ch-9. Recurrence Relations (C.L. Liu).

eg. For the numeric function $a = (3^0, 3^1, 3^2, \ldots, 3^r, \ldots)$

the f^n can be specified by expression,

$$a_r = 3^r \quad , \; r \geq 0$$

#Also the f^n can be specified by generating function,

$$A(z) = \frac{1}{1-3z}$$

The Recurrence Relation describes the same f^n,

$$a_r = 3\,a_{r-1} \quad \text{with } \boxed{a_0 = 1}$$

eg. sequence of Fibonacci Nos:-

1, 1, 2, 3, 5, 8, 13, 21, 34, - - - -

$a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{r+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{r+1}$

or $A(z) = \frac{1}{1 - z - z^2}$

or $a_r = a_{r-1} + a_{r-2}$ with $\boxed{a_0 = 1, \; a_1 = 1}$ → boundary conditions.

Recurrence Relation :- Also termed as difference equation.

Linear Recurrence Relations with constant coefficients

A Recurrence Relation of the form

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \dots + C_K a_{r-K} = f(r)$$

where c_i's are constants, is called Linear Rec. Rel. with constant coefficients.

The Eq. above is known as K^{th}-order Rec. Relation provided that both C_0 & C_K are non-zero.

for eg. 1) $2a_r + 3a_{r-1} = 2^r$ → first order.

eg. 2) $3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$ → second order

eg. 3) $a_r + 7a_{r-2} = 0$ → second order.

In second eg.

$$3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5, \boxed{a_3 = 3, a_4 = 6}$$

(given).

then $a_5 = \frac{-1}{3}\left[-5.6 + 2.3 - (5^2+5)\right] = 18$

then $a_6 = \frac{-1}{3}\left[-5.18 + 2.6 - (6^2+5)\right] = \frac{119}{3}$

& so on.

Also, we can compute, a_2, a_1, & a_0.

For a K-th order LRRWCC, K-boundary values are req. to determine numeric f^n uniquely.

Homogeneous Solutions ($f(r)$ should be $= 0$)

eg. $a_r = a_{r-1} + a_{r-2}$

$\Rightarrow a_r - a_{r-1} - a_{r-2} = 0$

characteristic Eq. is $\alpha^2 - \alpha - 1 = 0$

$$\alpha_1 = \left(\frac{1+\sqrt{5}}{2}\right), \quad \alpha_2 = \left(\frac{1-\sqrt{5}}{2}\right).$$

Homogeneous solⁿ, given by,

$$a_r = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r$$

Now, if $\boxed{a_0 = 1}$ & $\boxed{a_1 = 1}$ then A_1 & A_2 are evaluated.

If some roots are multiple roots.

eg. $a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$

characteristic eq. is :-

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$

Put $\alpha = -2$, $-8 + 24 - 24 + 8 = 0$. , Hence

$$
\begin{array}{r|l}
 & \alpha^2 + 4\alpha + 4 \\
\alpha+2 & \alpha^3 + 6\alpha^2 + 12\alpha + 8 \\
 & -\alpha^3 - 2\alpha^2 \\
\hline
 & 4\alpha^2 + 12\alpha \\
 & -4\alpha^2 - 8\alpha \\
\hline
 & 4\alpha + 8 \\
 & -4\alpha \pm 8 \\
\hline
 & 0
\end{array}
$$

$\therefore (\alpha + 2)(\alpha^2 + 4\alpha + 4) = 0$

$\alpha = -2, -2, -2.$

Thus,

$$a_r = (A_1 r^2 + A_2 r + A_3)(-2)^r$$

If α is the root of multiplicity m (in above eg $m=3$), then the corresponding homogeneous solⁿ. is.

$$a_r = \left(A_1 r^{m-1} + A_2 r^{m-2} + \dots + A_{m-1} r + A_m\right)\alpha^r$$

eg. $4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$

the char. Eq. is :-

$$4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$$

$$\begin{array}{r|l} & 4\alpha^2 - 4\alpha + 1 \\ \hline (\alpha-4) & 4\alpha^3 - 20\alpha^2 + 17\alpha - 4 \\ & 4\alpha^3 - 16\alpha^2 \\ & - \quad (+) \\ \hline & -4\alpha^2 + 17\alpha \\ & -4\alpha^2 + 16\alpha \\ & + \quad (-) \\ \hline & \alpha - 4 \\ & \alpha - 4 \\ \hline & \times \end{array}$$

Put $\alpha = 4$, $256 - 320 + 68 - 4$

$= 324 - 324 = 0$.

thus $(\alpha-4)(4\alpha^2 - 4\alpha + 1) = 0$

$\alpha = 4, \frac{1}{2}, \frac{1}{2}$.

(Here $m = 2$) thus

$$a_r = (A_1 r + A_2)\left(\frac{1}{2}\right)^r + A_3 4^r.$$

Particular solution

To find Particular solⁿ, we set up general form ~~of~~ acc. to the form of $f(r)$.

(I) # when $f(r)$ is of the form of a polynomial of degree t in r,

$$f_1 r^t + f_2 r^{t-1} + \dots + f_t r + f_{t+1}$$, the corresponding particular solution will be of the form,

$$P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1}.$$

eg. $$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1 \quad — (1)$$

Here $f(r) = 3r^2 - 2r + 1$

thus, corr. Particular sol^n is

$$a_r = P_1 r^2 + P_2 r + P_3 \quad — (2)$$

subs. (2) in (1), we get,

$$P_1 r^2 + P_2 r + P_3 + 5P_1(r-1)^2 + 5P_2(r-1) + 5P_3 + 6P_1(r-2)^2 + 6P_2(r-2) + 6P_3 = 3r^2 - 2r + 1$$

$$\Rightarrow 12P_1 r^2 - (34P_1 - 12P_2) r + (29P_1 - 17P_2 + 12P_3) = 3r^2 - 2r + 1$$

Equating both sides,

$$12P_1 = 3, \quad 34P_1 - 12P_2 = 2, \quad 29P_1 - 17P_2 + 12P_3 = 1$$

thus, $P_1 = \frac{1}{4}$, $P_2 = \frac{13}{24}$, $P_3 = \frac{71}{288}$. Hence the Particular Sol. is,

$$\boxed{a_r = \frac{1}{4} r^2 + \frac{13}{24} r + \frac{71}{288}}$$

eg. $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$ — ①

Here $f(r) = 3r^2$

or

$= 3r^2 + 0r + 0$

Hence, Particular solⁿ is,

$a_r = P_1 r^2 + P_2 r + P_3$ — ②

Put ② in ①, & simplyfying

$P_1 r^2 + P_2 r + P_3 + 5P_1(r-1)^2 + 5P_2(r-1) + 5P_3 + 6P_1(r-2)^2 + 6P_2(r-2) + 6P_3 = 3r^2$

Equating,

$P_1 = \frac{1}{4}$; $P_2 = \frac{17}{24}$, $P_3 = \frac{115}{288}$

∴ Particular solⁿ. is $a_r = \frac{1}{4}r^2 + \frac{17}{24}r + \frac{115}{288}$.

eg. $a_r - 5a_{r-1} + 6a_{r-2} = 1$ — ①

Here, $f(r) = 1$

or

$f(r) = 1 \cdot r^0$, Thus, Part. solⁿ is $a_r = P$ — ②

Put ② in ①, weget,

$P - 5P + 6P = 1$

$2P = 1$, $P = \frac{1}{2}$

Thus, $\boxed{a_r = \frac{1}{2}}$

(II) When $f(r)$ is of the form β^r, the corresponding particular solution is of the form $P\beta^r$, if β is not a characteristic root of the difference equation.

If $f(r)$ is of the form

$$\left(F_1 r^t + F_2 r^{t-1} + \dots + F_t r + F_{t+1}\right)\beta^r,$$ the corresponding particular solⁿ. is of the form,

$$\left(P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1}\right)\beta^r,$$ if β is not characteristic root of difference eq.

eg. $a_r + a_{r-1} = 3r.2^r$ — (1) char. eq: $\alpha + 1 = 0$, $\alpha = -1 \neq \beta$

The general form of particular solⁿ. is:-

$$a_r = (P_1 r + P_2)2^r \quad [\text{ie } (3r^1 + 0r^0)2^r] \quad -(2)$$

Put (2) in (1), we get,

$$(P_1 r + P_2)2^r + (P_1(r-1) + P_2)2^{r-1} = 3r.2^r$$

$$= P_1 r 2^r + P_2 2^r + \frac{(P_1 r - P_1 + P_2)2^r}{2} = 3r.2^r$$

$$= \underbrace{P_1 r 2^r} + P_2 2^r + \underbrace{\frac{1}{2}P_1 r 2^r} - \frac{P_1}{2}.2^r + \frac{P_2 2^r}{2} = 3r.2^r$$

$$= \frac{3}{2}P_1 r 2^r + 2^r\left(P_2 - \frac{P_1}{2} + \frac{P_2}{2}\right) = 3r.2^r$$

$$= \frac{3}{2}P_1 r 2^r + \left(\frac{3}{2}P_2 - \frac{P_1}{2}\right)2^r = 3r.2^r + 0.2^r$$

Equating both sides, $\frac{3}{2}P_1 = 3$ & $\left(\frac{3}{2}P_2 - \frac{P_1}{2}\right) = 0$

$$\therefore P_1 = 2, \quad P_2 = \frac{2}{3}$$

∴ Particular solⁿ. is $a_r = \left(2r + \frac{2}{3}\right)2^r$

(III) for the case that β is a characteristic root of multiplicity $(m-1)$, when $f(r)$ is of the form,

$$\left(F_1 r^t + F_2 r^{t-1} + \dots + F_t r + F_{t+1}\right)\beta^r,$$

the corresponding Particular solⁿ is of the form,

$$r^m \left(P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1}\right)\beta^r$$

eg.

$$a_r - 2a_{r-1} = 3.2^r \quad -(1)$$

$$f(r) = 3.2^r \quad \text{or} \quad 3.r^0.2^r. \quad [\beta = 2]$$

characteristic Eq. :- $\alpha - 2 = 0$

$\boxed{\alpha = 2 = \beta}$ & $\boxed{m = 1}$

then Particular solⁿ is :-

$$a_r = Pr.2^r \quad -(2)$$

Put (2) in (1), we get,

$$Pr2^r - 2\left(P(r-1).2^{r-1}\right) = 3.2^r$$

$$\Rightarrow Pr2^r - \frac{(2Pr - 2P)2^r}{2} = 3.2^r$$

$$\Rightarrow Pr.2^r - (Pr - P)2^r = 3.2^r$$

$$\Rightarrow \cancel{Pr.2^r} - \cancel{Pr2^r} + P2^r = 3.2^r$$

$$\boxed{P = 3}$$

∴ the Particular solⁿ. is

$$\boxed{a_r = 3r.2^r}$$

eg. $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$ — (1)

char. eq. is $\alpha^2 - 4\alpha + 4 = 0$

$\alpha = 2, 2 = \beta$ $\boxed{m = 2}$ Hence (III case)

Particular solⁿ is

$$a_r^p = r^2(P_1 r + P_2)2^r \quad — (2)$$

Put (2) in (1), we get,

$$6P_1 r 2^r = r.2^r$$

$$(-6P_1 + 2P_2)2^r = 2^r,$$

$$P_1 = \frac{1}{6}, \quad P_2 = 1$$

$$\therefore a_r^p = r^2\left(\frac{r}{6} + 1\right)2^r.$$

eg. $a_r = a_{r-1} + 7$

$\therefore a_r - a_{r-1} = 7r^0 1^r$ — (1)

char. Eq. is $\alpha - 1 = 0$

$\boxed{\alpha = 1 = \beta}$, $m = 1$.

case III,

$a_r^p = r^1 . P . 1^r$ — (2)

Put (2) in (1),

$rP - P(r-1) = 7$

$\cancel{rP} - \cancel{rP} + P = 7$

$\boxed{P = 7}$

$a_r^p = 7r$

$a_r = a_{r-1} + 7$

$\Rightarrow a_r - a_{r-1} = 7$ — (1)

Here $f(r) = 7.r^0$

$\therefore$ the Part. solⁿ. is

$a_r = P$ — (2)

$P - P = 7$ (Not true)

Hence consider

$f(r) = 7.r^0.1^r$

eg. $a_r - 2a_{r-1} + a_{r-2} = 7$ — (1)

Here $f(r) = 7.r^0$

Part. solⁿ is $a_r = P$ — (2)

Put (2) in (1)

$P - 2P + P = 7$ (Not true)

$\therefore$ Consider $f(r) = 7.r^0.1^r$

Now $\beta = 1$, find char. root.

char. eqⁿ. is $\alpha^2 - 2\alpha + 1 = 0$

$\alpha^2 - \alpha - \alpha + 1 = 0$

$\alpha(\alpha - 1) - 1(\alpha - 1) = 0$

$(\alpha - 1)(\alpha - 1) = 0$, $\boxed{\alpha = 1, 1 = \beta}$ $\boxed{m = 2}$

$\therefore a_r^p = r^2(P).1^r$ — (3)

Put (3) in (1)

$r^2P - 2(P(r-1)^2) + P(r-2)^2 = 7$

$\Rightarrow r^2P - 2P(r^2 - r + 1) + P(r^2 + 4 - 4r) = 7$

$= \cancel{r^2P} - \cancel{2Pr^2} + 2Pr - 2P + \cancel{Pr^2} + 4P - 4Pr = 7$

$\Rightarrow -2Pr + 2P = 7$, Equating,

$2P = 7$, $\boxed{P = \frac{7}{2}}$

$\therefore \boxed{a_r^p = r^2.\frac{7}{2}}$

$-2Pr \neq 0$

$-\frac{7}{2}r = 0$

$7r = 0$

eg. $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$ — ①

char. Eqn. is $\alpha^2 - 5\alpha + 6 = 0$

$\alpha^2 - 3\alpha - 2\alpha + 6 = 0$

$\alpha(\alpha - 3) - 2(\alpha - 3) = 0$

$(\alpha - 3)(\alpha - 2) = 0$, $\boxed{\alpha = 3, \alpha = 2}$

$\boxed{\alpha = 2 = \beta}$

Since atleast one root is equal to β, we consider $\boxed{fn = 1}$ & discard another root.

∴ The Particular sol^n is :-

$$a_r^p = \overbrace{P_1 r 2^r}^{2^r} + \overbrace{P_2 r + P_3}^{r} \quad — ②$$

Put ② in ①, we get,

$P_1 = -2$, $P_2 = \frac{1}{2}$, $P_3 = \frac{7}{4}$

$$\therefore a_r^p = -r2^{r+1} + \frac{1}{2}r + \frac{7}{4}$$

eg. $a_r - a_{r-1} = 7r \quad -①$

$f(r) = 7r$

Part. solⁿ. is $a_r^p = P_1 r + P_2 \quad -②$

Put ② in ①

$P_1 r + P_2 - (P_1(r-1)) - P_2 = 7r$

$\cancel{P_1 r} + \cancel{P_2} - \cancel{P_1 r} + P_1 - \cancel{P_2} = 7r$

$P_1 = 7r + 0, \quad \boxed{P_1 = 0}, \quad P_2 = N.D.$

$\therefore f(r) = 7r \cdot 1^r \quad \boxed{\beta = 1}$

char. eq :- $\alpha - 1 = 0, \quad \boxed{\alpha = 1 = \beta} \quad \boxed{m = 1}$

$a_r^p = r(P_1 r + P_2) 1^r \quad -③$, Put ③ in ①

$r(P_1 r + P_2) - \left[(r-1)(P_1(r-1) + P_2)\right] = 7r$

$= P_1 r^2 + P_2 r - \left((r-1)\left[P_1 r - P_1 + P_2\right]\right) = 7r$

$= P_1 r^2 + P_2 r - \left(P_1 r^2 - P_1 r + P_2 r - P_1 r + P_1 - P_2\right) = 7r$

$= \cancel{P_1 r^2} + \cancel{P_2 r} - \cancel{P_1 r^2} + P_1 r - \cancel{P_2 r} + P_1 r - P_1 + P_2 = 7r$

$= 2P_1 r - P_1 + P_2 = 7r$

Equating :-

$2P_1 = 7, \quad \boxed{P_1 = \frac{7}{2}}$, Also, $-P_1 + P_2 = 0$

$-\frac{7}{2} + P_2 = 0$

$\boxed{P_2 = \frac{7}{2}}$

$\therefore a_r^p = \frac{7}{2} r + \frac{7}{2}$

9.6. Total solutions

The total solution of a LDEWCC is the sum of two parts, the homogeneous solution & the particular solution.

eg. $a_r + 5a_{r-1} + 6a_{r-2} = 42.4^r$ — (1)

$\Rightarrow$ To find homogeneous solⁿ first, equate (1) to 0,

$a_r + 5a_{r-1} + 6a_{r-2} = 0$

chac. eq. is

$\alpha^2 + 5\alpha + 6 = 0$

$\alpha^2 + 3\alpha + 2\alpha + 6 = 0$

$\alpha(\alpha+3) + 2(\alpha+3) = 0$

$\alpha = -3, \alpha = -2.$

Thus, $a_r^{H} = A_1(-2)^r + A_2(-3)^r$ — (2)

$\Rightarrow$ To find particular solⁿ, $\boxed{\beta = 4}$

$a_r^{p} = Pr^{0}4^r$ [Because $\alpha \neq \beta$] — (3)

Put (3) in (1)

$P.4^r + 5(P4^{r-1}) + 6P4^{r-2} = 42.4^r$

$\Rightarrow P.4^r + \frac{5}{4}.P4^r + \frac{6}{16}P4^r = 42.4^r$

Equating $P + \frac{5}{4}P + \frac{3}{8}P = 42 \Rightarrow \frac{8P + 10P + 3P}{8} = 42$

$\Rightarrow \frac{21}{8}P = 42$, $P = \frac{\cancel{42}^{2} \times 8}{\cancel{21}} = 16.$

$\therefore a_r^{p} = 16.4^r$

Thus $a_r^{t} = a_r^{H} + a_r^{p} = A_1(-2)^r + A_2(-3)^r + 16.4^r.$

[Now, suppose boundary values are given, $a_2 = 278$, $a_3 = 962$.

solving, we get $A_1 = 1$, $A_2 = 2$, Thus, $a_r = (-2)^r + 2(-3)^r + \underline{16.4^r}$ Ans

9.7 Solution by the method of Generating functions

eg $a_r = 3a_{r-1} + 2 \qquad r \geq 1$

Solⁿ: Multiplying both sides, by z^r, we obtain,

$$a_r z^r = 3a_{r-1} z^r + 2z^r \quad , r \geq 1$$

Summing above eq for all $r \geq 1$, we obtain,

$$\sum_{r=1}^{\infty} a_r z^r = 3\sum_{r=1}^{\infty} a_{r-1} z^r + 2\sum_{r=1}^{\infty} z^r \qquad -(1)$$

consider L.H.S. first.

$$\sum_{r=1}^{\infty} a_r z^r = a_1 z^1 + a_2 z^2 + a_3 z^3 + \ldots = A(z) - a_0 z^0 \qquad -(2)$$

consider R.H.S., first Part,

$$3\sum_{r=1}^{\infty} a_{r-1} z^r = 3\left(a_0 z^1 + a_1 z^2 + \ldots\right)$$

or

$$3\sum_{r=1}^{\infty} a_{r-1} z^{r-1} . z = 3z\left(a_0 z^0 + a_1 z^1 + a_2 z^2 + \ldots\right) -$$

$$= 3z\left(A(z)\right) \qquad -(3)$$

Consider R.H.S, second Part,

$$2\sum_{r=1}^{\infty} z^r = 2\left(z^1 + z^2 + z^3 + \ldots\right)$$

$$= 2z\left(1^0.z^0 + 1^1.z^1 + 1^2.z^2 + \ldots\right)$$

$$= 2z.\left(\frac{1}{1-z}\right) \qquad -(4)$$

Put (2), (3) & (4) in Eq (1), we get,

$$A(z) - a_0 = 3z.A(z) + \frac{2z}{1-z}$$

$$\Rightarrow A(z) - 1 = 3z(A(z)) + \frac{2z}{1-z} \qquad [\, a_0 = 1 \,]$$

$$\Rightarrow A(z) - 3z\,A(z) = \frac{2z}{1-z} + 1$$

$$\Rightarrow A(z)(1-3z) = \frac{2z}{1-z} + 1$$

$$\Rightarrow A(z) = \frac{\cancel{2}z+1-z}{1-z} \times \frac{1}{1-3z}$$

$$= \frac{z+1}{1-z} \cdot \left(\frac{1}{1-3z}\right)$$

$$= \frac{2}{1-3z} - \frac{1}{1-z} \qquad [\text{Partial fractions}]$$

$$\therefore a_r = 2\,3^r - 1^r$$

or

$$\boxed{a_r = 2.3^r - 1}$$