第9章 控制系统的状态空间分析

Chapter 9 Analysis of Control Systems in State Space

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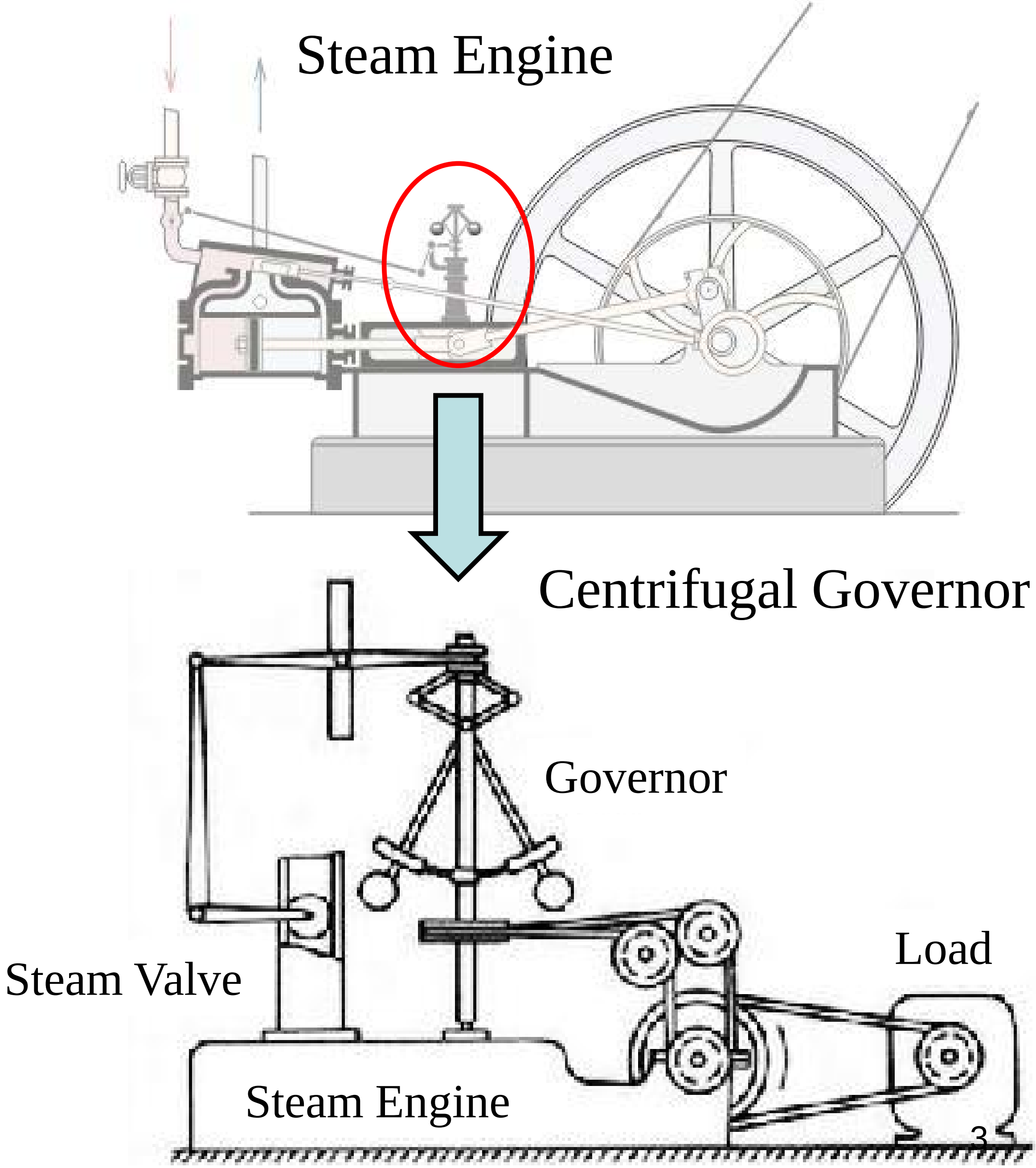
Chapter 9 Analysis of Control Systems in State Space

- 9.1 Introduction
- 9.2 State-Space and State-Equation
- 9.3 State-Space Establishing of Linear System
- 9.4 Solving the Linear Time-Invariant State Equation
- 9.5 Controllability and Observability
- 9.6 Feedback Structure and State-Observers

Mr. J. Watt


Steam Engine
Centrifugal Governor
Governor
Load
Steam Valve
Steam Engine

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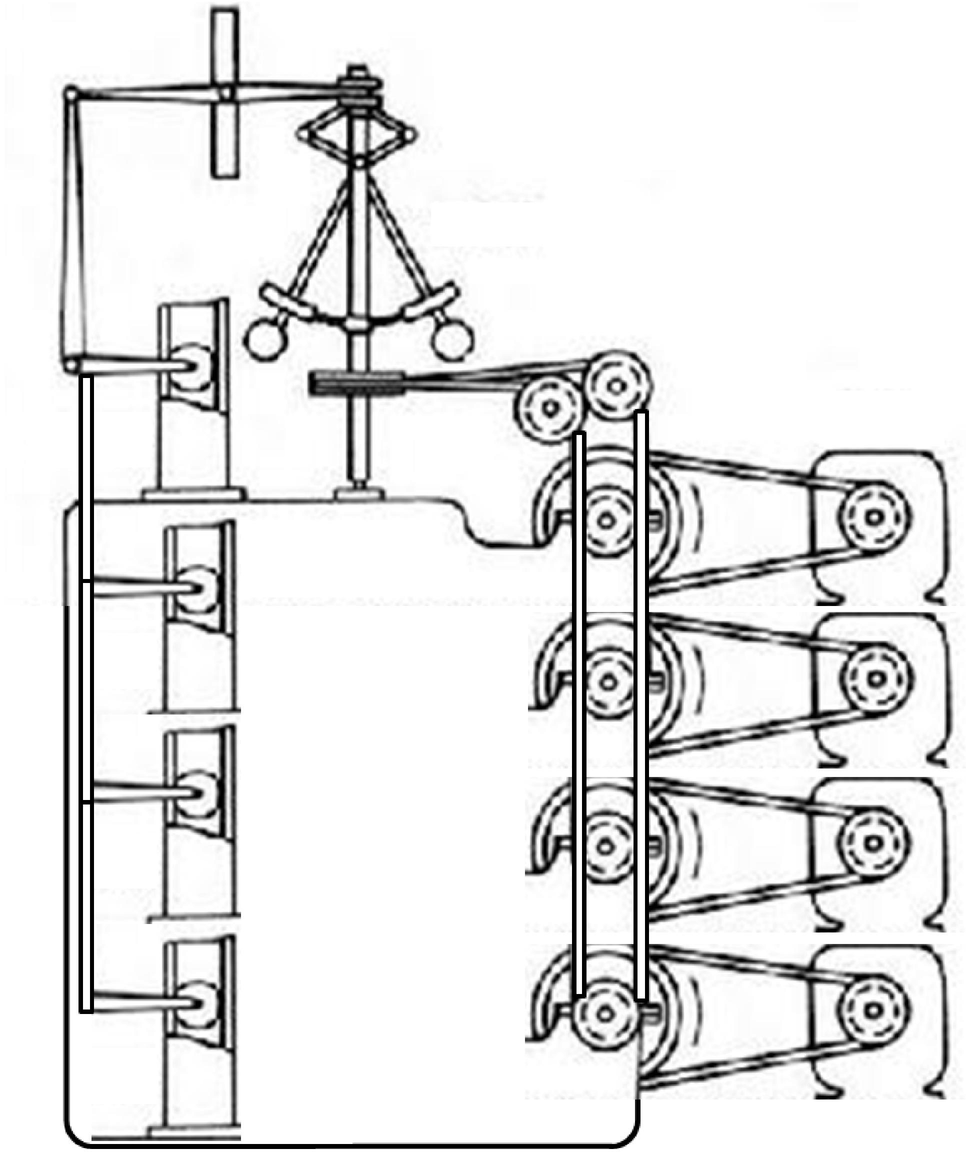

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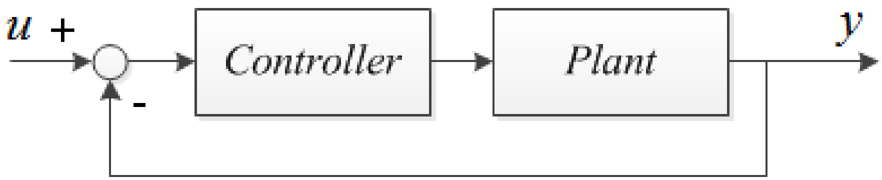
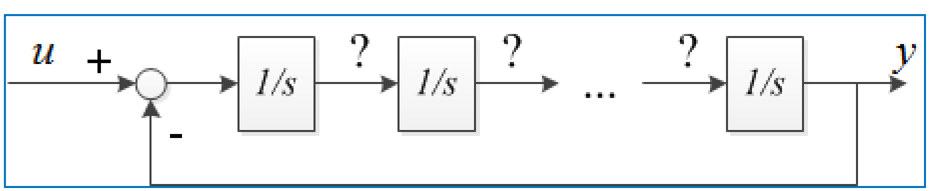
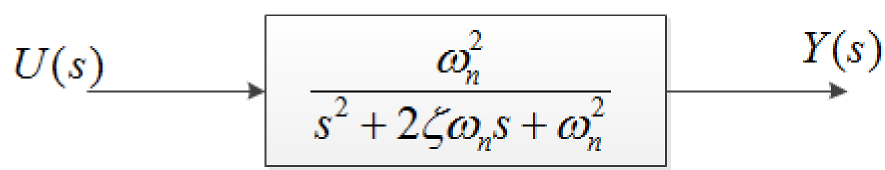
Why Modern Control Theory？

- 1. Result and Process



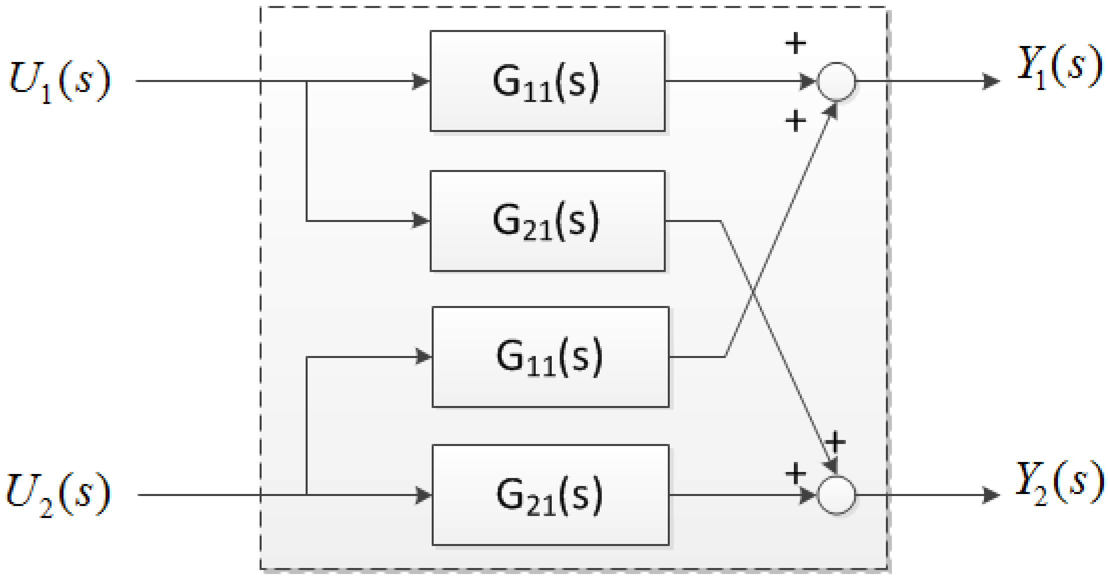
Why feedback？
How feedback?



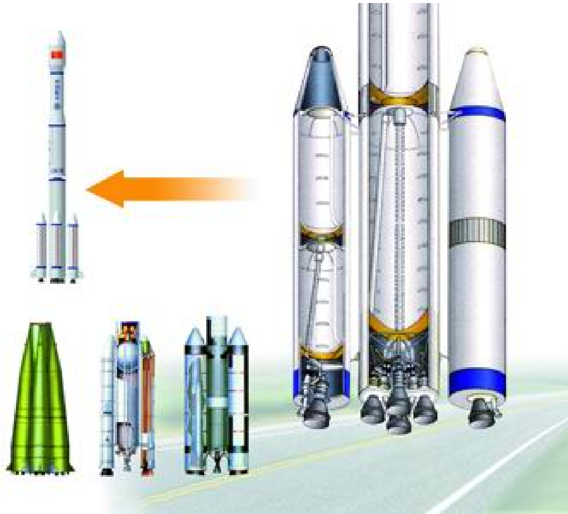


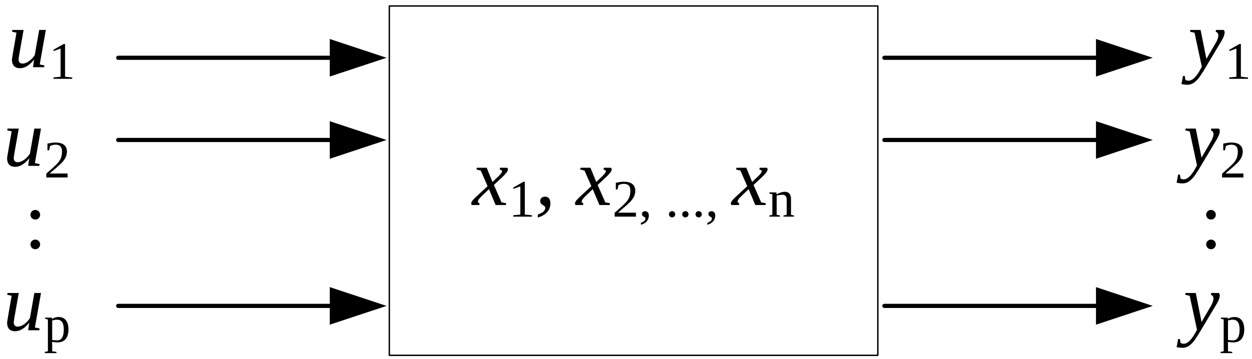
What about the intermediate states?

- 2. Internal relationship and Coupling(耦合)



- 3. Aerospace science requirement




u_1
u_2
$\vdots$
u_p
$x_1, x_2, \ldots, x_n$
y_1
y_2
$\vdots$
y_p

Areas:

- Aerospace Science;
- Robotics;
- Industry;

Benefit:

- Mathematical Tools
- Theory Fundation
- Research Method
- External Information(外部信息)(Input&Output Variable)

  Internal Information (内部信息)(State Variable)

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9.2 State Space and State Equation

9.2.1 Concepts

1. State: In time domain, a set (集合) of variables to describe System's motion and movement information

2. State-variable: the smallest set of variables that describe the "state" of a system. Intuitively, the state of a system describes enough about the system to determine its future behavior. In an n^{th}-order differential equation there should be n independent state variables.

Attention:

State-variable is Sufficient and Necessary to describe dynamic motion of the system.(NSC-必要充分条件)

State-variable is not unique

3. State-vector: n states variables: $x_1(t)$, $x_2(t)$, … , $x_n(t)$ to describe the observed states in State space.

$$X(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$$

4. State-space: The n dimension space based on the state-variable $x_1(t)$, $x_2(t)$, … , $x_n(t)$.
5. State-locus: At special time t_0, the state x(t_0) will be a point in the state-space; during a period of time t, the state $x(t)$ will be draw as a trajectory/locus.
6. State-equation: The mathematical relationship between two or more state functions, relationship between state variables and the input:

$$\dot{x}(t) = f[x(t), u(t)]$$

$$x(t_{k+1}) = f[x(t_k), u(t_k)]$$

7. Output-equation: Describe the relationship between the Output and State, and between Output and Input.

$$y(t) = g[x(t), u(t)]$$

$$y(t_k) = g[x(t_k), u(t_k)]$$

8. State-space Representation: Represent the system by the State-equations and Output-equations.

$$\begin{cases} \dot{x}(t) = f[x(t), u(t)] \\ y(t) = g[x(t).u(t)] \end{cases}$$

$$\begin{cases} x(t_{k+1}) = f[x(t_k), u(t_k)] \\ y(t_k) = g[x(t_k), u(t_k)] \end{cases}$$

- State-space Representation

$$\begin{cases} \dot{x}(t) = f[x(t), u(t)] \\ y(t) = g[x(t), u(t)] \end{cases}$$

f and *g* are linear function, for linear system.

State-equation: a first-order differential equation.

Output-equation: algebraic equation(代数方程) for vectors .

$$\begin{cases} \dot{X}(t) = A(t)x(t) + B(t)u(t) \\ Y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

$$\begin{cases} \dot{X}(t) = Ax(t) + Bu(t) \\ Y(t) = Cx(t) + Du(t) \end{cases}$$

Linear Time-Invariant System

➢ State-equation: Relationship between the Input and State

– For single-input linear time-invariant system

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1u(t) \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2u(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_nu(t) \end{cases}$$

The constant coefficients $a_{11},\ldots,a_{nn}$; $b_1,\ldots,b_n$ are decided by system characters.

Matrix expression: $\dot{x}(t) = Ax(t) + bu(t)$

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \dot{x}(t) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

– For multi-input Linear Time-Invariant System

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1p}u_p \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2p}u_p \\ \vdots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{np}u_p \end{cases}$$

Matrix expression: $\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$

In which

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & a_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}, \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix}$$

- Output-equation: relationship between Output & State, and Output & Input.

Output is decided by system task.

– General form Output-equation for single-out linear time-invariant system (单输出线性定常系统)

$$y(t)=c_1x_1(t)+c_1x_2(t)+\cdots+c_nx_n(t)+du(t)$$

The constant coefficients $c_1,c_2,\ldots,c_n$ and *d* are relative with the system character.

The Matrix representation:

$$y(t)=\mathbf{c}x(t)+du(t)$$

– General Output function of MIMO system:

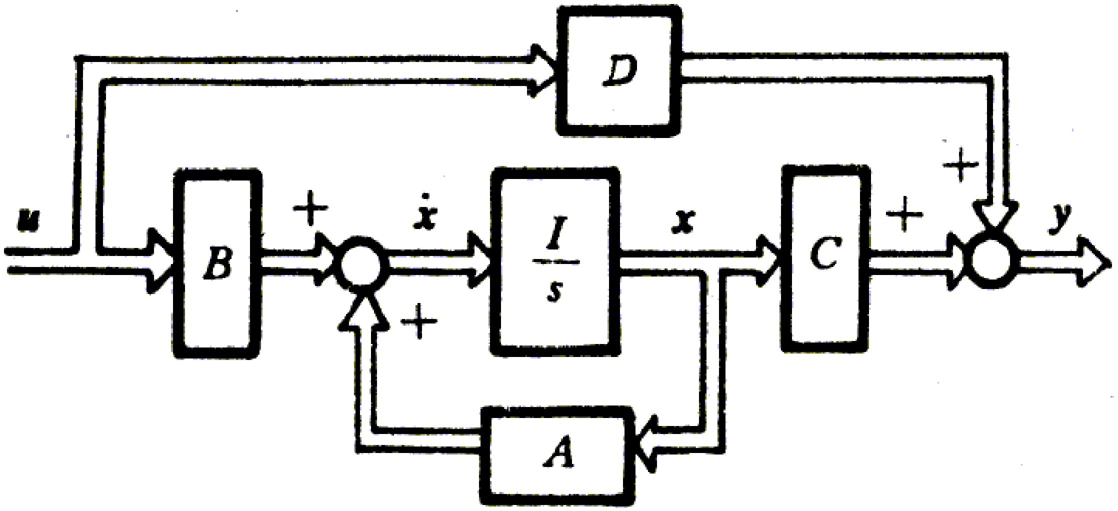
$$\begin{cases} y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots d_{1p}u_p \\ y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots d_{2p}u_p \\ \vdots \\ y_q = c_{q1}x_1 + c_{q2}x_2 + \cdots + c_{qn}x_n + d_{q1}u_1 + d_{q2}u_2 + \cdots d_{qp}u_p \end{cases}$$

The Matrix representation:

$$y = \mathbf{C}x + \mathbf{D}\boldsymbol{u}$$

State-space equations of Linear Time-Invariant (LTI) System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$



The Structure of the State-space

A – State Matrix (Systems matrix, coefficients matrix)

B – Input Matrix (Control Matrix)

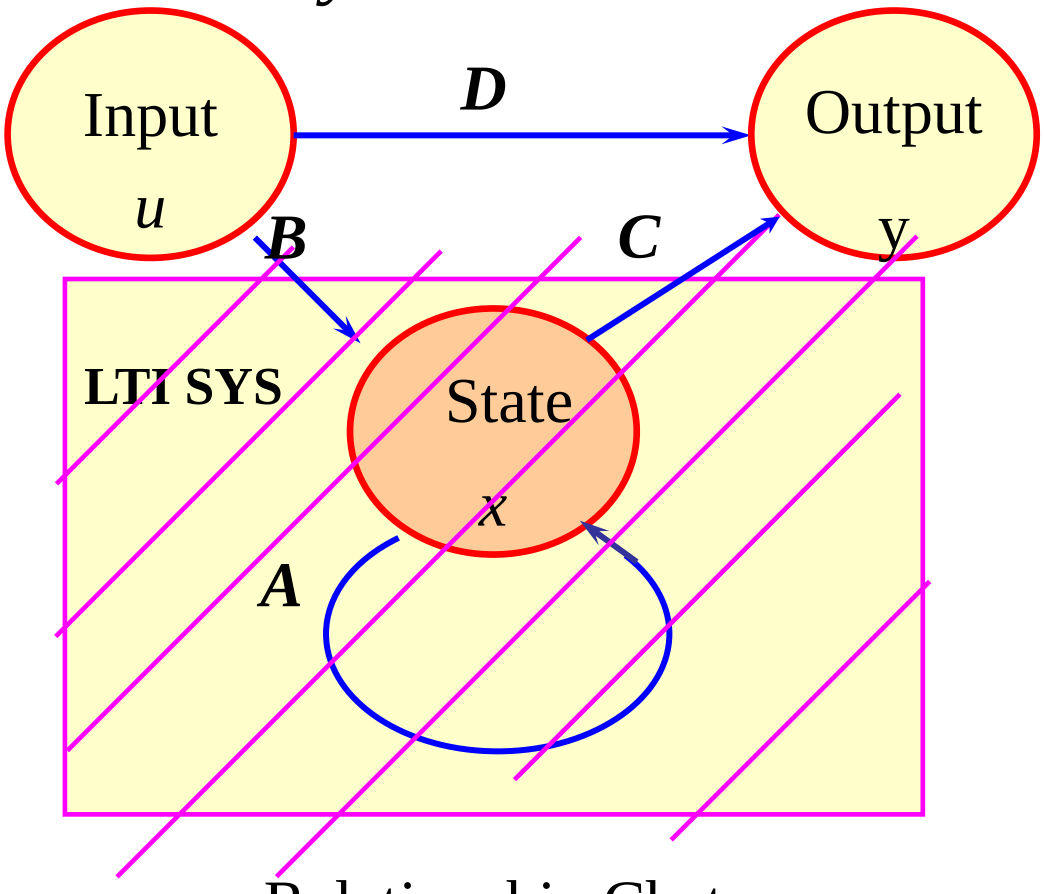
C – Observing Matrix (Output Matrix)

D – Feedforward Matrix (Directly Transfer Matrix)

The LTI system: {A, B, C, D}

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$



Relationship Chat

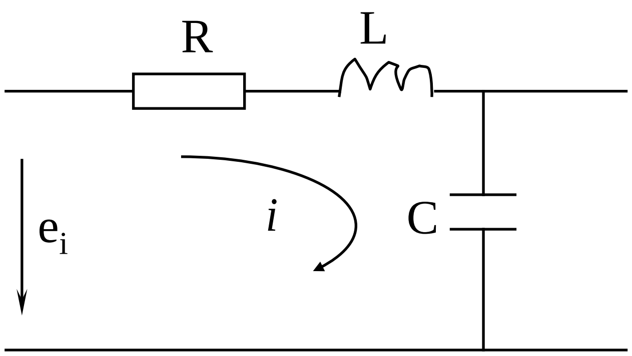
Advantage of State-space Analysis Method:

- **Computing:** Using computer to solve the first-order differential equations, easier than higher order equation;
- **Representation:** Using Vector Matrix to simplify the mathematic representation of the differential equations;
- **Field:** MIMO System, Time-Invariant System, Stochastic Process and Sample System, etc.
- **Special:** the use of the state space representation is not limited to systems with linear components and zero initial conditions.

9.2.2 Examples

Ex. 9-1 RLC electric circuit 's State-space model (figure).

Physics relationship:



$$L\frac{di}{dt}+Ri+\frac{1}{c}\int idt=e_i$$

If assume $e_i(t)$ is the input: u(t), $i(t)$ is the output: y(t), and select the proper state-variables $i(t)$ and $\int i(t)$dt:

$$x_1(t)=i(t)$$
$$x_2(t)=\int i(t)dt$$

$$\Longrightarrow$$

$$\frac{dx_1(t)}{dt}=-\frac{R}{L}x_1(t)-\frac{1}{LC}x_2(t)+\frac{1}{L}u(t)$$
$$\frac{dx_2(t)}{dt}=x_1(t)$$

$$\frac{dx_1(t)}{dt} = -\frac{R}{L}x_1(t) - \frac{1}{LC}x_2(t) + \frac{1}{L}u(t)$$

$$\frac{dx_2(t)}{dt} = x_1(t)$$

Rewrite to State equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{B} u(t)$$

$y(t)=i(t)=x_1(t)$, thus we have Output equation:

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t)$$

Not the end!!!!

Furthermore!!!!

$$L\frac{di}{dt}+Ri+\frac{1}{c}\int idt=e_i$$

If select some another state variables:

$$x_1=\frac{1}{C}\int idt+Ri,\ x_2=\frac{1}{C}\int idt$$

then $\quad x_1=x_2+Ri, \qquad L\frac{di}{dt}+x_1=e_i$

we have

$$\begin{cases}\dot{x}_1=\dot{x}_2+R\dfrac{di}{dt}=\dfrac{1}{RC}(x_1-x_2)+\dfrac{R}{L}(-x_1+e_i)\\ \dot{x}_2=\dfrac{1}{C}i=\dfrac{1}{RC}(x_1-x_2)\\ y=\dfrac{1}{R}x_1-\dfrac{1}{R}x_2\end{cases}$$

State-space representation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}\dfrac{1}{RL}-\dfrac{R}{L} & -\dfrac{1}{RC}\\ \dfrac{1}{RC} & -\dfrac{1}{RC}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\dfrac{R}{L}\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}\dfrac{1}{R} & -\dfrac{1}{R}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Comparison:

$$A=\begin{bmatrix}-\frac{R}{L} & -\frac{1}{LC}\\ 1 & 0\end{bmatrix} \qquad B=\begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix} \qquad C=\begin{bmatrix}1 & 0\end{bmatrix}$$

$$A=\begin{bmatrix}\frac{1}{RL}-\frac{R}{L} & -\frac{1}{RC}\\ \frac{1}{RC} & -\frac{1}{RC}\end{bmatrix} \qquad B=\begin{bmatrix}\frac{R}{L}\\ 0\end{bmatrix} \qquad C=\begin{bmatrix}\frac{1}{R} & -\frac{1}{R}\end{bmatrix}$$

Conclusion: State-space is non-unique.

- State-variable is not unique.
- State-equation is not unique.

Different state-equations
with
Different state-variable

1st selection: $x_1(t) = i(t)$

$$x_2(t) = \int i(t)dt$$

2nd selection: $\bar{x}_1 = \frac{1}{c}\int idt + Ri$

$$\bar{x}_2 = \frac{1}{c}\int idt$$

we have

$$x_1 = \frac{1}{R}\bar{x}_1 - \frac{1}{R}\bar{x}_2$$

$$x_2 = c\bar{x}_2$$

matrix representation:

$\boldsymbol{x} = \mathbf{P}\bar{\boldsymbol{x}}$ in which $\mathbf{P} = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ 0 & c \end{bmatrix}$

Infinite groups of state-variables x are available by any different nonsingular matrix (非奇异阵) **P**.

1st selection: $x_1(t) = i(t)$

$$x_2(t) = \int i(t)dt$$

2nd selection: $\bar{x}_1 = \frac{1}{c}\int idt + Ri$

$$\bar{x}_2 = \frac{1}{c}\int idt$$

we have

$$x_1 = \frac{1}{R}\bar{x}_1 - \frac{1}{R}\bar{x}_2$$

$$x_2 = c\bar{x}_2$$

matrix representation:

$$\boldsymbol{x} = \mathbf{P}\bar{\boldsymbol{x}} \quad \text{in which} \quad \mathbf{P} = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ 0 & c \end{bmatrix}$$

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