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Summations 'Appendix - A'. "CORMEN"

Given a sequence $a_1, a_2, a_3, \ldots a_n$,

Sum is $\sum_{K=1}^{n} a_K = a_1 + a_2 + \ldots + a_n$.

If $n = 0$, then $\sum_{K=1}^{n} a_K = 0$

Infinite sequence no. sum is,

$$\lim_{n \to \infty} \sum_{K=1}^{n} a_K$$

Linearity

for any real no. C & any finite seqs. $a_1, a_2, \ldots a_n$ & $b_1, b_2, \ldots, b_n$, we have,

$$\sum_{K=1}^{n} (c a_K + b_K) = c \sum_{K=1}^{n} a_K + \sum_{K=1}^{n} b_K.$$

Linearity property is also applied on Asymptotic Notations, for eg,

$$\sum_{K=1}^{n} \Theta(f(k)) = \Theta\left(\sum_{K=1}^{n} f(K)\right)$$

Useful Results

1) $\sum_{K=1}^{n} K = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = \Theta(n^2)$

2) $\sum_{K=0}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$

3) $\sum_{K=0}^{n} K^3 = \frac{n^2(n+1)^2}{4}.$

4) Geometric series

for real $x \neq 1$,

$$\sum_{K=0}^{n} x^K = 1 + x + x^2 + \dots + x^n.$$

5) When summation is infinite & $|x| < 1$, we have infinite decreasing geometric series,

$$\sum_{K=0}^{\infty} x^K = \frac{1}{1-x}$$

6) Harmonic series

for positive integers n, the n^{th} harmonic no. is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}.$$

$$= \sum_{K=1}^{n} \frac{1}{K} = \ln n + O(1).$$ [Proof later

7) Integrating and differentiating series

Differentiating both sides of infinite geometric series, ie

$$\sum_{K=0}^{\infty} x^K = \frac{1}{1-x}$$ & then multiplying by x, we g

$$\sum_{K=0}^{\infty} (K \cdot x^{K-1}) x. = \frac{x}{(1-x)^2} \quad \text{for } |x| < 1.$$

or

$$\sum_{K=0}^{\infty} K \cdot x^K = \frac{x}{(1-x)^2}$$

8) Telescoping series

(a) for any seq. $a_0, a_1, a_2, \dots a_n$, we have,

$$\sum_{K=1}^{n} (a_K - a_{K-1}) = a_n - a_0$$

for eg. 1 3 4 8 10 (random seq.).

a_0 a_1 a_2 a_3 a_4

$$\sum_{K=1}^{n} (a_K - a_{K-1}) = (\cancel{a_1} - a_0) + (\cancel{a_2} - \cancel{a_1}) + (\cancel{a_3} - \cancel{a_2}) + (a_4 - \cancel{a_3})$$

$$= 2 + 1 + 4 + 2 = 9$$

and

$$a_n - a_0 = 10 - 1 = 9.$$

Since each terms $a_1, a_2, a_3 \dots a_{n-1}$ is added exactly once & subtracted out exactely once, the remaining terms are a_n and $-a_0$.

Thus $a_n - a_0 = \sum_{K=1}^{n} (a_K - a_{K-1})$.

(b) Similarly, for any sequence $a_0, a_1, a_2, a_3 - a_n$,
1, 3, 4, 8, 10

we have,

$$\sum_{K=0}^{n-1} (a_K - a_{K+1}) = a_0 - a_n$$

Here $n = 4$,

$$(a_0 - \cancel{a_1}) + (\cancel{a_1} - \cancel{a_2}) + (\cancel{a_2} - \cancel{a_3}) + (\cancel{a_3} - a_4) = a_0 - a_4$$

$$\Rightarrow (-2) + (-1) + (-4) + (-2)$$

$$\Rightarrow a_0 - a_n = 1 - 10 = -9.$$

eg: $$\sum_{K=1}^{n-1} \frac{1}{K(K+1)}.$$

rewrite each term as,

$$\frac{1}{K(K+1)} = \frac{1}{K} - \frac{1}{(K+1)}$$ [Partial fractions]

Applying summation both sides,

$$\sum_{K=1}^{n-1} \frac{1}{K(K+1)} = \sum_{K=1}^{n-1}\left(\frac{1}{K} - \frac{1}{K+1}\right)$$

$$= \sum_{K=1}^{n-1} \frac{1}{K} - \sum_{K=1}^{n-1} \frac{1}{K+1}$$

$$\Rightarrow \left[\cancel{\left(\sum_{K=1}^{n} \frac{1}{K}\right)} - \frac{1}{n}\right] - \left[\cancel{\left(\sum_{K=1}^{n} \frac{1}{K}\right)} - 1\right]$$

$$\left[\because \sum_{K=1}^{n-1} \frac{1}{K} = 1 + \frac{1}{2} + \cdots + \frac{1}{4}\right.$$ [suppose $n = 5$]

$$\sum_{K=1}^{n} \frac{1}{K} = 1 + \frac{1}{2} + \cdots + \frac{1}{4} + \boxed{\frac{1}{5}}$$ → extra term.

similary, suppose $n = 5$,

$$\sum_{K=1}^{n-1} \frac{1}{K+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

extra term.

$$\sum_{K=1}^{n} \frac{1}{K} = \boxed{\frac{1}{1}} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$= 1 - \frac{1}{n}$$

9) Product series

The finite product $a_1 a_2 a_3 \cdots a_n$ can be written as

$$\prod_{K=1}^{n} a_K .$$

If $n = 0$, the product is defined to be 1.

We can convert product formula, to a summation formula by using the identity,

$$\log_2\left(\prod_{K=1}^{n} a_K\right) = \sum_{K=1}^{n} \log_2 a_K .$$

Exercise Questions

1). Find simple formula for,

$$\sum_{K=1}^{n} (2K - 1)$$

$$\Rightarrow \sum_{K=1}^{n} 2K - \sum_{K=1}^{n} 1$$

$$\Rightarrow 2 \sum_{K=1}^{n} K - n$$

$$\Rightarrow \cancel{2} \cdot \frac{n(n+1)}{\cancel{2}} - n$$

$$\Rightarrow n^2 + \cancel{n} - \cancel{n} = n^2 .$$

2) Show that

$$\sum_{K=1}^{n} \frac{1}{2K-1} = \ln(\sqrt{n}) + O(1)$$ by manipulating harmonic series.

$$= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1}$$

$$= \frac{1}{1} + \boxed{\frac{1}{2}} + \frac{1}{3} + \boxed{\frac{1}{4}} + \frac{1}{5} + \boxed{\frac{1}{6}} + \frac{1}{7} + \cdots + \frac{1}{2n-1} + \boxed{\frac{1}{2n}} - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$$

$$= \ln 2n + O(1) - \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$\left[\because \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} = \sum_{K=1}^{n} \frac{1}{K} = \ln n + O(1)\right]$$

$$= \ln 2n + \underline{O(1)} - \frac{1}{2}\left(\ln n + \underline{O(1)}\right)$$

$$= \ln 2 + \underline{\ln n - \frac{1}{2}(\ln n)} + O(1)$$

$$\left[\because \log_c(ab) = \log_c a + \log_c b\right]$$

$$\left[\begin{array}{l} \ln n = \log_e n \\ \text{ie } \ln 2 = \log_e 2 \\ \text{use } e = 2.71 \\ \text{ie } \log_{2.7} 2 < 1 \end{array}\right.$$

$$= \underset{(c<1)}{\text{constant}} + \frac{1}{2}\ln n + O(1)$$

$$= \ln n^{1/2} + O(1) \quad \text{[Discarding constant]}$$

$$= \ln \sqrt{n} + O(1)$$

(3) show that,

$$\sum_{K=0}^{\infty} K^2 x^K = \frac{x(1+x)}{(1-x)^3} \quad \text{for} \quad 0 < |x| < 1$$

Sol^n we know the eqn,

$$\sum_{K=0}^{\infty} K . x^K = \frac{x}{(1-x)^2}$$

Diff. above eq^n. both sides

$$\sum_{K=0}^{\infty} K^2 . x^{K-1} = \frac{1+x}{(1-x)^3}$$

multiplying both sides by x, we get,

$$\sum_{K=0}^{\infty} K^2 . x^K = \frac{x(1+x)}{(1-x)^3} = R.H.S.$$

(4) Show that, $\displaystyle\sum_{K=0}^{\infty} \frac{(K-1)}{2^K} = 0$.

$$L.H.S. \Rightarrow \sum_{K=0}^{\infty}\left(\frac{K}{2^K} - \frac{1}{2^K}\right)$$

$$= \sum_{K=0}^{\infty} K.\left(\frac{1}{2}\right)^K - \sum_{K=0}^{\infty}\left(\frac{1}{2}\right)^K$$

$$= \frac{1/2}{\left(1-\frac{1}{2}\right)^2} - \frac{1}{1-\frac{1}{2}} = 0$$

$$\therefore \left[\sum_{K=0}^{\infty} K . x^K = \frac{x}{(1-x)^2} \; , \quad \sum_{K=0}^{\infty} x^K = \frac{1}{1-x}\right]$$

⑤ Evaluate the sum,

$$\sum_{k=1}^{\infty} (2k+1)\, x^{2k}$$

Sol^n.

$$\sum_{k=1}^{\infty} 2.k.x^{2k} + \sum_{k=1}^{\infty} x^{2k}$$

$$(A) \quad + \quad (B).$$

(A) $\Rightarrow$

$$\sum_{k=1}^{\infty} 2k.x^{2k}$$

$$= \sum_{k=0}^{\infty} 2k.x^{2k} - 2.0.x^{2(0)}$$

$$= 2\sum_{k=0}^{\infty} k.(x^{2})^{k} = \frac{2x^2}{(1-x^2)^2} \qquad \left[\sum_{k=0}^{\infty} k.x^{k} = \frac{x}{(1-x)^2} \right.$$

(B) $\Rightarrow$

$$\sum_{k=1}^{\infty} x^{2k}.$$

$$= \sum_{k=0}^{\infty} x^{2k} - x^{2(0)}.$$

$$= \sum_{k=0}^{\infty} (x^2)^{k} - 1 \qquad \left[\sum_{k=0}^{\infty} x^{k} = \frac{1}{1-x}. \right.$$

$$= \frac{1}{1-x^2} - 1 = \frac{\cancel{1}-\cancel{1}+x^2}{1-x^2} = \frac{x^2}{1-x^2}.$$

$\therefore$ A + B $\Rightarrow$

$$\frac{2x^2}{(1-x^2)^2} + \frac{x^2}{1-x^2}$$

$\Rightarrow$ - - - -

Bounding Summations

Techniques used for Bounding the summations :-

(I) Mathematical Induction

(1) $$\sum_{K=1}^{n} K = \frac{n(n+1)}{2}$$

Sol^n. verify for $n=1$, $1 = \frac{1(2)}{2}$ $\therefore$ L.H.S = R.H.S.

make inductive assumption that it holds for n & prove that it holds for $n+1$. we have,

$$\sum_{K=1}^{n+1} K = \sum_{K=1}^{n} K + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) = \frac{1}{2}(n+1)(n+2)$$

Induction can be used to show bound as well.

Q. Prove that $\sum_{K=0}^{n} 3^K = O(3^n)$.

Sol^n. we have to Prove, $\sum_{K=0}^{n} 3^K \leq c.3^n$ for some constant $c > 0$,

verify for $n=0$, $\sum_{K=0}^{0} 3^0 \leq c.3^0$

ie $1 \leq c.1$ $\forall c \geq 1$

Assume that bound holds for n, Prove that it holds for $n+1$.

$$\sum_{K=0}^{n+1} 3^K = \sum_{K=0}^{n} 3^K + 3^{n+1}$$

$$\leq c.3^n + 3^{n+1}$$

$$\leq c.3^{n+1}\left(\frac{1}{3} + \frac{1}{c}\right)$$

$$\leq c.3^{n+1} \quad \text{[Discarding constants]}$$

$$= O(3^{n+1})$$

(II) Bounding the terms

A good upper bound on an arithmetic series can be obtained by bounding each term of the series, and it is often sufficient to use the largest term to bound the others. For eg:-

$$\sum_{K=1}^{n} K \leq \sum_{K=1}^{n} n \quad \left[\sum_{K=1}^{5} K \leq \sum_{K=1}^{5} n = 5+5+5+5+5 = 25 \text{ or } 5^2 = 25.(n^2)\right.$$

$$= n^2$$

when the given series is a geometric series, bounding the term is different. Given the series $\sum_{K=0}^{n} a_K$, find the ratio b/w two consecutive terms ie $(a_{K+1} / a_K) \leq \gamma$ for all $K \geq 0$, & $0 < \gamma < 1$ is constant. The sum can be bounded by an infinite decreasing geometric series ie

$$\sum_{K=0}^{n} a_K \leq \sum_{K=0}^{\infty} a_0 \gamma^K$$

$$\leq a_0 \sum_{K=0}^{\infty} \gamma^K$$

$$\leq a_0 \left(\frac{1}{1-\gamma}\right)$$

eg. $\sum_{K=1}^{\infty} \frac{K}{3^K}$.

Sol^n. In order to start the summation at $K=0$, we rewrite it as $\sum_{K=0}^{\infty} ((K+1)/3^{K+1})$. The first term (a_0) is $\frac{1}{3}$, & the ratio (γ) of consecutive terms is $\frac{(K+2)/3^{K+2}}{(K+1)/3^{K+1}}$

$$= \frac{1}{3} \cdot \frac{K+2}{K+1} = \frac{2}{3} \quad [\text{Put } K \geq 0]$$

$$\therefore \gamma = \frac{2}{3} \text{ \& it lies b/w } 0 < \gamma < 1$$

Thus, we have,

$$\sum_{K=1}^{\infty} \frac{K}{3^K} = \sum_{K=0}^{\infty} \frac{K+1}{3^{K+1}} \leqslant \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1$$

$\underline{\underline{Q.}}$ Find asymptotic upper bound on

$$\sum_{K=0}^{\infty} \frac{K^2}{2^K}$$

$\underline{Sol^n.}$ Ratio of two consecutive terms is,

$$\frac{(K+1)^2/2^{K+1}}{K^2/2^K} = \frac{(K+1)^2}{2K^2} \leqslant \frac{8}{9} \quad \text{if } K \geqslant 3.$$

$\because$ $K = 0 \Rightarrow \frac{(0+1)^2}{2.0} = N.D$

$K = 1 \Rightarrow \frac{(1+1)^2}{2.1^2} = \frac{4}{2} = 2$ [does not lie b/w 0 & 1]

$K = 2 \Rightarrow \frac{3^2}{2.4} = \frac{9}{8} = r, \; r > 1$

$K = 3 \Rightarrow \frac{4^2}{2.9} = \frac{\cancel{16}^{8}}{\cancel{18}_{9}} = \frac{8}{9} < 1$

$K = 4 \Rightarrow \frac{25}{2.16} = \frac{25}{32} < 1$

$\therefore$ true for $K \geqslant 3$, $\boxed{0 < r < 1}$

Thus split the summation,

$$\sum_{K=0}^{\infty} \frac{K^2}{2^K} = \sum_{K=0}^{2} \frac{K^2}{2^K} + \sum_{K=3}^{\infty} \frac{K^2}{2^K}$$

$$\leqslant \underbrace{\sum_{K=0}^{2} \frac{K^2}{2^K}}_{\text{constant}} + \frac{9}{8} \sum_{K=0}^{\infty} \left(\frac{8}{9}\right)^K$$ [check only for first two consecutive terms]

$$\leqslant \left(\frac{1}{2} + \frac{4}{4}\right) + \frac{9}{8} \cdot \left(\frac{1}{1-\frac{8}{9}}\right)$$

$$\leqslant \frac{6}{4} + \frac{9}{8}\left(\frac{1}{1} \times \frac{9}{1}\right)$$

$$\leq \frac{6}{4} + \frac{81}{8}$$

$$\leq \frac{12+81}{8} = \boxed{\frac{93}{8}}$$

or $\boxed{O(1)}$.

Because $\Rightarrow \frac{93}{8} = O(1)$

$$\frac{93}{8} \leq c.1$$

$$\frac{93}{8} \leq \frac{93}{8}.$$

$\frac{93}{8}n = O(n) \Rightarrow \frac{93}{8}\cancel{n} \leq c\cancel{n}$

$c = \frac{93}{8}$

Put $c = \frac{93}{8}$, $n_0 = 1$

Q. Show that $\sum_{K=1}^{n} \frac{1}{K^2}$ is bounded above by a constant.

Sol^n. $\sum_{K=1}^{n} \frac{1}{K^2}$

$$\leq \sum_{K=0}^{\infty} \frac{1}{(K+1)^2}$$

series $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$

$$= \sum_{K=0}^{\infty} \left(\frac{1}{K+1}\right)^2, \text{ we have, } a_0 = 1$$

$$\text{ratio} = \frac{1/4}{1} = \frac{1}{4}$$

since, $0 < \frac{1}{4} < 1$

Hence $a_0 \cdot \left(\frac{1}{1-r}\right) = 1 \cdot \left(\frac{1}{1-\frac{1}{4}}\right) = \frac{1}{\frac{3}{4}}$

$$= \boxed{\frac{4}{3} = c > 0}$$

$\therefore$ bounded above by a constant $\frac{4}{3}$