Discrete Numeric f^n and Generating f^n.

8.1

Numeric f^n :-

\# use bold face lower case letters for notation. For eg. '**a**'.

\# for numeric f^n '**a**', we use $a_0, a_1, a_2, \ldots, a_r, \ldots$ to denote the value of a f^n at $0, 1, 2, \ldots, r, \ldots$.

For eg:-

① $a_r = 7r^3 + 1 \quad , r \geq 0$

$$\begin{array}{ccccc} a_0, & a_1, & a_2, & \ldots & \\ (\cancel{8}, & 1, & 8, & 57, & \ldots) \end{array}$$

② $b_r = \begin{cases} 2r & 0 \leq r \leq 11 \\ 3^r - 1 & r > 11 \end{cases}$

$$\begin{array}{ccccccccccccc} a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & a_7, & a_8, & a_9, & a_{10}, & a_{11}, & a_{12} \ldots) \\ (0, & 2, & 4, & 6, & 8, & 10, & 12, & 14, & 16, & 18, & 20, & 22, & 3^{12}-1, \ldots) \end{array}$$

③ $c_r = \begin{cases} -4 & , r = 3, 5, 7 \\ 0 & , \text{otherwise} \end{cases}$

$$\begin{array}{ccccccccccc} a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & a_7, & a_8, & a_9 & \ldots \\ (0, & 0, & 0, & -4, & 0, & -4, & 0, & -4, & 0, & 0, & \ldots) \end{array}$$

④ $d_r = \begin{cases} 2 + r & 0 \leq r \leq 5 \\ 2 - r & r > 5, r \text{ is odd.} \\ 2/r & r > 5, r \text{ is even.} \end{cases}$

$$\begin{array}{cccccccccc} a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & a_7, & a_8, & a_9 \ldots \\ (2, & 3, & 4, & 5, & 6, & 7, & \frac{2}{6}, & -5, & \frac{2}{8}, & -7, \ldots) \end{array}$$

: When there is one simple expression for the value of the numeric f^n for every r (ie $r \geq 0$) such as a_r (above eg ①) then, we use notation $a = 7r^3 + 1$.

eg:

suppose we deposit \$ 100 in savings account at interest rate of 7% per year, compounded annually. Find numeric function that denote the money in A/c at the end of each year.

Solⁿ: At the end of first year :-

$$100 \times \frac{7}{100} = 7$$

$$\therefore 100 + 7 = \$107.$$

At the end of second year

$$107 \times \frac{7}{100} = 7.49.$$

$$\therefore 107 + 7.49 = \$114.49.$$

At the end of third year,

$$114.49 \times \frac{7}{100} = 8.01$$

$$\therefore 114.49 + 8.01 = \$122.50$$

Thus, the series is

$$(100, 107, 114.49, 122.50, \ldots)$$

or as $a_r = (100)(1.07)^r \quad , r \geq 0$

or as

$$a = (100)(1.07)^r$$

Comp. Int. formula :-

$$C.I = P\left(1 + \frac{r}{100}\right)^t$$

$$= 100\left(1 + \frac{7}{100}\right)^r$$

Scanned with CamScanner

(height above ground)

eg. Let a_r denote the altitude of an aircraft, in thousands of feet, at the r^{th} minute.

\# suppose the aircraft takes off after spending 10 min. on the ground, (ie 0 , $0 \le r \le 10$)

\# climbs up at a uniform speed to a cruising altitude of 30,000 feet in 10 min. (ie in 1 min, altitude is 3000 feet, after 2 min, alti. is 6,000 feet)

(ie $3(r-10)$, $11 \le r \le 20$) or [$3000(r-10)$]

\# starts to descend uniformly after 110 min of flying time (ie for 100 min, altitude is same)

[ie $30, 21 \le r \le 120$] or ($30{,}000$, $21 \le r \le 120$)

next to descend uniformlly, in (1 min, 3000 feet down)

ie ($3(130-r)$, $121 \le r \le 130$)

or

($3000(130-r)$, $121 \le r \le 130$

\# and lands 10 min later.

ie 0 , $r \ge 131$

$$\therefore \quad a_r = \begin{cases} 0 & , 0 \le r \le 10 \\ 3000(r-10), & 11 \le r \le 20 \\ 30{,}000 & , 21 \le r \le 120 \\ 3{,}000(130-r), & 121 \le r \le 130 \\ 0 & , r \ge 131 \end{cases}$$

8.2. Manipulation of Numeric functions

Sum of Numeric f^n :- is a numeric f^n whose value at r is equal to the sum of the values of the two numeric functions at r.

eg.
$$a_r = \begin{cases} 0 & , 0 \le r \le 2 \\ 2^{-r} + 5 & , r \geq 3 \end{cases}$$

$$(0, 0, 0, 2^{-3}+5, 2^{-4}+5, 2^{-5}+5 + \text{---})$$

$$b_r = \begin{cases} 3 - 2^r & , 0 \le r \le 1 \\ r+2 & , r \geq 2 \end{cases}$$

$$(3-2^0, 3-2^1, 4, 2+2, 3+2, \text{----})$$

or
$$c_r = a_r + b_r = \begin{cases} 3-2^r, & 0 \le r \le 1 \\ 4, & r = 2 \\ 2^{-r}+r+7, & r \geq 3 \end{cases}$$

Product of two numeric f^n is a numeric f^n whose value at r is equal to the product of the values of the two numeric functions at r.

for eg.

$$a_r = \begin{cases} 0, & 0 \le r \le 2 \\ 2^{-r} + 5, & r \ge 3 \end{cases}$$

$$b_r = \begin{cases} 3 - 2^r, & 0 \le r \le 1 \\ r + 2, & r \ge 2 \end{cases}$$

$$c_r = a_r \cdot b_r = \begin{cases} 0, & 0 \le r \le 2 \\ (2^{-r} + 5)(r + 2), & r \ge 3 \end{cases}$$

\# Let 'a' be a numeric function and 'i' a positive integer. we use $S^i a$ to denote a numeric function such that its value at r is 0 for $r = 0, 1, 2, \ldots, i-1$ and is a_{r-i} for $r \ge i$.

$$a_r = \begin{cases} 1 & 0 \le r \le 10 \\ 2 & r \ge 11 \end{cases}$$

$r=$	0	1	2	3	4	5	6	7	8	9	10
$a_r=$	1	1	1	1	1	1	1	1	1	1	1
$b_r=$	0	0	0	0	0	1	1				

Find $S^5 a$.

Let $S^5 a$ be b, Then $b_r = \begin{cases} 0, & 0 \le r \le 4 \\ 1, & 5 \le r \le 15 \\ 2, & r \ge 16 \end{cases}$

Here $i = 5$

Thus in aircraft example, $S^{17}a$ describes the altitude when take off is delayed by 17 minutes.

#. Let 'a' be a numeric function and i a positive integer, we use $S^{-i}a$ to denote a numeric function such that its value at r is a_{r+i} for $r \geq 0$.

eg: $$a_r = \begin{cases} 1 & , 0 \leq r \leq 10 \\ 2 & , r \geq 11 \end{cases}$$

find $S^{-7}a$. , Here $i = 7$.

sol^n. Let $S^{-7}a = b$.

$r =$	0	1	2	3	4	5	6	7	8	9	10	11	12
$a_r =$	1	1	1	1	1	1	1	1	1	1	1	2	2
$b_r =$	1	1	1	1	2	2	2	2	. . .				

ie $$b_r = \begin{cases} 1 & , 0 \leq r \leq 3 \\ 2 & , r \geq 4. \end{cases}$$

or we can say that in aircraft example, $S^{-7}a$ describes the altitude when takeoff is taken 7 minutes earlier

accumulated sum of a numeric function 'a' is a numeric function whose value at r is equal to

$$\sum_{i=0}^{r} a_i .$$

For eg:- Let 'a' describe the monthly earnings of an employee. Let 'b' be the accumulated sum of 'a'. Then 'b' gives ~~ea~~ employee's accumulated earnings by month.

eg. Let $a_r = 100(1.07)^r \quad , r \geq 0$

Let b be the accumulated sum of a. Then,

$$b_r = \sum_{i=0}^{r} a_i = \sum_{i=0}^{r} 100(1.07)^i =$$

$$\Rightarrow \quad = \frac{10,000}{7}\left[(1.07)^{r+1} - 1\right] \qquad r \geq 0$$

Now, if we deposit \$100 each year in a savings account at an annual comp. interest rate of 7%, then b_r is the total amount we have in the account after 'r' years.

Forward Difference

Forward Difference of a numeric function 'a' is denoted by Δa, whose value at r is equal to

$a_{r+1} - a_r$.

r	0	1	2	3	4	5
a_r	0	0	0	$2^{-3}+5$	$2^{-4}+5$	$2^{-5}+5$
b_r	0	0	$2^{-3}+5$			

For eg. $a_r = \begin{cases} 0 & 0 \le r \le 2 \\ 2^{-r}+5 & r \ge 3 \end{cases}$

$b_r = a_{r+1} - a_r$
$b_0 = a_1 - a_0$
$b_1 = a_2 - a_1$

then Δa or $b_r = \begin{cases} 0 & 0 \le r \le 1 \\ \frac{41}{8} & r = 2 \\ -2^{-(r+1)} & r \ge 3 \end{cases}$

Backward Difference

Backward Difference of a numeric f^n 'a' is denoted by ∇a, whose value is equal to a_0 at $r = 0$ &

$a_r - a_{r-1}$ at $r \ge 1$.

$r =$ 0	1	2	3	4	5
$a_r =$ 0	0	0	$2^{-3}+5$	$2^{-4}+5$	. . .
$C_r =$ 0	0	0	$2^{-3}+5$,		

eg. $a_r = \begin{cases} 0 & 0 \le r \le 2 \\ 2^{-r}+5 & r \ge 3 \end{cases}$

$C_r = a_r - a_{r-1}$
$C_1 = a_1 - a_0$
$C_3 = a_3 - a_2$
$C_4 = a_4 - a_3$
$= (2^{-4}+5) - (2^{-3}+5)$
$(2^{-r}+5) - (2^{-(r-1)}+5)$
$= -2^{-r}$

then ∇a or $C_r = \begin{cases} 0 & , 0 \le r \le 2 \\ \frac{41}{8} & r = 3 \\ -2^{-r} & r \ge 4 \end{cases}$

Note that $S^{-1}(\nabla a) = \Delta a$.

Convolution of 'a' and 'b'. $(a * b)$.

Let a & b be two numeric f^n, then convolution of a and b, denoted by $a * b$, is a numeric f^n c such that,

$$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_i b_{r-i} + \cdots + a_{r-1} b_1 + a_r b_0 = \sum_{i=0}^{r} a_i b_{r-i}$$

eg. Let a and b be two numeric functions such that,

$$a_r = 3^r \quad r \geq 0$$

and $$b_r = 2^r \quad r \geq 0$$

Then, for $c = a * b$

$$c_r = \sum_{i=0}^{r} 3^i 2^{r-i} \quad , r \geq 0.$$

ie $$c_0 = 3^0 . 2^0 = 1$$

$$c_1 = \sum_{i=0}^{1} 3^i . 2^{1-i} = 3^0 . 2^1 + 3^1 . 2^0 = 2 + 3 = 5$$

$$c_2 = \sum_{i=0}^{2} 3^i . 2^{2-i} = 3^0 . 2^2 + 3^1 . 2^1 + 3^2 . 2^0 = 4 + 6 + 9 = 19.$$

8.4 Generating functions

For a numeric f^n $(a_0, a_1, a_2, a_3, \ldots, a_r, \ldots)$, we define an infinite series,

$$a_0 + a_1 z^1 + a_2 z^2 + \ldots + a_r z^r + \ldots$$

which is called generating f^n of the numeric f^n a.

for numeric f^n 'a', Generating f^n is $A(z)$.

for eg:- Numeric f^n = $(3^0, 3^1, 3^2, 3^3, \ldots, 3^r, \ldots)$

generating f^n = $3^0 + 3^1 z + 3^2 z^2 + 3^3 z^3 + \ldots + 3^r z^r + \ldots$

OR

$$\frac{1}{1-3z} \quad \text{(in closed form)}$$

eg. Numeric f^n :- $a_r = 7 \cdot 3^r$, $r \geq 0$

generating f^n :- $A(z) = 7\left(\frac{1}{1-3z}\right)$

(# ie if, $b = \alpha a$, then $B(z) = \alpha A(z)$)

eg. Numeric f^n :- $a_r = 3^{r+2}$ $r \geq 0$

generating f^n :- $A(z) = 9\left(\frac{1}{1-3z}\right)$

If $c = a + b$, then $C(z) = A(z) + B(z)$.

eg. $a_r = 2^r + 3^r \quad , r \geq 0$

then $A(z) = \dfrac{1}{1-2z} + \dfrac{1}{1-3z}$

eg. generating f^n :- $A(z) = \dfrac{2 + 3z - 6z^2}{1-2z}$

$$A(z) = \frac{2}{1-2z} + \frac{3z\cancel{(1-2z)}}{\cancel{1-2z}}$$

$$A(z) = 3z + \frac{2}{1-2z}$$

$$A(z) = 3z + 2\left(2^0 z^0 + 2^1 z^1 + 2^2 z^2 + \dots + 2^r z^r + \dots\right)$$

$$= \left(2 \cdot 2^0 \cdot z^0 + (\underset{+3z}{2^1 \cdot 2 \cdot z^1}) + 2^2 \cdot 2 \cdot z^2 + \dots + 2^r \cdot 2 \cdot z^r + \dots\right)$$

$$a_r = \begin{cases} 2 & , r = 0 \\ 7 & , r = 1 \\ 2^{r+1} & , r \geq 2 \end{cases}$$

---

eg. $a_r = \begin{cases} 1 & , r = 0 \\ 4 & , r = 1 \\ 2^r & r \geq 2 \end{cases}$ Find generating f^n. (Reverse can Also be the Ques)

$$A(z) = 1 \cdot z^0 + 4 \cdot z^1 + 2^2 \cdot z^2 + 2^3 \cdot z^3 + \dots\dots$$

$$= 1 z^0 + \underline{\underline{2^1 z^1}} + 2^1 z^1 + 2^2 z^2 + 2^3 z^3 + \dots$$

$$= 2^1 z^1 + \frac{1}{1-2z} = \frac{1 + 2z - 4z^2}{1-2z}$$

Let 'a' be a numeric f^n and $A(z)$ its generating f^n.
ie $A(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$
Let 'b' be a numeric f^n such that,

$$b_r = \alpha^r a_r \quad \text{, for some constant } \alpha.$$

$\therefore$ generating f^n of b is,

$$B(z) = a_0 + \alpha^1 a_1 z^1 + \alpha^2 a_2 z^2 + \cdots + \alpha^r a_r z^r + \cdots$$

$$= a_0 + a_1 (\alpha z)^1 + a_2 (\alpha z)^2 + \cdots + a_r (\alpha z)^r + \cdots$$

ie $B(z) = A(\alpha z)$

eg. Numeric f^n :- $a_r = 1 \quad , r \geq 0$

generating f^n :- $A(z) = 1^0 z^0 + 1 z^1 + 1^2 z^2 + \cdots + 1^r z^r + \cdots$

$$A(z) = \frac{1}{1-z}$$

eg. Numeric f^n :- $a_r = \alpha^r \quad r \geq 0$

generating f^n :- $A(z) = \frac{1}{1-\alpha z}$

eg. generating f^n :- $A(z) = \frac{2}{1-4z^2}$

$$A(z) = \frac{2}{(1)^2 - (2z)^2} = \frac{2}{(1-2z)(1+2z)}$$

$$= \frac{1}{1-2z} + \frac{1}{1+2z}$$

$$a_r = (2)^r + (-2)^r \quad , r \geq 0$$

OR $a_r = \begin{cases} 0 & r \text{ is odd} \\ 2^{r+1} & r \text{ is even} \end{cases}$

for the case $C = ab$, there is no simple form in which we can express $C(z)$ in terms of $A(z)$ and $B(z)$.

Let $A(z)$ is the generating f^n of a, then $z^i A(z)$ is the generating f^n of $S^i a$ for i as any +ive integer.

For eg: $A(z) = \dfrac{z^4}{1-2z}$

$$\boxed{z^4(2^0z^0 + 2^1z^1 + 2^2z^2 + 2^3z^3 + \cdots) = 2^0z^4 + 2^1z^5 + 2^2z^6 + \cdots}$$

$\downarrow$

suppose $a = 2^r$ or $A(z) = \dfrac{1}{1-2z}$

then $z^4\left(\dfrac{1}{1-2z}\right) = S^4 a_r$

$$a_r = (2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6 \ldots)$$
$r = 0, 1, 2, 3, 4, 5, 6$

$$S^4 a_r = (0, 0, 0, 0, 2^0 2^4, 2^1 2^5, 2^2, 2^3, 2^4, \ldots)$$
$r = 0, 1, 2, 3, 4, 5, 6, 7, 8$

Thus $S^4 a = \begin{cases} 0 & , 0 \le r \le 3, \\ 2^{r-4} & , r \ge 4 \end{cases}$

$S^{-i} a$ is the numeric f^n. Then the corresponding generating function is $z^{-i}\left[A(z) - a_0 - a_1 z - a_2 z^2 - a_3 z^3 - \cdots - a_{i-1} z^{i-1}\right]$

eg. $a_r = 3^{r+2}$ $r \ge 0$

$= 3^r \cdot 3^2$ $r \ge 0$

$= 9\left(\dfrac{1}{1-3z}\right)$

suppose $a_r = 3^r$ $r \ge 0$

then $a_r = (3^0, 3^1, 3^2, 3^3, 3^4, \ldots)$

If $a_r = 3^{r+2}$, $r \ge 0$

then $a_r = (3^2, 3^3, 3^4, \ldots)$

ie f^n preponed by 2 minutes.

or $A(z) = z^{-2}\left[\dfrac{1}{1-3z} - 1 - 3z\right]$

$= \left(\dfrac{9}{1-3z}\right)$

Scanned with CamScanner

for the case $c = a * b$ (convolution), we can express generating f^n in the form,

$$C(z) = A(z).B(z).$$

eg. Let $C = a * b$

where $a_r = 3^r \quad r \geq 0$

$b_r = 2^r \quad r \geq 0$

since, $A(z) = \dfrac{1}{1-3z}$, $B(z) = \dfrac{1}{1-2z}$

we have,

$$C(z) = A(z).B(z) = \left(\frac{1}{1-3z}\right)\left(\frac{1}{1-2z}\right)$$

$$\Rightarrow \frac{3}{1-3z} - \frac{2}{1-2z}$$

ie $C_r = 3.3^r - 2.2^r = 3^{r+1} - 2^{r+1}$

eg. Let a be an arbitrary numeric f^n and b be the numeric function $(1, 1, 1, 1, \ldots)$. Let $c = a * b$.

so that $C_r = \sum_{i=0}^{r} a_i \, b_{r-i} = \sum_{i=0}^{r} a_i$.

Thus, c is the accumulated sum of a, and the generating f^n of c is :-

$$C(z) = A(z).\,B(z).$$

$$= A(z)\left(\frac{1}{1-z}\right)$$

Scanned with CamScanner

for eg, by letting $A(z)$ be $\frac{1}{1-z}$

ie $a_r = (1, 1, 1, 1, 1, \ldots)$

& $B(z)$ is $\left(\frac{1}{1-z}\right)$ (given)

ie $b_r = (1, 1, 1, 1, 1, \ldots)$

$C = a * b$, we have $C(z) = \left(\frac{1}{1-z}\right) A(z)$.

$$C_r = \sum_{i=0}^{r} a_i \, b_{r-i} = \sum_{i=0}^{r} a_i$$

ie $r = 3$,

$$C_r = \sum_{i=0}^{3} a_i = a_0 + a_1 + a_2 + a_3 = 1 + 1 + 1 + 1 = 4.$$

ie $C_3 = 4$.

if, to find $C_0 = a_0 b_0 = 1$

$C_1 = a_0 b_1 + a_1 b_0 = 1 + 1 = 2$

$C_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1 + 1 + 1 = 3$

$C_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 1 + 1 + 1 + 1 = 4.$

thus, $C_r = (1, 2, 3, 4, \ldots)$ for $r \geq 0$.

eg. Given $a * b = c$. Determine b.

$$a_r = \begin{cases} 1 & r=0 \\ 2 & r=1 \\ 0 & r \geq 2 \end{cases} \qquad c_r = \begin{cases} 1 & r=0 \\ 0 & r \geq 1 \end{cases}$$

Solⁿ.

$$c_0 = a_0 b_0$$

$$1 = 1 . b_0 \quad \Rightarrow b_0 = 1$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$0 = 1 . b_1 + 2 . 1 \quad \Rightarrow b_1 = -2$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$$0 = b_2 + 2.(-2) + 0.1$$

$$0 = b_2 - 4 \quad \Rightarrow b_2 = 4$$

$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$$

$$0 = 1 . b_3 + 2 . 4 + 0.(-2) + 0.1 \quad \Rightarrow b_3 = -8$$

$$c_4 = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0$$

$$0 = 1 . b_4 + 2.(-8) + 0.4 + 0.(-2) + 0.1$$

$$0 = b_4 - 16 \quad \Rightarrow b_4 = 16 \qquad \therefore b = (1, -2, 4, -$$

$$\therefore b_r = \begin{cases} 1 & r=0 \\ (-2)^r & r \geq 1 \end{cases}$$