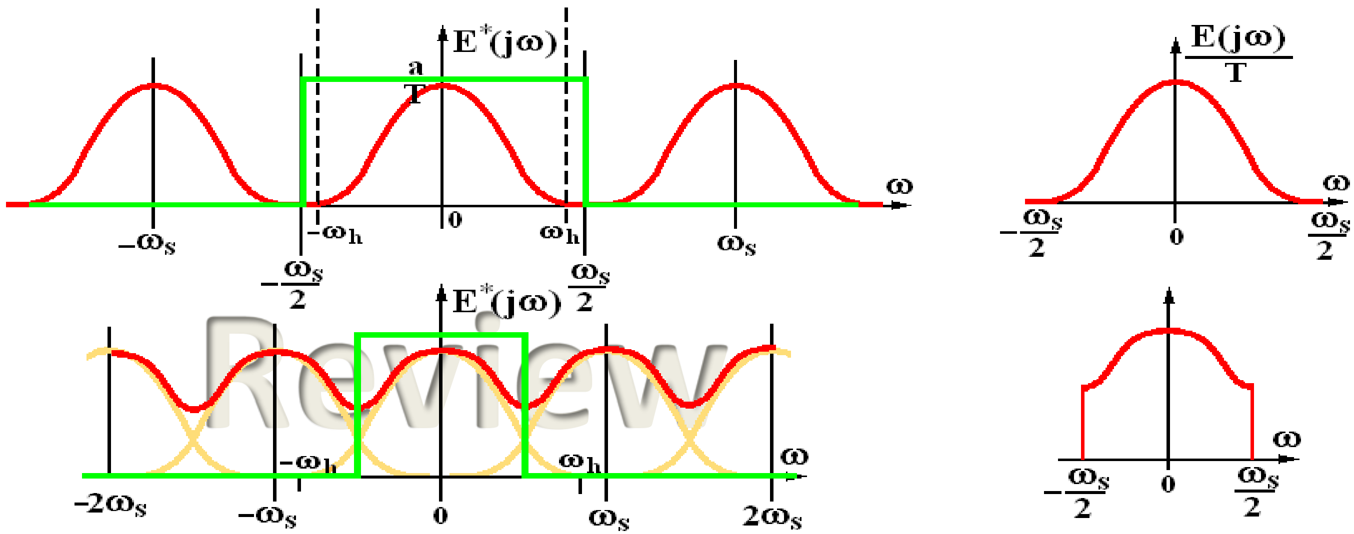
The Necessary Condition for signal recovery

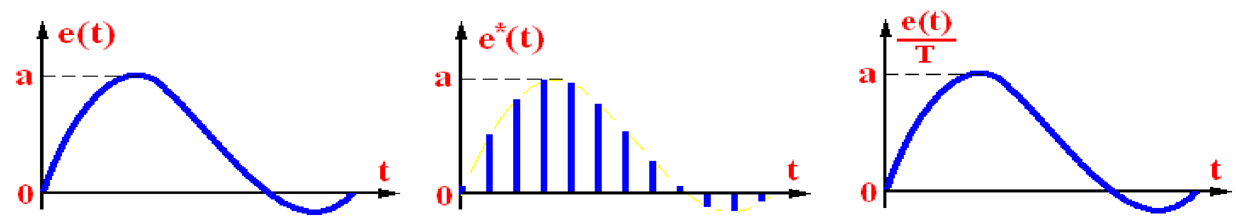
$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$

$$\omega_s = \frac{2\pi}{T} < 2\omega_h$$

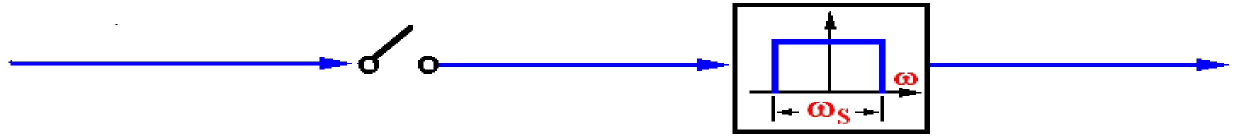
$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$

$$T < \frac{\pi}{\omega_h}$$





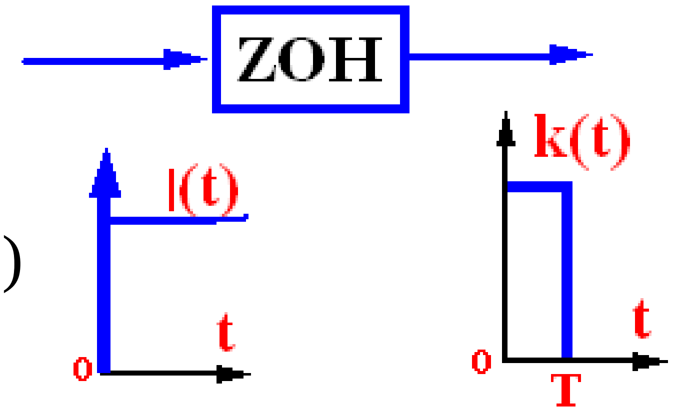
$$\omega_s > 2\omega_h$$



Sampling switch **Ideal filter**

7.3.2 Zero-Order Hold

$$k(t)=1(t)-1(t-T)$$



$$x_h(t)=\sum_{k=0}^{\infty}x(kT)(1(t-kT)-1(t-kT-T))$$

Using L-Transform, we get

$$x_h(s)=\underbrace{\sum_{k=0}^{\infty}x(kT)e^{-kTs}}_{x^*(s)}\left[\frac{1}{s}-\frac{1}{s}e^{-Ts}\right]$$

$$\frac{x_h(s)}{x^*(s)}=\frac{1-e^{-Ts}}{s}=G_h(s)$$

Frequency characteristics:

$$\boldsymbol{G_h(j\omega) = \frac{1-e^{-j\omega T}}{j\omega}} = \frac{e^{-j\frac{\omega T}{2}}(e^{j\frac{\omega T}{2}} - e^{-j\frac{\omega T}{2}})}{j\omega} = \frac{2e^{-j\frac{\omega T}{2}}\sin(\frac{\omega T}{2})}{\omega}$$

$$G_h(j\omega) = T\frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}}e^{-j\frac{\omega T}{2}}$$

$$\because T = \frac{2\pi}{\omega_s}$$

setting $S_a(x) = \sin x / x$

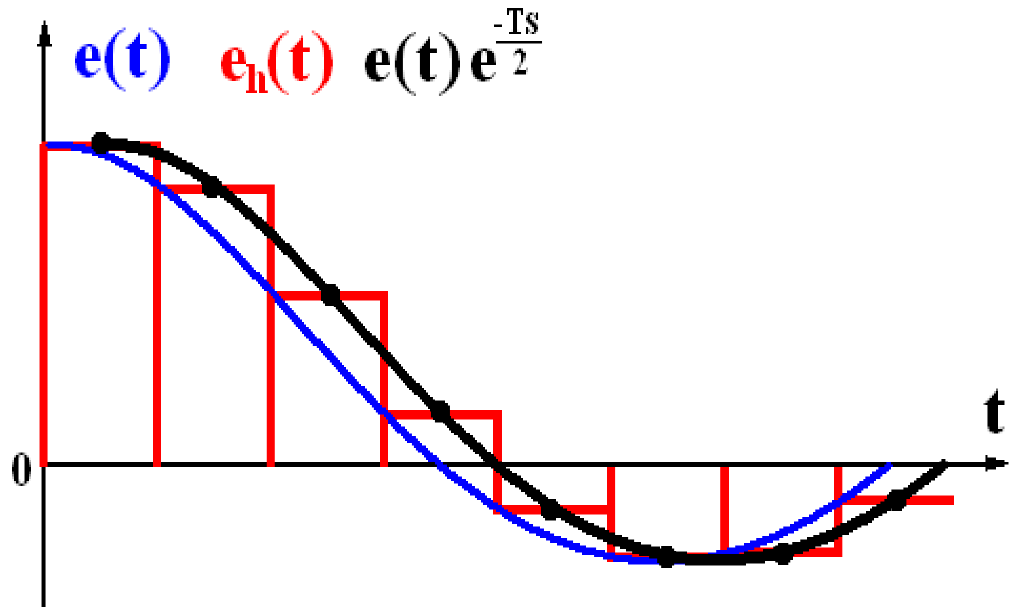
so the

$$G_h(j\omega) = \frac{2\pi}{\omega_s} \cdot S_a(\pi\omega/\omega_s) \cdot e^{-j\frac{\pi\omega}{\omega_s}}$$

Effect of zero-order holder on the system

$$G_h(s) = \frac{1 - e^{-Ts}}{s}$$

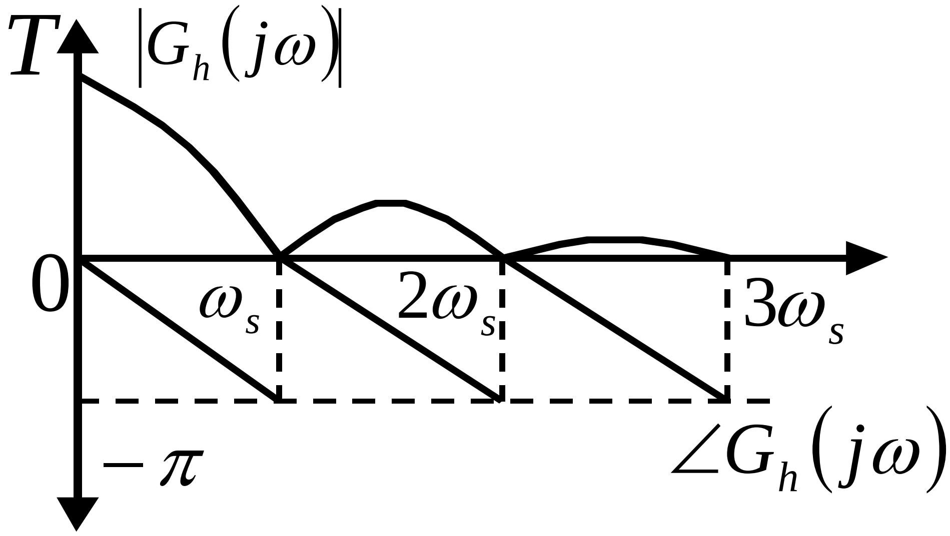
$$\approx T \cdot e^{-Ts/2}$$

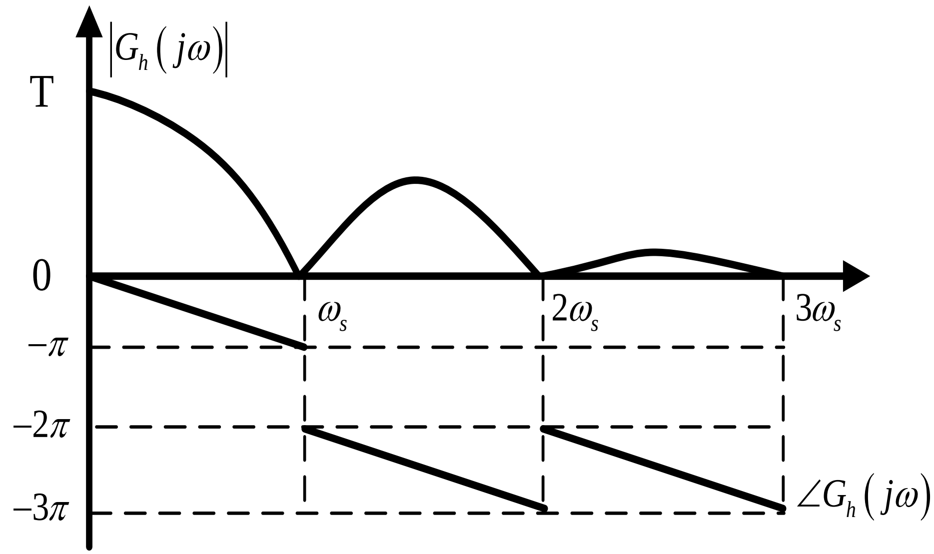


the amplitude $|G_h(j\omega)| = \dfrac{2\pi}{\omega_s} \cdot |S_a(\pi\omega/\omega_s)|$

phase angle $\angle G_h(j\omega) = \angle S_a(\pi\omega/\omega_s) - \dfrac{\pi\omega}{\omega_s}$

where $\angle S_a(\pi\omega/\omega_s) = \begin{cases} 0, & 2n\omega_s < \omega < (2n+1)\omega_s \\ -\pi, & (2n+1)\omega_s < \omega < (2n+2)\omega_s \end{cases}$

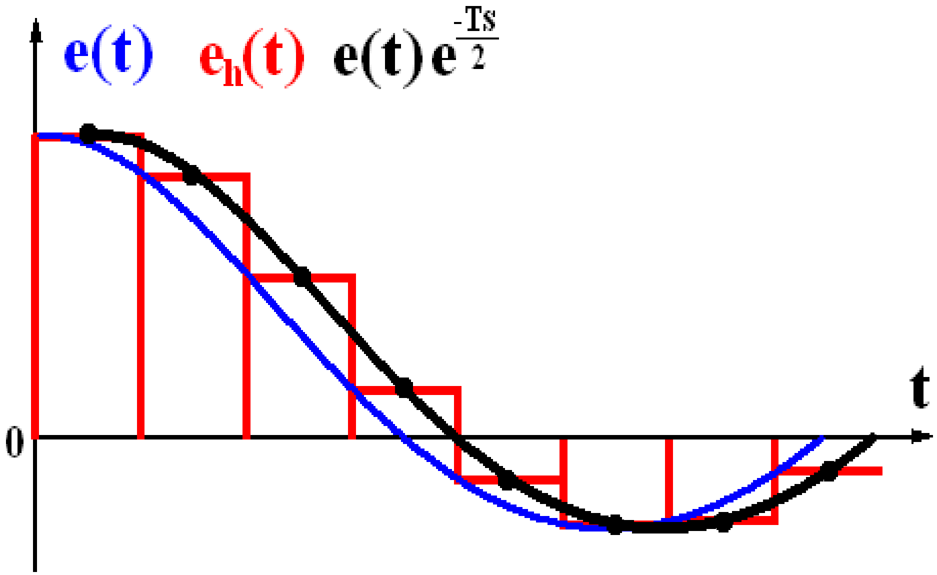


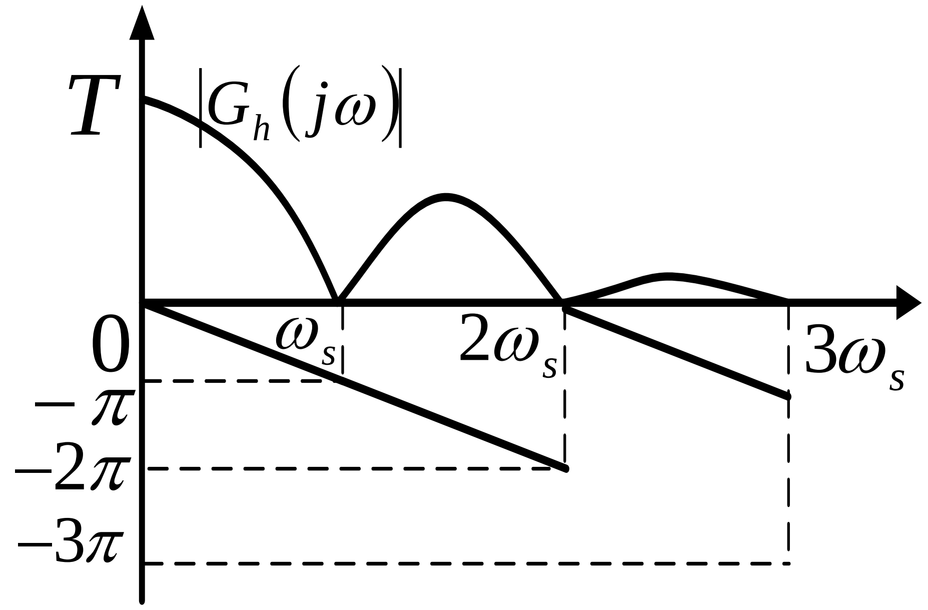


Which one do you think is correct?

Properties

- Low pass filter, but not the ideal filter. Existing Ripples – 纹波；
- Phase delay, reduce the stability.
- Time delay





Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

7.4 Z-Transform and Inverse Z Transform

7.4.1. Z-transform

Definition:

$$\because E^*(s) = \sum_{k=0}^{+\infty} e\ (kT) \cdot e^{-kTs}$$

Set

$$z = e^{Ts} \qquad\qquad s = \frac{1}{T} \ln z$$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

$$E(z) = E^*(s)\,|_{z=e^{Ts}}$$

$$E^*(s) = \sum_{k=0}^{\infty} e(kT)\cdot e^{-kTs}$$

$$z^{-1} = e^{-Ts}$$

$$E(z) = Z[e^*(t)] = E^*(s)\Big|_{z=e^{Ts}} = \sum_{n=0}^{\infty} e(kT)\cdot z^{-k}$$

Rmk:

- $E(z) = Z[e^*(t)] = Z[E(s)] = Z[E^*(s)] = Z[e(t)]$
- The z-transform is only for discrete signal.
- E(z) is only mapping to a unique e*(t), but not a unique e(t).

7.4.2 Methods of z-Transform

- **By the definition – summation of series**
- **Partial fraction expansion (部分分式展开).**

1. By the Definition

Example 1 $x_1(t) = 1(t)$ **and** $x_2(t) = \sum_{k=0}^{\infty} \delta(t - kT)$, **obtain** $X_1(z)$ and $X_2(z)$。

Solution:

$$X_1(z) = \sum_{k=0}^{\infty} x_1(kT) z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

$$X_2(z) = \sum_{k=0}^{\infty} x_2(kT) z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{z}{z - 1}$$

Tips: Though $x_1(t)$ and $x_2(t)$ are not same, they may have the same Z-transform.

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

Example 2 $\quad \boldsymbol{e(t)=\sin\omega t=\dfrac{e^{j\omega t}-e^{-j\omega t}}{2j}}$

$$E(z)=\sum_{k=0}^{\infty}\frac{1}{2j}\left[e^{j\omega kT}-e^{-j\omega kT}\right]\cdot z^{-k}\quad =\frac{1}{2j}\sum_{k=0}^{\infty}\left[(e^{j\omega T}z^{-1})^{k}-(e^{-j\omega T}z^{-1})^{k}\right]$$

$$=\frac{1}{2j}\left[\frac{1}{1-e^{j\omega T}z^{-1}}-\frac{1}{1-e^{-j\omega T}z^{-1}}\right]\ \boldsymbol{=\frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}}-\frac{z}{z-e^{-j\omega T}}\right]}$$

$$=\frac{1}{2j}\cdot\frac{z(e^{j\omega T}-e^{-j\omega T})}{z^{2}-(e^{j\omega T}+e^{-j\omega T})z+1}\ \boldsymbol{=\frac{z\sin\omega T}{z^{2}-2\cos\omega T\cdot z+1}}$$

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

Example 3 $e(t)=t$

Solution. $E(z)=\sum_{k=0}^{\infty} kT\cdot z^{-k}=T\left[z^{-1}+2z^{-2}+3z^{-3}+\ \cdots\ \right]$

$$=Tz\left[z^{-2}+2z^{-3}+3z^{-4}+\ \cdots\ \right]$$

$$=-Tz\left[\frac{d}{dz}z^{-1}+\frac{d}{dz}z^{-2}+\frac{d}{dz}z^{-3}+\ \cdots\ \right]$$

$$=-Tz\frac{d}{dz}\left[z^{-1}+z^{-2}+z^{-3}+\ \cdots\ \right]$$

$$=-Tz\frac{d}{dz}z^{-1}\left[1+z^{-1}+z^{-2}+\ \cdots\ \right]$$

$$=-Tz\frac{d}{dz}\left[\frac{1}{z}\cdot\frac{1}{1-z^{-1}}\right]=-Tz\frac{d}{dz}\left[\frac{1}{z-1}\right]=\frac{Tz}{(z-1)^2}$$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

Example 3(2) $\boldsymbol{e(t) = t}$

Solution: $$E(z) = \sum_{k=0}^{\infty} kT \cdot z^{-k} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots \qquad (1)$$

$$E(z)z - T = 2Tz^{-1} + 3Tz^{-2} + 4T\ z^{-3} + \cdots \qquad (2)$$

(2)-(1)

$$E(z)(z-1) - T = Tz^{-1} + Tz^{-2} + T\ z^{-3} + \cdots$$

$$= \frac{Tz^{-1}}{1 - z^{-1}} = \frac{T}{z-1}$$

$$E(z) = \frac{Tz}{(z-1)^2}$$

$$E(z)=\sum_{k=0}^{\infty}e(kT)\cdot z^{-k}$$

2. Partial Fraction Expansion

Example 4 $E(s)=\dfrac{1}{(s+a)(s+b)}$ **Obtain $E(z)$=?**

Solution：

$$E(s)=\frac{1}{a-b}\cdot\frac{(s+a)-(s+b)}{(s+a)(s+b)}=\frac{1}{a-b}\left[\frac{1}{s+b}-\frac{1}{s+a}\right]$$

$$e(t)=\frac{1}{a-b}\left[e^{-bt}-e^{-at}\right]$$

$$E(z)=\frac{1}{a-b}\sum_{k=0}^{\infty}\left[e^{-bkT}-e^{-akT}\right]\cdot z^{-k}$$

$$=\frac{1}{a-b}\left[\sum_{k=0}^{\infty}(e^{-bT}\cdot z^{-1})^{k}-\sum_{k=0}^{\infty}(e^{-aT}\cdot z^{-1})^{k}\right]$$

$$=\frac{1}{a-b}\left[\frac{1}{1-e^{-bT}z^{-1}}-\frac{1}{1-e^{-aT}z^{-1}}\right]=\frac{1}{a-b}\left[\frac{z}{z-e^{-bT}}-\frac{z}{z-e^{-aT}}\right]$$

The z-transform of typical functions

序号	拉氏变换 E(s)	时间函数 e(t)	Z 变换 E(z)
1	1	δ (t)	1
2	$\frac{1}{1-e^{-Ts}}$	$\delta_T(t)=\sum_{n=0}^{\infty}\delta(t-nT)$	$\frac{z}{z-1}$
3	$\frac{1}{s}$	$1(t)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{t^2}{2}$	$\frac{T^2z(z+1)}{2(z-1)^3}$
6	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	$\lim_{a\to 0}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial a^n}(\frac{z}{z-e^{-aT}})$
7	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$

8	$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
9	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
10	$\frac{b-a}{(s+a)(s+b)}$	$\boxed{e^{-at}-e^{-bt}}$	$\boxed{\frac{z}{z-e^{-aT}}-\frac{z}{z-e^{-bT}}}$
11	$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
12	$\frac{s}{s^2+\omega^2}$	$\cos\omega t$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
13	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin\omega t$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
14	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
15	$\frac{1}{s-(1/T)\ln a}$	$\boxed{a^{t/T}}$	$\boxed{\frac{z}{z-a}}$

- If $E(z) = Z\left[e(t)\right]$, try to proof:

- (1) $$E(\frac{z}{a}) = Z\left[a^k e(t)\right]$$

- (2) $$-Tz\frac{dE(z)}{dz} = Z\left[te(t)\right]$$

Review in one page

- Shannon Sampling Theorem—The Necessary Condition for signal recovery: $\omega_s \geq 2\omega_{\max}$
- $\longrightarrow \boxed{\text{ZOH}} \longrightarrow$ $\quad G_h(s) = \dfrac{1-e^{-Ts}}{s}$
- z-Transform $\quad E(z) = \sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$

  *if k<0, e(kT)=0

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

7.4.3 Properties of z-Transform

1. linear property $Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$

2. Real shift theorem 实位移定理

① Lag 延时定理 $Z\left[e(t-nT)\right]=z^{-n}E(z)$

Proof.

$$\mathrm{LHS}=\sum_{K=0}^{\infty} e(kT-nT)\cdot z^{-k}$$

$$\Big\downarrow\; j=k-n$$

$$=\sum_{j=-n}^{\infty} e(jT)\cdot z^{-(j+n)}=z^{-n}\sum_{j=0}^{\infty} e(jT)\cdot z^{-j}$$

$$=z^{-n}E(z)=\mathrm{RHS}$$

2. Real shift theorem 实位移定理

② Lead 超前定理

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

Proof.

$$\text{LHS}=\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-k}=z^{n}\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-(k+n)}$$

$$j=k+n$$

$$=z^{n}\sum_{j=n}^{\infty}e(jT)\cdot z^{-j}=z^{n}\left[\sum_{j=0}^{\infty}e(jT)\cdot z^{-j}-\sum_{j=0}^{n-1}e(jT)\cdot z^{-j}\right]$$

$$=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]=\text{RHS}$$

$$Z[e(t-nT)] = z^{-n}E(z)$$

$$Z[e(t+nT)] = z^{n}\left[E(z) - \sum_{k=0}^{n-1} e(kT)\cdot z^{-k}\right]$$

Example 5 $e(t) = t - T$

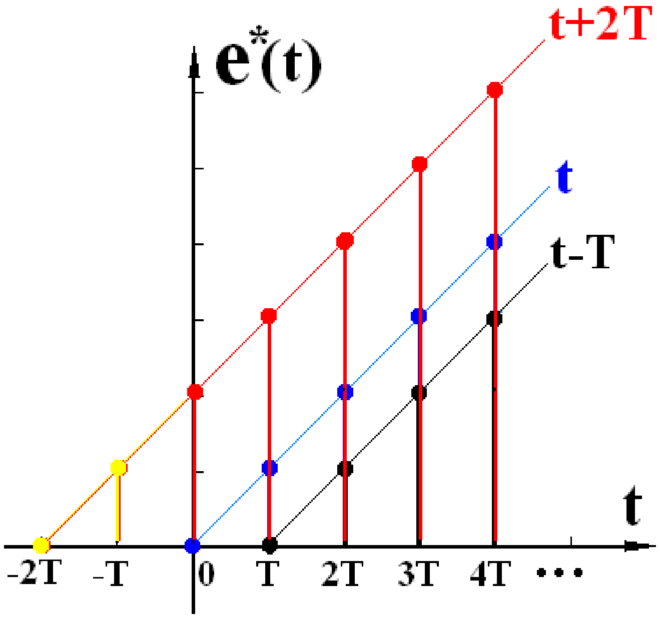
$$E(z) = Z[t-T] = z^{-1}Z[t] = z^{-1}\frac{Tz}{(z-1)^2} = \frac{T}{(z-1)^2}$$

Example 6 $e(t) = t + 2T$

$$E(z) = Z[t+2T]$$

$$= z^2\left\{Z[t] - \sum_{k=0}^{1} kT\cdot z^{-k}\right\}$$

$$= z^2\left[\frac{Tz}{(z-1)^2} - 0 - Tz^{-1}\right]$$



3. Complex shift theorem

复位移定理

$$Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$$

Proof.

$$\mathrm{LHS}=\sum_{k=0}^{\infty}e(kT)\cdot e^{\mp akT}z^{-k}\quad=\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}$$

$$z_1=z\cdot e^{\pm aT}$$

$$=\sum_{k=0}^{\infty}e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}=E(z_1)=E\left(z\cdot e^{\pm akT}\right)=\mathrm{RHS}$$

Example 7 $e(t)=t\cdot e^{-at}$

$$E(z_1)=Z[t]\big|_{z_1=z\cdot e^{aT}}=\frac{Tz_1}{(z_1-1)^2}=\frac{T(z\cdot e^{aT})}{(z\cdot e^{aT}-1)^2}=\frac{Tz\cdot e^{-aT}}{(z-e^{-aT})^2}$$

4	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$

4. Initial-value Theorem

$$\lim_{n\to 0} e(nT) = \lim_{z\to\infty} E(z)$$

Proof:

$$E(z) = \sum_{n=0}^{\infty} e(nT)\cdot z^{-n}$$

$$= \left[e(0) + e(1)\cdot z^{-1} + e(2)\cdot z^{-2} + e(3)\cdot z^{-3} + \cdots\right]$$

$$\lim_{z\to\infty} E(z) = e(0)$$

Example 8 $E(z) = \dfrac{0.792\cdot z^2}{(z-1)[z^2 - 0.416z + 0.208]}$

$$e(0) = \lim_{z\to\infty} E(z) = 0$$

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

5. Final value theorem $\lim_{n\to\infty} e(nT) = \lim_{z\to 1}(z-1)\cdot E(z)$

Proof： $Z\left[e(t+T)-e(t)\right]=z\left[E(z)-e(0)\right]-E(z)$

$$=(z-1)E(z)-z\cdot e(0)$$

$$(z-1)E(z)=z\cdot e(0)+Z\left[e(t+T)-e(t)\right]$$

$$\lim_{z\to 1}(z-1)E(z)=\lim_{z\to 1}\left\{z\cdot e(0)+\sum_{n=0}^{\infty}\left[e[(n+1)T]-e(nT)\right]\cdot z^{-n}\right\}$$

$$=e(0)+[e(1)-e(0)]+[e(2)-e(1)]+[e(3)-e(2)]+\cdots$$

$$=e(\infty T)$$

Example 9 $E(z)=\dfrac{0.792\cdot z^2}{(z-1)[z^2-0.416z+0.208]}$

$$e(\infty T)=\lim_{z\to 1}(z-1)E(z)=\lim_{z\to 1}\frac{0.792\cdot z^2}{[z^2-0.416z+0.208]}=\mathbf{1}$$

6. Convolution theorem

If $c^*(t)=e^*(t)*g^*(t)=\sum_{k=0}^{\infty}e(kT)\cdot g[(n-k)T]$

We have: $\boldsymbol{C(z)=E(z)\cdot G(z)}$

Proof：

$$E(z)=\sum_{k=0}^{\infty}e(kT)\cdot z^{-k}\qquad G(z)=\sum_{k=0}^{\infty}g(kT)\cdot z^{-k}$$

$$E(z)=\sum_{k=0}^{\infty}e(kT)\cdot z^{-k}$$

$$G(z)=\sum_{k=0}^{\infty}g(kT)\cdot z^{-k}$$

$$\boldsymbol{Z\left[e(t-nT)\right]=z^{-n}E(z)}$$

$$e^{*}(t)*g^{*}(t)=\sum_{k=0}^{\infty}e(kT)\cdot g[(n-k)T]$$

$$E(z)\cdot G(z)=\left(\sum_{k=0}^{\infty}e(kT)\cdot z^{-k}\right)G(z)$$

$$=\sum_{k=0}^{\infty}e(kT)\cdot\left(z^{-k}G(z)\right)$$

$$=\sum_{k=0}^{\infty}e(kT)\left\{Z\left[g(nT-kT)\right]\right\}$$

$$=\sum_{k=0}^{\infty}e(kT)\sum_{n=0}^{\infty}\left[g(nT-kT)\right]z^{-n}$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\infty}e(kT)\left[g(nT-kT)\right]\right\}z^{-n}$$

$$=\sum_{n=0}^{\infty}\left\{e(nT)*g(nT)\right\}z^{-n}$$

$$=\sum_{n=0}^{\infty}c(nT)z^{-n}=C(z)$$

Properties of z-Transform

1. linear property $$Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$$

2. Real shifting theorem
$$\begin{cases}\text{Lag} & Z\left[e(t-nT)\right]=z^{-n}E(z)\\ \textbf{Lead} & Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]\end{cases}$$

3. Complex shifting theorem
$$Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$$

4. Initial-value theorem
$$\lim_{n\to 0}e(nT)=\lim_{z\to\infty}E(z)$$

5. Final-value theorem
$$\lim_{n\to\infty}e(nT)=\lim_{z\to 1}(z-1)\cdot E(z)$$

6. Convolution theorem
$$c^*(t)=e^*(t)*g^*(t)\quad\Rightarrow\quad C(z)=E(z)\cdot G(z)$$

- Instant work:

- Calculate the initial and final values:

- $$E(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2}$$

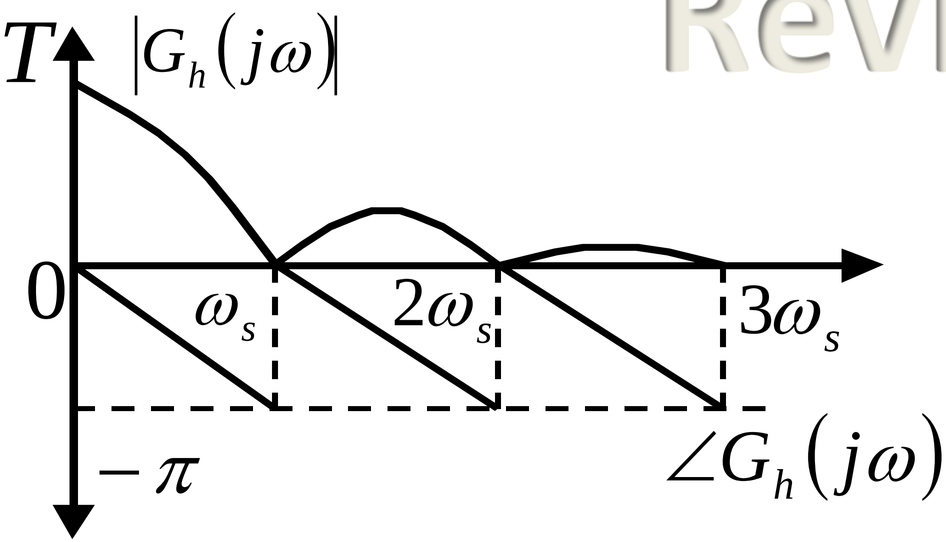
$$E(z) = \frac{z^2}{(z - 0.8)(z - 0.1)}$$

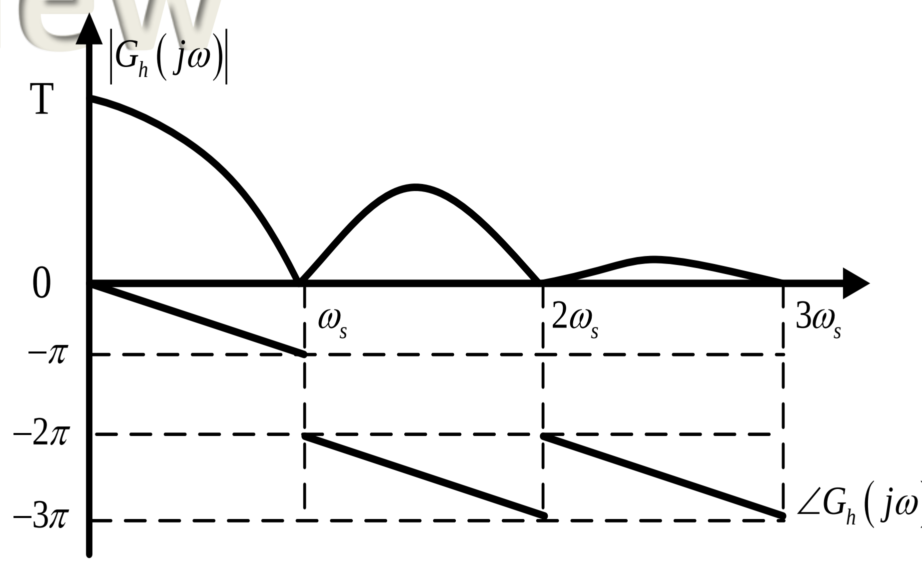
the amplitude $|G_h(j\omega)| = \dfrac{2\pi}{\omega_s} \cdot |S_a(\pi\omega/\omega_s)|$

phase angle $\angle G_h(j\omega) = \angle S_a(\pi\omega/\omega_s) - \dfrac{\pi\omega}{\omega_s}$

where $\angle S_a(\pi\omega/\omega_s) = \begin{cases} 0, & 2n\omega_s < \omega < (2n+1)\omega_s \\ \pi, & (2n+1)\omega_s < \omega < (2n+2)\omega_s \end{cases}$

Review





Which one do you think is correct?

7.4 Z-Transform and Inverse Z Transform

7.4.1. Z-transform

Definition:

$$\because E^*(s) = \sum_{k=0}^{+\infty} e(kT) \cdot e^{-kTs}$$

Review

Set

$$z = e^{Ts} \qquad s = \frac{1}{T} \ln z$$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

$$E(z) = E^*(s)|_{z=e^{Ts}}$$

7.4.4 Inverse z-Transform

$$Z^{-1}[X(z)] = x(nT)$$

Tips:

Inverse Z-transform can only provide discrete-time signal $x^*(t)$, instead of continuous signal x(t)。

- Long Division（长除法）
- Partial-Fraction expansion (PFE) Expansion of $\frac{E(z)}{z}$
- Residue（留数法） $e(nT) = \sum \mathrm{Res}\left[E(z) \cdot z^{n-1}\right]$

1. Long Division（长除法）/ Power Series

$$E(z)=\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}$$

Numerator is divided by denominator ,we get

$$E(z)=c_0 + c_1 z^{-1} + \cdots + c_k z^{-k} + \cdots = \sum_{k=0}^{\infty} c_k z^{-k} = \sum_{k=0}^{\infty} e(kT) z^{-k}$$

$$e^*(t)=c_0\delta(t)+c_1\delta(t-T)+\cdots+c_k\delta(t-kT)+\cdots$$

Example 10 $F(z)=\dfrac{z}{(z-2)(z-3)}$, **obtain f*(t).**

Solution:

Because $F(z)=\dfrac{z}{z^2-5z+6}=\dfrac{z^{-1}}{1-5z^{-1}+6z^{-2}}$

By long-division, we get that

$$F(z)=z^{-1}+5z^{-2}+19z^{-3}+65z^{-4}+\cdots$$

Thus

$$f(0)=0,\quad f(T)=1,\quad f(2T)=5,\quad f(3T)=19,\quad f(4T)=65,\cdots$$

Then

$$f^*(t)=\delta(t-T)+5\delta(t-2T)+19\delta(t-3T)+65\delta(t-4T)+\cdots$$

2. Partial fraction expansion (PFE)

序号	拉氏变换 E(s)	时间函数 e(t)	Z 变换 E(z)
1	1	δ (t)	1
2	$\frac{1}{1-e^{-Ts}}$	$\delta_T(t)=\sum_{n=0}^{\infty}\delta(t-nT)$	$\frac{z}{z-1}$
3	$\frac{1}{s}$	$1(t)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{t^2}{2}$	$\frac{T^2z(z+1)}{2(z-1)^3}$

Note: here, we expand $\frac{X(z)}{z}$, instead of z。

$$\frac{X(z)}{z}=\sum_{i=1}^{n}\frac{A_i}{z-z_i}$$

Consider

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n}$$

Then

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{a_0\prod_{i=1}^{n}(z-z_i)}$$

If there is no repeated root for the denominator， it generates

$$X(z)=z\left(\frac{A_1}{z-z_1}+\frac{A_2}{z-z_2}+\cdots+\frac{A_n}{z-z_n}\right)$$

Then look up each part in the transform table.

PS: the coefficients A_i **, are decided by:** $A_i=\left[(z-z_i)\frac{X(z)}{z}\right]\Big|_{z=z_i}$

Example 12 $$E(z)=\frac{z^2}{(z-0.8)(z-0.1)}$$ Obtain e*(t). (PFE & Residue)

PFE: $$\frac{E(z)}{z}=\frac{z}{(z-0.8)(z-0.1)}=\frac{C_1}{(z-0.8)}+\frac{C_2}{(z-0.1)}$$

$$C_1=\lim_{z\to 0.8}\frac{z}{(z-0.1)}=\frac{8}{7}\qquad C_2=\lim_{z\to 0.1}\frac{z}{(z-0.8)}=\frac{-1}{7}$$

$$=\frac{8/7}{(z-0.8)}-\frac{1/7}{(z-0.1)}$$

$$E(z)=\frac{8}{7}\cdot\frac{z}{(z-0.8)}-\frac{1}{7}\cdot\frac{z}{(z-0.1)}$$

$$e(t)=(8\times 0.8^{\frac{t}{T}}-0.1^{\frac{t}{T}})/7\qquad e(nT)=(8\times 0.8^{n}-0.1^{n})/7$$

$$e^*(t)=\sum_{n=0}^{\infty}\left[(8\times 0.8^{n}-0.1^{n})/7\right]\cdot\delta(t-nT)$$

Example 13 **Consider**

$$F(z)=\frac{z}{(z-1)(z-e^{-T})}$$

Obtain f*(t)。

Solution：

$$K_1=\lim_{z\to 1}\left(\frac{z-1}{z}\right)F(z)=\frac{1}{1-e^{-T}}$$

$$K_2=\lim_{z\to e^{-T}}\left(\frac{z-e^{-T}}{z}\right)F(z)=-\frac{1}{1-e^{-T}}$$

$$\frac{F(z)}{z}=\frac{K_1}{z-1}+\frac{K_2}{z-e^{-T}}$$

$$F(z)=\frac{1}{1-e^{-T}}\left(\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right)$$

$$\Rightarrow\quad f(nT)=\frac{1}{1-e^{-T}}\left(1-e^{-nT}\right)$$

$$f^*(t)=\frac{1}{1-e^{-T}}\sum_{k=0}^{+\infty}(1-e^{-kT})\delta(t-kT)$$

3、 Residue(留数法) / Inversion Integral

$$F(z)=\sum_{k=0}^{+\infty} f(kT)z^{-k}$$ **"Laurent Series"**

$$= f(0)+f(T)z^{-1}+f(2T)z^{-2}+\cdots+f(nT)z^{-n}+\cdots$$

$$F(z)z^{m-1}=\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}$$

$$= f(0)z^{m-1}+f(T)z^{m-2}+f(2T)z^{m-3}+\cdots+f(mT)z^{-1}+\cdots$$

"Cauchy's residual theorem"

Γ Encircle all the poles of $F(z)z^{k-1}$

$$\oint_\Gamma F(z)z^{m-1}dz=\oint_\Gamma\left[\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}\right]dz$$

$$=\oint_\Gamma f(0)z^{m-1}dz+\oint_\Gamma f(T)z^{m-2}dz+\cdots+\oint_\Gamma f(mT)z^{-1}dz+\cdots$$

"Complex function theorem"

$$\oint_\Gamma z^{m-k-1}dz=\begin{cases}0, & if\ \ m\neq k\\ 2\pi j, & if\ \ m=k\end{cases}$$

$$\oint_{\Gamma} F(z)z^{m-1}dz = \oint_{\Gamma} f(kT)z^{-1}dz = f(kT)\cdot 2\pi j$$

"Cauchy's residual theorem"

$$f(kT) = \frac{1}{2\pi j}\oint_{\Gamma} f(kT)z^{-1}dz = \sum_{i=1}^{n} Res[F(z)z^{k-1}, z_i]$$

$z_i, i = 1,2,\cdots,n$ **are all the poles of** $F(z)z^{k-1}$

其中***Res*****[]**表示函数的留数。

$$Res\left[z^{(k-1)}x(z)\right] = \lim_{z\to z_i}\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}\left[(z-z_i)^r z^{k-1}x(z)\right]$$

Example 14 For

$$F(z) = \frac{10z}{(z-1)(z-2)}$$

Obtain its inverse z-transform by residue Method.

Solution: $$F(z)z^{k-1} = \frac{10z^k}{(z-1)(z-2)}$$

Poles $z_1 = 1$ and $z_2 = 2$, and

$$res[F(z)z^{k-1},1] = \lim_{z \to 1}(z-1)F(z)z^{k-1} = -10$$

$$res[F(z)z^{k-1},2] = \lim_{z \to 2}(z-2)F(z)z^{k-1} = 10 \cdot 2^k$$

Then $$f(kT) = 10(2^k - 1) \qquad (k = 0,1,2,\cdots)$$

Example 15 $E(z)=\dfrac{5}{(z-a)^2}$ Obtain e*(t). (Residue)

Solution.

$$e(nT)=\sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]=\underset{z=a}{\mathrm{Res}}\left[\frac{5}{(z-a)^2}\cdot z^{n-1}\right]$$

$$e(nT)=\frac{1}{(2-1)!}\lim_{z\to a}\frac{d}{dz}\left[(z-a)^2\frac{5\cdot z^{n-1}}{(z-a)^2}\right]$$

$$=\lim_{z\to a}\frac{d}{dz}\left[5\cdot z^{n-1}\right]$$

$$=5\cdot\lim_{z\to a}\left[(n-1)\cdot z^{n-2}\right]$$

$$=5\cdot(n-1)\cdot a^{n-2}$$

$$e^*(t)=\sum_{n=0}^{\infty}\left(5(n-1)\cdot a^{n-2}\right)\cdot\delta(t-nT)$$

Example 16 $X(z)=\dfrac{az}{\sin mz}$ Obtain x*(t). (Residue)

Solution.

$$x(kT)=\sum \mathrm{Res}\left[\frac{az\cdot z^{k-1}}{\sin mz}, z=\frac{n\pi}{m}, n=0,1,\ldots\right]$$

$$Res\left[z^{(k-1)}x(z)\right]=\lim_{z\to z_i}\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}\left[(z-z_i)^r z^{k-1}x(z)\right]$$

$$x(kT)=\sum_{n=0}^{\infty}\lim_{z\to\frac{n\pi}{m}}(z-\frac{n\pi}{m})\frac{az^k}{\sin mz}=$$

$$=\sum_{n=0}^{\infty}\frac{az^k(z-n\pi/m)}{\sin mz}\Bigg|_{z\to\frac{n\pi}{m}}=\sum_{n=0}^{\infty}a(\frac{n\pi}{m})^k\frac{(z-n\pi/m)}{\sin mz}\Bigg|_{z\to\frac{n\pi}{m}}$$

$$=\sum_{n=0}^{\infty}a(\frac{n\pi}{m})^k\frac{1}{m\cos mz}\Bigg|_{z\to\frac{n\pi}{m}}=\sum_{n=0}^{\infty}(-1)^n\frac{a}{m}(\frac{n\pi}{m})^k$$

7.4.5 Explanation of z-Transform

(1) Uniqueness; $E_1(z) = E_2(z) \Leftrightarrow e_1^*(t) = e_2^*(t)$

$\not\Rightarrow e_1(t) = e_2(t)$

(2) Region of Convergence

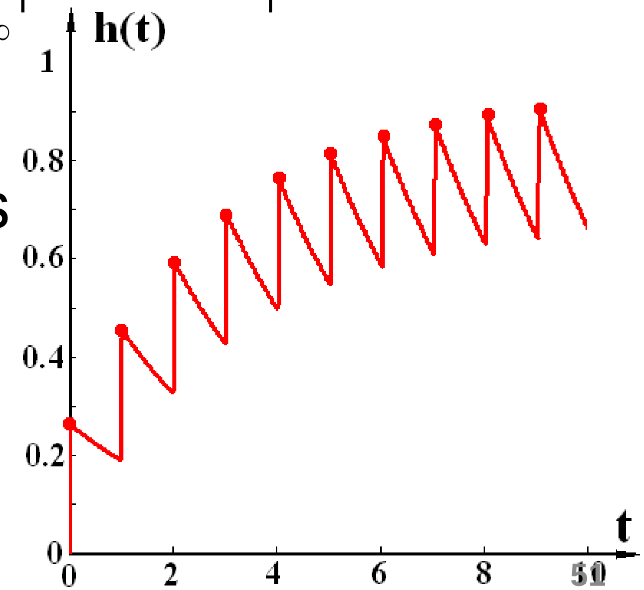
$$s = \sigma + j\omega \Rightarrow z = e^{sT} = e^{\sigma T} e^{j\omega T} \quad if: r = |z| = e^{\sigma T}, z = re^{j\omega T}$$

$$L: \int_0^{\infty} \left| e(t)e^{-\sigma t} \right| dt < \infty \Rightarrow Z: \sum_{n=-\infty}^{\infty} \left| e(nT)r^{-n} \right| < \infty$$

Some limitations of z-Transform

(1) only shows the information of samples

(2) In some cases, the continuous signal may jump on the sampling point.



- Homework:
- You should know.

$$\boldsymbol{z^{-1} = e^{-Ts}}$$

$$E(z) = Z[e^*(t)] = E^*(s)\Big|_{z=e^{Ts}} = \sum_{n=0}^{\infty} e(kT)\cdot z^{-k}$$

7.4.2 Methods of z-Transform

By the definition.

Partial fraction expansion.

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.1 Introduction

7.2 The Sampling Process and Sampling Theorem

7.3 Signal Recovery and Zero-Order Hold

7.4 Z-Transform and Inverse Z Transform

7.5 Mathematical Models of Discrete-Time Systems

7.6 Performance Analysis of Discrete-Time Systems

7.7 Digital Control Design for Discrete-Time Systems

7.5 Mathematical Models of Discrete-Time Systems

- **Difference Equation**
- **Impulse Transfer function**

7.5.1 Linear Time-Invariant Difference Equations

(1) Definition of difference e(kT) = e(k)

Forward difference

First-order $\Delta e(k) = e(k+1) - e(k)$

Second-order $\Delta^2 e(k) = \Delta e(k+1) - \Delta e(k)$

$= e(k+2) - 2e(k+1) + e(k)$

$\vdots$

nth-order $\Delta^n e(k) = \Delta^{n-1} e(k+1) - \Delta^{n-1} e(k)$

$$\dot{e}(t) = \frac{\mathrm{d}e(t)}{\mathrm{d}t} = \lim_{T\to 0} \frac{\Delta e(k)}{T} \approx \frac{\Delta e(k)}{T}$$

Backward difference

$$\begin{cases} \text{First-order} & \nabla e(k) = e(k) - e(k-1) \\ \text{Second-order} & \nabla^2 e(k) = \nabla e(k) - \nabla e(k-1) \\ \vdots & \qquad = e(k) - 2e(k-1) + e(k-2) \\ \text{nth-order} & \nabla^n e(k) = \nabla^{n-1} e(k) - \nabla^{n-1} e(k-1) \end{cases}$$

$$\dot{e}(t) = \frac{\mathrm{d}e(t)}{\mathrm{dt}} = \lim_{T\to 0} \frac{\nabla e(k)}{T} \approx \frac{\nabla e(k)}{T}$$

(2) Difference equation

The equation of the input, output and their higher order differences.

The (forward) difference equation of nth-order linear time-invariant discrete system.

$$c(k+n) + a_1 c(k+n-1) + a_2 c(k+n-2) + \cdots + a_{n-1} c(k+1) + a_n c(k)$$
$$= b_0 r(k+m) + b_1 r(k+m-1) + \cdots + b_{m-1} r(k+1) + b_m r(k)$$

or $$c(k+n) = -\sum_{i=1}^{n} a_i c(k+n-i) + \sum_{j=0}^{m} b_j r(k+m-j)$$

The (backward) differential equation of n-order linear time-invariant discrete system.

$$\boldsymbol{c(k) + a_1 c(k-1) + a_2 c(k-2) + \cdots + a_{n-1} c(k-n+1) + a_n c(k-n)}$$

$$\boldsymbol{= b_0 r(k) + b_1 r(k-1) + L + b_{m\text{-}1} r(k-m+1) + b_m r(k-m)}$$

or $$c(k) = -\sum_{i=1}^{n} a_i c(k-i) + \sum_{j=0}^{m} b_j r(k-j)$$

(3) To solve difference equations: $\begin{cases} \text{Iteration method (迭代)} \\ \text{Z-transform method} \end{cases}$

Example 1 The differential equation of a continuous system is:

$$\begin{cases} \ddot{e}(t) - 4\dot{e}(t) + 3e(t) = r(t) = 1(t) \\ e(t) = 0 \qquad (t \le 0) \end{cases}$$

Obtain the corresponding forward difference equation and its solution.

Solution.

$$\dot{e}(t) \approx \frac{\Delta e(k)}{T} = \frac{e(k+1) - e(k)}{T} \stackrel{T=1}{=} e(k+1) - e(k)$$

$$\ddot{e}(t) \approx \frac{\Delta^2 e(k)}{T^2} = \frac{\Delta e(k+1)/T - \Delta e(k)/T}{T} \stackrel{T=1}{=} e(k+2) - 2e(k+1) + e(k)$$

$$\begin{array}{rl} & e(k+2) - 2e(k+1) + e(k) \\ -4[& \qquad\qquad e(k+1) - e(k)] \\ +3[& \qquad\qquad\qquad\qquad e(k)] \\ \hline & e(k+2) - 6e(k+1) + 8e(k) = 1(k) \end{array}$$

$$\begin{cases} e(k+2) - 6e(k+1) + 8e(k) = 1(k) \\ e(k) = 0 \qquad (k \le 0) \end{cases}$$

Solution I of the difference equation —— Iteration method

$$\begin{cases} e(k+2)-6e(k+1)+8e(k)=1(k) \\ e(k)=0 \qquad (k\le 0) \end{cases}$$

Solution $$e(k+2)=6e(k+1)-8e(k)+1(k)$$

$$k=-1: \quad e(1)=6e(0)-8e(-1)+1(-1)=0$$

$$k=0: \quad e(2)=6e(1)-8e(0)+1(0)=0-0+1=1$$

$$k=1: \quad e(3)=6e(2)-8e(1)+1(1)=6-0+1=7$$

$$k=2: \quad e(4)=6e(3)-8e(2)+1(2)=6\times 7-8\times 1+1=35$$

$$\vdots \qquad\qquad \vdots$$

$$e^{*}(t)=\delta(t-2)+7\delta(t-3)+35\delta(t-4)+\cdots$$

Solution II of difference equation — Z-transform method

$$e(k+2)-6e(k+1)+8e(k)=1(k)$$

$$\begin{cases} e(k+2)-6e(k+1)+8e(k)=1(k) \\ e(k)=0 \qquad (k\le 0) \end{cases}$$

$$Z: \quad z^2[E(z)-e(0)z^0-e(1)z^{-1}]$$
$$-6\cdot z[E(z)-e(0)z^0]$$
$$+\quad 8\,[E(z)]$$

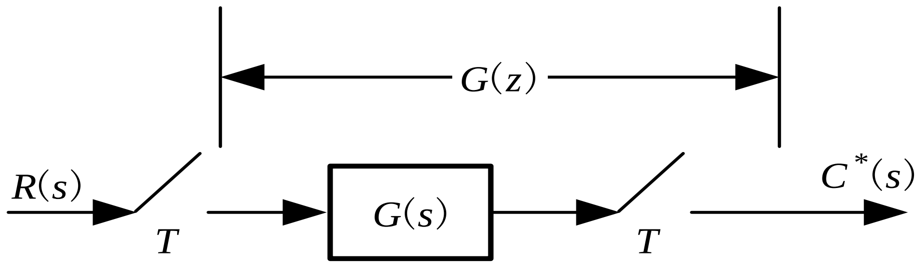
$$(z^2-6z+8)E(z)=Z[1(k)]=\frac{z}{z-1} \qquad E(z)=\frac{z}{(z-1)(z-2)(z-4)}$$

$$Z^{-1}: e(n)=\sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]$$

$$=\lim_{z\to1}\frac{z\cdot z^{n-1}}{(z-2)(z-4)}+\lim_{z\to2}\frac{z\cdot z^{n-1}}{(z-1)(z-4)}+\lim_{z\to4}\frac{z\cdot z^{n-1}}{(z-1)(z-2)}=\frac{1}{3}-\frac{2^n}{2}+\frac{4^n}{6}$$

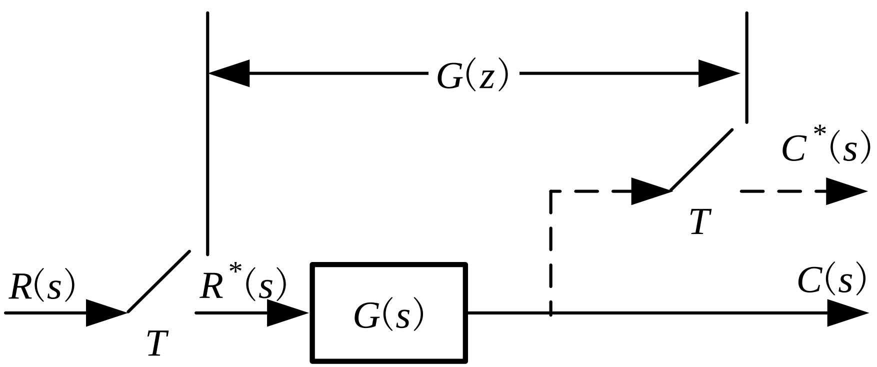
$$e^*(t)=\sum_{n=0}^{\infty}e(nT)\cdot\delta(t-nT)=\sum_{n=0}^{\infty}\left(\frac{1}{3}-\frac{2^n}{2}+\frac{4^n}{6}\right)\cdot\delta(t-nT)$$

7.5.2 Mathematical Models in Complex Domain — Impulse Transfer Function (脉冲传递函数)

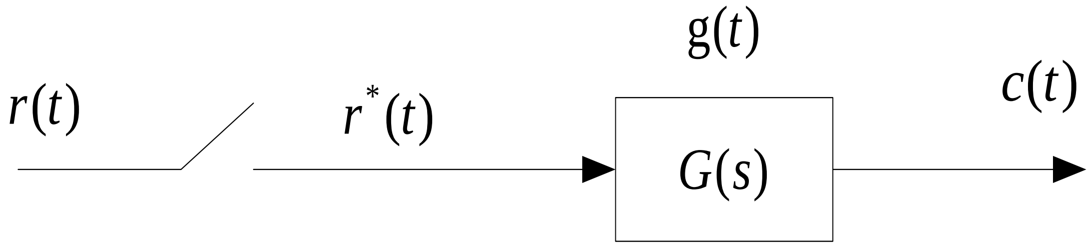


1. Definition

The ratio of the z-T. of the output to the z-T. of the input under zero initial condition.



$$G(z)=\frac{C(z)}{R(z)}=\frac{\sum_{k=0}^{\infty}c(kT)z^{-k}}{\sum_{k=0}^{\infty}r(kT)z^{-k}}$$



for a LTI system:

$$\left.\begin{aligned} &if\ : r(nT) = \delta(nT),\quad then: c(nT) = g(nT) \\ &if\ : r(nT) = \delta[(n-k)T],\quad then: c(nT) = g[(n-k)T] \end{aligned}\right\} \textbf{Weighted sequence}$$

Thus:

$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t-nT)$$

$$= r(0)\delta(t) + r(T)\delta(t-T) + \cdots + r(nT)\delta(t-nT) + \cdots$$

$$c(t) = r(0)g(t) + r(T)g[t-T] + \cdots + r(nT)g[t-nT] + \cdots$$

$$c(kT) = r(0)g(kT) + r(T)g[(k-1)T] + \cdots + r(nT)g[(k-n)T] + \cdots$$

$$= \sum_{n=0}^{\infty} r(nT)g[(k-n)T]$$

$$c(kT)=\sum_{n=0}^{\infty} r(nT)g[(k-n)T]$$

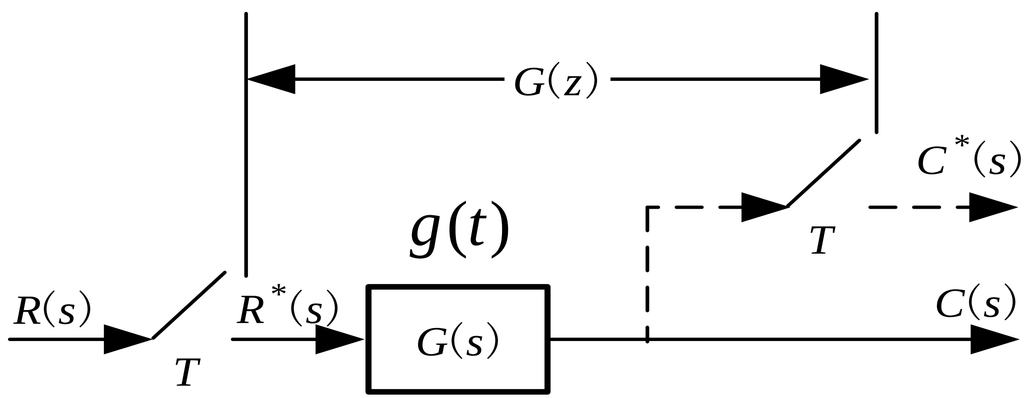
$$C(z)=\sum_{k=0}^{\infty} c(kT)z^{-k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} r(nT)g[(k-n)T]z^{-k}$$

$$=\sum_{n=0}^{\infty} r(nT)z^{-n}\sum_{k=0}^{\infty} g[(k-n)T]z^{-(k-n)}$$

The z-transform of unity impulse response sequence

$$\therefore G(z)=\frac{C(z)}{R(z)}=\sum_{k=n}^{\infty} g[(k-n)T]z^{-(k-n)} = \sum_{j=0}^{\infty} g(jT)z^{-j}$$

G(Z)=Z[g(t)]=Z[G(s)]



For a difference equation:

$$c(kT)+a_1c(kT-T)+\cdots+a_{n-1}c(kT-(n-1)T)+a_nc(kT-nT)$$
$$=b_0r(kT-(n-m)T)+\cdots+b_{m-1}r(kT-(n-1)T)+b_mr(kT-nT)$$

$$\Longleftrightarrow \quad c(kT)+\sum_{i=1}^{n}a_ic[(k-i)T]=\sum_{j=0}^{m}b_jr[(k-j)T]$$

z-T.

$$\Longleftrightarrow \quad C(z)+\sum_{i=1}^{n}a_iC(z)z^{-i}=\sum_{j=0}^{m}b_jR(z)z^{-j}$$

$$\Longleftrightarrow \quad G(z)=\frac{C(z)}{R(z)}=\frac{\sum_{j=0}^{m}b_jz^{-j}}{1+\sum_{i=1}^{n}a_iz^{-i}}$$

Example 1 Consider the discrete system shown in the figure with

$$G(s)=\frac{1}{s(0.1s+1)}$$

Obtain the impulse-transfer function $G(z)$。

Solution:

General Ideas: $G(s) \Rightarrow g(kT) \Rightarrow G(z)$

Method I. **The impulse response is:**

$$G(s)=\frac{1}{s(0.1s+1)} \Rightarrow \begin{aligned} g(t) &= (1-e^{-10t}) \qquad (t>0) \\ g(kT) &= 1-e^{-10kT} \end{aligned}$$

Then the impulse tranfer function is:

$$G(z)=\sum_{k=0}^{+\infty} g(kT)z^{-k} = \sum_{k=0}^{+\infty}\left(1-e^{-10kT}\right)z^{-k}$$

$$=\frac{z}{z-1}-\frac{z}{z-e^{-10T}} = \frac{z(1-e^{-10T})}{(z-1)(z-e^{-10T})}$$

Method II. **Because** $G(s)=\frac{1}{s}-\frac{1}{s+10}$

Then by G(Z)=Z[g(t)]=Z[G(s)], it derives

$$G(z)=\frac{z}{z-1}-\frac{z}{z-e^{-10T}}=\frac{z(1-e^{-10T})}{(z-1)(z-e^{-10T})}$$

The properties of impulse transfer function:

(1) $G(z)$ is a complex function of complex variable z;

(2) $G(z)$ depends only on the structure and parameters of the system;

(3) $G(z)$ has a relation with the difference equation of the system;

(4) $G(z)$ is equal to $Z[\, g^{*}(t)\,]$;

(5) $G(z)$ ~ zero-pole location in z plane.

The limitation of impulse-transfer functions

(1) It can not reflect the full information of the system response under non-zero initial conditions;

(2) It is only for SISO discrete systems;

(3) It is only for LTI (linear time-invariant) difference equations;

7.4.4 Inverse z-Transform

$$Z^{-1}\left[X(z)\right]=x(nT)$$

Tips:

Inverse Z-transform can only provide discrete-time signal $x^*(t)$, instead of continuous signal x(t)。

Review

- Long Division（**长除法**）
- Partial-Fraction expansion — Expansion of $\dfrac{E(z)}{z}$
- Residue（**留数法**） — $e(nT)=\sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]$

7.4.4 Inverse z-Transform

$$Z^{-1}[X(z)] = x(nT)$$

Tips:

Inverse Z-transform can only provide discrete-time signal $x^*(t)$, instead of continuous signal x(t)。

Review

- Long Division（长除法）
- Partial-Fraction expansion — Expansion of $\frac{E(z)}{z}$
- Residue（留数法） — $e(nT) = \sum \mathrm{Res}\left[E(z) \cdot z^{n-1}\right]$

$$Res\left[z^{(k-1)} x(z)\right] = \lim_{z \to z_i} \frac{1}{(r-1)!} \frac{d^{r-1}}{dz^{r-1}} \left[(z - z_i)^r z^{k-1} x(z)\right]$$

• Difference Equation

- The (forward) difference equation of nth-order linear time-invariant discrete system.

$$c(k+n)+a_1c(k+n-1)+a_2c(k+n-2)+\cdots+a_{n-1}c(k+1)+a_nc(k)$$

$$=b_0r(k+m)+b_1r(k+m-1)+\cdots+b_{m-1}r(k+1)+b_mr(k)$$

or

$$c(k+n)=-\sum_{i=1}^{n}a_ic(k+n-i)+\sum_{j=0}^{m}b_jr(k+m-j)$$

- The (backward) differential equation of n-order linear time-invariant discrete system.

$$c(k)+a_1c(k-1)+a_2c(k-2)+\cdots+a_{n-1}c(k-n+1)+a_nc(k-n)$$

$$=b_0r(k)+b_1r(k-1)+L+b_{m-1}r(k-m+1)+b_mr(k-m)$$

or

$$c(k)=-\sum_{i=1}^{n}a_ic(k-i)+\sum_{j=0}^{m}b_jr(k-j)$$

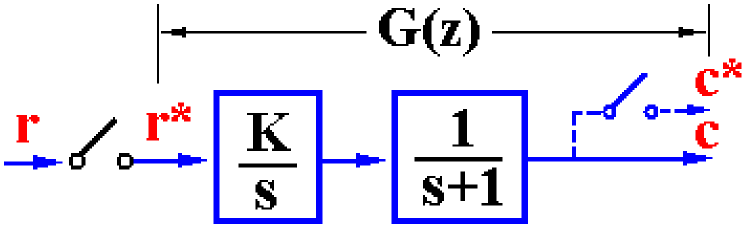
- **To solve difference equations:** $\begin{cases}\text{Iteration method（迭代)}\\ \text{Z-transform method}\end{cases}$

Example 2 Consider the discrete system shown in the figure (T=1). Obtain

(1) Impulse-transfer function of the system

(2) Zero-poles location in z plane;

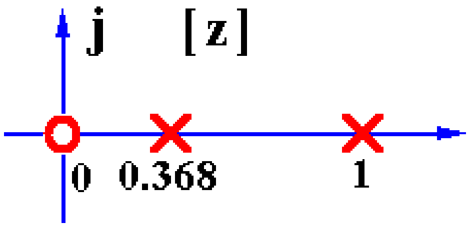
(3) Difference equation of the system.



Solution. (1) $$G(z)=\frac{C(z)}{R(z)}=Z\left[\frac{K}{s(s+1)}\right]=K\cdot Z\left[\frac{1}{s}-\frac{1}{s+1}\right]$$

$$=K\left[\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right]=\frac{(1-e^{-T})Kz}{(z-1)(z-e^{-T})}=\frac{(1-e^{-T})Kz}{z^2-(1+e^{-T})z+e^{-T}}$$

$$=\frac{0.632Kz^{-1}}{1-1.368z^{-1}+0.368z^{-2}}$$

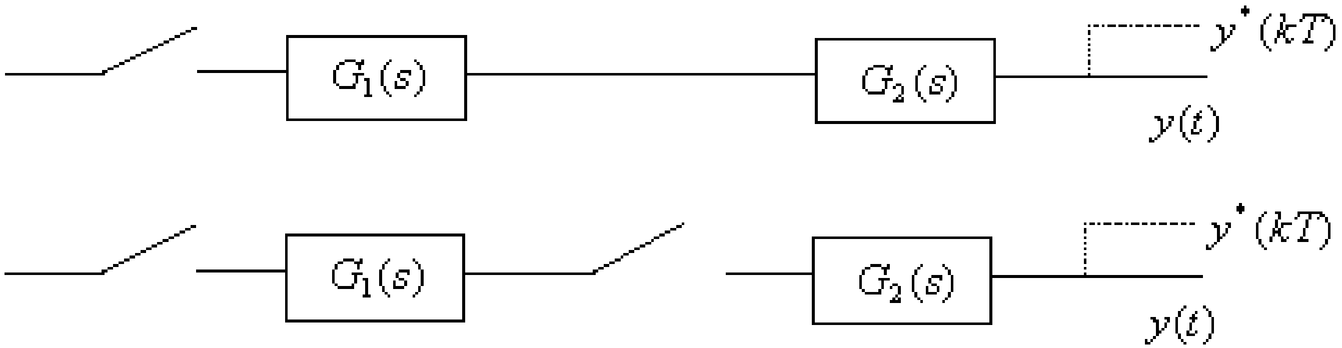


(2) Zero-poles location in z plane

(3) $$\left(1-1.368z^{-1}+0.368z^{-2}\right)C(z)=0.632Kz^{-1}R(z)$$

$$c(k)-1.368c(k-1)+0.368c(k-2)=0.632Kr(k-1)$$

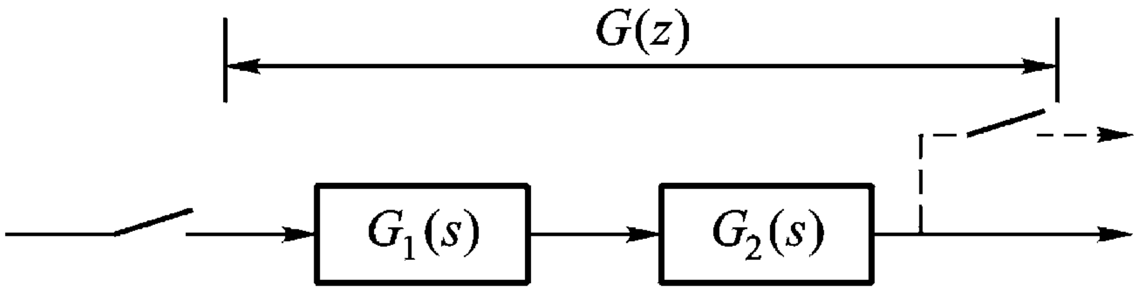
7.5.3 Impulse transfer function of Open-Loop Systems



(1) There is no sampler/switch between two components

$$G(s) = G_1(s)G_2(s)$$

$$G(z) = Z\left[G_1(s)G_2(s)\right] = G_1G_2(z)$$



图

Example 3 Consider the discrete system shown in the above figure， where

$$G_1(s) = \frac{1}{s+a} \qquad G_2(s) = \frac{1}{s+b}$$

Obtain the open-loop impulse transfer function。

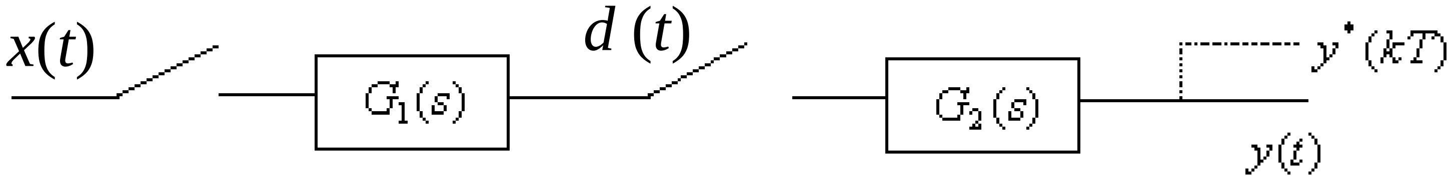
solution:

$$G_1(s)G_2(s)=\frac{1}{b-a}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]$$

$$G(z)=G_1G_2(z)$$

$$=\frac{1}{b-a}\left[\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}\right]$$

（2）There is a sampler/switch between two components



$$D(z) = G_1(z)X(z)$$

$$Y(z) = G_2(z)D(z) = G_1(z)G_2(z)R(z)$$

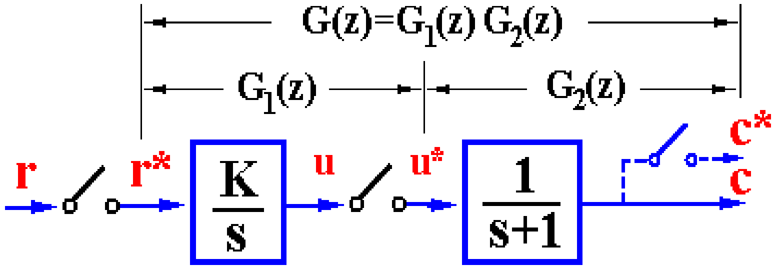
$$\therefore G(z) = G_1(z)G_2(z)$$

注 $G_1(z)G_2(z) \neq G_1G_2(z)$

(1) Switch between factors

$$G(z)=G_1(z)G_2(z)=Z\left[\frac{K}{s}\right]\cdot Z\left[\frac{1}{s+1}\right]$$

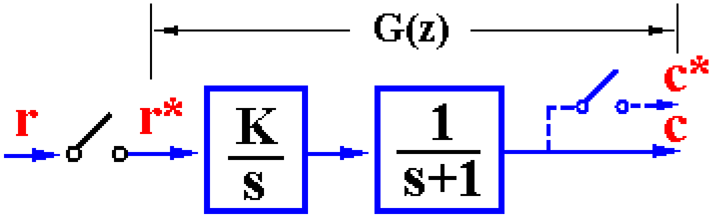
$$=\frac{Kz}{z-1}\cdot\frac{z}{z-e^{-T}}=\frac{Kz^2}{(z-1)(z-e^{-T})}$$



(2) No switch between factors

$$G(z)=Z[G_1(s)\cdot G_2(s)]=G_1G_2(z)$$

$$=K\left[\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right]=\frac{(1-e^{-T})Kz}{(z-1)(z-e^{-T})}$$



Note: the zeros of G(z), the poles of G(z).

Exercise: **Consider** $G_1(s) = \frac{1}{s}$, $G_2(s) = \frac{10}{s+10}$, **obtain G(z).**

Solution:

If there is no switch between the components,

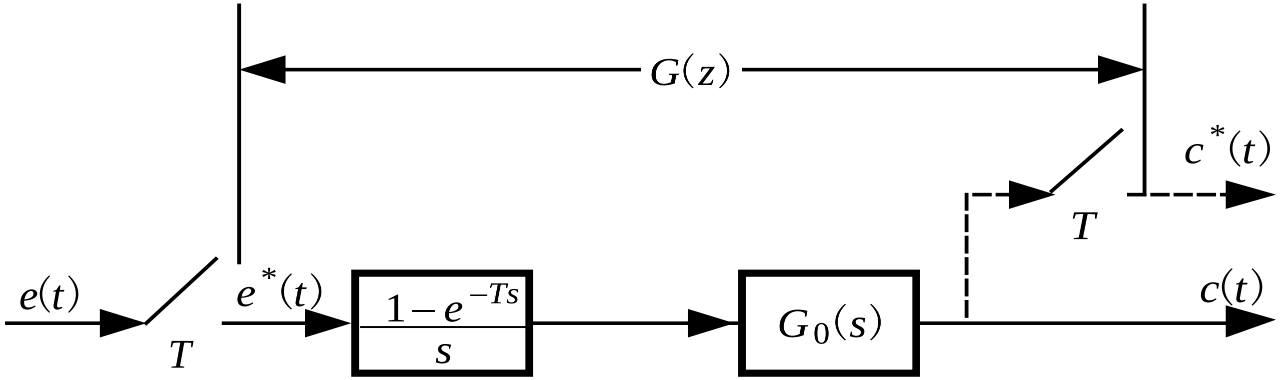
$$G(z) = G_1G_2(z) = Z[\frac{10}{s(s+10)}] = \frac{z(1-e^{-10T})}{(z-1)(z-e^{-10T})}$$

If there is a sampler between the components,,

$$G(z) = G_1(z)G_2(z) = Z[\frac{1}{s}]Z[\frac{10}{s+10}]$$

$$= \frac{z}{z-1}\frac{10z}{z-e^{10T}} = \frac{10z^2}{(z-1)(z-e^{-10T})}$$

（3）ZOH in the system

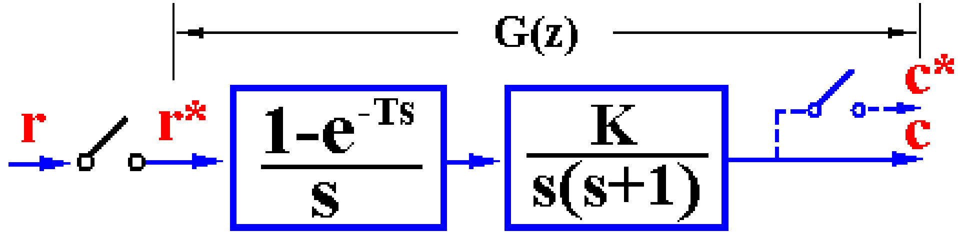


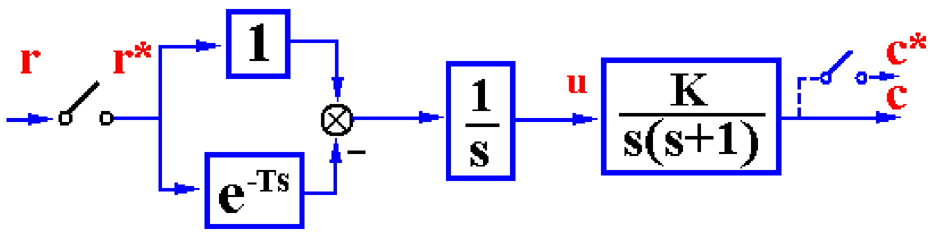
$$C(z) = Z\left[\frac{1-e^{-Ts}}{s}G_0(s)\right]R(z) = Z\left[\frac{1}{s}G_0(s) - \frac{e^{-Ts}}{s}G_0(s)\right]R(z)$$

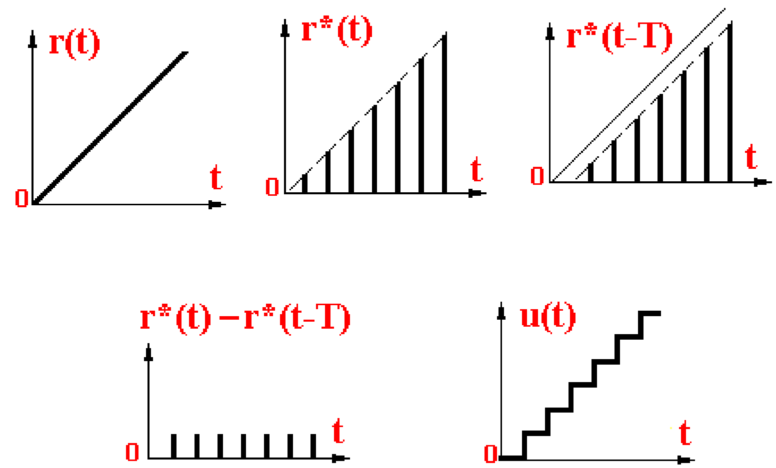
$$Z\left[\frac{e^{-Ts}}{s}G_0(s)\right] = z^{-1}Z\left[\frac{G_0(s)}{s}\right] \qquad C(z) = (1-z^{-1})Z\left[\frac{G_0(s)}{s}\right]R(z)$$

$$\boxed{G(z) = \frac{C(z)}{R(z)} = (1-z^{-1})Z\left[\frac{G_0(s)}{s}\right]}$$

Example 4 Consider the discrete system shown in the following figure, obtain its impulse transfer function.







$$G(z)=Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+1)}\right]$$

$$=K(1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right]$$

$$=K\frac{z-1}{z}Z\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right]$$

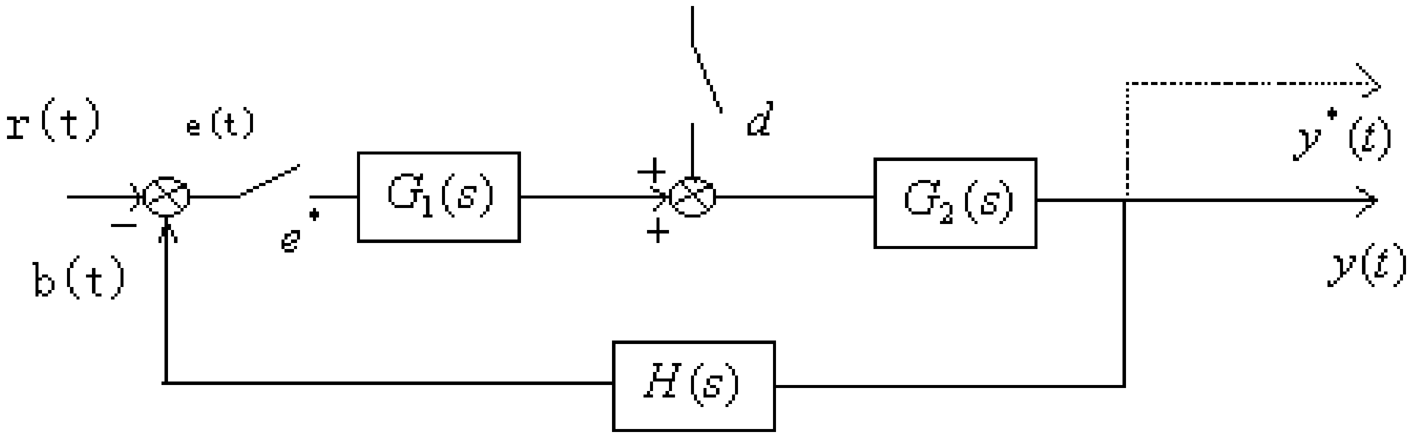
$$=K\frac{z-1}{z}\left[\frac{Tz}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-e^{-T}}\right]$$

$$=K\left[\frac{T}{z-1}-1+\frac{z-1}{z-e^{-T}}\right]$$

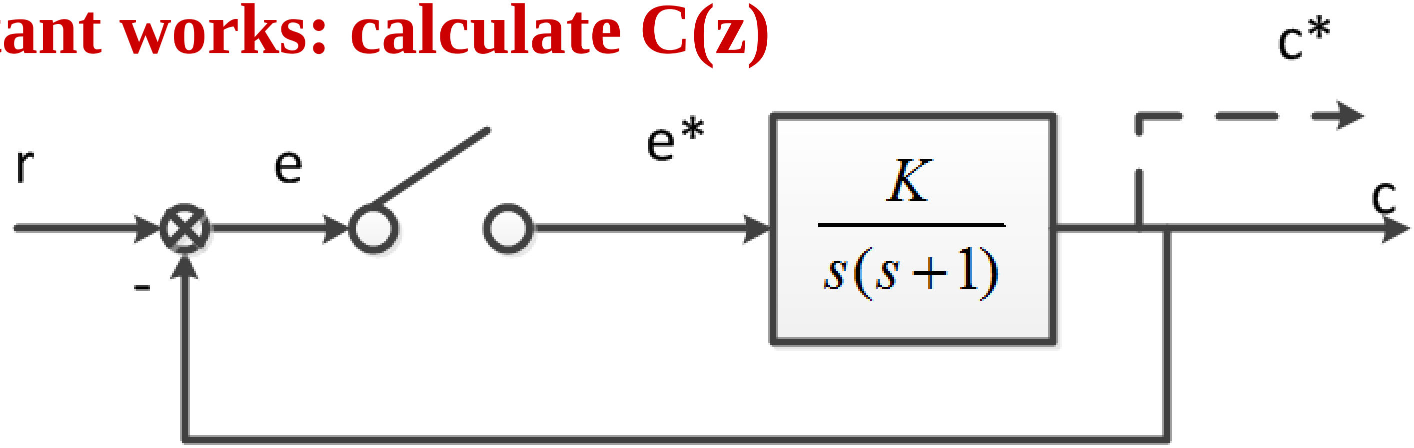
$$=K\frac{(T-1+e^{-T})z+(1-Te^{-T}-e^{-T})}{(z-1)(z-e^{-T})}$$

ZOH does not change the system order and O.-L. poles but changes the O.-L. zeros.

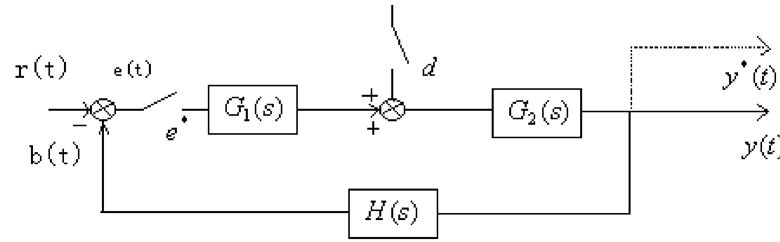
7.5.4、 Impulse transfer function of Closed-Loop Systems



Instant works: calculate C(z)



(1) Impulse Transfer Function for input to output



$$d=0$$

$$Y(z)=G_1G_2(z)E(z)$$

$$\left.\begin{aligned} &e(t)=r(t)-b(t) \\ &\Rightarrow E(z)=R(z)-B(z) \\ &B(z)=G_1G_2H(z)E(z) \end{aligned}\right\} \Rightarrow E(z)=\frac{R(z)}{1+G_1G_2H(z)}$$

Error impulse transfer function (误差脉冲传递函数):

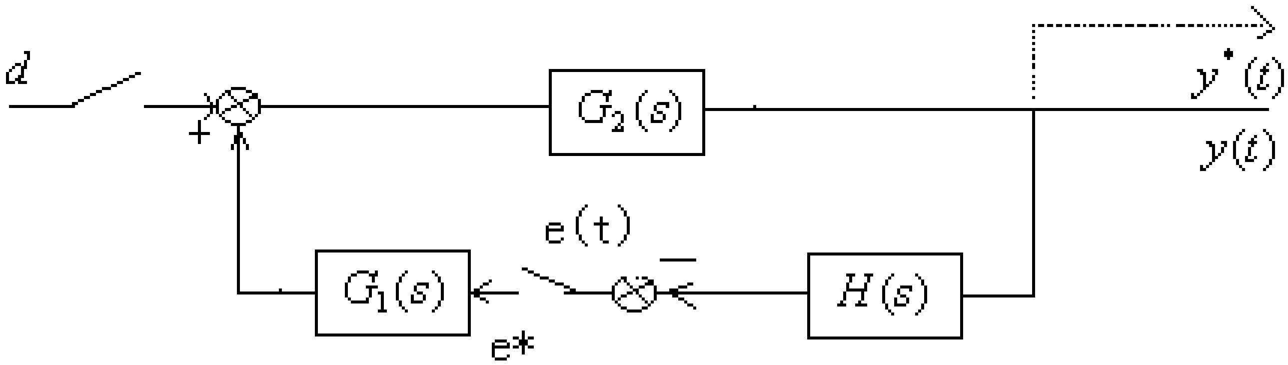
$$G_e(z)=\frac{E(z)}{R(z)}=\frac{1}{1+G_1G_2H(z)}$$

$$\Rightarrow Y(z)=G_1G_2(z)\frac{R(z)}{1+G_1G_2H(z)}$$

$$\therefore \Phi(z)=\frac{Y(z)}{R(z)}=\frac{G_1G_2(z)}{1+G_1G_2H(z)}$$

（2） Impulse Transfer Function for disturbance to output

$r(t) = 0$



$$Y(z) = G_2(z)D(z) + G_1G_2(z)E(z)$$

$$E(z) = -[G_2H(z)D(z) + G_1G_2H(z)E(z)]$$

$$\Rightarrow E(z) = -\frac{G_2H(z)}{1+G_1G_2H(z)}D(z)$$

$$\therefore Y(z) = G_2(z)D(z) - \frac{G_1G_2(z)G_2H(z)}{1+G_1G_2H(z)}D(z)$$

$$\Rightarrow \Phi_d(z) = \frac{Y(z)}{D(z)} = G_2(z) - \frac{G_1G_2(z)G_2H(z)}{1+G_1G_2H(z)}$$

E(z)：

D(z) passing through $G_2(z)$;

Loop of E(z)itself。

There is no switch/sampler for the error signal e(t)

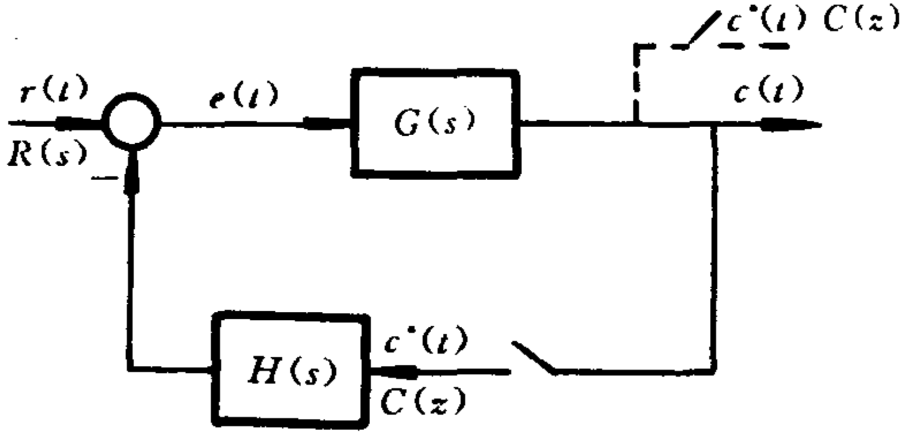


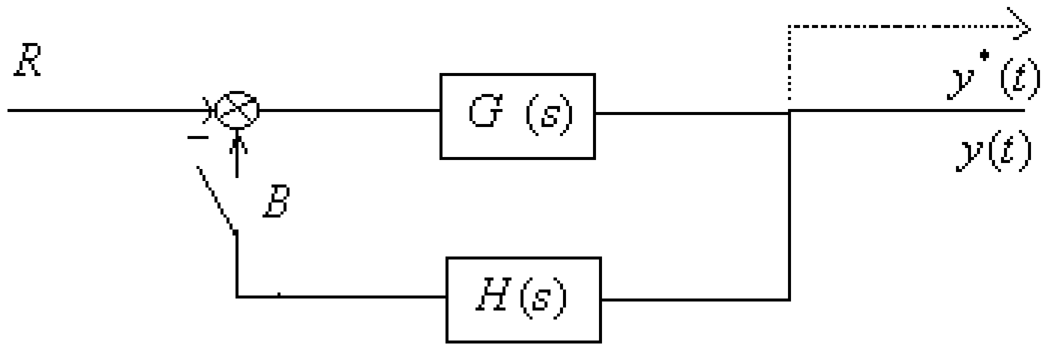
图 7-35 闭环离散系统

$$C(s)=G(s)R(s)-G(s)H(s)C^*(s)$$

$$C(z)=GR(z)-GH(z)C(z) \qquad \Rightarrow C(z)=\frac{GR(z)}{1+GH(z)}$$

Then, for this system, there exists no impulse transfer function.

Example **Consider the discrete-time system as shown in the figure, find the z-transform of the output y(t).**



Solution:

$$Y(z) = GR(z) - G(z)B(z)$$

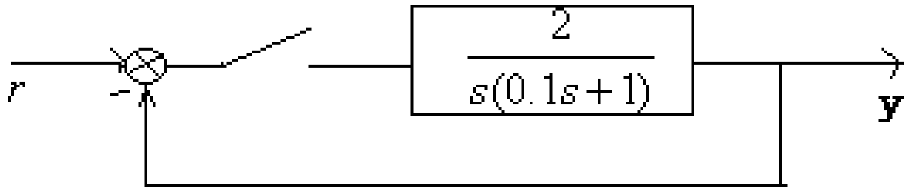
$$B(z) = GHR(z) - GH(z)B(z)$$

$$\therefore B(z) = \frac{GHR(z)}{1 + GH(z)}$$

$$\therefore Y(z) = GR(z) - \frac{G(z)GHR(z)}{1 + GH(z)}$$

There exists no impulse tranfer function.

Example **Consider the discrete-time system as shown in the figure, for T=0.1, find the unit step response of the system.**



Solution: $G(z) = Z\left[\frac{2}{s(0.1s+1)}\right] = \frac{2z}{z-1} - \frac{2z}{1-e^{-10T}}$

$$= \frac{2z - 0.736z}{(z-1)(z-0.368)} = \frac{1.264z}{z^2 - 1.368z + 0.368}$$

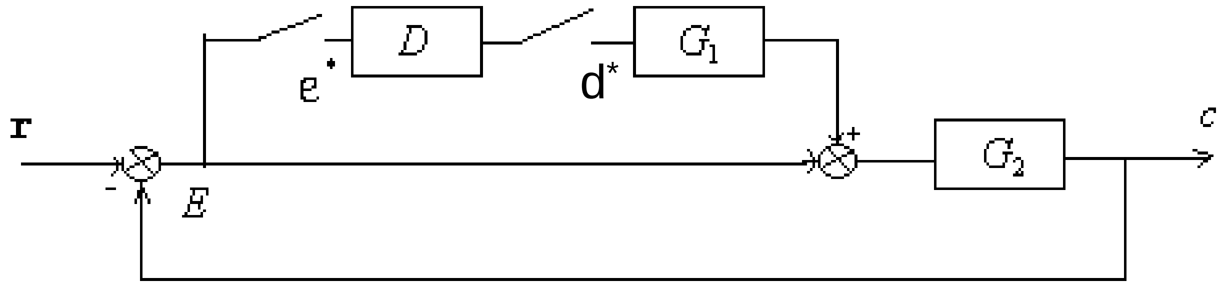
$$\therefore \Phi(z) = \frac{G(z)}{1+G(z)} = \frac{1.264z}{z^2 - 0.104z + 0.368}$$

$$\therefore Y(z) = \Phi(z)R(z) = \Phi(z)\frac{z}{z-1}$$

$$= 1.264z^{-1} + 1.396z^{-2} + 0.945z^{-3} + 0.849z^{-4} + \cdots$$

$$y^*(t) = 1.264\delta(t-0.1) + 1.396\delta(t-0.2) + \cdots$$

Example **Consider the discrete-time system as shown in the figure，find the expression of the output c.**



Solution: There exist both discrete and continuous signals, then employing L-Transform firstly，

$$C = G_1G_2 \cdot D^* \cdot E^* + G_2E = G_1G_2 \cdot D^* \cdot (R - C)^* + G_2(R - C)$$

$$= G_1G_2 \cdot D^* \cdot (R^* - C^*) + G_2(R - C)$$

$$C = \frac{G_2R}{1+G_2} + \frac{G_1G_2}{1+G_2} \cdot D^* \cdot R^* - \frac{G_1G_2}{1+G_2} \cdot D^* \cdot C^*$$

$$\boxed{C=\frac{G_2R}{1+G_2}+\frac{G_1G_2}{1+G_2}\cdot D^*\cdot R^*-\frac{G_1G_2}{1+G_2}\cdot D^*\cdot C^*}$$

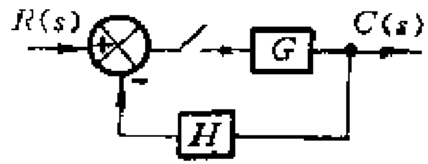
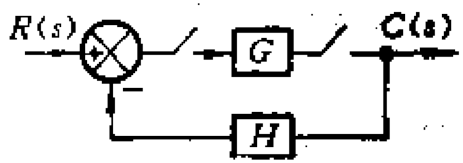
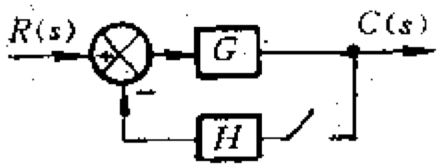
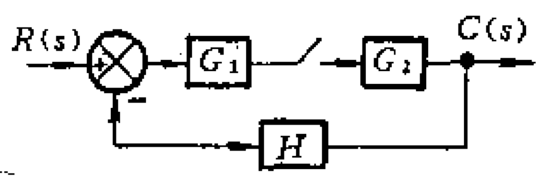
Discretize C to C*， then

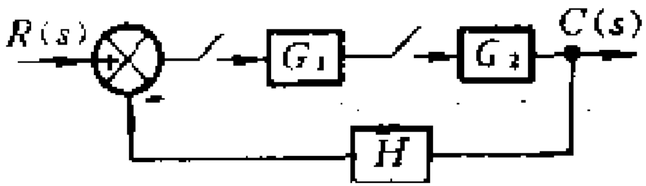
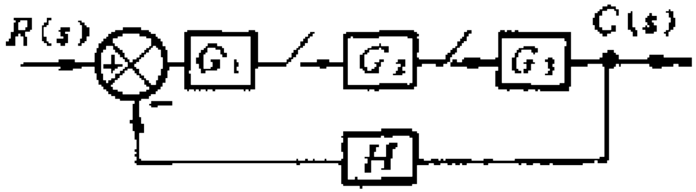
$$C^*=\left[\frac{G_2R}{1+G_2}\right]^*+\left[\frac{G_1G_2}{1+G_2}\right]^*\cdot D^*\cdot R^*-\left[\frac{G_1G_2}{1+G_2}\right]^*\cdot D^*\cdot C^*$$

PS: $\left[G_1(s)\cdot G_2(s)^*\right]^*=G_1(s)^*\cdot G_2(s)^*$

$$\therefore C^*=\frac{\left[\frac{G_2R}{1+G_2}\right]^*+\left[\frac{G_1G_2}{1+G_2}\right]^*D^*R^*}{1+\left[\frac{G_1G_2}{1+G_2}\right]^*D^*}$$

Typical diagram of C.L.discrete-time systems

	系统方框图	$C(z)$
1		$C(z)=\dfrac{G(z)}{1+HG(z)}R(z)$
2		$C(z)=\dfrac{G(z)}{1+G(z)H(z)}R(z)$
3		$C(z)=\dfrac{RG(z)}{1+HG(z)}$
4		$C(z)=\dfrac{RG_1(z)G_2(s)}{1+G_1G_2H(z)}$

5		$C(z)=\dfrac{G_1(z)G_2(z)}{1+G_1(z)HG_2(z)}R(z)$
6		$C(z)=\dfrac{G_2(z)G_3(z)RG_1(z)}{1+G_3(z)G_1G_3H(z)}$

作业：p256. 7-5, 7-8

7.5 Mathematical Models of Discrete-Time Systems

7.5.1 Linear Time-Invariant Difference Equations

(1) Definition of difference ① Forward ② Backward

(2) The difference equation and its solving method ① Iteration ② Z-transformation

7.5.2 Impulse-Transfer Function

(1) Definition (2) Properties (3) Limitation

Review

7.5.3 Impulse Transfer Function of Open-Loop Systems

- (1) Switch between factors
- (2) No switch between factors
- (3) With ZOH

7.5.4 Impulse Transfer Function of Closed-Loop Systems

Basic Principle

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.6 Performance Analysis of Discrete-Time Systems

- **Stability**
- **Dynamic Performance**
- **Steady-state Errors**

7.6.1 Stability of Discrete systems

1. Preliminaries

Stability is the most important performance of a system.

When we sampled a continuous systems, we still have a "continuous" system → the same properties hold as before:

A necessary and sufficient condition for a feedback system to be stable is that **all the poles of the system transfer function have negative real parts**.

Now, we introduced the variable $z=e^{Ts}$, how does stability look in the new variable?

First, we much understand the relationship between s-domain and z-domain.

2. s-Domain to z-Domain Mapping

Because $z = e^{sT},$ **let** $s = \sigma + j\omega$ **then** $|z| = e^{\sigma T}$

$\angle z = \omega T,$ **thus**

$$\begin{cases} \sigma > 0 & |z| > 1 \\ \sigma < 0 & |z| < 1 \\ \sigma = 0 & |z| = 1 \end{cases}$$

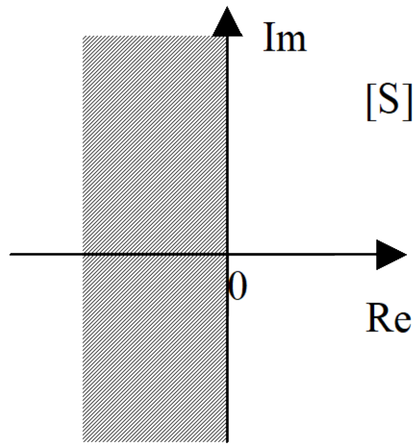
The mapping relationship can be described as in the following figures.

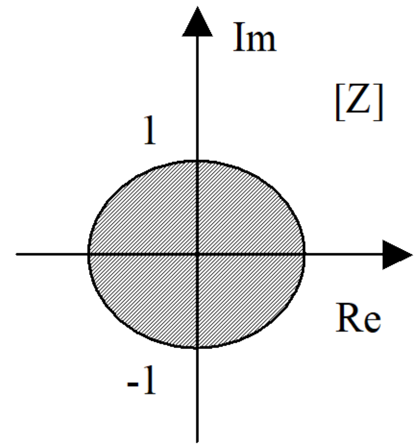
$$z = e^{Ts} = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{j\omega T}$$

$$\sigma < 0 \quad |z| = \frac{1}{e^{-T\sigma}} < 1$$

$$\sigma = 0 \quad |z| = 1$$

$$\sigma > 0 \quad |z| = e^{T\sigma} > 1$$





For a C.L. discrete-time system with unit feedback, the impulse transfer function is:

$$\Phi(z) = \frac{G(z)}{1 + G(z)}$$

Its characteristic function is: $1 + G(z) = 0$

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

——All poles of Φ(z) lie in the unit circle of z plane.

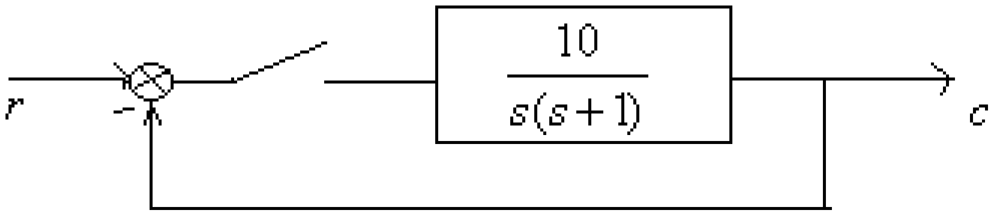
Prove:

$$\Phi(z)=\frac{M(z)}{D(z)}=\frac{\prod_{i=1}^{m}(z-\alpha_i)}{\prod_{j=1}^{n}(z-\beta_j)}=\sum_{j=1}^{n}\frac{C_j z}{z-\beta_j}=K(z)$$

$$c(k)=\sum_{j=1}^{n}C_j\beta_j^k \overset{k\to\infty}{=} 0 \qquad \left|\beta_j\right|<1 \qquad \text{— Necessity}$$

$$c^*(t)=\sum_{k=0}^{\infty}\left(\sum_{j=1}^{n}C_j\beta_j^k\right)\cdot\delta(t-kT) \qquad \text{— Sufficiency}$$

Example **The discrete-time system is shown as the following figure， suppose T=1， is the system stable?**



Solution: $$G(z) = Z[\frac{10}{s(s+1)}] = \frac{6.32z}{(z-1)(z-0.368)}$$

$$1 + G(z) = 0 \Rightarrow z^2 + 4.952z + 0.368 = 0$$

$$\Rightarrow z_1 = -0.076 \quad z_2 = -4.876$$

$\because |z_2| > 1$ So the system is unstable.

3. The Stability Criterion of Discrete-Time Systems

- **For continuous-time systems , we can use Routh criterion to determine the stability of the system, where the stable area is on LHP (left-hand-plane) of [s]-domain.**

- **Unfortunately, for discrete-time systems, the stable area is unit circle, not LHP of [z]-domain, we cannot directly apply the Routh criterion as we have to test on something else than LHP.**

w-transformation and Routh criterion in w-domain

We find a transformation that maps the unit circle back onto the LHP while maintaining the algebraic structure of rational functions.

A particular transformation that will accomplish this would be the bilinear transformation:

$$z = \frac{w+1}{w-1}, \quad w = \frac{z+1}{z-1}$$

Suppose $$\boldsymbol{z = x + j\,y \quad w = u + j\,v}$$

Then

$$w = \frac{z+1}{z-1} = \frac{(x+1) + jy}{(x-1) + jy}$$

$$u + jv = \frac{(x^2 + y^2) - 1}{(x-1)^2 + y^2} + j\frac{2y}{(x-1)^2 + y^2}$$

$$\therefore u = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2}$$

$$w = \frac{z+1}{z-1} = \frac{x+1+jy}{x-1+jy} = \frac{x^2 - 1 + y^2 - j2y}{(x-1)^2 + y^2} = u + jv$$

[w] imaginary axis $u = 0 = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2} = 0$

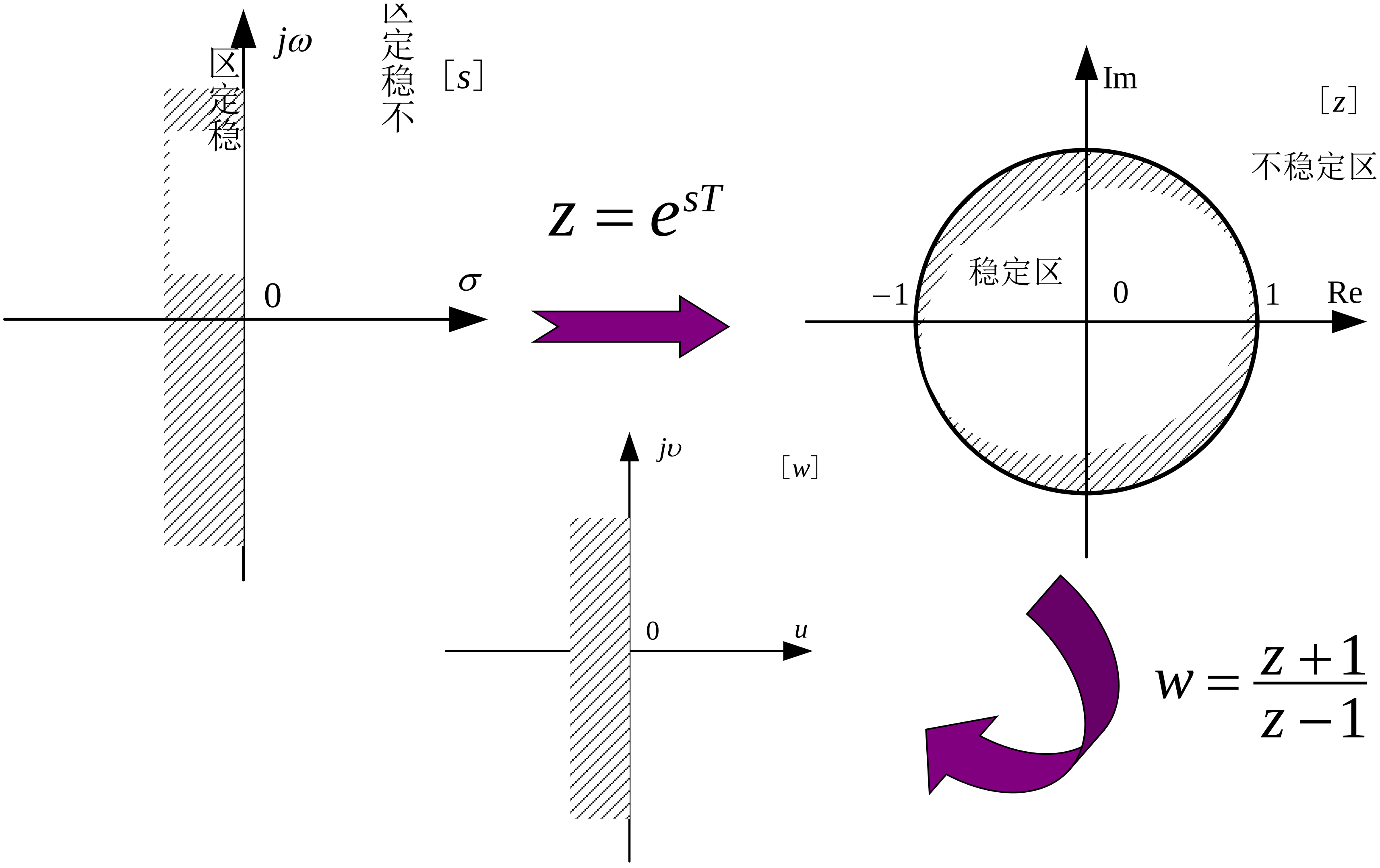
$$\Rightarrow \quad x^2 + y^2 = 1$$ **[z] unit circle**

Points $\begin{cases} \text{inside} \\ \text{outside} \end{cases}$ of the unit circle

$$\Rightarrow \quad x^2 + y^2 \begin{cases} < 1 \\ > 1 \end{cases}$$ in z-plane

$$\Rightarrow \quad \begin{cases} u < 0 \\ u > 0 \end{cases}$$ in s-plane

we’ve learned three methods to determine the stability of a discrete-time systems.



Example 1 Determine the stability from the characteristic equation of a discrete system.

$$D(z)=45z^3-117z^2+119z-39=0$$

$$\Big\downarrow \quad z=(w+1)/(w-1)$$

$$=45\left(\frac{w+1}{w-1}\right)^3-117\left(\frac{w+1}{w-1}\right)^2+119\left(\frac{w+1}{w-1}\right)-39=0$$

$$D(w)=45(w+1)^3-117(w+1)^2(w-1)\ +119(w+1)(w-1)^2-39(w-1)^3=0$$

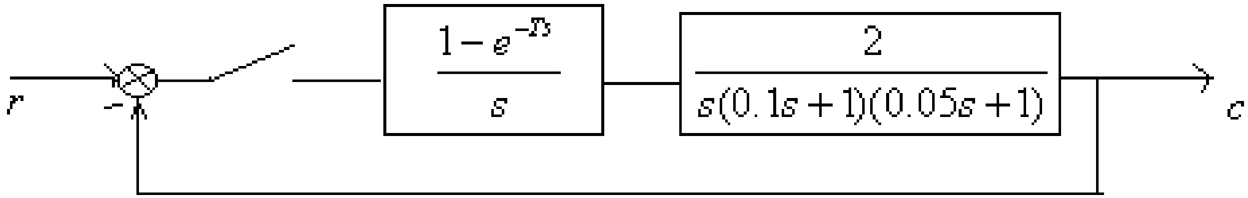
$$D(w)=w^3+2w^2+2w+40=0$$

Routh

w^3	1	2
w^2	2	40
w^1	-18	
w^0	40	

Unstable!

Example 2 Consider the discrete-time system as shown in the figure， if $T=0.1$, determine the stability of the system.



Solution:

$$G(z)=\frac{z-1}{z}Z\left[\frac{2}{s^2(0.1s+1)(0.05s+1)}\right]$$

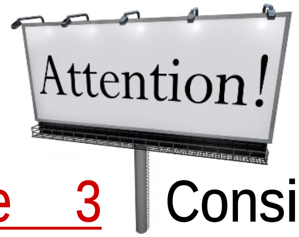
$$=\frac{z-1}{z}\left[-\frac{0.3z}{z-1}+\frac{0.4z}{(z-1)^2}+\frac{0.4z}{z-e^{-10T}}-\frac{0.1z}{z-e^{-20T}}\right]$$

$$\boxed{1+G(z)=0} \qquad z^3-1.001z^2+0.3356z+0.0535=0$$

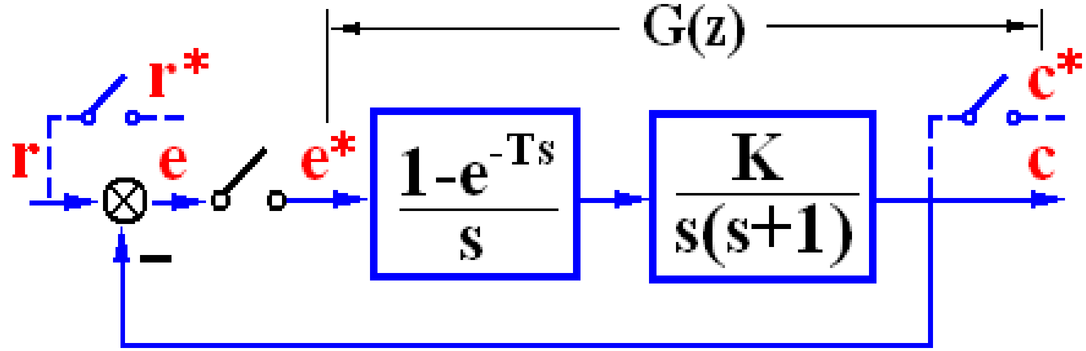
$$\boxed{z=\frac{w+1}{w-1}} \qquad 2.33w^3+3.68w^2+1.65w+0.34=0$$

w^3	2.33	1.65
w^2	3.68	0.34
w	1.43	0
w^0	0.34	

The elements in the first column are all positive， the system is stable.



Example 3 Consider the system shown in the figure (T=1). Determine the stable range of K.



Routh criterion in w domain

$$G(z)=Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+1)}\right]=(1-z^{-1})K\cdot Z\left[\frac{1}{s^2(s+1)}\right]$$

$$=\frac{(z-1)K}{z}\cdot Z\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right]=\frac{(z-1)K}{z}\cdot\left[\frac{Tz}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-e^{-T}}\right]$$

$$=K\left[\frac{(T-1+e^{-T})z+(1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}\right]\overset{T=1}{=}\frac{0.368K(z+0.718)}{(z-1)(z-0.368)}$$

$$\Phi(z)=\frac{G(z)}{1+G(z)}=\frac{0.368K(z+0.718)}{z^2+(0.368K-1.368)z+(0.264K+0.368)}$$

$$\Phi(z)=\frac{G(z)}{1+G(z)}=\frac{0.368K(z+0.718)}{z^2+(0.368K-1.368)z+(0.264K+0.368)}$$

$$D(z)=z^2+(0.368K-1.368)z+(0.264K+0.368)=0$$

$$z=\frac{w+1}{w-1}$$

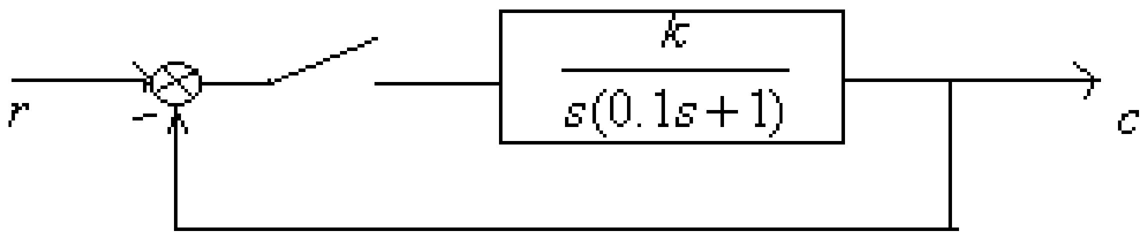
$$=(\frac{w+1}{w-1})^2+(0.368K-1.368)(\frac{w+1}{w-1})+(0.264K+0.368)=0$$

$$(w+1)^2+(0.368K-1.368)(w+1)(w-1)+(0.264K+0.368)(w-1)^2=0$$

$$D(w)=0.632Kw^2+(1.264-0.528K)w+(2.736-0.104K)=0$$

$$\begin{cases} K>0 \\ 1.264-0.528K>0 \\ 2.736-0.104K>0 \end{cases} \qquad \begin{cases} K>0 \\ K<2.394 \\ K<26.3 \end{cases} \qquad \boxed{0<K<2.394}$$

Exercise Consider the system shown in the figure (T=0.1). Determine the stable range of K. If it is a continuous-time system, Determine the stable range of K



Solution:

$$G(z) = Z\left[\frac{k}{s(0.1s+1)}\right] = \frac{0.632kz}{z^2 - 1.368z + 0.368}$$

$$1 + G(z) = 0 \qquad z^2 + (0.632k - 1.368)z + 0.368 = 0$$

$$z = \frac{w+1}{w-1} \qquad 0.632kw^2 + 1.264w + 2.736 - 0.632k = 0$$

For continuous-time system, $0 < k$, then system is stable;

For discrete-time system, $0 < k < 4.33$, stable.

Summary

s-Domain to z-Domain Mapping

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

— All poles of Φ(z) lie in the unit circle of z plane

Routh criterion in w domain (Generalized Routh Criterion)

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

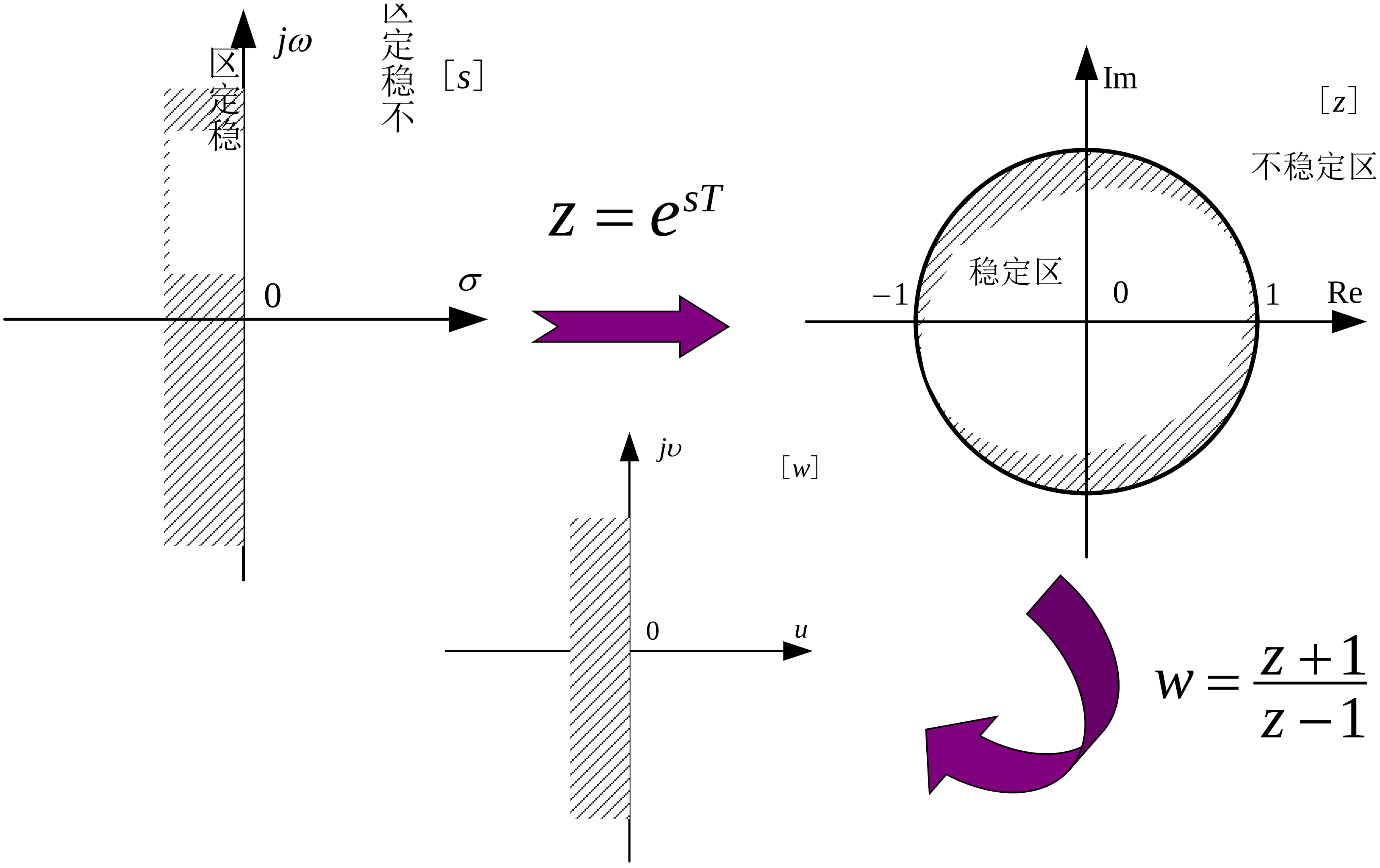
s-Domain to z-Domain Mapping

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

— All poles of Φ(z) lie in the unit circle of z plane

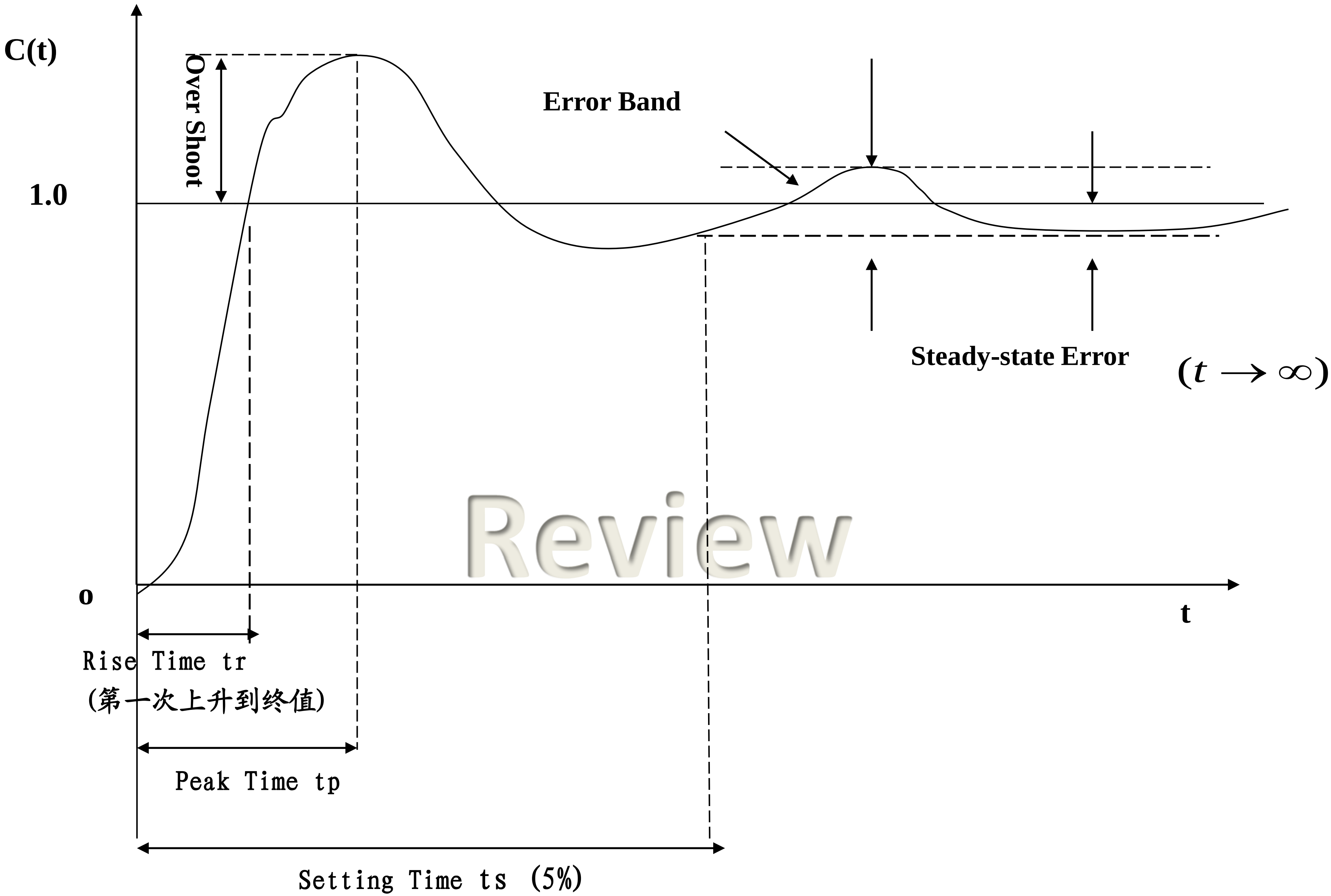
Routh criterion in w domain (Generalized Routh Criterion)

we’ve learned three methods to determine the stability of a discrete-time systems.



7.6 Performance Analysis of Discrete-Time Systems

- **Stability**
- **Dynamic Performance**
- **Steady-state Errors**

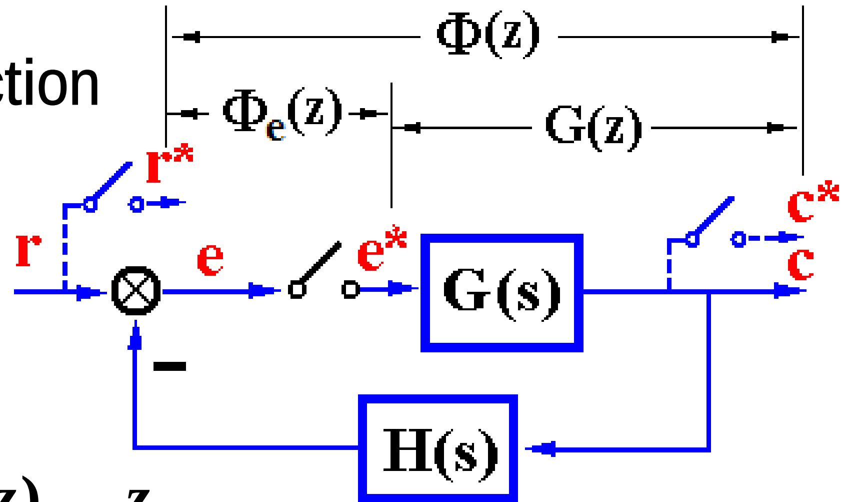


Performance Index

7.6.2 Dynamic Performance Analysis of Discrete-Time Systems

1. General algorithm to obtain the dynamic performance

(1) Obtain the impulse transfer function



Let
$$\begin{cases} GH(z) = Z\left[G(s)H(s)\right] \\ \Phi(z) = \dfrac{G(z)}{1+GH(z)} = \dfrac{M(z)}{D(z)} \end{cases}$$

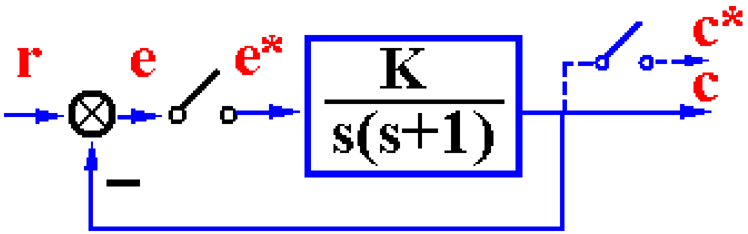
(2) Obtain $C(z) = \Phi(z)R(z) = \dfrac{M(z)}{D(z)} \cdot \dfrac{z}{z-1}$

$$= c(0) + c(T)z^{-1} + c(2T)z^{-2} + \cdots$$

(3) $c^*(t) = c(0)\delta(t) + c(T)\delta(t-T) + c(2T)\delta(t-2T) + \cdots$

(4) Determine the specifications $\sigma\%$, t_s .

Example 1 Consider the system shown in the figure, T=K=1. Obtain the dynamic specifications. (**σ %**, t_s).



Solution. $$G(z)=Z\left[\frac{K}{s(s+1)}\right]=\frac{K(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

$$\overset{K=T=1}{=}\frac{0.632z}{(z-1)(z-0.368)}$$

$$\Phi(z)=\frac{G(z)}{1+G(z)}=\frac{0.632z}{z^2-0.736z+0.368}$$

$$c(\infty T)=\lim_{z\to 1}(z-1)\cdot\Phi(z)\cdot\frac{z}{z-1}=1$$

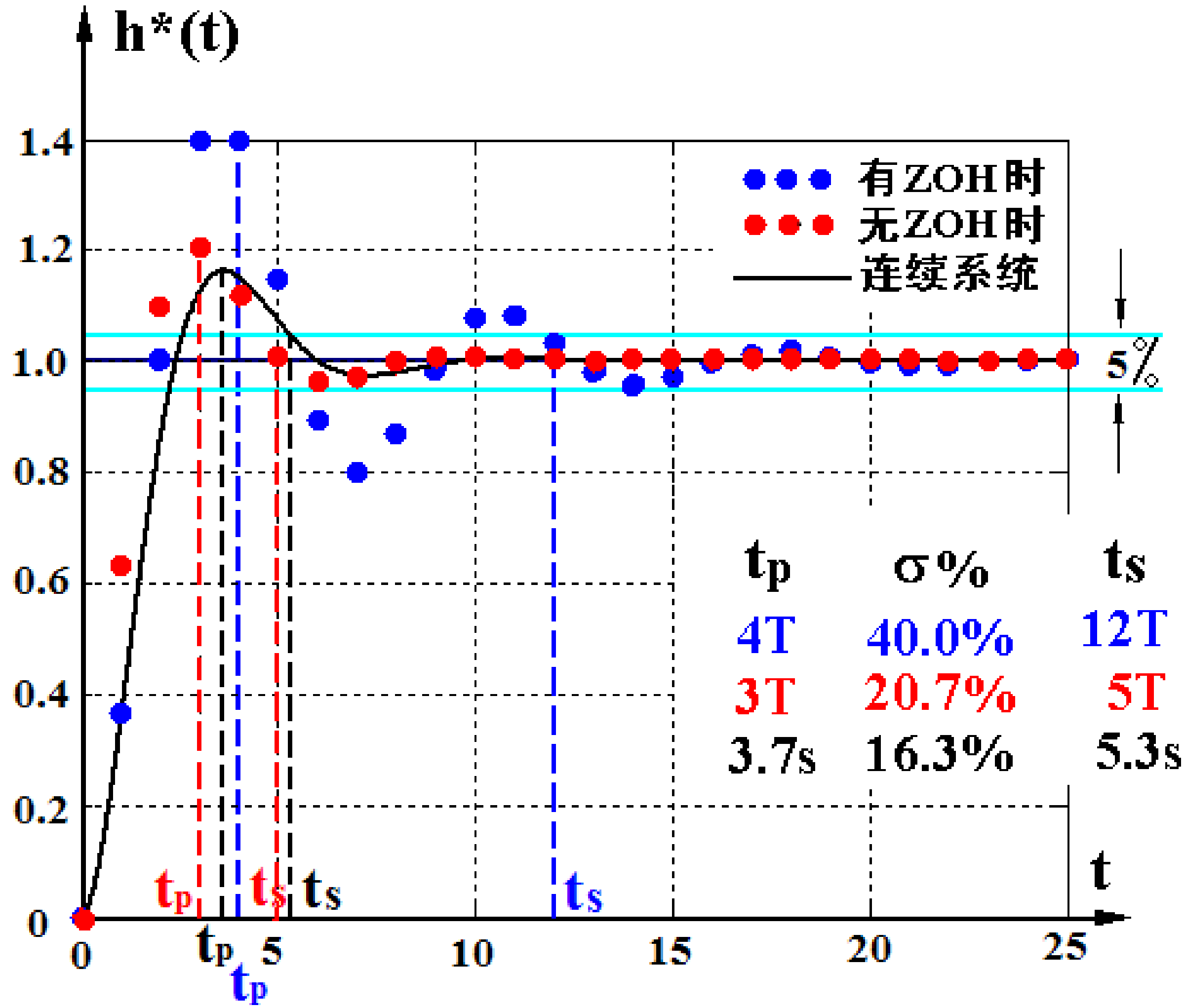
$$C(z)=\Phi(z)\cdot\frac{z}{z-1}=\frac{0.632z^2}{z^3-1.736z^2+1.104z-0.368}$$

h(0)= 0
h(1)= 0.632
h(2)= 1.097
h(3)= 1.207
h(4)= 1.117
h(5)= 1.014
h(6)= 0.964
h(7)= 0.970
h(8)= 0.991
h(9)= 1.004
h(10)=1.007
h(11)=1.003
h(12)=1.000
⋮

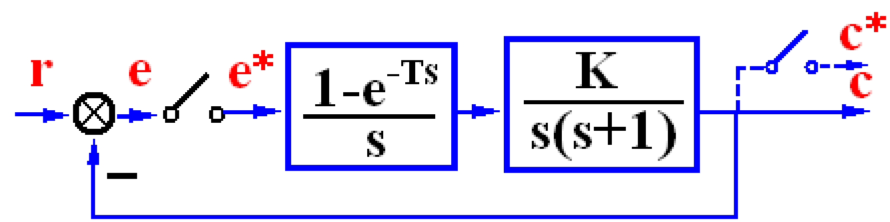
$t_P=3T$
$\sigma\%=20.7\%$

$t_s=5T$

Obtain the unit step response series h(k) by long division method.


h*(t)
有ZOH时
无ZOH时
连续系统
5%
tp
σ%
ts
4T
40.0%
12T
3T
20.7%
5T
3.7s
16.3%
5.3s
t
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25

Example 1 Consider the system shown in the figure, T=K=1. Obtain the dynamic specifications. (σ %, t_s).



Solution. $$G(z)=K\frac{z-1}{z}Z\left[\frac{1}{s^2(s+1)}\right]$$

$$=K\frac{(T-1+e^{-T})z+(1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}$$

$$\overset{K=T=1}{=}\frac{0.368z+0.264}{(z-1)(z-0.368)}$$

$$\Phi(z)=\frac{G(z)}{1+G(z)}=\frac{0.368z+0.264}{z^2-z+0.632}$$

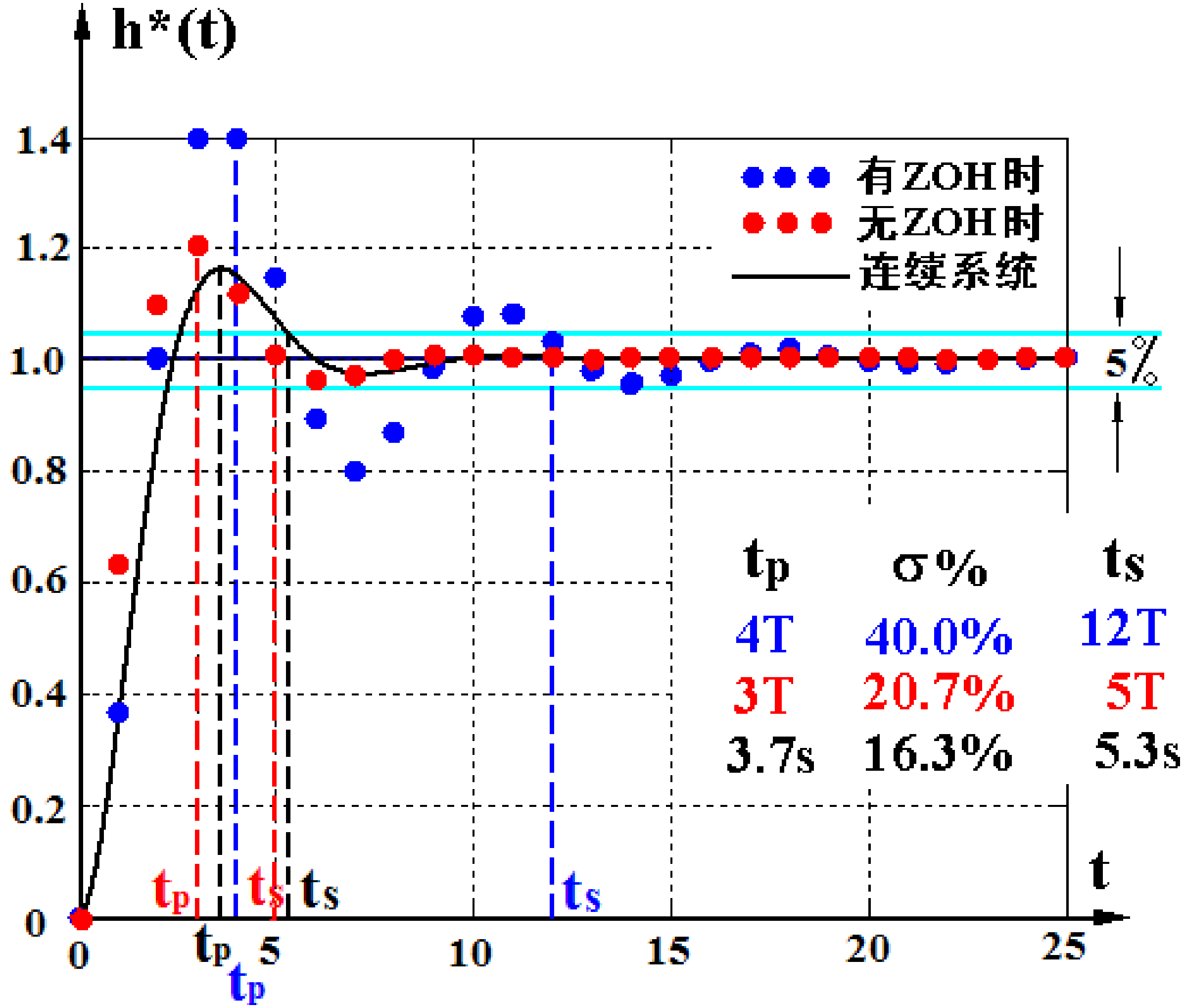
$$c(\infty T)=\lim_{z\to 1}(z-1)\cdot\Phi(z)\cdot\frac{z}{z-1}=1$$

$$C(z)=\Phi(z)\cdot\frac{z}{z-1}=\frac{(0.368z+0.264)z}{z^3-2z^2+1.632z-0.632}$$

h(0)=0
h(1)=0.3679
h(2)=1.0000
h(3)=1.3996
h(4)=1.3996
h(5)=1.1470
h(6)=0.8944
h(7)=0.8015
h(8)=0.8682
h(9)=0.9937
h(10)=1.0770
h(11)=1.0810
h(12)=1.0323
h(13)=0.9811
h(14)=0.9607
⋮

$\begin{cases} t_P=4T \\ \sigma\%=40\% \end{cases}$

$t_s=12T$



(1) Sampler can reduce the Peak-time tp and Setting-time ts, but increase the Over-shoot σ%.

(2) ZOH can increase tp, ts, σ% and also oscillating times.

$$\lim_{n\to\infty} e(nT) = \lim_{z\to 1}(z-1)\cdot E(z)$$

2. Relationship between dynamic response and closed-loop poles

$$\Phi(z) = \frac{M(z)}{D(z)} = \frac{b_m}{a_n}\frac{\prod_{i=1}^{m}(z-z_i)}{\prod_{k=1}^{n}(z-p_k)} \qquad m \le n$$

$$C(z) = \Phi(z)R(z) = \frac{M(z)}{D(z)}\cdot\frac{z}{z-1}$$

$$= \underbrace{\boxed{\frac{M(1)}{D(1)}\cdot\frac{z}{z-1}}}_{\text{Steady response}} + \underbrace{\boxed{\sum_{k=1}^{n}\frac{c_k z}{z-p_k}}}_{\text{Transient response}}$$

(1) Single closed-loop poles on the real axis

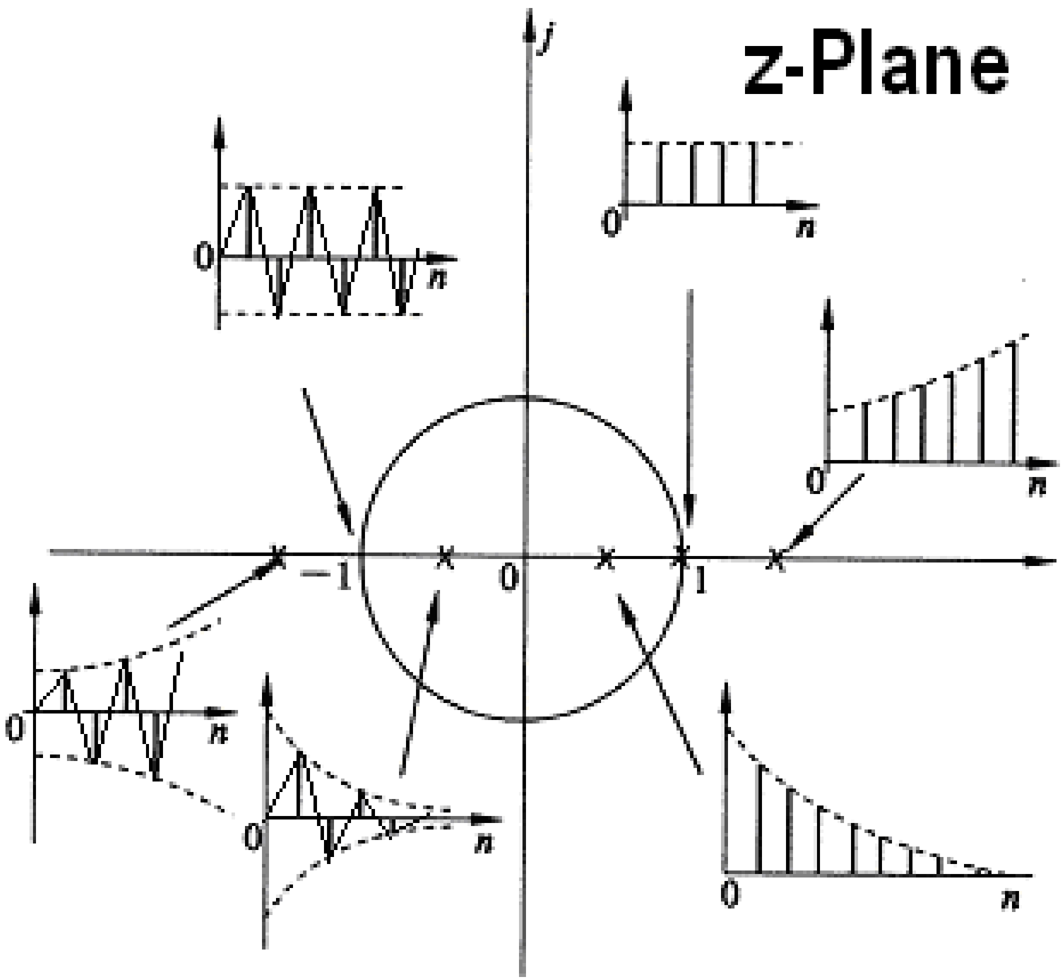
$$c_k^*(t) = Z^{-1}\left[\frac{c_k z}{z-p_k}\right] \qquad k = 1, 2, \cdots\ n$$

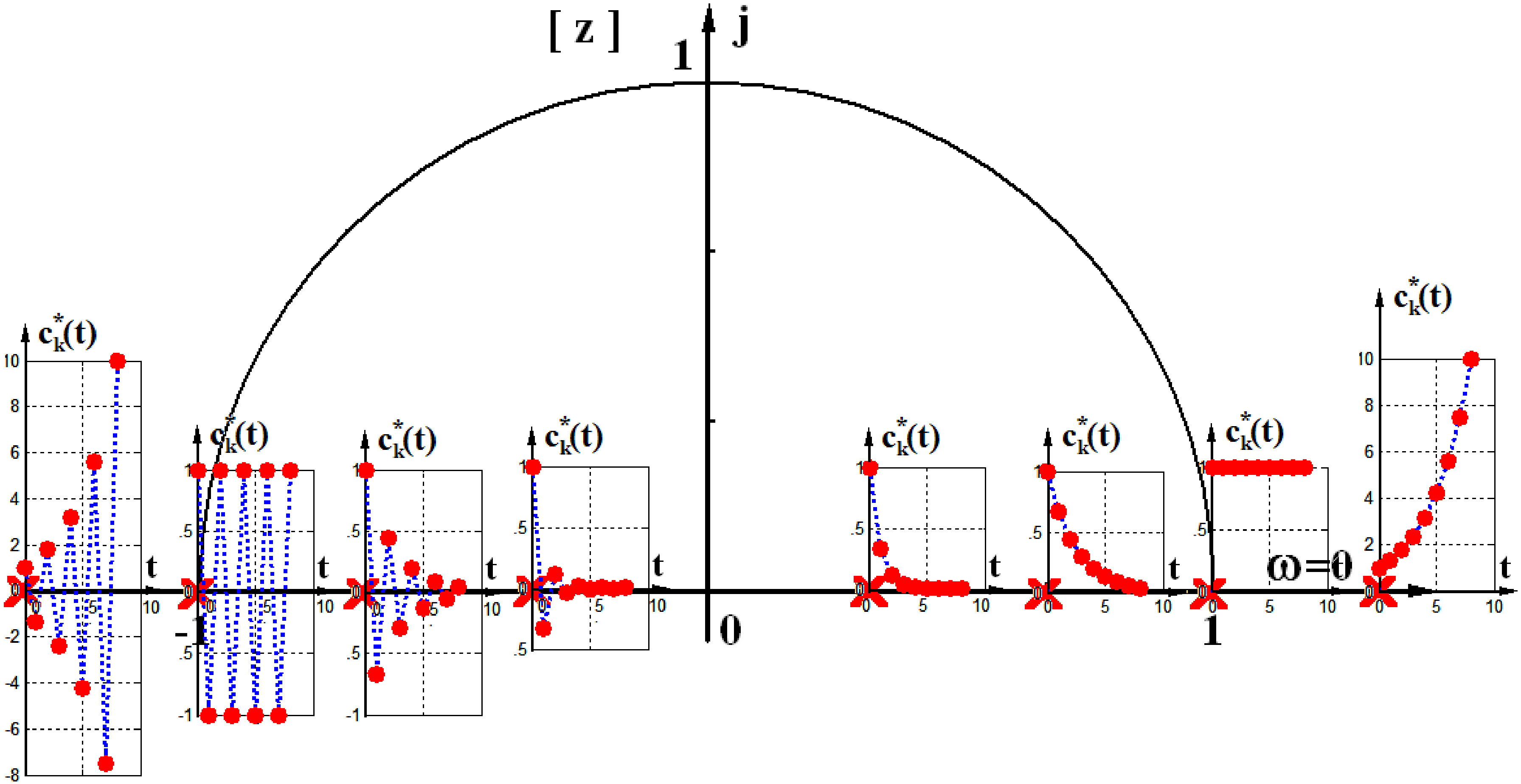
$$c_k(nT) = c_k\, p_k^n \qquad k = 1, 2, \cdots\ n$$

$$c_k(nT) = c_k p_k^n \qquad k = 1, 2, \cdots\ n$$

$p_k > 0$: $p_k > 1$ $\quad p_k = 1$ $\quad p_k < 1$

$p_k < 0$:




[z]
j
1
0
-1
1
ω=0
c*k(t)
t

(2) Closed-loop Complex conjugate(共轭) poles

$$p_k = |p_k| e^{j\theta_k} \qquad \overline{p}_k = |p_k| e^{-j\theta_k}$$

$$c^*_{k,k}(k) = Z^{-1}\left[\frac{c_k z}{z - p_k} + \frac{\overline{c}_k z}{z - \overline{p}_k}\right]$$

$$\begin{cases} a = \dfrac{1}{T}\ln|p_k| \\ \omega = \dfrac{\theta_k}{T} \\ 0 < \theta_k < \pi \end{cases}$$

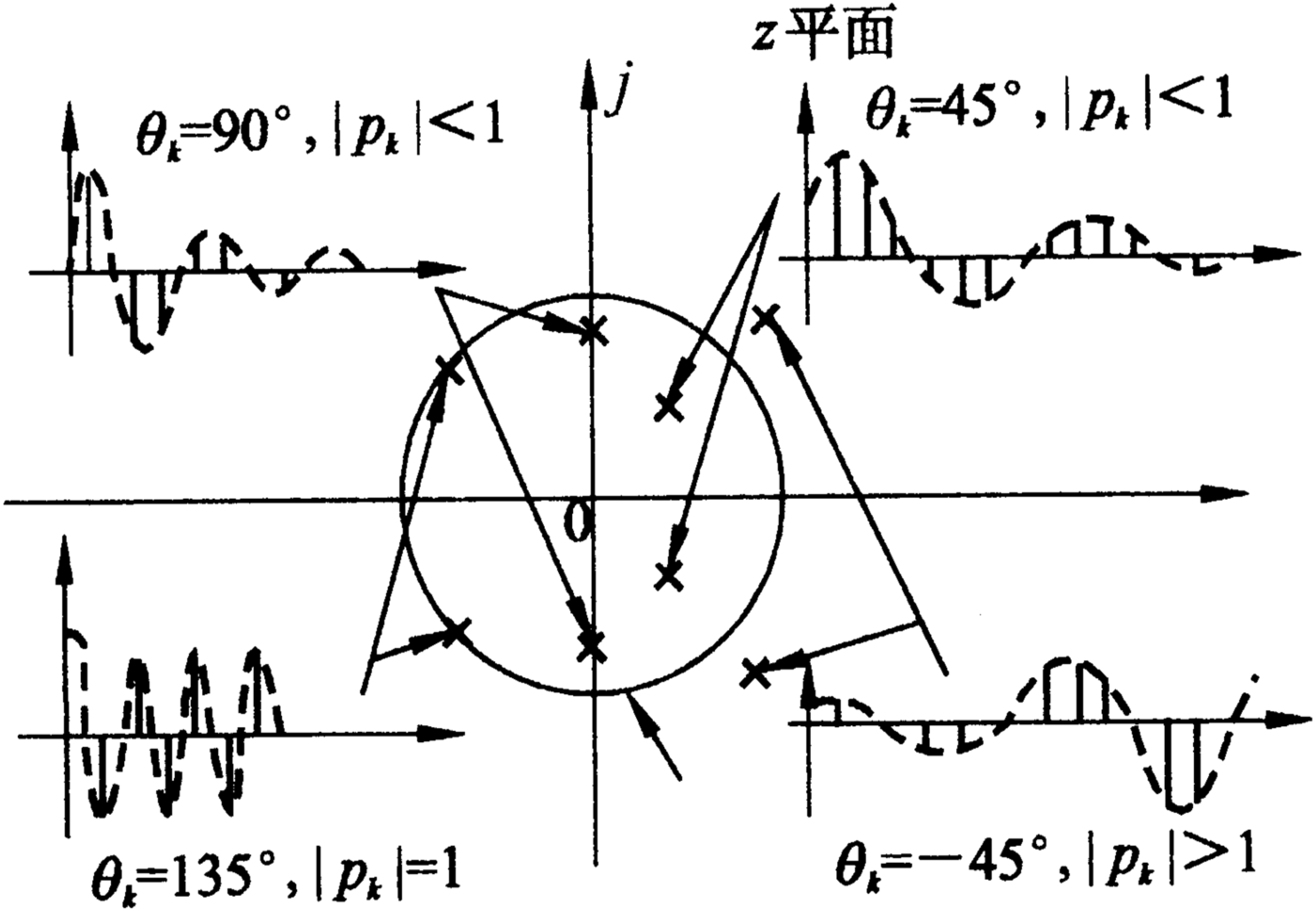
$$c_{k,k}(nT) = c_k p_k{}^n + \overline{c}_k \overline{p}_k{}^n$$

$$= c_k e^{a_k nT} + \overline{c}_k e^{\overline{a}_k nT}$$

$$= |c_k| e^{j\varphi_k} e^{(a+j\omega)nT} + |c_k| e^{-j\varphi_k} e^{(a-j\omega)nT}$$

$$= 2|c_k| e^{anT}\cos(n\omega T + \varphi_k) = 2|c_k||p_k|^n \cos(n\theta_k + \varphi_k)$$

$|p_k|<1$ $\left|p_k\right|>1$ $c_{k,k}(nT)=2\left|c_k\right|\left|p_k\right|^n\cos(n\theta_k+\varphi_k)$



Summary:

(1) General method

$$\begin{cases} G(z) \to \Phi(z) \longrightarrow C(z) = \sum_{n=0}^{\infty} c(nT) z^{-n} \\ c^*(t) = \sum_{n=0}^{\infty} c(nT)\delta(t-nT) \longrightarrow \text{Obtain } \sigma\%,\ t_s \text{ by definition} \end{cases}$$

(2) Closed-loop poles $p_k \longrightarrow$ Response $c_k(nT) = C_k p_k^n$

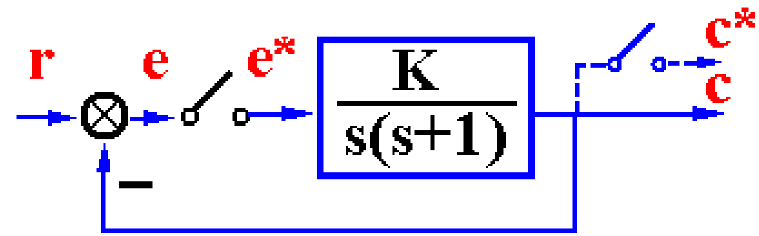
7.6 Performance Analysis of Discrete-Time Systems

- **Stability**
- **Dynamic Performance**
- **Steady-state Errors**

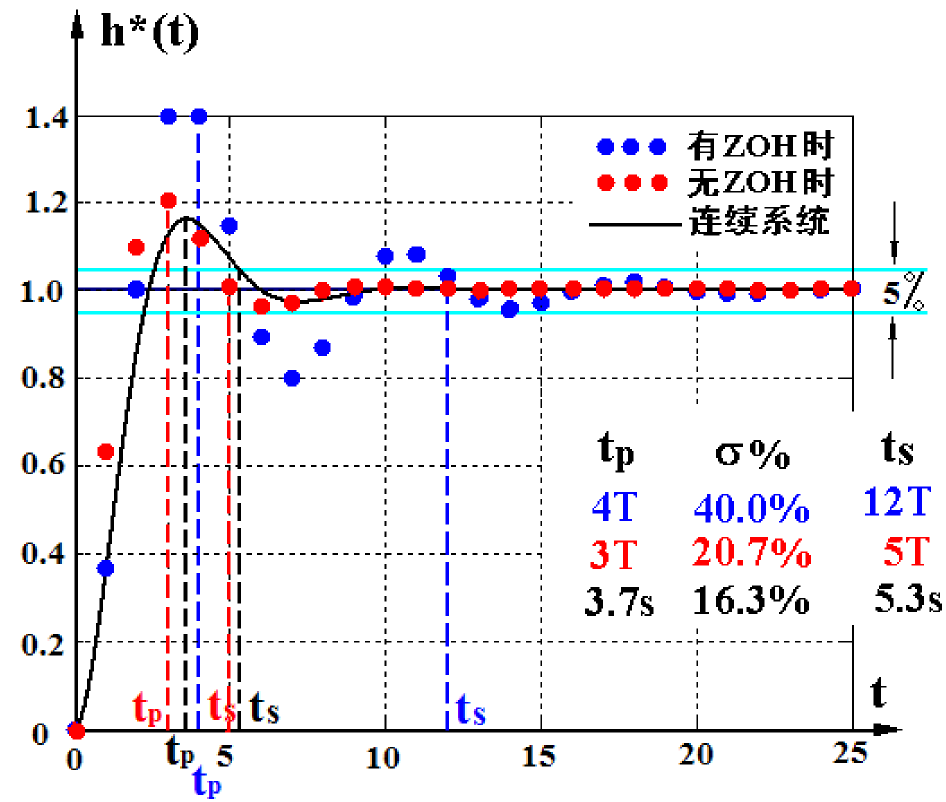
7.6.3 Steady-state error

1. General method to obtain steady-state error

Example Consider the system shown in the figure, T=K=1. Obtain the dynamic specifications. (σ %, t_s).

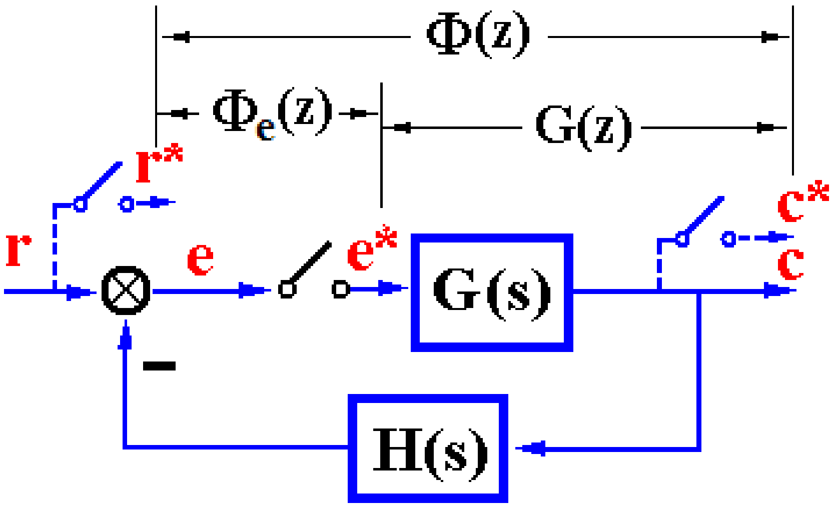


$h(0)=0$
$h(1)=0.632$
$h(2)=1.097$
$h(3)=1.207$
$h(4)=1.117$
$h(5)=1.014$
$h(6)=0.964$
$h(7)=0.970$
$h(8)=0.991$
$h(9)=1.004$
$h(10)=1.007$
$h(11)=1.003$
$h(12)=1.000$
$\vdots$



2. Using final value theorem to obtain steady-state error

Note: Only the stable system can have the steady-state error.



Algorithm:

(1) Determine the stability

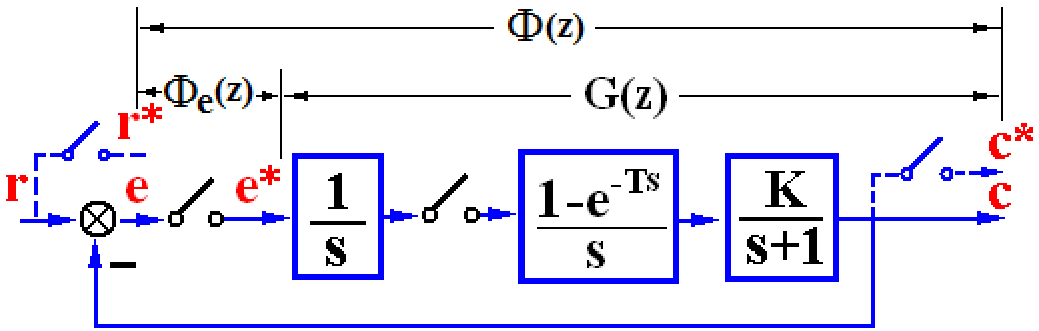
(2) Obtain the impulse transfer function from E(z) to R(z).

$$\Phi_e(z)=\frac{E(z)}{R(z)}=\frac{1}{1+GH(z)}$$

(3) Obtain $e(\infty)$ by the final value theorem

$$e(\infty)=\lim_{z\to 1}(z-1)\,\Phi_e(z)\,R(z)\ =\lim_{z\to 1}(z-1)\cdot R(z)\cdot\frac{1}{1+GH(z)}$$

Example 1 Consider the discrete system shown in the figure, K=2, T=1; Obtain e(∞) for r(t)=1(t), t, $t^2/2$.



$$G(z) = Z\left[\frac{1}{s}\right] \cdot Z\left[\frac{1-e^{-Ts}}{s} \cdot \frac{K}{s+1}\right]$$

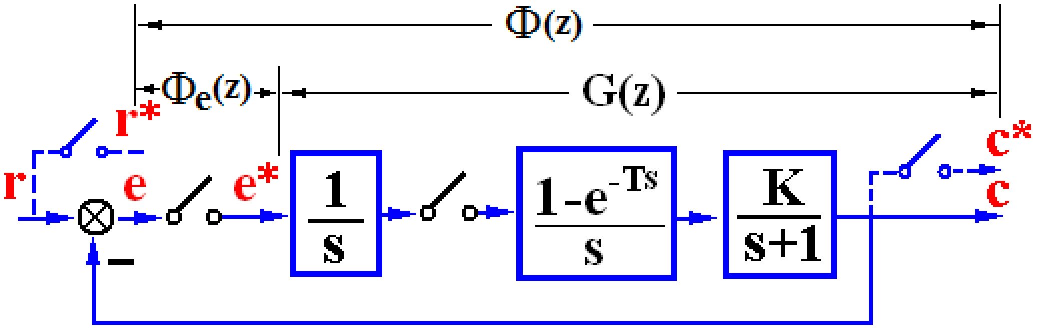
$$= \frac{K(1-e^{-T})z}{(z-1)(z-e^{-T})} \qquad v = 1$$

$$\Phi_e(z) = \frac{1}{1+\dfrac{K(1-e^{-T})z}{(z-1)(z-e^{-T})}} = \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T})+K(1-e^{-T})z}$$

$$D(z) = z^2 + [K(1-e^{-T}) - (1+e^{-T})]z + e^{-T} = 0$$

$$0 < K < \frac{2(1+e^{-T})}{(1-e^{-T})} \overset{T=1}{=} 4.33$$

Example 1 Consider the discrete system shown in the figure, K=2, T=1; Obtain e(∞) for r(t)=1(t), t, t²/2 .



$$0 < K < 4.33$$

$$e(\infty) = \lim_{z\to 1}(z-1)R(z)\Phi_e(z)$$

$$\Phi_e(z) = \frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z}$$

$r_1(t) = 1(t)$ $$e_1(\infty) = \lim_{z\to 1}(z-1)\frac{z}{z-1}\cdot\frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = 0$$

$r_2(t) = t$ $$e_2(\infty) = \lim_{z\to 1}(z-1)\frac{Tz}{(z-1)^2}\cdot\frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = \frac{T}{K}$$

$r_3(t) = \frac{t^2}{2}$ $$e_3(\infty) = \lim_{z\to 1}(z-1)\frac{Tz(z+1)}{2(z-1)^3}\cdot\frac{(z-1)(z-e^{-T})}{(z-1)(z-e^{-T}) + K(1-e^{-T})z} = \infty$$

3. Static Error Constant

shows how e(∞) changes with r(t)

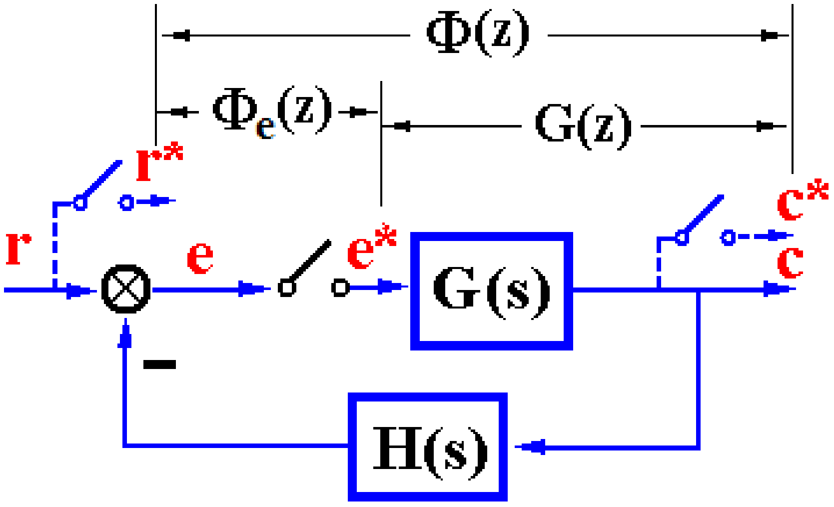
(For stable linear discrete systems subject to r(t) and sampled at the error signal)

Let $\begin{cases} GH(z) = Z[G(s)H(s)] = \dfrac{1}{(z-1)^v} GH_0(z) \\ \lim\limits_{z\to 1} GH_0(z) = K \neq 0 \end{cases}$ v: System type

$$\Phi_e(z) = \frac{E(z)}{R(z)} = \frac{1}{1+GH(z)}$$

$$e(\infty) = \lim_{z\to 1}(z-1)\,\Phi_e(z)\,R(z)$$

$$= \lim_{z\to 1}(z-1)\cdot R(z)\cdot\frac{1}{1+GH(z)}$$



$$e(\infty T)=\lim_{z\to1}(z-1)\,\Phi_e(z)\,R(z)=\lim_{z\to1}(z-1)\cdot R(z)\cdot\frac{1}{1+GH(z)}$$

$r(t)=A\cdot 1(t)$ $$e(\infty T)=\lim_{z\to1}(z-1)\cdot\frac{Az}{z-1}\cdot\frac{1}{1+GH(z)}=\frac{A}{1+\lim_{z\to1}GH(z)}=\frac{A}{K_p}$$

Static position error constant $K_p=1+\lim_{z\to1}GH(z)$

$r(t)=A\cdot t$ $$e(\infty T)=\lim_{z\to1}(z-1)\cdot\frac{ATz}{(z-1)^2}\cdot\frac{1}{1+GH(z)}=\frac{AT}{\lim_{z\to1}(z-1)GH(z)}=\frac{A}{K_v}$$

Static velocity error constant $K_v=\frac{1}{T}\lim_{z\to1}(z-1)GH(z)$

$r(t)=\frac{A}{2}t^2$ $$e(\infty T)=\lim_{z\to1}(z-1)\cdot\frac{AT^2z(z+1)}{2(z-1)^3}\cdot\frac{1}{1+GH(z)}=\frac{AT^2}{\lim_{z\to1}(z-1)^2GH(z)}=\frac{A}{K_a}$$

Static acceleration error constant $K_a=\frac{1}{T^2}\lim_{z\to1}(z-1)^2GH(z)$

Similar to the continuous system, we can divide the discrete-time system as type 0, type I, type II,… according to the numbers of the pole z=1 of the <u>impulse transfer function</u>.

Open-looped T.F.

$r(t) = A \cdot 1(t)$

Static position error constant $K_p = 1 + \lim_{z \to 1} GH(z)$

Type 0: K_p=constant

Type >=1: Kp= ∞, e(∞)=A/Kp=0

$r(t) = A \cdot t$

Static velocity error constant $K_v = \frac{1}{T} \lim_{z \to 1} (z-1) GH(z)$

Type 0: K_v=0, e(∞)=A/K_v= ∞

Type =1: K_v= constant,

Type >=2: K_v=∞, e(∞)=A/K_v =0

$r(t) = \frac{A}{2} t^2$ Static acceleration error constant $K_a = \frac{1}{T^2} \lim_{z \to 1} (z-1)^2 GH(z)$

Type 0,1: K_a=0, e(∞)=A/K_a = ∞

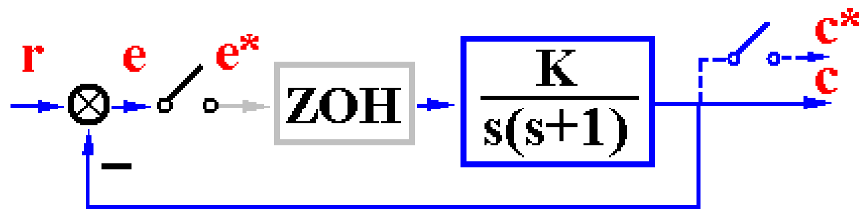
Type =2: K_a= constant,

Type >=3: K_a= ∞, e(∞))=A/K_a =0

$$\begin{cases} GH(z) = \dfrac{1}{(z-1)^v} GH_0(z) \\ \lim\limits_{z\to 1} GH_0(z) = K \end{cases}$$

型别	Static Error Constant			Steady-State Error		
v	$K_p = 1 + \lim\limits_{z\to 1} GH(z)$	$K_v = \frac{1}{T}\lim\limits_{z\to 1}[(z-1)GH(z)]$	$K_a = \frac{1}{T^2}\lim\limits_{z\to 1}[(z-1)^2 GH(z)]$	$r(t) = A\cdot 1(t)$	$r(t) = A\cdot t$	$r(t) = \frac{A}{2}t^2$
0	K_p	0	0	$\frac{A}{K_p}$	∞	∞
I	∞	K_v	0	0	$\frac{A}{K_v}$	∞
II	∞	∞	K_a	0	0	$\frac{A}{K_a}$

Example 2 Consider the stable discrete system shown in the figure. When r(t)=2t, obtain **e(∞)** with/without ZOH.



Solution.

no ZOH

$$\begin{cases} G(z)=Z\left[\frac{K}{s(s+1)}\right]=\frac{K(1-e^{-T})z}{(z-1)(z-e^{-T})} \\ K_v=\frac{1}{T}\lim_{z\to 1}(z-1)G(z)=\frac{1}{T}\lim_{z\to 1}\frac{K(1-e^{-T})z}{(z-e^{-T})}=\frac{K}{T} \end{cases}$$

$$e(\infty)=\frac{A}{K_v}=\frac{2T}{K}$$

— dependent on T

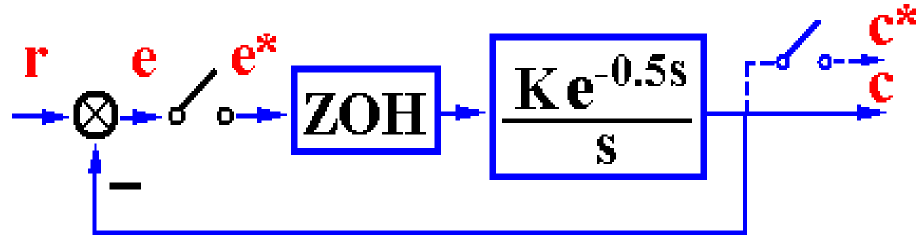
with ZOH

$$\begin{cases} G(z)=Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+1)}\right]=K\frac{z-1}{z}\cdot Z\left[\frac{1}{s^2(s+1)}\right] \\ \quad =K\frac{(T-1+e^{-T})z+(1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})} \\ K_v=\frac{1}{T}\lim_{z\to 1}(z-1)G(z)=\frac{1}{T}\lim_{z\to 1}\frac{K(T-Te^{-T})}{z-e^{-T}}=K \end{cases}$$

$$e(\infty)=\frac{A}{K_v}=\frac{A}{K}=\frac{2}{K}$$

— independent to T

Example 3 Consider the system shown in the figure, T=0.25. When $r(t)=2\cdot 1(t)+t$, obtain the range of K for $e(\infty)<0.5$.



Solution. The stable range of K is

$$0 < K < 2.472$$

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{Ke^{-2Ts}}{s}\right]$$

$$= K(1-z^{-1})z^{-2}Z\left[\frac{1}{s^2}\right] = Kz^{-2}\frac{z-1}{z}\cdot\frac{Tz}{(z-1)^2} = \frac{KT}{z^2(z-1)} \quad v=1$$

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = \frac{1}{T}\lim_{z\to 1}(z-1)\frac{KT}{z^2(z-1)} = K$$

$$r_1(t) = 2\cdot 1(t) \qquad e_1(\infty) = 0$$

$$r_2(t) = t \qquad e_2(\infty) = A/K_v = 1/K$$

$$e(\infty) = e_1(\infty) + e_2(\infty) = 1/K < 0.5 \quad \Rightarrow \quad K > 2$$

$$2 < K < 2.472$$

Summary:

(1) General method: obtain system response

(2) Final value theorem $\begin{cases} G(z) \to \Phi_e(z) \\ D(z) \to \text{Stability} \\ e(\infty) = \lim\limits_{z\to 1}(z-1)R(z)\Phi_e(z) \end{cases}$

(3) Static error constant $\begin{cases} G(z) \to v,\ K_p,\ K_v,\ K_a \\ \text{Obtain } e(\infty) \end{cases}$

Homework:

7-10

7.6.3 Steady-state error of discrete systems

(1) General method: obtain system response

(2) Final value theorem
$$\begin{cases} G(z) \to \Phi_e(z) \\ D(z) \to \text{Stability} \\ e(\infty) = \lim\limits_{z\to 1}(z-1)R(z)\Phi_e(z) \end{cases}$$

(3) Static error constant
$$\begin{cases} G(z) \to v,\ K_p,\ K_v,\ K_a \\ \text{Obtain } e(\infty) \end{cases}$$

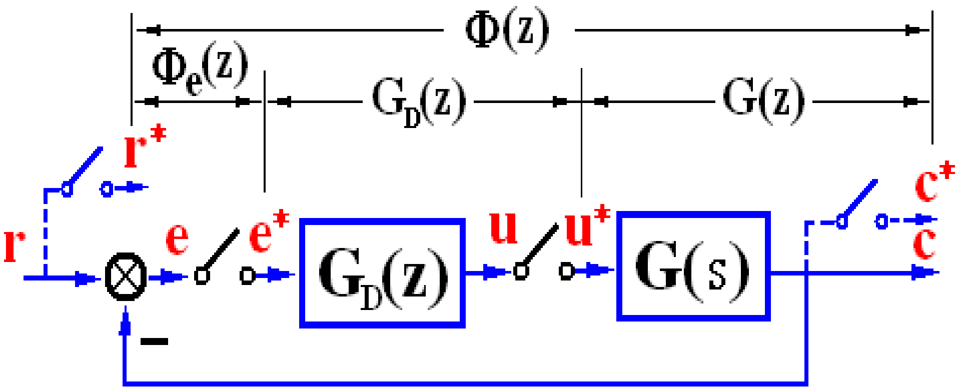
Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

Design for discrete-time systems can be done in s-domain, z-domain and w-domain, respectively.

7.7.1 The Impulse Transfer Function for the Digital Controller

$$\Phi(z)=\frac{G_D(z)\cdot G(z)}{1+G_D(z)\cdot G(z)}$$

$$\Phi_e(z)=\frac{1}{1+G_D(z)\cdot G(z)}=1-\Phi(z)$$



$$G_D(z)\cdot G(z)=\frac{\Phi(z)}{1-\Phi(z)}=\frac{\Phi(z)}{\Phi_e(z)}$$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

$$\Phi_e(z)=1-\Phi(z),\ \mathrm{E}(z)=\Phi_e(z)R(z)$$

7.7.2 Deadbeat Control Design 最少拍控制

Deadbeat Control Systems: Matching a particular test input within a number of steps. —— No steady-state error on the sampling point.

(典型输入作用下，能在有限拍内结束响应过程且在采样点上无稳态误差的系统。）

1. A unified description of typical test inputs

$$r(t)=\begin{cases}1(t)\\ t\\ t^2/2\end{cases}\qquad R(z)=\left.\begin{cases}\dfrac{z}{z-1}=\dfrac{1}{1-z^{-1}}\\[2ex] \dfrac{Tz}{(z-1)^2}=\dfrac{Tz^{-1}}{(1-z^{-1})^2}\\[2ex] \dfrac{T^2z(z+1)}{2(z-1)^3}=\dfrac{T^2z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}\end{cases}\right\}\boxed{\dfrac{A(z)}{(1-z^{-1})^{\nu}}}$$

ν	$A(z)$
1	1
2	Tz^{-1}
3	$\dfrac{T^2z^{-1}(1+z^{-1})}{2}$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

Design Idea: **Obtain $G_D(z)$ by constructing $\Phi(z)$ so that the output can match the typical test signal within the minimum steps.**

No $\begin{cases}\text{Zeros}\\ \text{Poles}\end{cases}$ of G(z) on or beyond the unit circle, except for (1, j0)

$$\mathrm{R}(z)=\frac{A(z)}{(1-z^{-1})^{\nu}}$$

$$\mathrm{E}(z)=\Phi_e(z)R(z),\quad \Phi_e(z)=1-\Phi(z)$$

$$e(\infty)=\lim_{z\to 1}(z-1)\,\Phi_e(z)\,R(z)$$

From the design idea, we know that $e(\infty T)=0$

$$E(z)=\Phi_e(z)\cdot R(z)=\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)$$

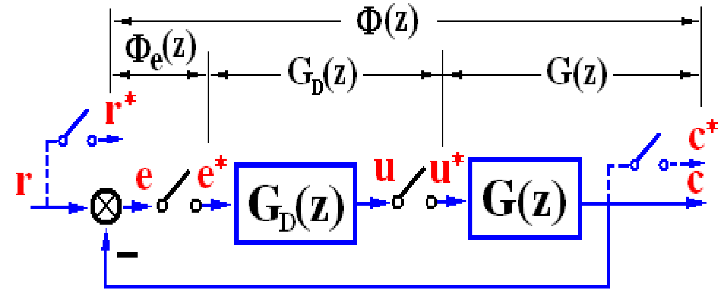
$$e(\infty T)=\lim_{z\to 1}(1-z^{-1})\frac{A(z)}{(1-z^{-1})^{\nu}}\Phi_e(z)=0$$

$$\Rightarrow \Phi_e(z)=(1-z^{-1})^{\nu}F(z^{-1})$$

To make the D(z) simplest and of the lowest-order, we can choose $F(z^{-1})$ as 1.

$$\Phi_e(z)=(1-z^{-1})^{\nu}F(z)\overset{F(z)=1}{=}(1-z^{-1})^{\nu}$$

$$\Phi(z)=1-\Phi_e(z)=1\text{-}(1\text{-}z^{-1})^{\nu}$$



Hence:

$$\Phi(z)=1-\Phi_e(z)=1-(1-z^{-1})^{\nu}=b_1z^{-1}+b_2z^{-2}+\cdots+b_{\nu}z^{-\nu}$$

$$=\frac{b_1z^{\nu-1}+b_2z^{\nu-2}+\cdots+b_{\nu}}{z^{\nu}}$$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)\cdot G(z)}$$

The rule to construct $\Phi(z)$: All poles of $\Phi(z)$ are located on the origin of z-plane.

2. $\Phi(z)$ for typical test inputs

（1） for $r(t) = 1(t)$

- The C. L. impulse transfer function:

$$\nu = 1 \qquad \Phi(z) = z^{-1}$$

$$E(z) = \sum_{n=0}^{\infty} e(nT) z^{-n}$$

$$= e(0) + e(T) z^{-1} + \ldots = 1$$

The system can track the input by 1 step only。

（2） for $r(t) = t \cdot 1(t)$

- The C.L. impulse transfer function:

$$v = 2 \quad \Phi(z) = 2z^{-1} - z^{-2}$$

$$E(z) = \sum_{n=0}^{\infty} e(nT) z^{-n}$$

$$= e(0) + e(T) z^{-1} + \ldots = Tz^{-1}$$

The system can track the input by 2 steps。

（3） for $r(t)=\frac{1}{2}t^2\cdot 1(t)$

- **The C.L.impulse transfer function:**

$$\nu=3 \qquad \Phi(z)=3z^{-1}-3z^{-2}+z^{-3}$$

$$E(z)=\sum_{n=0}^{\infty}e(nT)z^{-n}$$

$$=e(0)+e(T)z^{-1}+\ldots=\frac{1}{2}T^2z^{-1}+\frac{1}{2}T^2z^{-2}$$

The system can track the input by 3 steps。

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z) \cdot G(z)}$$

Deadbeat Control Design Table

$r(t)$	$R(z)$	$\Phi_e(z) = (1 - z^{-1})^v$	$\Phi(z) = 1 - \Phi_e(z)$	$G_D(z)$	t_s
$1(t)$	$\frac{1}{1 - z^{-1}}$	$1 - z^{-1}$	z^{-1}	$\frac{z^{-1}}{(1 - z^{-1}) \cdot G(z)}$	T
t	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$	$(1 - z^{-1})^2$	$2z^{-1} - z^{-2}$	$\frac{z^{-1}(2 - z^{-1})}{(1 - z^{-1})^2 G(z)}$	$2T$
$\frac{t^2}{2}$	$\frac{T^2 z^{-1}(1 + z^{-1})}{2(1 - z^{-1})^3}$	$(1 - z^{-1})^3$	$3z^{-1} - 3z^{-2} + z^{-3}$	$\frac{z^{-1}(3 - 3z^{-1} + z^{-2})}{(1 - z^{-1})^3 G(z)}$	$3T$

3. Algorithm for Deadbeat Control Design

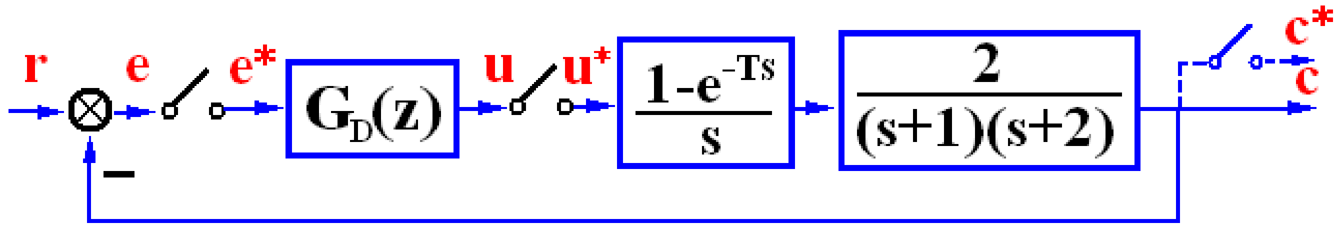
① **Obtain G(z) — Suppose there are no poles and zeros of G(z) on or beyond the unit circle.**

② **Determine $\Phi_e(z)$ for the particular test input by v**

$$r(t) \quad \Rightarrow \quad R(z) = \frac{A(z)}{(1-z^{-1})^{\nu}} \quad \Rightarrow \quad \Phi_e(z) = (1-z^{-1})^{\nu}$$

③ **Obtain** $\Phi(z) = 1 - \Phi_e(z)$

④ **Achieve** $G_D(z) = \dfrac{\Phi(z)}{\Phi_e(z) \cdot G(z)}$



Example 1. Consider the system shown in the above figure (T=1). Design deadbeat controllers $G_D(z)$ for r(t)=1(t)、t .

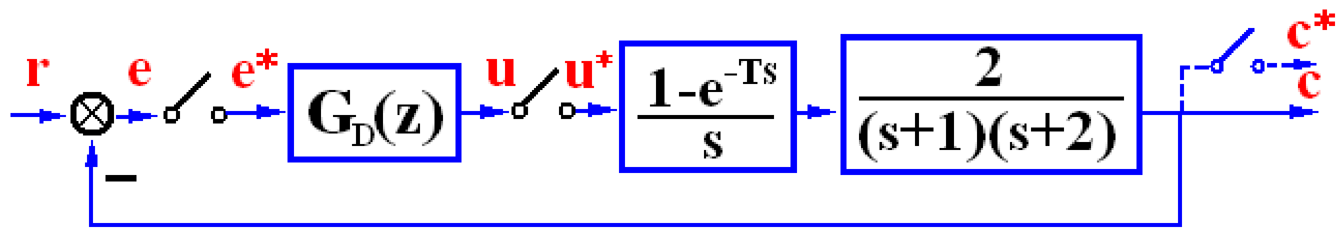
Solution.
$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{2}{(s+1)(s+2)}\right] = 2(1-z^{-1})\cdot Z\left[\frac{C_0}{s}-\frac{C_1}{s+1}+\frac{C_2}{s+2}\right]$$

$$= 2\cdot\frac{z-1}{z}\cdot Z\left[\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{s+1}+\frac{1}{2}\cdot\frac{1}{s+2}\right]$$

$$= \frac{z-1}{z}\left[\frac{z}{z-1}-\frac{2z}{z-e^{-T}}+\frac{z}{z-e^{-2T}}\right] = 1-\frac{2(z-1)}{z-e^{-T}}+\frac{z-1}{z-e^{-2T}}$$

$$= \frac{(1+e^{-2T}-2e^{-T})z+(e^{-3T}+e^{-T}-2e^{-2T})}{(z-e^{-T})(z-e^{-2T})}$$

$$\overset{T=1}{=} \frac{0.4(z+0.365)}{(z-0.368)(z-0.136)}$$



Referring to the result for $r(t) = 1(t)$ in the Design Table

$$R(z) = \frac{z}{z-1} \qquad \text{Choose} \begin{cases} \Phi_e(z) = 1 - z^{-1} \\ \Phi(z) = 1 - \Phi_e(z) = z^{-1} \end{cases}$$

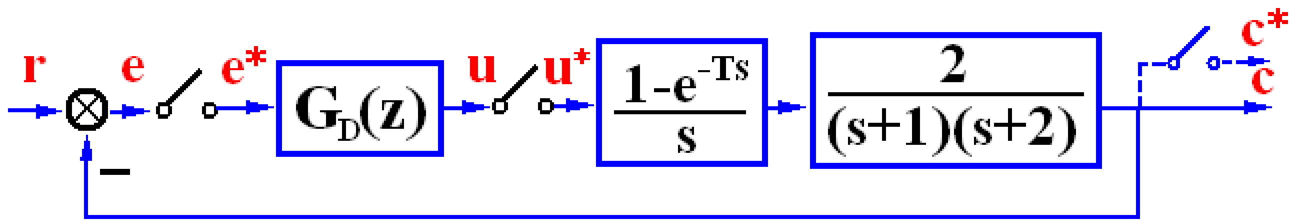
$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z) \cdot G(z)} = \frac{z^{-1}}{1-z^{-1}} \cdot \frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$= \frac{2.5(z-0.368)(z-0.136)}{(z-1)(z+0.365)}$$

$$C(z) = \Phi(z)R(z) = z^{-1} \cdot \frac{1}{1-z^{-1}}$$

$$= z^{-1}[1 + z^{-1} + z^{-2} + \cdots] = z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$E(z) = \Phi_e(z)R(z) = (1-z^{-1}) \cdot \frac{1}{1-z^{-1}} = 1$$



For $r(t) = t$

$$R(z) = \frac{Tz^{-1}}{(1-z^{-1})^2} \qquad \text{Choose} \begin{cases} \Phi_e(z) = (1-z^{-1})^2 \\ \Phi(z) = 1-\Phi_e(z) = 2z^{-1} - z^{-2} \end{cases}$$

$$G_D(z) = \frac{\Phi(z)}{\Phi_e(z)\cdot G(z)} = \frac{2z^{-1}-z^{-2}}{(1-z^{-1})^2}\cdot\frac{(z-0.368)(z-0.136)}{0.4(z+0.365)}$$

$$= \frac{5(z-0.5)(z-0.368)(z-0.136)}{(z-1)^2(z+0.365)}$$

$$E(z) = \Phi_e(z)\cdot R(z) = Tz^{-1}$$

$$C(z) = \Phi(z)R(z) = (2z^{-1}-z^{-2})\cdot\frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$= R(z) - E(z) = 2Tz^{-2} + 3Tz^{-3} + 4Tz^{-4} + \cdots$$

- **Attention:**
- The setting time (the minimum steps) of the system is decided by designed $\Phi(z)$, rather than the typical input signal: 1(t), t, $t^2/2$.
- Such as: if we use the Deadbeat system of velocity input r(t)=t.
- We have $\Phi(z) = 2z^{-1} - z^{-2}$
- For r(t)=1(t)

$$R(z) = \frac{1}{1-z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$C(z) = \frac{2z^{-1} - z^{-2}}{1-z^{-1}} = 0 + 2z^{-1} + z^{-2} + z^{-3} + \ldots$$

- For r(t)=t

$$R(z)=\frac{Tz^{-1}}{(1-z^{-1})^{2}}=0+Tz^{-1}+2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\dots$$

$$C(z)=\frac{Tz^{-1}(2z^{-1}-z^{-2})}{(1-z^{-1})^{2}}=0+0+2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\dots$$

- For r(t)=$t^2/2$

$$R(z)=\frac{T^{2}z^{-1}(1+z^{-1})}{2(1-z^{-1})^{3}}=0+0.5T^{2}z^{-1}+2T^{2}z^{-2}+4.5T^{2}z^{-3}+8T^{2}z^{-4}+\dots$$

$$C(z)=\frac{T^{2}z^{-1}(1+z^{-1})(2z^{-1}-z^{-2})}{2(1-z^{-1})^{3}}=0+0+T^{2}z^{-2}+3.5T^{2}z^{-3}+7T^{2}z^{-4}+\dots$$

4. G(z) has poles or zeros on or beyond the unit circle

suppose

$$G(z)=\frac{z^{-v}\prod_{i=1}^{L}(1-z_i z^{-1})}{\prod_{i=1}^{n}(1-p_i z^{-1})}$$

where Z_i **is the zero of G(z);**

p_i **is the pole of G(z).**

v **is delay.**

Then

$$\mathrm{G}_D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi_e(z)}$$

$$G_D(z)=\frac{\Phi(z)}{\Phi_e(z)G(z)}=\frac{z^{v}\prod_{i=1}^{n}(1-p_i z^{-1})\Phi(z)}{\prod_{i=1}^{L}(1-z_i z^{-1})\Phi_e(z)}$$

①If there is Z^{v} in G_D(z)， G_D(z) is un-realizable.

Thus, we have to ensure that there exists Z^{-v} in Φ(z)，which promises G_D(z) is realizable.

② If there is z_i on or beyond the unit circle, G_D(z) is unstable.

Then, those z_i will be designed as the zeros of Φ(z).

③**Note that** $\Phi(z) = G_D(z)G(z)\Phi_e(z)$

$$= \frac{z^{-v}\prod_{i=1}^{L}\left(1 - z_i z^{-1}\right)}{\prod_{i=1}^{n}\left(1 - p_i z^{-1}\right)} G_D(z)\Phi_e(z)$$

If there are p_i on or beyond the unit circle,

$\Phi(z)$ will be unstable,

Then **those p_i will be designed as the zeros of $\Phi_e(z)$.**

④ **Determine the relative parameters by**

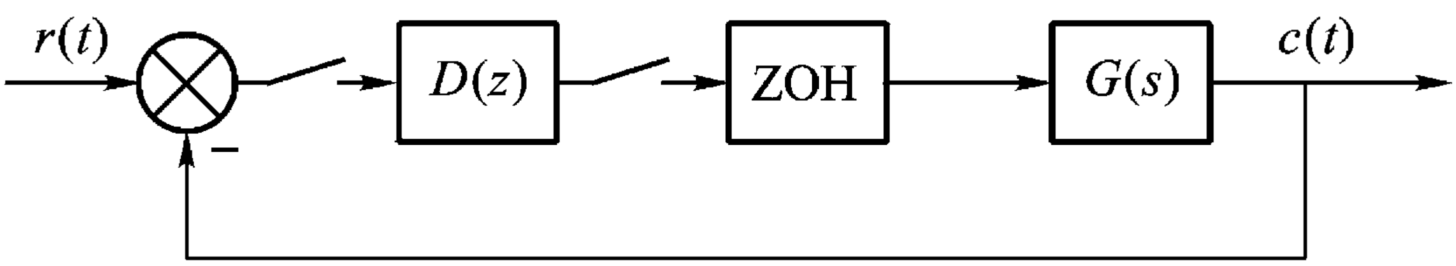
$$\Phi(z) = 1 - \Phi_e(z)$$

Example **Given the discrete system described as in the Following figure, where**

$$G(s)=\frac{10}{s(0.1s+1)(0.05s+1)},\quad G_{zoh}(s)=\frac{1-e^{-Ts}}{s}$$

with $$\mathrm{T}=0.2s$$

Design a deadbeat controller for $r(t)=1(t)$



Solution: the O. L. impulse transfer function is

$$G(z) = Z[G_{zoh}(s)G(s)] = \frac{0.76z^{-1}(1+0.05z^{-1})(1+1.065z^{-1})}{(1-z^{-1})(1-0.135z^{-1})(1-0.0185z^{-1})}$$

For $r(t) = 1(t)$ **, we can design**

$$\varphi_e(z) = 1 - z^{-1} \qquad (1)$$

$$\varphi(z) = z^{-1} \qquad (2)$$

Because there exists *z* = − 1.065 (beyond the unit circle),

Thus, z should also be the zero of Φ(z)

There exist z^{-1} ***in G(z)*** **,** z^{-1} **should be in Φ(z), thus**

$$\varphi(z) = z^{-1}(1+1.065z^{-1}) \qquad (3)$$

Because that

$$\varphi(z) = 1 - \varphi_e(z) \qquad (4)$$

from (3), $\varphi(z)$ **is now a polynomial on** z^{-1} **of order 2,**

To satisfy（4）, $\varphi_e(z)$ **must be a polynomial on** z^{-1} **of order 2，thus based on (1), we redesign:**

$$\varphi_e(z) = (1 - z^{-1})\,(1 + a_1 z^{-1}) \qquad (5)$$

Where a_1 **is a constant to be chosen later。**

Thus multiplied by a constant b_1 to be designed later, we get

$$\varphi(z) = b_1 z^{-1}(1 + 1.065 z^{-1}) \qquad (6)$$

From（5）and（6），we get:

$$a_1 = 0.516 \qquad b_1 = 0.484$$

Thus,

$$\varphi_e(z) = \left(1 - z^{-1}\right)\left(1 + 0.516z^{-1}\right) \tag{7}$$

$$\varphi(z) = 0.484z^{-1}\left(1 + 1.065z^{-1}\right) \tag{8}$$

Then the deadbeat controller is

$$D(z) = \frac{1 - \varphi_e(z)}{G(z)\varphi_e(z)}$$

$$= \frac{1 - \left(1 - z^{-1}\right)\left(1 + 0.516z^{-1}\right)}{\dfrac{0.76z^{-1}\left(1 + 0.05z^{-1}\right)\left(1 + 0.065z^{-1}\right)}{\left(1 - z^{-1}\right)\left(1 - 0.135z^{-1}\right)\left(1 - 0.0185z^{-1}\right)}\left(1 - z^{-1}\right)\left(1 + 0.516z^{-1}\right)}$$

$$D(z)=\frac{0.637(1-0.0185z^{-1})\,(1-0.135z^{-1})}{(1+0.05z^{-1})\,(1+0.516z^{-1})}$$

Then the *Z*-transform is

$$C(z)=\varphi(z)R(z)=0.484z^{-1}(1+1.085z^{-1})\frac{1}{1-z^{-1}}$$
$$=0.484z^{-1}+z^{-2}+z^{-3}+\cdots+z^{-4}+\cdots$$

System can follow the input at the 2nd step, which is 2 step later.

Although the deadbeat control system tracks a particular test input accurately within a number of steps, it has the following disadvantages:

(1) It is designed only for a particular input.

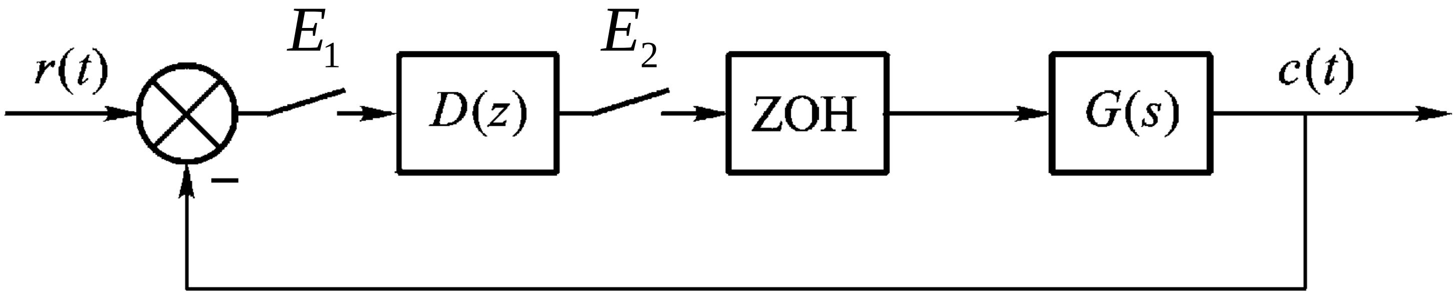
(2) Although there are no errors on the sampling points, the output has ripples between the sampling points.

(3) The high-order controller will make the control process (the output of controller) changes drastically.

5. Ripple-free deadbeat control design

Ripple: **though the system outputs are stable at the sampling time, they are varying between two sampling time. Fig. 7-50 in page 230.**

Objective: **Not only tracking the input at the sampling time, but also the one between two sampling point. Then, the outputs are ripple-free.**



Example: $G(s) = \dfrac{10}{s(s+1)} \qquad r(t) = t$

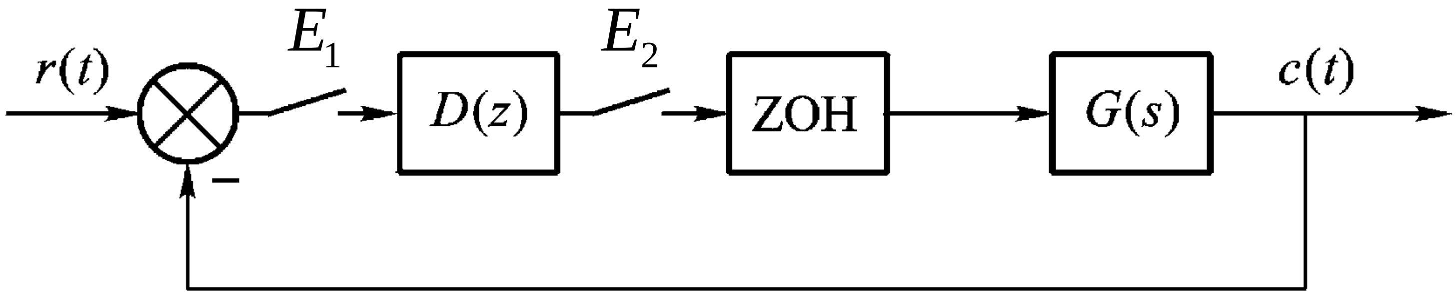
$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\frac{10}{s(s+1)}\right] \overset{T=1}{=} \frac{3.68z^{-1}(1+0.718z^{-1})}{(1-z^{-1})(1-0.368z^{-1})}$$

$$D(z) = \frac{0.543(1-0.5z^{-1})(1-0.368z^{-1})}{(1-z^{-1})(1+0.718z^{-1})}$$

$$E_1(z) = \Phi_e(z)R(z) = (1-z^{-1})^2\frac{z^{-1}}{(1-z^{-1})^2} = z^{-1}$$

$$E_2(z) = D(z)E_1(z) = \frac{0.543(1-0.5z^{-1})(1-0.368z^{-1})}{(1-z^{-1})(1+0.718z^{-1})}z^{-1}$$

$$= 0.543z^{-1} - 0.319z^{-2} + 0.39z^{-3} - 0.119z^{-4} + 0.246z^{-5} - \cdots$$



$$\Phi_{e2}(z)=\frac{E_2(z)}{R(z)}=D(z)\Phi_e(z)=\frac{D(z)}{1+D(z)G(z)}=\frac{\Phi(z)}{G(z)}=\frac{P_\Phi(z)}{z^r}\frac{Q(z)}{P(z)}$$

Solution: to ensure $E_2(z)$ achieving the stable state in finite steps.

$$E_2(z)=\Phi_{e2}R(z),\quad \Phi_{e2}=D(z)\phi_e(z)=\frac{P_\Phi(z)}{z^r}\frac{Q(z)}{P(z)}$$

$\phi_{e2}(z)$ should have best dynamic performance ➔

$\phi_{e2}(z)=D(z)\Phi_e(z)=(*)/z^r$, that is P(z) {the zero of G(z)} must be the zeros of Φ(z).

Example: P231 7-33

最少拍设计中，$\Phi(z)$和$\Phi_e(z)$选取时应遵循的原则:

1。D(z)零点的数目不能大于极点的数目；

2。$\Phi_e(z)$应把G(z)在单位圆上及单位圆外的极点作为自己的零点;

3。$\Phi(z)$应把G(z)在单位圆上及单位圆外的零点作为自己的零点;

4。当G(z)含有z^{-1}因子时，要求$\Phi(z)$也含有z^{-1}的因子;

5。因为$\Phi(z)=1-\Phi_e(z)$，他们应该是关于z^{-1}同样阶次的多项式，而且$\Phi_e(z)$还应包含常数项1。

6。当最小拍系统还有无纹波要求时，闭环脉冲传函$\Phi(z)$的零点应抵消G(z)的全部零点（因为最少拍系统设计中G(z)单位圆上及单位圆外的零极点已经被补偿，因此在无纹波的设计中只需抵消G(z)单位圆内的零点）。

Homework:

p238 7-15, 7-16

7-15. Consider the system as shown in Fig 7-69, T=1s, design deadbeat controller D(z) for r(t)=t. Draw $r^*(t)$, $e_1^*(t)$, $e_2^*(t)$, x(t), y(t) and $y^*(t)$.

7-16. Furthermore, design a ripple-free deadbeat controller for the system in 7-15.