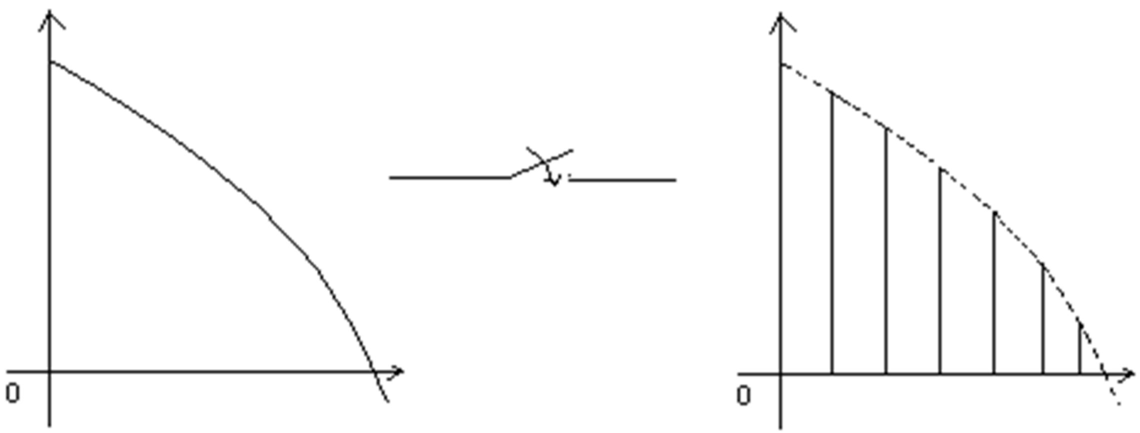
7.1 Discrete-Time Control Systems

Discrete-systems: There is one or more impulse series or digital signals in the system.

Sampled-Data System: a system that is continuous except for one or more sampling operations.



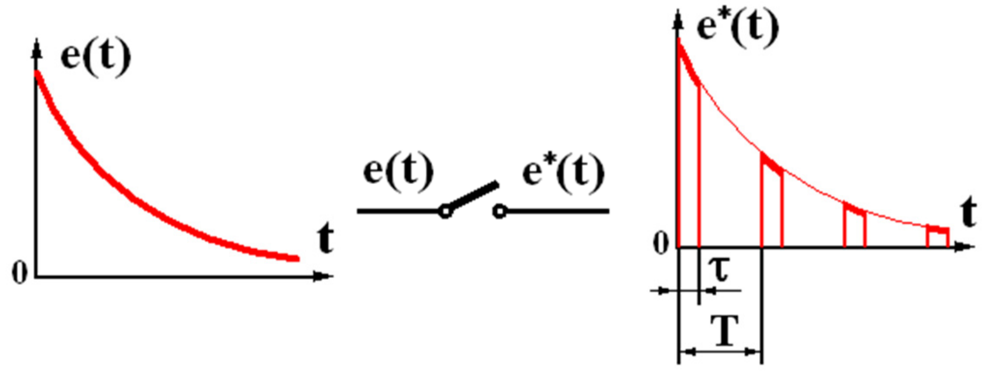
(a) (b) (c)

A/D : analog to digital converter

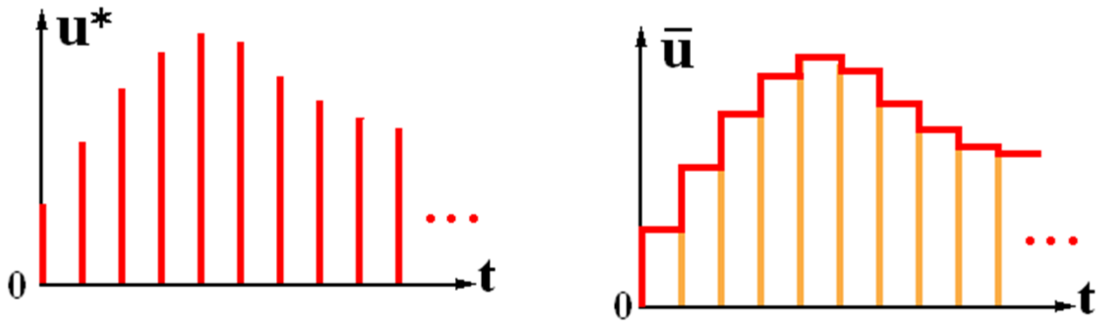
D/A : digital to analog converter

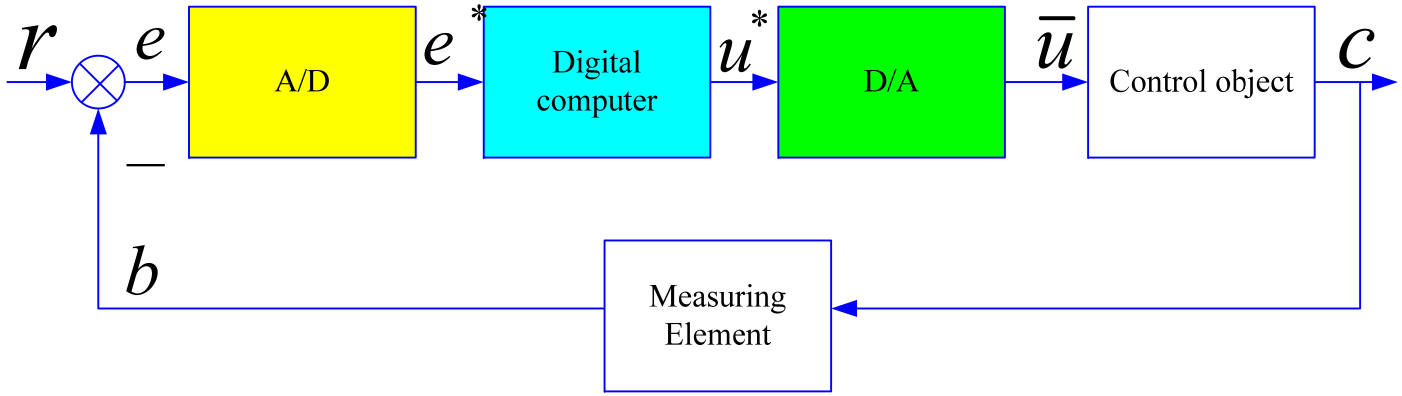
A/D process

- **Sampling** — Time sampled
- **Quantization** — Value quantized

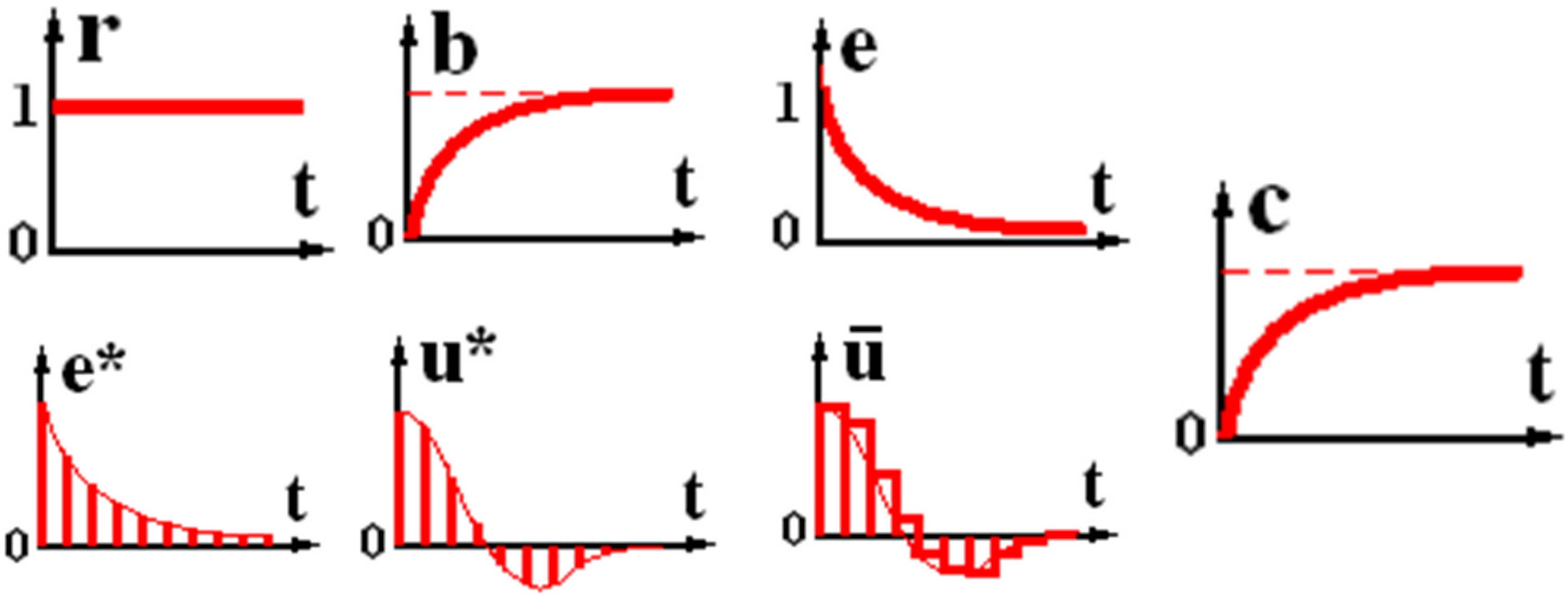


D/A process





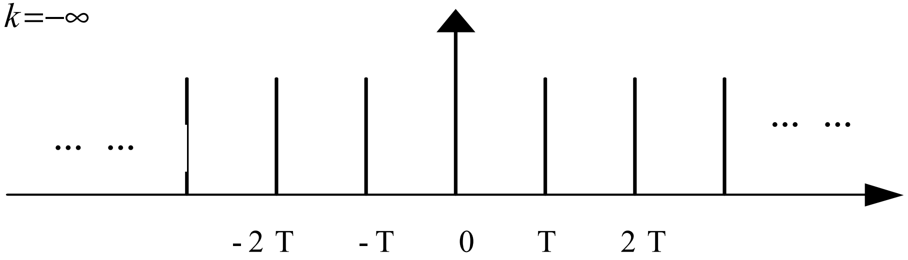
Computer Controlled Systems



7.2 Signal Sampling and Shannon Theorem

3、Unit Impulse Sequence (Unit Impulse Train)

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = \delta(t) + \delta(t-T) + \cdots + \delta(t-kT) + \cdots$$



Unit Impulse sequence

4、Sampling Signal

$$e^*(t) = \sum_{k=-\infty}^{\infty} e(t)\delta(t-kT)$$

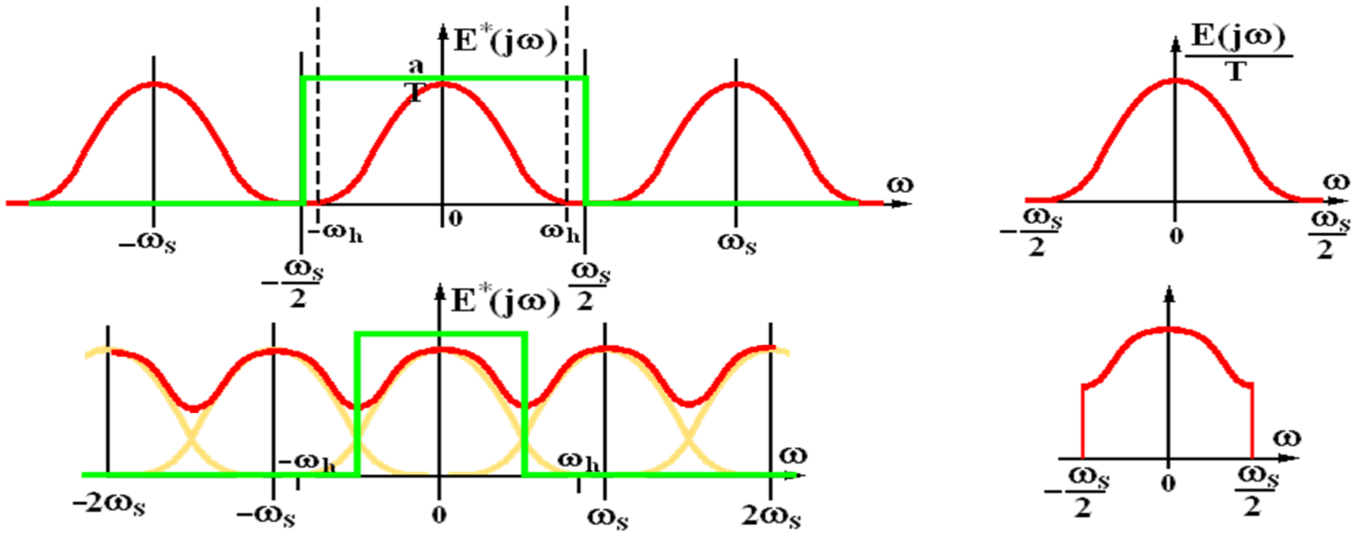
$$E^*(s) = L\left[e^*(t)\right] = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

$$E^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} E\left[j\left(\omega + k\omega_s\right)\right]$$

Shannon Sampling Theorem—

The Necessary Condition for signal recovery

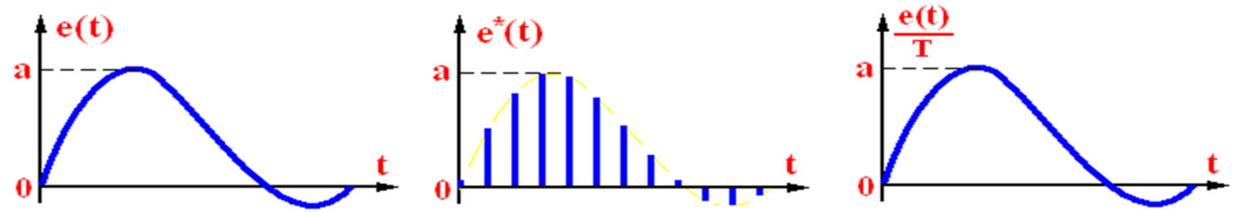
$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$



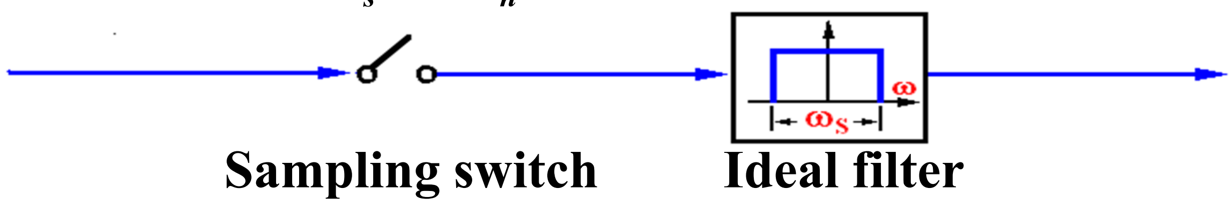
$$\omega_s = \frac{2\pi}{T} < 2\omega_h$$

$$\omega_s = \frac{2\pi}{T} > 2\omega_h$$

$$T < \frac{\pi}{\omega_h}$$



$$\omega_s > 2\omega_h$$



Example Consider $e(t)=e^{-t}$， determine the sampling frequency ω_s according to Shannon Sampling Theorem.

Solution： L-Transform of e(t) is: $E(s)=\dfrac{1}{s+1}$

Frequency characteristic $E(j\omega)=\dfrac{1}{j\omega+1}$

thus

$$\left|E(j\omega)\right|=\frac{1}{\sqrt{\omega^2+1}}$$

Take $\left|E(j\omega)\right|=0.05\left|E(0)\right|$,

$$\frac{1}{\sqrt{\omega_h^2+1}}=0.05,\quad \omega_{\mathrm{h}}=20rad/s=\omega_{\max}$$

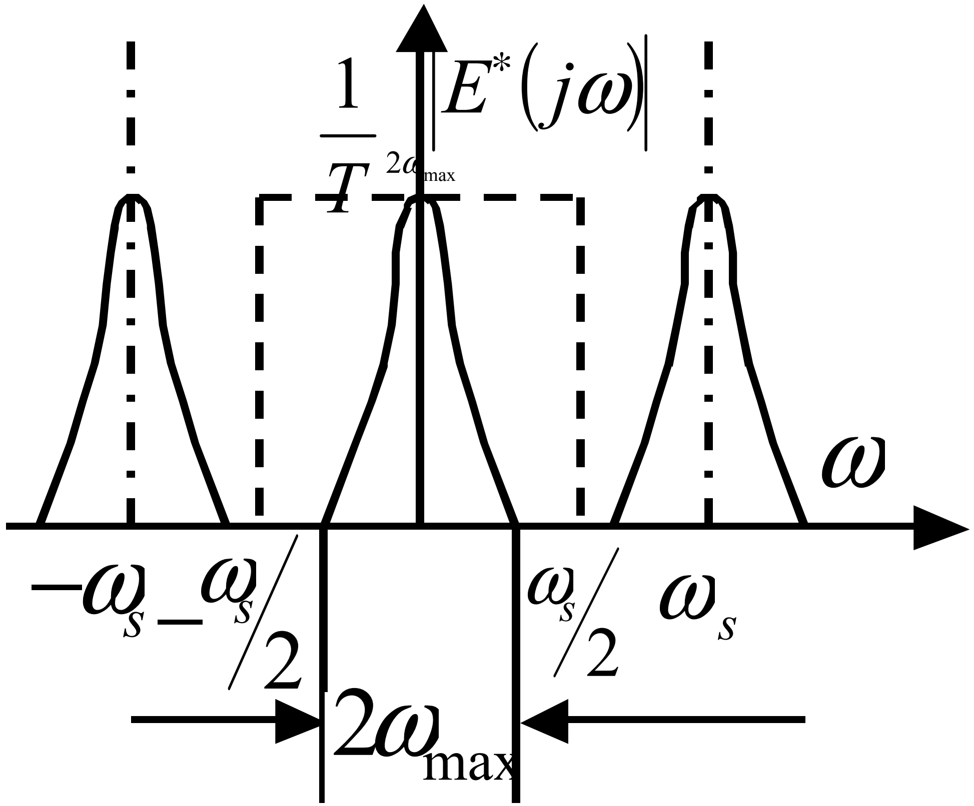
$$\omega_s\geq 2\omega_{\max}=40rad/s$$

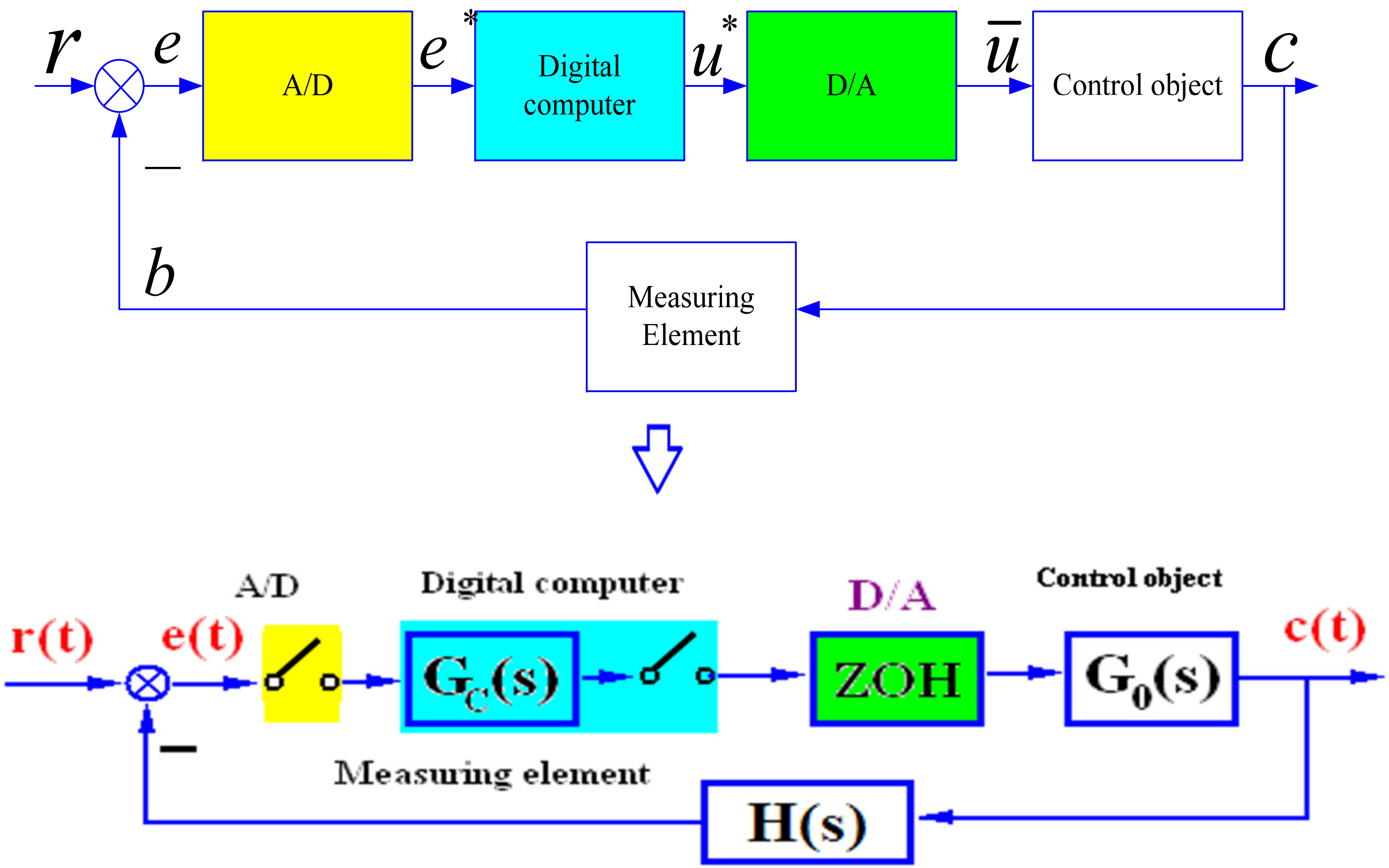
7.3 Signal Recovery and Zero-Order Hold

7.3.1 Signal recovery

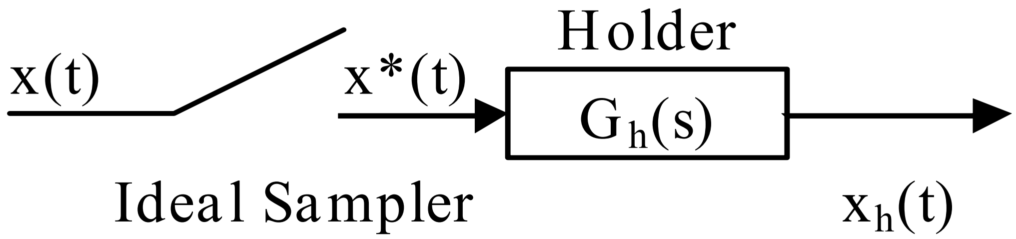
The ideal filter is illustrated as the dotted line in the figure.

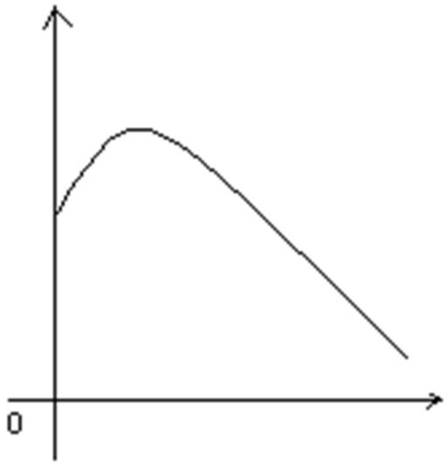
0



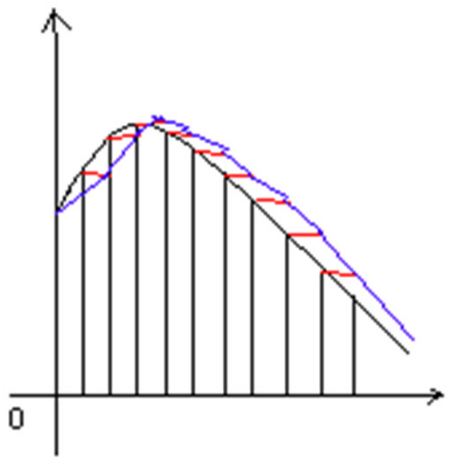


Computer Control System





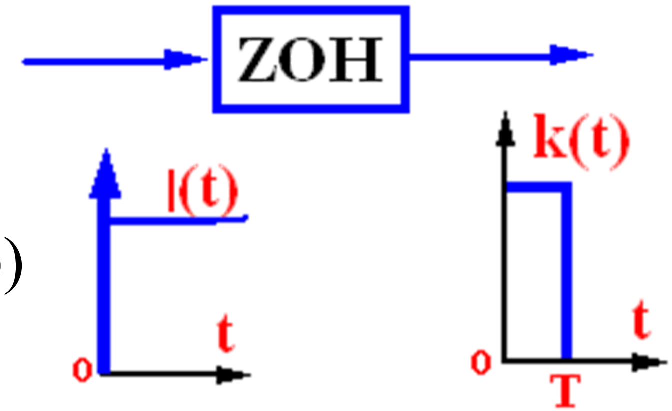
Continuous Signal



Recovered Signal after ZOH

7.3.2 Zero-Order Hold

$$\boldsymbol{k(t) = 1(t) - 1(t - T)}$$



$$x_h(t) = \sum_{k=0}^{\infty} x(kT)(1(t-kT) - 1(t-kT-T))$$

Using L-Transform, we get

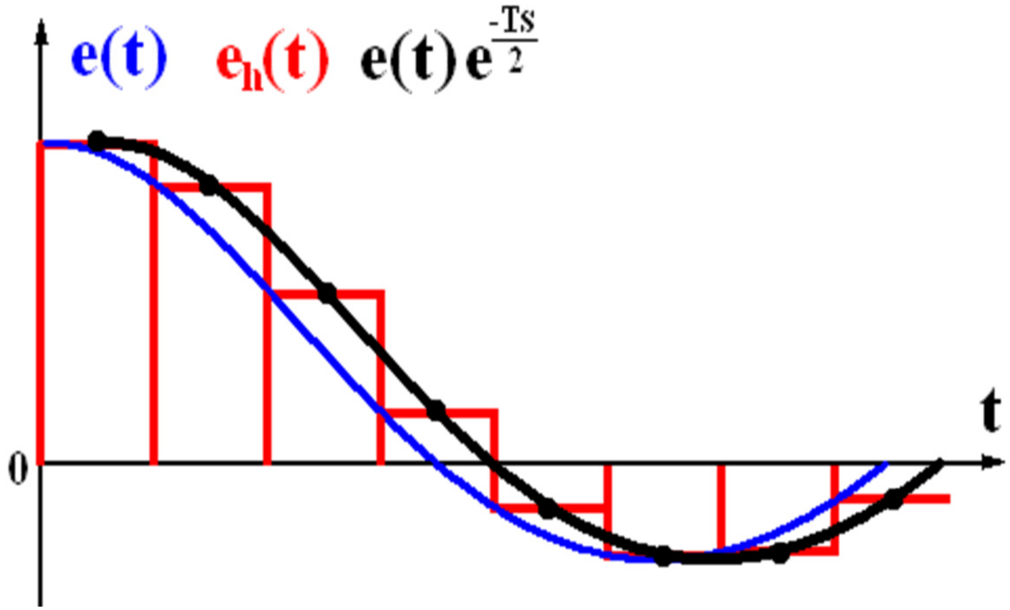
$$x_h(s) = \underbrace{\sum_{k=0}^{\infty} x(kT)e^{-kTs}}_{x^*(s)} \left[\frac{1}{s} - \frac{1}{s}e^{-Ts}\right]$$

$$\frac{x_h(s)}{x^*(s)} = \frac{1-e^{-Ts}}{s} = G_h(s)$$

Effect of zero-order holder on the system

$$G_h(s) = \frac{1-e^{-Ts}}{s}$$

$$\approx T \bullet e^{-Ts/2}$$



Frequency characteristics:

$$G_h(j\omega)=\frac{1-e^{-j\omega T}}{j\omega}=\frac{e^{-j\frac{\omega T}{2}}(e^{j\frac{\omega T}{2}}-e^{-j\frac{\omega T}{2}})}{j\omega}=\frac{2e^{-j\frac{\omega T}{2}}\sin(\frac{\omega T}{2})}{\omega}$$

$$G_h(j\omega)=T\frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}}e^{-j\frac{\omega T}{2}}$$

$$\because T=\frac{2\pi}{\omega_s}$$

setting $S_a(x)=\sin x/x$

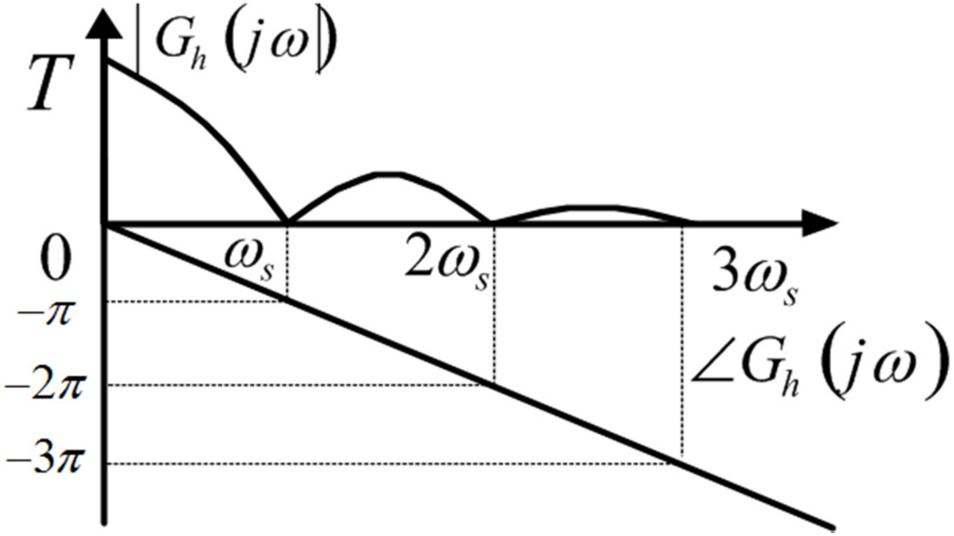
so the

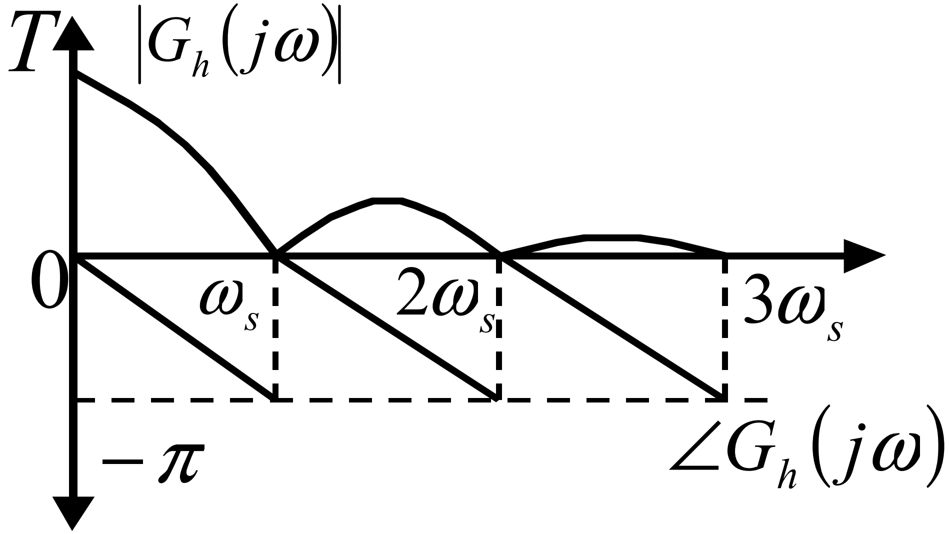
$$G_h(j\omega)=\frac{2\pi}{\omega_s}\cdot S_a(\pi\omega/\omega_s)\cdot e^{-j\frac{\pi\omega}{\omega_s}}$$

the amplitude $|G_h(j\omega)| = \dfrac{2\pi}{\omega_s} \cdot |S_a(\pi\omega/\omega_s)|$

phase angle $\angle G_h(j\omega) = \angle S_a(\pi\omega/\omega_s) - \dfrac{\pi\omega}{\omega_s}$

where $\angle S_a(\pi\omega/\omega_s) = \begin{cases} 0, & 2n\omega_s < \omega < (2n+1)\omega_s \\ \pi, & (2n+1)\omega_s < \omega < (2n+2)\omega_s \end{cases}$

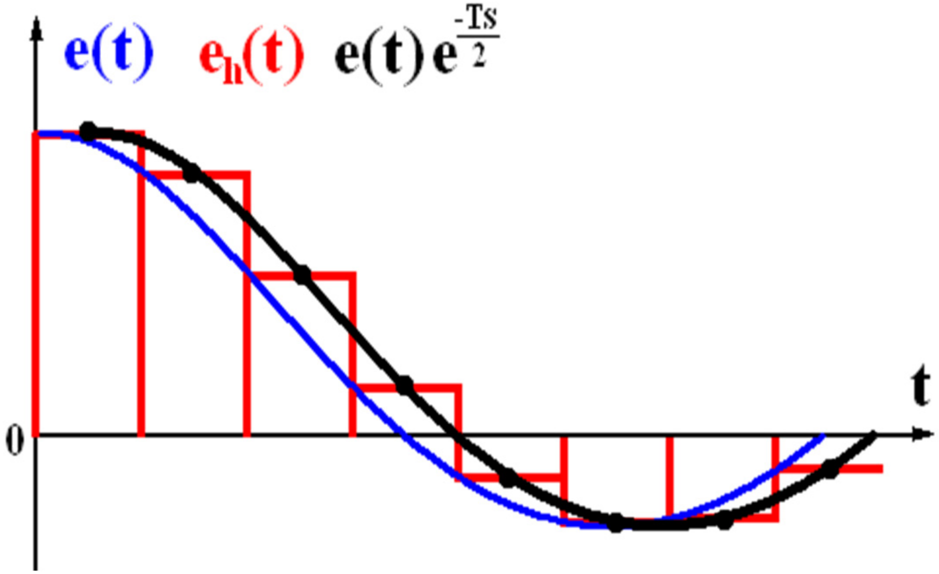


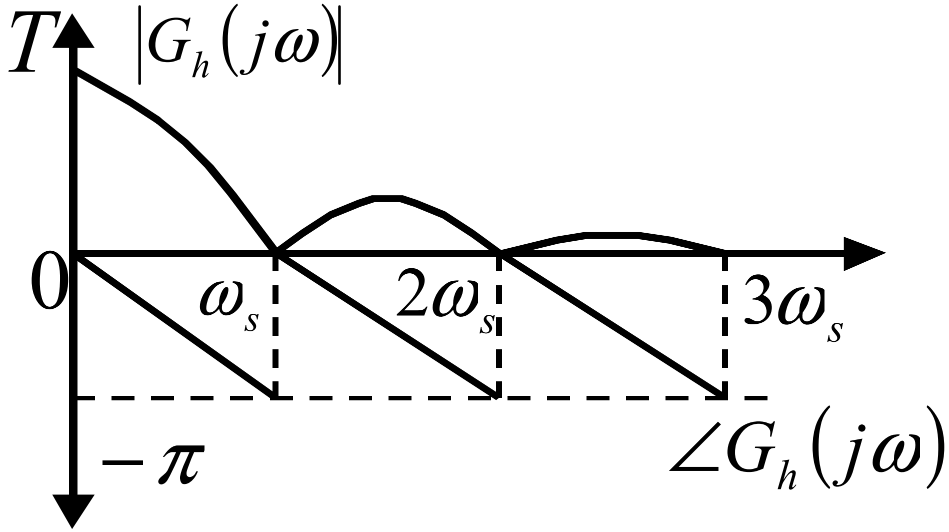


Which one do you think is correct?

Properties

- Low pass filter, but not the ideal filter. Existing Ripples – 纹波；
- Phase delay, reduce the stability.
- Time delay





Review

- **Unit Impulse Sequence**
- **Sampling Signal**
- **Laplace Transformation**
- **Shannon Sampling Theorem**
- **Zero-Order Hold**

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.4 Z-Transform and Inverse Z Transform

7.4.1. Z-transform

Definition:

$$\because E^*(s) = \sum_{k=0}^{+\infty} e(kT) \cdot e^{-kTs}$$

Set

$$z = e^{Ts} \qquad s = \frac{1}{T} \ln z$$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

$$E(z) = E^*(s) \,\big|_{z=e^{Ts}}$$

$$E^*(s) = \sum_{k=0}^{\infty} e(kT) \cdot e^{-kTs}$$

$$z^{-1} = e^{-Ts}$$

$$E(z) = Z[e^*(t)] = E^*(s)\Big|_{z=e^{Ts}} = \sum_{n=0}^{\infty} e(kT) \cdot z^{-k}$$

Rmk:

- $E(z) = Z[e^*(t)] = Z[E(s)] = Z[E^*(s)] = Z[e(t)]$
- The z-transform is only for discrete signal.
- E(z) is only mapping to a unique e*(t), but not a unique e(t).

7.4.2 Methods of z-Transform

- By the definition – summation of series
- Partial fraction expansion (部分分式展开).

1. By the Definition

Example 1 $x_1(t) = 1(t)$ and $x_2(t) = \sum_{k=0}^{\infty} \delta(t - kT)$, obtain $X_1(z)$ and $X_2(z)$。

Solution:

$$X_1(z) = \sum_{k=0}^{\infty} x_1(kT) z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

$$X_2(z) = \sum_{k=0}^{\infty} x_2(kT) z^{-k} = 1 + z^{-1} + z^{-2} + \cdots = \frac{z}{z - 1}$$

Tips: Though $x_1(t)$ and $x_2(t)$ are not same, they may have the same Z-transform.

$$E(\mathrm{z}) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

Example 2 $$e(t) = \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$E(z) = \sum_{k=0}^{\infty} \frac{1}{2j} \left[e^{j\omega kT} - e^{-j\omega kT} \right] \cdot z^{-k} \quad = \frac{1}{2j} \sum_{k=0}^{\infty} \left[(e^{j\omega T} z^{-1})^k - (e^{-j\omega T} z^{-1})^k \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega T} z^{-1}} - \frac{1}{1 - e^{-j\omega T} z^{-1}} \right] = \frac{1}{2j} \left[\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}} \right]$$

$$= \frac{1}{2j} \cdot \frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - (e^{j\omega T} + e^{-j\omega T})z + 1} = \frac{z \sin \omega T}{z^2 - 2\cos \omega T \cdot z + 1}$$

$$E(\mathrm{z}) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

Example 3 $e(t) = t$

Solution. $E(z) = \sum_{k=0}^{\infty} kT \cdot z^{-k} = T\left[z^{-1} + 2z^{-2} + 3z^{-3} + \ \cdots \right]$

$$= Tz\left[z^{-2} + 2z^{-3} + 3z^{-4} + \ \cdots \right]$$

$$= -Tz\left[\frac{d}{dz}z^{-1} + \frac{d}{dz}z^{-2} + \frac{d}{dz}z^{-3} + \ \cdots \right]$$

$$= -Tz\frac{d}{dz}\left[z^{-1} + z^{-2} + z^{-3} + \ \cdots \right]$$

$$= -Tz\frac{d}{dz}z^{-1}\left[1 + z^{-1} + z^{-2} + \ \cdots \right]$$

$$= -Tz\frac{d}{dz}\left[\frac{1}{z} \cdot \frac{1}{1 - z^{-1}}\right] = -Tz\frac{d}{dz}\left[\frac{1}{z-1}\right] = \frac{Tz}{(z-1)^2}$$

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

Example 3(2) $\boldsymbol{e(t) = t}$

Solution:
$$E(z) = \sum_{k=0}^{\infty} kT \cdot z^{-k} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots \quad \textbf{(1)}$$

$$E(z)z - T = 2Tz^{-1} + 3Tz^{-2} + 4T\,z^{-3} + \cdots \quad \textbf{(2)}$$

(2)-(1)

$$E(z)(z-1) - T = Tz^{-1} + Tz^{-2} + T\,z^{-3} + \cdots$$

$$= \frac{Tz^{-1}}{1 - z^{-1}} = \frac{T}{z-1}$$

$$E(z) = \frac{Tz}{(z-1)^2}$$

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

2. Partial Fraction Expansion

Example 4 $E(s)=\dfrac{1}{(s+a)(s+b)}$ **Obtain $E(z)$=?**

Solution:

$$E(s)=\frac{1}{a-b}\cdot\frac{(s+a)-(s+b)}{(s+a)(s+b)}=\frac{1}{a-b}\left[\frac{1}{s+b}-\frac{1}{s+a}\right]$$

$$e(t)=\frac{1}{a-b}\left[e^{-bt}-e^{-at}\right]$$

$$E(z)=\frac{1}{a-b}\sum_{k=0}^{\infty}\left[e^{-bkT}-e^{-akT}\right]\cdot z^{-k}$$

$$=\frac{1}{a-b}\left[\sum_{k=0}^{\infty}(e^{-bT}\cdot z^{-1})^k-\sum_{k=0}^{\infty}(e^{-aT}\cdot z^{-1})^k\right]$$

$$=\frac{1}{a-b}\left[\frac{1}{1-e^{-bT}z^{-1}}-\frac{1}{1-e^{-aT}z^{-1}}\right]=\frac{1}{a-b}\left[\frac{z}{z-e^{-bT}}-\frac{z}{z-e^{-aT}}\right]$$

The z-transform of typical functions

序号	拉氏变换 E(s)	时间函数 e(t)	Z 变换 E(z)
1	1	δ (t)	1
2	$\frac{1}{1-e^{-Ts}}$	$\delta_T(t)=\sum_{n=0}^{\infty}\delta(t-nT)$	$\frac{z}{z-1}$
3	$\frac{1}{s}$	$1(t)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{t^2}{2}$	$\frac{T^2z(z+1)}{2(z-1)^3}$
6	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	$\lim_{a\to 0}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial a^n}(\frac{z}{z-e^{-aT}})$
7	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$

8	$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
9	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
10	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$\frac{z}{z-e^{-aT}}-\frac{z}{z-e^{-bT}}$
11	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\frac{z\sin \omega T}{z^2-2z\cos \omega T+1}$
12	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\frac{z(z-\cos \omega T)}{z^2-2z\cos \omega T+1}$
13	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$\frac{ze^{-aT}\sin \omega T}{z^2-2ze^{-aT}\cos \omega T+e^{-2aT}}$
14	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$\frac{z^2-ze^{-aT}\cos \omega T}{z^2-2ze^{-aT}\cos \omega T+e^{-2aT}}$
15	$\frac{1}{s-(1/T)\ln a}$	$a^{t/T}$	$\frac{z}{z-a}$

- If $E(z) = \mathrm{Z}[e(t)]$, try to proof:

- (1) $E(\frac{z}{a}) = \mathrm{Z}\left[a^k e(t)\right]$

- (2) $-Tz\frac{dE(z)}{dz} = \mathrm{Z}[te(t)]$

Review in one page

- Shannon Sampling Theorem—The Necessary Condition for signal recovery: $\omega_s \geq 2\omega_{\max}$

- $\longrightarrow \boxed{\text{ZOH}} \longrightarrow$ $$G_h(s) = \frac{1 - e^{-Ts}}{s}$$

- z-Transform
  *if k<0, e(kT)=0

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k}$$

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

7.4.3 Properties of z-Transform

1. linear property $$Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$$

2. Real shift theorem 实位移定理

① Lag 延时定理 $$Z\left[e(t-nT)\right]=z^{-n}E(z)$$

Proof. $$\text{LHS}=\sum_{K=0}^{\infty} e(kT-nT)\cdot z^{-k}$$

$$j=k-n$$

$$=\sum_{j=-n}^{\infty} e(jT)\cdot z^{-(j+n)}=z^{-n}\sum_{j=0}^{\infty} e(jT)\cdot z^{-j}$$

$$=z^{-n}E(z)=\text{RHS}$$

2. Real shift theorem 实位移定理

② Lead 超前定理

$$Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]$$

Proof.

$$\mathrm{LHS}=\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-k}=z^{n}\sum_{k=0}^{\infty}e(kT+nT)\cdot z^{-(k+n)}$$

$$\downarrow\; j=k+n$$

$$=z^{n}\sum_{j=n}^{\infty}e(jT)\cdot z^{-j}=z^{n}\left[\sum_{j=0}^{\infty}e(jT)\cdot z^{-j}-\sum_{j=0}^{n-1}e(jT)\cdot z^{-j}\right]$$

$$=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]=\mathrm{RHS}$$

$$Z[e(t-nT)] = z^{-n}E(z)$$

$$Z[e(t+nT)] = z^{n}\left[E(z) - \sum_{k=0}^{n-1} e(kT)\cdot z^{-k}\right]$$

Example 5 $e(t) = t - T$

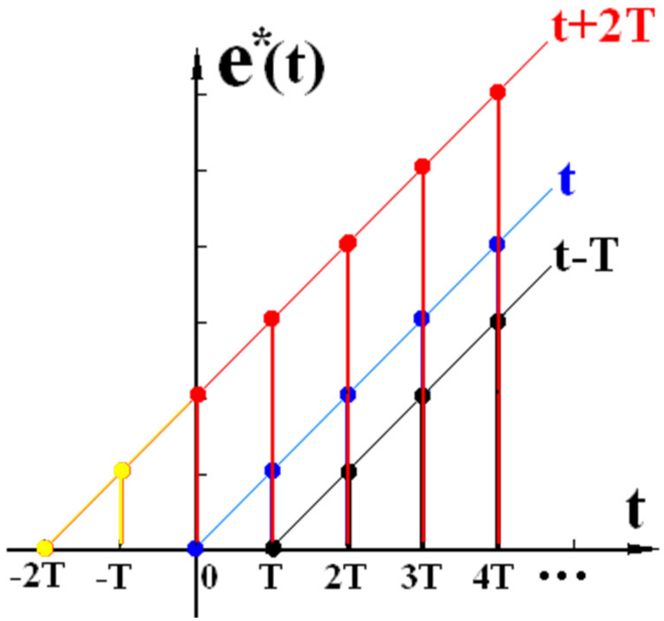
$$E(z) = Z[t-T] = z^{-1}Z[t] = z^{-1}\frac{Tz}{(z-1)^2} = \frac{T}{(z-1)^2}$$

Example 6 $e(t) = t + 2T$

$$E(z) = Z[t+2T]$$

$$= z^2\left\{Z[t] - \sum_{k=0}^{1} kT\cdot z^{-k}\right\}$$

$$= z^2\left[\frac{Tz}{(z-1)^2} - 0 - Tz^{-1}\right]$$



3. Complex shift theorem

复位移定理

$$\boldsymbol{Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)}$$

Proof.

$$\mathrm{LHS}=\sum_{k=0}^{\infty} e(kT)\cdot e^{\mp akT} z^{-k} \quad =\sum_{k=0}^{\infty} e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k}$$

$$\Big\downarrow \; z_1 = z\cdot e^{\pm aT}$$

$$=\sum_{k=0}^{\infty} e(kT)\cdot\left(z\cdot e^{\pm aT}\right)^{-k} = \boldsymbol{E(z_1)} = E\left(z\cdot e^{\pm akT}\right)=\mathrm{RHS}$$

<u>Example 7</u> $\boldsymbol{e(t)=t\cdot e^{-at}}$

$$\boldsymbol{E(z_1)=Z[t]\big|_{z_1=z\cdot e^{aT}}=\frac{Tz_1}{(z_1-1)^2}=\frac{T(z\cdot e^{aT})}{(z\cdot e^{aT}-1)^2}=\frac{Tz\cdot e^{-aT}}{(z-e^{-aT})^2}}$$

4	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$

4. Initial-value Theorem

$$\lim_{n\to 0} e(nT) = \lim_{z\to\infty} E(z)$$

Proof:

$$E(z) = \sum_{n=0}^{\infty} e(nT)\cdot z^{-n}$$

$$= \left[e(0) + e(1)\cdot z^{-1} + e(2)\cdot z^{-2} + e(3)\cdot z^{-3} + \cdots\right]$$

$$\lim_{z\to\infty} E(z) = e(0)$$

Example 8 $E(z) = \dfrac{0.792\cdot z^2}{(z-1)[z^2 - 0.416z + 0.208]}$

$$e(0) = \lim_{z\to\infty} E(z) = 0$$

$$Z[e(t+nT)] = z^n\left[E(z) - \sum_{k=0}^{n-1} e(kT)\cdot z^{-k}\right]$$

5. Final value theorem $\lim_{n\to\infty} e(nT) = \lim_{z\to 1}(z-1)\cdot E(z)$

Proof： $Z[e(t+T)-e(t)] = z\,[E(z)-e(0)] - E(z)$

$$= (z-1)E(z) - z\cdot e(0)$$

$$(z-1)E(z) = z\cdot e(0) + Z[e(t+T)-e(t)]$$

$$\lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}\left\{ z\cdot e(0) + \sum_{n=0}^{\infty}\big[e[(n+1)T] - e(nT)\big]\cdot z^{-n} \right\}$$

$$= e(0) + [e(1)-e(0)] + [e(2)-e(1)] + [e(3)-e(2)] + \cdots$$

$$= e(\infty T)$$

Example 9 $E(z) = \dfrac{0.792 \cdot z^2}{(z-1)[z^2 - 0.416z + 0.208]}$

$$e(\infty T) = \lim_{z \to 1}(z-1)E(z) = \lim_{z \to 1} \frac{0.792 \cdot z^2}{[z^2 - 0.416z + 0.208]} = \mathbf{1}$$

6. Convolution theorem

If $c^*(t) = e^*(t) * g^*(t) = \sum_{k=0}^{\infty} e(kT) \cdot g[(n-k)T]$

We have: $\boldsymbol{C(z) = E(z) \cdot G(z)}$

Proof:

$$E(z) = \sum_{k=0}^{\infty} e(kT) \cdot z^{-k} \qquad G(z) = \sum_{k=0}^{\infty} g(kT) \cdot z^{-k}$$

$$E(z)=\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}$$

$$G(z)=\sum_{k=0}^{\infty} g(kT)\cdot z^{-k}$$

$$\boldsymbol{Z[e(t-nT)]=z^{-n}E(z)}$$

$$e^*(t)*g^*(t)=\sum_{k=0}^{\infty} e(kT)\cdot g[(n-k)T]$$

$$E(z)\bullet G(z)=\left(\sum_{k=0}^{\infty} e(kT)\cdot z^{-k}\right)G(z)$$

$$=\sum_{k=0}^{\infty} e(kT)\cdot\left(z^{-k}G(z)\right)$$

$$=\sum_{k=0}^{\infty} e(kT)\{\mathrm{Z}[g(nT-kT)]\}$$

$$=\sum_{k=0}^{\infty} e(kT)\sum_{n=0}^{\infty}[g(nT-kT)]z^{-n}$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\infty} e(kT)[g(nT-kT)]\right\}z^{-n}$$

$$=\sum_{n=0}^{\infty}\{e(nT)*g(nT)\}z^{-n}$$

$$=\sum_{n=0}^{\infty} c(nT)z^{-n}=C(z)$$

Properties of z-Transform

1. linear property $$Z\left[a\cdot e_1^*(t)\pm b\cdot e_2^*(t)\right]=a\cdot E_1(z)\pm b\cdot E_2(z)$$

2. Real shifting theorem
$$\begin{cases}\text{Lag} & Z\left[e(t-nT)\right]=z^{-n}E(z)\\ \text{Lead} & Z\left[e(t+nT)\right]=z^{n}\left[E(z)-\sum_{k=0}^{n-1}e(kT)\cdot z^{-k}\right]\end{cases}$$

3. Complex shifting theorem $$Z\left[e(t)\cdot e^{\mp at}\right]=E\left(z\cdot e^{\pm aT}\right)$$

4. Initial-value theorem $$\lim_{n\to 0}e(nT)=\lim_{z\to\infty}E(z)$$

5. Final-value theorem $$\lim_{n\to\infty}e(nT)=\lim_{z\to 1}(z-1)\cdot E(z)$$

6. Convolution theorem $$c^*(t)=e^*(t)*g^*(t)\quad\Rightarrow\quad C(z)=E(z)\cdot G(z)$$

- Instant work:

- Calculate the initial and final values:

- $$E(z) = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$E(z) = \frac{z^2}{(z-0.8)(z-0.1)}$$

7.4.4 Inverse z-Transform

$$Z^{-1}[X(z)] = x(nT)$$

Tips:

Inverse Z-transform can only provide discrete-time signal $x^*(t)$, instead of continuous signal $x(t)$。

$$\begin{cases} \text{Long Division（长除法）} \\ \text{Partial-Fraction expansion (PFE)} \quad \text{Expansion of } \dfrac{E(z)}{z} \\ \text{Residue（留数法）} \quad e(nT) = \sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right] \end{cases}$$

1. Long Division（长除法）/ Power Series

$$E(z)=\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}$$

Numerator is divided by denominator ,we get

$$E(z)=c_0 + c_1 z^{-1} + \cdots + c_k z^{-k} + \cdots = \sum_{k=0}^{\infty} c_k z^{-k} = \sum_{k=0}^{\infty} e(kT) z^{-k}$$

$$e^*(t)=c_0\delta(t)+c_1\delta(t-T)+\cdots+c_k\delta(t-kT)+\cdots$$

Example 10 $F(z)=\dfrac{z}{(z-2)(z-3)}$, **obtain f*(t).**

Solution:

Because $F(z)=\dfrac{z}{z^2-5z+6}=\dfrac{z^{-1}}{1-5z^{-1}+6z^{-2}}$

By long-division, we get that

$$F(z)=z^{-1}+5z^{-2}+19z^{-3}+65z^{-4}+\cdots$$

Thus

$$f(0)=0,\ \ f(T)=1,\ \ f(2T)=5,\ \ f(3T)=19,\ \ f(4T)=65,\cdots$$

Then

$$f^*(t)=\delta(t-T)+5\delta(t-2T)+19\delta(t-3T)+65\delta(t-4T)+\cdots$$

2. Partial fraction expansion (PFE)

序号	拉氏变换 E(s)	时间函数 e(t)	Z 变换 E(z)
1	1	$\delta(t)$	1
2	$\frac{1}{1-e^{-Ts}}$	$\delta_T(t)=\sum_{n=0}^{\infty}\delta(t-nT)$	$\frac{z}{z-1}$
3	$\frac{1}{s}$	$1(t)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{t^2}{2}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$

Note: here, we expand $\frac{X(z)}{z}$, instead of z。

$$\frac{X(z)}{z}=\sum_{i=1}^{n}\frac{A_i}{z-z_i}$$

Consider

$$X(z)=\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}$$

Then

$$X(z)=\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{a_0 \prod_{i=1}^{n} (z - z_i)}$$

If there is no repeated root for the denominator, it generates

$$X(z) = z\left(\frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \cdots + \frac{A_n}{z-z_n}\right)$$

Then look up each part in the transform table.

PS: the coefficients A_i **, are decided by:** $A_i = \left[(z-z_i)\frac{X(z)}{z} \right]\Big|_{z=z_i}$

Example 12 $E(z)=\dfrac{z^2}{(z-0.8)(z-0.1)}$ **Obtain e*(t). (PFE & Residue)**

PFE: $$\frac{E(z)}{z}=\frac{z}{(z-0.8)(z-0.1)}=\frac{C_1}{(z-0.8)}+\frac{C_2}{(z-0.1)}$$

$$C_1=\lim_{z\to 0.8}\frac{z}{(z-0.1)}=\frac{8}{7}\qquad C_2=\lim_{z\to 0.1}\frac{z}{(z-0.8)}=\frac{-1}{7}$$

$$=\frac{8/7}{(z-0.8)}-\frac{1/7}{(z-0.1)}$$

$$E(z)=\frac{8}{7}\cdot\frac{z}{(z-0.8)}-\frac{1}{7}\cdot\frac{z}{(z-0.1)}$$

$$e(t)=(8\times 0.8^{\frac{t}{T}}-0.1^{\frac{t}{T}})/7\qquad e(nT)=(8\times 0.8^n-0.1^n)/7$$

$$e^*(t)=\sum_{n=0}^{\infty}\left[(8\times 0.8^n-0.1^n)/7\right]\cdot\delta(t-nT)$$

Example 13 **Consider**

$$F(z)=\frac{z}{(z-1)(z-e^{-T})}$$

Obtain f*(t)。

Solution:

$$K_1=\lim_{z\to 1}\left(\frac{z-1}{z}\right)F(z)=\frac{1}{1-e^{-T}}$$

$$K_2=\lim_{z\to e^{-T}}\left(\frac{z-e^{-T}}{z}\right)F(z)=-\frac{1}{1-e^{-T}}$$

$$\frac{F(z)}{z}=\frac{K_1}{z-1}+\frac{K_2}{z-e^{-T}}$$

$$F(z)=\frac{1}{1-e^{-T}}\left(\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right)$$

$$\Rightarrow\quad f(nT)=\frac{1}{1-e^{-T}}\left(1-e^{-nT}\right)$$

$$f^*(t)=\frac{1}{1-e^{-T}}\sum_{k=0}^{+\infty}(1-e^{-kT})\delta(t-kT)$$

3、 Residue(留数法) / Inversion Integral

$$F(z)=\sum_{k=0}^{+\infty} f(kT)z^{-k}$$ **"Laurent Series"**

$$= f(0)+f(T)z^{-1}+f(2T)z^{-2}+\cdots+f(nT)z^{-n}+\cdots$$

$$F(z)z^{m-1}=\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}$$

$$= f(0)z^{m-1}+f(T)z^{m-2}+f(2T)z^{m-3}+\cdots+f(mT)z^{-1}+\cdots$$

"Cauchy's residual theorem"

Γ Encircle all the poles of $F(z)z^{k-1}$

$$\oint_\Gamma F(z)z^{m-1}dz=\oint_\Gamma\left[\sum_{k=0}^{+\infty} f(kT)z^{m-k-1}\right]dz$$

$$=\oint_\Gamma f(0)z^{m-1}dz+\oint_\Gamma f(T)z^{m-2}dz+\cdots+\oint_\Gamma f(mT)z^{-1}dz+\cdots$$

"Complex function theorem"

$$\oint_\Gamma z^{m-k-1}dz=\begin{cases}0, & if\ \ m\neq k\\ 2\pi j, & if\ \ m=k\end{cases}$$

$$\oint_{\Gamma} F(z) z^{m-1} dz = \oint_{\Gamma} f(kT) z^{-1} dz = f(kT) \cdot 2\pi j$$

"Cauchy's residual theorem"

$$f(kT) = \frac{1}{2\pi j} \oint_{\Gamma} f(kT) z^{-1} dz = \sum_{i=1}^{n} Res[F(z) z^{k-1}, z_i]$$

$z_i, i = 1, 2, \cdots, n$ **are all the poles of** $F(z) z^{k-1}$

其中***Res*[]**表示函数的留数。

$$Res\left[z^{(k-1)} x(z)\right] = \lim_{z \to z_i} \frac{1}{(r-1)!} \frac{d^{r-1}}{dz^{r-1}} \left[(z - z_i)^r z^{k-1} x(z)\right]$$

Example 14 For

$$F(z) = \frac{10z}{(z-1)(z-2)}$$

Obtain its inverse z-transform by residue Method.

Solution： $$F(z)z^{k-1} = \frac{10z^k}{(z-1)(z-2)}$$

Poles $z_1 = 1$ and $z_2 = 2$, and

$$res[F(z)z^{k-1},1] = \lim_{z\to 1}(z-1)F(z)z^{k-1} = -10$$

$$res[F(z)z^{k-1},2] = \lim_{z\to 2}(z-2)F(z)z^{k-1} = 10\cdot 2^k$$

Then $$f(kT) = 10(2^k - 1) \qquad (k = 0,1,2,\cdots)$$

Example 15 $E(z)=\dfrac{5}{(z-a)^2}$ **Obtain e*(t). (Residue)**

Solution.

$$e(nT)=\sum \mathrm{Res}\left[E(z)\cdot z^{n-1}\right]=\underset{z=a}{\mathrm{Res}}\left[\frac{5}{(z-a)^2}\cdot z^{n-1}\right]$$

$$e(nT)=\frac{1}{(2-1)!}\lim_{z\to a}\frac{d}{dz}\left[(z-a)^2\,\frac{5\cdot z^{n-1}}{(z-a)^2}\right]$$

$$=\lim_{z\to a}\frac{d}{dz}\left[5\cdot z^{n-1}\right]$$

$$=5\cdot\lim_{z\to a}\left[(n-1)\cdot z^{n-2}\right]$$

$$=5\cdot(n-1)\cdot a^{n-2}$$

$$e^{*}(t)=\sum_{n=0}^{\infty}\left(5(n-1)\cdot a^{n-2}\right)\cdot\delta(t-nT)$$

Example 16 $X(z)=\dfrac{az}{\sin mz}$ **Obtain x*(t). (Residue)**

Solution.

$$x(kT)=\sum \mathrm{Res}\left[\frac{az\cdot z^{k-1}}{\sin mz}, z=\frac{n\pi}{m}, n=0,1,\ldots\right]$$

$$Res\left[z^{(k-1)}x(z)\right]=\lim_{z\to z_i}\frac{1}{(r-1)!}\frac{d^{r-1}}{dz^{r-1}}\left[(z-z_i)^r z^{k-1}x(z)\right]$$

$$x(kT)=\sum_{n=0}^{\infty}\lim_{z\to\frac{n\pi}{m}}(z-\frac{n\pi}{m})\frac{az^k}{\sin mz}=$$

$$=\sum_{n=0}^{\infty}\frac{az^k(z-n\pi/m)}{\sin mz}\bigg|_{z\to\frac{n\pi}{m}}=\sum_{n=0}^{\infty}a(\frac{n\pi}{m})^k\frac{(z-n\pi/m)}{\sin mz}\bigg|_{z\to\frac{n\pi}{m}}$$

$$=\sum_{n=0}^{\infty}a(\frac{n\pi}{m})^k\frac{1}{m\cos mz}\bigg|_{z\to\frac{n\pi}{m}}=\sum_{n=0}^{\infty}(-1)^n\frac{a}{m}(\frac{n\pi}{m})^k$$

7.4.5 Explanation of z-Transform

(1) Uniqueness; $E_1(z) = E_2(z) \Leftrightarrow e_1^*(t) = e_2^*(t)$

$\not\Rightarrow e_1(t) = e_2(t)$

(2) Region of Convergence

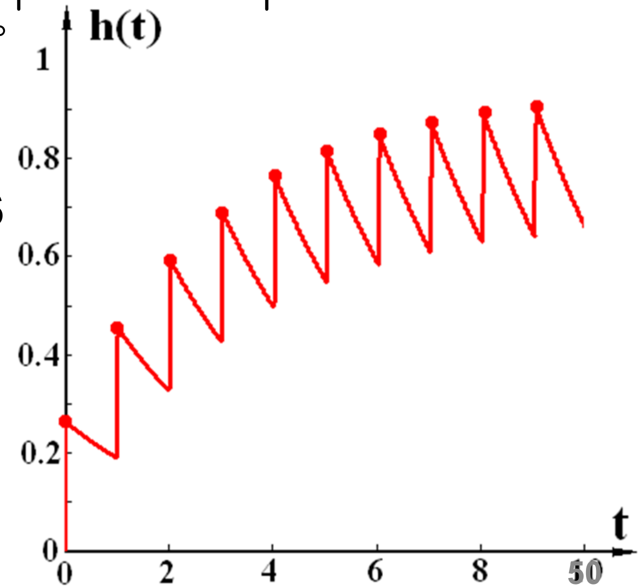
$$s = \sigma + j\omega \Rightarrow z = e^{sT} = e^{\sigma T} e^{j\omega T} \quad if: r = |z| = e^{\sigma T}, z = re^{j\omega T}$$

$$L: \int_0^{\infty} \left| e(t) e^{-\sigma t} \right| dt < \infty \Rightarrow Z: \sum_{n=-\infty}^{\infty} \left| e(nT) r^{-n} \right| < \infty$$

Some limitations of z-Transform

(1) only shows the information of samples

(2) In some cases, the continuous signal may jump on the sampling point.



- Homework:
- You should know.