DATA 605: Computational Mathematics

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Contents

Library

```
library(dplyr)
library(e1071)
library(ggplot2)
library(MASS)
library(readr)
```

Final Examination: Business Analytics and Data Science

Instructions:

You are required to complete this take-home final examination by the end of the last week of class. Your solutions should be uploaded in **pdf** format as a knitted document (with graphs, content, commentary, etc. in the pdf). This project will showcase your ability to apply the concepts learned throughout the course.

The dataset you will use for this examination is provided as retail data.csv, which contains the following variables:

- Product_ID: Unique identifier for each product.
- Sales: Simulated sales numbers (in dollars).
- Inventory_Levels: Inventory levels for each product.
- Lead_Time_Days: The lead time in days for each product.
- Price: The price of each product.
- Seasonality_Index: An index representing seasonality.

Problem 1:

Business Risk and Revenue Modeling

Context: You are a data scientist working for a retail chain that models sales, inventory levels, and the impact of pricing and seasonality on revenue. Your task is to analyze various distributions that can describe sales variability and forecast potential revenue.

Data Load

```
retail_df <- read_csv("synthetic_retail_data.csv")
```

Part 1:

Empirical and Theoretical Analysis of Distributions (5 Points)

Task:

1.

Generate and Analyze Distributions:

- **X ~ Sales:** Consider the Sales variable from the dataset. Assume it follows a Gamma distribution and estimate its shape and scale parameters using the fitdistr function from the MASS package.
- **Y ~ Inventory Levels:** Assume that the sum of inventory levels across similar products follows a Lognormal distribution. Estimate the parameters for this distribution.
- **Z ~ Lead Time:** Assume that Lead_Time_Days follows a Normal distribution. Estimate the mean and standard deviation. Calculate Empirical Expected Value and Variance:

Calculate the empirical mean and variance for all three variables. Compare these empirical values with the theoretical values derived from the estimated distribution parameters.

```
head(retail_df)
```

```
## # A tibble: 6 x 6
##   Product_ID Sales Inventory_Levels Lead_Time_Days Price Seasonality_Index
##        <dbl> <dbl>            <dbl>          <dbl> <dbl>             <dbl>
## 1          1  158.             367.           6.31  18.8             1.18
## 2          2  279.             427.           5.80  26.1             0.857
## 3          3  699.             408.           3.07  22.4             0.699
## 4          4 1832.             392.           3.53  27.1             0.698
## 5          5  460.             448.          10.8   18.3             0.841
## 6          6 1693.             547.          10.1   23.5             1.13
```

```
glimpse(retail_df)
```

```
## Rows: 200
## Columns: 6
## $ Product_ID        <dbl> 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 1~
## $ Sales             <dbl> 158.43952, 278.99020, 698.85868, 1832.39467, 459.703~
## $ Inventory_Levels  <dbl> 367.4421, 426.6512, 407.6394, 392.3912, 448.3120, 54~
## $ Lead_Time_Days    <dbl> 6.314587, 5.800673, 3.071936, 3.534253, 10.802241, 1~
## $ Price             <dbl> 18.795197, 26.089636, 22.399985, 27.092013, 18.30782~
## $ Seasonality_Index <dbl> 1.1839497, 0.8573051, 0.6986774, 0.6975404, 0.840725~
```

```
summary(retail_df)
```

```
##    Product_ID         Sales         Inventory_Levels Lead_Time_Days
##  Min.   :  1.00   Min.   :  25.57   Min.   : 67.35   Min.   : 0.491
##  1st Qu.: 50.75   1st Qu.: 284.42   1st Qu.:376.51   1st Qu.: 5.291
##  Median :100.50   Median : 533.54   Median :483.72   Median : 6.765
##  Mean   :100.50   Mean   : 636.92   Mean   :488.55   Mean   : 6.834
##  3rd Qu.:150.25   3rd Qu.: 867.58   3rd Qu.:600.42   3rd Qu.: 8.212
##  Max.   :200.00   Max.   :2447.49   Max.   :858.79   Max.   :12.722
##      Price        Seasonality_Index
##  Min.   : 5.053   Min.   :0.3305
##  1st Qu.:16.554   1st Qu.:0.8475
##  Median :19.977   Median :0.9762
##  Mean   :19.560   Mean   :0.9829
##  3rd Qu.:22.924   3rd Qu.:1.1205
##  Max.   :29.404   Max.   :1.5958
```

```
# Isolate Sales data
sales_retail_df <- retail_df$Sales
summary(sales_retail_df)
```

X ~ Sales

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   25.57  284.42  533.54  636.92  867.58 2447.49
```

```
sum(sales_retail_df<0)
```

```
## [1] 0
```

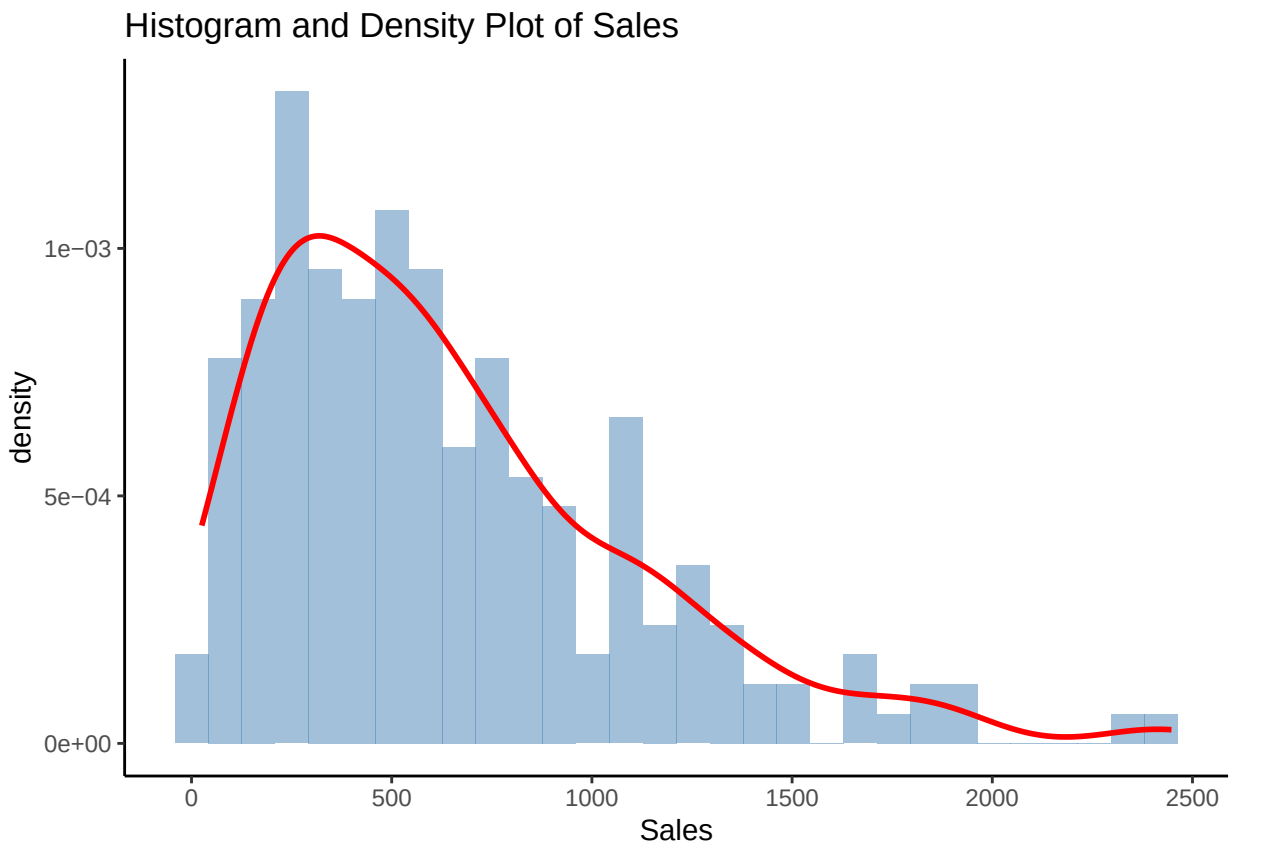
```
sum(is.na(sales_retail_df))
```

```
## [1] 0
```

```
shapiro.test(sales_retail_df)
```

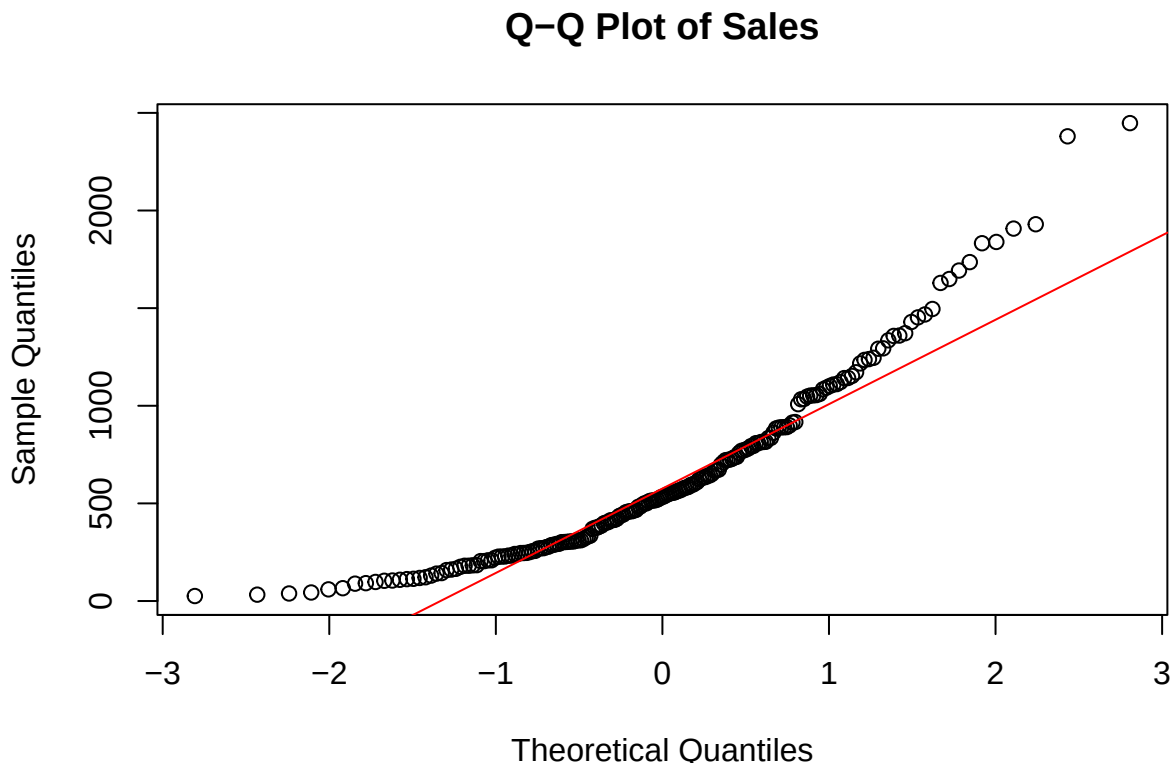
```
##
##  Shapiro-Wilk normality test
##
## data:  sales_retail_df
## W = 0.90377, p-value = 4.397e-10
```

```
ggplot(retail_df, aes(x = Sales)) +
  geom_histogram(aes(y = ..density..),
                 bins = 30, fill = "steelblue",
                 alpha = 0.5) +
  geom_density(color = "red", size = 1) +
  ggtitle("Histogram and Density Plot of Sales") +
  theme_classic()
```

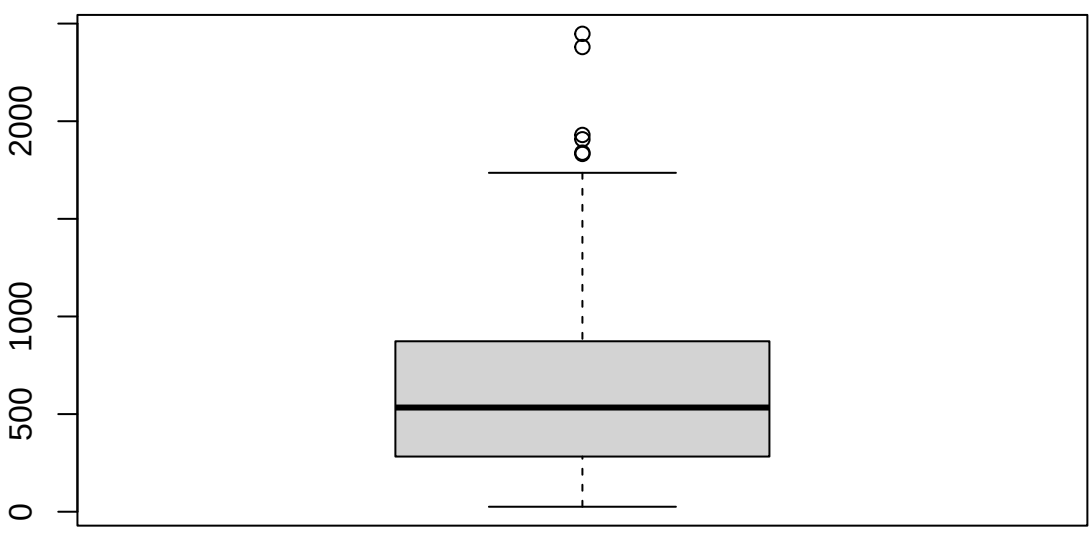



```
qqnorm(sales_retail_df,
       main = "Q-Q Plot of Sales")
qqline(sales_retail_df,
```

```
        col = "red")
```




```
boxplot(sales_retail_df,
        main = "Boxplot of Sales Data")
```



Initial analysis

- For our *Sales* data our $Mean > Median$ (636.92 > 533.54) which indicates that our data is right skewed and not normalized. This is supported by our Histogram, our Q-Q plot and the Shapiro test's $p - value$ of less than 0.05.

- No *NAs* are noted with the *Sales* data
- Our range for the values within *Sales* is 25.57 to 2447.49, encompassing a wide range.
- Our Box plot indicates that there are outliers, primarily for values > 1000

fitdstr Sales Solution Assume $X \sim Gamma(\alpha, \beta)$ the parameter *"gamma"* will be used with *fitdistr()*.

```
sales_gamma_fit <- fitdistr(sales_retail_df, "gamma")
```

```
## Warning in densfun(x, parm[1], parm[2], ...): NaNs produced
```

```
print(sales_gamma_fit)
```

```
##       shape           rate
##   1.8349640762   0.0028810166
##  (0.1511756159) (0.0002556985)
```

Considering no NAs or negative values were noted in our original data set, the *NaNs produced* warning, is likely a result of the right-skewed data or from our outliers. I will remove the outliers to see if it removes the warning. Regardless, dealing with these outliers should improve precision.

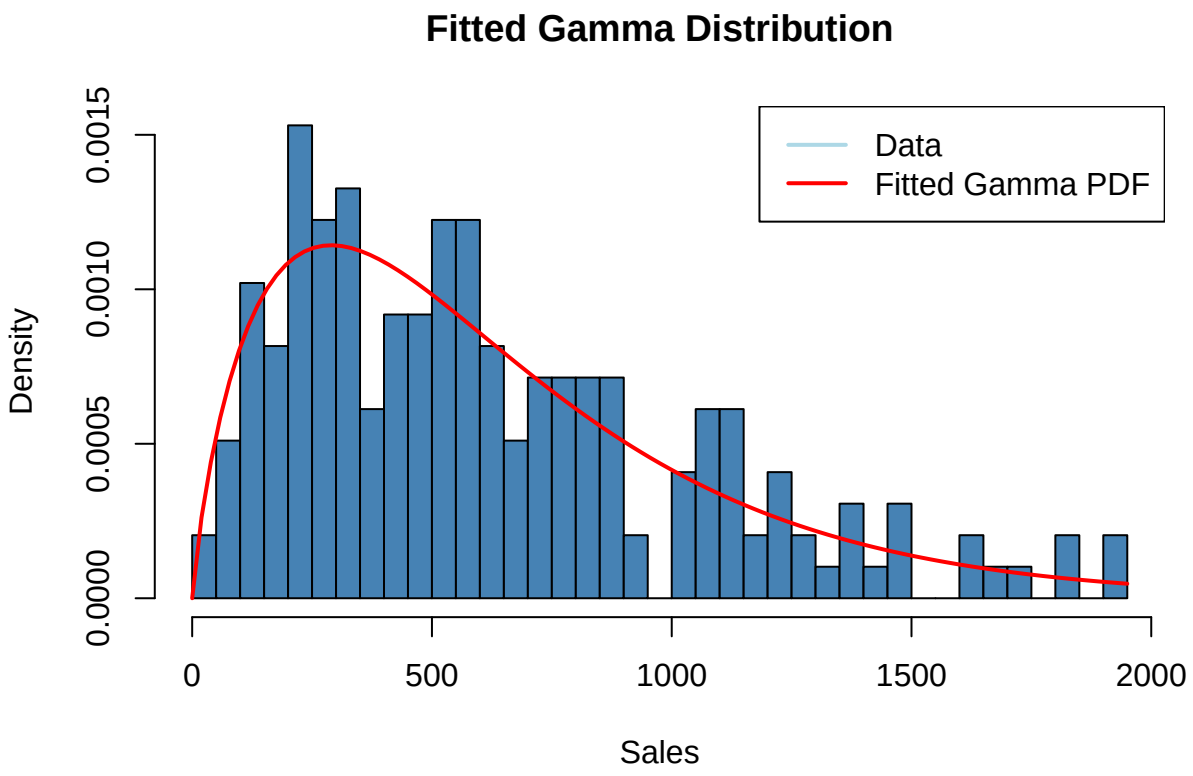
The below steps should remove values above our 99% quantile or below the 1%

```
# compute quantiles at 1% and and 99%
sales_retail_quantiles <-
  quantile(sales_retail_df, probs =c(0.01,0.99))
# remove outliers below the 1% and above 99%
sales_retail_df_clean<- sales_retail_df[
  sales_retail_df >= sales_retail_quantiles[1] &
    sales_retail_df <= sales_retail_quantiles[2]
]
sales_gamma_fit_clean <- fitdistr(sales_retail_df_clean, "gamma")
print(sales_gamma_fit_clean)
```

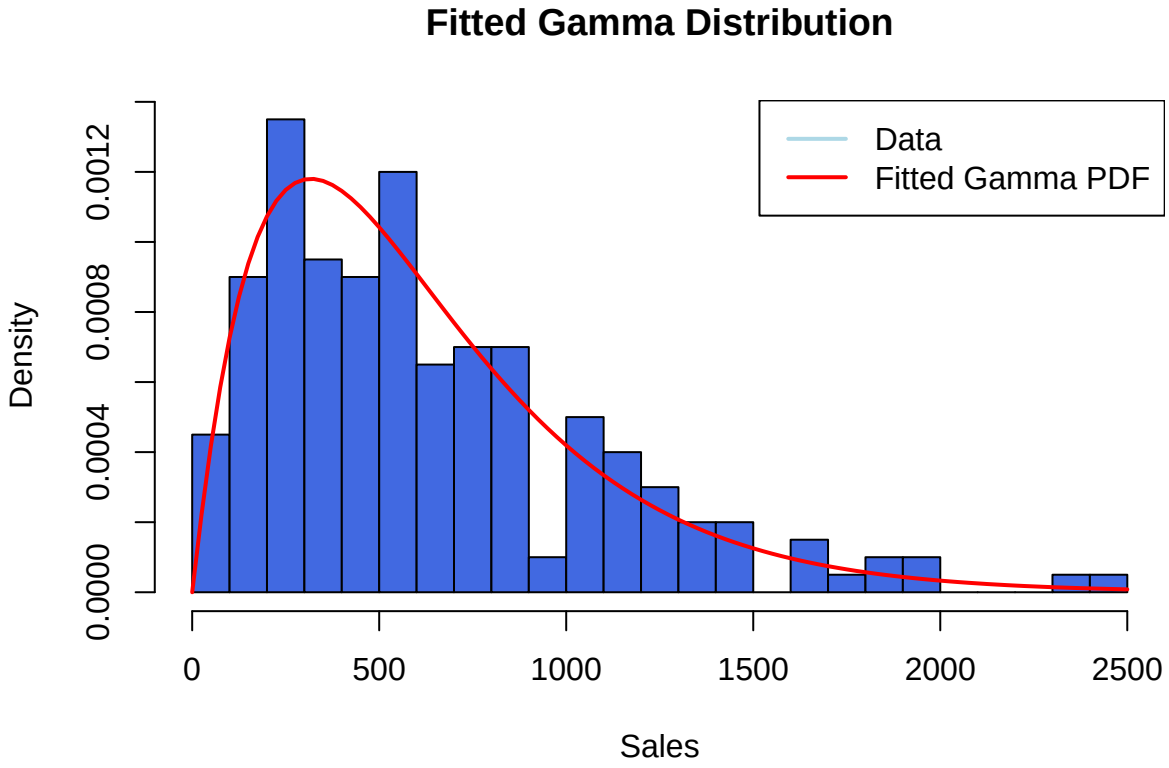
```
##       shape           rate
##   2.0323543224   0.0032518379
##  (0.1724396139) (0.0002975715)
```

The cleaned model still creates an error therefore I would like to see visually how well the values fit.

```
hist(sales_retail_df_clean,
     breaks = 30,
     probability = TRUE,
     main = "Fitted Gamma Distribution",
     xlab = "Sales",
     col = "steelblue")

curve(dgamma(x,
             shape = 1.8349640762,
             rate = 0.0028810166),
      col = "red",
      lwd = 2,
      add = TRUE)

legend("topright",
       legend = c("Data",
                  "Fitted Gamma PDF"),
       col = c("lightblue",
               "red"), lwd = 2)
```



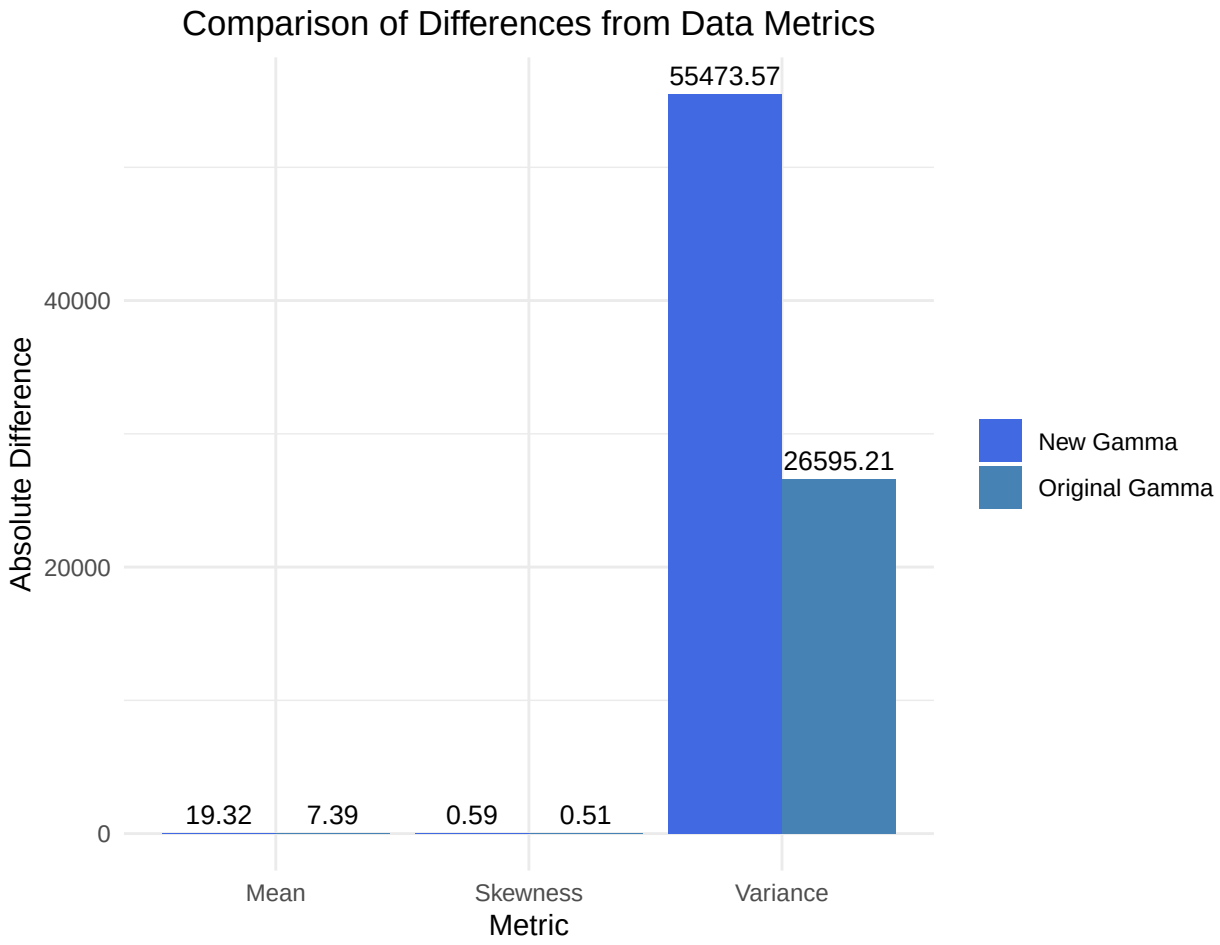

```
hist(sales_retail_df,
     breaks = 30,
     probability = TRUE,
     main = "Fitted Gamma Distribution",
     xlab = "Sales",
     col = "royalblue")

curve(dgamma(x,
             shape = 2.0323543224,
             rate = 0.0032518379 ),
      col = "red",
      lwd = 2,
      add = TRUE)

legend("topright",
       legend = c("Data",
                  "Fitted Gamma PDF"),
       col = c("lightblue",
               "red"), lwd = 2)
```



Visually, both values seem to match the *Sales* behavior. By calculating the absolute difference in mean, skewness and variance, I might get a better indication on which gamma distribution and gamma parameters, better emulates the datas behavior.

```
cmp_metrics <- c("Mean", "Variance", "Skewness")

cmp_data_values <- c(617.595, 165599.6, 0.8891198)

# Original Gamma differences
cmp_shape1 <- 2.0323543224

cmp_rate1 <- 0.0032518379

cmp_original_gamma_values <-
  c(cmp_shape1 / cmp_rate1,
    cmp_shape1 / (cmp_rate1^2),
    2 / sqrt(cmp_shape1))

cmp_original_differences <-
  abs(cmp_original_gamma_values - cmp_data_values)

# New Gamma differences
cmp_shape2 <- 1.8349640762
cmp_rate2 <- 0.0028810166

cmp_new_gamma_values <-
  c(cmp_shape2 / cmp_rate2,
    cmp_shape2 / (cmp_rate2^2),
    2 / sqrt(cmp_shape2))
cmp_new_differences <-
```

```r
  abs(cmp_new_gamma_values - cmp_data_values)

# Prepare data for ggplot
plot_data <- data.frame(
  Metric = rep(cmp_metrics,
               times = 2),
  Difference =
    c(cmp_original_differences,
                cmp_new_differences),
  Gamma = rep(c("Original Gamma",
                "New Gamma"),
              each = length(cmp_metrics))
)

# Create the clustered bar plot
ggplot(plot_data, aes(x = Metric, y = Difference, fill = Gamma)) +
  geom_bar(stat = "identity", position = position_dodge(width = 0.9)) +
  geom_text(aes(label = round(Difference, 2)),
            position = position_dodge(width = 0.9),
            vjust = -0.5, size = 3.5) +
  labs(
    title = "Comparison of Differences from Data Metrics",
    x = "Metric",
    y = "Absolute Difference"
  ) +
  scale_fill_manual(values = c("Original Gamma" = "steelblue", "New Gamma" = "royalblue")) +
  theme_minimal() +
  theme(
    plot.title = element_text(hjust = 0.5),
    legend.title = element_blank()
  )
```



Based on these results the original gamma parameters of 1.8349640762 and 0.0028810166 are the more accurate fit.

Mean and Variance (X~Sales) Our Empirical mean and variance is just

```
sales_empirical_mean<-mean(sales_retail_df)
sales_empirical_var<-var(sales_retail_df)
```

and our theoretical mean and variance is calculate as

$\mu_{gamma} = \frac{\alpha}{\beta}$

$\sigma^2_{gamma} = \frac{\alpha}{\beta^2}$ or $\frac{shape}{rate^2}$

```
sales_gamma_shape <- sales_gamma_fit$estimate["shape"]
sales_gamma_rate <- sales_gamma_fit$estimate["rate"]

sales_theoretical_gamma_mean <- sales_gamma_shape / sales_gamma_rate
sales_theoretical_gamma_var <- sales_gamma_shape/ (sales_gamma_rate^2)
```

```
comparison_Sales <- data.frame(
  Metric = c("Mean", "Variance"),
  Empirical = c(sales_empirical_mean, sales_empirical_var),
  Theoretical = c(sales_theoretical_gamma_mean, sales_theoretical_gamma_var)
)
```

Answer **The shape parameter of 1.8349640762 and the rate parameter of 0.0028810166 define the best-fit Gamma distribution for the data.**

Our Empirical and Theoretical, Variance and Mean are as follow:

```
##     Metric   Empirical Theoretical
## 1     Mean    636.9162    636.9155
## 2 Variance 214831.7509 221073.1799
```

```
rm(list = ls(pattern = "^comparison"))
```

```
inv_retail_df <- retail_df$Inventory_Levels
summary(inv_retail_df)
```

Y ~ Inventory Levels

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   67.35  376.51  483.72  488.55  600.42  858.79
```

```
sum(inv_retail_df<0)
```

```
## [1] 0
```

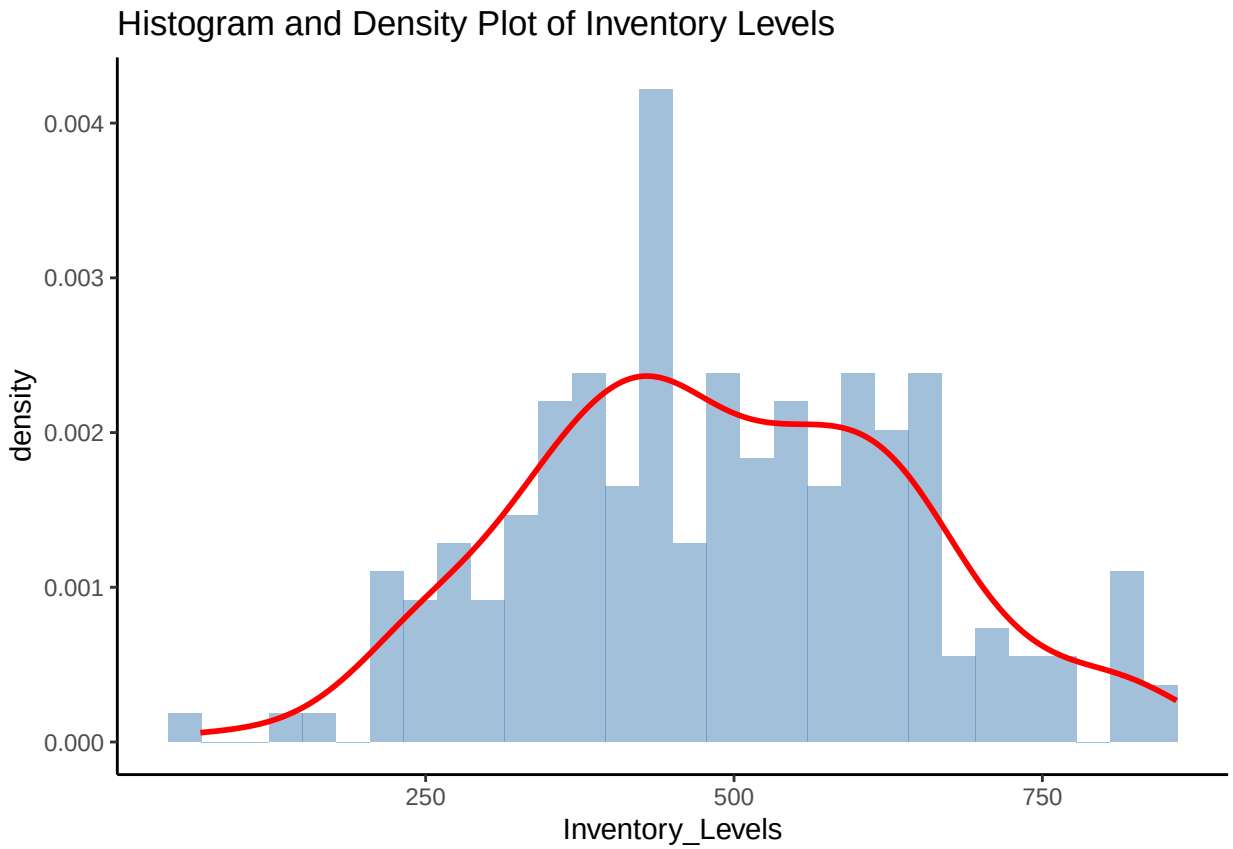
```
sum(is.na(inv_retail_df))
```

```
## [1] 0
```

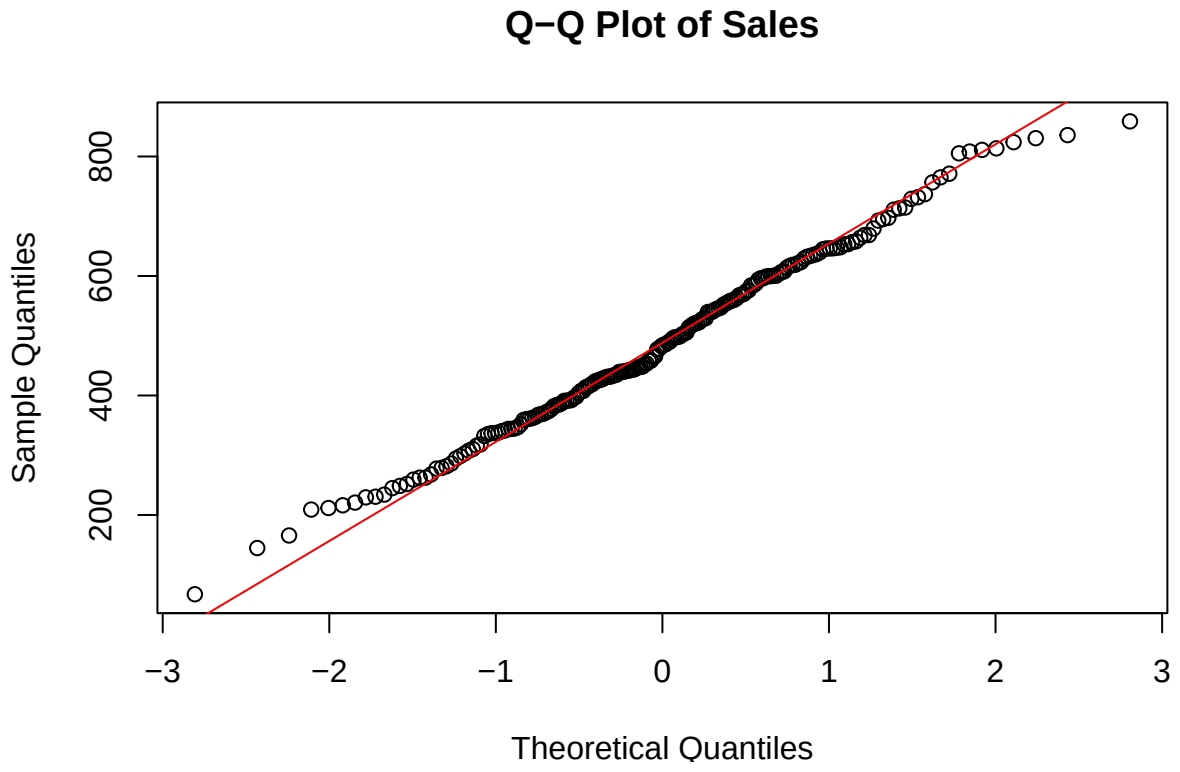
```
shapiro.test(inv_retail_df)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  inv_retail_df
## W = 0.99303, p-value = 0.4646
```

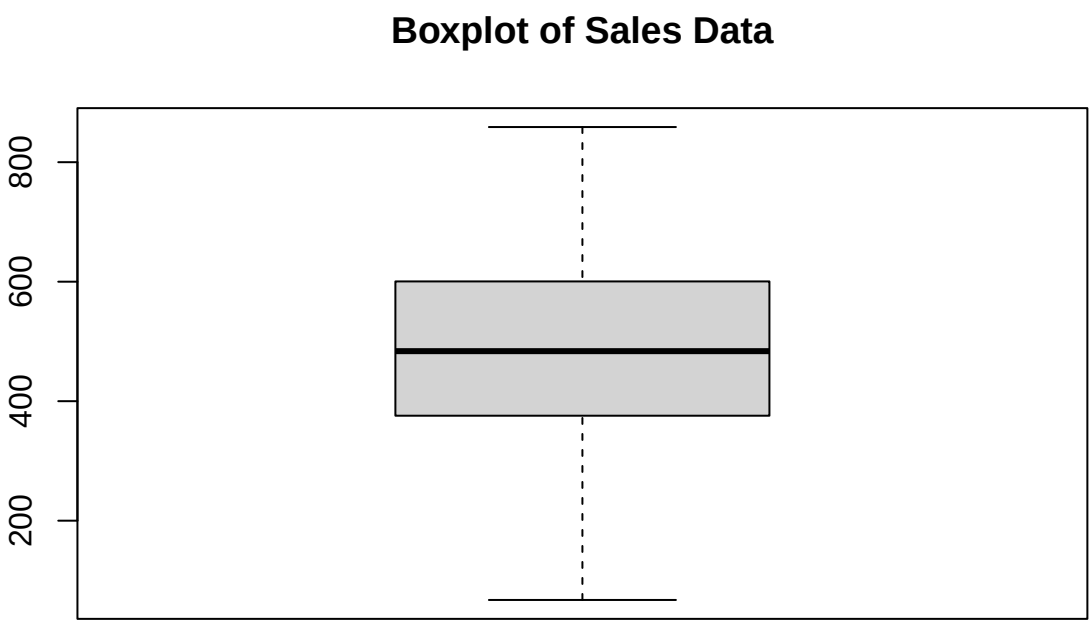
```
ggplot(retail_df, aes(x = Inventory_Levels )) +
  geom_histogram(aes(y = ..density..),
                 bins = 30, fill = "steelblue",
                 alpha = 0.5) +
  geom_density(color = "red", size = 1) +
  ggtitle("Histogram and Density Plot of Inventory Levels") +
  theme_classic()
```




```
qqnorm(inv_retail_df,
       main = "Q-Q Plot of Sales")
qqline(inv_retail_df,
       col = "red")
```




```
boxplot(inv_retail_df,
        main = "Boxplot of Sales Data")
```



Initial analysis

- No NAs found in *Inventory_Levels*
- No values below 0 for the *Inventory_Levels* values.
- 488.55 ($Mean$) > 483.72 ($Median$) suggests the data may be right skewed.
- Shapiro test had a $p - value = 0.4646$. This is below 0.5, suggesting it is not normalized, however it is relatively close to being normal.

- Q-Q plot and, Histogram and Density plot, show the data as near normal.

fitdistr Inventory Levels Since we are assuming the Inventory Levels across products follows a Lognormal distribution ($Y \sim Lognormal(\mu, \sigma^2)$), the parameter for *fitdistr()* we use is *lognormal*. Since we are explicitly looking for the sum of inventory levels across similar products, we will *sum* can consider finding the values for the individual *Product_ID*'s before evaluating the distribution.

```
retail_df %>%
  group_by(Inventory_Levels) %>%
  summarise(Count = n()) %>%
  filter(Count > 1)
```

```
## # A tibble: 0 x 2
## # i 2 variables: Inventory_Levels <dbl>, Count <int>
```

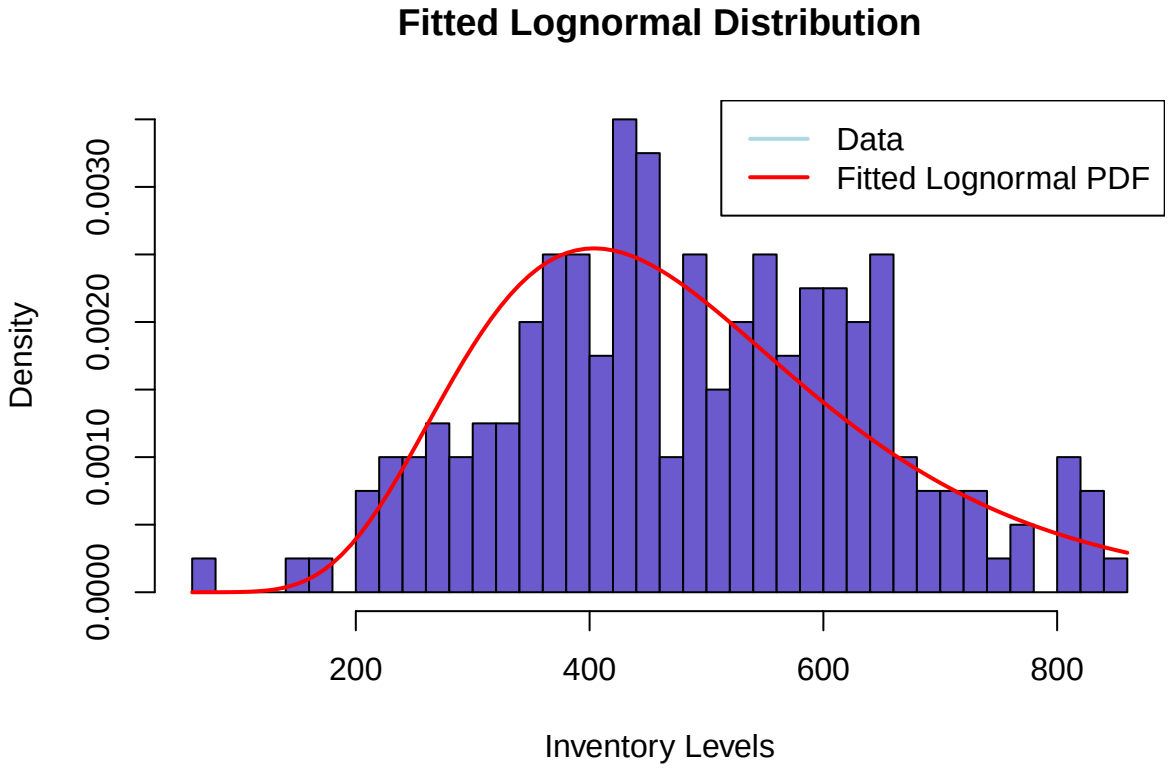
Since it appears that all *Product_ID*'s are unique I will refrain from using the sum.

The parameters of this distribution will ultimately be μ_{log} and σ_{log}

```
inv_lognormal <-
  fitdistr(inv_retail_df,"lognormal")

inv_log_mean <- inv_lognormal$estimate["meanlog"]
inv_log_var <- inv_lognormal$estimate["sdlog"]

print(inv_lognormal)
```

```
##     meanlog       sdlog
##   6.13303680   0.36332727
##  (0.02569112) (0.01816636)
```

```
# Histogram for Inventory Levels
hist(inv_retail_df,
     breaks = 30,
     probability = TRUE,
     main = "Fitted Lognormal Distribution",
     xlab = "Inventory Levels",
     col = "slateblue")

# Overlay the fitted lognormal curve
curve(dlnorm(x,
             meanlog = inv_log_mean,
             sdlog = inv_log_var),
      col = "red",
      lwd = 2,
      add = TRUE)

# Add legend
legend("topright",
       legend = c("Data", "Fitted Lognormal PDF"),
       col = c("lightblue", "red"),
       lwd = 2)
```



Mean and Variance (Y~Inventory Levels) *Theoretical Mean* or $E[x]$ of a random variable $X \sim Lognormal(\mu, \sigma^2)$ is equal to $e^{\mu+\frac{\sigma^2}{2}}$ and its theoretical variance is $(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$

```r
# Empirical statistics
inv_empirical_mean <- mean(retail_df$Inventory_Levels)
inv_empirical_variance <- var(retail_df$Inventory_Levels)

# Theoretical mean and variance for lognormal distribution
inv_theoretical_mean <-
  exp(inv_log_mean +
        (inv_log_var^2) / 2)

inv_theoretical_variance <-
  (exp(inv_log_var^2) - 1) *
  exp(2 * inv_log_mean + inv_log_var^2)

# # Print results
# cat("Empirical Mean:", empirical_mean, "\n")
# cat("Theoretical Mean:", theoretical_mean, "\n")
# cat("Empirical Variance:", empirical_variance, "\n")
# cat("Theoretical Variance:", theoretical_variance, "\n")

comparison_Inventory_Level <- data.frame(
  Metric = c("Mean", "Variance"),
  Empirical = c(inv_empirical_mean, inv_empirical_variance),
  Theoretical = c(inv_theoretical_mean, inv_theoretical_variance)
)
```

Answer The parameters for the Inventory Level distribution, ($X \sim Lognormal(\mu, \sigma^2)$) are $\mu_{log} = 6.13303680$ and $\sigma^2_{log} = 0.3633273$

Our Empirical and Theoretical, Variance and Mean are as follow:

```
print(comparison_Inventory_Level)
```

```
##           Metric  Empirical Theoretical
## meanlog     Mean   488.5472    492.2763
## sdlog   Variance 24039.4464  34197.4741
```

```
rm(list = ls(pattern = "^comparison"))
```

```
ltd_retail_df <- retail_df$Lead_Time_Days
summary(ltd_retail_df)
```

Z ~ Lead Time

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   0.491   5.291   6.765   6.834   8.212  12.722
```

```
sum(ltd_retail_df<0)
```

```
## [1] 0
```

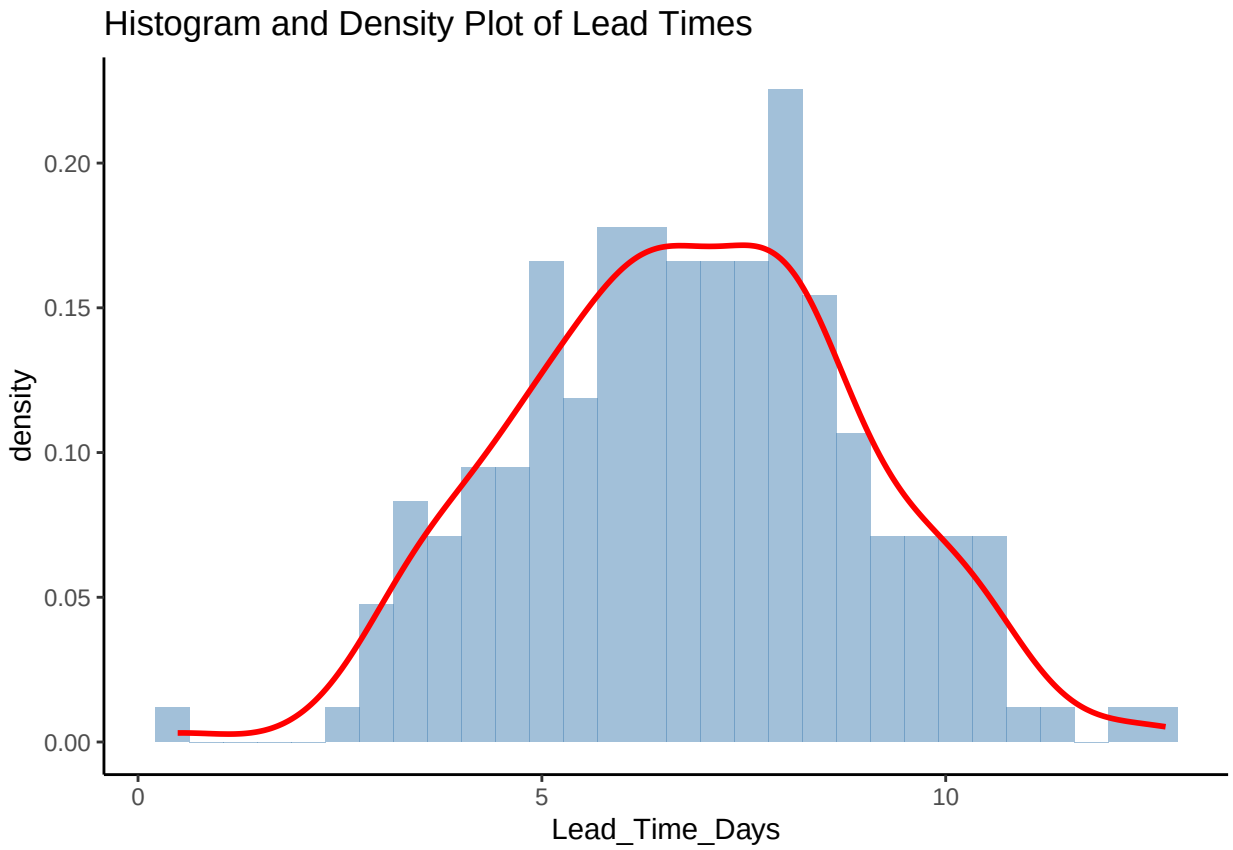
```
sum(is.na(ltd_retail_df))
```

```
## [1] 0
```

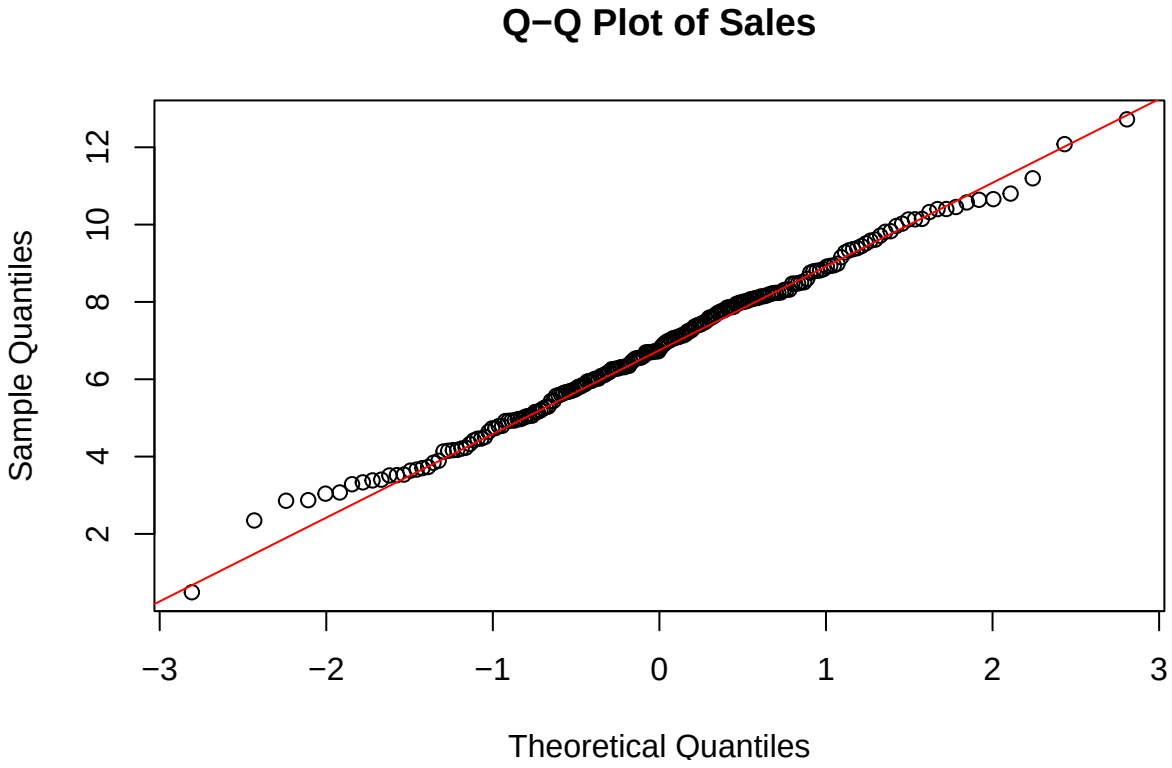
```
shapiro.test(ltd_retail_df)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  ltd_retail_df
## W = 0.99618, p-value = 0.9026
```

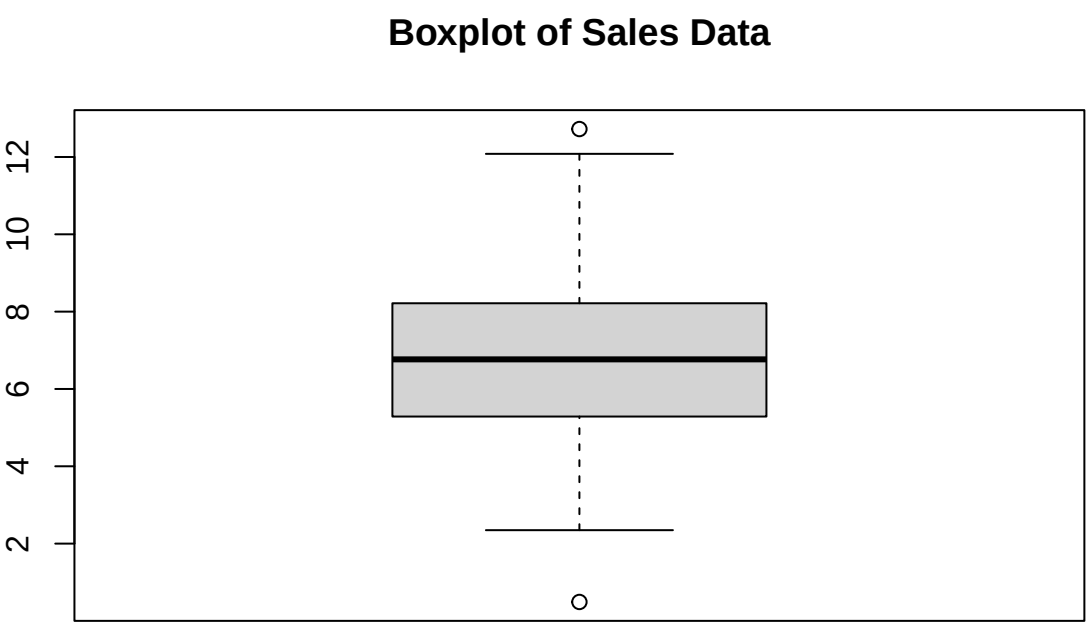
```
ggplot(retail_df, aes(x = Lead_Time_Days )) +
  geom_histogram(aes(y = ..density..),
                 bins = 30, fill = "steelblue",
                 alpha = 0.5) +
  geom_density(color = "red", size = 1) +
  ggtitle("Histogram and Density Plot of Lead Times") +
  theme_classic()
```




```
qqnorm(ltd_retail_df,
       main = "Q-Q Plot of Sales")
qqline(ltd_retail_df,
       col = "red")
```




```
boxplot(ltd_retail_df,
        main = "Boxplot of Sales Data")
```



This data is basically normalized, with a slight right skewed and outliers that do not deviate much from the mean.

Mean and Variance Lead Time We are using the same data with our theoretical Mean and Variance. Considering this, finding the mean and variance for our Lead Times should be the same. Another way of deriving our varian would be to square our standard deviation or σ. I've applied a few methods to show this.

1. Finding the standard deviation and squaring it

2. Using the *var()* function directly
3. Using the *fitdistr()* function to conclude our mean and standard are the same (alternatively I called on the *estimate* attribute for my *fitdistr* which is a redundant call to this value)

```
ltd_empirical_mean <- mean(retail_df$Lead_Time_Days)
sd(retail_df$Lead_Time_Days)
```

```
## [1] 2.088441
```

```
sd(retail_df$Lead_Time_Days)^2
```

```
## [1] 4.361587
```

```
var(retail_df$Lead_Time_Days)
```

```
## [1] 4.361587
```

```
ltd_fitdistr<-fitdistr(ltd_retail_df,"normal")
print(ltd_fitdistr)
```

```
##      mean         sd
##   6.8342981   2.0832137
##  (0.1473055) (0.1041607)
```

```
ltd_fitdistr$estimate
```

```
##     mean       sd
## 6.834298 2.083214
```

Answer The estimated mean is 6.834298 and standard deviation is 2.083214 Calculate Empirical Expected Value is 6.834298 and Variance is 4.361587

2.

Part 2:

Probability Analysis and Independence Testing (5 Points)

Task:

1.

Empirical Probabilities: For the Lead_Time_Days variable (assumed to be normally distributed), calculate the following empirical probabilities:

- $P(Z > \mu | Z > \mu - \sigma)$
- $P(Z > \mu + \sigma | Z > \mu)$
- $P(Z > \mu + 2\sigma | Z > \mu)$

Notes

- We assume the **standard normal distribution in context to these probabilities, because they do not specify a random variable X with its own mean μ and standard deviation σ
- The standard normal distribution also known as the z distribution, can have the notation $N(\mu, \sigma)$ where N signifies the distribution is normal, while μ, σ and σ^2 retains its known definition as the mean, standard deviation and variance of the distribution. Please note **Reference** *vii* for a more in depth explanation.

i.

- $P(Z > \mu | Z > \mu - \sigma) = \frac{P(Z>\mu \ \cap \ Z>\mu-\sigma)}{Z>\mu-\sigma}$
- Our numerator can be simplified to $P(Z > \mu)$. Stated plainly, $P(Z > \mu - \sigma)$ is encompassed with $P(Z > \mu)$, but both conditions cannot be met if we only satisfy $P(Z > \mu - \sigma)$ therefore we only need to satisfy the second condition $P(Z > \mu)$.
- This simplifies our conditional probability $P(Z > \mu | Z > \mu - \sigma) = \frac{P(Z>\mu \ \cap \ Z>\mu-\sigma)}{Z>\mu-\sigma}$ to $\frac{Z>\mu}{Z>\mu-\sigma}$
- $N(\mu, \sigma)$ for the standard normal distribution is $Z' \sim N(0, 1)$, since the mean of a standard normal distribution is $\mu = 0$ with a standard deviation of $\sigma = 1$ (Reference *vii*).
- $P(Z > \mu)$ represents 50% of the probability as it is the mean $\therefore P(Z > \mu) = 0.5$
- We can use the standardization formula $F_X(x) = P(X \leq x) = F_z(\frac{x-\mu}{\sigma})$ or just $Z = \frac{x-\mu}{\sigma}$ to substitute based on our second conditions probability (Grinstead and Snell's Introduction to Probability pg. 214):
  - random variable $X = \mu - \sigma$
  - $X \sim N(\mu, \sigma^2) \rightarrow Z = \frac{(\mu-\sigma)-\mu}{\sigma} = -\frac{\sigma}{\sigma} = -1$
  - This transforms our formula from $Z > \mu - \sigma$ to $Z > -1$

we can find use *pnorm()* to get the value for $Z = -1$

```
pnorm(-1)
```

```
## [1] 0.1586553
```

Since $P(Z > -1) = 1 - P(Z \leq -1)$ we can solve by subtracting these values.

```
1-pnorm(-1)
```

```
## [1] 0.8413447
```

Again we are trying to solve for $\frac{Z>\mu}{Z>\mu-\sigma}$ and by substituting we get $\frac{0.5}{0.8413447}$ which is

Conclusion

```
0.5/(1-pnorm(-1))
```

```
## [1] 0.5942867
```

ii.

- For $P(Z > \mu + \sigma | Z > \mu)$ we can use a similar conditional probability as above ($P(A|B) = \frac{P(A \cap B)}{P(B)}$) to get $\frac{P((Z>\mu+\sigma) \ \cap \ (Z>\mu))}{P(Z>\mu)}$
- In this probability the $P(Z > \mu + \sigma)$ the condition encompasses $P(Z > \mu)$, since for $P((Z > \mu + \sigma) \ \cap \ (Z > \mu)$ to be true, $P(Z > \mu + \sigma)$ would be required.
- Therefore $\frac{P((Z>\mu+\sigma) \ \cap \ (Z>\mu))}{P(Z>\mu)}$ can be written as $\frac{P(Z>\mu+\sigma)}{P(Z>\mu)}$
- $P(Z > \mu) = 0.5$ as per our last probabilities conclusions.
- Again we can use the standardization formula $F_X(x) = P(X \leq x) = F_z(\frac{x-\mu}{\sigma})$ or just $Z = \frac{x-\mu}{\sigma}$ to substitute based on our second conditions probability (Grinstead and Snell's Introduction to Probability pg. 214):
- random variable $X = \mu + \sigma$
  - $X \sim N(\mu, \sigma^2) \rightarrow Z = \frac{(\mu+\sigma)-\mu}{\sigma} = \frac{\sigma}{\sigma} = 1$
  - This transforms our formula from $Z > \mu + \sigma$ to $Z > 1$

Again we solve for $Z = 1$ this using *pnorm()* and we understand $P(Z > 1) = 1 - P(Z < 1)$ because we are computing for the tail of this probability

```
1-pnorm(1)
```

```
## [1] 0.1586553
```

Which we will use substitute and solve for $\frac{P(Z>\mu+\sigma)}{P(Z>\mu)}$

Conclusion

```
(1-pnorm(1))/0.5
```

```
## [1] 0.3173105
```

iii.

$P(Z > \mu + 2\sigma | Z > \mu) = \frac{P(Z>\mu+2\sigma \cap Z>\mu)}{Z>\mu}$ where $P(Z > \mu + 2\sigma \cap Z > \mu)$ implies $Z > \mu + 2\sigma$ for our numerator and $Z > \mu = 0.5$ based on previous work. - Substituting for the standardization formula we get $X \sim N(\mu, \sigma^2) \rightarrow Z = \frac{(\mu+2\sigma)-\mu}{\sigma} = \frac{2\sigma}{\sigma} = 2$ - $P(Z > 2) = 1 - P(Z < 2) = \frac{1-P(Z<2)}{0.5}$ which is

Conclusion

```
(1-pnorm(2))/0.5
```

```
## [1] 0.04550026
```

Answers

- $P(Z > \mu | Z > \mu - \sigma) \approx 0.594$
- $P(Z > \mu + \sigma | Z > \mu) \approx 0.317$
- $P(Z > \mu + 2\sigma | Z > \mu) \approx 0.0455$

2.

Correlation and Independence:

- Investigate the correlation between Sales and Price. Create a contingency table using quartiles of Sales and Price, and then evaluate the marginal and joint probabilities.
- Use Fisher's Exact Test and the Chi-Square Test to check for independence between Sales and Price. Discuss which test is most appropriate and why.

Problem 2

Advanced Forecasting and Optimization (Calculus) in Retail

Context: You are working for a large retail chain that wants to optimize pricing, inventory management, and sales forecasting using data-driven strategies. Your task is to use regression, statistical modeling, and calculus-based methods to make informed decisions.

Part 1

Descriptive and Inferential Statistics for Inventory Data (5 Points)

Task:

1.

Inventory Data Analysis:

- Generate univariate descriptive statistics for the Inventory_Levels and Sales variables.
- Create appropriate visualizations such as histograms and scatterplots for Inventory_Levels, Sales, and Price.
- Compute a correlation matrix for Sales, Price, and Inventory_Levels.
- Test the hypotheses that the correlations between the variables are zero and provide a 95% confidence interval.

Univariate Descriptive Statistics

I surprisingly did this for the first part, but will repeat this for completion sake

```
summary(sales_retail_df)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   25.57  284.42  533.54  636.92  867.58 2447.49
```

```
sum(sales_retail_df < 0)
```

```
## [1] 0
```

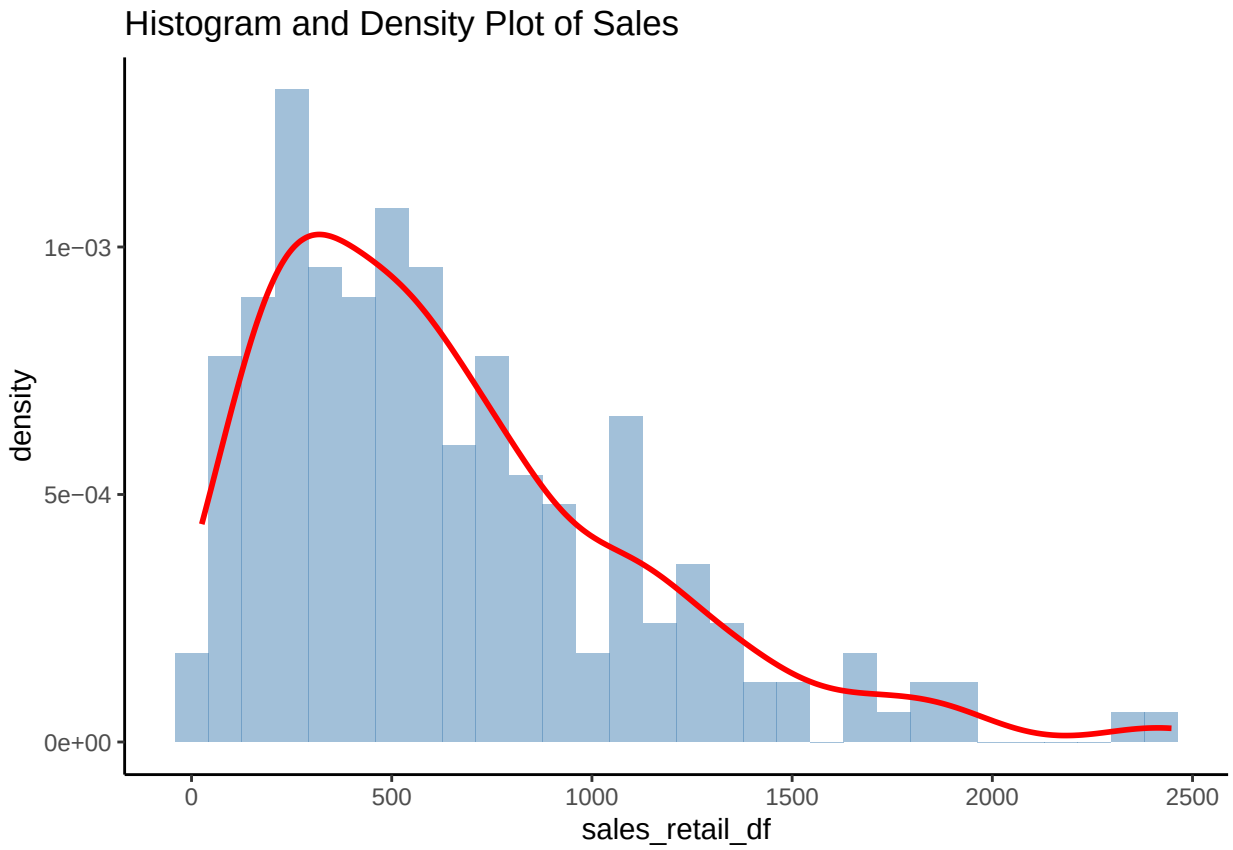
```
sum(is.na(sales_retail_df))
```

```
## [1] 0
```

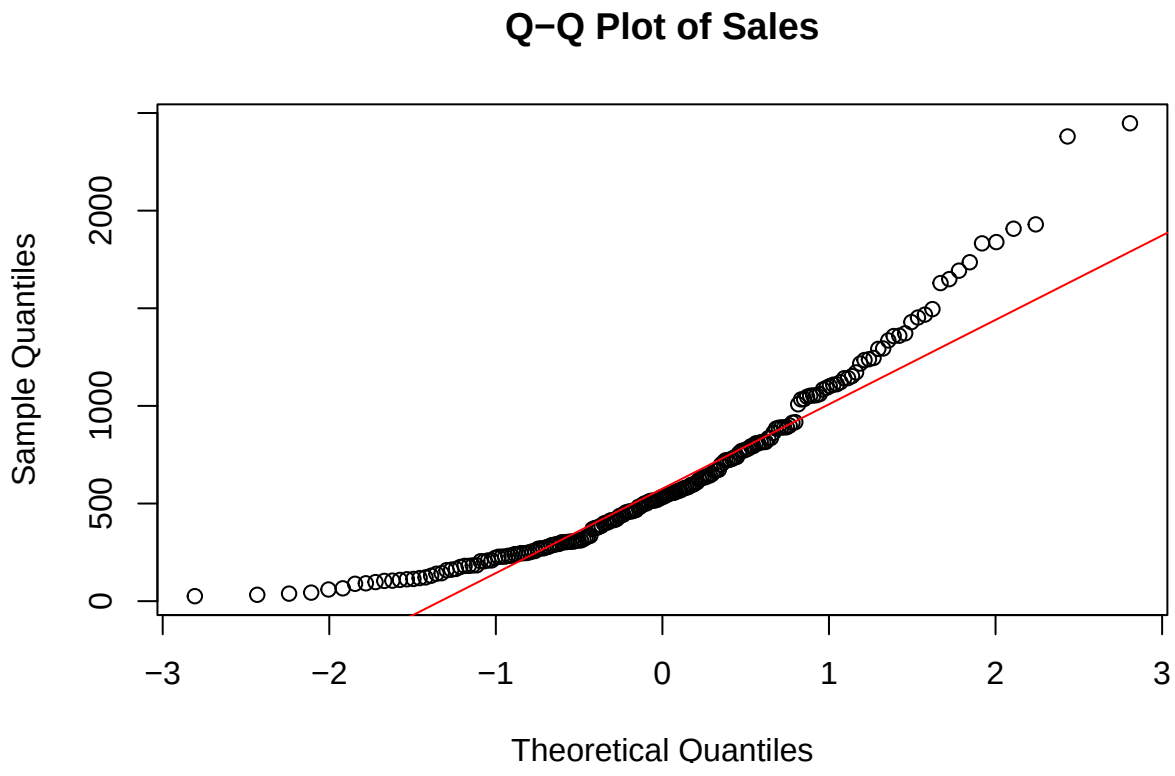
```
shapiro.test(sales_retail_df)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  sales_retail_df
## W = 0.90377, p-value = 4.397e-10
```

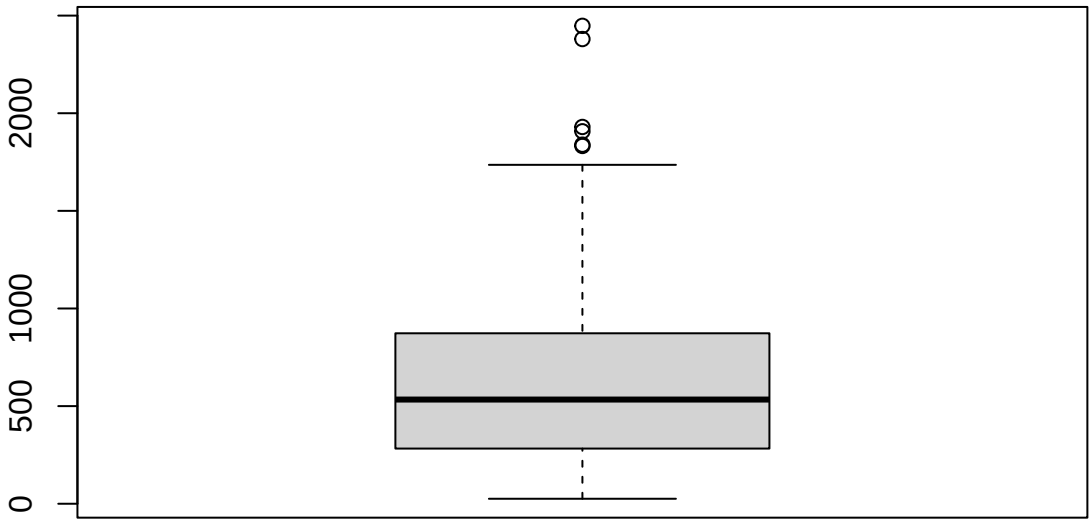
```
ggplot(retail_df, aes(x = sales_retail_df)) +
  geom_histogram(aes(y = ..density..),
                 bins = 30, fill = "steelblue",
                 alpha = 0.5) +
  geom_density(color = "red", size = 1) +
  ggtitle("Histogram and Density Plot of Sales") +
  theme_classic()
```




```r
qqnorm(sales_retail_df, main = "Q-Q Plot of Sales")
qqline(sales_retail_df, col = "red")
```



```r
boxplot(sales_retail_df, main = "Boxplot of Sales")
```




```r
summary(inv_retail_df)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   67.35  376.51  483.72  488.55  600.42  858.79
```

```r
sum(inv_retail_df < 0)
```

```
## [1] 0
```

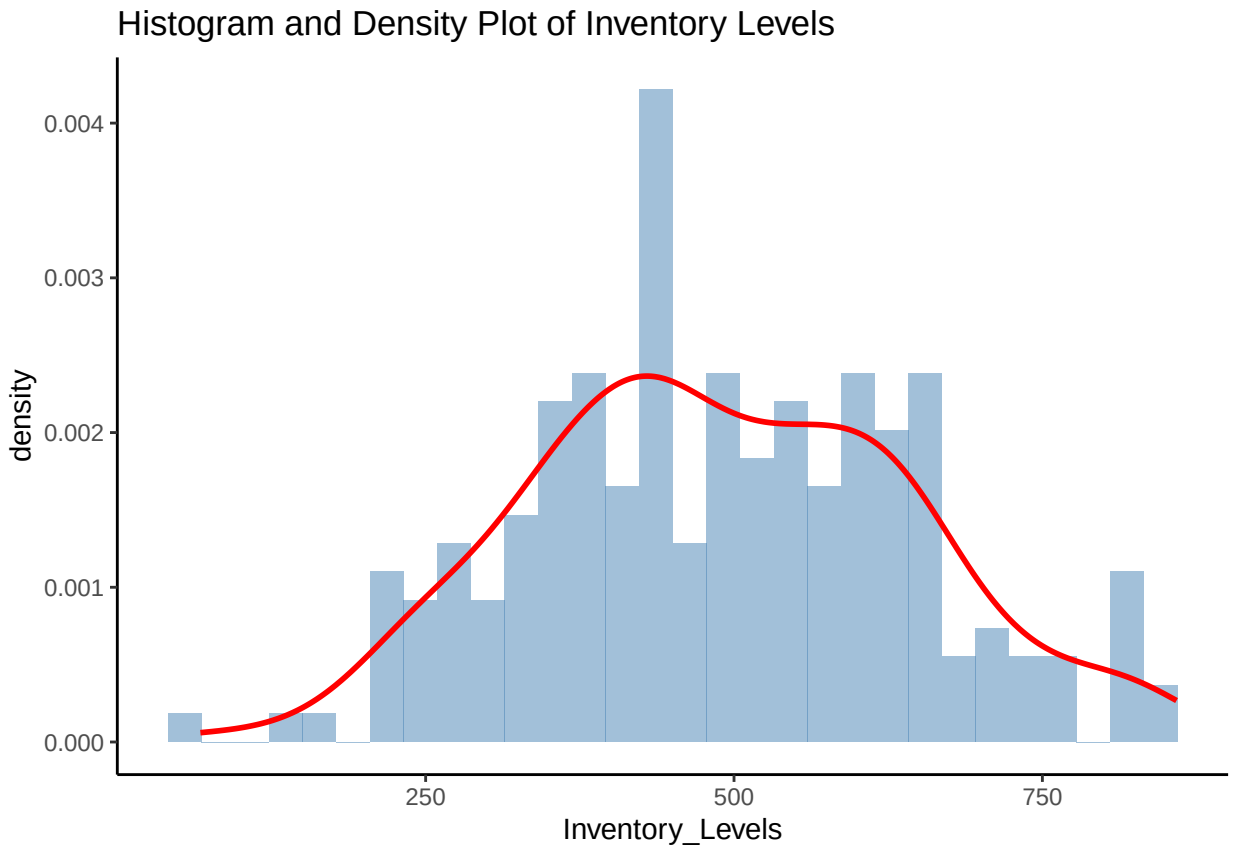
```r
sum(is.na(inv_retail_df))
```

```
## [1] 0
```

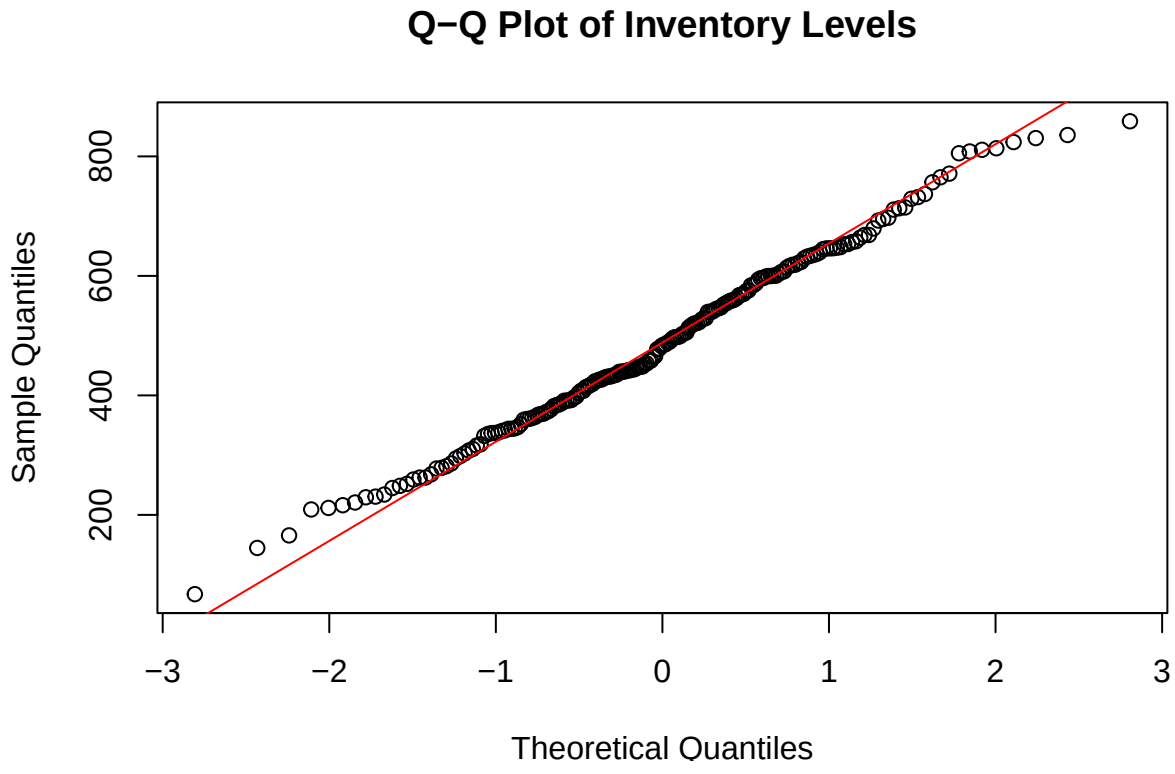
```r
shapiro.test(inv_retail_df)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  inv_retail_df
## W = 0.99303, p-value = 0.4646
```

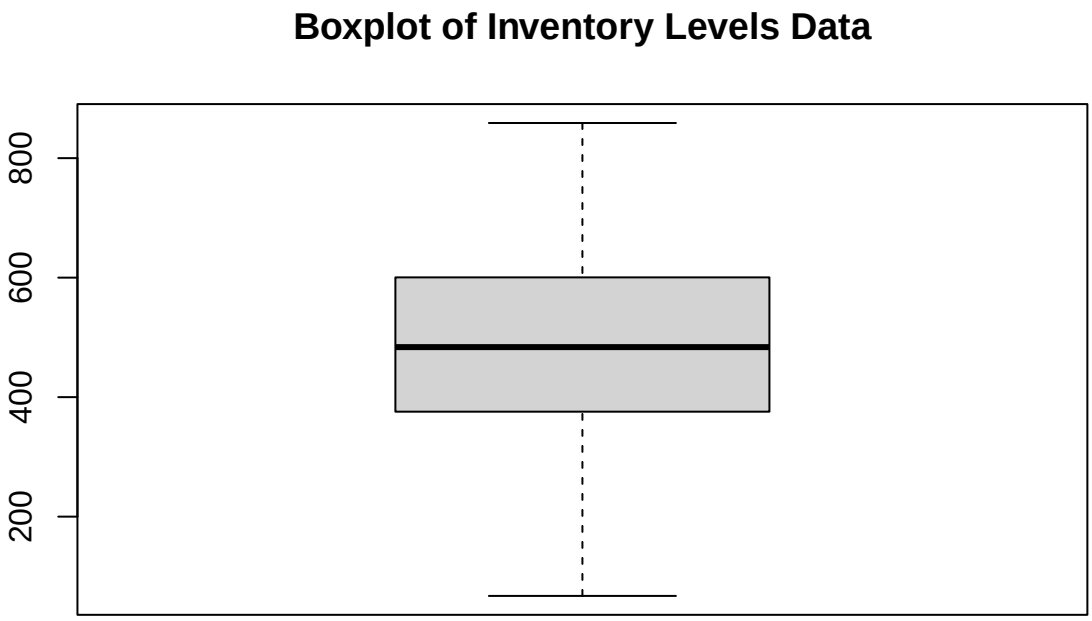
```r
ggplot(retail_df, aes(x = Inventory_Levels)) +
  geom_histogram(aes(y = ..density..),
                 bins = 30, fill = "steelblue",
                 alpha = 0.5) +
  geom_density(color = "red", size = 1) +
  ggtitle("Histogram and Density Plot of Inventory Levels") +
  theme_classic()
```




```r
qqnorm(inv_retail_df, main = "Q-Q Plot of Inventory Levels")
qqline(inv_retail_df, col = "red")
```



```r
boxplot(inv_retail_df, main = "Boxplot of Inventory Levels Data")
```



Repeating the conclusions from my initial analysis

Sales Analysis

- For our *Sales* data our $Mean > Median$ ($636.92 > 533.54$) which indicates that our data is right skewed and not normalized. This is supported by our Histogram, our Q-Q plot and the Shapiro test's $p - value$ of less than 0.05.
- No NAs are noted with the *Sales* data
- Our range for the values within *Sales* is 25.57 to 2447.49, encompassing a wide range.
- Our Box plot indicates that there are outliers, primarily for values > 1000

Inventory Levels Analysis

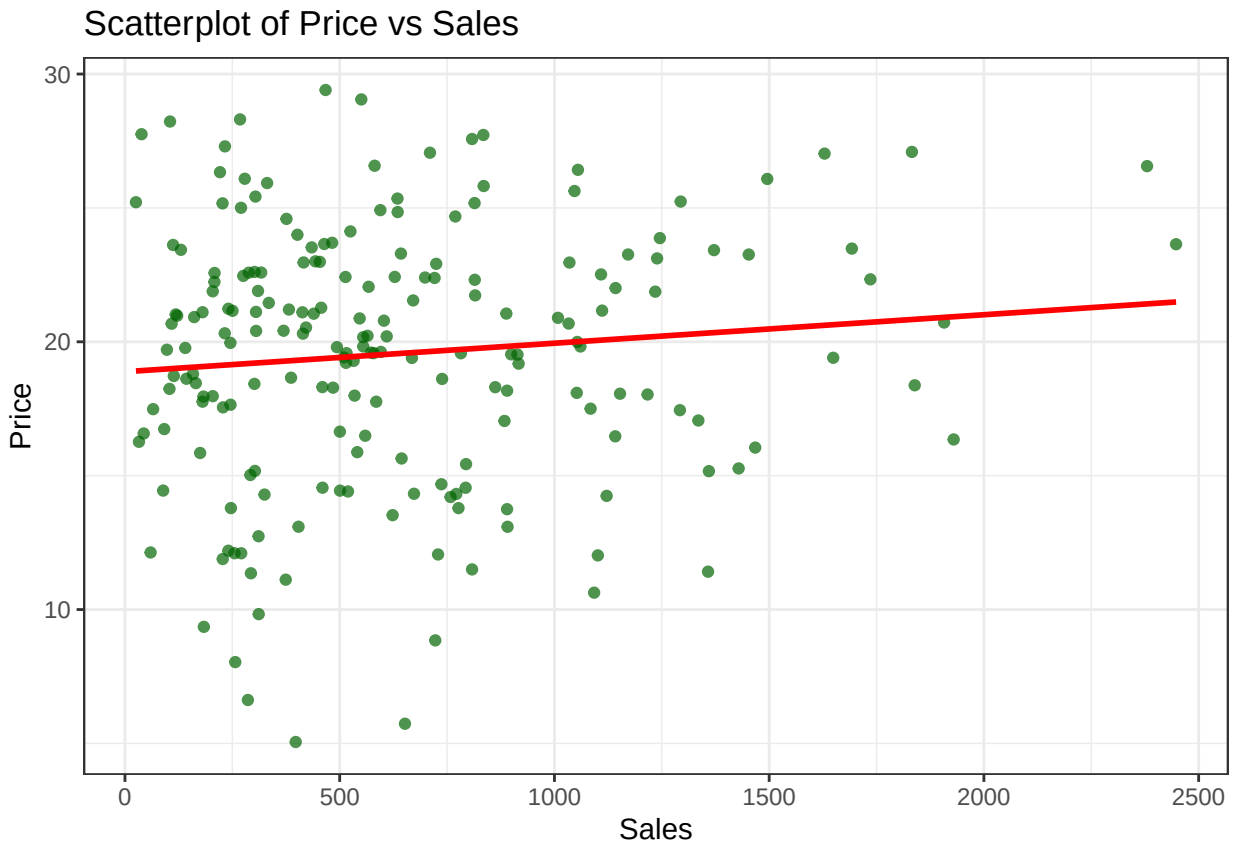
- No NAs found in *Inventory_Levels*
- No values below 0 for the *Inventory_Levels* values.
- 488.55 $(Mean) > 483.72$ $(Median)$ suggests the data may be right skewed.
- Shapiro test had a $p - value = 0.4646$. This is below 0.5, suggesting it is not normalized, however it is relatively close to being normal.
- Q-Q plot and, Histogram and Density plot, show the data as near normal.

Scatterplots and Price

Considering I already generated a histogram for *Inventory_Levels* and *Sales* I will only create a histogram for *Price* to visualize the relationships

```r
# Scatterplot: Price vs Sales
ggplot(retail_df, aes(x = Sales, y = Price)) +
  geom_point(alpha = 0.7, color = "darkgreen") +
  geom_smooth(method = "lm", se = FALSE, color = "red") +
  ggtitle("Scatterplot of Price vs Sales") +
  xlab("Sales") +
  ylab("Price") +
  theme_bw()
```

```
## `geom_smooth()` using formula = 'y ~ x'
```

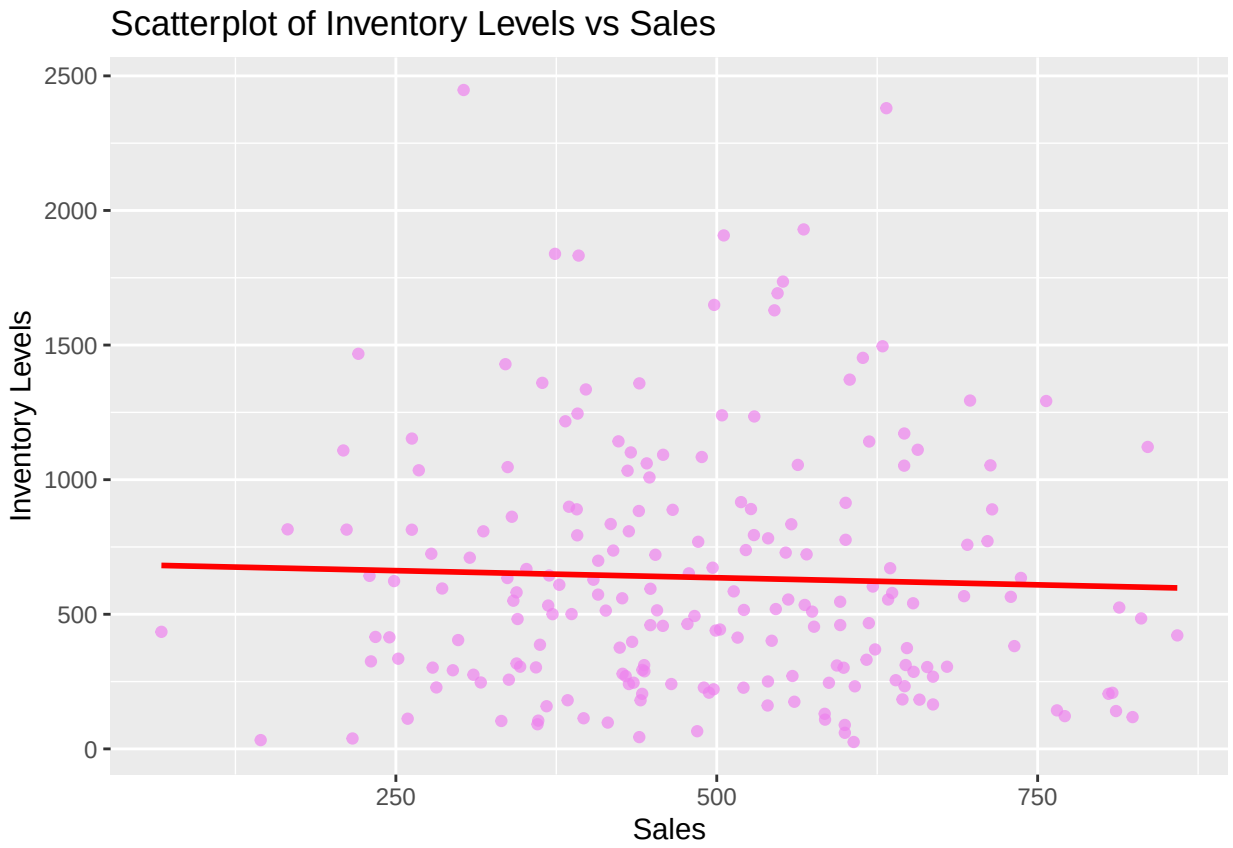



```
# Scatterplot: Price vs Inventory Levels
ggplot(retail_df, aes(x = Inventory_Levels, y = Price)) +
  geom_point(alpha = 0.7, color = "orange") +
  geom_smooth(method = "lm", se = FALSE, color = "red") +
  ggtitle("Scatterplot of Price vs Inventory Levels") +
  xlab("Inventory Levels") +
  ylab("Price") +
  theme_gray()
```

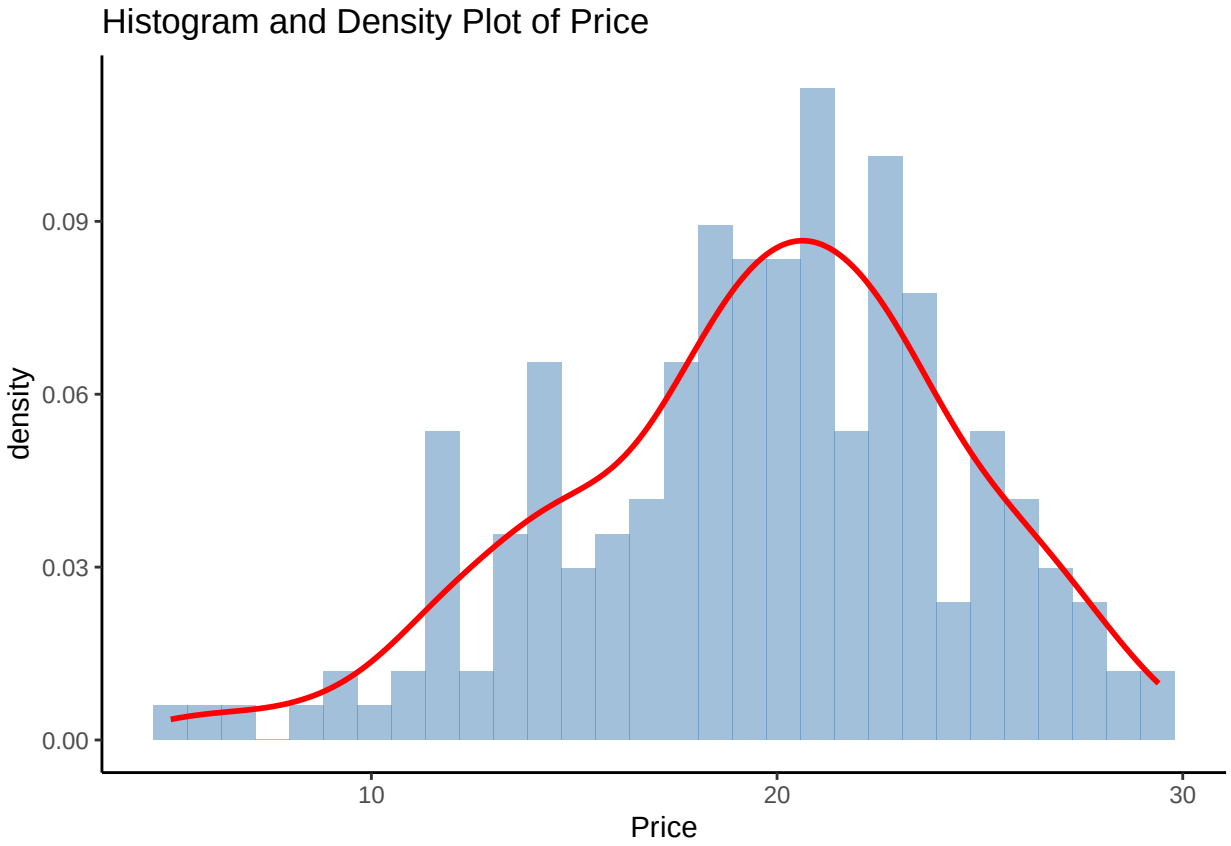
```
## `geom_smooth()` using formula = 'y ~ x'
```


```
# Scatterplot: Inventory Levels vs Price
ggplot(retail_df, aes(x = Inventory_Levels, y = Sales)) +
  geom_point(alpha = 0.7, color = "violet") +
  geom_smooth(method = "lm", se = FALSE, color = "red") +
  ggtitle("Scatterplot of Inventory Levels vs Sales") +
  xlab("Sales") +
  ylab("Inventory Levels") +
  theme_gray()
```

```
## `geom_smooth()` using formula = 'y ~ x'
```




```
ggplot(retail_df, aes(x = Price)) +
  geom_histogram(aes(y = ..density..),
                 bins = 30, fill = "steelblue",
                 alpha = 0.5) +
  geom_density(color = "red", size = 1) +
  ggtitle("Histogram and Density Plot of Price") +
  theme_classic()
```



- Scatterplot of There does not seem to be a strong relationship between these variables, as the Price hardly changes based on Inventory_Level or Sales

2.

Discussion:

- Explain the meaning of your findings and discuss the implications of the correlations for inventory management. Would you be concerned about multicollinearity in a potential regression model? Why or why not?

Part 2

Linear Algebra and Pricing Strategy (5 Points)

Task:

1.

Price Elasticity of Demand:

- Use linear regression to model the relationship between Sales and Price (assuming Sales as the dependent variable).
- Invert the correlation matrix from your model, and calculate the precision matrix.
- Discuss the implications of the diagonal elements of the precision matrix (which are variance inflation factors).
- Perform LU decomposition on the correlation matrix and interpret the results in the context of price elasticity.

Part 3:

Calculus-Based Probability & Statistics for Sales Forecasting (5 Points)

Task:

1.

Sales Forecasting Using Exponential Distribution:

- Identify a variable in the dataset that is skewed to the right (e.g., Sales or Price) and fit an exponential distribution to this data using the fitdistr function.
- Generate 1,000 samples from the fitted exponential distribution and compare a histogram of these samples with the original data's histogram.
- Calculate the 5th and 95th percentiles using the cumulative distribution function (CDF) of the exponential distribution.
- Compute a 95% confidence interval for the original data assuming normality and compare it with the empirical percentiles.

2.

Discussion:

- Discuss how well the exponential distribution models the data and what this implies for forecasting future sales or pricing. Consider whether a different distribution might be more appropriate.

Part 4

Regression Modeling for Inventory Optimization (10 Points)

Task:

1.

Multiple Regression Model:

- Build a multiple regression model to predict Inventory_Levels based on Sales, Lead_Time_Days, and Price.
- Provide a full summary of your model, including coefficients, R-squared value, and residual analysis.

2.

Optimization:

- Use your model to optimize inventory levels for a peak sales season, balancing minimizing stockouts with minimizing overstock.

References

i. Statology fitdistr-r

ii. rdocumentation MASS package fitdistr

iii. Statology - fit gamma distribution to dataset in r

iv. Wiki Gamma Distribution

v. rdocumentation qualtile

vi. statlect lognormal distribution

vii. Penn State University Online Stat 200 book