ch-1. (Liu). Sets

set :- collection of distinct objects.

$S = \{a, b, c\}$

$S = \{a, a, b, c\}$ // redundent Representation.

eg: $S = \{2, 4, 6, 8, 10\}$

$S = \{x \mid x$ is an even +tive integer not larger than $10\}$.

empty set = $\{\ \}$ or ϕ

$S = \{a, b, c\}$ // three elements.

$S = \{\{a, b, c\}, d\}$ // two elements.

Subset $(P \subseteq Q)$:- for two set P & Q, P is subset of Q if every element in P is also an element in Q

EQual :- Two sets are equal if they contain same collection of elements. ($P \subseteq Q$ & $Q \subseteq P$)

Proper subset :- If P is not equal to Q & is a subset of Q, then P is a proper subset of Q. ie there is atleast one element in Q that is not in P. (denoted by $P \subset Q$).

union :- $(P \cup Q)$:- $\{a, b\} \cup \{c, d\} = \{a, b, c, d\}$

$\{a, b\} \cup \{a, c\} = \{a, b, c\}$

$\{a, b\} \cup \phi = \{a, b\}$

$\{a, b\} \cup \{\{a, b\}, c\} = \{a, b, c, \{a, b\}\}$

Intersection :- $(P \cap Q)$

$\{a, b\} \cap \{a, c\} = \{a\}$

$\{a, b\} \cap \{c, d\} = \phi$

$\{a, b\} \cap \phi = \phi$

Properties.

$P \cup Q = Q \cup P$

$P \cap Q = Q \cap P$

$P \cup Q \cup R = (P \cup Q) \cup R = \cancel{P \cup (Q \cup R)}$

$P \cap Q \cap R = (P \cap Q) \cap R$

$R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$

$R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$

$R \cap (P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_k) = (R \cap P_1) \cup (R \cap P_2) \cup \ldots \cup (R \cap P_k)$

$R \cup (P_1 \cap P_2 \cap P_3 \cap \ldots \cap P_k) = (R \cup P_1) \cap (R \cup P_2) \cap \ldots \cap (R \cup P_k)$

difference $(P-Q)$:- $(P-Q)$ is the set containing exactly those elements in P that are not in Q.

$\{a,b,c\} - \{a\} = \{b,c\}$

$\{a,b,c\} - \{d,e\} = \{a,b,c\}$

$\{a,b,c\} - \{a,d\} = \{b,c\}$

complement

If Q is a subset of P, the set P − Q is also the complement of Q with respect to P.

$P = \{a,b,c,d\}$, $Q = \{c,d\}$

$P - Q := \overline{Q} = \{a,b\}$ // complement of Q w.r.t. P.

Symmetric difference $(P \oplus Q)$:- $P \oplus Q$ is the set containing exactly all the elements that are in P or in Q but not in both. $P \oplus Q = (P \cup Q) - (P \cap Q)$

eg. $\{a,b\} \oplus \{a,c\} = \{b,c\}$

$\{a,b\} \oplus \phi = \{a,b\}$

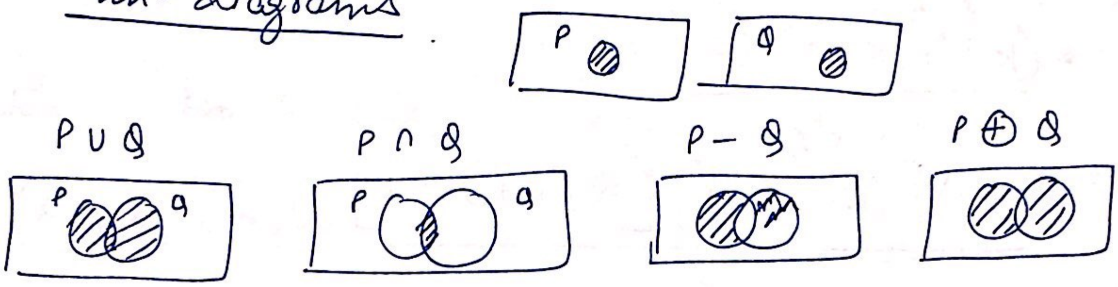
$\{a,b\} \oplus \{a,b\} = \phi$

Power set of A. denoted by $\mathcal{P}(A)$ is the set containing exactly all the subsets of A.

eg. $P(\{a, b\}) = \{\{\}, \{a\}, \{b\}, \{a,b\}\}$

for any set A, $\{\} \in P(A)$

Venn-diagrams.



Mathematical Induction

for a statement involving a natural number n,

Basis 1. The statement is true for $n = n_0$ and

Ind. hypothesis 2. The statement is true for $n = K+1$, assuming that the statement is true for $n = K$, $(K \geq n_0)$.

eg. show that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \geq 1$

ans. 1. Basis, for $n = 1$, $1^2 = \frac{1(1+1)(2+1)}{6} = 1$

2. Induction step :- Assume that,

$$1^2 + 2^2 + \dots + K^2 = \frac{K(K+1)(2K+1)}{6} \text{ (given)}$$

Thus, Prove for, $1^2 + 2^2 + \dots + K^2 + (K+1)^2 = \frac{K(K+1)(2K+1)}{6} + (K+1)^2$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Principle of Inclusion & Exclusion

we know that,

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$(A_1 \cup A_2 \cup A_3) = A_1 + A_2 + A_3 - (A_1 \cap A_2) - (A_1 \cap A_3) - (A_2 \cap A_3)$$
$$+ (A_1 \cap A_2 \cap A_3)$$

Principle :- for the sets, $A_1, A_2, \ldots, A_r$, we have,

$$(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_r) = \sum_{i=1}^{r} A_i - \sum_{1 \le i < j \le r} (A_i \cap A_j)$$
$$+ \sum_{1 \le i < j < k \le r} (A_i \cap A_j \cap A_k) + \ldots + (-1)^{r-1} (A_1 \cap A_2 \cap \ldots A_r)$$

Proof: Proof is by induction on the no. of sets r.

1. Basis of Induction :- $(A_1 \cup A_2) = A_1 + A_2 - (A_1 \cap A_2)$ — ①
2. Induction hypothesis :- Assume that the Principle is valid for $r-1$ sets. & Prove it for 'r' sets.

Now, consider,

$(A_1 \cup A_2 \cup \ldots \cup A_{r-1})$ and A_r as two sets.

then acc. to ①, we have

$$(A_1 \cup A_2 \cup \ldots \cup A_{r-1}) \cup A_r = (A_1 \cup A_2 \cup \ldots \cup A_{r-1}) + A_r$$
$$- (A_r \cap (A_1 \cup A_2 \cup \ldots \cup A_{r-1})) \quad — ②$$

Now, consider the last part of Eq. ② first,

$$(A_r \cap (A_1 \cup A_2 \cup \ldots \cup A_{r-1})) = (A_r \cap A_1) \cup (A_r \cap A_2) \cup \ldots \cup (A_r \cap A_{r-1})$$

According to Induction hypothesis, for the $r-1$ sets $\Rightarrow$ ie, $(A_r \cap A_1), (A_r \cap A_2), (A_r \cap A_3) \ldots, (A_r \cap A_{r-1})$, we have

$$(A_r \cap A_1) \cup (A_r \cap A_2) \cup \ldots \cup (A_r \cap A_{r-1})$$
$$= (A_r \cap A_1) + (A_r \cap A_2) + \ldots + (A_r \cap A_{r-1})$$
$$- ((A_r \cap A_1) \cap (A_r \cap A_2)) - ((A_r \cap A_1) \cap (A_r \cap A_3)) - \ldots$$
$$- \ldots\ldots$$
$$+ ((A_r \cap A_1) \cap (A_r \cap A_2) \cap (A_r \cap A_3)) + \ldots\ldots$$
$$- \ldots\ldots$$
$$+ (-1)^{r-2} ((A_r \cap A_1) \cap (A_r \cap A_2) \cap \ldots \cap (A_r \cap A_{r-1}))$$

$$= (A_r \cap A_1) + (A_r \cap A_2) + \ldots + (A_r \cap A_{r-1})$$
$$- (A_r \cap A_1 \cap A_2) - (A_r \cap A_1 \cap A_3) - \ldots$$
$$- \ldots\ldots$$
$$+ (A_r \cap A_1 \cap A_2 \cap A_3) + \ldots\ldots$$
$$- \ldots\ldots$$
$$+ (-1)^{r-2} (A_r \cap A_1 \cap A_2 \cap \ldots \cap A_{r-1}) \quad — \ 3$$

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Now, consider the first part of Eq (2) ie,

$(A_1 \cup A_2 \cup \ldots \cup A_{r-1})$

Also, Acc. to induction hypothesis, we have,

$$(A_1 \cup A_2 \cup \ldots \cup A_{r-1}) = A_1 + A_2 + \ldots + A_{r-1}$$
$$- (A_1 \cap A_2) - (A_1 \cap A_3) - (A_1 \cap A_4) \ldots\ldots$$
$$+ \ldots\ldots$$
$$+ (-1)^{r-2} (A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_{r-1}) \quad (4)$$

substituting (3) & (4) in (2), we get,

Eq. (4)
$$= \left[(A_1 + A_2 + \ldots + A_{r-1}) - (A_1 \cap A_2) - (A_1 \cap A_3) - \ldots - (A_1 \cap A_{r-1}) + \ldots\ldots + (-1)^{r-2} (A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_{r-1}) \right.$$

Eq (2)
$$\left\{ + A_r - \Big((A_r \cap A_1) + (A_r \cap A_2) + \ldots + (A_r \cap A_{r-1}) - (A_r \cap A_1 \cap A_2) \ldots\ldots + (A_r \cap A_1 \cap A_2 \cap A_3) + \ldots\ldots + (-1)^{r-2} (A_r \cap A_1 \cap A_2 \cap \ldots \cap A_{r-1}) \Big) \right.$$

= Principle of Inclusion - Exclusion.

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Eg. on Inclusion - Exclusion Principle

Q. 30 cars assembled in factory.

options available are radio, air-conditioner & white wall tires.

15 cars have radios, 8 has A.C, 6 have white wall tires.

3 cars have all three options.

Find atleast how many cars do not have any options.

Sol^n. we know,

$$A_1 = 15, \quad A_2 = 8, \quad A_3 = 6$$

Also, $A_1 \cap A_2 \cap A_3 = 3$

we know that,

$$(A_1 \cup A_2 \cup A_3) = A_1 + A_2 + A_3 - (A_1 \cap A_2) - (A_1 \cap A_3) - (A_2 \cap A_3) + (A_1 \cap A_2 \cap A_3)$$

$$\text{ie} = 15 + 8 + 6 - (A_1 \cap A_2) - (A_1 \cap A_3) - (A_2 \cap A_3) + 3$$

$$= 32 - (A_1 \cap A_2) - (A_1 \cap A_3) - (A_2 \cap A_3)$$

since,

$$(A_1 \cap A_2) \geq (A_1 \cap A_2 \cap A_3)$$

$$(A_1 \cap A_3) \geq (A_1 \cap A_2 \cap A_3)$$

$$(A_2 \cap A_3) \geq (A_1 \cap A_2 \cap A_3)$$

we have,

$$(A_1 \cup A_2 \cup A_3) \leq 32 - 3 - 3 - 3 = 23$$

ie atmost '23' cars have one or more options.

$\therefore$ $30 - 23 = 7$, ie atleast 7 cars do not have any options.