(90 marks) Mathematical Logic ch-1. (Trembley)

CONNECTIVES

①

1). **Negation** :- If 'P' denotes a statement then negation of 'P' is written as '$\neg P$' & read as 'not P.'

Truth Table

P	$\neg P$
T	F
F	T

2). **Conjunction** :- The conjunction of two statements P and Q is the statement '$P \wedge Q$' which is read as P and Q. The statement $P \wedge Q$ has the truth value T whenever both P & Q have the truth value 'T' otherwise it has truth value F.

Truth Table

P	Q	$P \wedge Q$
T	T	T
T	F	~~T~~ F
F	T	~~T~~ F
F	F	F

3). **Disjunction** :- The Disjunction of two statements P and Q is the statement '$P \vee Q$' has the truth value F only when both P & Q have the truth value F, otherwise it is true.

Truth Table

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

__Atomic / Primary / Simple statements__ :- Those statements which do not contain any connectives are called simple statements.

__Molecular / Composite / Compound statements__ :- Statements that contain one or more primary statements & some connectives

eg :- $\neg P$, $P \vee Q$, $(P \vee Q) \vee (\neg P)$ . . . etc.

__Truth Table__ :- Our basic concern is to determine the truth values of the statement formula for each possible combination of the truth values of the component statements. A table showing all such truth values is called the Truth Table of the formula.

In general, if there are 'n' distinct components in a statement formula, we need to consider 2^n possible combinations of truth values in order to obtain the truth table.

__Conditional statement $(P \to Q)$ OR $(\neg P \vee Q)$__

If P & Q are two statements, then the statement __$P \to Q$__ read as __"If P, then Q"__ is called a conditional statement. The statement $P \to Q$ has the truth value 'F' when Q has the truth value 'F' and 'P' has the truth value 'T', otherwise it has the truth value 'T'.

- $P \to$ antencedent
- $Q \to$ consequent in $(P \to Q)$

Truth Table $(P \to Q)$ (OR) Truth Table $(\neg P \vee Q)$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

=

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

eg. Write the foll. statements in symbolic form :- ②

①. If either Jerry takes calculus or Ken takes sociology, then Larry will take english.

Sol^n: Denoting the statements as

J : Jerry takes calculus.

K : Ken takes sociology.

L : Larry takes english.

the above statement can be symbolized as

$\boxed{(J \vee K) \rightarrow L.}$

②. The crop will be destroyed if there is a flood.

Sol^n. C : Crop will be destroyed

F : There is a flood.

$\boxed{F \rightarrow C.}$

For the Proposition $P \rightarrow Q$,

converse is :- $q \rightarrow p$

contrapositive is :- $\bar{q} \rightarrow \bar{P}$

inverse is :- $\bar{P} \rightarrow \bar{q}$

③. Construct Truth Table for $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F.	T	F
F	T	T	F	F
F	F	T	T	T

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$(P \leftrightarrow Q)$

Biconditional statements $\left[(P \rightleftarrows Q) \text{ OR } (P \to Q) \wedge (Q \to P) \right]$

If 'P' and 'Q' are two statements, then the statement '$P \rightleftarrows Q$' which is read as "P if and only if Q" ("P iff Q") has the truth value T whenever both P and Q have same truth values.

Truth Table

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

eg:- Construct the T.T. for $\neg(P \wedge Q) \rightleftarrows (\neg P \vee \neg Q)$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(P \vee Q) \rightleftarrows (\neg P \vee \neg Q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T.

Tautology:- A statement formula which is true regardless of the truth values of the statements which replace the variable in it is called a universally valid formula or a tautology or a logical truth.

Contradiction:- A statement formula which is false regardless of the truth values of the statements which replace the variables in it is called a contradiction

eg:-

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Well formed formulas. (wff)

A well formed formula can be generated by the following rules :-

1). A statement variable standing alone is a wff.

2). If A is a wff, then $\neg A$ is a wff.

3). If A and B are wff, then $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$ are wff.

4). A string of symbols containing the statement variable, connectives, and parentheses is a wff, iff it can be obtained by finitely many applications of the rules 1, 2 and 3.

eg:- The following are wffs:-

$\neg(P \wedge Q)$, $\neg(P \vee Q)$, $(P \rightarrow (P \vee Q))$, $(P \rightarrow (Q \rightarrow R))$,

$(((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R))$.

The following are not wffs:-

1. $\neg P \vee Q$ [A wff will be either $(\neg P \vee Q)$ or $\neg(P \vee Q)$.
2. $(P \rightarrow Q) \rightarrow (\wedge Q)$ [not wff because $\wedge Q$ is not].
3. $(P \rightarrow Q$. [Note that $(P \rightarrow Q)$ is a wff.]
4. $(P \wedge Q) \rightarrow Q)$ [one of the '(' in the beginning is missing]

   [so, $((P \wedge Q) \rightarrow Q)$ is a wff.]

   while $(P \wedge Q) \rightarrow Q$ is still not a wff.

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Substitution Instance

A formula 'A' is called a substitution Instance of another formula 'B' if 'A' can be obtained from 'B' by substituting formulas for some variables of 'B', with the condition that the same formula is substituted for the same variable each time it occurs.

eg:- $B : P \to (J \wedge P)$

Substitute $R \rightleftarrows S$ for P in B,

$A : (R \rightleftarrows S) \to (J \wedge (R \rightleftarrows S))$.

then A is a substitution Instance of B.

Equivalence of formulas

If the truth value of A is equal to the truth value of B for every one of the 2^n possible sets of truth values assigned to $P_1, P_2, \ldots, P_n$ then A and B are said to be equivalent.

[written as $A \Leftrightarrow B$]

eg 1. $(P \to Q) \Leftrightarrow (\neg P \vee Q)$

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\Leftrightarrow$

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

2.

$$P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R.$$

$\Rightarrow$ $P \to (\neg Q \vee R)$

$\Leftrightarrow$ $\neg P \vee (\neg Q \vee R)$

$\Leftrightarrow$ $(\neg P \vee \neg Q) \vee R$

$\Leftrightarrow$ $\neg (P \wedge Q) \vee R$ $[\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ (Demorgan's law)$]$

$\Leftrightarrow$ $(P \wedge Q) \to R. = R.H.S.$

3. $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$ $\bar{P}(\bar{Q}R) + QR + PR$

$\Leftrightarrow$ $(\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$

$\Leftrightarrow$ $((\neg P \wedge \neg Q) \wedge R)) \vee ((Q \vee P) \wedge R)$

$\Leftrightarrow$ $(\neg (Q \vee P) \wedge R) \vee ((Q \vee P) \wedge R)$

$\Leftrightarrow$ $((\neg (Q \vee P) \vee (Q \vee P)) \wedge R)$

$\Leftrightarrow$ $T \wedge R$ $[P \vee \neg P = T]$

$\Leftrightarrow$ R $[P \wedge T = P]$

4. $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology

$\Leftrightarrow$ $(P \vee Q) \wedge \neg(\neg P \wedge \neg(Q \wedge R)) \vee (\neg(P \vee Q) \vee \neg(P \vee R))$

$\Leftrightarrow$ $(P \vee Q) \wedge \neg(\neg(P \vee (Q \wedge R))) \vee (\neg((P \vee Q) \wedge (P \vee R)))$

$\Leftrightarrow$ $(P \vee Q) \wedge (P \vee (Q \wedge R)) \vee (\neg((P \vee Q) \wedge (P \vee R)))$

$\Rightarrow$ $(P \vee Q) \wedge (P \vee Q) \wedge (P \vee R) \vee \neg((P \vee Q) \wedge (P \vee R))$

$\Leftrightarrow$ $((P \vee Q) \wedge (P \vee R)) \vee (\neg((P \vee Q) \wedge (P \vee R)))$

which is a substitution Instance of

$P \vee \neg P = \boxed{T}$

Equivalent formulas :-

1) **Idempotent Laws** or $P \wedge P \equiv P$

$P \vee P \Leftrightarrow P$ $\qquad$ $P \wedge P \Leftrightarrow P$

2). **Associative Laws**

$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ $\qquad$ $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3). **Commutative Laws**

$P \vee Q \Leftrightarrow Q \vee P$ $\qquad$ $P \wedge Q \Leftrightarrow Q \wedge P$

4). **Distributive Laws**

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $\qquad$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5). $P \vee F \Leftrightarrow P$ $\qquad$ $P \wedge T \Leftrightarrow P$

6). $P \vee T \Leftrightarrow T$ $\qquad$ $P \wedge F \Leftrightarrow F$

7) $P \vee \neg P \Leftrightarrow T$ $\qquad$ $P \wedge \neg P \Leftrightarrow F$

8) **Absorption Laws**

$P \vee (P \wedge Q) \Leftrightarrow P$ $\qquad$ $P \wedge (P \vee Q) \Leftrightarrow P$

9). **Demorgan's Laws**

$\neg (P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ $\qquad$ $\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$.

$A + 1 = 1$ $\qquad$ $\overline{A} + \overline{B} = \overline{(A \cdot B)}$

$A \cdot 1 = A$ $\qquad$ $\overline{A} \cdot \overline{B} = \overline{(A + B)}$

Duality Law

Two formulas, A and A^* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. If the formula A contains the special variable T or F, then A^*, its dual is obtained by replacing T by F and F by T in addition to above.

eg:- write the duals of (1 mark).

ⓐ $(P \vee Q) \wedge R$ [Nov-2008]

Soln $(P \wedge Q) \vee R$

ⓑ $(P \wedge Q) \vee T$

Soln $(P \vee Q) \wedge F$

ⓒ $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$

Soln $\neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg S))$

Truth Table for ⓐ.

$(P \vee Q) \wedge R$

P	Q	R	$P \vee Q$	$(P \vee Q) \wedge R$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

$(P \wedge Q) \vee R$

R	$P \wedge Q$	$(P \wedge Q) \vee R$
T	T	T
F	T	T
T	F	T
F	F	F
T	F	T
F	F	F
T	F	T
F	F	F

'other connectives'

1) **Exclusive - OR** :- Let P and Q be any two formulas. Then the formula $P \bar{\vee} Q$, in which the connective $\bar{\vee}$ is called 'exclusive OR', is true whenever either P or Q, but not both, is true.

It is also called 'exclusive disjunction'

Truth Table

P	Q	$P \bar{\vee} Q$
T	T	F
T	F	T
F	T	T
F	F	F

The foll. equivalences follow from its definition:-

① $P \bar{\vee} Q \Leftrightarrow Q \bar{\vee} P$ [symmetric]

②. $(P \bar{\vee} Q) \bar{\vee} R \Leftrightarrow P \bar{\vee} (Q \bar{\vee} R)$ (associative)

③. $P \wedge (Q \bar{\vee} R) \Leftrightarrow (P \wedge Q) \bar{\vee} (P \wedge R)$ (distributive)

④. $P \bar{\vee} Q \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

⑤. $(P \bar{\vee} Q) \Leftrightarrow \neg (P \rightleftarrows Q)$.

2) **NAND** :- is a combination of "NOT" and "AND" where "NOT" stands for negation & "AND" stands for the conjunction. The connective NAND is denoted by the symbol $\uparrow$.

ie $P \uparrow Q = \neg (P \wedge Q)$

Truth Table

P	Q	$\neg(P \wedge Q)$ or $P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

Normal forms # in end

I) **Disjunctive Normal form** (SOP form) :- A formula which is equivalent to a given formula & which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

eg:- obtain DNF of :-

ⓐ $P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

ⓑ. $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\Leftrightarrow \cancel{(\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))}$

$\Leftrightarrow \cancel{((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))}$

$\Leftrightarrow \cancel{((P \vee Q) \wedge (\neg(P \wedge Q))) \vee ((P \vee Q) \wedge (\neg(P \vee Q)))}$

$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg(P \wedge Q)))$

$[\because R \rightleftarrows S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)]$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

III ⇒ NOR

:- Combination of NOT and OR. where 'OR' stands for the disjunction. The connective NOR is denoted by the symbol $\downarrow$.

ie $P \downarrow Q \Leftrightarrow \neg(P \vee Q)$

Truth Table

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

eg. Express $P \rightarrow (\neg P \rightarrow Q)$ in terms of $\uparrow$ only. [$P \uparrow Q = \neg(P$

$\Leftrightarrow \neg P \vee (\neg P \rightarrow Q)$

$\Leftrightarrow \neg P \vee (P \vee Q)$

~~$P \vee (P \vee Q)$~~

~~$\neg P \vee (P \vee Q)$~~

$T \vee Q$

$\Leftrightarrow \bar{P} + P + Q$

$\Leftrightarrow \bar{P}(Q + \bar{Q}) + P + Q$

$\Leftrightarrow \bar{P}\bar{Q} + \bar{P}Q + P + Q$

$\Leftrightarrow P + \bar{P}\bar{Q} + Q(1 + \bar{P})$

$\Rightarrow P + \bar{P}\bar{Q} + Q$

$\Leftrightarrow (P + Q) + \bar{P}\bar{Q}$

$\Leftrightarrow (P + Q) + (\overline{P + Q})$

$\Leftrightarrow \overline{\overline{(P + Q) + (\overline{P + Q})}}$

$\Leftrightarrow$

Express $P \rightarrow (P \rightarrow \neg Q)$ using $\downarrow$ operator only.

$\Rightarrow \neg P \vee (P \rightarrow \neg Q)$

$\Rightarrow \neg P \vee \neg P \vee \neg Q$

$\Rightarrow \neg P \vee \neg Q$

$\Rightarrow \neg\neg(\neg P \vee \neg Q)$

$\Rightarrow \neg(\neg P \downarrow \neg Q)$

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II) Conjunctive Normal form (POS form).

A formula which is equivalent to a given formula & which consists of a product of elementary sums is called a conjunctive Normal form.

eg:- obtain CNF for the foll.:-

ⓐ $P \wedge (P \to Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$

ⓑ $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \to (P \wedge Q)) \wedge ((P \wedge Q) \to \neg(P \vee Q))$

[using $R \rightleftarrows S \Leftrightarrow (R \to S) \wedge (S \to R)$]

$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee (\neg P \wedge \neg Q))$

$\Leftrightarrow ((P \vee Q \vee P) \wedge (P \vee Q \vee Q)) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$

$\Leftrightarrow ((P \vee Q \vee P) \wedge (P \vee Q \vee Q)) \wedge ((\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q))$

$[\because (P+Q) + PQ = (P+Q+P).(P+Q+Q)]$

$(P+Q)+(PQ) \,.\, (\overline{PQ} + \bar{P}.\bar{Q})$

$= (P+Q+P).(P+Q+Q).[(\bar{P}+\bar{Q}) + \bar{P}.\bar{Q}]$

$= (P+Q+P)(P+Q+Q).(\bar{P}+\bar{Q}+\bar{P}).(\bar{P}+\bar{Q}+\bar{Q})$

Principal Disjunctive Normal forms

An equivalent formula consisting of disjunctions of 'minterms' only is known as its principal disjunctive normal form. Such a normal form is also called the "Sum-of-products canonical form."

eg-1: obtain PDNF of these formulas :-

(a) $P \to Q \Leftrightarrow \neg P \vee Q$

$[A + \bar{A} = 1]$

$[A . 1 = A]$

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$.

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

IInd method. Truth Table for $P \to Q$.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

(b). $P \vee Q \Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$.

$[\overline{A+B} = \bar{A} . \bar{B}]$

(c). $\neg(P \wedge Q) \Leftrightarrow \neg((P \wedge (Q \vee \neg Q)) \wedge (Q \wedge (P \vee \neg P))$

$\overline{AB} = \bar{A} + \bar{B}$

$\bar{A} . \bar{B} = \overline{A+B}$

$\Leftrightarrow \neg((P \wedge Q) \vee (P \wedge \neg Q) \wedge (Q \wedge P) \vee (Q \wedge \neg P))$

$\Leftrightarrow \neg(P \wedge Q) . \neg(P \wedge \neg Q) . \neg(Q \wedge P) . \neg(Q \wedge \neg P)\}$

(c) $\neg(P\wedge Q) \Leftrightarrow \neg P \vee \neg Q$ $\qquad \overline{AB} = \overline{A}+\overline{B}$ (8)

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (\neg Q \wedge (P \vee \neg P))$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee \{(\neg Q \wedge P) \vee (Q \wedge \neg P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (\neg Q \wedge P)$.

eg 2.

(a). $\neg P \vee Q \Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$.

(b). $(P\wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.

$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q)) \vee (Q \wedge R \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q)$

$\vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$.

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$.

eg: Show the foll. are equivalent formulas :-

(a) $P \vee (P \wedge Q) \Rightarrow P$. [we write PDNF of these formulas & compare them.].

LHS.

$\Rightarrow (P \wedge (Q \vee \neg Q)) \vee (P \wedge (Q \vee \neg Q) \wedge Q \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (((P \wedge Q) \vee (P \wedge \neg Q)) \wedge ((Q \wedge P) \vee (Q \wedge \neg P)))$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee ((P \wedge Q) \vee ((P \wedge \neg Q) \wedge (\neg P \wedge Q)))$

$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q) \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$ — ①

RHS.

$P \Leftrightarrow P \wedge (Q \vee \neg Q) \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$ — ②

① = ② Hence Proved.

(b). $P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q$

L.H.S:

$P \vee (\neg P \wedge Q)$

$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee ((\neg P \wedge (Q \vee \neg Q)) \wedge (Q \wedge (P \vee \neg P)))$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \wedge ((Q \wedge P) \vee (Q \wedge \neg P)))$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$.

R.H.S. $P \vee Q$.

$\Rightarrow P(Q \vee \bar{Q}) \vee Q(P \vee \bar{P})$

$\Rightarrow (P \wedge Q) \vee (P \wedge \bar{Q}) \vee (P \wedge Q) \vee (Q \wedge \neg P)$

$\Rightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$.

L.H.S = R.H.S. Hence Proved.

exer. obtain PDNF of

$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$. Solⁿ = $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

Principal Conjunctive Normal forms.

For a given formula, an equivalent formula consisting of conjunctions of the maxterms only is known as its principal conjunctive normal form. This normal form is also called the product of sums cononical form.

eg:- obtain PCNF of the formula S given by:-

$(\neg P \to R) \wedge (Q \rightleftarrows P)$.

Solⁿ :- $(\neg P \to R) \wedge (Q \rightleftarrows P)$.

$\Leftrightarrow (P \vee R) \wedge (Q \to P) \wedge (P \to Q)$

$\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \vee R \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P \vee (R \wedge \neg R)) \wedge (\neg P \vee Q \vee (R \wedge \neg R))$

$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\Rightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$ Ans

To obtain the conjunctive form of $\neg S$ can easily be obtained by writing the conjunction of the remaining maxterms; thus $\neg S$ has the PCNF :-

$$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R) \quad \text{Ans}$$

By Considering $\neg\neg S$, we obtain

$$\neg(P \vee Q \vee \neg R) \wedge \neg(\neg P \vee \neg Q \vee R) \wedge \neg(\neg P \vee \neg Q \vee \neg R)$$

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$$

which is the principal Disjunctive Normal form of S.

eg-2. The T.T. for a formula A is given. Determine its DNF & CNF.

P	Q	R	A
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

CNF ← (F rows); DNF ← (T rows)

By choosing the minterms corresponding to each T value of A,

$\boxed{\text{DNF}}$ $A \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

$\boxed{\text{CNF}}$ $A \Leftrightarrow$ The maxterms appearing in the normal form correspond to the F values of A. (The maxterm are written down by including the variable if its truth value is F & its negation if the value is T.)

$$A \Leftrightarrow (\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee \neg R)$$

eg. obtain PCNF of :- $P \to Q$.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\therefore$ $\neg P \vee Q$. Ans

Completely Parenthesized Infix Notation and Polish Notation

Precedence Assumed :- $\neg, \wedge, \vee, \to, \rightleftarrows$

Infix.	Suffix	Prefix.
A	A	A.
$A \vee B$	$AB\vee$	$\vee AB$.
$A \vee B \vee C$	$AB\vee C\vee$	$\vee\vee ABC$
$A \vee (B \vee C)$	$ABC\vee\vee$	$\vee A \vee BC$
$A \vee B \wedge C$	$ABC\wedge\vee$	$\vee A \wedge BC$
$A \wedge (B \vee C)$	$ABC\vee\wedge$	$\wedge A \vee BC$
$A \wedge B \wedge C$	$AB\wedge C\wedge$	$\wedge\wedge ABC$
$A \wedge \neg B$	$AB\neg\wedge$	$\wedge A \neg B$
$A \wedge (B \vee \neg C)$	$ABC\neg\vee\wedge$	$\wedge A \vee B \neg C$.

Method to evaluate Prefix expressions can be summarized by the foll. 4 rules :-

1). Find the rightmost operator in the expression.
2). Select the two operands immediately to the right of the operator found.
3). Perform the indicated operation.
4). Replace the operator & operands with the result.

eg:- $\wedge A \vee B \neg C$
$\Rightarrow \wedge A \vee B (\neg C)$ [$\neg$ is unary operator]
$\Rightarrow \wedge A (B \vee \neg C)$
$\Rightarrow A \wedge (B \vee \neg C)$ =

Method to evaluate Postfix expression :-

1) find the leftmost operator in the expression.
2). Select the two operands immediately to the left of the operator found.
3). Perform the indicated operation.
4). Replace the operator and operands with the result.

eg:- $A B C \neg \vee \wedge$

$\Rightarrow A\ B\ (\neg C) \vee \wedge$

$\Rightarrow A\ (B \vee \neg C) \wedge$

$\Rightarrow A \wedge (B \vee \neg C)$.

Theory of Inference for statement Calculus start here.

Validity using Truth Tables

We look for the rows in which all $H_1, H_2, \ldots H_m$ have the value ~~T~~ T. If for every such row, C also has the value T, then the foll condition holds:-

$$H_1 \wedge H_2 \wedge H_3 \wedge \ldots \wedge H_m \Rightarrow C. \quad —(1)$$

Alternatively, we may look for the rows in which C has the value 'F'. If, in every such row, atleast one of the values of $H_1, H_2, \ldots, H_m$ is F, then (1) also holds.

eg: Determine whether the conclusion C follows logically from the premises H_1 and H_2.

ⓐ $H_1 : P \rightarrow Q$, $H_2 : P$, $C : Q$.

P	Q	$P \rightarrow Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$P \rightleftarrows Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

ⓑ $H_1 : P \rightarrow Q$ $H_2 : \neg P$ $C : Q$ ✗

ⓒ. $H_1 : P \rightarrow Q$ $H_2 : \neg(P \wedge Q)$ $C : \neg P$

ⓓ. $H : \neg P$ $H_2 : P \rightleftarrows Q$ $C : \neg(P \wedge Q)$.

ⓔ. $H_1 : P \rightarrow Q$ $H_2 : Q$ $C : P$ ✗.

Rules of Inference.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceeding formulas in the derivation.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

$\rightarrow \quad \leftrightarrows \quad \Leftrightarrow \quad \Rightarrow$

Implications

I_1 $P \wedge Q \Rightarrow P$

I_2 $P \wedge Q \Rightarrow Q$

I_3 $P \Rightarrow P \vee Q$

I_4 $Q \Rightarrow P \vee Q$

I_5 $\neg P \Rightarrow P \rightarrow Q$

I_6 $Q \Rightarrow P \rightarrow Q$

I_7 $\neg(P \rightarrow Q) \Rightarrow P$

I_8 $\neg(P \rightarrow Q) \Rightarrow \neg Q$

I_9 $P, Q \Rightarrow P \wedge Q$

I_{10} $\neg P, P \vee Q \Rightarrow Q$

I_{11} $P, P \rightarrow Q \Rightarrow Q$

I_{12} $\neg Q, P \rightarrow Q \Rightarrow \neg P$

I_{13} $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$

I_{14} $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$

I_{15} $Q, \overline{Q} \vee R \Rightarrow R$

or

$Q, Q \rightarrow R$

Equivalences

E_1 $\neg\neg P \Leftrightarrow P$

E_2 $P \wedge Q \Leftrightarrow Q \wedge P$

E_3 $P \vee Q \Leftrightarrow Q \vee P$

E_4 $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

E_5 $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

E_6 $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

E_7 $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

E_8 $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

E_9 $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

E_{10} $P \vee P \Leftrightarrow P$

E_{11} $P \wedge P \Leftrightarrow P$

E_{12} $R \vee (P \wedge \neg P) \Leftrightarrow R$

E_{13} $R \wedge (P \vee \neg P) \Leftrightarrow R$

E_{14} $R \vee (P \vee \neg P) \Leftrightarrow T$

E_{15} $R \wedge (P \wedge \neg P) \Leftrightarrow F$

E_{16} $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

E_{17} $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$

E_{18} $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

E_{19} $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

E_{20} $\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$

E_{21} $P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

E_{22} $(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \vee \neg Q)$

eg 1. Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$, and P.

Sol^n.

		Rule
(1)	$P \to Q$	P
(2)	P	P
(3)	Q	T, 1, 2, and I_{11}
(4)	$Q \to R$	P
(5)	R	T, 3, 4, I_{11}

eg 2. Show that '$R \vee S$' follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$, and $(A \wedge \neg B) \to (R \vee S)$

Sol^n.

		Rule
(1)	$(C \vee D) \to \neg H$	P
(2)	$\neg H \to (A \wedge \neg B)$	P
(3)	$(C \vee D) \to (A \wedge \neg B)$	T, 1, 2 and I_{13}.
(4)	$(A \wedge \neg B) \to (R \vee S)$	P
(5)	$(C \vee D) \to (R \vee S)$	T, 3, 4 and I_{13}
(6)	$C \vee D$	P
(7)	$R \vee S$	T, 5, 6 and I_{11}

eg. Show that q is a valid inference from the Premises $P \to q$, $P \vee q$, $\bar{q}$.

Sol^n.

		Rule
(1)	$P \to Q$	P
(2)	$\bar{Q}$	P
(3)	$\bar{P}$	T, (1), (2) & I_{12}
(4)	$P \vee Q$	P
(5)	Q	T, (3), (4) & I_{10}

eg 3. Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

Soln

		Rule.
(1)	$P \vee Q$	P
(2)	$\neg P \to Q$	T, 1, E_{16}
(3)	$Q \to S$	P
(4)	$\neg P \to S$	T, 2, 3 and I_{13}.
(5)	$\neg S \to P$	T, 4 and E_{18}
(6)	$P \to R$	P
(7)	$\neg S \to R$	T, 5, 6 and I_{13}
(8)	$S \vee R$	T, 7 and E_{16}.

eg 4. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Soln.

		Rule.
(1)	$P \to M$	P
(2)	$\neg M$	P
(3)	$\neg P$	T, 1, 2, and I_{12}
(4)	$P \vee Q$	P
(5)	Q	T. 3, 4 and I_{10}.
(6)	$Q \to R$	P
(7)	R	T, 5, 6 and I_{11}
(8)	$R \wedge (P \vee Q)$	T, 4, 7 and I_9.

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eg 5 Show I_{12} : $\neg Q$, $P \to Q \Rightarrow \neg P$.

Sol^n:

		Rule.
(1)	$P \to Q$	P
(2)	$\neg Q \to \neg P$	T, 1 and E_{18}
(3)	$\neg Q$	P
(4)	$\neg P$	T, 2, 3, and I_{11}.

IIIrd Inference Rule : CP (Rule of Conditional Proof)

Rule CP : If we can derive S from R and a set of Premises then we can derive $R \to S$ from the set of premises alone.

Rule CP is equivalent to E_{19} which states that:

$$(P \wedge R) \to S \Leftrightarrow P \to (R \to S)$$

Here, let P denote the conjunction of the set of premises and let R be any formula. The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \to S$ can be derived from the premises P alone.

Rule CP is also called the deduction theorem & generally used if the conclusion is of the form $R \to S$. In such cases, R is taken as an additional premise & S is derived from the given premises and R.

eg 6. Show that $R \rightarrow S$ can be derived from the premise $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$ and Q.

Sol^n: Instead of deriving $R \rightarrow S$, we shall introduce 'R' as an additional premise and show S first.

		Rule.
(1)	$\neg R \vee P$	P
(2)	R	P (assumed Premise)
(3)	P	T, 1, 2 and I_5
(4)	$P \rightarrow (Q \rightarrow S)$	P
(5)	$Q \rightarrow S$	T, 3, 4 and I_{11}
(6)	Q	P
(7)	S	T, 5, 6, & I_{11}
(8)	$R \rightarrow S$	1, 4, 6, & CP

eg 7. "If there was a ball game, then traveling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no ball game". Show that these statements constitute a valid argument.

Sol^n: Let P: There was a ball game.
Q: Travelling was difficult.
R: They arrived on time.

$\therefore$ $P \rightarrow Q$, $R \rightarrow \neg Q$ and R are given.
$\neg P$ (conclusion - to be shown through derivations)

		Rule
(1)	$R \rightarrow \neg Q$	P
(2)	R	P
(3)	$\neg Q$	T, 1, 2, & I_{11}
(4)	$P \rightarrow Q$	P
(5)	$\neg P$	T 3, 4 & I

def^n first! $\Longrightarrow$

Consistency of Premises and Indirect Method of Proof

eg1. Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Sol^n. We introduce $\neg\neg(P \wedge Q)$ as an additional premise & show that this additional premise leads to a contradiction.

		Rule.
(1)	$\neg\neg(P \wedge Q)$	P (assumed)
(2)	$P \wedge Q$	T, 1 & E_1
(3)	P	T, 2 & I_1
(4)	$\neg P \wedge \neg Q$	P
(5)	$\neg P$	T, 4 & I_1
(6)	$P \wedge \neg P$	T, 3, 5 & I_9. $P, Q \Rightarrow P \wedge Q$

eg2. Show that the following Premises are inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated
4. Jack misses many classes through illness & reads a lot of books.

Sol^n.

$\overline{P}\,\overline{Q} = \overline{(P+Q)}$

E: Jack misses many classes.

S: Jack fails high school.

A: Jack reads a lot of books.

H: Jack is uneducated.

The Premises are: $E \to S$, $S \to H$, $A \to \neg H$ & $E \wedge A$.

		Rule.
(1)	$E \to S$	P
(2)	$S \to H$	P
(3)	$E \to H$	T, 1, 2 & I_{13}.
(4)	$A \to \neg H$	P
(5)	$H \to \neg A$	T, 4 & E_{18}
(6)	$E \to \neg A$	T, 3, 5 & I_{13}
(7)	$\neg E \vee \neg A$	T, 6 & E_{16}
(8)	$\neg(E \wedge A)$	T, 7 & E_8
(9)	$E \wedge A$	P
(10)	$(E \wedge A) \wedge \neg(E \wedge A)$	T, 8, 9 & I_9.

Imp. '**Method of Indirect Proof**'

A set of formulas $H_1, H_2, \ldots, H_m$ is inconsistent if their conjunction implied a contradiction, that is,

$$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \wedge \neg R.$$

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	F	F	F	T	F
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	$\boxed{T}$	F	$\boxed{T}$

Predicate Calculus (1.15)

Quantifiers :-

1) universal Quantifier.
2) Existential Quantifier.

I) **Universal Quantifier** :- The symbol $(\forall x)$ or (x) are called universal Quantifier. It is used to translate expressions such as " for all", "every" and " for any".

eg1:- $(x)\, M(x)$ is a statement translated as :-

$M(x)$:- x is a Man.

$\therefore (x) M(x)$:- For all x, x is a Man.

eg2. $M(x)$:- x is a man , $H(x)$: x is a mortal.

$\therefore$ we can write,

$$(x)(M(x) \rightarrow H(x)) \quad \text{OR} \quad (\forall x)(M(x) \rightarrow H(x)).$$

OR

$\Leftrightarrow (y)(M(y) \rightarrow H(y))$ are equal if 'x' is replaced by y throughout.

eg3. $G(x, y)$: x is taller than y.

This statement can be symbolized as

$$(x)(y)(G(x, y) \rightarrow \neg G(y, x)).$$

II) **Existential Quantifier** :- The symbol $(\exists x)$ is called the existential Quantifier, which symbolizes expression such as " there is atleast one x such that " OR " there exists an x such that " or " for some x".

eg:- $M(x)$: x is a Man.
$C(x)$: x is clever.

$(\exists x)(M(x) \wedge C(x))$

$(\exists x)(M(x) \wedge C(x))$

$(\exists x)(M(x) \rightarrow C(x))$

eg. $R_1(x)$: x is a real number.
$R_2(x)$: x is rational

$(\exists x)(R_1(x) \wedge R_2(x))$.

Free And Bound Variables (1.17)

Given a formula containing a part of the form $(x)P(x)$ or $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of a formula is called bound occurrence of x, while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence.

egs:-

1. $(x)\,P(x,y)$:- $P(x,y)$ is the scope of the Quantifier & both occurrences of x are bound occurrences, while the occurrence of y is a free occurrence.

2. $(x)(P(x) \to Q(x))$:- The scope of universal Quantifier is $P(x) \to Q(x)$ & all occurrences of x are bound occurrences.

3. $(x)(P(x) \to (\exists y)R(x,y))$:- The scope of x is $P(x) \to (\exists y)R(x,y)$, while scope of $(\exists y)$ is $R(x,y)$. All occurrences of both x and y are bound occurrences.

4. $(x)(P(x) \to R(x)) \vee (x)(P(x) \to Q(x))$:- The scope of the first Quantifier is $P(x) \to R(x)$ & the scope of second Quantifier is $P(x) \to Q(x)$. All occurrences of x are bound occurrences.

5. $(\exists x)(P(x) \wedge Q(x))$:- Scope of $(\exists x)$ is $P(x) \wedge Q(x)$.

6. $(\exists x)\,P(x) \wedge Q(x)$:- The scope of $(\exists x)$ is $P(x)$ & the last occurrence of x in $Q(x)$ is free.

7. $(x)\,P(y)$:- occurrence of y is free & the scope of (x) does not contain an x.

eg: Let $P(x)$: x is a person.

$F(x, y)$: x is the father of y.

$M(x, y)$: x is the mother of y.

Write the predicate "x is the father of the mother of y".

Solⁿ: In order to symbolize the predicate, we name a person called z as the mother of y. Obviously, we want to say that x is the father of z & z the mother of y. It is assumed that such a person z exists. we symbolize the Predicate as :-

$$(\exists z)(P(z) \wedge F(x, z) \wedge M(z, y)).$$

Not in course acc. to new syllabus.

Equivalences and Implications involving Quantifiers

Equivalences

E_{23} $(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A x \vee (\exists x) B(x)$

E_{24} $(x)(A(x) \wedge B(x)) \Leftrightarrow (x)A(x) \wedge (x) B(x)$.

E_{25} $\neg(\exists x) A(x) \Leftrightarrow (x) \neg A(x)$

E_{26} $\neg(x) A(x) \Leftrightarrow (\exists x) \neg A(x)$

E_{27} $(x)(A \vee B(x)) \Leftrightarrow A \vee (x) B(x)$

E_{28} $(\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x) B(x)$

E_{29} $(x) A(x) \rightarrow B \Leftrightarrow (\exists x)(A(x) \rightarrow B)$

E_{30} $(\exists x) A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$

E_{31} $A \rightarrow (x) B(x) \Leftrightarrow (x)(A \rightarrow B(x))$

E_{32} $A \rightarrow (\exists x) B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$.

E_{33} $(\exists x)(A(x) \rightarrow B(x)) \Leftrightarrow (x) A(x) \rightarrow (\exists x) B(x)$

E_{34} $(x)(A(x) \rightarrow B(x)) \Leftrightarrow (\exists x) A(x) \rightarrow (x) B(x)$.

Implications

I_{15} $(x) A(x) \vee (x) B(x) \Rightarrow (x)(A(x) \vee B(x))$

I_{16} $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x) A(x) \wedge (\exists x) B(x)$

Q. show that the following system is inconsistent.

$P \to Q,\ P \to R,\ Q \to \overline{R},\ P$

Solⁿ

		Rule
(1)	$P \to Q$	P
(2)	P	P
(3)	Q	T, (1), (2) & I_{11}
(4)	$P \to R$	P
(5)	R	T, (2), (4) & I_{11}
(6)	$Q \wedge R$	T, (3), (5) & I_9
(7)	$Q \to \overline{R}$	P
(8)	$\overline{Q} \vee \overline{R}$	T, (7) & E_{16}
(9)	$\overline{(Q \wedge R)}$	T, (8) & E_8
(10)	$(Q \wedge R) \wedge \overline{(Q \wedge R)}$	T, (6), (9) & I_9

which is contradiction, Hence the given premises are inconsistent.

Q. using indirect method of Proof, show that,

$(R \to \overline{Q},\ R \vee S,\ S \to \overline{Q},\ P \to Q) \to \overline{P}$

Solⁿ

		Rule
(1)	$P \to Q$	P
(2)	P	P (assumed Premises)
(3)	Q	T, (1), (2) & I_{11}
(4)	$S \to \overline{Q}$	P
(5)	$\overline{S}$	T, (3), (4) & I_{12}
(6)	$R \vee S$	P
(7)	R	T, (5), (6) & I_{10}
(8)	$R \to \overline{Q}$	P
(9)	$\overline{R} \vee \overline{Q}$	T, (8) & E_{16}
(10)	$\neg(R \wedge Q)$	T, (9) & E_8
(11)	$R \wedge Q$	T, (7) (3) & I_9
(12)	$\neg(R \wedge Q) \wedge (R \wedge Q)$	T, (10) (11) & I_9

Hence, it leads to a contradiction.

$\overline{A.B} = \overline{A} + \overline{B}$

Q. Show that $\overline{P}$ is tautologically implied by $\overline{(P \wedge \overline{Q})}$, $\overline{Q} \vee R$, $\overline{R}$.

Sol^n

		Rule
(1)	$\overline{P \wedge \overline{Q}}$	P
(2).	$\overline{P} \vee Q$	T, (1) & E_8 (Demorgans law).
(3).	$P \to Q$	T, (2) & E_{16}
(4)	$\overline{Q} \vee R$	P
(5)	$Q \to R$	T, (4) & E_{16}
(6)	$P \to R$	T, (3), (5), & I_{13}
(7)	$\overline{R}$	P
(8)	$\overline{P}$	T, 6, 7 & I_{12}

Q. Derive the following using the CP rule :-

$$(\overline{P} \vee Q,\ \overline{Q} \vee R,\ R \to S) \longrightarrow (P \to S)$$

Sol^n. According to CP Rule, here, we will include P as an additional Premise & show S first.

		Rule
(1).	$\overline{P} \vee Q$	P
(2).	$P \to Q$	T, (1) & E_{16}
(3)	P	P (assumed premises)
(4)	Q	T, (2), (3) & I_{11}
(5).	$\overline{Q} \vee R$	P
(6)	R	T, (4), (5) & I_{15}
(7)	$R \to S$	P
(8)	S	T, (6), (7) & I_{11}
(9)	$P \to S$	CP, (1), (5) & 7 & Rule CP.

Q. Write the following predicate in symbolic form:
" Someone in your school has visited Agra".

Sol^n. Let $S(x)$: x is in your school.
$A(x)$: x has visited Agra.

then, $(\exists x)\,(S(x) \wedge A(x))$

Q. Symbolize the expression :-
" Everyone has exactly one favorite language."

Sol^n. Let $M(y,x)$: y is favorite language of x.
then, the statement is :-

$$(\forall x)(\exists y)\left(M(y,x) \wedge (\forall z)(z \neq y) \rightarrow \bar{M}(z,x)\right).$$

If 'y' is the fav. lang. of x, then all other lang. are suppose in set 'z'.
Since x should have exactly one fav. lang then all the lang. in set z must not be fav. lang of, x. ie $\bar{M}(z,x)$ is 'z is not the fav. lang. of x'.

Universe of Discourse (in course)

If the Quantifiers represent only those objects which belong to specific domain or class, then the restriction is known as universe of Discourse.

eg ① Symbolize " All dogs are animals".

Sol^n. Let $D(x)$: x is a dog

$A(x)$: x is an animal.

then $(\forall x)(D(x) \rightarrow A(x))$. [without universe of Discourse]

\# If universe of Discourse is taken into Account & universe is the set of all dogs, then,

$(\forall x)\ A(x)$

eg ② Symbolize " Everyone in final year class has a cellular phone".

Sol^n. $C(x)$: x is in final year class.

$P(x)$: x has a cellular phone.

then $(\forall x)\left(C(x) \rightarrow P(x)\right)$ [without universe of Discourse]

\# Let universe be set of all students in final year class, then, $(\forall x)\ P(x)$.

eg 3. Consider the Predicate:

$P(x)$: x is greater than 2.

& the statements

ⓐ $(x)\ P(x)$ ⓑ $(\exists x)\ P(x)$

universe of Discourse is :-

	statement ⓐ $(x)\ P(x)$	ⓑ $(\exists x)\ P(x)$
(i) $\{-5, -3, 0, 1, 2\}$	False	False
(ii) $\{3, 5, 7, 10\}$	True	True
(iii) $\{-1, 0, 2, 6\}$	False	True.

eg 4. Predicate :- $C(x)$: x is a cat.

$A(x)$: x is an animal.

universe :- $E : \{\underbrace{\text{cuddle}, \text{ginger}}_{\text{Name of cats.}}, 0, 1\}$

statement : All cats are animals.

ⓐ $(x)\left(C(x) \rightarrow A(x)\right)$

sol^n:

$$\Big(C(\text{cuddle}) \rightarrow A(\text{cuddle})\Big) \wedge \Big(C(\text{ginger}) \rightarrow A(\text{ginger})\Big) \wedge \Big(C(0) \rightarrow A(0)\Big) \wedge \Big(C(1) \rightarrow A(1)\Big)$$

$\Rightarrow (T \rightarrow T) \wedge (T \rightarrow T) \wedge (F \rightarrow F) \wedge (F \rightarrow F)$

$\Rightarrow T \wedge T \wedge T \wedge T$

$\Rightarrow$ True

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

(b) $(x)\ (C(x) \wedge A(x))$

$\Rightarrow (C(\text{cuddle}) \wedge A(\text{cuddle})) \wedge (C(\text{ginger}) \wedge A(\text{ginger})) \wedge (C(0) \wedge A(0))$
$\wedge (C(1) \wedge A(1))$

$\Rightarrow (T \wedge T) \wedge (T \wedge T) \wedge (F \wedge F) \wedge (F \wedge F)$

$\Rightarrow T \wedge T \wedge F \wedge F$

$\Rightarrow$ False.

Predicate :- some cats are black.

$B(x)$: x is black.

statement :- $(\exists x)\ (C(x) \wedge B(x))$

sol^n. $(C(\text{cuddle}) \wedge B(\text{cuddle})) \vee (C(\text{ginger}) \wedge B(\text{ginger})) \vee (C(1) \wedge B(0))$
$\vee (C(0) \wedge B(0))$

$= F \vee F \vee F \vee F =$ False

(Assume both are white cats)

statement $(\exists x)\ (C(x) \rightarrow B(x))$

sol^n. $(C(\text{cuddle}) \rightarrow B(\text{cuddle})) \vee (C(\text{ginger}) \rightarrow B(\text{ginger})) \vee (C(0) \rightarrow B(0))$
$\vee (C(1) \rightarrow B(1))$

$\Rightarrow (T \rightarrow F) \vee (T \rightarrow F) \vee (F \rightarrow F) \vee (F \rightarrow F)$

$\Rightarrow$ False $\vee$ False $\vee$ True $\vee$ True

$\Rightarrow$ True.

Q. Universe of Discourse is the set $\{a, b, c\}$.

Eliminate the Quantifiers :-

(a) $(x)\ P(x)$

sol^n. $P(a) \wedge P(b) \wedge P(c)$

(b) $(\exists x)\ P(x)$

sol^n. $P(a) \vee P(b) \vee P(c)$

(c) $(x)\ R(x) \wedge (\exists x)\ S(x)$

$\Rightarrow R(a) \wedge R(b) \wedge R(c) \wedge (S(a) \vee S(b) \vee S(c))$

Q. Find the truth values of :-

$(x)\ (P(x) \vee Q(x))$

(a) $P(x) : x = 1$

$Q(x) : x = 2$

$E : \{1, 2\}$ universe of discourse.

sol^n. $(P(1) \vee Q(1)) \wedge (P(2) \vee Q(2))$

$\Rightarrow (T \vee F) \wedge (F \vee T) = (T \wedge T) = \underline{\underline{True}}$

(b) $(x)\ (P \rightarrow Q(x)) \vee R(a)$

$P : 2 > 1$

$Q(x) : x \leq 3$

$R(x) : x > 5$

$a : 5$, $E = \{-2, 3, 6\}$

\# Precedence of $\wedge$ and $\vee$ is same, start evaluating from left to Right which ever comes first.

sol^n. $((P \rightarrow Q(-2)) \wedge (P \rightarrow Q(3)) \wedge (P \rightarrow Q(6))) \vee R(5)$

$\Rightarrow ((T \rightarrow T) \wedge (T \rightarrow T) \wedge (T \rightarrow F)) \vee F$

$\Rightarrow T \wedge T \wedge F \vee F = False.$

Q. Test the validity of the following arguments:

"If I go to my office tomorrow, then I must get up before 7 a.m. & if I attend the dinner party at the club, I will return home late. If I return home late & get up before 7.a.m., I'll not sleep well. I want to sleep well. Therefore, either I'll not go to office or I'll not attend the dinner party."

Ans. O : I'll go to office tommorrow.

B : I'll get up before 7.a.m.

D : I'll attend the dinner party.

H : I'll return home late.

S : I'll not sleep well.

Premises are :-

$O \to B$, $D \to H$, $(H \wedge B) \to S$, $\neg S$

Conclusion :

$\neg O \vee \neg D$

1.	$(H \wedge B) \to S$	P
2.	$\neg(H \wedge B) \vee S$	T, (1)
3.	$\neg H \vee \neg B \vee S$	T, (2)
4.	$\neg B \vee (\neg H \vee S)$	T, (3)
5.	$B \to (\neg H \vee S)$	T, (4)
6.	$O \to B$	P
7.	$O \to (\neg H \vee S)$	T, 5, 6.
8.	$\neg O \vee \neg H \vee S$	T, 7
9.	$\neg H \vee (\neg O \vee S)$	T, 8
10.	$H \to (\neg O \vee S)$	T, 9
11.	$D \to H$	P
12.	$D \to (\neg O \vee S)$	T, 10, 11
13.	$\neg D \vee \neg O \vee S$	T, 12
14.	$(\neg D \vee \neg O) \vee S$	T, 13
15.	$\neg S$	P
16.	$\neg D \vee \neg O$	T, 14, 15, I_{10}

Q. Express the statement:

"If a person is female & is a parent, then this person is someone's mother", as a logical expression involving predicates, quantifiers with universe of discourse consisting of all people.

Ans $F(x)$: x is female person.

$P(x)$: x is parent.

$M(x, y)$: x is mother of y.

$$(\exists y)(\exists x)\left(F(x) \wedge P(x) \rightarrow M(x,y)\right)$$

Assignment.

1) Prove Equivalency:-

ⓐ $P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\bar{P} \wedge \bar{Q})$

ⓑ $\neg(P \Leftrightarrow Q) \equiv P \Leftrightarrow \neg Q$

ⓒ $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

ⓓ $\neg P \Leftrightarrow Q \equiv P \Leftrightarrow \neg Q$

ⓔ $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$

ⓕ $(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$

ⓖ $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R) \equiv T$

ⓗ $(P \rightarrow Q) \rightarrow R \not\equiv P \rightarrow (Q \rightarrow R)$

ⓘ $(P \rightarrow Q) \rightarrow (R \rightarrow S) \not\equiv (P \rightarrow R) \rightarrow (Q \rightarrow S)$

ⓙ $(P \vee Q) \wedge (\neg P \vee R) \rightarrow (Q \vee R) \equiv T$

2) show the validity of following arguments:-

ⓐ $B \wedge C,\ (B \Leftrightarrow C) \rightarrow H \vee G \Rightarrow H \vee G$

ⓑ $\neg J \rightarrow (M \vee N),\ (H \vee G) \rightarrow \neg J,\ H \vee G \Rightarrow M \vee N$

ⓒ $\neg S,\ P \vee (Q \wedge R),\ P \Leftrightarrow S \Rightarrow Q$

ⓓ $P \vee Q,\ P \rightarrow R,\ S \wedge \neg R \Rightarrow Q$ (Prove by contradiction)

ⓔ $(P \rightarrow P \wedge Q) \Rightarrow P \rightarrow Q$

contd ....

Ans 2(d). Prove

$P \vee Q, \; P \to R, \; S \wedge \neg R \Rightarrow Q$

$\overline{A.B} = \bar{A} + \bar{B}$

$[\; \overline{S \wedge \neg R} = \overline{\neg S \vee R} \;]$

(Demorgan's law)

1.	$S \wedge \neg R$	P
2.	$\neg(\neg S \vee R)$	T, 1, E
3.	$\neg(S \to R)$	T, 2, E
4.	$\neg R$	T, 3, I_8
5.	$P \to R$	P
6.	$\neg P$	T, 4, 5, I_{12}
7.	$P \vee Q$	P
8.	$\neg P \to Q$	T, 7, E
9.	Q	T, 6, 8, I_{11}

2(d). $P \vee Q, \; P \to R, \; S \wedge \neg R \Rightarrow Q$ using contradiction

1.	$S \wedge \neg R$	P
2.	$\neg(\neg S \wedge R)$	T, 1, E
3.	$\neg(S \to R)$	T, 2, E
4.	$\neg R$	T, 3, I_8
5.	$P \to R$	P
6.	$\neg P$	T, 4, 5, I_{12}
7.	$\neg Q$	P (assumed Premise)
8.	$\neg P \wedge \neg Q$	T, 6, 5, I_9
9.	$\neg(P \vee Q)$	T, 8, ~~E~~ Demorgan's law
10.	$P \vee Q$	P
11.	$(P \vee Q) \wedge \neg(P \vee Q)$	T, 10, 9, I_9

which leads to contradiction.

Q. Prove Tautology:-

$$(P \to Q) \to (\neg Q \to \neg P)$$

$$= (\neg P \vee Q) \to (Q \vee \neg P)$$

$$= \neg(\neg P \vee Q) \vee (Q \vee \neg P)$$

$$= \cancel{\neg\neg P \vee}$$

$$= (P \wedge \neg Q) \vee Q \vee \neg P$$

$$= (P \vee Q) \wedge (\neg Q \vee Q) \vee \neg P$$

$$= (P \vee Q) \vee \neg P$$

$$= (P \vee \neg P) \vee Q$$

$$= 1 \vee Q = 1 = T$$