Chapter 8 Nonlinear Systems Theory

Nonlinear Models and Nonlinear Phenomena

Examples:

Pendulum Equation:

l denotes the length of the rigid rod with zero mass,

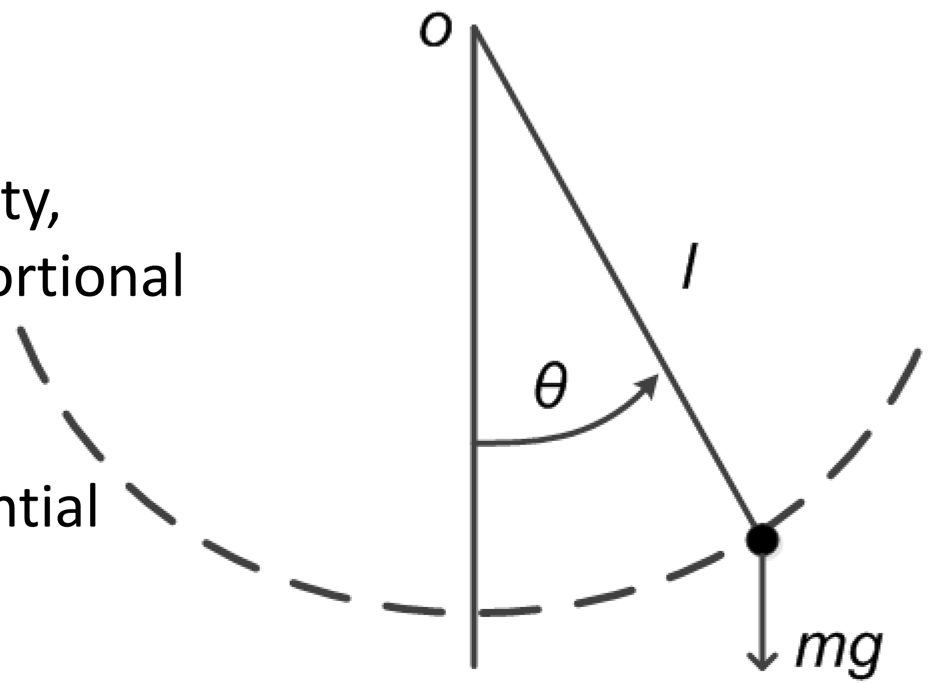
m denotes the mass of the bot,

g is the acceleration due to gravity,

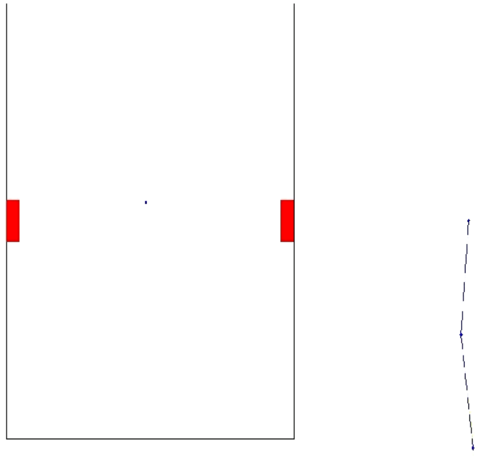
k is a coefficient of friction proportional to the speed of the bob.

From **Newton's second law**, the equation of motion in the tangential direction is:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$



Interesting Examples of Nonlinearity

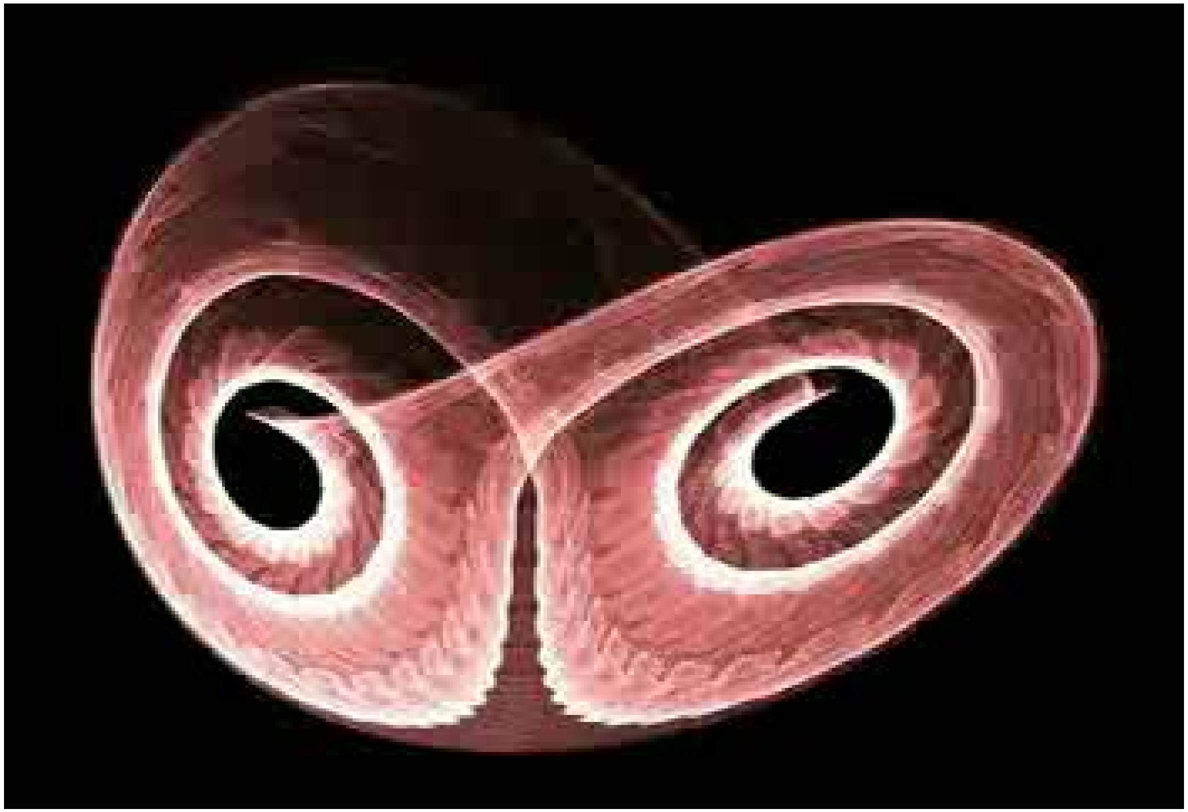


Other pendulums: Acrobot Robots

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = F$$
$$q = (x, \theta_1, \theta_2)$$

Biomimetic robots

Interesting Examples of Nonlinearity



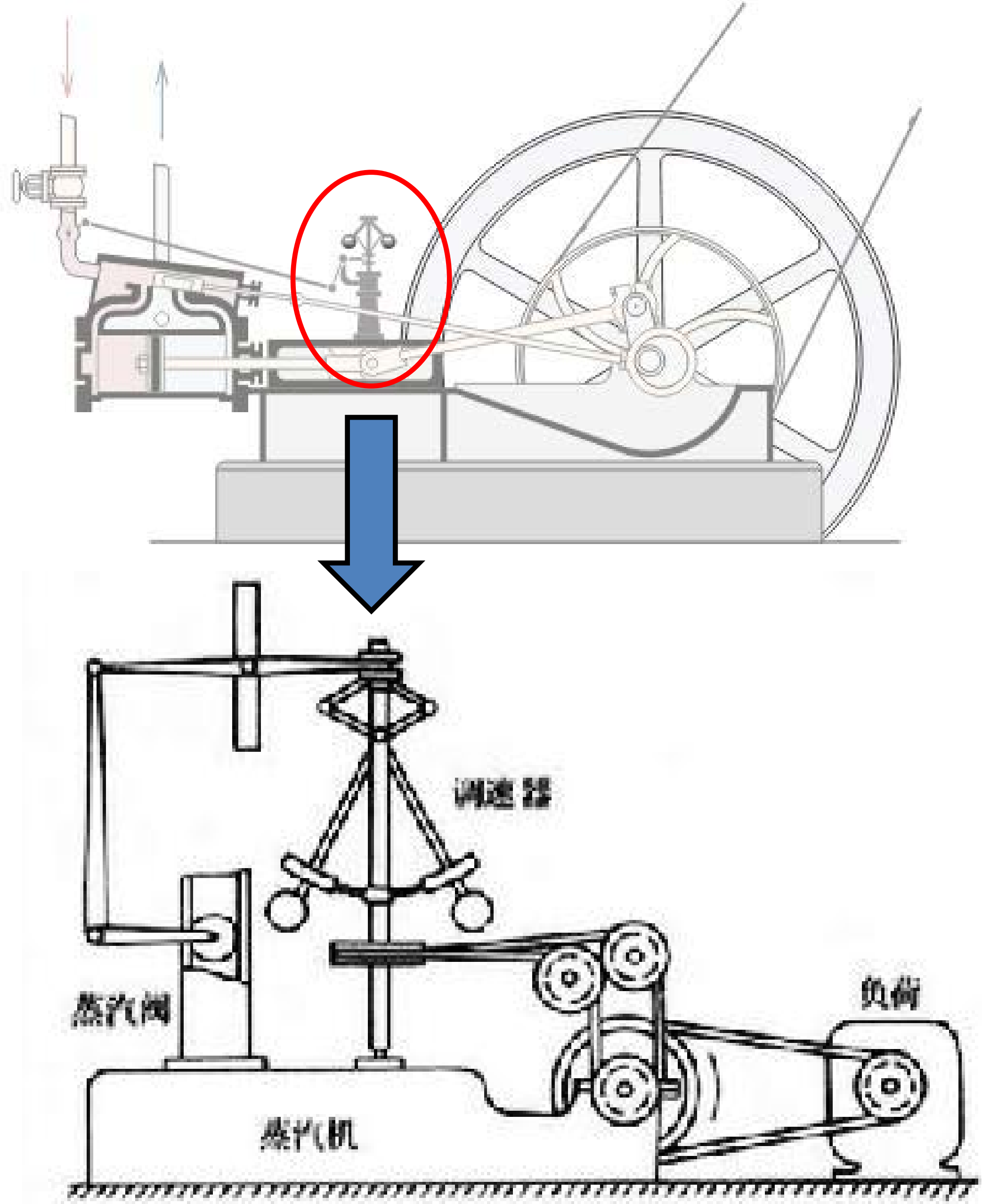
$$\frac{dx}{dt} = -c(x-y)$$

$$\frac{dy}{dt} = ax - y - xz$$

$$\frac{dz}{dt} = b(xy-z)$$

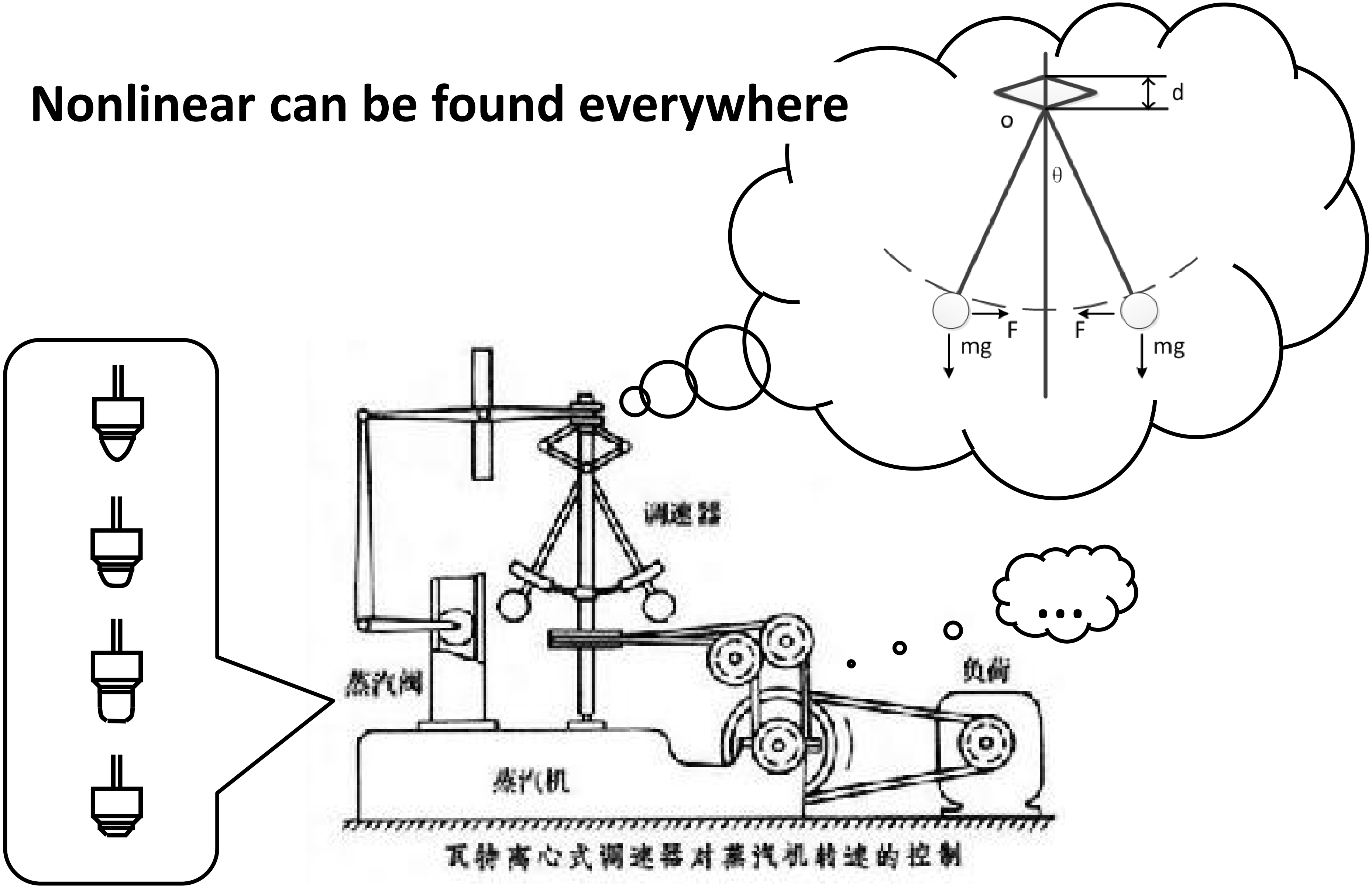
Lorenz Chaotic Attractor

Mr. J. Watt



瓦特离心式调速器对蒸汽机转速的控制

Nonlinear can be found everywhere



§ 8.1 Overview of Nonlinear Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y + \varepsilon \cdot f(y, y', \cdots y^{(n)}) = x$$

If ε→0, it is a linear System;

If ε cannot be ignored, it is a Nonlinear System.

Such as： $\ddot{y} + \dot{y}y + y = \sin\omega t$

$$(\ddot{y})^2 + 3\dot{y} + y = e^t$$

$$\ddot{y} + 3\dot{y} + y^2 = x$$

Thinking: How to distinguish nonlinearity？

1. Significance of Studying Non-linear Systems

1) There are **no** systems without nonlinearity.

- ***Dead-zone*** characteristics of *measurement element*;
- ***Saturation*** characteristics of *amplification element*;
- ***Dead-zone*** and ***saturation*** characteristics of *actuator*;
- ***Gap*** Characteristics of actuating unit and so on…

2) The ***inherent nonlinearities*** make the linear system theory cannot be applied in analyzing the actual systems. The influences of nonlinear factors can not be explained by linear system theory.

3) The nonlinear characteristics do not always have negative impacts on systems. ***Optimal control laws are often nonlinear laws***. The relay(继电器) and waveform generator(波形发生器) are also widely used.

2. Features of Nonlinear Systems

Comparing with linear control systems，non-linear systems have many *new features*:

1. **A linear systems satisfies the *principle of superposition*(叠加原理), while a non-linear system does not, always.**

(1) Additivity(叠加性):

$$y = f(x) \Longrightarrow f(x_1 + x_2) = f(x_1) + f(x_2)$$

$f(x) = ax$ Obviously, a linear function satisfies the principle of Superposition.

Consider： $f(x) = ax + b$

(2) Multiplicativity (均匀性,可乘性):

$$f(ax) = af(x)$$

- Nonlinear systems may be additive (rarely), but it is completely not multiplicative.

Fig. 8—1 Nonlinear system with filters

$$X_1 \rightarrow Y_1, \quad X_2 \rightarrow Y_2$$

Additivity:

$$X_1 + X_2 \rightarrow Y_1 + Y_2$$

Multiplicativity :

$$nX_1 \nrightarrow nY_1$$

2. **<u>The stabilities of non-linear systems depend on not only the inherent structure and parameters of control systems, but also the *initial conditions* and the *inputs*.</u>**

Example: An nonlinear systems described by the non-linear differential equation: $\dot{x} = -x(1-x)$

which has two equilibrium points, obviously, $x_1=0$ and $x_2=1$.

The equation equals to $$\frac{dx}{x(1-x)} = -dt$$

Integrating both sides: $$\ln\frac{cx}{1-x} = -t \Rightarrow \frac{cx}{1-x} = e^{-t}$$

- Assume the initial state of the system be x_0,
- if t = 0, then: $$c = \frac{1-x_0}{x_0}$$

$$\therefore \quad x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

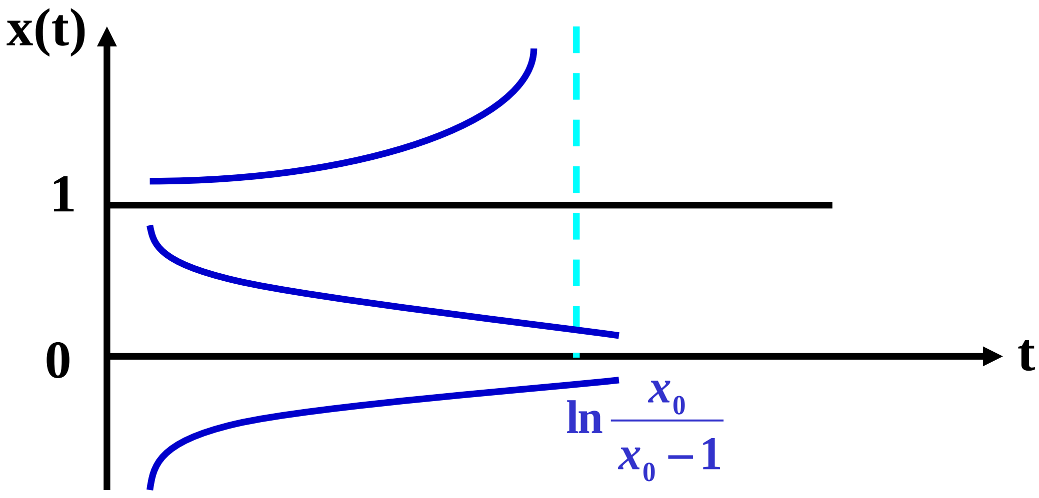


Fig. 8-2 First-order non-linear systems

If $x_0 < 1,\ t \to \infty$, then $x \to 0$

Initial conditions will affect stability of the system!

If $x_0 > 1$, when $t = \ln \dfrac{x_0}{x_0 - 1}$, we have $x \to \infty$

- The initial conditions can be even used as a key to the ***encryption*** of transmission signals in Chaotic systems

Useful signals

Chaotic signals
with certain
initial conditions

$\xrightarrow{\text{Transmission}}$ ***Decryption*** by the known initial conditions

3. *Periodic oscillation* **does not exist in an actual physical linear systems , while it may occur in a nonlinear system.**

4. **A stable linear system under a *periodic input* → output with the same frequency;**
   A nonlinear system under a periodic input → many complex cases of the outputs

   Distortion

(1) *Jump resonance* and *Multi-valued response*

Input signals with constant amplitude, then the *amplitude frequency characteristics* of the output is：

$$\omega\uparrow:\ 1\rightarrow 2\rightarrow 3 \vdots \overset{\updownarrow}{\rightarrow} 4 \vdots \rightarrow 5$$

$$\omega\downarrow:\ 5\rightarrow 4\rightarrow 4' \vdots \overset{\updownarrow}{\rightarrow} 2' \vdots \rightarrow 1$$

Hysteresis Loop Characteristics

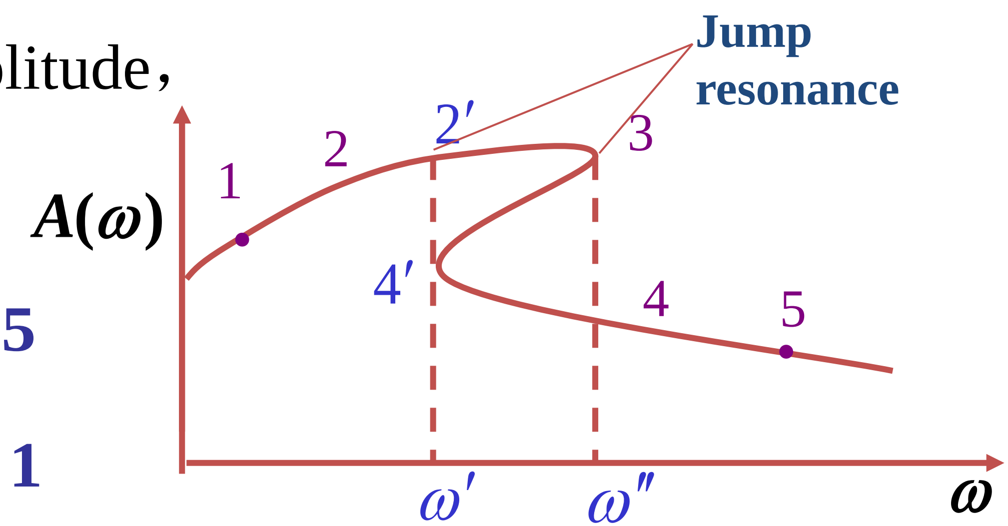


Fig. 8-3 Amplitude frequency characteristic of output of non-linear spring

(2) ***Harmonic Oscillation:*** *double frequency and dividing frequency Oscillation*

Steady-state outputs of non-linear systems can be divided into *double frequency* ***oscillation*** **and** *dividing frequency* ***oscillation*****. When the input signal is sinusoidal, showed in Fig. 8-4**

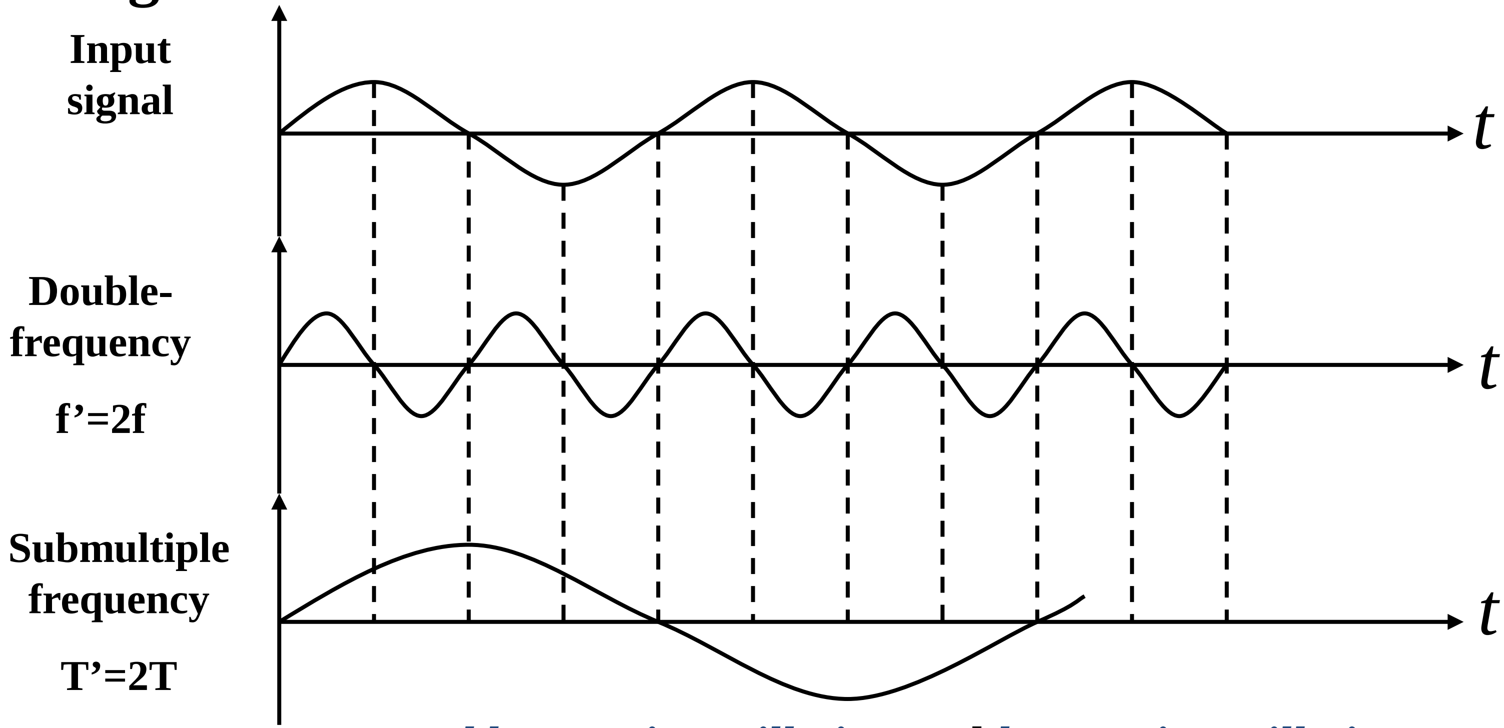


Fig. 8-4 ***subharmonic oscillation*** **and** ***harmonic oscillation***

3. Methods of Studying Nonlinear Systems

1） Phase-Plane Analysis is the graphical method used to analyze first-order and second-order Nonlinear systems. It analyzes the features of Nonlinear systems through drawing *phase portrait* to find all the solutions of the differential equations in any initial condition. It is the generalization and application of time-domain analyzing method in non-linear systems. *It can only be used in the first- and second-order nonlinear systems.*

2） Describing Function Approach is a kind of method for analyzing nonlinear systems inspired by frequency method of linear systems. It is the generalization of frequency method in nonlinear systems, and *is not restricted by the system order*.

3） Numerical Solution is a kind of numerical methods to solve the nonlinear differential equation using high-speed computers. It is almost the only effective method for analyzing and designing *complex nonlinear systems*.

Note:

- **It should be pointed out that: the above methods aim at solving the "*analysis*" problems of nonlinear systems based on analyzing the system stability.**

- The achievement of "*synthesis*" methods in nonlinear systems is much less than stability problem. There are **NO** general approaches can be used to design arbitrary nonlinear systems so far.

Review of last class

- What is the nonlinear system?
- What is the special feature of the nonlinear system:
- Typical nonlinear system?

§ 8.2 Typical Nonlinear characteristics and Their Mathematical Description

8.2.1 Saturation characteristics

8.2.2 Dead-zone characteristics

8.2.3 Gap characteristics

8.2.4 Relay characteristics

1. Saturation

A common nonlinearity in electronic amplifiers

Mathematical description of saturation features :

$$x(t) = \begin{cases} ke(t), & |e(t)| < e_0 \\ ke_0 sign[e(t)], & |e(t)| \geq e_0 \end{cases}$$

$sign[e(t)]$ **is the *sign* function**

$$sign[e(t)] = \begin{cases} 1, & e(t) \geq 0 \\ -1, & e(t) < 0 \end{cases}$$

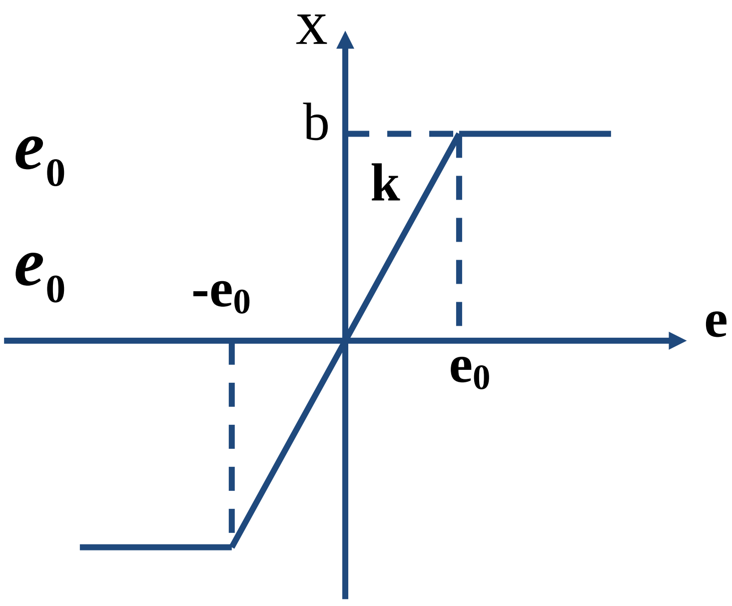


Fig. 8-5 Saturation characteristics

2. Dead-zone

Dead-zone can be also called neutral zone , its mathematical description is:

$$x(t)=\begin{cases}0, & |e(t)|\le e_0\\ k\left[e(t)-e_0 sign\left[e(t)\right]\right], & |e(t)|>e_0\end{cases}$$

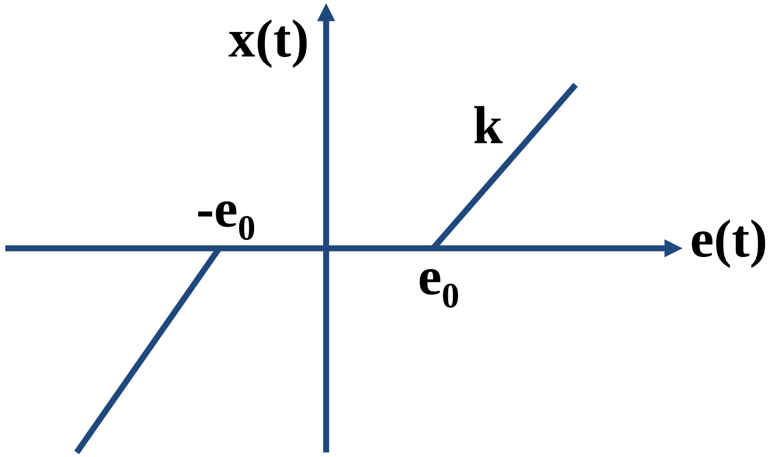


Fig. 8-6 Dead-zone characteristics

3. Gap

Mechanical transmission devices are based on gears, there must exist some gaps for sliding and reversing transmission, that means the gears have to pass a few distances when reversing transmission is needed.

Its mathematical description is :

$$x(t)=\begin{cases} k\left[e(t)-e_0\right], & \dot{x}(t)>0 \\ k\left[e(t)+e_0\right], & \dot{x}(t)<0 \\ b\,sign\left[e(t)\right], & \dot{x}(t)=0 \end{cases}$$

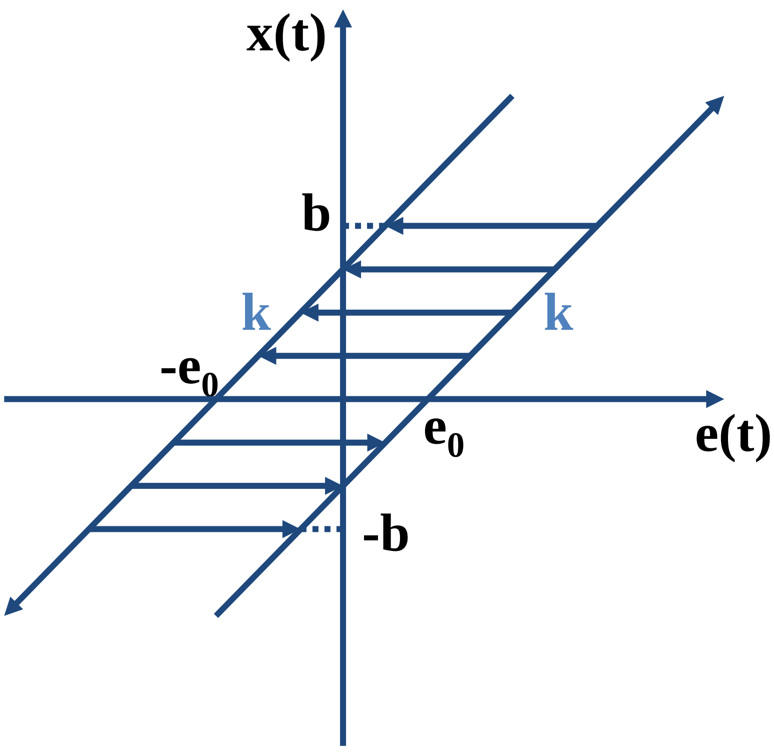
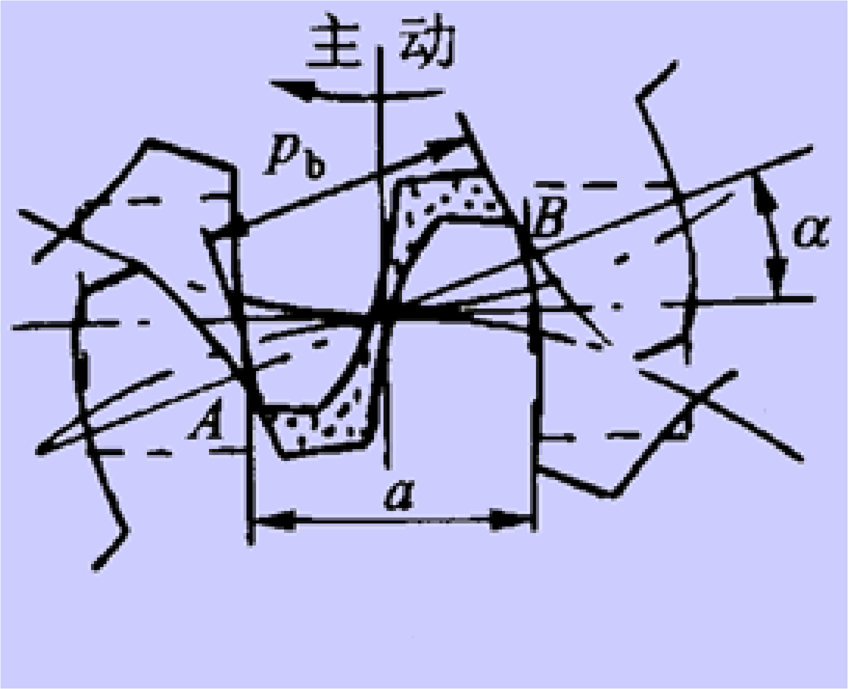
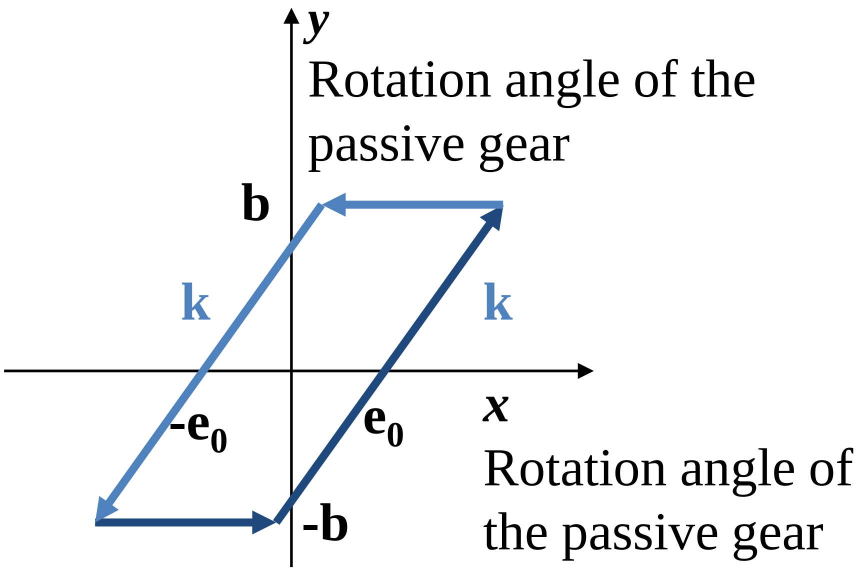


Fig. 8-7 Gap characteristics





Input x is the rotation angle of the driven gear
Output y is the rotation angle of the passive gear

4. Relay(继电特性)

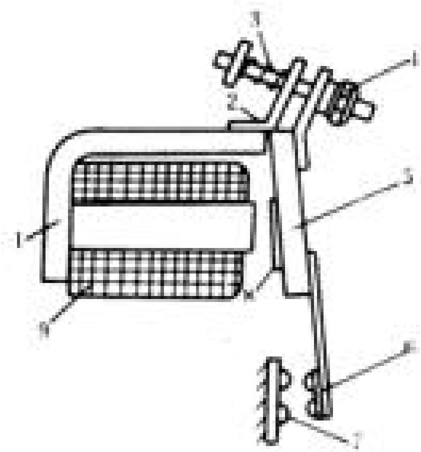


图1 电磁式继电器原理图

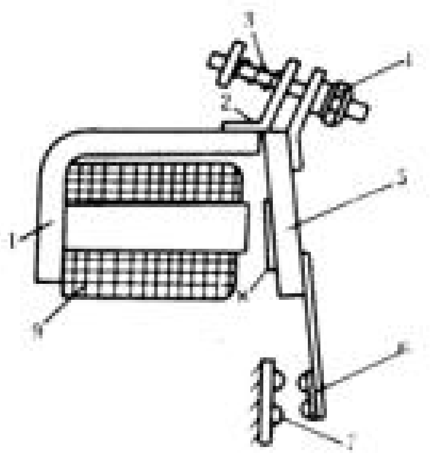
1- 铁心 2-旋转棱角 3-释放弹簧 4-调节螺母 5--衔铁

6-动触点 7-静触点 8-非磁性垫片 9-线圈

***Principle of relay*:**

Input voltage → Current in coil → generates the electromagnetic force → Close the relay contact

If the input voltage is e_0, the electromagnetic force generated by current in coil is enough to make the switch to be closed, then e_0 is called ***Operation Voltage*.**



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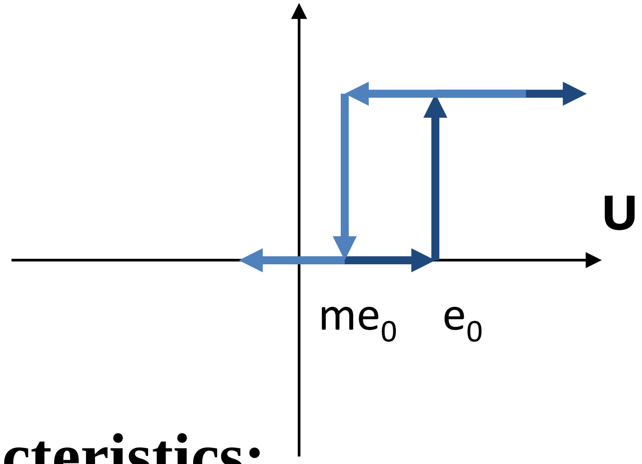
The relay contact will not release when the input voltage is reduced to e_0 because of the influence of Hysteresis.

When it is further reduced to me_0 (m<1), the relay contact will be released.

Then me_0 is called Release Voltage.



There are four forms of relay characteristics:

1. Ideal relay characteristics

$$x(t)=\begin{cases} M, & e>0 \\ -M, & e<0 \end{cases}$$

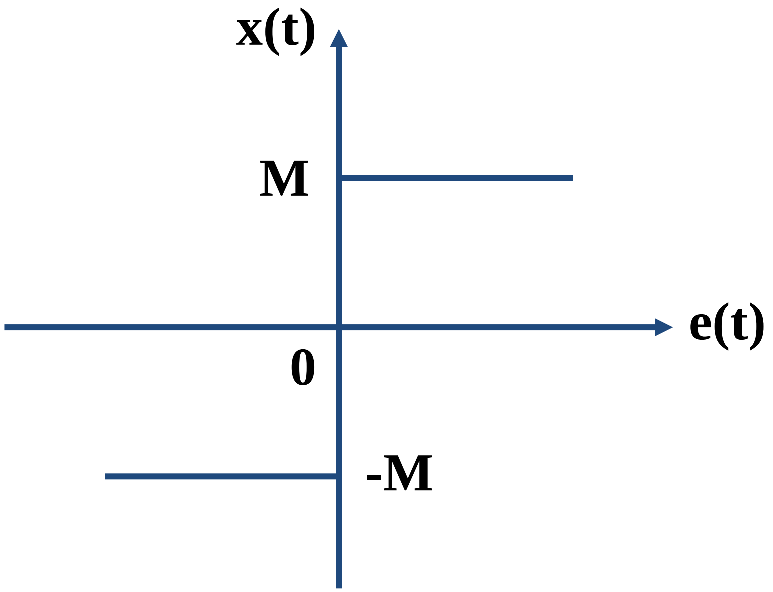


Fig. 8-8(a) Perfect relay characteristics

2. Relay characteristics with Dead-zone

$$x(t)=\begin{cases} M, & e(t)>e_0 \\ 0, & -e_0\le e(t)\le e_0 \\ -M, & e(t)<-e_0 \end{cases}$$

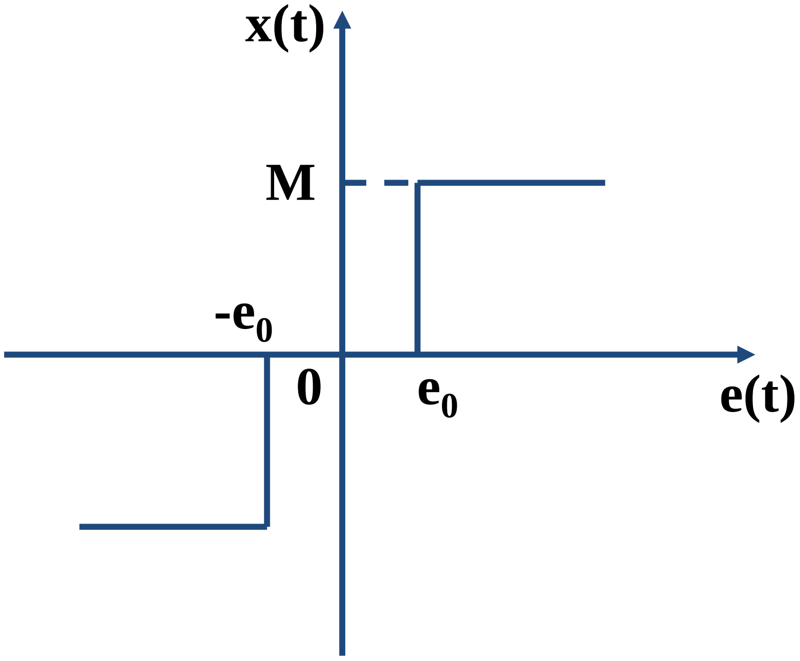


Fig. 8-8(b) Relay characteristics with Dead-zone

3. Relay characteristics with Hysteresis loop

$$x(t)=\begin{cases} M, & \dot{e}>0, e>e_0 \\ & \dot{e}<0, e>-e_0 \\ -M, & \dot{e}>0, e<e_0 \\ & \dot{e}<0, e<-e_0 \end{cases}$$

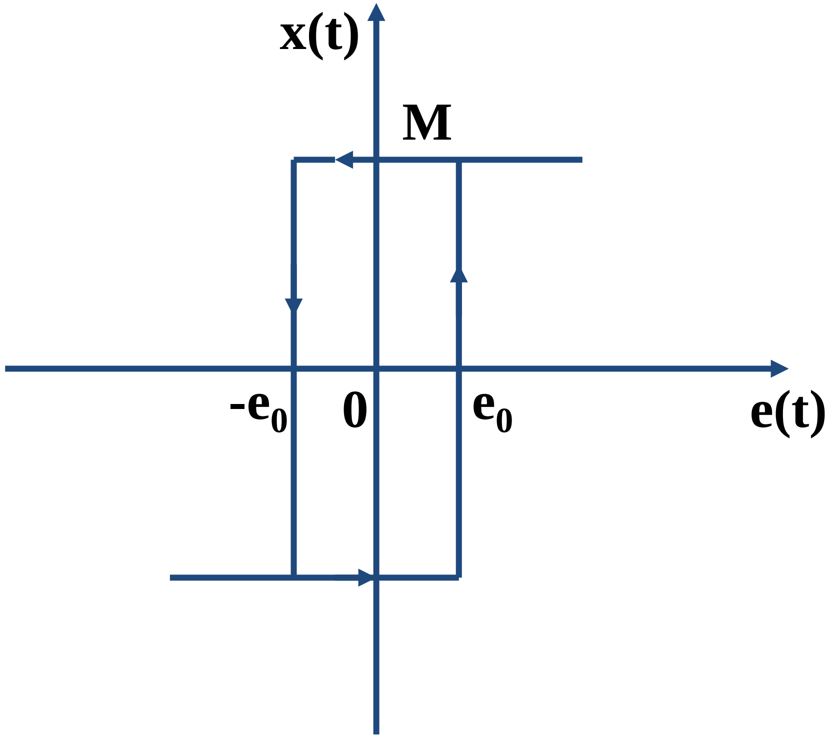


Fig. 8-8(c) Relay characteristics with Hysteresis loop

4. Relay characteristics with Dead-zone and Hysteresis loop

$$x(t) = \begin{cases} M, & \dot{e} > 0, e \geq e_0 \\ & \dot{e} < 0, e > me_0 \\ 0, & \dot{e} > 0, -me_0 < e < e_0 \\ & \dot{e} < 0, -e_0 < e < me_0 \\ -M, & \dot{e} > 0, e < -me_0 \\ & \dot{e} < 0, e \leq -e_0 \end{cases}$$

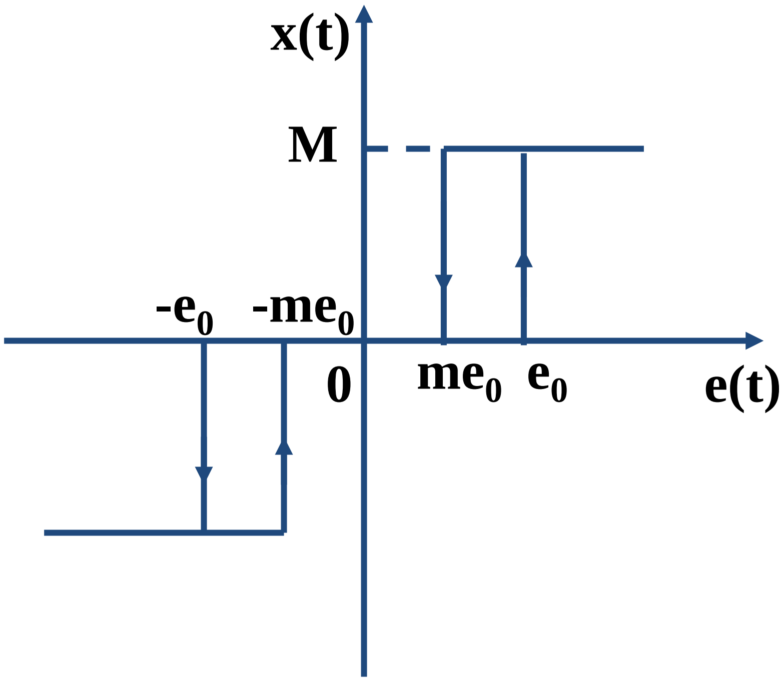


Fig. 8-8(d) Relay characteristics with Dead-zone and Hysteresis loop