7.5 Mathematical Models of Discrete-Time Systems

7.5.1 Linear Time-Invariant Difference Equations

(1) Definition of difference   ① **Forward**   ② **Backward**

(2) The difference equation and its solving method   ① **Iteration**   ② **Z-transformation**

7.5.2 Impulse-Transfer Function

(1) Definition   **(2) Properties**   **(3) Limitation**

7.5.3 Impulse Transfer Function of Open-Loop Systems
$$\begin{cases} \text{(1) Switch between factors} \\ \text{(2) No switch between factors} \\ \text{(3) With ZOH} \end{cases}$$

7.5.4 Impulse Transfer Function of Closed-Loop Systems   **Basic Principle**

Chapter 7 Analysis and Design of Linear Discrete-Time System （Sampled-data System）

7.6 Performance Analysis of Discrete-Time Systems

➢ **Stability**

➢ **Dynamic Performance**

➢ **Steady-state Errors**

7.6.1 Stability of Discrete systems

1. Preliminaries

Stability is the most important performance of a system.

When we sampled a continuous systems, we still have a "continuous" system → the same properties hold as before:

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

Now, we introduced the variable $z=e^{Ts}$, how does stability look in the new variable?

First, we much understand the relationship between s-domain and z-domain.

2. s-Domain to z-Domain Mapping

Because $z = e^{sT}$, **let** $s = \sigma + j\omega$ **then** $\mid z \mid = e^{\sigma T}$

$\angle z = \omega T$, **thus**

$$\begin{cases} \sigma > 0 & \mid z \mid > 1 \\ \sigma < 0 & \mid z \mid < 1 \\ \sigma = 0 & \mid z \mid = 1 \end{cases}$$
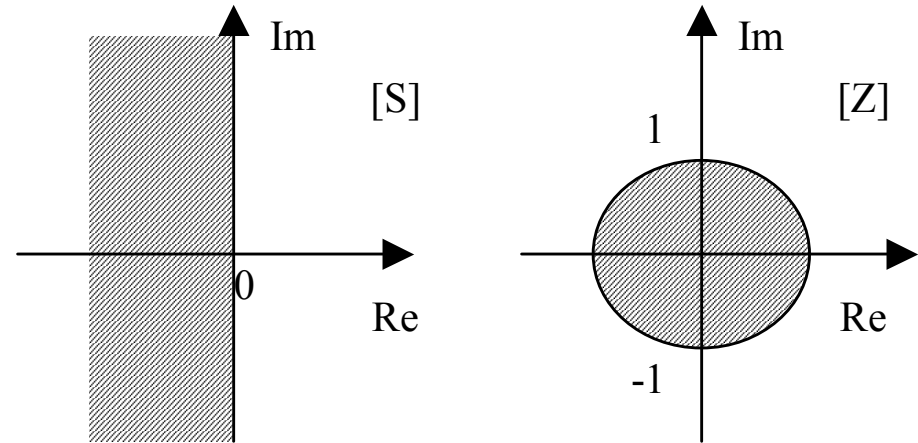
The mapping relationship can be described as in the following figures.

$$z = e^{Ts} = e^{T(\sigma + j\omega)} = e^{T\sigma}e^{j\omega T}$$

$$\sigma < 0 \quad |z| = \frac{1}{e^{-T\sigma}} < 1$$

$$\sigma = 0 \qquad |z| = 1$$

$$\sigma > 0 \quad |z| = e^{T\sigma} > 1$$

For a C.L. discrete-time system with unit feedback, the impulse transfer function is:

$$\Phi(z) = \frac{G(z)}{1 + G(z)}$$

Its characteristic function is: $\quad 1 + G(z) = 0$

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

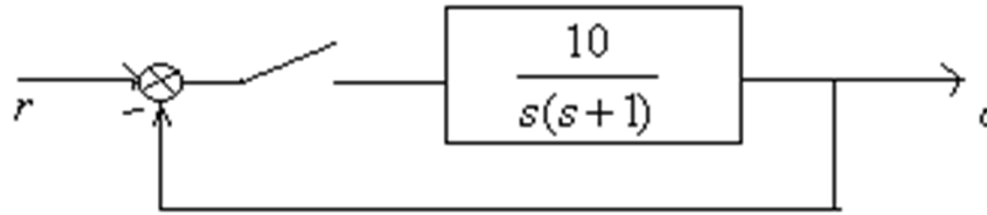**——All poles of Φ(z) lie in the unit circle of z plane.**

Prove：

$$\Phi(z) = \frac{M(z)}{D(z)} = \frac{\prod_{i=1}^{m}(z-\alpha_i)}{\prod_{j=1}^{n}(z-\beta_j)} = \sum_{j=1}^{n}\frac{C_j z}{z-\beta_j} = K(z)$$

$$c(k) = \sum_{j=1}^{n} C_j \beta_j^k \overset{k\to\infty}{=} 0 \qquad\qquad |\beta_j| < 1 \qquad \text{— Necessity}$$

$$c^*(t) = \sum_{k=0}^{\infty}\left(\sum_{j=1}^{n} C_j \beta_j^k\right)\cdot \delta(t-kT) \qquad \text{— Sufficiency}$$

Example The discrete-time system is shown as the following figure, suppose T=1, is the system stable?

Solution: $G(z) = Z[\dfrac{10}{s(s+1)}] = \dfrac{6.32z}{(z-1)(z-0.368)}$

$$1 + G(z) = 0 \Rightarrow z^2 + 4.952z + 0.368 = 0$$

$$\Rightarrow z_1 = -0.076 \quad z_2 = -4.876$$

$$\because \left| z_2 \right| > 1 \quad \text{So the system is unstable.}$$

3. The Stability Criterion of Discrete-Time Systems

- For continuous-time systems，we can use Routh criterion to determine the stability of the system, where the stable area is on LHP (left-hand-plane) of [s]-domain.

- Unfortunately, for discrete-time systems, the stable area is unit circle, not LHP of [z]-domain, we cannot directly apply the Routh criterion as we have to test on something else than LHP.

w-transformation and Routh criterion in w-domain

We find a transformation that maps the unit circle back onto the LHP while maintaining the algebraic structure of rational functions.

A particular transformation that will accomplish this would be the bilinear transformation:

$$z = \frac{w+1}{w-1} \qquad w = \frac{z+1}{z-1}$$

Suppose $$z = x + jy \qquad w = u + jv$$

Then

$$w = \frac{z+1}{z-1} = \frac{(x+1)+jy}{(x-1)+jy}$$

$$u + jv = \frac{(x^2+y^2)-1}{(x-1)^2+y^2} + j\frac{2y}{(x-1)^2+y^2}$$

$$\therefore u = \frac{x^2+y^2-1}{(x-1)^2+y^2}$$

$$w = \frac{z+1}{z-1} = \frac{x+1+jy}{x-1+jy} = \frac{x^2-1+y^2-j2y}{(x-1)^2+y^2} = u+jv$$

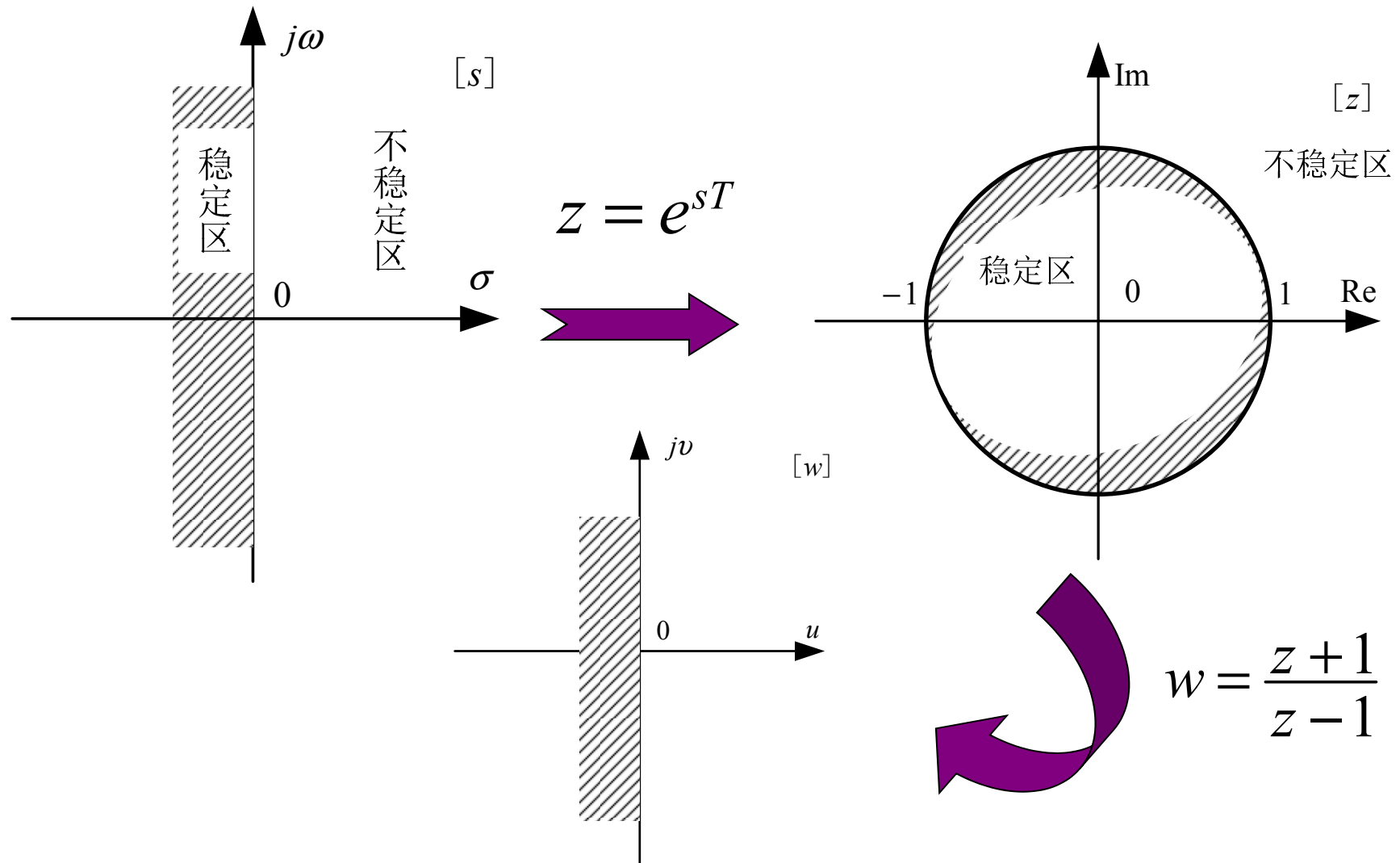**[w] imaginary axis** $u = 0 = \dfrac{x^2+y^2-1}{(x-1)^2+y^2} = 0$

$$\Rightarrow x^2 + y^2 = 1 \quad \text{[z] unit circle}$$

$$\text{Points} \begin{cases} \text{inside} \\ \text{outside} \end{cases} \text{of the unit circle}$$

$$\Longrightarrow x^2 + y^2 \begin{cases} < 1 \\ > 1 \end{cases} \quad \text{in z-plane}$$

$$\Longrightarrow \begin{cases} u < 0 \\ u > 0 \end{cases} \quad \text{in s-plane}$$

we've learned three methods to determine the stability of a discrete-time systems.

$$z = e^{sT}$$

$$w = \frac{z+1}{z-1}$$

Example 1 Determine the stability from the characteristic equation of a discrete system.

$$D(z) = 45z^3 - 117z^2 + 119z - 39 = 0$$

$$\downarrow \quad z = (w+1)/(w-1)$$

$$= 45(\frac{w+1}{w-1})^3 - 117(\frac{w+1}{w-1})^2 + 119(\frac{w+1}{w-1}) - 39 = 0$$

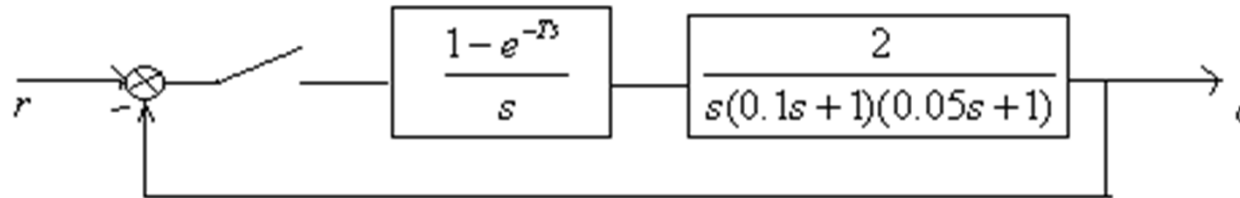$$D(w) = 45(w+1)^3 - 117(w+1)^2(w-1) + 119(w+1)(w-1)^2 - 39(w-1)^3 = 0$$

$$D(w) = w^3 + 2w^2 + 2w + 40 = 0$$

Routh

w^3	1	2
w^2	2	40
w^1	-18	
w^0	40	

Unstable!

Example 2 Consider the discrete-time system as shown in the figure, if T=0.1, determine the stability of the system.

Solution:

$$G(z) = \frac{z-1}{z} Z\left[\frac{2}{s^2(0.1s+1)(0.05s+1)}\right]$$

$$= \frac{z-1}{z}\left[-\frac{0.3z}{z-1} + \frac{0.4z}{(z-1)^2} + \frac{0.4z}{z-e^{-10T}} - \frac{0.1z}{z-e^{-20T}}\right]$$

$\boxed{1+G(z)=0}$ $z^3 - 1.001z^2 + 0.3356z + 0.0535 = 0$

$\boxed{z = \dfrac{w+1}{w-1}}$ $2.33w^3 + 3.68w^2 + 1.65w + 0.34 = 0$

w^3	*2.33*	*1.65*
w^2	*3.68*	*0.34*
w	*1.43*	*0*
w^0	*0.34*	

The elements in the first column are all positive, the system is stable.

Example 3 Consider the system shown in the figure (T=1). Determine the stable range of K.
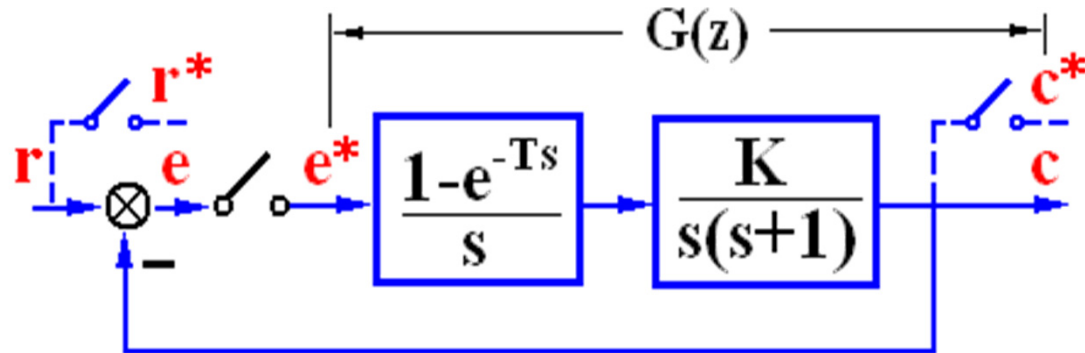
Routh criterion in w domain

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s} \cdot \frac{K}{s(s+1)}\right] = (1-z^{-1})K \cdot Z\left[\frac{1}{s^2(s+1)}\right]$$

$$= \frac{(z-1)K}{z} \cdot Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right] = \frac{(z-1)K}{z} \cdot \left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= K\left[\frac{(T-1+e^{-T})z + (1-e^{-T}-Te^{-T})}{(z-1)(z-e^{-T})}\right] \overset{T=1}{=} \frac{0.368K(z+0.718)}{(z-1)(z-0.368)}$$

$$\Phi(z) = \frac{G(z)}{1+G(z)} = \frac{0.368K(z+0.718)}{z^2 + (0.368K-1.368)z + (0.264K+0.368)}$$

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$$\Phi(z) = \frac{G(z)}{1 + G(z)} = \frac{0.368K(z + 0.718)}{z^2 + (0.368K - 1.368)z + (0.264K + 0.368)}$$

$$D(z) = z^2 + (0.368K - 1.368)z + (0.264K + 0.368) = 0$$

$$\Bigg| \quad z = \frac{w+1}{w-1}$$

$$= (\frac{w+1}{w-1})^2 + (0.368K - 1.368)(\frac{w+1}{w-1}) + (0.264K + 0.368) = 0$$

$$(w+1)^2 + (0.368K - 1.368)(w+1)(w-1) + (0.264K + 0.368)(w-1)^2 = 0$$

$$D(w) = 0.632Kw^2 + (1.264 - 0.528K)w + (2.736 - 0.104K) = 0$$

$$\begin{cases} K > 0 \\ 1.264 - 0.528K > 0 \\ 2.736 - 0.104K > 0 \end{cases} \qquad \begin{cases} K > 0 \\ K < 2.394 \\ K < 26.3 \end{cases} \qquad 0 < K < 2.394$$

Exercise Consider the system shown in the figure (T=0.1).
Determine the stable range of K. If it is a continuous-time system,
Determine the stable range of K

Solution:

$$G(z) = Z\left[\frac{k}{s(0.1s+1)}\right] = \frac{0.632kz}{z^2 - 1.368z + 0.368}$$

$$1 + G(z) = 0 \qquad z^2 + (0.632k - 1.368)z + 0.368 = 0$$

$$z = \frac{w+1}{w-1} \qquad 0.632kw^2 + 1.264w + 2.736 - 0.632k = 0$$

For continuous-time system, $\boldsymbol{0 < k}$, then system is stable;

For discrete-time system, $\boldsymbol{0 < k < 4.33}$, stable.

Summary

s-Domain to z-Domain Mapping

Necessary and Sufficient Condition for Stability of Linear Discrete-Time Systems

— All poles of $\Phi(z)$ lie in the unit circle of z plane

Routh criterion in w domain (Generalized Routh Criterion)

Homework: 7-14