Permutations And Combinations (Liu).

Rule of Product :- If one experiment has 'm' possible outcomes & another experiment has 'n' possible outcomes, then there are $m \times n$ possible outcomes when both of these experiments takes place.

Rule of Sum :- If one experiment has m possible outcomes & another experiment has n possible outcomes, then there are $m + n$ possible outcomes when exactly one of these experiments takes place.

For eg :- 1^{ST} experiment → tossing a coin. → 2 outcomes (H, T)
2^{nd} experiment → rolling a dice. → 6 outcomes (1, 2, 3, 4, 5, 6).

Rule of Product :- If both experiment takes place the outcomes are $2 \times 6 = 12$. ie

H, 1		T, 1
H, 2		T, 2
H, 3	or	T, 3
H, 4		T, 4
H, 5		T, 5
H, 6		T, 6.

Rule of Sum :- If exactly one of these experiment takes place then possible no. of outcomes are $2 + 6 = 8$.

If 1^{ST} exp. takes place, outcome are → 2

If 2^{nd} exp takes place, outcome are → 6.

∴ Total no. of possible outcomes are $2 + 6 = 8$.

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Permutations

Q. Ten boxes are there. Three ball are there to be placed in these 10 boxes, coloured Red, blue & green. Find the no. of ways in which balls can be placed in the boxes, if each box can hold only one ball at a time.

Sol^n. first ball can be placed in any of the 10 boxes.
second ball " " " " " " " 9 boxes.
third ball " " " " " " " 8 boxes.

$\therefore$ Total no. of ways to place 3 balls in 10 boxes are :-

$$10 \times 9 \times 8 = 720.$$

OR

$$P(n,r) = \frac{n!}{(n-r)!} \text{ ie } P(10,3) = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} = 720.$$

Q. In how many ways can three exams be organised within a five day period so that no two exams are on the same day.

Ans. first exam $\rightarrow$ 5 days
second " $\rightarrow$ 4 days
third " $\rightarrow$ 3 days.

or $P(5,3) = \frac{5!}{2!}$

$\therefore 5 \times 4 \times 3 = 60$ ways

$$= \frac{5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 60$$

Q. find a four-digit decimal numbers that contain no repeated digits.

Ans. Total 10 digits are there $\rightarrow 0, 1, 2, \ldots, 9$.

_ _ _ _

$\downarrow$

first place can be occupied by $(1, 2, 3, \ldots 9)$ any of the 9 nos.
second " " " " " $(0, \ldots 9)$ any of 10 nos.
except the one used at first place.

$\therefore 9 \times 9 \times 8 \times 7 = 4536$ OR

OR.

consider it as a problem of arranging 4 of the 10 digits $(0, 1, \ldots, 9)$.

ie $P(10,4) = 5040$.

Among these 5040 nos. $9 \times 8 \times 7 = 504$ of them have a leading 0. ie

$$\underset{1}{\underline{0}} \times \underset{9}{\underline{\;}} \times \underset{8}{\underline{\;}} \times \underset{7}{\underline{\;}} = 504.$$

ie $5040 - 504 = \underline{\underline{4536}}$.

Q. Find no. of ways in which we can makeup strings of four distinct letters followed by three distinct digits.

Ans. $P(26,4) \times P(10,3) = 258,336,000$

OR

$$\underset{26}{\underline{\;}} \times \underset{25}{\underline{\;}} \times \underset{24}{\underline{\;}} \times \underset{23}{\underline{\;}} \times \underset{10}{\underline{\;}} \times \underset{9}{\underline{\;}} \times \underset{8}{\underline{\;}} = 258,336,000$$

Q. place 3 distinct coloured balls in 10 boxes if a box can hold as many balls as we wish.

Soln. $10 \times 10 \times 10 = 1000$

first ball → any 10 boxes

second " → " " "

third " → " " "

$\therefore 10^3 = 1000$

or

n^r ways to place r colored balls into n numbered boxes if a box can hold any no of balls.

Q. organize 3 exams in 5-days with no restriction on the no. of exams organised on each day.

Solⁿ. $5^3 = 125$, $\underset{\downarrow}{\_}\ \_\ \_$

1^{st} exam → 5 days time
2^{nd} " → " " "
3^{rd} " → " " " ∴ $5 \times 5 \times 5 = 125$.

Consider the Problem of Placing 4 balls (unique color) in 10 boxes! the result is $P(10, 4) = 5040$.

		Box 1	Box 2	Box 3	Box 4
Case I	one arrangement is :-	Light Red	Dark Red	Blue	green.
Case II	another " "	Dark Red	Light Red	Blue	green.

Now, if there is a single Red color, then above two cases maps to a single cell. ie to half.

∴ 5040 placements can be paired off in a simpler way so that every pair of placements becomes one when we do not differentiate the two shades of red.

$$\therefore \frac{5040}{2} = 2520 \text{ ways.}$$

Q. Place 3 Red balls, one blue ball & ~~one~~ two white ball in 10 boxes.

Solⁿ.

$$\frac{P(10, 6)}{3!\ 2!}$$

Generalized Result is

$$\frac{P(n, r)}{q_1!\ q_2! \ldots q_t!}$$

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In terms of arrangement of objects, we say that there are $\frac{n!}{q_1! \, q_2! \ldots q_t!}$ ways to arrange n objects, where q_1 of them are of one kind, q_2 of second kind, ... & q_t are of t^{th} kind.

Q. find the no. of ways to paint 12 offices so that 3 of them will be green, 2 of them pink, 2 of them yellow & rest of them white.

Ans.

$$\frac{12!}{3! \; 2! \; 2! \; 5!} = 166,320.$$

Q. find the no. of different messages that can be represented by seq. of three dashes & two dots is:-

Ans.

$$\frac{5!}{3! \; 2!} = 10$$

'Combinations'

Q. Place 3 Red colored balls in 10 boxes (one box can hold 1 ball)

Ans $\frac{P(n,r)}{3!} = \frac{P(10,3)}{3!} = \frac{10\times9\times8}{3!} =$

No. of ways of placing r balls of same color in n numbered boxes is $= \frac{n!}{(n-r)!\,r!}$ or $\frac{P(n,r)}{r!}$ or $c(n,r)$.
(1 box → 1 ball)

ie $c(n,r) = \frac{n!}{(n-r)!\ r!}$

Q. How many binary sequences of length 32 are there that contain exactly 7 one's.

Ans. if 7 one's are there then ~~25~~ 25 zero's will be there.

ie $\frac{32!}{7!\ 25!}$ or $c(32,7) = \frac{32!}{25!\ 7!}$

In general, the no. of ways of placing r balls of the same color in n numbered boxes, ~~$\frac{n!}{r!\,(n-r)!}$ or $c(n,r)$~~ allowing as many balls in a box as we wish, is ⇒

$$\frac{(n+r-1)!}{r!\ (n-1)!} = c(n+r-1,\ r).$$

Q. No. of ways to choose 3 out of 7 days (with repetition allowed) is

Solⁿ. $$c(7+3-1, 3) = c(9,3) = 84.$$

Q. No. of ways to choose seven out of 3 days (with repetitions necessarily allowed) is,

$$c(3+7-1, 7) = c(9,7) = 36.$$

Q. When three dices are rolled, the no. of different outcomes is

$$c(6+3-1, 3) = c(8,3) = 56.$$

because rolling a dice is equivalent to selecting three nos. from the six nos 1, 2, 3, 4, 5, 6. with repetitions allowed.

Q. Determine the no. of ways to seat five boys in a row of 12 chairs.

Ans. $$\frac{12!}{7!\ 1!\ 1!\ 1!\ 1!\ 1!} = \frac{12!}{7!}$$

The problem can be viewed as that of arranging 12 objects that are of six different kinds, with each boy being an object of a distinct kind & the 7 unoccupied chairs being objects of the same kind. Thus $\frac{12!}{7!}$

Or

suppose, we first arrange 5 boys in a row ($\because$ 5! ways) and then distribute 7 unoccupied chairs arbitrarily, either b/w the boys or at the two ends. The problem becomes that of placing 7 balls ~~in~~ of same color in six boxes.

one of the case $\rightarrow$ C B C B C B C B C B C C

~~Thus, the no. of ways~~

another case $\rightarrow$ B B B B B C C C C C C C

Thus, the no. of ways to do so is,

$$5! \times C(6+7-1, 7) = \cancel{5!} \times \frac{12!}{7!\cancel{5!}} = \frac{12!}{7!}$$

If we want to seat the boys so that no two boys are next to each other. then,

$$5! \times C(6+3-1, 3) = 5! \times \frac{8!}{3!5!} = \frac{8!}{3!}$$

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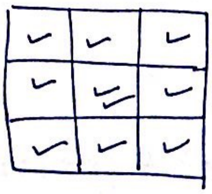
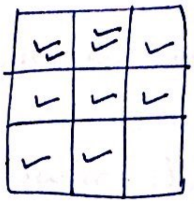
'Pigeon-Hole Principle' (Rosen - 5.2).

If there are more pigeons than pigeonholes, then there must be atleast one pigeonhole with at least two pigeons in it.

eg:- 9 Pigeonholes
10 Pigeons



atleast 1 Pigeonhole



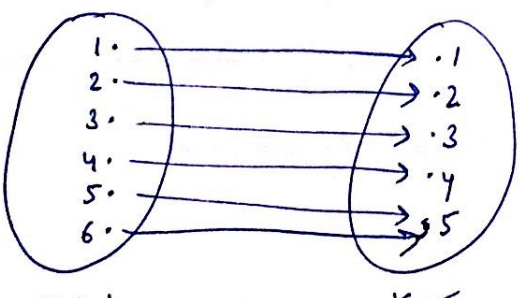
at least 1 Pigeonhole
(here 2).

Theorem If K is a +tive integer and K+1 or more objects are placed into K boxes, then there is atleast one box containing two or more of the objects.

Proof. Pigeonhole Principle is proved using Proof by contraposition. Suppose that none of the K boxes contains more than 1 object. Then the total no. of objects would be atmost K. This is a contradiction, because there are atleast K+1 objects.

Corollary A function f from a set with K+1 or more elements to a set with K elements is not one-to-one.

eg:- $K=5$



K+1 K=5

Hence Not one-to-one.

eg 1 Among any group of 367 people, there must be atleast two with the same birthday, because there are only 366 possible birthdays.

eg 2. In any group of 27 english words, there must be atleast two that begin with the same letter, because there are 26 letters in english alphabet.

eg 3. How many students must be in a class to gaurantee that atleast two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

sol^n There are 101 possible scores on the final. The pigeonhol principle shows that among any 102 students there must be atleast ~~#~~ 2 students with the same score.

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