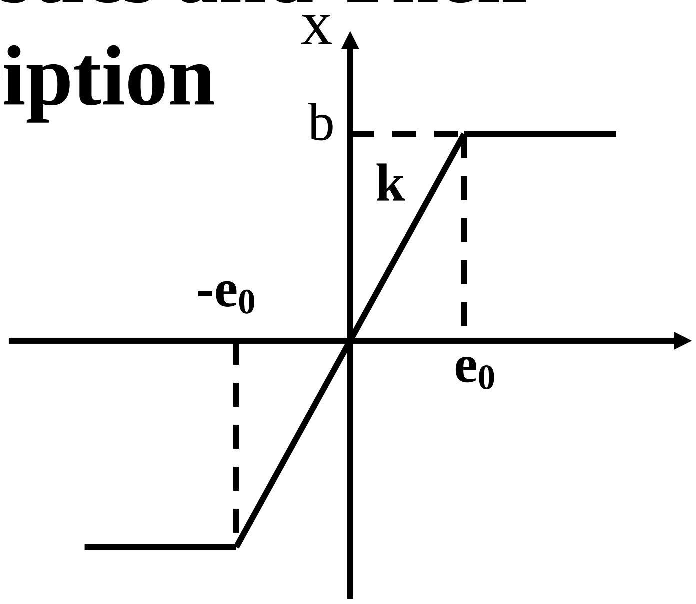
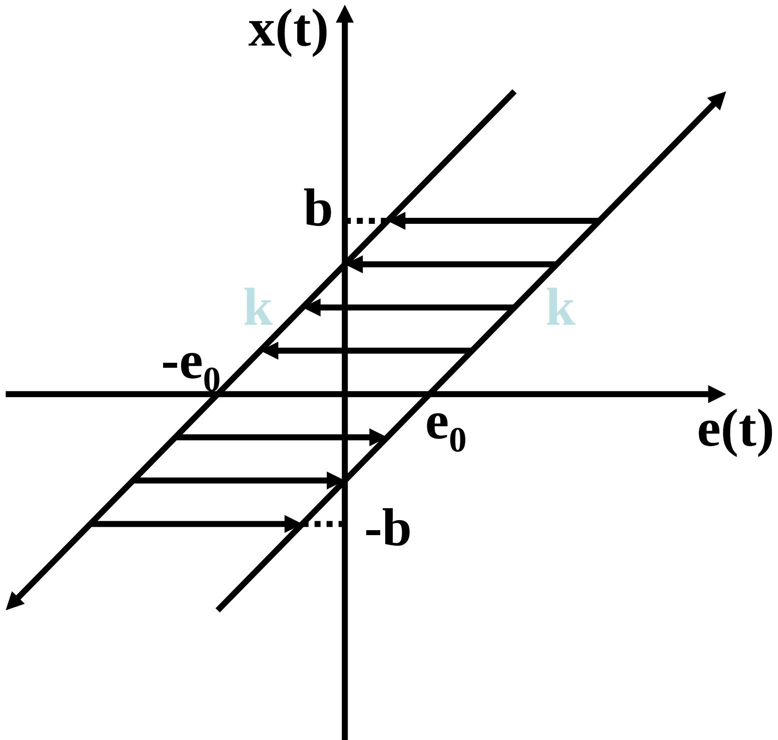
Review

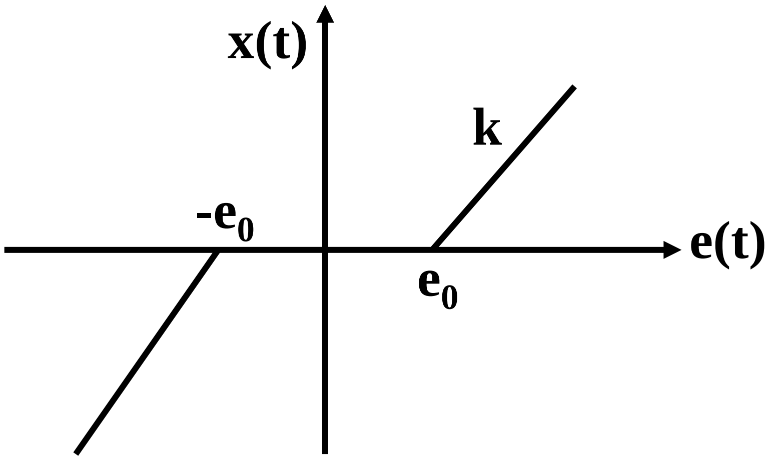
- Features of nonlinear systems
  - principle of superposition(叠加原理) is not available
  - The stability of a nonlinear system depends on not only the inherent structure and parameters of control systems, but also the initial conditions and the inputs.
  - Periodic oscillation
  - Jump resonance and Multi-valued response
  - harmonic oscillation
- Phase Plane Method
  - Singular Point and Limit Cycle
  - Time domain analysis of typical nonlinear systems

Typical Nonlinear characteristics and Their Mathematical Description

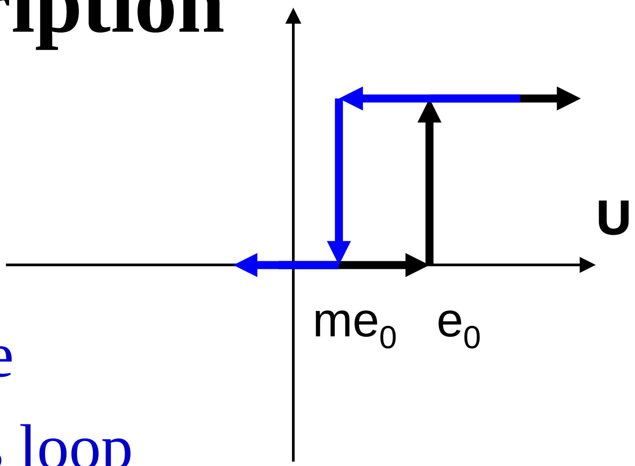
- Saturation characteristics
- Dead-zone characteristics
- Gap characteristics



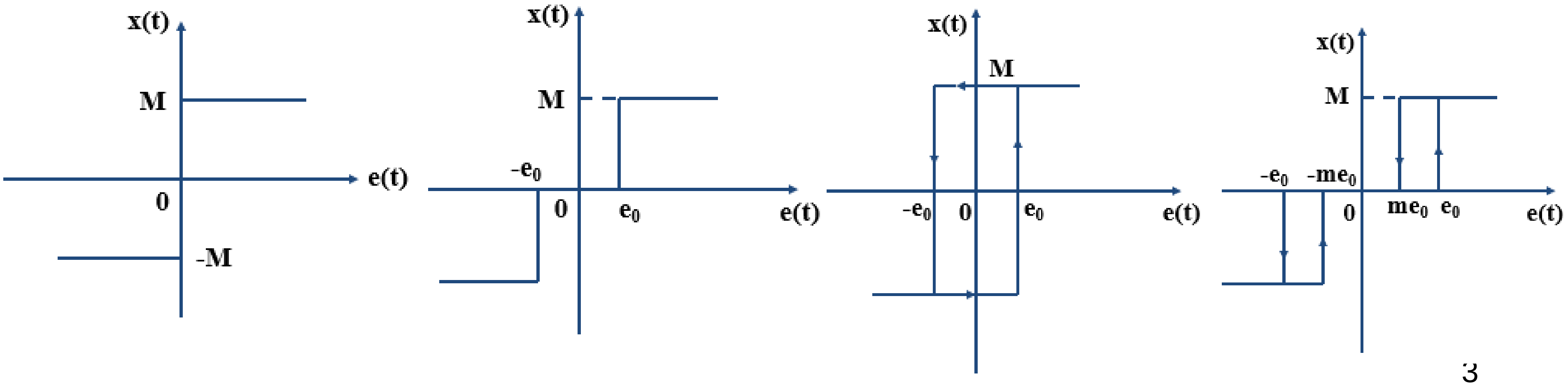




Typical Nonlinear characteristics and Their Mathematical Description



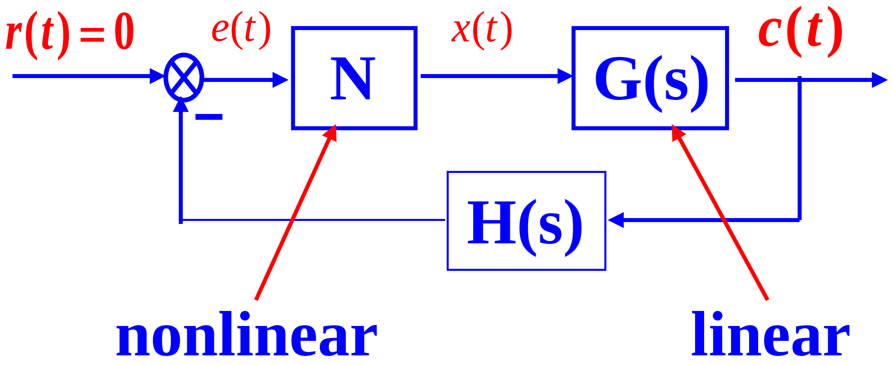
- Relay characteristics
  - Ideal relay characteristics
  - Relay characteristics with Dead-zone
  - Relay characteristics with Hysteresis loop
  - Relay characteristics with Dead-zone and Hysteresis loop



§ 8.3 Describing function method
(Harmonic Linearizing method)

Basic idea

For the nonlinear system



Typical structure of the nonlinear systems

$If\ : e(t) = A\sin\omega t$ $\Longrightarrow$ **a sinusoidal input,**

***x*(*t*), maybe it is not a sinusoidal but a periodic function, can be expressed as a Fourier series.**

Assumption:

The harmonic of *x*(*t*) could be neglected, then:

$$x(t) \approx x_1 \sin(\omega t + \phi_1) \Longrightarrow$$

output frequency is equal to input frequency approximately.

Similar to the *frequency analysis* of linear system, we can perform frequency analysis for the nonlinear system based on the assumption.

$\Longrightarrow$ **Describing function**

- The describing function method is mainly used to analyze the ***stability*** and ***self-oscillation*** of the nonlinear system without external excitation.

- Advantage:
  - It is not confined by ***the order of the system***.

- Disadvantages:
  - It is an ***approximate analysis method***.
  - It can only be used to study the system ***frequency characteristics***.

8.3.1 Concept of describing function

The describing function method can be applied to nonlinear systems with the following features:

1. The linear part and the nonlinear part can be separated.

Shown in Fig. 8-9，NL is a *nonlinear part*，G is the transfer function of the *linear part*.

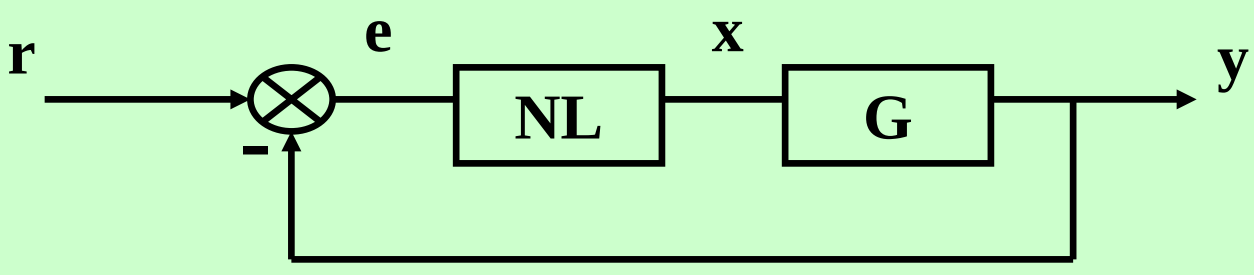


Fig. 8-9 Typical structure of nonlinear system

2. The system has an *odd-symmetric nonlinearity*, and the input-output relationship of the nonlinear part is static (without energy storage elements).

- **If so, sinusoidal input → periodic output**
- **The output can be expanded into Fourier series with a *zero D. C. component*.**

$$f(x)=-f(-x)$$

3. The linear part is a good low-pass filter

We can suppose the *higher-oder harmonic* is filtered out.

⇓

There is only a *fundamental component* in the the output .

If all the conditions above are satisfied, we can describe the nonlinear components by the frequency response like as that we did in the linear systems.

So we have:

Definition of the describing function

The describing function *N*(*A*) of the nonlinear element is: the *complex ratio* of the fundamental component of the output *x*(*t*) and the sinusoidal input *e*(*t*), that is:

$$For\ e(t) = A\sin\omega t,$$

$$x(t) \approx A_1\cos\omega t + B_1\sin\omega t$$

$$= x_1\sin(\omega t + \phi_1) \Longrightarrow N(A) = \frac{x_1 e^{j\phi_1}}{A}$$

Assume the input of nonlinear is sinusoidal $e(t) = A\sin\omega t$

Normally, the output is periodic, which can be expressed as a *Fourier series*:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\omega t + B_n \sin n\omega t\right)$$

The nonlinearity is *odd-symmetric*(奇对称).

$$\Rightarrow \quad A_0 = 0$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos n\omega t \; d(\omega t)$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin n\omega t \; d(\omega t)$$

For the fundamental component, we have

$$\boxed{A_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos\omega t \; d(\omega t)}$$

$$\boxed{B_1 = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin\omega t \; d(\omega t)}$$

Thus, the fundamental component is

$$\boxed{x_1(t) = A_1\cos\omega t + B_1\sin\omega t = x_1\sin(\omega t + \varphi_1)}$$

where

$$\boxed{x_1 = \sqrt{A_1^2 + B_1^2}} \qquad \boxed{\varphi_1 = arctg\frac{A_1}{B_1}}$$

The describing function is then given by

$$N(A) = \frac{X_1}{A} e^{j\varphi_1}$$

Obviously, the describing function is a function of the input amplitude *A*. So we can regard it as a *variable gain amplifier*.

$$\boxed{N(A) = \frac{\sqrt{A_1^2 + B_1^2}}{A} e^{j\, arctg \frac{A_1}{B_1}} = \frac{B_1}{A} + j\frac{A_1}{A}}$$

Replacing the nonlinear part by N(A), we can extend the *frequency response method* of linear system to the nonlinear system so as to analyze the *frequency characteristics* of nonlinear system.

Remarks:

Normally, the describing function N is *a function of the amplitude and the frequency of input signal*, it should be expressed as $N(A,\omega)$.

In most of the nonlinear components, there are no energy storage elements. The frequencies of output and input are then independent. So the describing function N of common nonlinear components is *only a function of the amplitude* of input, which can be expressed as $N(A)$。

Remarks: (cont.)

If the nonlinearity is *single-valued odd-symmetric* ⇒

The output $x(t)$ is *an odd function*.

$$A_1 = 0 \qquad N(A) = B_1 / A$$

The describing function is a *real function* of input amplitude A.

If the nonlinearity is not *single-valued odd-symmetric* ⇒

The output $x(t)$ is *neither an odd nor even*.

$$A_1 \neq 0, B_1 \neq 0$$

The describing function is a *complex function* of input amplitude A.

8.3.2 Describing function of typical nonlinear characteristic

1. Saturation

Assume the input is $e(t) = A\sin\omega t$

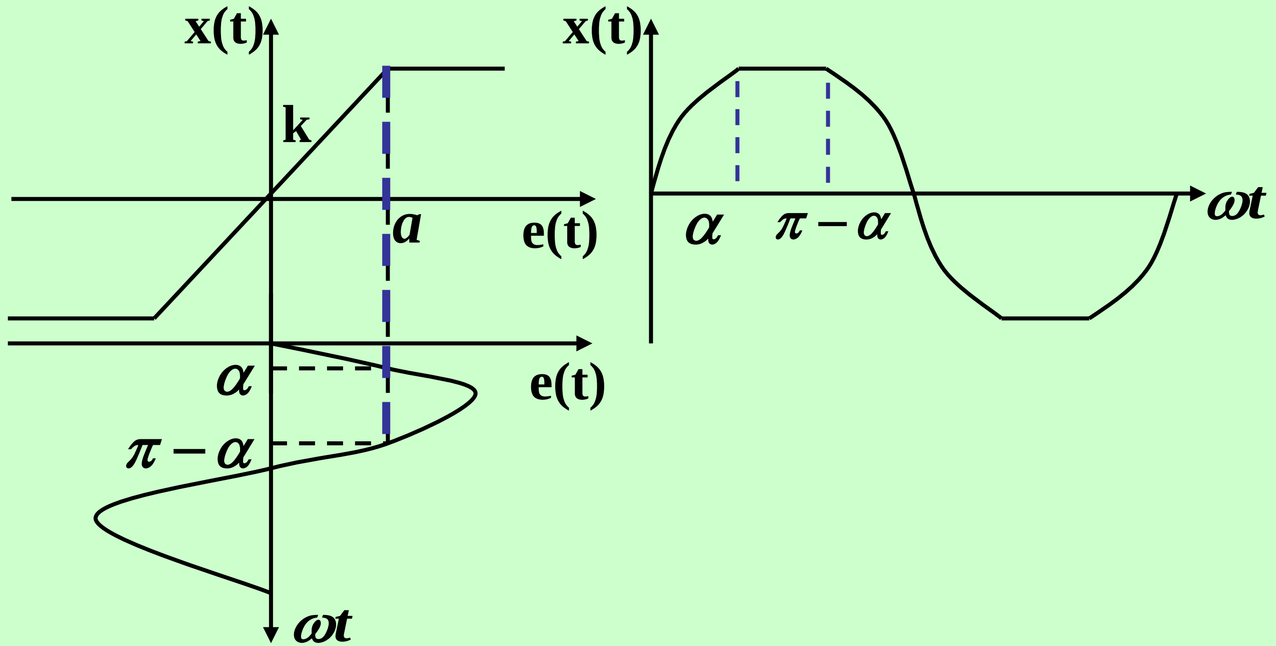


Fig 8-10 Input and output waveform of a saturation nonlinearity

when $A > a$**, the output** $x(t)$ **is**

$$x(t) = \begin{cases} KA\sin\omega t, & 0 \le \omega t \le \alpha \\ Ka, & \alpha < \omega t \le \pi - \alpha \\ KA\sin\omega t, & \pi - \alpha < \omega t \le \pi \end{cases}$$

where,

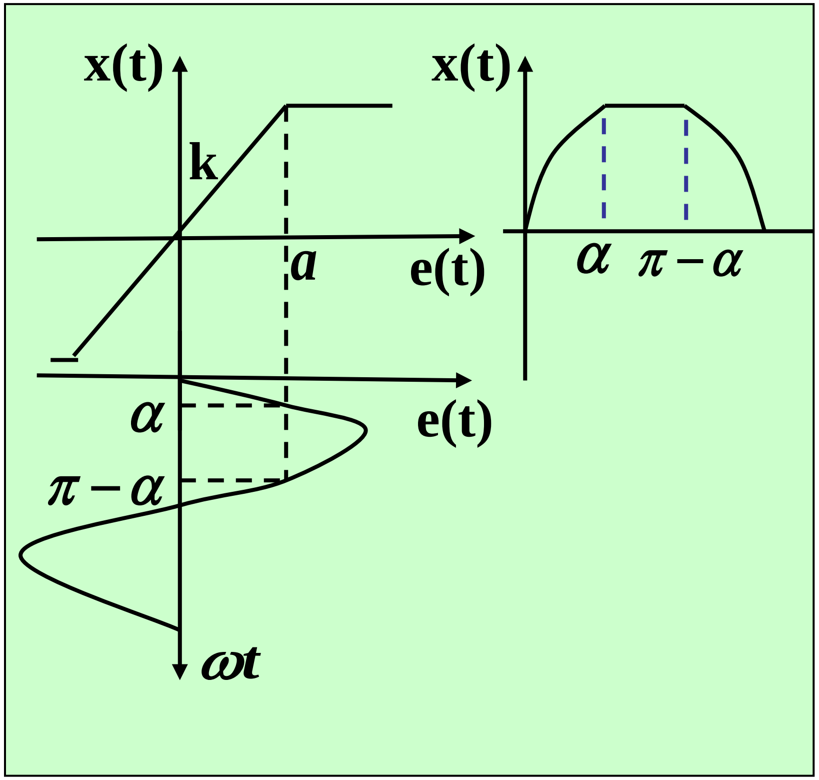
$$A\sin\alpha = a, \therefore \alpha = \sin^{-1}\frac{a}{A}$$

Because the output is odd,

$A_1 = 0$ (*single-valued odd-symmetric*)

$$\phi_1 = tg^{-1}\frac{A_1}{B_1} = 0$$

$$\boxed{N(A) = \frac{B_1}{A} + j\frac{A_1}{A} = \frac{B_1}{A}}$$



$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t \; d(\omega t)$$

$$= \frac{2}{\pi}\left[\int_0^{\alpha} KA\sin^2\omega t d(\omega t) + \int_{\alpha}^{\pi-\alpha} Ka\sin\omega t d(\omega t) + \int_{\pi-\alpha}^{\pi} KA\sin^2\omega t d(\omega t)\right]$$

$$= \frac{2}{\pi}KA\left[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function of saturation nonlinearity is:

$$N(A) = \frac{B_1}{A} = \frac{2}{\pi}K\left[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

***N*(*A*) is a *nonlinear real function* of input amplitude *A*.**
It can be regarded as a *variable gain amplifier*.

2. Dead-zone

Assume the input is $e(t) = A\sin\omega t$ **,**

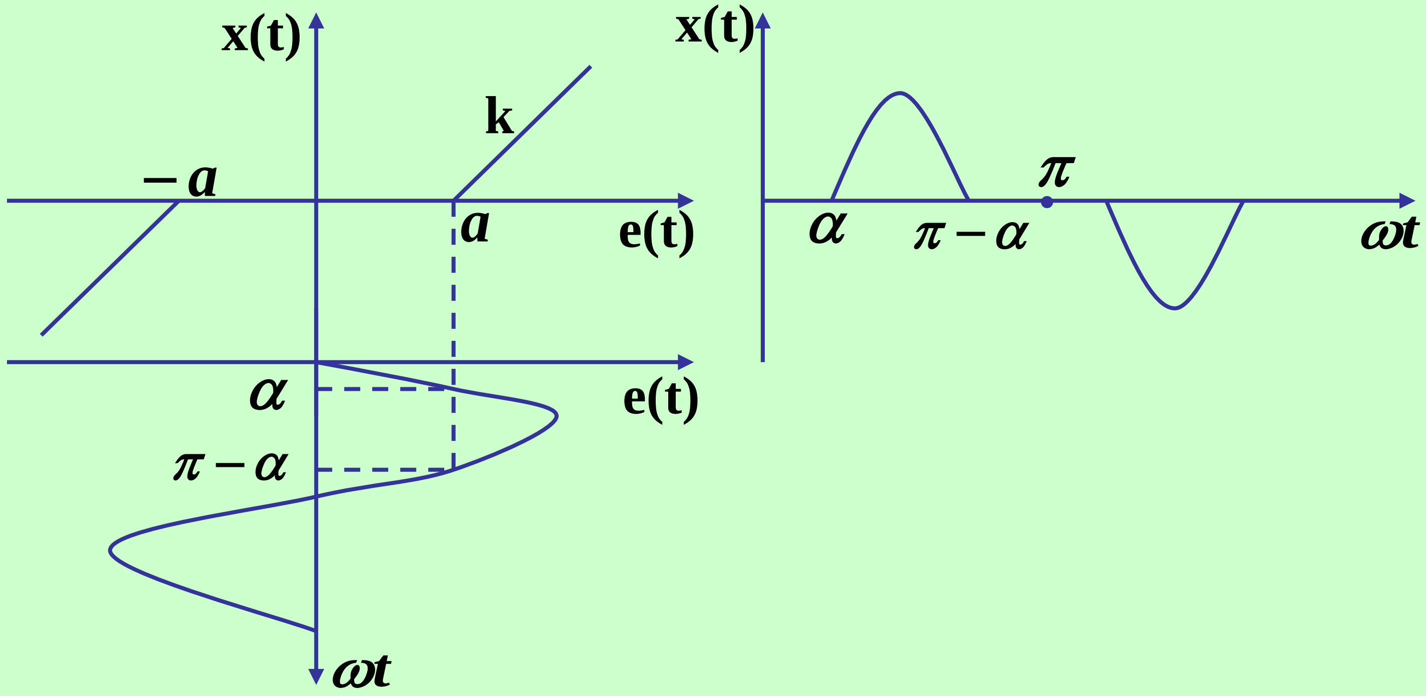


Fig8-11 input and output waveforms of dead zone nonlinearity

When $A > a$, the output of dead zone nonlinearity is

$$x(t) = \begin{cases} 0, & 0 \le \omega t \le \alpha \\ K(A\sin\omega t - a), & \alpha < \omega t \le \pi - \alpha \\ 0, & \pi - \alpha < \omega t \le \pi \end{cases}$$

where, $A\sin\alpha = a, \therefore \alpha = \sin^{-1}\dfrac{a}{A}$

The output is odd→$A_1 = 0$, $\phi_1 = 0$

$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t d(\omega t)$$

$$= \frac{2}{\pi}\int_{\alpha}^{\pi-\alpha} K\left(A\sin\omega t - a\right)\sin\omega t d(\omega t)$$

$$B_1 = \frac{2}{\pi} KA\left[\frac{\pi}{2} - \sin^{-1}\frac{a}{A} - \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function of dead zone nonlinearity is

$$N(A) = \frac{B_1}{A} = \frac{2}{\pi} K\left[\frac{\pi}{2} - \sin^{-1}\frac{a}{A} - \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

Note:

(1) When a/A is very small, i.e., the non-sensible zone is small, $N(A)$ is approximately equal to K;

(2) $N(A)$ decreases as a/A becomes larger

(3) $a/A = 1 \quad \Rightarrow \quad N(A) = 0$

3. Gap

Assume the input is $e(t) = A\sin\omega t$,

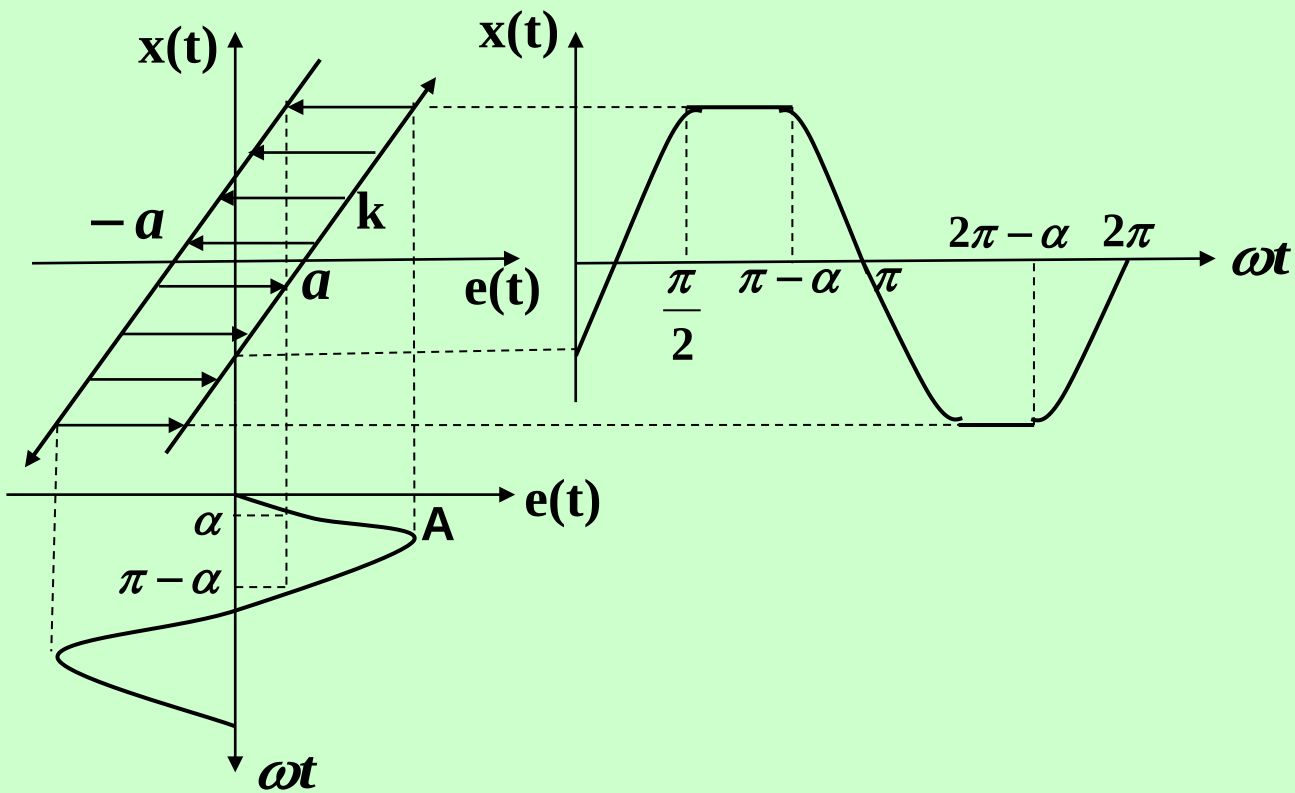


Fig 8-12 input and output waveforms of gap nonlinearity

From the mathematical description of gap nonlinearity, the *x*(*t*) is given by

$$x(t)=\begin{cases} K(A\sin\omega t-a), & 0\le \omega t<\dfrac{\pi}{2} \\ K(A-a), & \dfrac{\pi}{2}\le \omega t<\pi-\alpha \\ K(A\sin\omega t+a), & \pi-\alpha\le \omega t\le \pi \end{cases}$$

where, $A\sin(\pi-\alpha)=A-2a, \ \therefore \alpha=\sin^{-1}\dfrac{A-2a}{A}$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\cos\omega t\ d(\omega t)$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} K(A\sin\omega t - a)\cos\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\alpha} K(A - a)\cos\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\pi-\alpha}^{\pi} K(A\sin\omega t + a)\cos\omega t d(\omega t) = \frac{4KA}{\pi}\left[\left(\frac{a}{A}\right)^2 - \frac{a}{A}\right]$$

$$B_1 = \frac{2}{\pi}\int_0^{\pi} x(t)\sin\omega t\ d(\omega t) \quad = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} K(A\sin\omega t - a)\sin\omega t d(\omega t)$$

$$+ \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi-\alpha} K(A - a)\sin\omega t d(\omega t) + \frac{2}{\pi}\int_{\pi-\alpha}^{\pi} K(A\sin\omega t + a)\sin\omega t d(\omega t)$$

$$= \frac{KA}{\pi}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{A-2a}{A}\right) + \frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right]$$

So, we can obtain the describing function N(A) of gap nonlinearity as follows:

$$N(A)=\frac{B_1}{A}+j\frac{A_1}{A}$$

$$=\frac{K}{\pi}\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{A-2a}{A}\right)+\frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right]+j\frac{4K}{\pi}\left[\frac{a(a-A)}{A^2}\right]$$

$$=\left|N(A)\right|e^{j\varphi_1}$$

$$\left|N(A)\right|=\sqrt{\left[\frac{4K}{\pi}\left(\frac{a(a-A)}{A^2}\right)\right]^2+\left[\frac{K}{\pi}\left(\frac{\pi}{2}+\sin^{-1}\frac{A-2a}{A}+\frac{A-2a}{A}\sqrt{1-\left(\frac{A-2a}{A}\right)^2}\right)\right]^2}$$

$$\varphi_1=tg^{-1}\frac{4\dfrac{a(a-A)}{A^2}}{\left[\dfrac{\pi}{2}+\sin^{-1}\left(\dfrac{A-2a}{A}\right)+\dfrac{A-2a}{A}\sqrt{1-\left(\dfrac{A-2a}{A}\right)^2}\right]}$$

4. Relay

Assume the input is $e(t) = A\sin\omega t$,

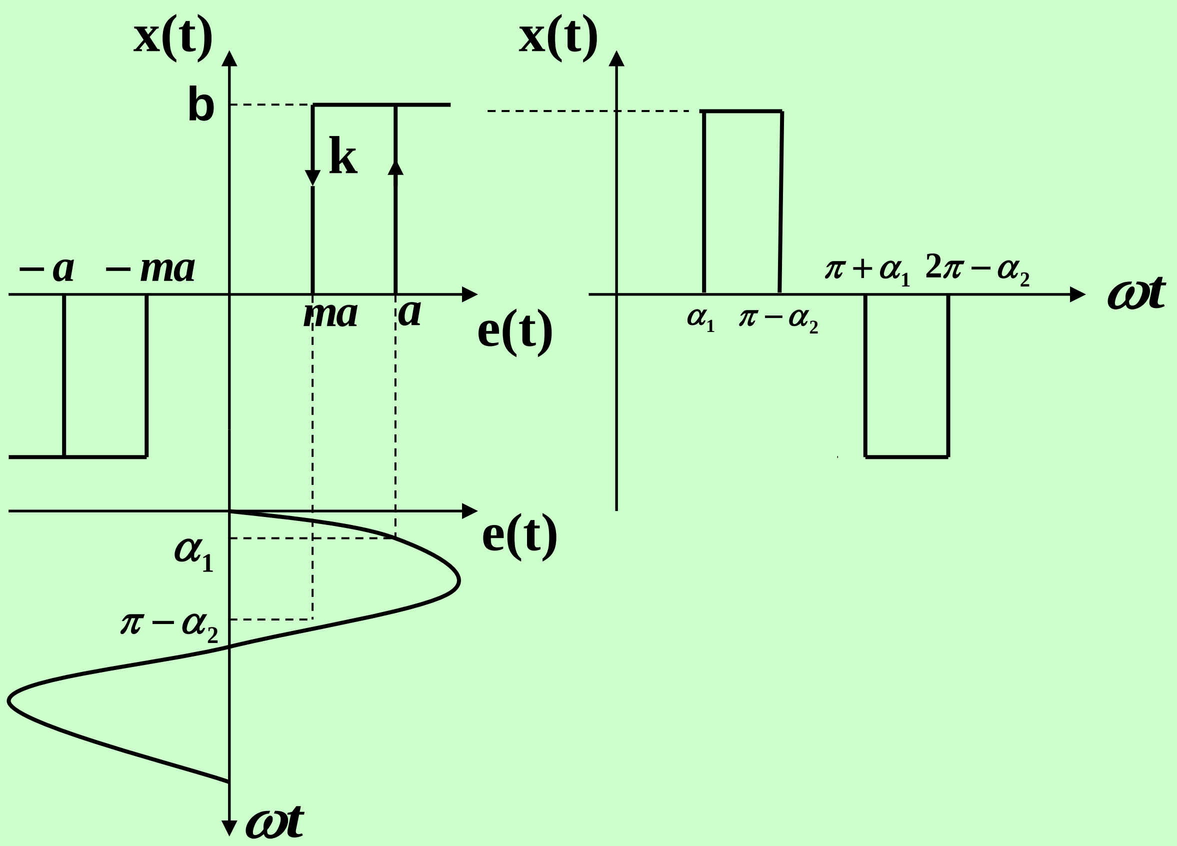


Fig 8-13 input and output waveforms of relay nonlinearity with dead zone and hysteresis ring

The output of the relay characteristic is :

$$x(t) = \begin{cases} 0, & 0 \le \omega t < \alpha_1 \\ b, & \alpha_1 \le \omega t < \pi - \alpha_2 \\ 0, & \pi - \alpha_2 \le \omega t \le \pi \end{cases}$$

where,

$$A \sin \alpha_1 = a, \ \therefore \alpha_1 = \sin^{-1} \frac{a}{A}$$

$$A \sin(\pi - \alpha_2) = ma, \ \therefore \alpha_2 = \sin^{-1} \frac{ma}{A}$$

$$A_1 = \frac{2}{\pi}\int_{\alpha_1}^{\pi-\alpha_2} b\cos\omega t d(\omega t)$$

$$= \frac{2b}{\pi}(\sin\alpha_2 - \sin\alpha_1) = \frac{2ab(m-1)}{\pi A}$$

$$B_1 = \frac{2}{\pi}\int_{\alpha_1}^{\pi-\alpha_2} b\sin\omega t d(\omega t)$$

$$= \frac{2b}{\pi}(\cos\alpha_2 + \cos\alpha_1) = \frac{2b}{\pi}\left[\sqrt{1-\left(\frac{ma}{A}\right)^2} + \sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$

The describing function N(A) of relay nonlinearity with dead zone and hysteresis ring is

$$N(A)=\left|N(A)\right|e^{j\phi_1}=\sqrt{\left(\frac{A_1}{A}\right)^2+\left(\frac{B_1}{A}\right)^2}\,e^{jtg^{-1}\frac{A_1}{B_1}}$$

$$\left|N(A)\right|=\frac{2b}{\pi A}\sqrt{2\left[1-m\left(\frac{a}{A}\right)^2+\sqrt{1+m^2\left(\frac{a}{A}\right)^4-(m^2+1)\left(\frac{a}{A}\right)^2}\right]}$$

$$\phi_1=tg^{-1}\frac{(m-1)\left(\frac{a}{A}\right)}{\sqrt{1-m^2\left(\frac{a}{A}\right)^2}+\sqrt{1-\left(\frac{a}{A}\right)^2}}$$

Corollary:

When a = 0, we can obtain the describing function of a *ideal relay nonlinearity*

$$N(A) = \frac{4b}{\pi A}$$

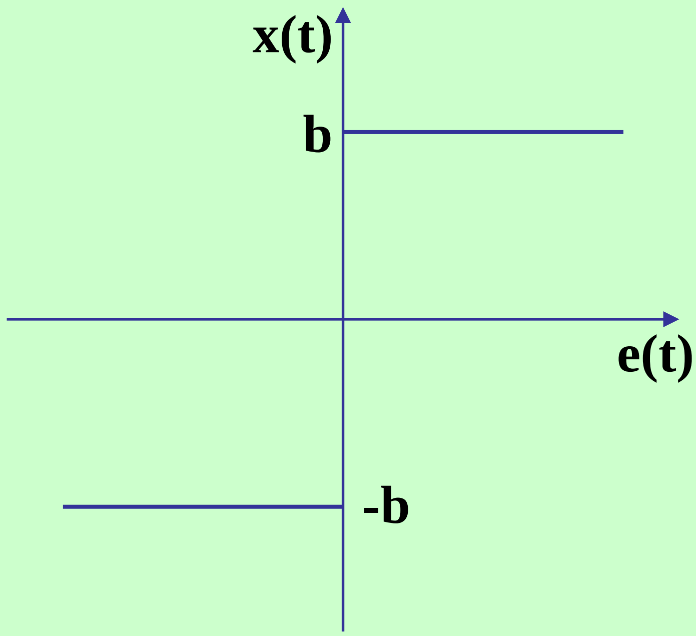


Fig 8-14 Ideal relay nonlinearity

When m = 1 and a≠0, we can obtain the describing function of a relay with dead zone

$$N(A)=\frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$

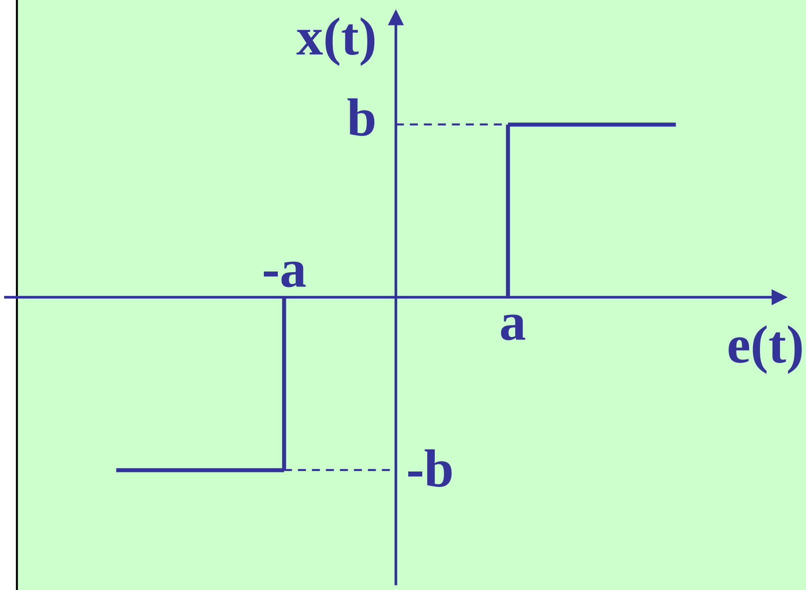


Fig 8-15 A relay with dead zone

When m = − 1, we can obtain the describing function of a relay with hysterics

$$N(A)=\frac{4b}{\pi A}e^{\,jtg^{-1}\frac{-\left(\frac{a}{A}\right)}{\sqrt{1-\left(\frac{a}{A}\right)^{2}}}}$$

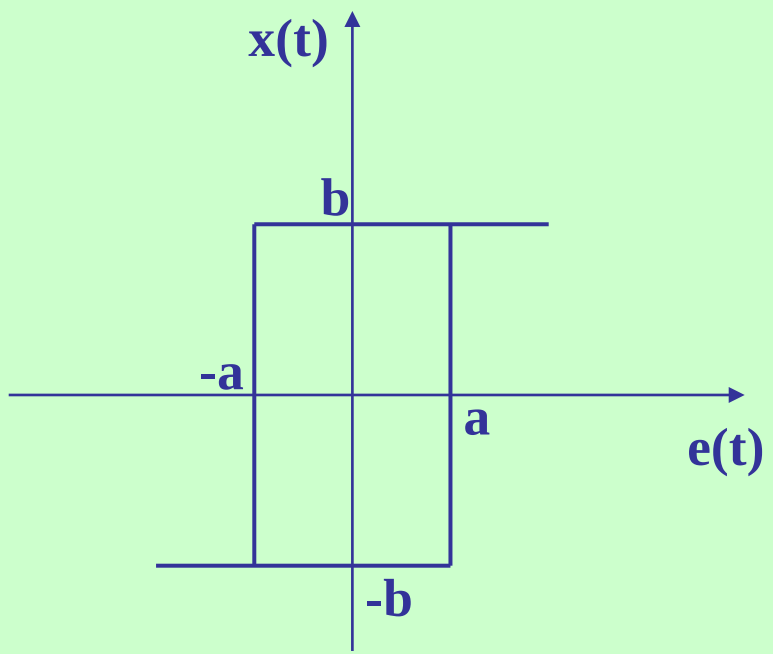


Fig 8-16 A relay with hysteresis

- Summary:
  - Nonlinear system analysis by the Describing Function Method:
    1. Draft of x-y, x-t, y-t;
    2. Decide the odd/even quality of y(t);
    3. Decide the symmetry property of y(t);
    4. Calculate A1, B1 - by integration;
    5. Calculate N(A).

- 2课时结束

8.3.3 Describing function of multiple nonlinearities

1. Cascaded nonlinear system

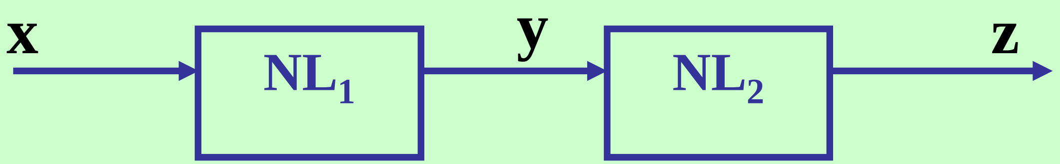
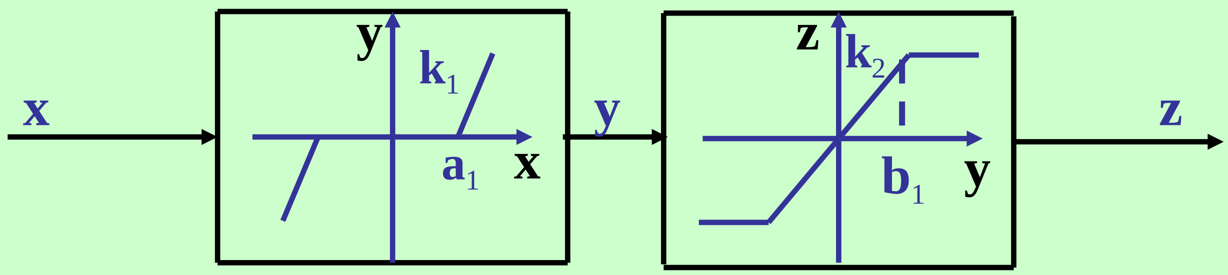


Fig 8-17 cascaded nonlinear system

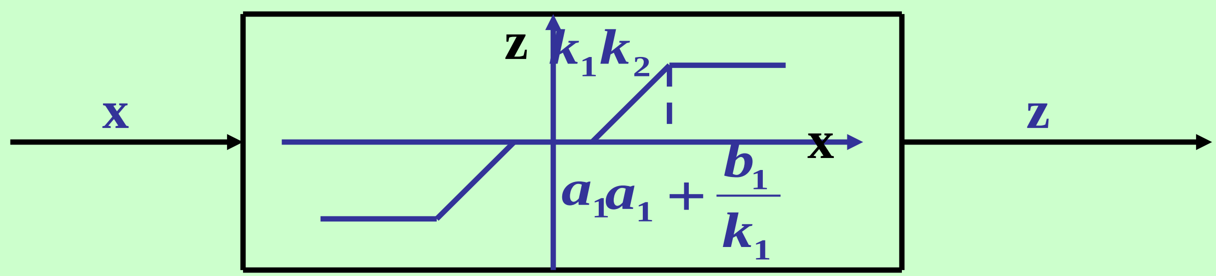
The describing function of a cascaded nonlinear system *is not equal to* the product of the two describing functions of each nonlinear elements.

$$N(A) \neq N_1(A) \cdot N_2(A)$$

assume NL_1 is a dead zone nonlinearity, NL_2 is a saturation nonlinearity, the composite nonlinearity of the cascaded nonlinear system is shown in Fig.8-18.



(a) cascaded nonlinear system



(b) composite nonlinearity

Fig 8-18 the cascaded nonlinear system and its composite nonlinearity

2. Parallel nonlinear system

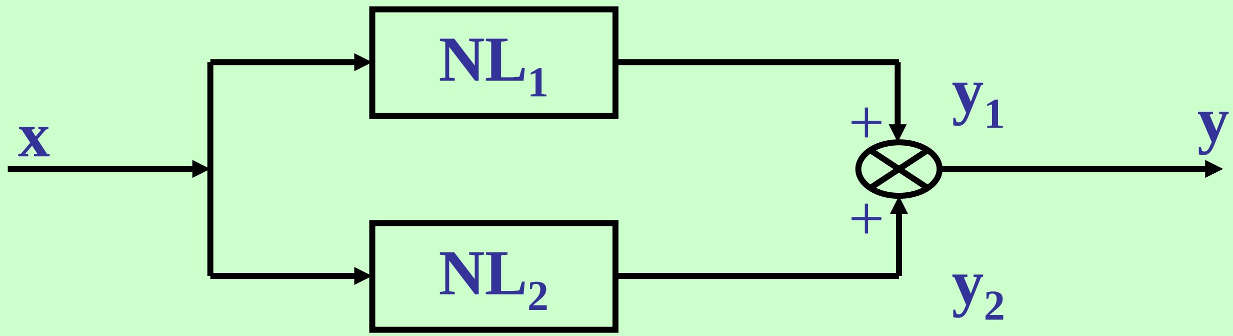


Fig 8-19 Parallel nonlinear system

According to the definition of describing function, the describing function N(A) of the output y and the input x is equal to the sum of two describing functions:

$$N(A) = N_1(A) + N_2(A)$$

[Example 1] **Obtain the describing function of the nonlinear system shown in the following figure.**

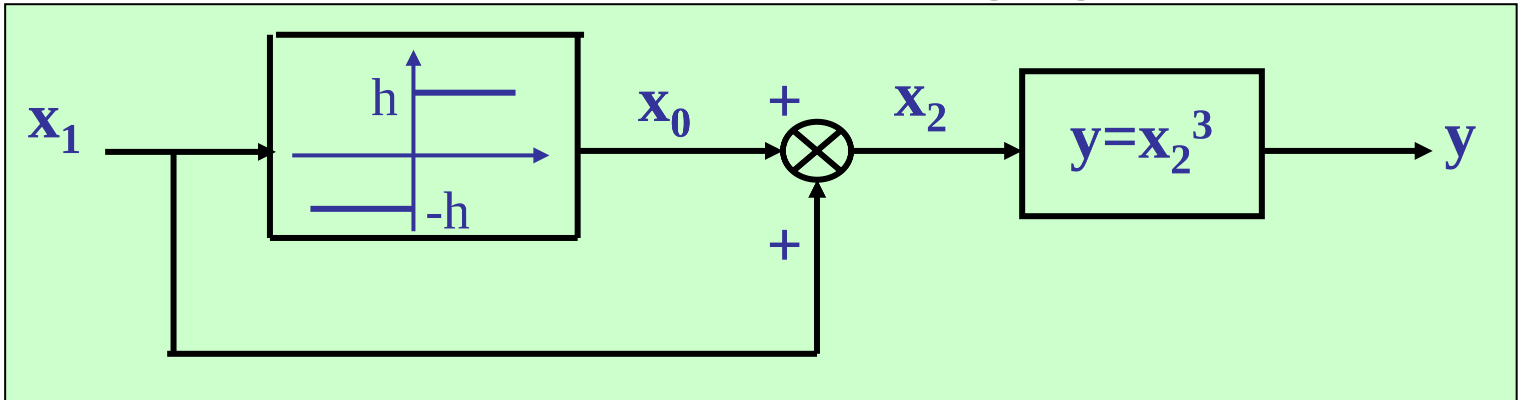


Fig 8-20 multiple nonlinearities

Solution: $y = x_2^3 = (x_0 + x_1)^3 = x_0^3 + 3x_0^2 x_1 + 3x_0 x_1^2 + x_1^3$

then $N(A) = N_1(A) + N_2(A) + N_3(A) + N_4(A)$

assume $x_1 = A\sin\omega t$

NL$_1$ is an ideal relay nonlinearity when a=0,

$\therefore \quad A_1 = 0$

Obtain $N_1(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} h^3 \sin\omega t d(\omega t) = \frac{4h^3}{\pi}$$

$$\therefore \quad N_1(A) = \frac{B_1}{A} = \frac{4h^3}{\pi A}$$

Obtain $N_2(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} 3h^2 A \sin\omega t \cdot \sin\omega t d(\omega t) = 3h^2 A$$

$$\therefore \quad N_2(A) = 3h^2$$

Obtain $N_3(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} 3hA^2 \sin^2 \omega t \cdot \sin \omega t d(\omega t) \;= \frac{8hA^2}{\pi}$$

$$\therefore \quad N_3(A) = \frac{8hA}{\pi}$$

Obtain $N_4(A)$:

$$B_1 = \frac{2}{\pi}\int_0^{\pi} A^3 \sin^3 \omega t \cdot \sin \omega t d(\omega t) \qquad \text{suppose } \theta = \omega t$$

$$= \frac{2}{\pi}\int_0^{\pi} -A^3 \sin^3 \theta \cdot d(\cos\theta)$$

$$= \frac{2A^3}{\pi}\left[\left(-\sin^3\theta\cos\theta\right)_0^{\pi} + \int_0^{\pi} 3\sin^2\theta\cos^2\theta d\theta\right] = \frac{3}{4}A^3$$

$$\therefore \quad N_4(A) = \frac{3}{4}A^2$$

Then, the describing function of the multiple nonlinearity is

$$N(A) = \frac{4h^3}{\pi A} + 3h^2 + \frac{8hA}{\pi} + \frac{3}{4}A^2$$

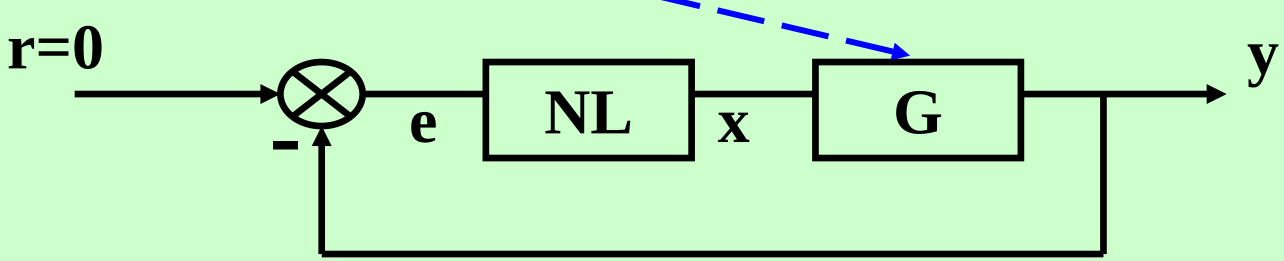
8.3.4 Analyze nonlinear system with describing function method

$$N(A) = \frac{\textit{Fundamental component}\ \text{of the output}\ x(t)}{\text{sinusoidal input}\ e(t)}$$

It can only reflect *partial dynamic characteristics* of the system.

If a *self-excited oscillation* occurs in the system, any input $x(t)$ can be regarded as a *sinusoidal* signal because the linear part is a *low-pass filter*.

The describing function method is then *applicable*.



Typical structure of nonlinear system

Assume the input of a nonlinear system is zero,
N(A) is the describing function of a nonlinear element,
analyze the *condition for self-exited oscillation*.

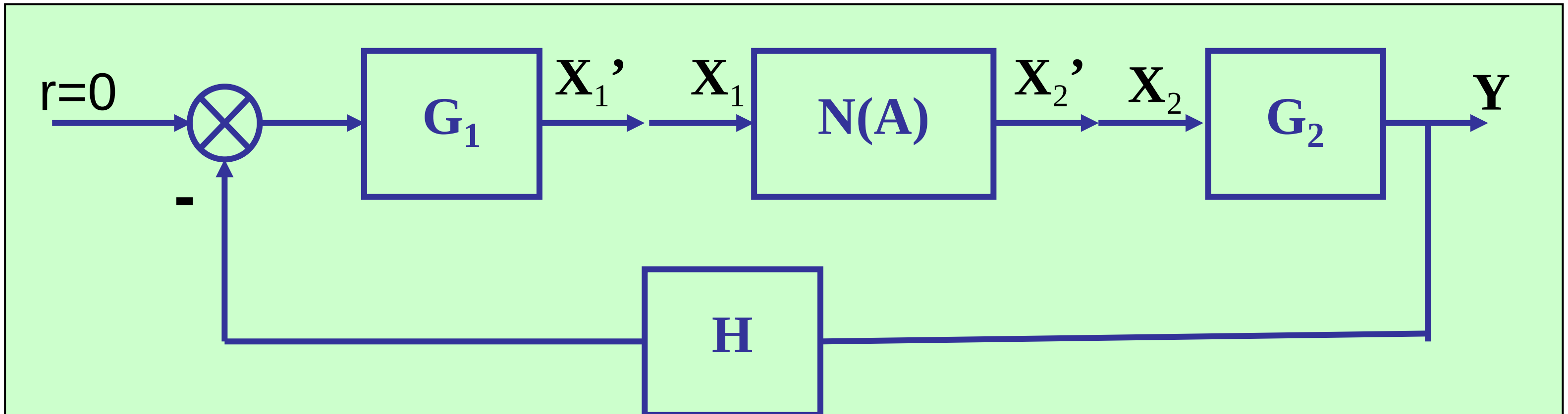


Fig. 8-21 nonlinear system

Assume $X_2 = A_2 \sin \omega t$

Then $X_1' = -\left|G_1(j\omega)G_2(j\omega)H(j\omega)\right| A_2 \sin(\omega t + \theta)$

where: $\boldsymbol{\theta = \angle G_1(j\omega) + \angle G_2(j\omega) + \angle H(j\omega)}$

Assume: $N(A)=|N(A)|e^{j\phi}$

then $x_2'(t)=-|N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)|A_2\sin(\omega t+\theta+\phi)$

Note: $x_2'(t)=x_2(t)$ $\Longleftrightarrow$ **The self-oscillation occurs.**

Considering we have $x_2=A_2\sin\omega t$ **,**

the ***condition of self-oscillation*** **is**

$$\begin{cases} |N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)|=1 \\ \theta+\phi=(2n+1)\pi \end{cases}$$

$$\begin{cases} |N(A)||G_1(j\omega)G_2(j\omega)H(j\omega)| = 1 \\ \theta + \phi = (2n+1)\pi \end{cases}$$

Suppose the transfer function of linear part satisfies

$$G(s) = G_1(s)G_2(s)H(s)$$

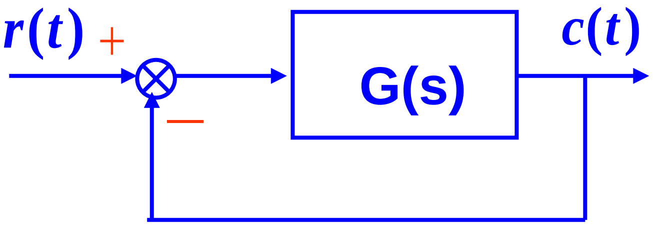
We can conclude that the *condition of self-excited oscillation* is

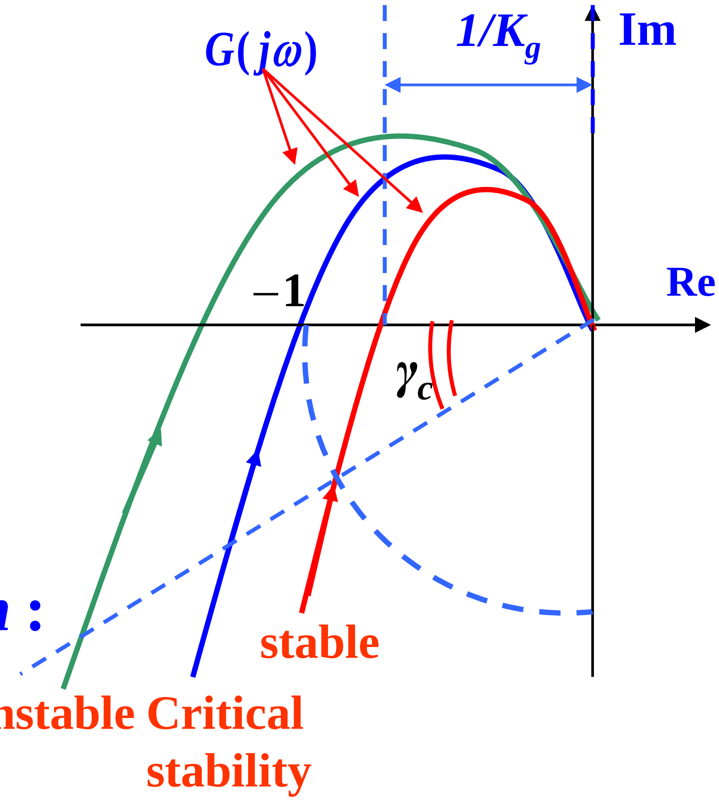
$$G(j\omega) = -\frac{1}{N(A)}$$

$$\text{or}\quad 1 + N(A)G(j\omega) = 0$$

Review of Nyquist criterion

For the linear system:





The characteristic equation of the system :

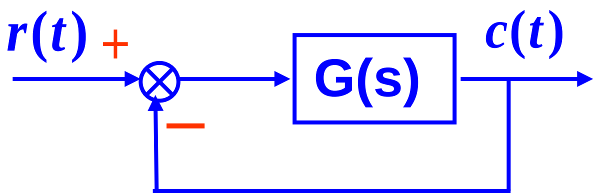
$$1 + G(j\omega) = 0$$

$$\Rightarrow G(j\omega) = -1 + j0$$

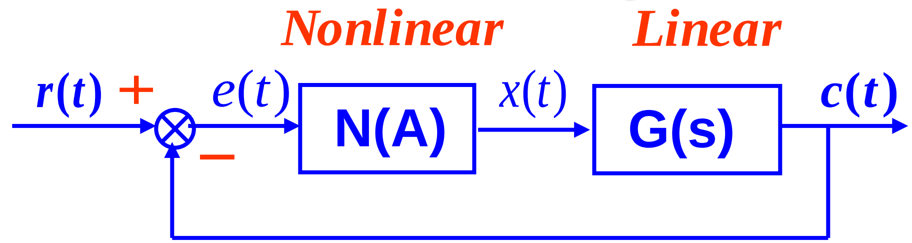
If $G(s)$ is a minimum phase transfer function, the necessary and sufficient condition of the stable system is:

$G(j\omega)$ *does not circle the point* $(-1, j0)$

Compare the nonlinear system with the linear system



Linear system



nonlinear system

Transfer function of the system:

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}$$

$$\phi(j\omega)=\frac{C(j\omega)}{R(j\omega)}=\frac{N(A)G(j\omega)}{1+N(A)G(j\omega)}$$

Characteristic equation:

$$1+G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-1$$

$$1+N(A)G(j\omega)=0$$

$$\Rightarrow G(j\omega)=-\frac{1}{N(A)}$$

In the $G(j\omega)$ plane

A point

A curve

Because the describing function N(A) actually is a linearized frequency response, we can expand the Nyquist criterion to the nonliear system:

Stability analysis of the nonlinear system

(For example the minimum phase system)

(1) $G(j\omega)$ don't circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear system is stable;

(2) $G(j\omega)$ circle the $-\dfrac{1}{N(A)}$ curve, the nonlinear system is unstable;

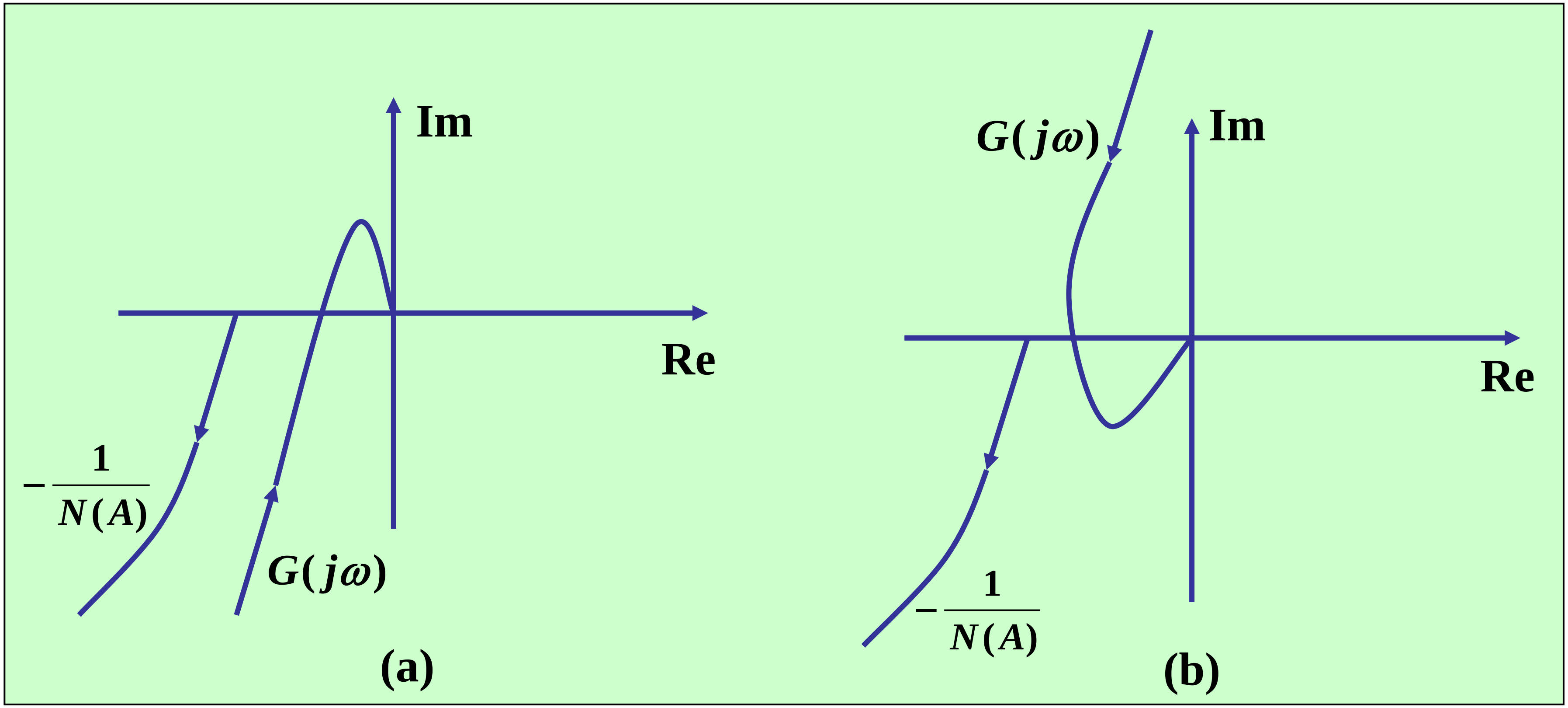
(3) $G(j\omega)$ intersect with the $-\dfrac{1}{N(A)}$ curve, there will be a self-oscillation in the nonlinear system .

compare with linear system

(1) **$G(j\omega)$ don' t circle the point $(-1, j\omega)$, the system is stable;**

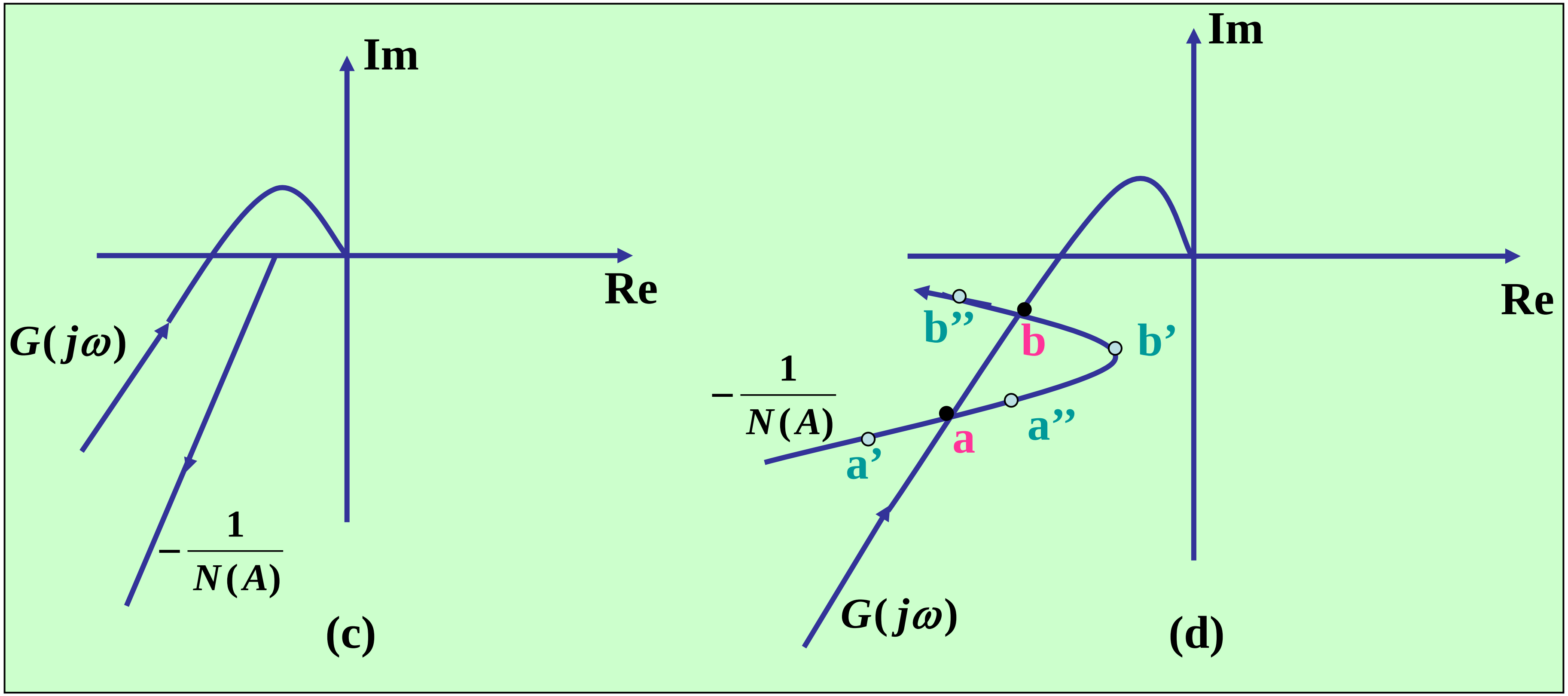
(2) **$G(j\omega)$ circle the point $(-1, j\omega)$, the system is unstable;**

(3) **$G(j\omega)$ intersect with the point $(-1, j\omega)$, the system is in the critical stability.**



$G(j\omega)$ **do not circle －1/N(A)**
(Stable)

$G(j\omega)$ **circle －1/N(A)**
(Unstable)



Unstable

$G(j\omega)$ **intersect with －1/N(A)**

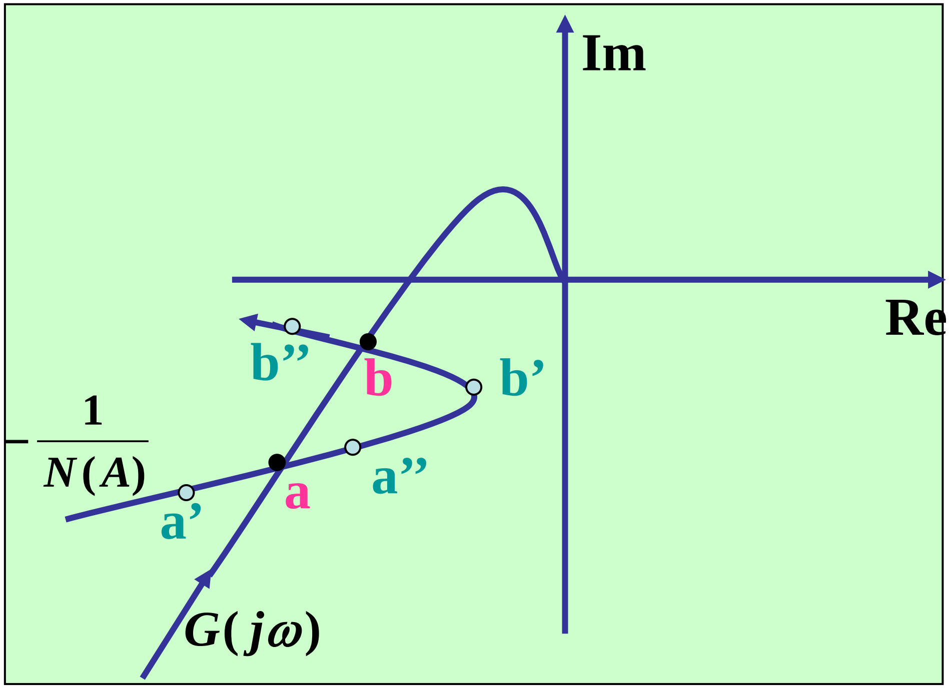
(Self-oscillation)

Analysis:

(1) Self-oscillation occurs at point a and point b.

(2) The self-oscillation at point a is unstable while the self-oscillation at point b is stable.

(3) There is only one stable self-oscillation in an actual physical system.



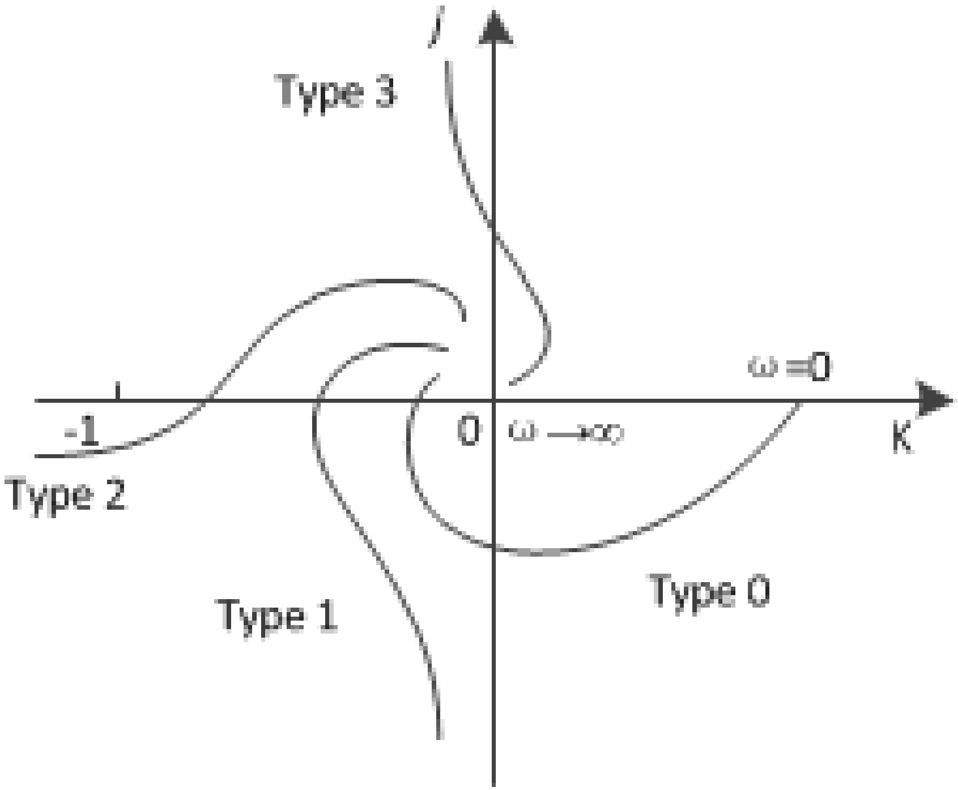
The ***amplitude*** and ***frequency*** of the self-oscillation can be obtained by solving

$$\left|G(j\omega)N(A)\right| = 1$$

$$\theta + \phi = -\pi$$

Additional Summary of G(s)

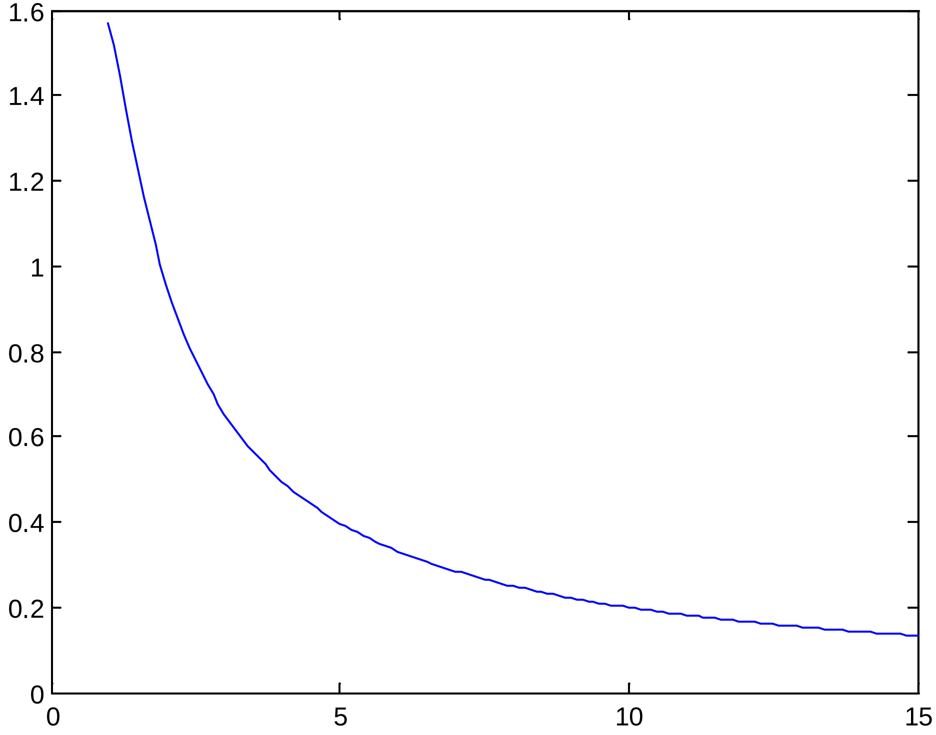
- **Linear system** $G(s) = \dfrac{K(\tau_1 s+1)\cdots(\tau_w s+1)}{s^v (T_1 s+1)\cdots(T_u s+1)}$



Additional Summary of N(A)

- **Saturation**

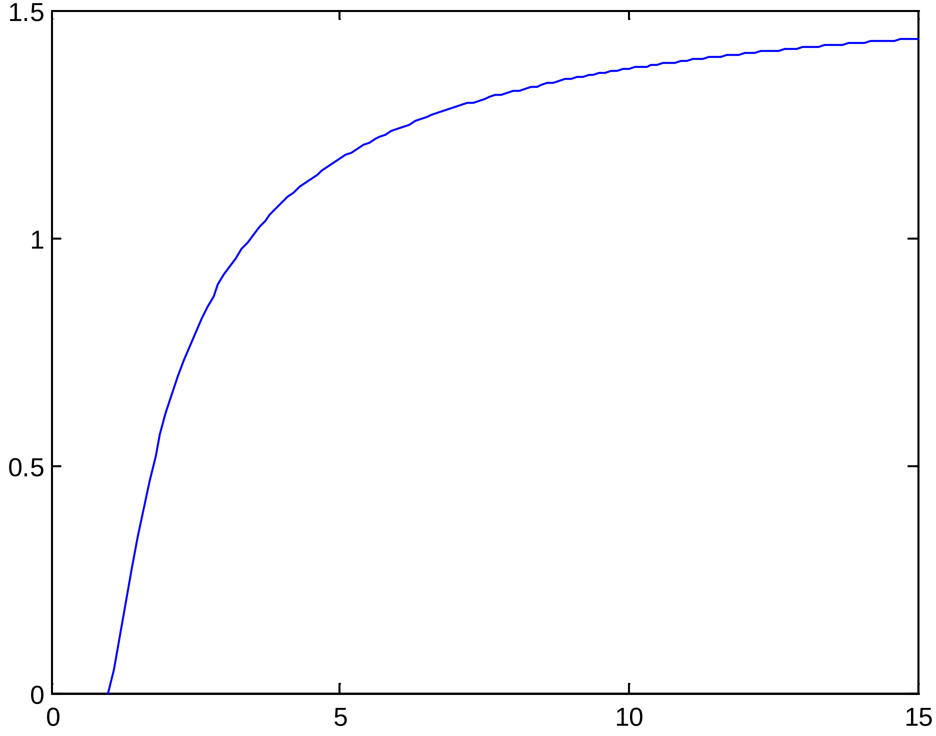
$$N(A)=\frac{2}{\pi}K\left[\sin^{-1}\frac{a}{A}+\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^{2}}\right]$$



Additional Summary of N(A)

- **Dead-zone**

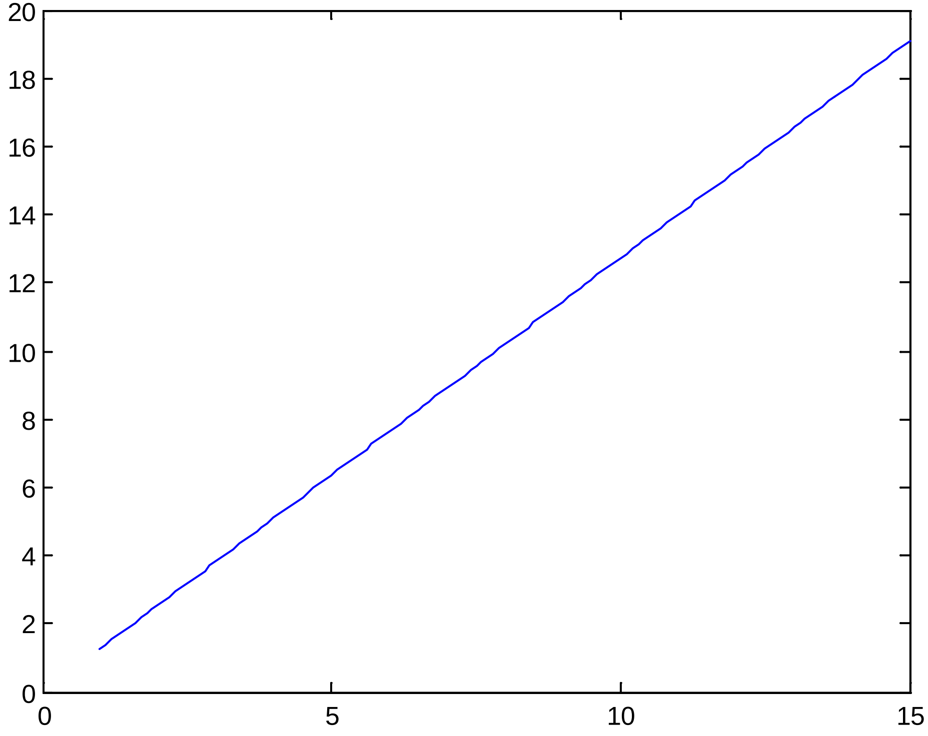
$$N(A)=\frac{2}{\pi}K\left[\frac{\pi}{2}-\sin^{-1}\frac{a}{A}-\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]$$



Additional Summary of N(A)

- **Ideal relay**

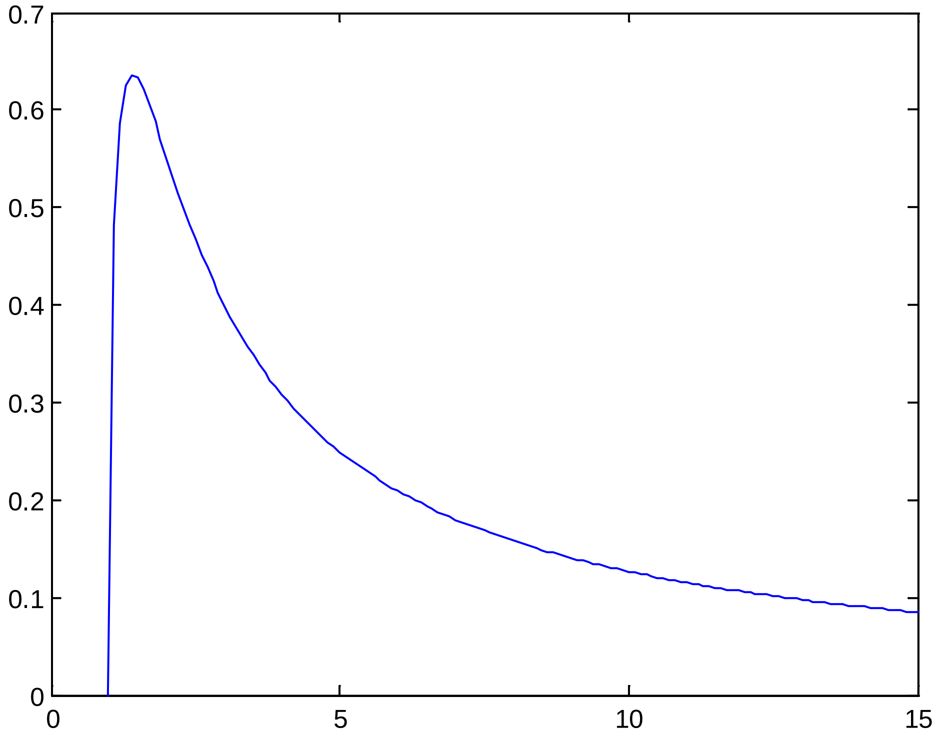
$$N(A)=\frac{4b}{\pi A}$$



Additional Summary of N(A)

- **Relay with dead zone**

$$N(A)=\frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$



Additional Summary of N(A)

- **Analysis:** $$-\frac{1}{N(A)} = -\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}$$

Try to analyze its *extreme value*.

[Example 2] A relay control system structure is shown in Fig. 8-23. Suppose a = 1, b = 3.

(1) Is there a self-oscillation in the system? If there is, obtain the amplitude and frequency of the oscillation.

(2) How to adjust the parameters if you want to eliminate the self-oscillation?

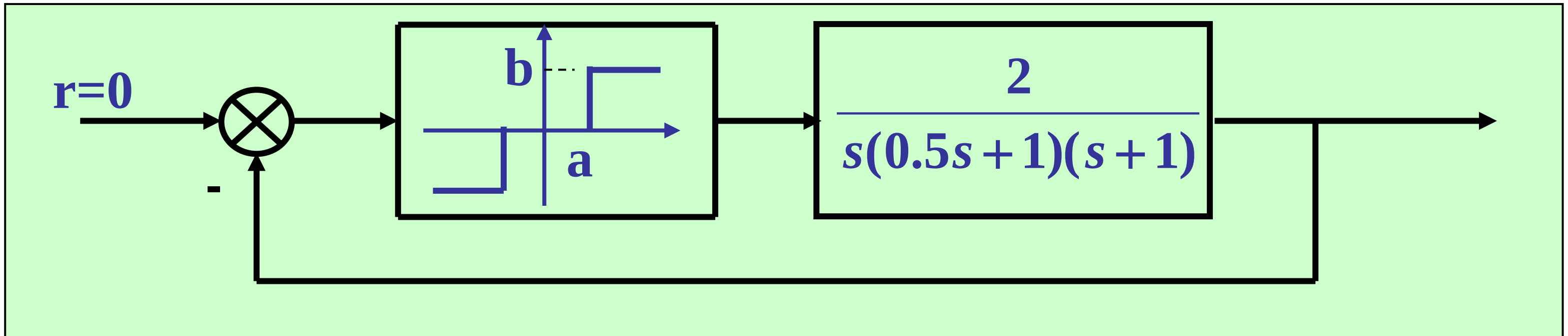


Fig.8-23 relay control sytem

Solution: the describing function of a relay nonlinearity with dead zone is

$$N(A)=\frac{4b}{\pi A}\sqrt{1-\left(\frac{a}{A}\right)^2}$$

$$\therefore\ -\frac{1}{N(A)} = -\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}$$

$$when \qquad A=a, -\frac{1}{N(A)} \to -\infty$$

$$when \qquad A \to \infty, -\frac{1}{N(A)} \to -\infty$$

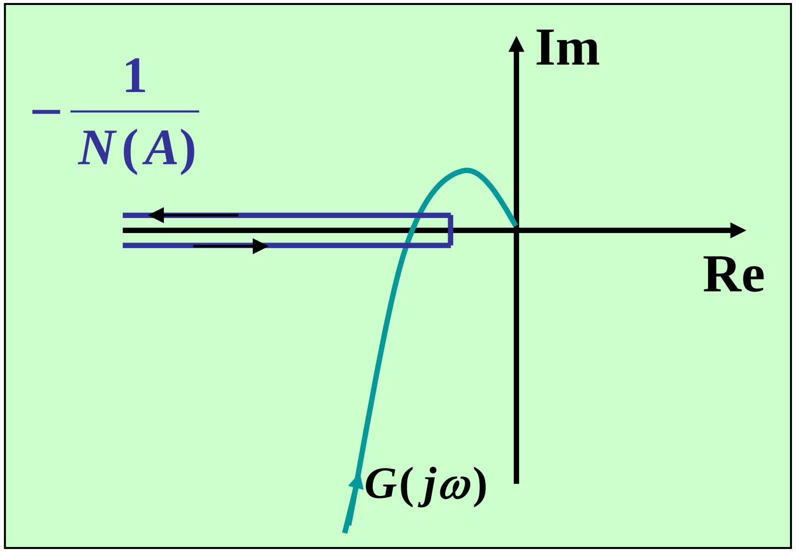
There is an *extreme value* of function $-\frac{1}{N(A)}$ **on the real axis.**

$$\frac{d}{dA}\left(-\frac{1}{N(A)}\right) = 0 \Rightarrow 1-2\left(\frac{a}{A}\right)^2 = 0 \qquad \therefore\ A=\sqrt{2}a$$

Substitute a = 1, b = 3 into above equations, we have

$$A=\sqrt{2}$$

$$-\frac{1}{N(A)}\bigg|_{A=\sqrt{2}} = -\frac{\pi}{6} \approx -0.52$$



$$G(s) = \frac{2}{s(0.5s+1)(s+1)}$$

$$G(j\omega) = -\frac{3\omega}{\omega(0.25\omega^4+1.25\omega^2+1)} - j\frac{2(1-0.5\omega^2)}{\omega(0.25\omega^4+1.25\omega^2+1)}$$

Set the imaginary part to zero, we have $\omega = \sqrt{2}$

Substituting $\omega = \sqrt{2}$ **into the real part, we have**

$$\mathrm{Re}\,G(j\omega)\Big|_{\omega=\sqrt{2}} = -\frac{1}{1.5} \approx -0.66$$

$$\text{Let} \quad -\frac{1}{N(A)} = \frac{-\pi A}{12\sqrt{1-\left(\frac{1}{A}\right)^2}} = -\frac{1}{1.5}$$

We can obtain two amplitude: $\underbrace{A_1 = 1.11}_{\text{(not exist in the reality)}}$, $A_2 = 2.3$

There is a self-oscillation in the system with the amplitude 2.3 and the frequency $\sqrt{2}$.

(2) To eliminate the self-oscillation, let

$$-\frac{1}{N(A)} = -\left.\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}\right|_{A=\sqrt{2}a} \le -\frac{1}{1.5}$$

We can get the ratio of relay parameters

$$\frac{b}{a} \le \frac{1.5\pi}{2} \approx 2.36$$

Adjust the ratio of a and b to b = 2a, we can eliminate the self-oscillation.

[Example 3] **A multi-loop control system is shown in Fig. 8-25. If $G_1(s)=1$, the natural oscillation frequency and damping ratio of system is $\omega_n = 2$ and $\varsigma = 1$ when working on the linear area of the saturation. If we suppose $G_1(s) = 1 + \frac{1}{8s}$, try to find the minimal ratio T_1/T_2 when the system is stable.**

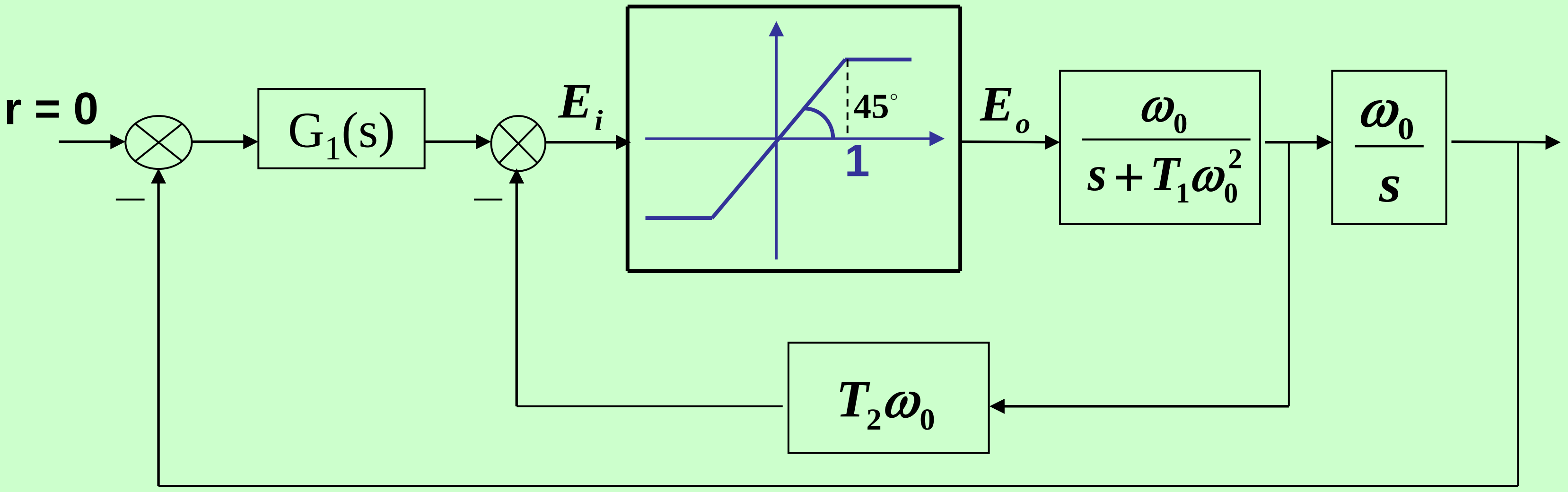


Fig.8-25 multi-loop control system

Solution: When $G_1(s) = 1$, the closed-loop transfer function of the inner loop is:

$$G_{内}(s)=\frac{\dfrac{\omega_0}{s+T_1\omega_0^2}}{1+\dfrac{T_2\omega_0^2}{s+T_1\omega_0^2}}=\frac{\omega_0}{s+(T_1+T_2)\omega_0^2}$$

The closed-loop transfer function of the whole system is

$$G_B(s)=\frac{\dfrac{\omega_0^2}{s[s+(T_1+T_2)\omega_0^2]}}{1+\dfrac{\omega_0^2}{s[s+(T_1+T_2)\omega_0^2]}}=\frac{\omega_0^2}{s^2+(T_1+T_2)\omega_0^2 s+\omega_0^2}$$

$$\therefore\ \begin{cases}\omega_0=\omega_n=2\\ T_1+T_2=\varsigma=1\end{cases}$$

when $G_1(s) = 1 + \dfrac{1}{8s}$ **, the transfer function of inner loop is**

$$G_{内}(s) = \frac{\dfrac{\omega_0}{s + T_1\omega_0^2} N(A)}{1 + \dfrac{T_2\omega_0^2}{s + T_1\omega_0^2} N(A)} = \frac{\omega_0 N(A)}{s + T_1\omega_0^2 + T_2\omega_0^2 N(A)}$$

The open-loop transfer function of the whole system

$$G(s) = G_1(s)G_{内}(s)\frac{\omega_0}{s} = \frac{\omega_0^2(1 + \dfrac{1}{8s})N(A)}{s[s + T_1\omega_0^2 + T_2\omega_0^2 N(A)]}$$

From the characteristic equation of closed-loop system $1 + G(s) = 0$**, we have**

$$s^2 + T_1\omega_0^2 s + T_2\omega_0^2 sN(A) + \omega_0^2(1 + \frac{1}{8s})N(A) = 0$$

Substituting $\omega_0 = 2$ to the above equation, we can obtain

$$8s^3 + 32T_1 s^2 + (32T_2 s^2 + 32s + 4)N(A) = 0$$

$$\therefore \quad -\frac{1}{N(A)} = \frac{8T_2 s^2 + 8s + 1}{2s^3 + 8T_1 s^2} = \frac{8T_2 s^2 + 8s + 1}{s^2(2s + 8T_1)}$$

Note that the describing function of saturation nonlinearity is

$$N(A) = \frac{2}{\pi}\left[\sin^{-1}\frac{1}{A} + \frac{1}{A}\sqrt{1 - \left(\frac{1}{A}\right)^2}\right]$$

So the function of $-\dfrac{1}{N(A)}$ **is given by**

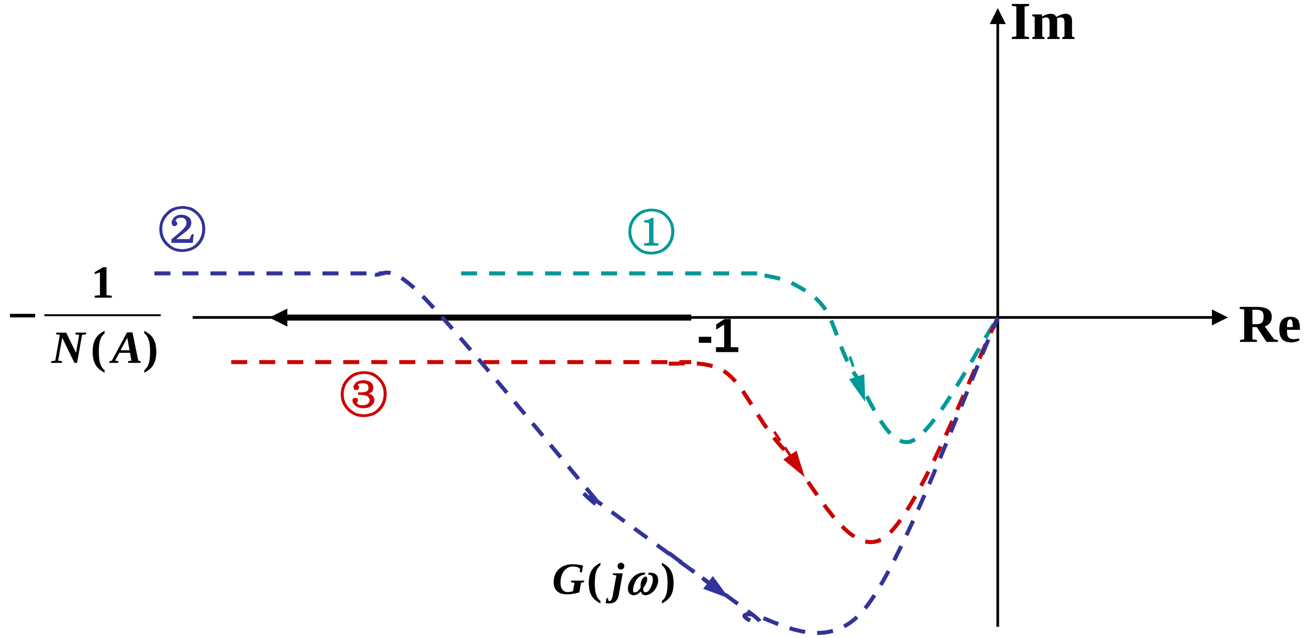
$$-\frac{1}{N(A)} = -\frac{\pi}{2}\frac{1}{\left[\sin^{-1}\frac{1}{A}+\frac{1}{A}\sqrt{1-\left(\frac{1}{A}\right)^2}\right]}$$

The open-loop transfer function of the whole system is

$$G(s) = -\frac{1}{N(A)} = \frac{8T_2s^2+8s+1}{s^2(2s+8T_1)}$$

$$\therefore\ \ G(j\omega) = \frac{-8T_2\omega^2+j8\omega+1}{-\omega^2(j2\omega+8T_1)} = u(\omega)+jv(\omega)$$

$$G(j\omega) = -\frac{4T_1 - 32\omega^2 T_1 T_2 + 8\omega^2}{2\omega^2(\omega^2 + 16T_1^2)} + j\frac{32T_1 - 1 + 8T_2\omega^2}{2\omega(\omega^2 + 16T_1^2)}$$



To guarantee the system to be stable, we should choose parameters T_1 and T_2 which make $G(j\omega)$ has no intersection with the negative real axis. So we choose curve ③

Let $$v(\omega)=\frac{32T_1-1+8T_2\omega^2}{2\omega(\omega^2+16T_1^2)}=0$$

Then we have $$\omega=\sqrt{\frac{1-32T_1}{8T_2}}$$

If $T_1>\dfrac{1}{32}$ **holds,** $G(j\omega)$ **has no intersection with the negative real axis.**

On the other hand, $T_1+T_2=1$

$$\therefore\ T_2<\frac{31}{32} \qquad \therefore\ \frac{T_1}{T_2}\geq\frac{1}{31}$$

So, the minimal ratio of $\dfrac{T_1}{T_2}$ **to guarantee the system to be stable is** $\dfrac{T_1}{T_2}=\dfrac{1}{31}$

Additional Summary of N(A)

- **Relay with dead zone** $-\frac{1}{N(A)} = -\frac{\pi A}{4b\sqrt{1-\left(\frac{a}{A}\right)^2}}$

$$when \qquad A = a, -\frac{1}{N(A)} \to -\infty$$

$$when \qquad A \to \infty, -\frac{1}{N(A)} \to -\infty$$

There is an *extreme value* of function $-\frac{1}{N(A)}$ **on the real axis.**

$$\frac{d}{dA}\left(-\frac{1}{N(A)}\right) = 0 \Rightarrow 1-2\left(\frac{a}{A}\right)^2 = 0 \qquad \therefore\ A = \sqrt{2}a$$

- Homework

Review:

- What is the Describing Function?
  - Input e(t) and Output x(t)
  - Assumptions 1, 2, 3

- How to solve the Describing Function problem?
  - A_0, A_1, B_1

注：傅里叶（**Fourier**）变换的概念和原理（参考《复变函数与积分变换》）

1804 年，傅里叶首次提出“在有限区间上由任意图形定义的任意函数都可以表示为单纯的正弦和余弦之和”，但没有给出严格的证明

1829 年，法国数学家狄利克雷（**Dirichlet**）证明了以下定理，为傅里叶级数奠定了理论基础：

设 $f_T(t)$是以 **T** 为周期的实值函数，且在$\left[-\frac{T}{2},\frac{T}{2}\right]$上满足狄利克雷条件，即 $f_T(t)$在$\left[-\frac{T}{2},\frac{T}{2}\right]$上满足：

（**1**）连续或只有有限个第一类间断点（左右极限均存在）；

（**2**）只有有限个极限点。

则在 $f_T(t)$的连续点处有

$$f_T(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t\right),$$

其中$\omega_0=\dfrac{2\pi}{T}$,

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f_T(t)\cos n\omega_0 tdt\quad(n=0,1,2,\ldots)$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f_T(t)\sin n\omega_0 tdt\quad(n=1,2,\ldots)$$

注：三角函数满足正交特性：

$$\int_{t_0}^{t_0+T}\cos(n\omega t)\sin(m\omega t)dt=0$$

$$\int_{t_0}^{t_0+T}\sin n\omega t\sin m\omega tdt=\begin{cases}\frac{T}{2} & (m=n)\\ 0 & (m\neq n)\end{cases}$$

$$\int_{t_0}^{t_0+T}\cos n\omega t\cos m\omega tdt=\begin{cases}\frac{T}{2} & (m=n)\\ 0 & (m\neq n)\end{cases}$$

对于非线性系统的输出：$x(t)$ 如果是奇对称函数，则 $X\left(e(t)\right)$ 也是一个奇对称函数，则

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} X\left(e(\omega t)\right) d\omega t = 0$$

$x(t)$ 如果是奇对称单值函数，则当输入为正弦信号时，非线性环节的输出 $X\left(e(t)\right)$ 也是一个与输入同频率（满足静态特性的假设）的奇对称的周期函数（类似于正弦函数），根据三角函数的正交特性有：

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} X\left(e(\omega t)\right)\cos\omega t dt = 0$$

$$b_1 = \frac{2}{T}\int_{-T/2}^{T/2} X\left(e(\omega t)\right)\sin\omega t dt \neq 0$$

如果 $x(t)$ 如果不是一个奇对称单值函数，则当输入为正弦信号时，非线性环节的输出 $X\left(e(t)\right)$ 是一个与输入同频率（满足静态特性的假设）的周期函数但不满足奇对称性，则：

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} X\left(e(\omega t)\right)\cos\omega t dt \neq 0$$

$$b_1 = \frac{2}{T}\int_{-T/2}^{T/2} X\left(e(\omega t)\right)\sin\omega t dt \neq 0$$

另外由于$X\left(e(\omega t)\right)$，$\sin\omega t$和$\cos\omega t$都是周期为 T 的周期信号，所以可以等周期的求解积分，因此有

$$a_1=\frac{2}{T}\int_{-T/2}^{T/2}X\left(e(\omega t)\right)\cos\omega t dt=\frac{2}{T}\int_0^T X\left(e(\omega t)\right)\cos\omega t dt$$

$$b_1=\frac{2}{T}\int_{-T/2}^{T/2}X\left(e(\omega t)\right)\sin\omega t dt=\frac{2}{T}\int_0^T X\left(e(\omega t)\right)\sin\omega t dt$$

$$\boxed{A_1=\frac{1}{\pi}\int_0^{2\pi}x(t)\cos\omega t\ d(\omega t)}$$

$$\boxed{B_1=\frac{1}{\pi}\int_0^{2\pi}x(t)\sin\omega t\ d(\omega t)}$$

$$\boxed{\begin{aligned}x_1(t)&=A_1\cos\omega t+B_1\sin\omega t\\&=x_1\sin(\omega t+\varphi_1)\end{aligned}}$$

$$\boxed{\begin{aligned}N(A)&=\frac{\sqrt{A_1^2+B_1^2}}{A}e^{j\,arctg\frac{A_1}{B_1}}\\&=\frac{B_1}{A}+j\frac{A_1}{A}\end{aligned}}$$