9.4 the solution of linear time-invariant system state equation

Mathematical Representation

Quantitative(定量) Analysis

Qualitative(定性) Analysis

Solving the dynamic mathematical model equations;
The solutions analysis.

<u>Transfer function</u>

<u>State Space – modern control theory</u>

Matrices Operation (State transition Matrix)

Dynamic Response

State equation (Model) $\Longrightarrow$ Dynamic analysis (solve state equation)

Insuring the existence and uniqueness of the solution: the elements in A and B are bounded.

9.4.1 Solution of Linear Time-invariant Continual System

1. The solution of homogeneous state equation(齐次状态方程):

$\dot{x} = Ax$ is homogeneous state equation, and there are general 2 solutions:

- Power Series Method (幂级数法)

Assume the solution of above equation is a vector power series (幂级数) of t

$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{t} + \boldsymbol{b}_2\boldsymbol{t}^2 + \cdots + \boldsymbol{b}_k\boldsymbol{t}^k + \cdots$$

$\boldsymbol{x}, \boldsymbol{b}_0, \boldsymbol{b}_1, \cdots \boldsymbol{b}_k \cdots$ are n dimensional vectors.

Calculate the derivative of above equation:

$$\dot{x} = b_1 + 2b_2 t + \cdots + k b_k t^{k-1} + \cdots = A(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots)$$

Assume the coefficients with the same power are uniform.

$$b_1 = A b_0$$

$$b_2 = \frac{1}{2} A b_1 = \frac{1}{2} A^2 b_0$$

$$b_3 = \frac{1}{3} A b_2 = \frac{1}{3 \times 2} A^3 b_0$$

$$\vdots$$

$$b_k = \frac{1}{k} A b_{k-1} = \frac{1}{k!} A^k b_0$$

$$\vdots$$

$$\because \qquad x(0) = b_0$$

$$\therefore \qquad x(t) = (I + At + \frac{1}{2} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots) x(0)$$

Define: $$e^{At} = I + At + \frac{1}{2} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$x(t) = e^{At} x(0)$$

e^{At} —— Matrix exponential function, called state transition matrix: Φ(t).

- Laplace transformation for $\dot{x} = Ax$

$$sx(s) - x(0) = Ax(s)$$

$$(Is - A)x(s) = x(0)$$

$$x(s) = (Is - A)^{-1}x(0)$$

Inverse Laplace Transformation:

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0)$$

Compare with the power series method:

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$\Uparrow$$

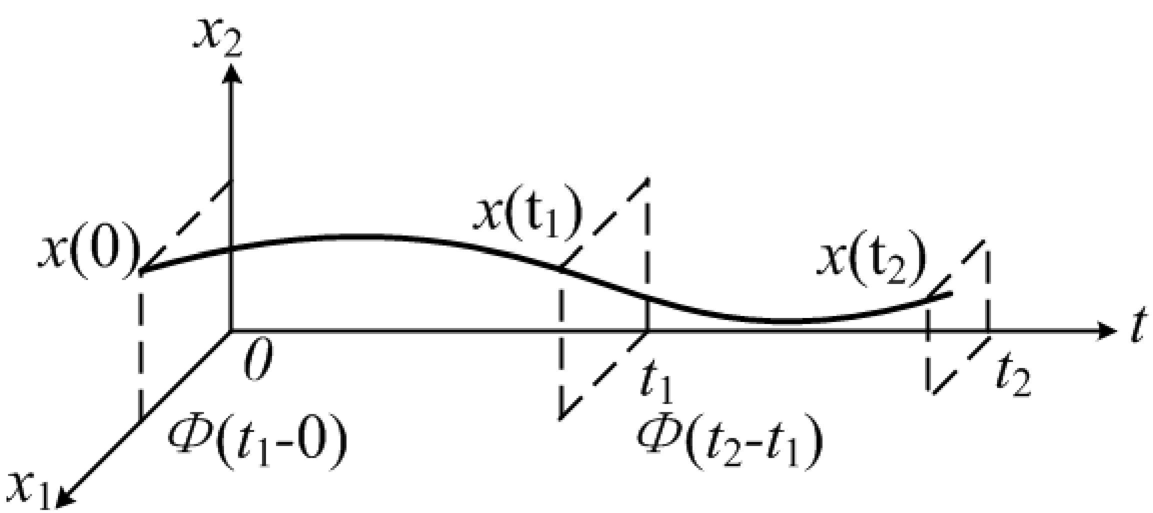
The closed form (闭合形式) (analytic form 解析形式) of the state transition matrix, which is convergent (收敛的).

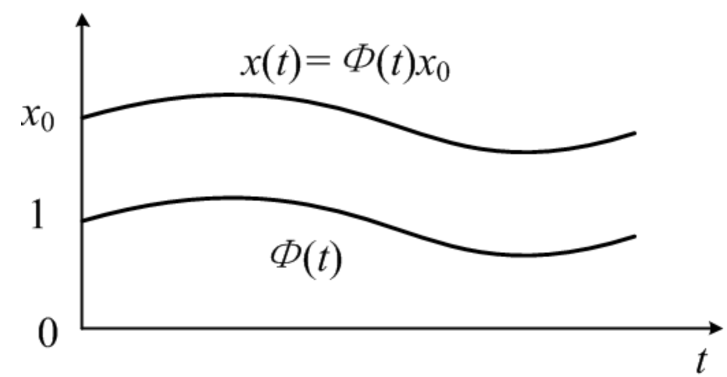
solve homogeneous state equation $\Longrightarrow$ calculate state transition matrix

Discussion:

$$\dot{x} = Ax \Rightarrow x(t) = e^{At}x(0) \quad \text{OR} \quad x(t) = e^{A(t-t_0)}x(t_0)$$

- The solution of homogeneous state equation describe a freedom motion (自由运动) of the system without the input *u*(*t*), which is the transition of the initial state only based on the state transition matrix $\mathbf{e}^{\mathbf{A(t-t0)}}$.





2. The solution of non-homogeneous state equation:

Give the non-homogeneous state equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \qquad \boldsymbol{x}(t) \in R^n, \boldsymbol{u}(t) \in R^r, A \in R^{n\times n}, B \in R^{n\times r}$$

(1) Direct method（Integral method 积分法） $\dot{x}(t) - Ax(t) = Bu(t)$

left multiply e^{-At} simultaneously: $\underline{\underline{e^{-At}[\dot{x}(t) - Ax(t)]}} = e^{-At}Bu(t)$

$$\boxed{\frac{d}{dt}[e^{-At}x(t)]}$$

$$\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu(t)$$

$$e^{-At}x(t) - x(0) = \int_o^t e^{-A\tau}Bu(\tau)d\tau$$

$$\boldsymbol{x}(t)\big|_{t=0} = \boldsymbol{x}(0)$$

$$x(t) = e^{At}x(0) + \int_o^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \Phi(t)x(0) + \int_o^t \Phi(t-\tau)Bu(\tau)d\tau$$

response of zero initial condition

Response of input u(t)

(2) Laplace transformation method

$$sX(s)-x(0)=AX(s)+Bu(s)$$

$$(sI-A)X(s)=x(0)+Bu(s)$$

$$X(s)=(sI-A)^{-1}x(0)+(sI-A)^{-1}Bu(s)$$

then $$x(t)=L^{-1}[(sI-A)^{-1}x(0)]+L^{-1}[(sI-A)^{-1}Bu(s)]$$

(from $e^{At}=L^{-1}[(sI-A)^{-1}]$, we have)

$$x(t)=e^{At}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\Phi(t)x(0)+\int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

Discussion:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \Longrightarrow x(t) = \Phi(t-t_0)x(t_0) + \int_{t_0}^{t} \Phi(t-\tau)Bu(\tau)d\tau$$

- The solution of non-homogeneous state equation is composed by two parts
  - the freedom motion of the initial state: $\boldsymbol{\Phi}(t\text{-}t_0)x(t_0)$, which is called zero-input response;
  - the controlled motion by the input: $\int_0^t \boldsymbol{\Phi}(t\text{-}\tau)\boldsymbol{Bu}(\tau)\mathrm{d}\tau$, which is called zero-state response.

9.4.2 Properties of state transition matrix

$$e^{At}=I+At+\frac{1}{2}A^2t^2+\cdots+\frac{1}{k!}A^kt^k+\cdots=\sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k$$

1. Initial state: $\Phi(0)=I$
2. $\dot{\Phi}(t)=A\Phi(t)=\Phi(t)A \qquad \dot{\Phi}(0)=A$
3. Linear relationship: $\Phi(t_1\pm t_2)=\Phi(t_1)\Phi(\pm t_2)=\Phi(\pm t_2)\Phi(t_1)$
4. Reversibility: $\Phi^{-1}(t)=\Phi(-t),\ \Phi^{-1}(-t)=\Phi(t)$
5. $x(t_2)=\Phi(t_2-t_1)x(t_1)$
6. Segmentation: $\Phi(t_2-t_0)=\Phi(t_2-t_1)\Phi(t_1-t_0)$
7. $[\Phi(t)]^k=\Phi(kt)$
8. if $AB=BA,\quad e^{(A+B)t}=e^{At}e^{Bt}=e^{Bt}e^{At};$

   if $AB\neq BA,\quad e^{(A+B)t}\neq e^{At}e^{Bt}\neq e^{Bt}e^{At}$

9. if $\Phi(t)$ is state transition matrix of $\dot{x}(t) = Ax(t)$, the newly state tranition matrix after non-singular transform $x = P\overline{x}$ is:

$$\overline{\Phi}(t) = P^{-1}e^{At}P$$

10. Two common state transition matrices

If A is n-order Diagonal Matrix,

$$A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \qquad \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

If A is m-order Jordan Matrix,

$$A = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}_{m\times m}, \quad \Phi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \dfrac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & te^{\lambda t} \\ 0 & \cdots & 0 & e^{\lambda t} \end{bmatrix}$$

9.4.3 Calculation of matrix transition function e^{At}

- Method One: Direct method (matrix power function)

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

For any constant matrix A and limited t, the above infinite series is convergent.

- Method Two: Linear transform method (diagonal form method and Jordan form method)

If the matrix A can be transited to the diagonal form, e^{At} can be given as:

$$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

In the above equation, P is the non-singular linear transform matrix for A.

Similarly, if matrix A can be transformed to Jordan form, e^{At} can be given as:

$$e^{At} = Se^{Jt}S^{-1}$$

- Method Three: Laplace transform method

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

For e^{At}, it is essential to calculate the inverse of (sI-A). Generally, the Recursive Algorithm (递推算法) can be used when the order of system matrix A is high.

Ex.9-13 Consider following system matrix, try to find the proper eAt by linear transform method and Laplace transform method.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Solution:

Linear transform method: the eigenvalues of A is 0 and -2 (λ_1=0, λ_2=-2), thus, the transform matrix P is:

$$P = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

from $$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

we have, $$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} e^{o} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

<u>Laplace transform method:</u>

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}$$

We have:

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}$$

thus:

$$\mathrm{e}^{\mathrm{At}} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 1 & \dfrac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Ex.9-14 **find the state transition matrix Φ(t) and its inverse of following linear time-invariant system**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\Phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Then calculate the inverse of state transition matrix $\Phi^{-1}(t)$.

According to $\Phi^{-1}(t)=\Phi(-t)$, the inverse of state transition matrix is:

$$\Phi^{-1}(t) = e^{-At} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

Ex.9-15 try to find the time response relationship of following system, in which, the input $u(t)$=**1**(t), the unit step function at t=0.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$ (according to Ex.9-14)

$$x(t) = e^{At}x(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1(t)\,d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero: **x**(0)=**0**, **x**(t) can be simplified as:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Ex.9-16 Assume the dynamic equation is: $\ddot{y}+(a+b)\dot{y}+aby=\dot{u}+cu$

With a, b and c are real constants. Try to find:

(1) the state space equation of the system;

(2) the state transition matrix Φ(t).

Solution:

(1)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{s+c}{s^2+(a+b)s+ab}=\frac{s+c}{(s+a)(s+b)}=\frac{c-a}{b-a}\cdot\frac{1}{s+a}+\frac{c-b}{a-b}\cdot\frac{1}{s+b}$$

$$\begin{cases}\dot{x}=\begin{bmatrix}-a & 0\\ 0 & -b\end{bmatrix}x+\begin{bmatrix}1\\1\end{bmatrix}u\\ y=\begin{bmatrix}\frac{c-a}{b-a} & \frac{c-b}{a-b}\end{bmatrix}x\end{cases}$$

(2)

$$\Phi(t)=L^{-1}[(sI-A)^{-1}]$$

$$=L^{-1}\left[\begin{pmatrix}s+a & 0\\ 0 & s+b\end{pmatrix}^{-1}\right]=L^{-1}\begin{bmatrix}\frac{1}{s+a} & 0\\ 0 & \frac{1}{s+b}\end{bmatrix}=\begin{bmatrix}e^{-at} & 0\\ 0 & e^{-bt}\end{bmatrix}$$

9.4.4 Establishing and solution of linear discrete system state space representation

➢ **1. The state space description of discrete-time linear system:**

The discrete-time linear time-variant system:

$$X(k+1) = A(k)x(k) + B(k)u(k)$$
$$Y(k) = C(k)x(k) + D(k)u(k)$$

The discrete-time linear time-invariant system:

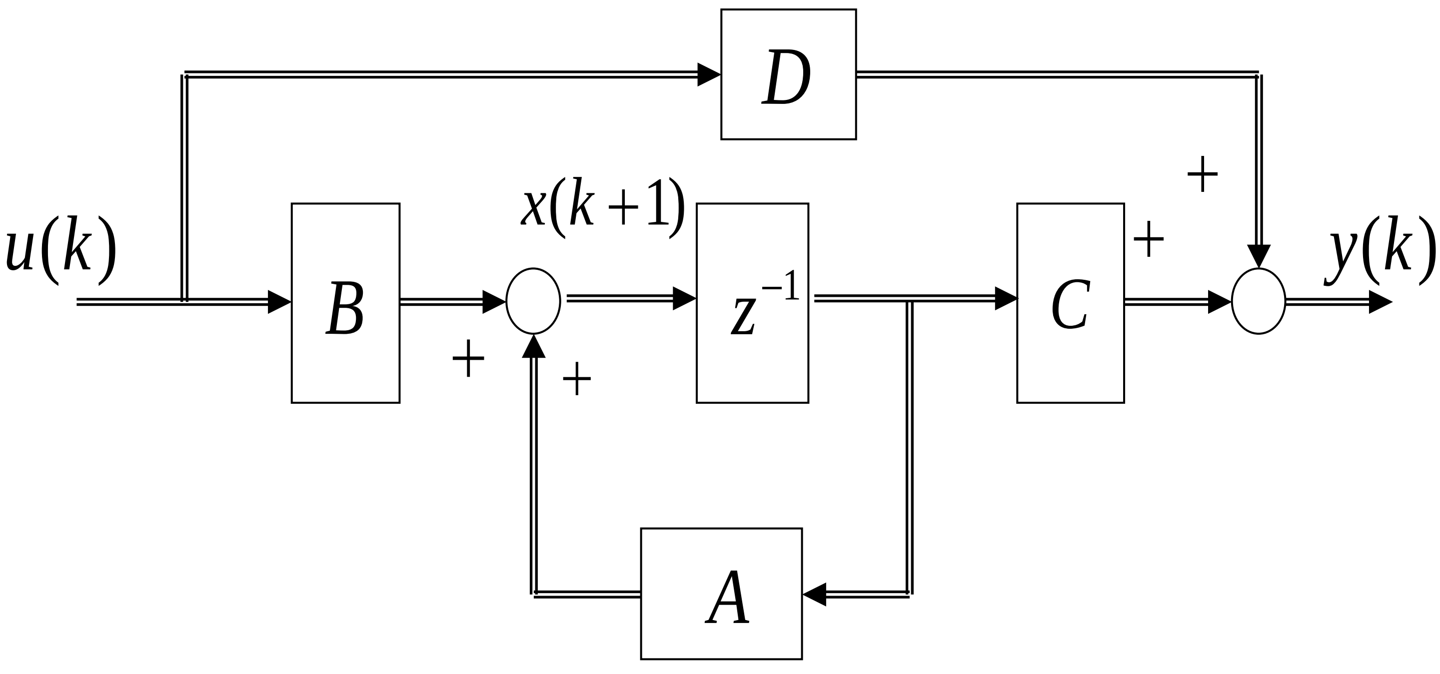
$$X(k+1) = Ax(k) + Bu(k)$$
$$Y(k) = Cx(k) + Du(k)$$

A_{nxn}: system matrix; B_{nxp}: input matrix

C_{qxn}: output matrix; D_{qxp}: transfer matrix

State space representation of linear time-invariant MIMO discrete system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$



➢ 3. Discretization of continual system state space expression

The solution of the time-invariant continual state equation $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ under the input u(t)

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \Phi(t-t_0)x(t_0) + \int_{t_0}^t \Phi(t-\tau)Bu(\tau)d\tau$$

assume $t_0 = kT, \quad x(t_0) = x(kT) = x(k)$

$$t = (k+1)T, \quad x(t) = x[(k+1)T] = x(k+1)$$

at $t \in [k, k+1], \quad$ u(k)=u(k+1) is constant

$$\boldsymbol{x}(k+1) = \Phi[(k+1)T - kT]\boldsymbol{x}(k) + \boxed{\left(\int_{kT}^{(k+1)T} \Phi[(k+1)T-\tau]Bd\tau\right)}\boldsymbol{u}(k)$$

$$G(T) = \int_{kT}^{(k+1)T} \Phi[(k+1)T-\tau]Bd\tau$$

Variable replacement $(k+1)T - \tau = \tau'$

then

$$G(T) = \int_0^T \Phi(\tau)Bd\tau$$

$$\boldsymbol{x}(k+1)=\Phi(T)\boldsymbol{x}(k)+\left(\int_0^T \Phi(\tau)Bd\tau\right)\boldsymbol{u}(k)$$

State equation of discrete system is:

$$\boldsymbol{x}(k+1)=\Phi(T)\boldsymbol{x}(k)+G(T)\boldsymbol{u}(k)$$

The relationship between $\Phi(T)$ and state transition matrix $\Phi(t)$ of continual system:

$$\Phi(T)=\Phi(t)\Big|_{t=T}$$

The output equation of discrete system is:

$$y(k)=C\boldsymbol{x}(k)+D\mathrm{u}(k)$$

Ex.9-17 **find the discrete state equation with T=1s from following continual system**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution: from Ex.9-15, the state transition matrix Φ(t) of above continual system is:

$$\Phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\Phi(T) = \Phi(t)\big|_{t=T=1} = \begin{bmatrix} 0.6004 & 0.2325 \\ -0.4651 & -0.0972 \end{bmatrix}$$

$$G(T) = \int_0^T \Phi(\tau) B d\tau = \int_0^T \begin{pmatrix} e^{-\tau} - e^{-2\tau} \\ -e^{-\tau} + 2e^{-2\tau} \end{pmatrix} d\tau = \begin{bmatrix} 1/2 - e^{-T} + 1/2e^{-2T} \\ e^{-T} - e^{-2T} \end{bmatrix}$$

$$G(T)\big|_{T=1} = \begin{bmatrix} 0.1998 \\ 0.2325 \end{bmatrix}$$

$$\boldsymbol{x}(k+1) = \Phi(T)\boldsymbol{x}(k) + G(T)\boldsymbol{u}(k)$$