ch-8 Graphs

'K. Rosen'

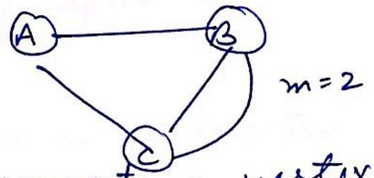
GRAPH :- A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) & E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Infinite Graph :- Infinite vertex set

finite Graph :- finite vertex set.

Simple Graph (linear Graph) :- A graph in which each edge connects two different vertices & where no two edges connect the same pair of vertices is called simple graph.

multigraph :- Graphs that have multiple edges connecting the same vertices are called multigraphs. If there are 'm' different edges connecting the same pair of vertices, then these are the edges of multiplicity m.



loop :- Edge that connects a vertex to itself is called a loop.

Pseudograph :- Graphs that may include loops &/or multiple edges are sometimes called Pseudographs.

undirected Graphs :- Simple graph / Pseudograph / multigraph whose edges are undirected.

Directed Graph (or digraph) (V, E) consists of non-empty set of vertices V & set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Simple Directed Graph :- Digraph having no loops & no multiple directed edges is called simple Digraph.

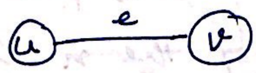
Directed Multigraphs :- Digraphs having multiple directed edges. Where there are 'm' directed edges, each associated to an ordered pair of vertices (u, v), we say (u, v) is an edge of multiplicity m.

Mixed Graphs :- A graph with both directed & undirected edges is called a mixed graph.

Graph Terminology

Adjacent (neighbors) :- Two vertices u and v in an undirected graph G are called adjacent in G if u & v are endpoints of an edge in G.

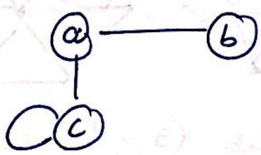
Incident :- If e is associated with $\{u, v\}$, the edge e is called incident with the vertices u and v.



degree of a vertex :- No. of ~~vertices~~ edges incident on a vertex.

\# loop at a vertex contributes twice to the degree of that vertex.

Denoted by $deg(a) = 2$, $deg(b) = 1$, $deg(c) = 3$



isolated :- vertex of degree zero.

pendant :- vertex of degree one.

Th 1. Let $G = (V, E)$ be an undirected graph with e edges.

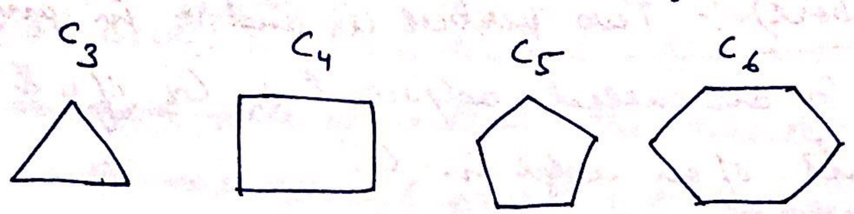
Then $2e = \sum_{v \in V} deg(v)$

\# Also true even if multiple edges or loops are present.

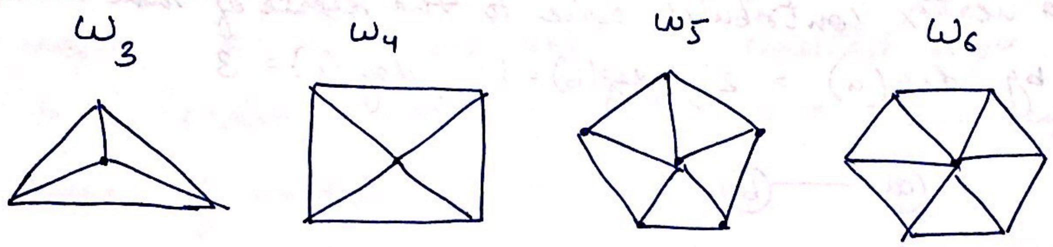
Q. Find no. of edges is G with $V = 10$ each of degree six.

$\therefore \; 2e = 10 \times 6 \Rightarrow 2e = 60 \Rightarrow e = \underline{30}$ Ans.

Cycles :- C_n , $n \geq 3$ consist of n vertices $(1, 2, \ldots n)$ & edges $\{1,2\}, \{2,3\}, \ldots \{n-1, n\}, \{n, 1\}$.



Wheels :- W_n :- we obtain wheel W_n when we add additional vertex to the cycle C_n for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edge.

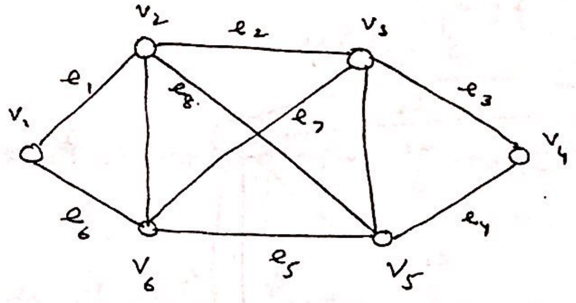


GRAPHS

20 marks. (1)
(N.K. Das & Liu).

Def^n:- A simple graph or an undirected graph $G = (V, E)$ consists of a finite non-empty set V of vertices and an edge set E such that no two edge connects the same pairs of vertices.

eg:-



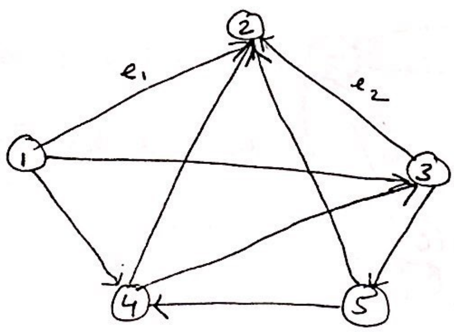
$V = \{ v_1, v_2, v_3, v_4, v_5, v_6 \}$

$E = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \}$ where,

$e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$ ....

Directed Graph :- $G = (V, E)$ defines a directed graph or digraph, if edges in it are associated with an ordered pair of $V \times V$.

eg:-



$e_1 = (1, 2)$

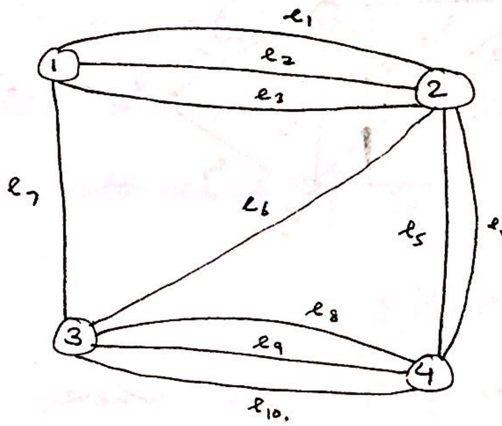
$e_2 = (3, 2)$ .... & so on.

edges such as (2,3), (2,5) are not defined.

Multigraph :- (opposite of multigraph - linear graph)

$G = (V, E)$ defines a multigraph with V as a set of vertices, E as a set of edges and a function f from E to $\{(x, y) \mid x, y \in V, x \neq y\}$. The edges e_i and e_j are termed as parallel or multiple if $f(e_i) = f(e_j)$.

eg:-

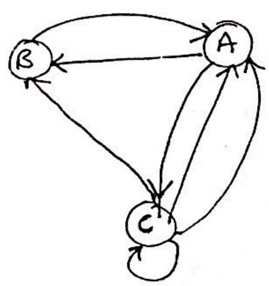


Parallel edges :- e_1, e_2, e_3

- e_4, e_5

- e_8, e_9, e_{10}.

Directed Multigraph :- A directed multigraph $G = (V, E)$ is such that a function f from E to $\{(x, y) \mid x, y \in V\}$ defines multiple edges if $f(e_i) = f(e_j)$.



Two vertices u & v are called adjacent if an edge $e = \{u, v\}$ connects them. The vertices u & v are called end-points of the edge.

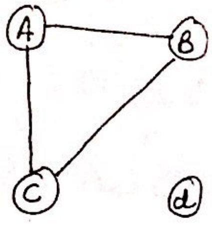
degrees of vertex :- no. of edges incident on a vertex. & is denoted by deg (v).

A vertex of degree 0 is called isolated.

A vertex of degree 1 is called pendant.

#: $\sum_{v \in V} deg(v) = 2|E|$

eg.



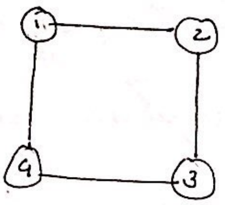
$deg(A) = 2$
$deg(B) = 2$
$deg(C) = 2$
$deg(d) = 0$

$2+2+2 = 2(3)$.
$6 = 6$.

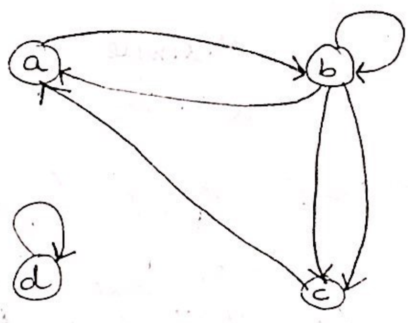
Regular Graph:- A simple graph, if each vertex has the same degree.

⇒ If the degree of each vertex of a regular graph G is m, then G is also called m-regular.

eg:-



In-degree/Out degree:- In a directed graph, in-degree of a vertex 'u' corresponds to no. of edges terminating at u & out-degree of u is no. of outgoing edges with 'u' as the initial vertex.



Vertices	a	b	c	d
In-degree	2	3	4	1
out degree	1	3	2	1

In a directed graph $G = (V, E)$

$$\sum_{v \in V} \text{in-deg}(v) = \sum_{v \in V} \text{out-deg}(v) = |E|$$

Walk :- In a graph $G = (V, E)$ a walk is a finite alternating sequence of vertices and edges ie $\omega = v_0, e_1, v_1, e_2, \dots, e_n, v_n$ where $v_0, v_1, \dots, v_n$ are vertices and $e_1, e_2, \dots e_n$ are edges joining the vertices v_{i-1} & v_i, $1 \le i \le n$.

A walk from a vertex v_0 to v_n in a graph G may be denoted as $v_0 - v_n$. v_0 & v_n are termed as end or terminal vertices while all other vertices are termed as internal vertices of the walk $v_0 - v_n$. In a walk, edges & vertices may appear more than once. A walk is called open when the terminal vertices are distinct.

For the same terminal/end vertices it is called closed.

⇒ A walk/Path is called a trail or simple if all its edges are distinct.

~~An open trail is called a path~~

~~A closed trail is called a circuit.~~

⇒ A walk is called elementary if it does not ~~include~~ meet same vertex twice.

⇒ The no. of edges in a path/circuit defines its length.

A circuit in which only repeated vertex is the first vertex is called a cycle.

+ circuit is simple if edges are distinct.

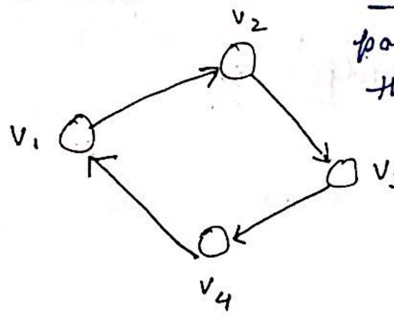
+ circuit is elementary if vertex are distinct except terminal one.

⇒ A graph is called connected if a path in it exists from a vertex to any other vertex. If such a path does not exist, the graph is said to be disconnected & various connected pieces of a disconnected graph are called its components.

Strongly connected :- A directed graph $G = (V, E)$ is said to be strongly connected if for every pair of vertices v_i and v_j, there is a path connecting v_i & v_j and also a path that connects v_j and v_i.

weekly connected :- iff there is a path b/w ~~every~~ any two vertices when the directions of edges are disregarded.

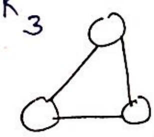
(need to ask).



∴ Any strongly connected digraph is also weakly connected.

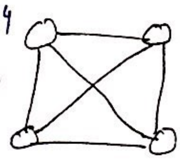
Complete graph (K_n) :- A graph $G = (V, E)$ is complete if each of its vertex is connected to every other vertex with exactly one edge

K_3

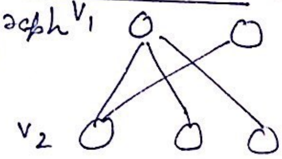


$e = \frac{n(n-1)}{2}$

K_4



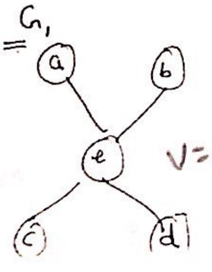
Bipartite graph eg.



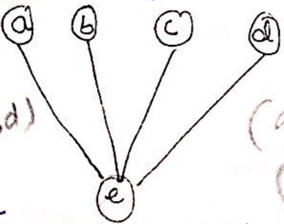
Bipartite graphs :- A simple graph $G = (V, E)$ is bipartite if it's vertex set V can be divided or partitioned into subsets V_1 and V_2 such that ~~each~~ edge in E is incident on a vertex in V_1 and a vertex in V_2. (If each edge is incident then it is Complete Bipartite) → $K_{2,4}$ = 8 edges & V = 6 Ver. (2+4)

egs of complete Bipartite graphs

G_1



V = (a, b, c, d)

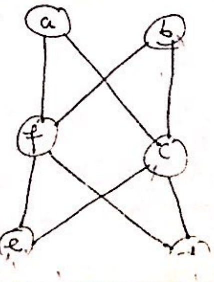


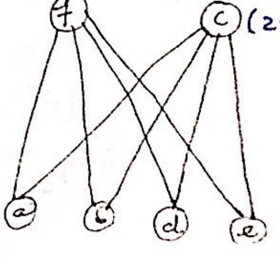
$K_{1,4}$ = 4 edges

(a, b, e, f)

(f, c)

G_2





Scanned with CamScanner

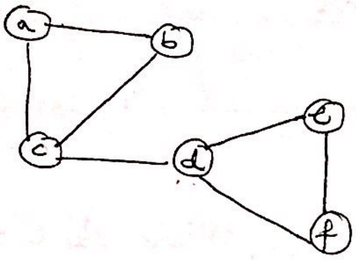
1) Every tree is bipartite ~~2) A graph is bipartite if and only if~~

2) Cycle graphs with an even no. of vertices are bipartite

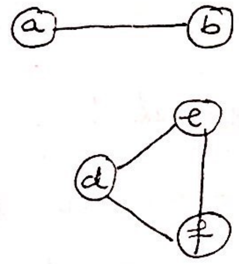
3) Every planar graph whose faces all have even length is bipartite.

cut-vertices :- The removal of a vertex from a graph may produce a sub-graph with more connected components. Such vertices are termed as cut-vertices (or articulation points).

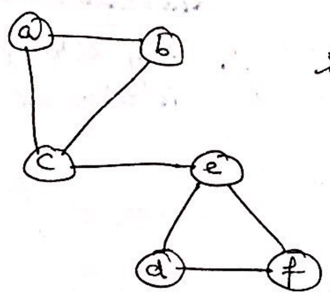
eg.



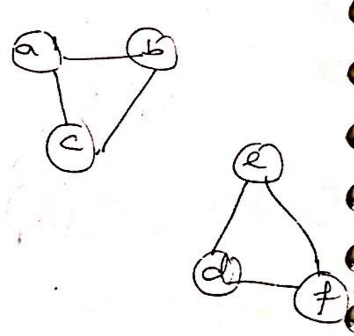
removing 'c'



cut-edges :- Removal of an edge from a graph may also produce a sub-graph which is not connected. Such edges are called cut-edges (or bridge)



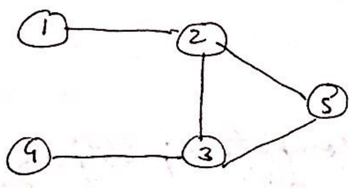
removing (c, e)

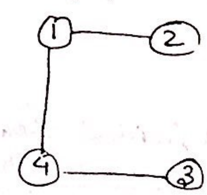


union :- union of two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ is a new graph $G = (V, E)$ written as $G = G_1 \cup G_2$ such that $V = V_1 \cup V_2$ & $E = E_1 \cup E_2$

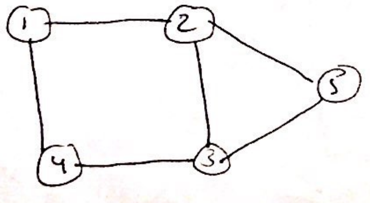
Intersection :- of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a new graph $G = (V, E)$ written as $G = G_1 \cap G_2$, such that $V = V_1 \cap V_2$ & $E = E_1 \cap E_2$

eg :-

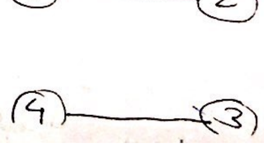




union



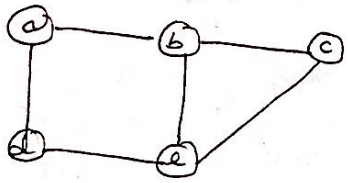
Intersection



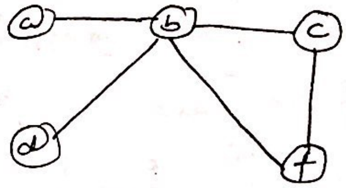
Ring Sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph consisting of the vertex set $V_1 \cup V_2$ and edges either in G_1 or G_2 but not is both. we denote the ring sum of two graphs G_1 and G_2 as $G_1 \oplus G_2$.

eg.

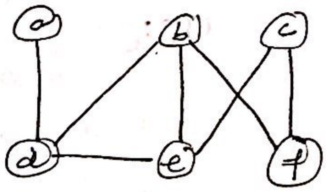
G_1



G_2

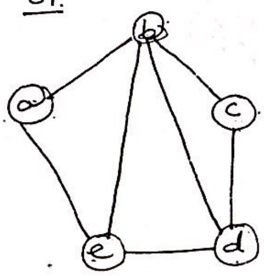


$G_1 \oplus G_2$

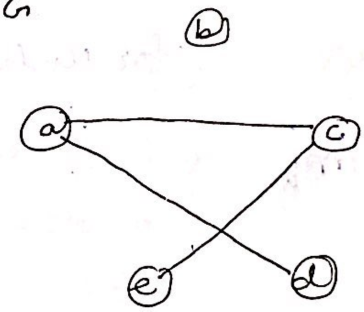


Complement :- A graph $G' = (V, E')$ is said to be a complement of a simple graph $G = (V, E)$ if the edge $(V_i, V_j) \in E$, if and only if it is not in E'.

eg:- G.

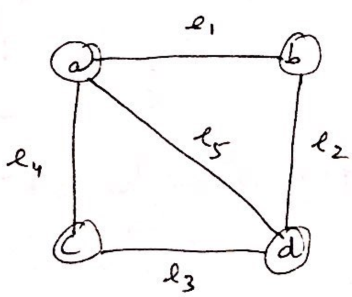


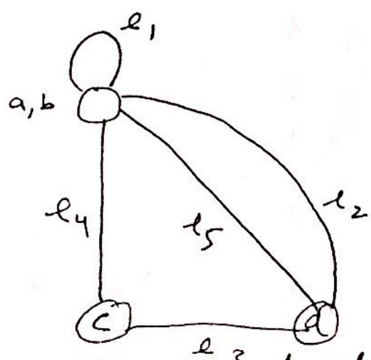
G'



fusion of a graph :- A pair of vertices a and b in a graph G are said to be fused (merged) if the two vertices are replaced by a new vertex such that edges incident on a and b in G are now incident on the new vertex.

eg:- merge the vertices 'a' and 'b'.

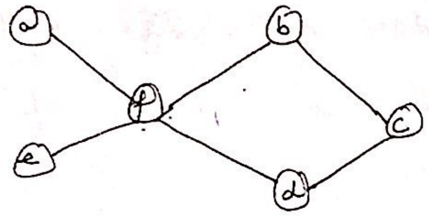




fusion of two vertices doesn't alter the no. of edges. but it reduces the no. of vertices by one.

Rep. of Graphs

Adjacency ~~list~~ ~~matrix~~ list.



vertex	adjacent vertices.
a	f
b	f, c
c	b, d
d	f, c
e	f
f	a, b, e, d.

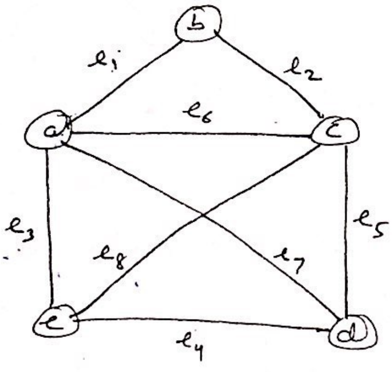
Adj. Matrix Rep. (undirected)

$$m_{ij} = \begin{cases} 1 & \text{if an edge connects } v_i \text{ to } v_j \\ 0 & \text{if no edge connects } v_i \text{ to } v_j \end{cases}$$

	a	b	c	d	e	f
a	0	0	0	0	0	1
b	0	0	1	0	0	1
c	0	1	0	1	0	0
d	0	0	1	0	0	1
e	0	0	0	0	0	1
f	1	1	0	1	1	0

Incidence Matrix :- (for undirected graph)

$$m_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is incident with vertex } v_i \\ 0 & \text{otherwise.} \end{cases}$$

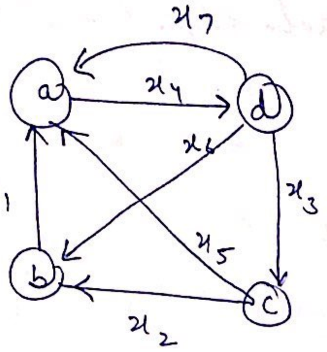


	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
a	1	0	1	0	0	1	1	0
b	1	1	0	0	0	0	0	0
c	0	1	0	0	1	1	0	1
d	0	0	0	1	1	0	1	0
e	0	0	1	1	0	0	0	1

Rep. of Graphs

Incidence matrix (for digraphs) edges

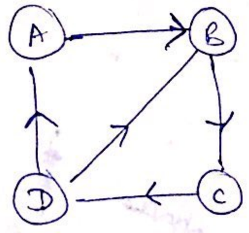
$$M_{ij} = \begin{cases} 1 \text{ out}, & \text{if the } j^{th} \text{ edge } e_j \text{ is incident out of the } i^{th} \text{ vertex } V_i \\ 0, & \text{if the } j^{th} \text{ edge is not incident on the } i^{th} \text{ vertex } V_i \\ -1 \text{ in}, & \text{if the } j^{th} \text{ edge } e_j \text{ is incident into the } i^{th} \text{ vertex } V_i \end{cases}$$



	x_1	x_2	x_3	x_4	x_5	x_6	x_7
a	-1	0	0	1	-1	0	-1
b	1	-1	0	0	0	-1	0
c	0	1	-1	0	1	0	0
d	0	0	+1	-1	0	1	1

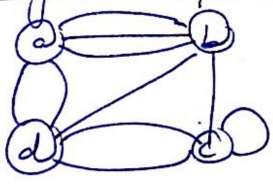
Adjacency vertex matrix (for digraphs)

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge directed from } i^{th} \text{ vertex to } j^{th} \text{ vertex} \\ 0, & \text{otherwise} \end{cases}$$



	a	b	c	d
a	0	1	0	0
b	0	0	1	0
c	0	0	0	1
d	1	1	0	0

Adjacency matrix for Pseudograph

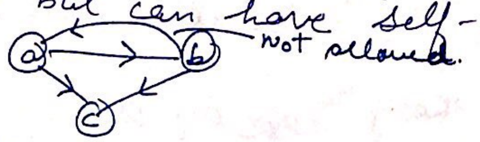


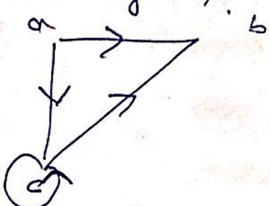
$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Simple graph :- A graph having no self-loops or parallel edges.

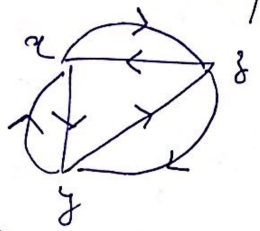
Asymmetric Digraph :- Digraphs having only one directed edge b/w each pair of vertices but can have self-loops.

ie



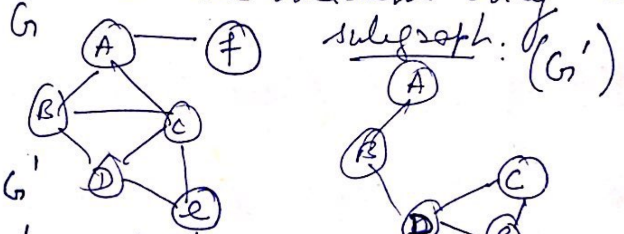


Symmetric Digraph :- Digraphs in which for each edge (x, y) there is also an edge (y, x).



Subgraph :- If $G = (V, E)$ is a graph then $G' = (V', E')$ is said to be subgraph of G if E' is a subset of E & V' is a subset of V such that edges in E' are incident only with the vertices in V'.

eg.



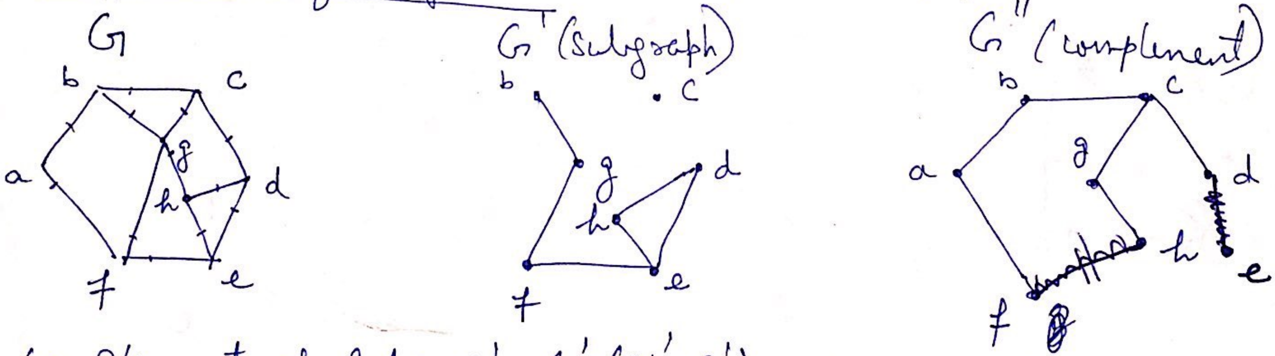
or $V' \subseteq V$ & $E' \subseteq E$

Subgraph is proper subgraph if $G \neq G'$

Spanning Subgraph :- A subgraph of G is said to be spanning subgraph if it contains all the vertices of G.

no cycle

Complement of Subgraph (with respect to Graph)



Complement of Subgraph $G' = (V', E')$ with respect to graph G is another subgraph $G'' = (V'', E'')$ such that E'' is equal to $E - E'$ & V'' contains only the vertices with which the edges in E'' are incident.

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Precedence graph.

A directed edge from one vertex to another means the necessity of executing former statement before the latter statement.

eg: S_1 : $x = 0$

S_2 : $x = x+1$

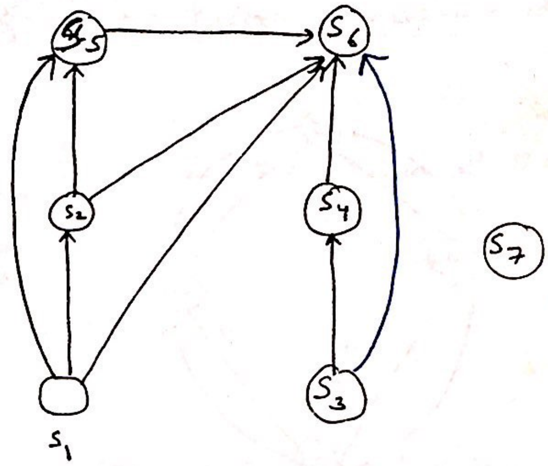
S_3 : $y = 2$

S_4 : $z = y$

S_5 : $x = x+2$

S_6 : $i = x+z$

S_7 : $z = 4$



A path is said to be <u>simple</u> if it does not include the same edge twice.

→ A Path is said to be <u>elementary</u> if it does not meet the same vertex twice.

- A circuit is a path $(e_{i_1}, e_{i_2}, ---, e_{i_k})$ in which terminal vertex of e_{i_k} coincides with the initial vertex of e_{i_1}.

- A circuit is said to be <u>simple</u> if it does not include the same edge twice.

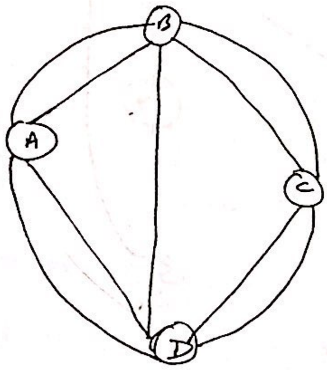
- A circuit is said to be <u>elementary</u> if it does not meet the same vertex twice.

Euler Graphs [Nov-2008].

Eulerian Path :- in a connected graph $G = (V, E)$ is a path that traverses each edge in G only once.

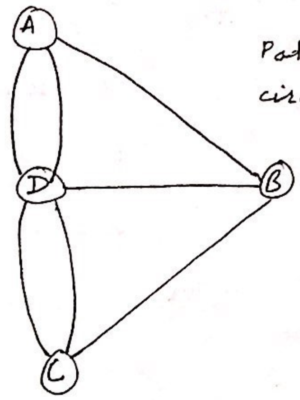
An Eulerian circuit is a Eulerian Path which is a circuit.

eg:-



E. Path :- ✓

E. circuit :- ×.



Path ×

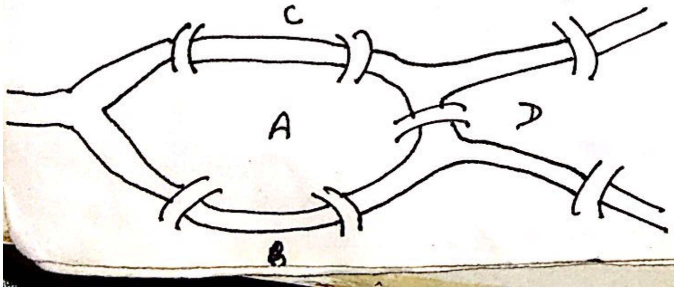
circuit ×

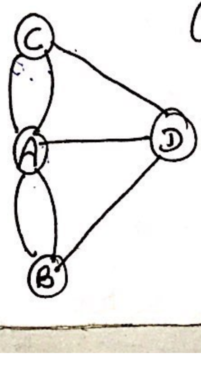
criteria to determine whether a graph has a E.C. or E.Path.

1). A connected multigraph has a Eulerian circuit iff every vertex in it has even degree. [Nov-2008].

2). A connected multigraph has an Eulerian Path but not a circuit iff it has ~~either zero~~ or two of its vertices which are of odd degree.

Q.1 Can someone cross all the bridges shown in this map exactly one & return to the starting point.





(consider Reg. as vertex & bridges as edges).

Thus, No Path & No circuit.

Hamilton graphs

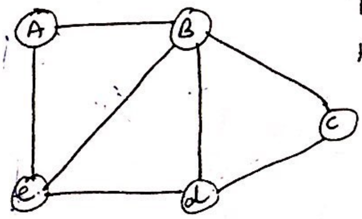
H. Path :- in a graph $G=(V,E)$ is such that it passes through all the vertices in it only once.

H. circuit is a Hamilton Path which is a circuit.

⇒ Let G be a linear graph of n vertices. If the sum of the degrees for each pair of vertices in G is $n-1$ or larger, then then there exists a hamiltonian path in G.

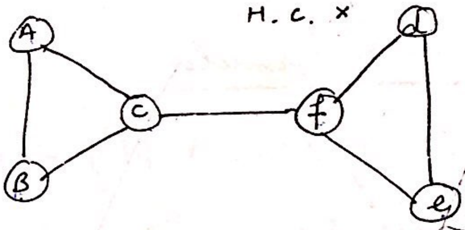
⇒ There is always a hamiltonian path in a directed complete graph.

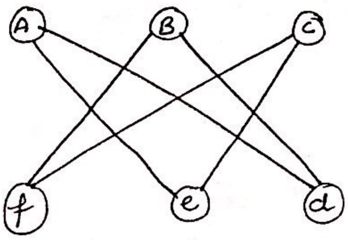
eg :-



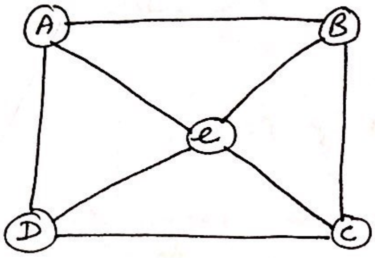
H. P. ✓
H. C. ✓

H. P. ✓
H. C. ×





H. P. ✓
H. C. ✓

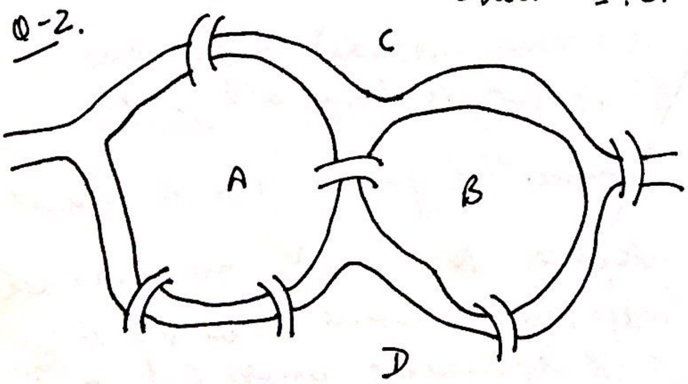


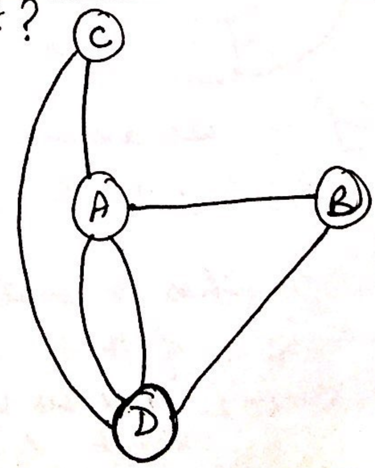
H. P. ✓
H. C. ✓.

---

check E.C. exist?

Q-2.





E.C. ✓

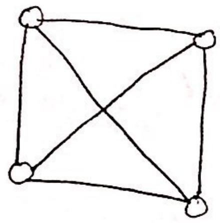
since every vertex has even degree

Planar Graphs

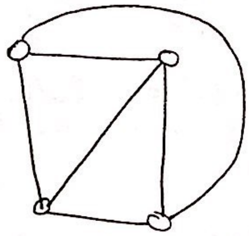
Drawing a graph G in a plane is said to be planar if no two edges intersect each other except only at a vertex.

eg.

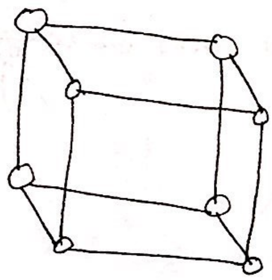
(1)



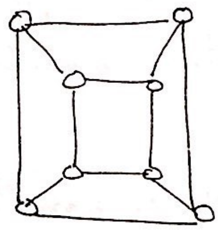
⇒



(2)



⇒



Euler formulae :-

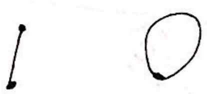
$$|F| = |E| - |V| + 2. \qquad \text{—— (1)}$$

Th. Prove for any connected Planar graph

$v - e + r = 2$. where v = no. of vertices, e = edges, r = regions.

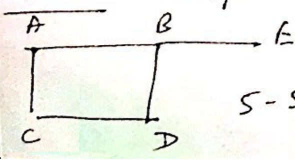
Pf :- Proof is by PMI,

basis :- The two graph shown below, satisfies (1),



Induction step :- we assume that (1) is satisfied in all graphs with $n-1$ edges. Let 'G' be a connected graph with 'n' edges.

Case I If G has a vertex of degree 1 (i.e. E). on removal of this vertex, edge is also removed, thus V & E decreases by 1 & R remains unaffected. so (1) is satisfied ...



$5-5+2=2$

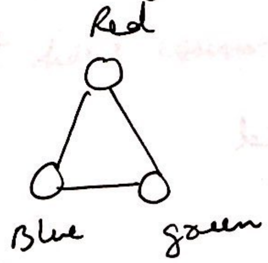
8.8. Graph Coloring (Rosen).

Def^n:- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

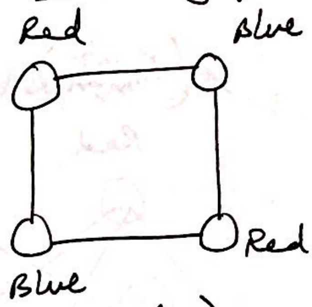
chromatic number of a graph is the least no. of colors needed for a coloring of this graph. The chromatic no. of a graph G is denoted by $\chi(G)$. [chi (G)]

eg.

$\chi(G) = 2$

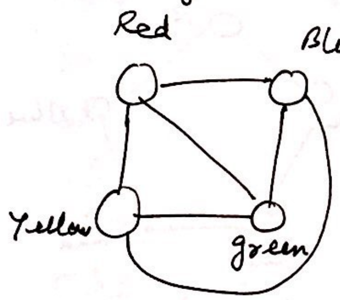


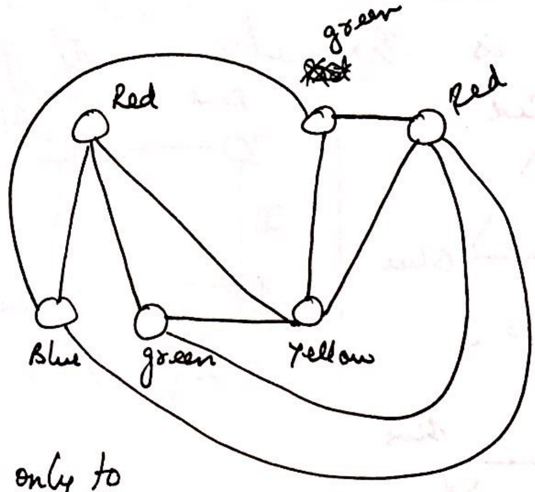
$\chi(G) = 3$



$\chi(G) = 2$

Th. Four Color Theorem:- Chromatic no. of a Planar graph is no greater than 4.





Four color Theorem applies only to planar graphs.

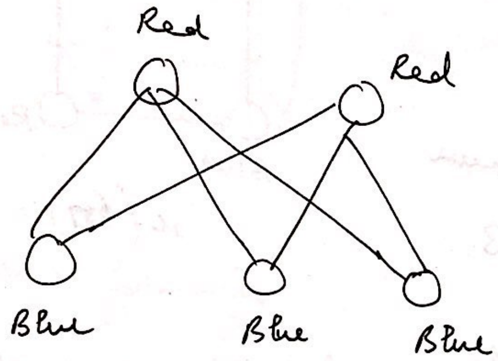
eg. What is chromatic no. of K_n.

Ans. n. [since each & every vertex is connected to each other, a new color is needed every time].

ie $\chi(K_n) = n$

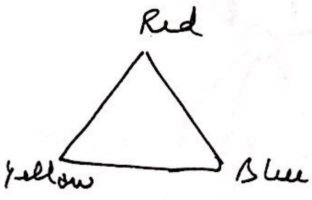
eg. Find chromatic no. of a complete Bipartite graph $K_{m,n}$, where m, n are +tive integers.

Ans. $\chi(K_{m,n}) = 2$.

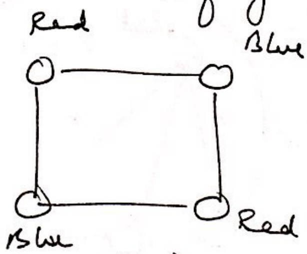


eg. What is chromatic no. of graph C_n, where $n \geq 3$.

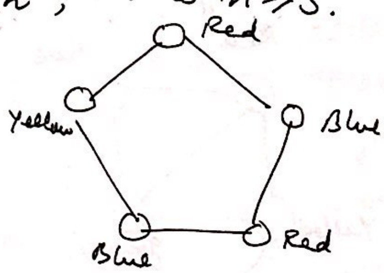
rule apply to Cycle graph only



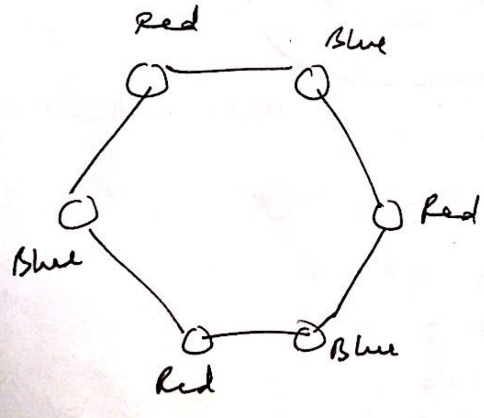
$\chi(C_3) = 3$



$\chi(C_4) = 2$



$\chi(C_5) = 3$



$\chi(C_6) = 2$.

	chromatic no.
ie n = odd & $n>1$	3
n = even	2.

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Application of graph coloring.

Scheduling Exams :- schedule exams so that no student has two exams at the same time.

vertices represent subjects

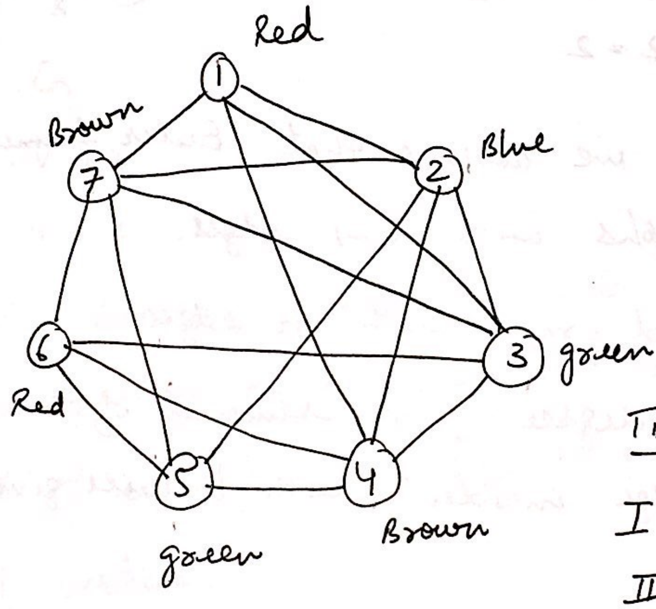
edge b/w two vertices → common student in the subjects.

Each time slot for a final exam is rep. by a different color.

eg:- suppose subjects are no. 1 – 7.

foll. pair of subject have common students:

1 & 2
1 & 3
1 & 4
1 & 7
2 & 3
2 & 4
2 & 5
2 & 7
3 & 4
3 & 6
3 & 7
4 & 5
4 & 6
5 & 6
5 & 7
6 & 7



$\therefore \chi(G) = 4$

$\therefore$ four time slots are needed.

Time slot	courses
I	1, 6
II	2
III	3, 5
IV	4, 7.

Th. For any connected Planar Graph, Euler's formula $R = E - V + 2$.

Proof :- The Proof proceeds by induction on the no. of edges.

As the basis of induction, we observe that for the two graphs with a single edge shown in figure is satisfied.

$R = E - V + 2$ $\qquad\qquad$ $R = E - V + 2$

$1 = 1 - 2 + 2$ $\qquad\qquad$ $2 = 1 - 1 + 2$

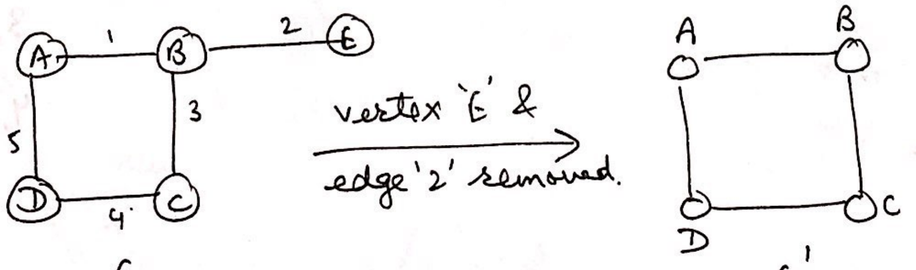
$1 = 1$ $\qquad\qquad$ $2 = 2$

As induction step, ~~as~~ we assume that Euler formula is satisfied in all graphs with $n-1$ edges.

Let G be a connected graph with n edges.

Case I

If G has a vertex of degree 1, the removal of this vertex together with the edge incident with it will give a connected graph G'.



G $\qquad\qquad$ G'

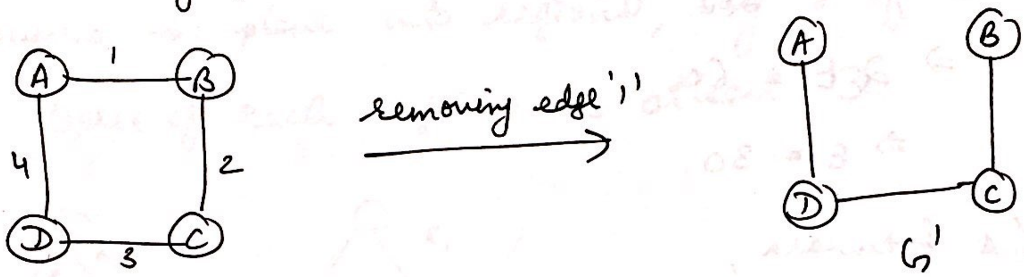
$R = E - V + 2$ $\qquad\qquad$ $R = E - V + 2$

$2 = 5 - 5 + 2$ $\qquad\qquad$ $2 = 4 - 4 + 2$

Since Euler's formula is satisfied in G', it is also satisfied in G, because putting the removed edge and vertex back into G' will increase the count of vertices & edges by 1, but will not change the count of regions.

Case II:

If G has no vertex of degree 1, The removal of any edge in the boundary of a finite region will give a connected graph G'.



G $\qquad\qquad$ G'

$R = E - V + 2$

$1 = 3 - 4 + 2$

$R = E - V + 2$

$2 = 4 - 4 + 2$

Since Euler formula is satisfied in G', it is also satisfied in G, because putting the removed edge back into G' will increase the count of edges & regions by 1, but will not change the count of vertices.

Hence Euler formula is satisfied for all connected planar graphs.

Q. A connected Planar simple Graph has 20 vertice, each of degree 3. Into how many regions does a represen-tation of this planar graph split the plane.

Sol^n

$V = 20$

$$\sum_{v \in V} deg(v) = 20 \times 3 = 60.$$

$$\therefore 2E = \sum_{v \in V} deg(v)$$

$$\Rightarrow 2E = 60$$

$$\Rightarrow E = 30.$$

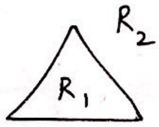
Euler's formula,

$$R = E - V + 2$$

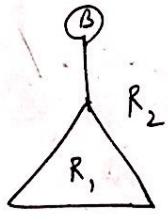
$$R = 30 - 20 + 2 = \boxed{12}$$

degree of a region :- No. of edges on the boundary of a region. When an edge occurs twice on the boundary (so that it is traced out twice when the boundary is traced out), it contributes two to the degree.

For egs :-

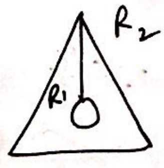


$R_1 = 3$
$R_2 = 3$

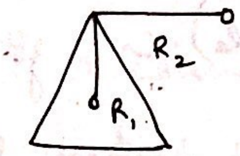


$R_1 = 3$
$R_2 = 5$

[consider it to be starting from vertex B and going back to vertex B



$R_2 = 3$
$R_1 = 5$



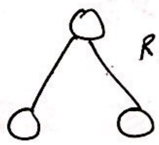
$R_1 = 5$
$R_2 = 5$

Th. If G is a connected planar simple graph with e edges & v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Proof:- A connected planar simple graph drawn in the plane divides the plane into regions, say r of them.
The degree of each region is atleast 3.

eg.

$R_1 = 3$
$R_2 = 3$



$R_1 = 4$

That is, degree of the unbounded region is atleast 3 (eg 1) because there are at least three vertices in the graph.

Now, we know that, sum of the degrees of the regions is exactly twice the no. of edges in the graph, because each edge occurs on the boundary of a region exactly twice. (either in two diff. regions, or twice in the same region).
Because each region has degree greater than or equal to three, it follows that,

$$2e = \sum_{r \in R} \deg(r) \geq 3r \text{, } \cancel{\text{Hence}}$$

$$\text{Hence } \frac{2}{3} e \geq r \text{,}$$

using $r = E - V + 2$ (Euler's formula) in above Eq, we obtain,

$$e - v + 2 \leq \frac{2}{3} e$$

$$e - \frac{2}{3} e \leq v - 2$$

$$\frac{3e - 2e}{3} \leq v - 2$$

$$\frac{e}{3} \leq v - 2$$

$$e \leq 3v - 6$$

Hence Proved.

eg. Show that K_5 is non-planar.

Sol^n. For any connected planar Graph, we know,

$$e \leq 3v - 6. \quad — (1).$$

∴ Eq (1) must not be satisfied if K_5 is non-planar.

for K_5, $V = 5$, $E = 10$.

Put in (1)

$$10 \leq 3.5 - 6$$

$$10 \leq 9. \quad \text{false}$$

Hence Proved. K_5 is non planar.

Th. If a connected planar simple graph has e edges & v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v-4$.

Proof:- No circuit of length 3 means, degree of each region must be at least four.

$\therefore$ we can write,

$$2e = \sum_{r \in R} \deg(r) \geq 4r$$

Hence $2e \geq 4r$

$$2e \geq 4(e-v+2) \quad [\text{Euler's formula}]$$

$$2e \geq 4e - 4v + 8$$

$$4v - 8 \geq 2e$$

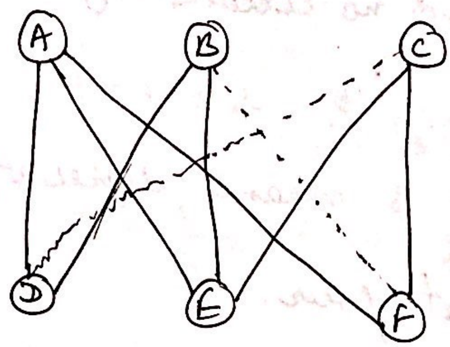
$$2v - 4 \geq e$$

or

$$\boxed{e \leq 2v-4}$$

Hence Proved.

eg show that $K_{3,3}$ is non planar.



Since $K_{3,3}$ has no circuit of length 3 & $v \geq 3$, then $e \leq 2v-4$ must be satisfied if it is planar.

$K_{3,3}$ $\quad V = 6$

$E = 9$

$e \leq 2v - 4$

$9 \leq 2.6 - 4$

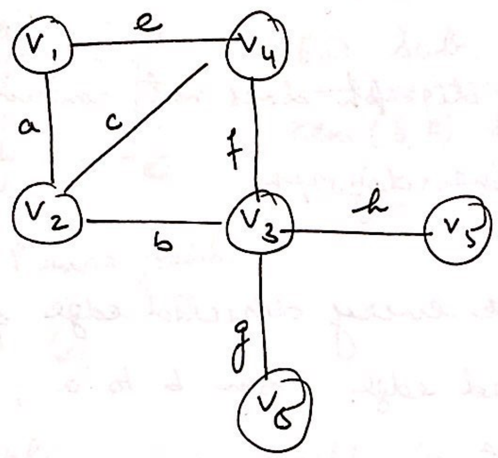
$9 \leq 8$ $\boxed{\text{False}}$

Hence $K_{3,3}$ is not planar.

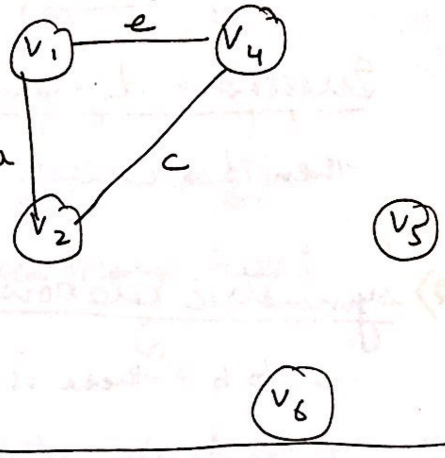
Deletion operation on a Graph

If V_i is a vertex in Graph G then $G - V_i$ denotes a subgraph of G obtained by deleting V_i from G. Deletion of a vertex implies the deletion of all the edges incident on that vertex. But deletion of an edge does'nt imply deletion of its end vertices.

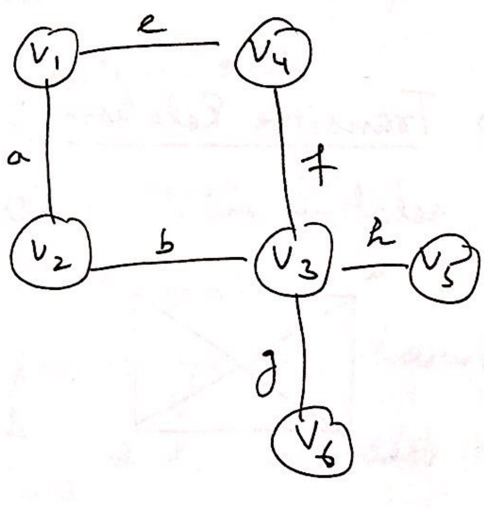
eg.



After deleting vertex V_3

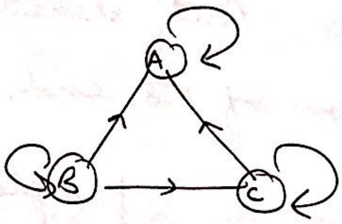


After deleting edge 'c'



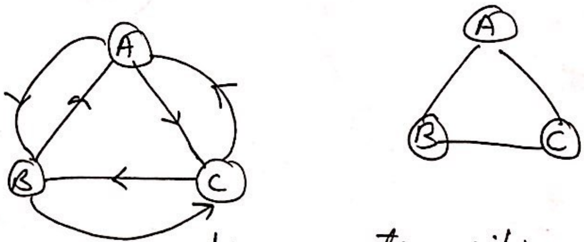
Digraphs And Relations

1) **Reflexive Relation** :- A digraph of a reflexive relation will have a self-loop at each of its vertices. Also called reflexive digraph.

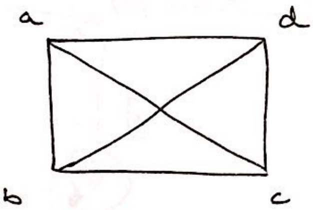


Irreflexive digraph :- If a digraph does not contain any self-loop then it is called irreflexive digraph.

2) **symmetric Relation** :- If for every directed edge from vertex a to b there is a directed edge from b to a, then it is called symmetric digraph.



3) **Transitive Relation** :- A graph representing a transitive relation on its vertex set is called trasitive graph.

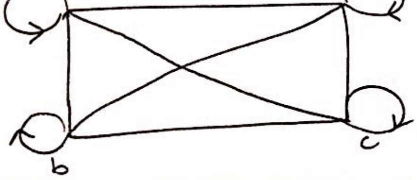


edge → (a, b) → (b, c) then (a, c) is there

edge → (a, d) → (d, b) then (a, b) is there

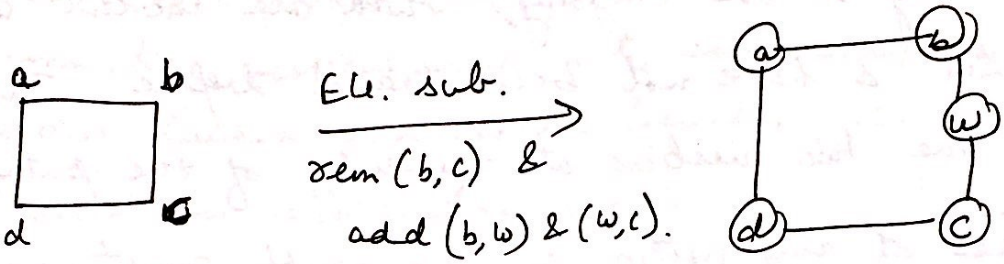
& so on.

4) **Equivalence Relation** :- A Binary relation is equivalence if it is reflexive, symmetric & transitive.



Elementary subdivision :- If a Graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex 'w' together with edges $\{u, w\}$ & $\{w, v\}$. Such an operation is called an elementary sub-division.

For eg.



Planar graph

G_1

Also. Planar Graph.

G_2

Homeomorphic :- The graph $G_2 = (V_2, E_2)$ is called Homeomorphic if it can be obtained from the same graph by a sequence of elementary subdivisions.

Kuratowski's Theorem :- A Graph is nonplanar iff it contains a subgraph Homeomorphic to $K_{3,3}$ or K_5.

Proof. Since $K_{3,3}$ and K_5 are non-planar graph (shown abov Hence Graph Homeomorphic to $K_{3,3}$ & K_5 are also non-planar.

Th: An undirected graph possess an Eulerian path iff it is connected & has either zero or two vertices of odd degree.

Proof. firstly the graph must be connected. When the Eulerian path is traced, we observe that every time the path meets a vertex, it goes through two edges (one incoming & one outgoing) which are incident with the vertex & have not been traced before. Thus except for the two vertices at the ends of the path, the degree of any vertex in the graph must be even. If the two vertices at the end of the path are distinct, they have odd degree. If they coincide the end vertex degree becomes even & euler circuit is also formed.

Th:- An undirected graph possesses an Eulerian circuit iff it is connected & its vertices are all of even degree.

Proof:- Same as above.

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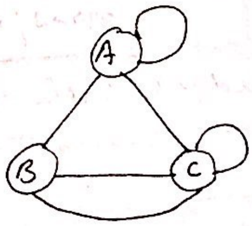
Pseudograph :- A network with a node connected to itself in terms of a Pseudograph. (A graph with self loops).



Intersection graph :- Let $A_1, A_2, A_3, \ldots, A_n$ be a collection of sets. If we represent each of these sets by vertices or nodes and draw an edge connecting vertices, such that, sets corresponding to a pair of vertices have a non-empty intersection, the resulting graph is called an intersection graph.

eg:

$A_1 = \{0, 2, 4, 6, 8\}$

$A_2 = \{0, 1, 2, 3, 4\}$

$A_3 = \{1, 3, 5, 7, 9\}$

$A_4 = \{5, 6, 7, 8, 9\}$

$A_5 = \{0, 1, 8, 9\}$

$A_1 \cap A_2 = \{0, 2, 4\}$

$A_1 \cap A_4 = \{6, 8\}$

$A_1 \cap A_5 = \{0, 8\}$

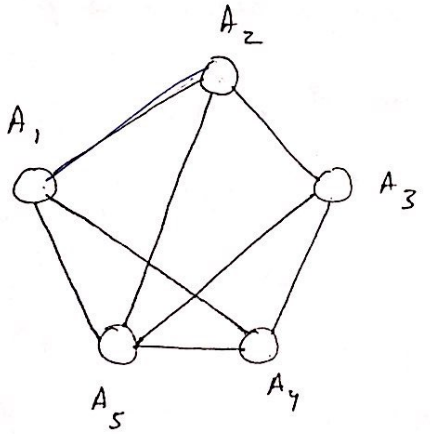
$A_2 \cap A_3 = \{1, 3\}$

$A_2 \cap A_5 = \{0, 1\}$

$A_3 \cap A_4 = \{5, 7, 9\}$

$A_3 \cap A_5 = \{1, 9\}$

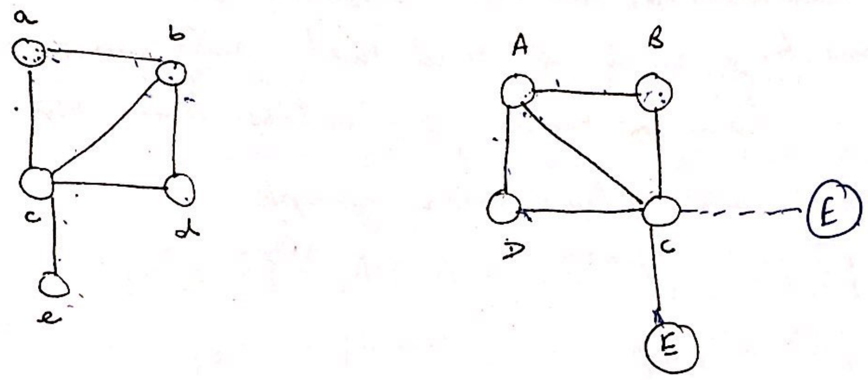
$A_4 \cap A_5 = \{8, 9\}$



Isomorphism of Graphs.

The two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one ~~and onto function f from V_1 to V_2 such that v_i and v_j are adjacent in G_1 iff $f(v_i)$ and $f(v_j)$ are adjacent in G_2 for all $v_i, v_j \in V_1$. The function f defines isomorphism.~~ correspondance b/w their vertices & b/w their edges such that incidence are preserved

eg.



vertex correspondance : $a \Leftrightarrow B$, $b \Leftrightarrow A$, $c \Leftrightarrow C$, $d \Leftrightarrow D$, $e \Leftrightarrow E$

edge correspondance :- $a,b \Leftrightarrow (B,A)$, $c,b \Leftrightarrow (C,A)$, $(b,d) \Leftrightarrow (A,D)$,

$a,c \Leftrightarrow (B,C)$, $c,e \Leftrightarrow (C,E)$

The two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are isomorphic if :-

1). The two graphs must have same no. of vertex. $|V_1| = |V_2|$

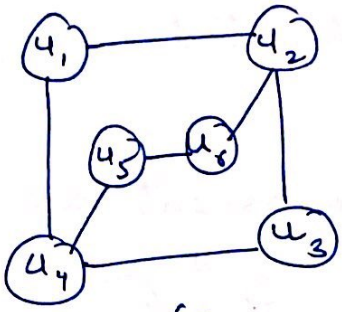
2). The two graphs must have same no. of edges. $|E_1| = |E_2|$

3). If the degree of vertex V_i in G_1 is m, then the degree of the vertex $f(V_i)$ in G_2 must also be m.

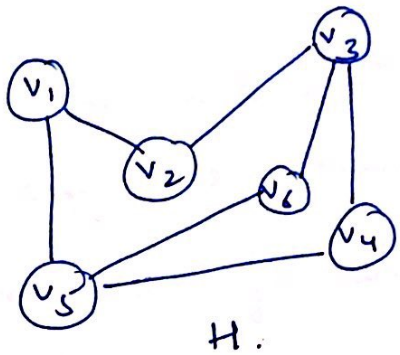
4) If f is isomorphic from graph G_1 to G_2 then f^{-1} defines ~~the~~ an isomorphism from G_2 to G_1

The above rules are necessary not sufficient for proving two simple graphs as isomorphic.

Q. Determine whether the Graphs G & H are isomorphic.



G



H.

Sol^n. first no. of vertices are same in G & H i.e 6.

no. of edges " " " i.e 7.

4 vertices have degree 2.
2 vertices have degree 3] same in G & H.

Thus G & H can be isomorphic.

Prove it using Adjacency matrix.

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$u_1 \leftrightarrow v_6$
$u_2 \leftrightarrow v_3$
$u_3 \leftrightarrow v_4$
$u_4 \leftrightarrow v_5$
$u_5 \leftrightarrow v_1$
$u_6 \leftrightarrow v_2$

$$A_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Graphs.

(20)

Q. A graph containing m edges {$e_1, e_2, \ldots, e_m$} can be decomposed into $2^{m-1}-1$ different ways into pairs of sub-graphs g_1 and g_2. (Pg - 195 Liu)

Th. In a (directed or undirected) graph with n vertices, if there is a path from v_1 to v_2, then there is a path no more than n-1 edges from vertex v_1 to vertex v_2 $(E = V-1)$ (Pg-197 Liu). (Pg-12)(Notes)

Th. An undirected graph possesses an eulerian path iff it is connected & has either zero or two vertices of odd degree. (Pg 213, Liu)

Th. An undirected graph possesses an eulerian circuit iff it is connected & its vertices are all of even degree. (Pg 214-Liu)

Th. Let G has n vertices. If the sum of the degrees for each pair of vertices in G is n-1 or larger, then there exist a hamilton path. (Pg 218, Liu).

Th. There is always a hamilton path in a directed complete ~~path~~ graph. (Pg 220-Liu).

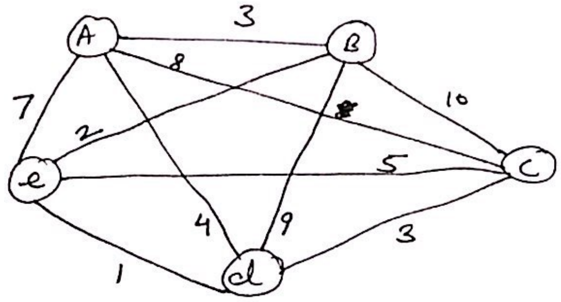
Th. For any connected planar graph, $v - e + r = 2$ (Pg 232-Liu) (Pg-7 Notes)

Th. For any connected planar graph that has no loops & has two or more edges, $e \leq 3v-6$. (Pg 232-Liu).

⇒ No. of hamilton circuits if 'n' vertices given:-

$$\frac{(n-1)!}{2}.$$

eg:-



∴ 5 vertices ∴ no. of H. circuits $(5-1)!/2 = 12$.

		Length
1).	c, d, a, b, e, c	3 + 4 + 3 + 2 + 5 = 17 ⇐
2)	c, d, a, e, b, c	3 + 4 + 7 + 2 + 10 = 26
3)	c, d, e, b, a, c	3 + 1 + 2 + 3 + 8 = 17 ⇐
4)	c, d, e, a, b, c	3 + 1 + 7 + 3 + 10 = 24
5).	c, d, b, a, e, c	3 + 9 + 3 + 7 + 5 = 27
6).	c, d, b, e, a, c	3 + 9 + 2 + 7 + 8 = 29.
7).	c, b, d, a, e, c	10 + 9 + 4 + 7 + 5 = 35
8).	c, b, d, e, a, c	10 + 9 + 1 + 7 + 8 = 35
9).	c, b, e, d, a, c	10 + 2 + 1 + 4 + 8 = 25
10).	c, b, a, d, e, c	10 + 3 + 4 + 1 + 5 = 23
11)	c, a, b, d, e, c	8 + 3 + 9 + 1 + 5 = 26
12)	c, a, d, b, e, c	8 + 4 + 9 + 2 + 5 = 28.

5! = 1×2×3×4×5 (6, 20) 120 ×5 600

Th. An undirected graph possess an eulerian path iff it is connected & has either zero or two vertices of odd degree.

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Proof. same as above.

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