SMA 104: CALCULUS I

BACKGROUND INFORMATION

Functions

Example 1:

Suppose we have 2 functions, *f* and *g*, both having $\Re$ as a domain and suppose one of them squares each member of a domain and the other doubles each member of a domain.

We wrote *f(x)* to represent the image of *x* under the function *f* and *g(x)* to represent the image of *g(x)* under *g*.

i.e. $f(x) = x^2$ and $g(x) = 2x$. In this case,

$f(5) = 25$ and $g(5) = 10$, $f(a) = a^2$, $g(k) = 2k$, $f(a+h) = (a+h) = (a+h)^2$ and so on.

Example 2: Given that $h(x) = x^2 - x$, find the value of

a) $h(10)$ b) $h(-3)$ c) $h(t+1)$

Solution:

(a) $h(x) = x^2 - x$

$h(10) = 100 - 10 = 90$

(b) $h(-3) = 9 + 3 = 12$

(c) $h(t+1) = (t+1)^2 - (t+1)$

$= t^2 + 2t + 1 - t - 1$

$= t^2 + t$

Composite functions

Example 3: Given that $f(x) = 10 + x$ and $g(x) = x^3$. Find (a) *fg* b) *gf*

Solution:

a) $g(x) = x^3$ $f(g(x)) = f(x^3) = 10 + x^3$ $\therefore fg = 10 + x^3$

b) *gf* *f(x)=10+x* ; $g\{f(x)\} = g[10 + x] = (10 + x)^3$

Example 4: Given that *f(x)=5x+1* and that $g(x) = x^2$, express the composite functions (a) *fg* b) *gf* in their simplest possible forms.

Solution:

(a) $g(x) = x^2$; $f[g(x)] = f(x^2) = 5x^2 + 1$; $fg = 5x^2 + 1$

(b) $f(x) = 5x + 1$; $g[f(x)] = g(5x + 1) = (5x + 1)^2 = 25x^2 + 10x + 1$

The Inverse of a function

Consider the function $f(x) = \frac{1}{8}x^3 + 1$

If we are given a member of the range say *f(x)=9* ,it is possible to find the corresponding member of the domain.

Example 5: Find the inverse of $f(x) = \frac{1}{8}x^3 + 1$

Solution:

Let $y = \frac{1}{8}x^3 + 1$

$\frac{1}{8}x^3 + 1 = y$

$\frac{1}{8}x^3 = y - 1$

$x^3 = 8(y-1)$

$x = \sqrt[3]{8(y-1)}$

$\therefore f^{-1}(x) = \sqrt[3]{8(x-1)}$

e.g. If *f(x)=9*; , we have *y = 9, i.e.*

$\frac{1}{8}x^3 + 1 = 9$

$\frac{1}{8}x^3 = 8$

$x^3 = 64$

$x = 4$

We may also use the inverse function

i.e. $f^{-1}(x) = \sqrt[3]{8(x-1)}$; $f^{-1}(9) = \sqrt[3]{8(9-1)} = \sqrt[3]{64} = 4$

Example 6: (beginning of lesson on 16/1/12)

Let $f(x) = \frac{5x+7}{3x+2}$. Find $f^{-1}(x)$

Solution:

$f(x) = \frac{5x+7}{3x+2}$; $y = \frac{5x+7}{3x+2}$; $y(3x+2) = 5x+7$; $3xy + 2y = 5x + 7$

$3xy - 5x = 7 - 2y$; $x(3y-5) = 7 - 2y$; $x = \frac{7-2y}{3y-5}$; $f^{-1}(x) = \frac{7-2x}{3x-5}$

Example 7: Given that *f(x)=5x+1* ,find the values of (a) $f^{-1}(36)$, (b) $f^{-1}(0)$

Solution:

Let *y=5x+1*

$5x = y - 1$; $x = \frac{y-1}{5}$; $f^{-1}(x) = \frac{x-1}{5}$

(a) $f^{-1}(36) = \frac{35}{5} = 7$; (b) $f^{-1}(0) = \frac{-1}{5}$

Example 8: Given that *f(x)=10x* and *g(x)=x+3*, find

a) $fg(x)$ b) $(fg)^{-1}x$

Solution:

a) $g(x) = x+3$; $f(g(x)] = f(x+3) = 10x + 30$; $(fg)(x) = 10x + 30$

b) $(fg)(x) = 10x + 30$;

Let $y = 10x + 30$ $y - 30 = 10x$; $x = \frac{y-30}{10}$; $(fg)^{-1}x = \frac{x-30}{10} = \frac{x}{10} - 3$

Exercise 1

Find the inverse of the following functions

1. $f(x)=\frac{5}{9}(x-32)$

2. $f(x)=180(x-2)$

3. $f(x)=\frac{5(x+7)}{3}-9$

4. $f(x)=\frac{1}{x^2}$

Answers to Exercise 1

1. $f(x)=\frac{5}{9}(x-32)$

$y=\frac{5}{9}(x-32)$

$9y=5x-160$

$x=\frac{9y+160}{5}$

$f^{-1}(x)=\frac{9x+160}{5}$

$=\frac{9x}{5}+32$

2. $f(x)=180(x-2)$

$y=180x-360$

$x=\frac{y+360}{180}$

.

$f^{-1}(x)=\frac{x+360}{180}=\frac{x}{180}+2$

3. $f(x)=\frac{5(x+7)}{3}-9$; $y=\frac{5x+35}{3}-9$; $y+9=\frac{5x+35}{3}$

$3(y+9)=5x+35$; $x=\frac{3y+27-35}{5}$; $x=\frac{3y-8}{5}$; $f^{-1}(x)=\frac{3x-8}{5}$

4. $f(x)=\frac{1}{x^2}\Rightarrow y=\frac{1}{x^2}\Rightarrow x^2=\frac{1}{y}\Rightarrow x=\pm\sqrt{\frac{1}{y}}$; $f^{-1}(x)=\pm\sqrt{\frac{1}{x}}$

Injections (One-to-one functions)

Let $f:A\to B$ be a function, then f is said to be $1-1$(injective) if x', $x\in A$ and $x\neq x'$, then $f(x)\neq f(x')$ or If $f(x)=f(x')$ implies $x=x'$ $\forall x,x'\in A$

That is distinct elements in the domain have distinct images.

Example 9: Let $f:\Re\to\Re$ be defined by $f(x)=x^2$.Is f $1-1$?

Solution:

f is not $1-1$ since $-1\neq 1$ but $f(-1)=f(1)$.

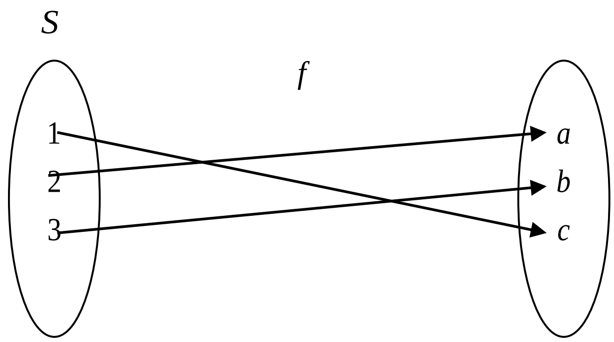
Example 10: Let $f:\Re\to\Re$ be defined by $f(x)=2x+1$.Is f $1-1$?

Solution:

f is $1-1$ since if $a,b\in\Re$ such that $f(a)=f(b)$, then

$$2a+1=2b+1$$
$$\Rightarrow 2a=2b$$
$$\Rightarrow a=b$$

Example 11:



$\therefore f \, is \, 1-1$

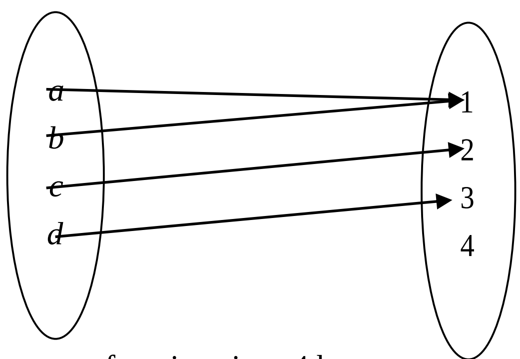
Onto functions (surjective functions)

Let $f: A \to B$. Then we say that f is an onto (or surjective function) if for every $y \in B$,there exists $x \in A$ such that $y = f(x)$.

i.e $f(A) = \operatorname{Im} f = B$ i.e the image of f is the entire codomain B.In this case,we say that f is a function from A onto B.

Example 12:

Let



This is not an onto function since 4 has no corresponding image.

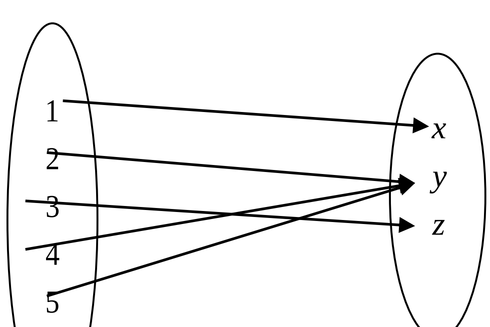
Example 13: Let $f: \Re \to \Re$ be defined by $f(x) = x^2$.Is f onto?

Solution:

f is not onto since $-1 \in \Re$ and -1 has no pre-image in $\Re$.

Example 14:

Let



f is onto

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Example 15: Let $f:\Re \to \Re$ be defined by $f(x)=2x+1$.Is f onto?
Solution: f is onto.

Bijective functions
A bijective function is a function which is both $1-1$ and onto
A function $f: A \to B$ is invertible if and only if it is a bijection i.e a $1-1$ and onto function.
Example 16: Let $f:\Re \to \Re$ be defined by $f(x)=x^2+1$. Is f onto? Is f $1-1$? Is f a bijection?
Solution:
a) f is not onto since for example $0 \in \Re$ has no corresponding preimage.
b) f is not $1-1$ since $f(-1)=f(1)=2$ and $-1 \neq 1$.
c) f is not a bijection since it is not an onto and $1-1$ function.

Example 17: Let $f:\Re \to \Re$ be defined by $f(x)=2x+1$.Find the formula for f^{-1}.
Solution:
Let $y=2x+1$; making x the subject,

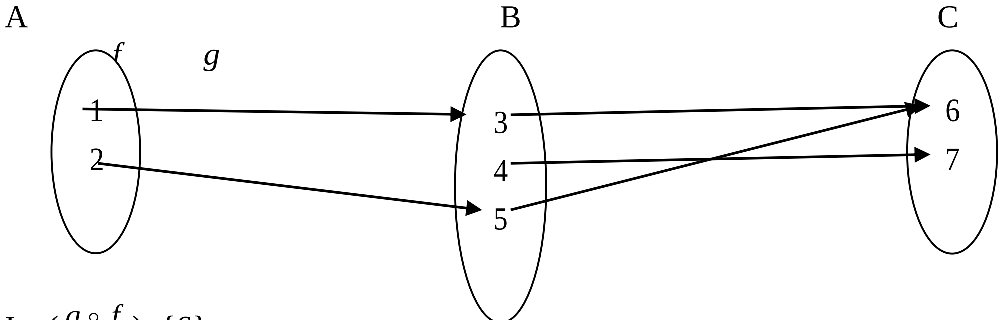
$$2x = y-1; \quad x=\frac{1}{2}(y-1)$$

Hence $f^{-1}(x)=\frac{1}{2}(x-1)$

Example 18:.Let $f(x)=\frac{3x-5}{7}$ .Find a formula for f^{-1}.
Solution:
Let $y=\frac{3x-5}{7}$, then

$$> y = 3x-5 \Rightarrow 3x = 7y-5; \quad x=\frac{7y-5}{3}; \quad \therefore f^{-1}(x)=\frac{7x-5}{3}$$

Composition of functions
Let $f: A \to B$ and $g: B \to C$. The composite function $g \circ f$ of is the function from A to C defined by $(g\ o f)(x)=g(f(x))\ for\ all\ x \in A$
Example 19: Let $A=\{1,2\}$, $B=\{3,4,5\}$ *and* $C=\{6,7\}$ and let $f: A \to B$ and $g: B \to C$ be defined by $f(1)=3, f(2)=5, g(3)=6, g(4)=7, g(5)=6$.
Find image i.e. im $(g \circ f)$
Solution:



Im ($g \circ f$)={6}

Example 20: Let $f:\Re \to \Re$ and $g:\Re \to \Re$ be defined by $f(x)=2x+1$ and $g(x)=x^2-2$.Find the formula defining the composition

(i) $g\ of$ (ii) $fof = f^2$ (iii) fog (iv) $(g\ of)(2)$

Solution:

$$(i)(g\ of)(x)=g(f(x))=g(2x+1)$$
$$=(2x+1)^2-2$$
$$=4x^2+4x+1-2$$
$$=4x^2+4x-1$$

$$(ii)\ (fof)(x)=f(f(x))=f(2x+1)$$
$$=2(2x+1)+1$$
$$=4x+2+1$$
$$=4x+3$$

$$(iii)\ (fog)(x)=f(g(x))=f(x^2-2)$$
$$=2(x^2-2)+1$$
$$=2x^2-4+1$$
$$=2x^2-3$$

$$(iv)\ (g\ of)(x)=4x^2+4x-1$$

$$(g\ of)(2)=4(2^2)+4(2)-1$$
$$=4(4)+8-1$$
$$=16+8-23$$

Exercise 2

1.Let $g(x)=5x-3$ and $f(x)=x^3+1$ for $x\in\Re$.Find the composite function $h=g\ of$.

2.Let $f:\Re\to\Re$ be defined by $f(x)=x^3-x$ .Show that f is not injective.

Limits

The concept of limits of a function is one of the fundamental ideas that distinguishes Calculus from other areas of mathematics e.g. Algebra or Geometry.

If *f(x)* becomes arbitrarily close to a single number *L* as *x* approaches *a* from either side, then the limit of *f(x)* as *x* approaches *a* is *L* written as $\lim_{x\to a} f(x)=L$.

Consider a function *y=f(x)*

$\lim_{x \to a} f(x) = L$ means the limit of *f(x)* as *x* approaches *a* is equal to a number *L* i.e. as *x* gets closer and closer to *a* ($x \neq a$), *f(x)* gets closer and closer to *L*.

Example 21: Let $f(x) = x^2$. Find $\lim_{x \to 2} f(x)$

Solution:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 2^2 = 4$$

Example 22: Let *f(x)* = *5x – 3*. Find $\lim_{x \to 2} 5x - 3$

Solution:

$$\lim_{x \to 2} 5x - 3 = (5 \times 2 - 3) = 7$$

Example 23: Let

$$f(x) = \frac{1}{x}.$$

Find

$$\lim_{x \to \infty} \frac{1}{x}$$

Solution:

$$\lim_{x \to 0} \frac{1}{x} = \infty \quad (undefined)$$

Properties of limits

1. $\lim_{x \to a} k = k$
2. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
3. $\lim_{x \to a} f(x) \times g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad provided\,that \lim g(x) \neq 0$

5. $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

$$e.g \quad \lim_{x \to a} x^{\frac{1}{2}} = \left(\lim_{x \to a} x\right)^{\frac{1}{2}}$$

Example 24:

$$\lim_{x\to5} x^2-4x+3 = \lim_{x\to5} x^2 - \lim_{x\to5} 4x + \lim_{x\to5} 3$$

$$= 5^2 - 4\times5+3$$
$$= 25-20+3$$
$$= 8$$

Example 25:

$$\lim_{x\to2}\frac{3x+5}{5x+7} = \frac{\lim_{x\to2} 3x+5}{\lim_{x\to2} 5x+7} = \frac{3\times2+5}{5\times2+7} = \frac{11}{17}$$

Example 26:

$$\lim_{x\to2}\frac{x^2-4}{x-2} \neq \frac{\lim_{x\to2} x^2-4}{\lim_{x\to2} x-2} \qquad \text{since } \lim_{x\to2} x-2=0$$

Hence $\displaystyle\lim_{x\to2}\frac{x^2-4}{x-2} = \lim_{x\to2}\frac{(x+2)(x-2)}{(x-2)} = \lim_{x\to2}(x+2) = 4$

Example 27:

$$\lim_{x\to8}\frac{x^{\frac{2}{3}}+3\sqrt{x}}{4-\frac{16}{x}} = \frac{\lim_{x\to8} x^{\frac{2}{3}} + \lim_{x\to8} 3\sqrt{x}}{\lim_{x\to8} 4 - \lim_{x\to8}\frac{16}{x}}$$

$$= \frac{8^{\frac{2}{3}}+3\sqrt{8}}{4-\frac{16}{8}}$$

$$= \frac{4+6\sqrt{2}}{2}$$

$$= 2+3\sqrt{2}$$

Example 28:

$$\lim_{x\to\infty}\frac{3x+5}{6x-8} = \lim_{x\to\infty}\frac{\frac{3x}{x}+\frac{5}{x}}{6-\frac{8}{x}} = \lim_{x\to\infty}\frac{3+\frac{5}{x}}{6-\frac{8}{x}}$$

$$\frac{\lim_{x\to\infty} 3 + \lim_{x\to\infty}\frac{5}{x}}{\lim_{x\to\infty} 6 - \lim_{x\to\infty} 8} = \frac{3+0}{6-0} = \frac{1}{2}$$

Example 29: $\displaystyle\lim_{x\to\infty}\frac{4x^2-x}{2x^3-5}$ Divide by the highest power of *x*.

$$\lim_{x \to \infty} \left(\frac{\frac{4}{x} - \frac{1}{x^2}}{2 - \frac{5}{x^3}} \right) = \frac{0-0}{2-0} = \frac{0}{2} = 0$$

Example 30:

$$\lim_{x \to \infty} \frac{\sqrt{x^2+2}}{3x-6} = \frac{\lim_{x \to \infty} \sqrt{x^2\left(1+\frac{2}{x^2}\right)}}{3x-6}$$

$$= \lim_{x \to \infty} \frac{x\left(1+\frac{2}{x^2}\right)^{\frac{1}{2}}}{3x-6}$$

$$= \lim \frac{x \sqrt{\left(1+\frac{2}{x^2}\right)}}{x\left(3-\frac{6}{x}\right)}$$

$$= \frac{\lim_{x \to \infty} \sqrt{1+\frac{2}{x^2}}}{\lim_{x \to \infty} \left(3-\frac{6}{x}\right)}$$

$$= \frac{1}{3}$$

Example 31:

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$$

$$\lim_{x \to 1} \frac{(x^2+x+1)(x-1)}{(x-1)} = \lim_{x \to 1} x^2+x+1$$

$$= 1+1+1$$

$$= 3$$

Example 32:

$$\lim_{x \to 2}\left(\frac{x^3-8}{x-2}\right)=\frac{0}{0}$$

$$\lim_{x \to 2}\frac{(x-2)(x^2+2x+4)}{x-2}$$

$$\lim_{x \to 2} x^2+2x+4=4+4+4=12$$

Exercise 3

1. $\lim_{x \to \infty}\frac{5x+1}{10+2x}$
2. $\lim_{x \to 5}\frac{x^2-4x-5}{x-5}$
3. $\lim_{x \to 5}\frac{x^2-25}{x-5}$
4. $\lim_{x \to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

5.The domain of the functions $f(x)=\frac{x}{5}$ and $g(x)=7-x$ is $\Re$.Write down as simply as possible.

a. $f^{-1}(x)$ b. $g^{-1}(x)$ c. $fg(x)$ d. $(fg)^{-1}(x)$

Solutions to Exercise 3

1. $\lim_{x \to \infty}\frac{5x+1}{10+2x}=\lim_{x \to \infty}\frac{5+\frac{1}{x}}{\frac{10}{x}+2}=2\frac{1}{2}$

2. $\lim_{x \to 5}\frac{x^2-4x-5}{x-5}=\lim_{x \to 5}\frac{(x-5)(x+1)}{(x-5)}$

$$=\lim_{x \to 5} x+1=6$$

Or $\lim_{x \to 5}\frac{2x-4}{1}=2(5)-4=6$

3. $\lim_{x \to 5}\frac{\times 2-25}{x-5}=\lim_{x \to 5}\frac{(x+5)(x-5)}{(x-5)}=10$

Or $\lim\limits_{x \to 5} \frac{2x}{1} = 2(5) = 10$

4. $\lim\limits_{x \to 0} \frac{\sqrt{2-x}-\sqrt{2}}{x} \times \frac{\sqrt{2-x}+\sqrt{2}}{\sqrt{2}-x+\sqrt{2}}$

$= \frac{2-x-2}{x\left(\sqrt{2}+\sqrt{2-x}\right)}$

$= \frac{-x}{x\left(\sqrt{2}+\sqrt{2-x}\right)} = \frac{-1}{\sqrt{2}+\sqrt{2-x}}$

$\lim\limits_{x \to 0} \frac{-1}{\sqrt{2}+\sqrt{2-x}} = \frac{-1}{\sqrt{2}+\sqrt{2}} = \frac{-1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{2\sqrt{2}} = \frac{2\sqrt{2}}{4 \times 2} = \frac{\sqrt{2}}{4}$

$L^{'}$ Hospital Rule

$\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \; or \; \infty$

Then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f^{'}(x)}{g^{'}(x)}$

e.g

1. $\lim\limits_{x \to 1} \frac{x^3-1}{x-1} = \lim\limits_{x \to 1} \frac{3x^2}{1}$

$= 3 \times 1$
$= 3$

2. $\lim\limits_{x \to 2} \frac{x^3-8}{x-2} = \frac{0}{0} \lim\limits_{x \to 2} \frac{3x^2}{1} = 12$

3. $\lim\limits_{x \to 0} \frac{\cos x - 2x - 1}{3x} = \lim\limits_{x \to 0} \frac{-\sin x}{3} = \frac{-2}{3}$

Continuity

Continuity at a point.

A function is considered continous if the following conditions are met.

1. $f(a)$ is defined.
2. $\lim\limits_{x \to a} f(x)$ exists.
3. $\lim\limits_{x \to a} f(x) = f(a)$

Otherwise it is discontinuous.

Example 33: Show that $f\left(x\right)=\dfrac{x^2-4}{x-2}$ is not continous at x=2

Solution:

Condition 1: $f\left(2\right)=\dfrac{4-4}{2-2}=\dfrac{0}{0}$, which is undefined

Condition 2: $\lim\limits_{x\to 2} f\left(x\right)=\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$

$$=\lim_{x\to 2}\frac{\left(x+2\right)\left(x-2\right)}{x-2}$$

$$=\lim_{x\to 2} x+2=4$$

Therefore, $\lim\limits_{x\to 2} f\left(x\right)$ exists.

Condition 3: $\lim\limits_{x\to 2} f\left(x\right)=4$, but $f(2)$ is undefined

$$\therefore \lim_{x\to 2} f\left(x\right)\neq f\left(2\right)$$

Therefore f(x) is not continous at x=2

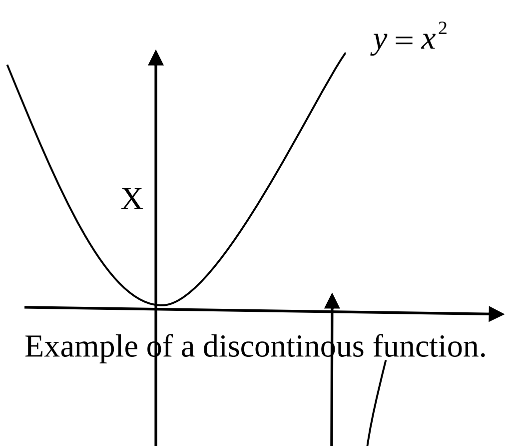
Note: It is possible to redefine f(x) to make it continous at x=2, as follows:

$$f\left(x\right)=\begin{cases}\dfrac{x^2-4}{x-2}, x\neq 2\\ \\ 4, \quad x=2\end{cases}$$

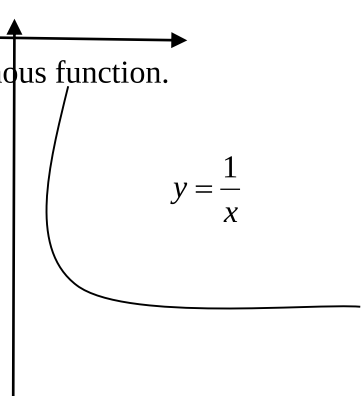
$\lim\limits_{x\to 2} f(x)=4$, i.e. $\lim\limits_{x\to 2}$ *exists*, we redefine f(x) so that

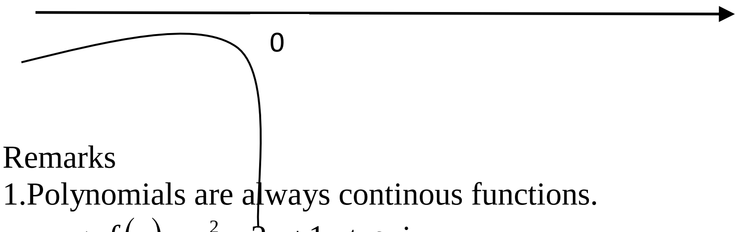
$$\lim_{x\to 2} f\left(x\right)=f\left(2\right)=4$$

Example of a continous function.



Example of a discontinous function.





Remarks

1.Polynomials are always continous functions.

e.g $f(x)=x^2-2x+1$ at c since

Condition 1: $f(c)$ is defined i.e. $f(c)=c^2-2c+1$

Condition 2: $\lim_{x\to c} f(x) = \lim_{x\to c} x^2-2x+1 = c^2-2x+1$ exists.

Condition3: $\lim_{x\to c} f(x) = c^2-2c+1 = f(c)$

2 .Discontinuity means a function breaks at a particular point.

Example 34: Discuss the continuity of $f(x)$ if

$$f(x)=\begin{cases}\dfrac{x^3+27}{x+3}; & x\neq -3\\ 27; & x=-3\end{cases}$$

Solution:Condition 1: f(-3)=27, therefore f(x) is defined at x=3

Condition 2: $\lim_{x\to -3}\dfrac{x^3+27}{x+3} = \lim_{x\to -3}\dfrac{(x+3)(x^2-3x+9)}{(x+3)}$

$= \lim_{x\to -3} x^2-3x+9$

$=9+9+9$

$=27$

Condition 3: $\lim_{x\to -3} f(x) = f(-3)=27$

$\therefore f(x)$ is continous.

Example 35:Determine whether or not the function below is continous at x=1

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & if\ x\neq 1\\ 2 & if\ x=1\end{cases}$$

Solution:

Condition 1: $f(1)=2$ hence $f(1)$ is defined.

Condition 2: $\lim_{x\to 1} f(x) = \lim_{x\to 1}\dfrac{x^2-1}{x-1} = \lim_{x\to 1}\dfrac{(x+1)(x-1)}{(x-1)} = 2$ Therefore $\lim_{x\to 1} f(x)$ exists.

Condition3: $\lim f(x) \lim_{x \to 1} \frac{x^2-1}{x-1} = f(1)$, hence $f(x)$ is continous at x=1

Example 36: .Discuss the continuity of $f(x)$ if

$$f(x)=\begin{cases} \frac{x^2-4}{x-2}, x \neq 2 \\ 3 \quad x=2 \end{cases}$$

Solution:

Condition 1: f(2) = 3, so f(x) is defined at x=2

Condition 2:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2}$$
$$= \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} \quad \text{hence } \lim f(x) \text{ exists.}$$
$$= 2$$

$\therefore f(x)$ Condition 3: $f(2)=3$ but $\lim_{x \to 2} f(x) = 2 \therefore \lim_{x \to 2} f(x) \neq f(2)$ Thus *f(x)* is discontinuous at $x=2$.

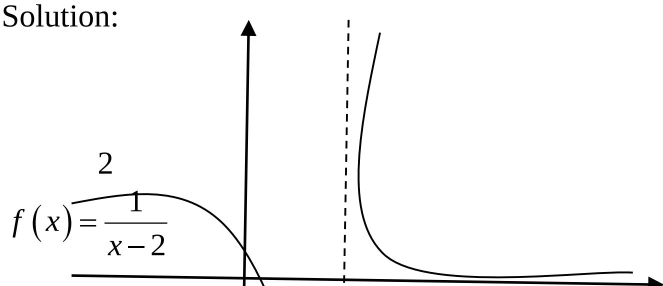
Exercise

Define the continuity of a real valued function $f(x)$ at a point x=a. Hence determine if the following function is continous at *x=1*.

$$f(x)=\begin{cases} \frac{x^3-1}{x-1}, x \neq 1 \\ 3, \quad x=1 \end{cases}$$

Example37:Show that $f(x)=\frac{1}{x-2}$ for is $x \neq 2$ is not continous at *x =2*.

Solution:



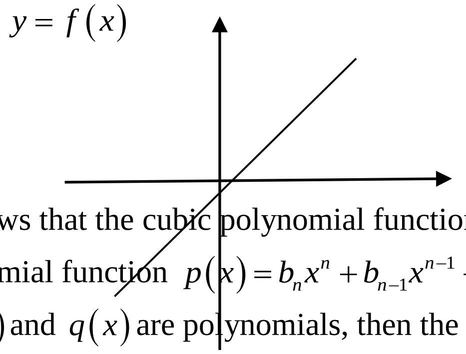
Because f is not defined at the point $x=2$ it is not continous there. Moreover f has what might be called an infinite discontinuity at $x=2$

<u>Combinations of continous Functions.</u>

Any sum or product of continous functions is continous. That is, if the functions f and g are continous at $x=a$, then so are $f+g$ and $f \cdot g$ e.g if f and g are continous at $x=a$, then

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)=f(a)+g(a)$$

Example 38: $f(x) = x$ is continous everywhere,i.e.



It follows that the cubic polynomial function $f(x) = x^3 - 3x^2 + 1$ is continous everywhere. More generally every polynomial function $p(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$ is continous at each point of the real line.

If $p(x)$ and $q(x)$ are polynomials, then the quotient law for limits and the continuity of polynomials imply that $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)} = \dfrac{\lim\limits_{x \to a} p(x)}{\lim\limits_{x \to a} q(x)} = \dfrac{p(a)}{q(a)}$ provided $q(a) \neq 0$. Thus every rational function $f(x) = \dfrac{p(x)}{q(x)}$ is continous wherever it is defined.

The point $x = a$ where the function f is discontinuous is called a removable discontinuity of f provided that there exists a function F such that $F(x) = f(x)$ for all $x \neq a$ in the domain of f, and this new function F is continous at $x = a$.

Example 39: Suppose that $f(x) = \dfrac{x-2}{x^2 - 3x + 2}$

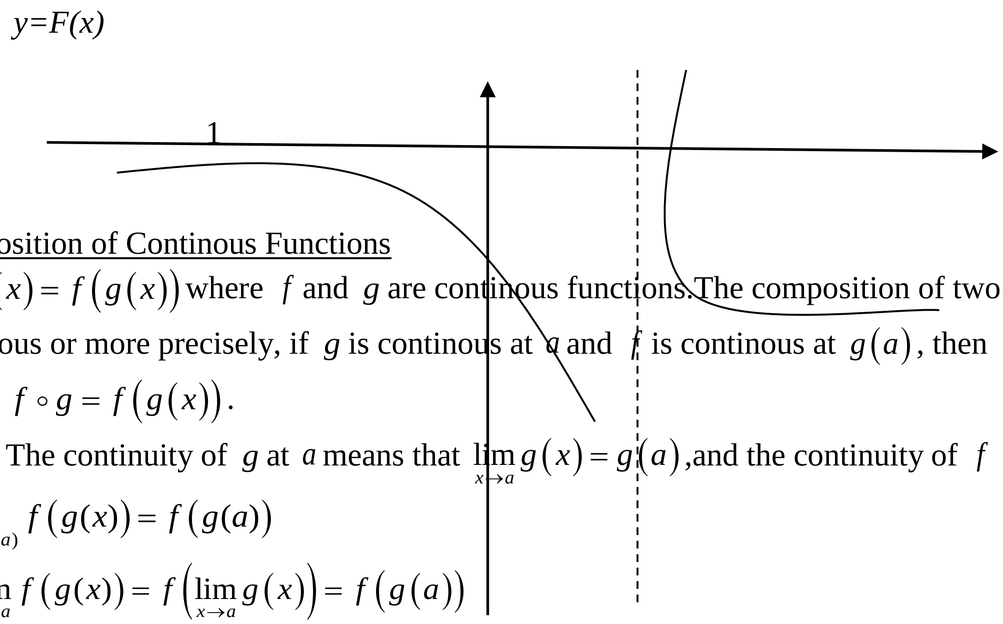
$x^2 - 3x + 2 = (x-1)(x-2)$

$\therefore f(x) = \dfrac{x-2}{(x-1)(x-2)}$

This shows that f is not defined at $x = 1$ and $x = 2 \Rightarrow f(x)$ is continous except at these points.

But $f(x) = \dfrac{x-2}{(x-1)(x-2)} = \dfrac{1}{x-1}$. The new function $F(x) = \dfrac{1}{x-1}$ is continous at $x = 2$, where $F(2) = 1$.

Therefore f has a removable discontinuity at $x = 2$; the discontinuity at $x = 1$ is not removable.



<u>Composition of Continous Functions</u>

Let $h(x) = f(g(x))$ where f and g are continous functions.The composition of two continous functions is continous or more precisely, if g is continous at a and f is continous at $g(a)$, then $f \circ g$ is continous at a where $f \circ g = f(g(x))$.

<u>Proof:</u> The continuity of g at a means that $\lim\limits_{x \to a} g(x) = g(a)$,and the continuity of f at $g(a)$ implies that

$\lim\limits_{g(x) \to g(a)} f(g(x)) = f(g(a))$

i.e. $\lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right) = f(g(a))$

Example 40: Show that the function $f\left(x\right)=\left(\dfrac{x-7}{x^2+2x+2}\right)^{\frac{2}{3}}$ is continous on the whole real line.

Solution: Consider the denominator

$x^2+2x+2=\left(x+1\right)^2+1>0$ for all value of *x*. Hence the rational function

$r\left(x\right)=\dfrac{x-7}{x^2+2x+2}$ is defined and continous everywhere. Thus $f\left(x\right)=\left(\left[r\left(x\right)\right]^2\right)^{\frac{1}{3}}$ is continous everywhere.

One-sided limits

Let $S\subseteq \mathbb{R}$ and $f:S\to\mathbb{R}$ be a function. If for every $x\in S,\ f\left(x\right)\to L$ as $x\to a$ and $x>a$ always, then we say that $x\to a$ from the right and write $x\to \overset{+}{a}$ or we say $\lim\limits_{x\to a^+} f\left(x\right)=L$.

Similarly, if $f\left(x\right)\to L$ as $x\to a$ and $x<a$ always, we say that $x\to a$ from the left and write $x\to \bar{a}$ or we say $\lim\limits_{x\to a^-}=L$.

The limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f\left(x\right)$ are called one-sided limits of *f* and *a*

Remarks

1. $\lim\limits_{x\to a} f\left(x\right)=L$ iff $\lim\limits_{x\to a^+} f\left(x\right)=\lim\limits_{x\to a^-} f\left(x\right)=L$

i.e the limit of a function $f\left(x\right)$ exists if the right hand side limit = left-hand side limit.

2. If $\lim\limits_{x\to a^+}\neq\lim\limits_{x\to a^-} f\left(x\right)$, then $\lim\limits_{x\to a} f\left(x\right)$ does not exist.

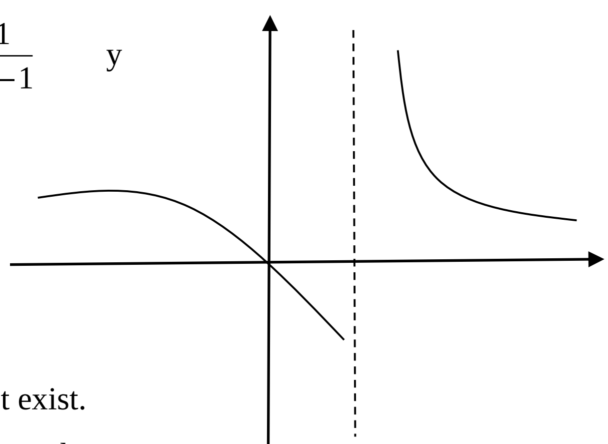
Example 41: Given $f(x)=\dfrac{x}{x-1}$,Find $\lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1^-} f(x)$

Solution:

0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
0	-1	∞	3	2

Also consider the graph of $f\left(x\right)=\dfrac{1}{x-1}$ y

0 1 x



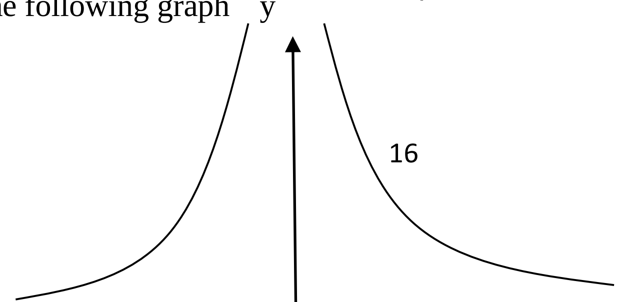
$\lim\limits_{x\to 1^-} f\left(x\right)=-\infty\ if\ x<1$

$\lim\limits_{x\to 1^+} f\left(x\right)-\infty^1\ if\ x>1$

$\therefore \lim\limits_{x\to 1} f\left(x\right)=\infty \Rightarrow \lim\limits_{x\to 1} f\left(x\right)$ does not exist.

Example 42: Consider the following graph y

$y=f\left(x\right)=\dfrac{1}{x^2}$



$y = \frac{1}{x^2}$

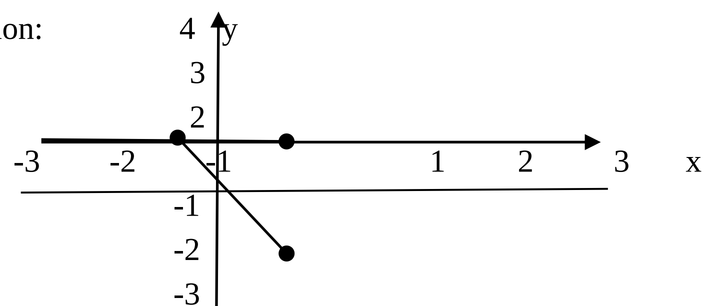
x

$\lim_{x\to 0^+} f(x) = \infty \qquad \lim_{x\to 0^+} f(x) = \infty$

Example 43: Draw the graph of

$$f(x) = \begin{cases} 1, & if\ x = 1 \\ -x, & if\ -1 < x < 1 \\ -1, & if\ x > 1 \end{cases}$$

Solution:



Example 44: Evaluate $\lim_{x\to 2^+}\left(1+\sqrt{x-2}\right)$ and $\lim_{x\to 2^-}\left(1+\sqrt{x-2}\right)$

Solution:

$\lim_{x\to 2^+}\left(1+\sqrt{x-2}\right)$

$= \lim_{x\to 2^+} 1 + \lim_{x\to 2^+}\sqrt{x-2}$

$= 1 + 0 = 1$

On the other hand, $\lim_{x\to 2^-}\left(1+\sqrt{x-2}\right)$ does not exist (is not real).

Definition: A function f is said to be continous from the right at $x = p$ if $\lim_{x\to p^+} f(x) = f(p)$.

We say that f is continous from the left at q if $\lim_{x\to p^-} f(x) = f(q)$

A function is said to be continous if its continous from the right and from the left i.e

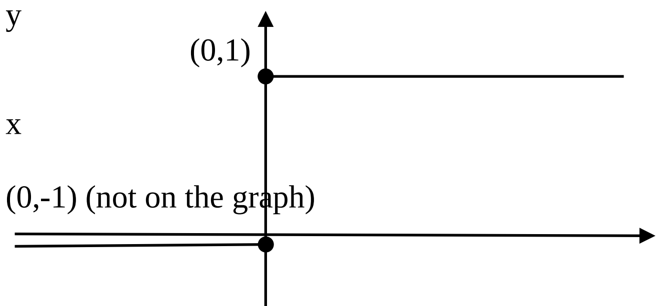
$\lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x) = f(p)$

Example 45: Discuss the continuity of $g(x) = \sin x = \begin{cases} +1 & if\ x \ge 0 \\ -1 & if\ x < 0 \end{cases}$

Solution:

$\lim_{x\to 0^-} g(x) = -1$ and $\lim_{x\to 0^+} g(x) = -1$ .Therefore Its left-hand and right-hand limits at $x = 0$ are unequal

Thus $g(x)$ has no limit as $x \to 0$. Hence the function g is not continous at $x=0$, it has what might be called a finite jump discontinuity there. (see the graph below)

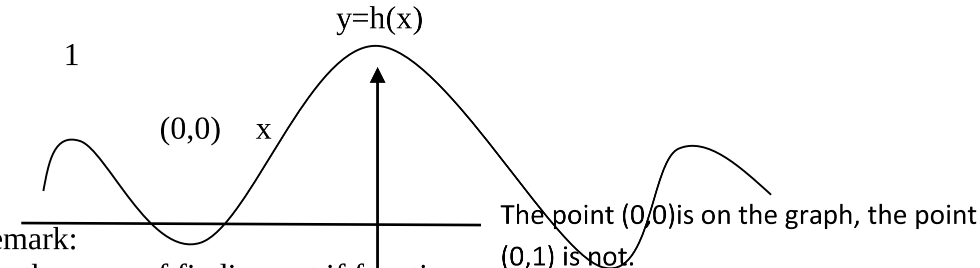


Example 46: Discuss the continuity of $h(x)=\begin{cases} \dfrac{\sin x}{x} & if\ \ x \neq 0 \\ 0 & if\ \ x=0 \end{cases}$

Solution:

$\lim\limits_{x\to 0} h(x)=\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ whereas $h(0)=0$

$\Rightarrow$ the limit and the value of h at $x=0$ are not equal.
Thus the function h is not continous there (see the graph below)



Remark:
Another way of finding out if functions

1. Checking if $f(a)$ is defined.
2. Checking if $\lim\limits_{x\to a^+} f(x)=\lim\limits_{x\to a^-} f(x)$ and exist and are equal.
3. Ensuring that both are equal to $f(a)$.

Example 47: Find the value of c such that $f(x)=\begin{cases} x+c & if\ \ x<0 \\ 4-x^2 & if\ \ x\geq 0 \end{cases}$ is continous at $x = 0$.

Solution:
Condition 1: Is $f(x)$ defined at $x=0$?

Yes, $f(x)=4-x^2$
$\therefore f(0)=4-0^2=4$

Condition 2: Does $\lim\limits_{x\to 0} f(x)$ exist? In other words,

Does $\lim\limits_{x\to 0^-} f(x)$ exist?

Yes, $\lim\limits_{x \to 0^-} f(x) = 0 + c = c$

Does $\lim\limits_{x \to 0^+}$ exist?

Yes, $\lim\limits_{x \to 0^+} f(x) = 4 - x^2 = 4 - 0^2 = 4$

(c) Is $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$.

For them to be equal, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) \quad \Rightarrow c = 4$

Thus, $\lim\limits_{x \to 0} f(x)$ exists, i.e. $\lim\limits_{x \to 0} f(x) = 4$

Condition 3: Is $\lim\limits_{x \to 0} f(x) = f(0)$

Yes, $\lim\limits_{x \to 0} f(x) = f(0) = 4$

Conclusion: for *f(x)* to be continous at $x = 0$, then $c = 4$.

Exercise: Evaluate

1. $\lim\limits_{x \to 3} \dfrac{x^2 - 2x}{x+1}$ 2. $\lim\limits_{x \to -\infty} \dfrac{x-2}{x^2+2x+1}$ 3. $\lim\limits_{x \to 6^+} \dfrac{y+6}{y^2-36}$ 4. $\lim\limits_{x \to +\infty} \dfrac{2-y}{\sqrt{7+6y^2}}$ 5. $\lim\limits_{x \to 3^+} \dfrac{x}{x-3}$ 6. $\lim\limits_{x \to 2^-} \dfrac{x}{x^2-4}$

7. $\lim\limits_{y \to 4} \dfrac{4-y}{2-\sqrt{y}}$ 8. $\lim\limits_{x \to \infty} \sqrt{\dfrac{5x^2-2}{x+3}}$ 9. $\lim\limits_{x \to \pi} \sin\left(\dfrac{x^2}{\pi + x}\right)$

10. For the following problems find the points where given function is not defined and therefore not continous. For each such point *a*, tell whether this discontinuity is removable.

a) $f(x) = \dfrac{x}{(x+3)^3}$

b) $f(x) = \dfrac{x-2}{x^2-4}$

c) $f(x) = \dfrac{1}{1-|x|}$ d) $f(x) = \dfrac{x-17}{|x-17|}$

e) $f(x) = \begin{cases} -x & if\ x < 0 \\ x^2 & if\ x > 0 \end{cases}$ f) $f(x) = \begin{cases} 1+x^2 & if\ x < 0 \\ \dfrac{\sin x}{x} & if\ x > 0 \end{cases}$

11. For the following problems find a value of the constant *c* so that the function $f(x)$ is continous for all *x*.

a) $f(x) = \begin{cases} x+c & if\ x < 0 \\ 4-x^2 & if\ x \ge 0 \end{cases}$ Answer: c=4 b) $f(x) = \begin{cases} 2x+c & if\ x \le 3 \\ 2c-x & if\ x > 3 \end{cases}$ Answer: c=9

c) $f(x) = \begin{cases} c^2-x^2 & if\ x < 0 \\ 2(x-c)^2 & if\ x \ge 0 \end{cases}$ Answer: c=0 d) $f(x) = \begin{cases} c^3-x^3 & if\ x \le \pi \\ c\sin x & if\ x > \pi \end{cases}$ Answer: $c = \pi$

THE DERIVATIVE OF FUNCTIONS

Definition: The derivative of the function f is the function f' defined by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$$ for all x for which this limit exists.

The function f is differentiable at $x=a$ if $\lim_{x\to a} f(x)=f(a)$ exists.

The process of finding the derivative f' is called differentiation of f.

Solution:

Example 48: Apply the definition of the derivative directly to differentiate the function $f(x)=\frac{x}{x+3}$.

Solution:

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{\frac{x+h}{x+h+3}-\frac{x}{x+3}}{h}\\
&=\lim_{h\to 0}\frac{(x+h)(x+3)-x(x+h+3)}{h(x+h+3)(x+3)}\\
&=\lim_{h\to 0}\frac{x^2+3x+hx+3h-x^2-hx-3x}{h(x+h+3)(x+3)}\\
&=\lim_{h\to 0}\frac{3h}{h(x+h+3)(x+3)}\\
&=\lim_{h\to 0}\frac{3}{(x+h+3)(x+3)}\\
&=\frac{3}{(x+3)(x+3)}=\frac{3}{(x+3)^2}
\end{aligned}$$

This process is known as differentiation from first principles.

Differentiation of Quadratic Functions

Example 49: Let $f(x)=ax^2+bx+c$, where a,b and c are constants. Show from first principles that $f'(x)=2ax+b$

Solution:

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{\left[a(x+h)^2+b(x+h)+c\right]-\left[ax^2+bx+c\right]}{h}\\
&=\lim_{h\to 0}\frac{\left(ax^2+2ahx+ah^2+bx+bh+c-ax^2-bx-c\right)}{h}\\
&=\lim_{h\to 0}\frac{2ahx+ah^2+bh}{h}\\
&=\lim_{h\to 0}(2ax+ah+b)\\
&=2ax+b
\end{aligned}$$

Example 50: Show from first principles that If $f(x)=3x^2-7x+7$, then $f'(x)=6x-7$

Differential Notation

$$\Delta x=h\ ;\Delta y=f(x+\Delta x)-f(x);\ \frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

If $y=f(x)$, we often write $\frac{dy}{dx}=f'(x)$ e.g. If $y=ax^2+bx+c$,then $\frac{dy}{dx}=f'(x)=2ax+b$

Examples: Find the derivatives of the following functions from first principles.

Example 51: $f(x)=x^2$

Solution:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{2hx+h^2}{h}$$

$$=\lim_{h\to 0}2x+h\ =2x$$

Example 52: $f(x)=\frac{1}{x}$

Solution: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{x-x-h}{h(x+h)x}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h)x}=\frac{-1}{x^2}$$

Example 53: $f(x)=\sqrt{x}$

Solution: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Exercise: Differentiate the following functions from first principles.

1) $y = x^3$ 2) $f(x) = x^2 + 3x - 2$ 3) $f(t) = kt$

Basic Differentiation Rules

The derivative of a constant

If $f(x) = c$ (a constant) for all *x*, then $f'(x) = 0$ for all *x*. That is $\frac{dc}{dx} = f'(x) = 0$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

The Power Rule

If *n* is a positive integer and $f(x) = x^n$, then $f'(x) = nx^{n-1}$

Proof: $f'(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

But $(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n$

$$\therefore f'(x) = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \ldots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \frac{n(n-1)}{21}x^{n-2}h + \ldots + h^{n-1} \quad = nx^{n-1}$$

Example 54: Find (a) $f'(x)$ if $f(x) = 6x^5$ (b) Find $\frac{dy}{dt}$ if $y = t^{17}$

Solution:

(a) $f(x) = 6x^5 \Rightarrow f'(x) = 30x^4$ (b) $y = t^{17} \Rightarrow \frac{dy}{dt} = 17t^{16}$

The derivative of a linear combination

If f and g are differentiable functions and a and b are fixed real numbers, then

$$\frac{d}{dx}\left[af(x) + bg(x)\right] = af'(x) + bg'(x)$$

Proof: Let $k(x) = af(x) + bg(x)$

$$\therefore k'(x)=\lim_{h\to 0}\frac{k(x+h)-k(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left[af(x+h)+bg(x+h)\right]-\left[af(x)+bg(x)\right]}{h}$$
$$=a\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]+b\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$
$$=af'(x)+bg'(x)$$

Example 55: Let $y=36+24x+8x^5-6x^{10}$. Find $\frac{dy}{dx}$.

Solution: $y=36+24x+8x^5-6x^{10}\Rightarrow\frac{dy}{dx}=k'(x)=24+40x^4-60x^5$

The derivative of a Polynomial

Let $y=f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_2$

$$f'(x)=\frac{dy}{dx}=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\ldots+3a_3x^2+2a_2x^2+a_1$$

if $y=f(x)=7x^3-6x^2+4x+2$ then $\frac{dy}{dx}=f'(x)=21x^2-12x+4$

5. The Product Rule and Quotient Rule

(a)The Product Rule

If f and g are differentiable at x ,then fg is differentiated at x ,then $\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$

Proof: Let $k(x)=f(x)g(x)$

$$\therefore k'(x)=\lim_{h\to 0}\frac{k(x+h)-k(x)}{h}$$
$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

Add and subtract at $f(x)g(x+h)$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$
$$k'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}$$
$$=\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\left(\lim_{h\to 0}g(h+x)\right)+f(x)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$$
$$=f'(x)g(x)+f(x)g'(x)$$

The product rule says that the derivative of the product of two functions is formed by multiplying the derivative of each by the other and then adding the results.

Example 56: Find the derivative of $f(x)=(1-6x^3)(4x^2-6x+2)$

Solution

$$\begin{aligned} f'(x) &= \left(-18x^2\right)\left(4x^2-6x+2\right)+\left(1-6x^3\right)(8x-12) \\ &= -72x^4+108x^3-36x^2+8x-12-48x^4+36x \\ &= 120x^4+144x^3-36x^2+8x-12 \end{aligned}$$

Now, suppose $k(x) = f_1(x)\, f_2(x) \ldots f_n(x)$

$$\begin{aligned} k'(x) = {} & f_1'(x)\, f_2(x) \ldots f_n(x) \\ & + f_1(x)\, f_2'(x) \ldots f_n(x) \\ & \vdots \\ & + f_1(x)\, f_2(x) \ldots f'(x) \end{aligned}$$

Example 57: Let $k(x) = (x-2)\left(x^2+6\right)\left(x^4+1\right)$. Find $k'(x)$.

Solution:

$$\begin{aligned} \therefore k'(x) &= (1)\left(x^2+6\right)\left(x^4+1\right)+(x-2)(2x)\left(x^4+1\right)+(x-2)\left(x^2+6\right)(4x) \\ &= x^6+x^2+6x^4+6+2x\left(x^5+x-2x^4-2\right)+4x\left(x^3+6x-2x^2-12\right) \\ &= x^6+x^2+6x^4+6+2x^6+2x^2-4x^5-4x+4x^4+6x^2-8x^3-48x \\ &= 3x^6-4x^5+10x^4-8x^3+9x^2-52x+6 \end{aligned}$$

(b)The Reciprocal Rule

If f is differentiable at x and $f(x) \neq 0$, then $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = -\dfrac{f'(x)}{\left[f(x)\right]^2}$

Proof: Let $k(x) = \dfrac{1}{f(x)} \Rightarrow k'(x) = \lim\limits_{h \to 0} \dfrac{k(x+h)-k(x)}{h}$

$$\begin{aligned} &= \lim_{h \to 0} \frac{\dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}}{h} \\ &= \lim_{h \to 0} \frac{f(x) - f(x+h)}{hf(x+h)\, f(x)} \\ &= -\left(\lim_{h \to 0} \frac{1}{f(x+h)\, f(x)}\right)\left(\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}\right) \\ &= -\frac{f'(x)}{\left[f(x)\right]^2} \end{aligned}$$

Example 58: Find $k'(x)$ if $k(x) = \dfrac{1}{x^2+1}$

Solution: $k'(x) = \dfrac{-\dfrac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} = \dfrac{-2x}{\left(x^2+1\right)^2}$

(c)Power rule for a negative integer n

If n is a negative integer, then $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$

Proof: Let $m = -n$, so that m is a positive integer. Then

$$\frac{d}{dx}\left(x^{n}\right)=\frac{d}{dx}\left(x^{-m}\right)=\frac{d}{dx}\left(\frac{1}{x^{m}}\right)=\frac{\frac{d}{dx}\left(x^{m}\right)}{\left(x^{m}\right)^{2}} \quad =-\frac{mx^{m-1}}{x^{2m}}=\left(-m\right)x^{-m-1}=nx^{n-1}$$

Example 59: Find $f^{'}(x)$ if $f(x)=\dfrac{5x^{4}-6x+7}{2x^{2}}$

Solution:

$$f(x)=\frac{5x^{4}-6x+7}{2x^{2}}$$

$$=\frac{5}{2}x^{2}-\frac{3}{x}+\frac{7}{2x^{2}}$$

: $$=\frac{5}{2}x^{2}-3x^{-1}+\frac{7}{2}x^{-2}$$

$$\therefore f^{'}(x)=\frac{5}{2}(2x)-3\left(-x^{-2}\right)+\frac{7}{2}\left(-2x^{-3}\right)$$

$$=5x+\frac{3}{x^{2}}-\frac{7}{x^{3}}$$

The Quotient Rule

If f and g are differentiable at x and $g(x)\neq 0$ then $\dfrac{f}{g}$ is differentiable x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f^{'}(x)g(x)-f(x)g^{'}(x)}{\left[g(x)\right]^{2}}$$

Proof: Let $k(x)=\dfrac{f(x)}{g(x)}$

$$k^{'}(x)=\lim_{h\to 0}\frac{k(x+h)-k(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x)-g(x+h)f(x)}{hg(x+h)g(x)}$$

Add and subtract $f(x)g(x)$

$$\lim_{h\to 0}\frac{g(x)f(x+h)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h\big(g(x)g(x+h)\big)}$$

$$=\lim_{h\to 0}\frac{g(x)f(x+h)-f(x)g(x)}{h\big(g(x)g(x+h)\big)}+\lim_{h\to 0}\frac{f(x)g(x)-f(x)g(x+h)}{h\big(g(x)g(x+h)\big)}$$

$$=\lim_{h\to 0}g(x)\frac{\left[f(x+h)-f(x)\right]}{hg(x)g(x+h)}+\lim_{h\to 0}f(x)\frac{\left[g(x)-g(x+h)\right]}{h\big(g(x)g(x+h)\big)}$$

$$=\frac{g(x)\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}}{\lim\limits_{h\to 0}g(x)g(x+h)}-\frac{f(x)\lim\limits_{h\to 0}\dfrac{\left[g(x+h)-g(x)\right]}{h}}{\lim\limits_{h\to 0}g(x)g(x+h)}$$

$$=\frac{g(x)f^{'}(x)}{(g(x)]^{2}}-\frac{f(x)g^{'}(x)}{(g(x)]^{2}}$$

$$=\frac{g(x)f^{'}(x)-f(x)g^{'}(x)}{(g(x)]^{2}}$$

Slope of a tangent

Let M be a slope of a tangent line at point P. Then $M=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

Example 60: .If $f(x)=x^2$, find the slope of tangent line at at the point $P\left(a,a^2\right)$.

Solution: $f(a+h)=(a+h)^2$

$$M=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$=\lim_{h\to 0}\frac{(a+h)^2-a^2}{h}=\lim_{h\to 0}\frac{a^2+2ah+h^2-a^2}{h}=\lim_{h\to 0}2a+h=2a$$

Example 61: Find the slope and the equation of the tangent line to a graph of $f(x)=x^3$ at the point $P(3,27)$.

Solution:

$$M=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$=\lim_{h\to 0}\frac{(a+h)^3-a^3}{h}=\lim_{h\to 0}\frac{a^3+3a^2h+3ah^2+h^3-a^3}{h}$$

$$=\lim_{h\to 0}3a^2+3ah+h^2$$

$$=3a^2$$

But $a=3$

Let $a=3\times 3^2=27$; $M=\dfrac{y-y_0}{x-x_0}$; $27=\dfrac{y-27}{x-3}$; $y-27=27x-81$; $y=27x-54$

In general, consider $y = f(x)$,the slope of the tangent line at any arbitrary point $P(x,y)$on the curve is given by $m = f^{'}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$,where $f^{'}(x)$is a derived function of $f(x)$. $f^{'}(x)$is read as "*f* prime of *x*"

Example 62: Let $f(x) = x^2 + 1$,find $f(x)$.Use this result to find the slope of the tangent line $y = x^2 + 1$at point $x = 2, x = 0$ and at $x = -2$.

Solution:

$$f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^2 + 1 - x^2 - 1}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h\to 0} 2x + h = 2x$$

Therefore $f(x) = 2x$

When $x = 2$, $f^{'}(2) = 4$; When $x = 0$, $f^{'}(0) = 0$; when $x = -2$, $f^{'}(-2) = -4$

Definition: The function $f^{'}(x)$defined by the format $f^{'}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is called the derivative of $f(x)$ with respect to *x*.

The derivative can also be defined in various other equivalent ways e.g

$$f^{'}(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

Note:

1 .A function is said to be differentiable at a point $x = x_0$ if it has a derivative at this point, i.e $f(x_0)$exists. If $f(x)$is differentiable at $x = x_0$ it must be continous there.

2. If $y = f(x)$, $f'(x) = \frac{d\,f(x)}{dx} = \frac{dy}{dx}$ =derivative of *y* with respect to *x*.

   y-dependent variable
   x-independent variable

3. The process of finding a derivative is called differentiation.
4. If you ever required to differentiate a given function from first principles, you should always start the proof by quoting the formula below

$$f^{'}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Examples

1.Find the derivative of $f(x) = 3x^2 - 5x + 4$ from first principles

$$f^{'}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h)^2 - 5(x+h) + 4 - 3x^2 + 5x - 4}{h}$$
$$= \lim_{h\to 0} \frac{3\left[x^2 + 2xh + h^2\right] - 5x - 5h + 4 - 3x^2 + 5x - 4}{h}$$
$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 4 - 3x^2 + 5x - 4}{h}$$
$$= \lim_{h\to 0} 6x + 3h - 5$$
$$= 6x - 5$$

Differentiate $f(x) = \dfrac{x}{x-9}$ from 1^{st} principles

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-9} - \dfrac{x}{x-9}}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)(x-9) - x(x+h-9)}{h(x-9)(x+h-9)}$$
$$= \lim_{h\to 0} \frac{x^2 - 9x + xh - 9h - x^2 - xh + x9}{h(x-9)(x+h-9)}$$
$$= \lim_{h\to 0} \frac{x - 9 - x}{(x-9)(x+h-9)} = \frac{-9}{(x-9)^2}$$

Confirm: If $y = \dfrac{x}{x-9}$, $\quad \dfrac{dy}{dx} = \dfrac{-9}{(x-9)^2}$

Differentiate (a) $\sqrt{x+2}$ (b) $f(x) = \sqrt{x-2}$ from 1^{st} principles

Solution:

$$y = \sqrt{x+2} = f(x)$$
$$f(x') = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$\lim_{h\to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)+2-(x+2)}{h[\sqrt{(x+h)+2}+\sqrt{x+2}]}$$

$$= \lim_{h \to 0} \frac{h}{h[\sqrt{(x+h)+2}+\sqrt{x+2}]}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)+2}+\sqrt{x+2}}$$

$$\frac{1}{\sqrt{x+2}+\sqrt{x+2}}$$
$$= \frac{1}{2\sqrt{x+2}}$$

(b) $f(x) = \sqrt{x-2}$

Further examples

Differentiate the following with respect to x

$y = x^5$

Solution:

$$\frac{d\left(x^5\right)}{dx} = 5x^4 \quad \Rightarrow \frac{dy}{dx} = 5x^4$$

y $= x^3$

Solution:

$$\frac{d}{dx}\left(5x^3\right) = 5\frac{d\left(x^3\right)}{d} = 5 \times 3x^2 = 15x^2$$

3. $y = -7x^{10}$

Solution:

$$\frac{dy}{dx} = -70x^9$$

3. $y = f(x) = 24x^2$

Solution:

$$\frac{dy}{dx} = 48x .$$

4.

$y = f(x) = 8x^3 - 4x^2 + x - 5$

Solution

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}\left(8x^3 - 4x^2 + x - 5\right)$$

$$= \frac{d}{dx}\left(8x^3\right) - \frac{d}{dx}\left(4x^2\right) + \frac{d}{dx}(x) - \frac{d}{dx}(5)$$
$$= 24x^2 - 8x + 1 - 0$$
$$= 24x^2 - 8x + 1$$

5.

$f(x) = y = (x+3)^4$

Solution

$1x^4 + 4(3)x^3 + 6(3)^2 x^2 + 4(3)^3 x + 1(3)^4$

$= x^4 + 12x^3 + 54x^2 + 108x + 81$

$$\frac{dy}{dx} = 4x^3 + 36x^2 + 108x + 108$$

6\. $y = \dfrac{2}{x^3}$

Solution:

$$\frac{dy}{dx} = 2.-3(x^{-3-1}) = -6x^{-3-1} = -6x^{-4} = \frac{-6}{x^4}$$

7\. y = $\dfrac{1}{\sqrt{x}}$

Solution:

$$y = x^{-\frac{1}{2}};\ \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2}\times\frac{1}{x^{\frac{3}{2}}} = -\frac{1}{2}\frac{1}{\sqrt{x^3}}$$

8\. $y = x^{\frac{1}{2}}$

Solution: $\dfrac{dy}{dx} = \dfrac{1}{2}x^{\frac{1}{2}-1} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

9\. $(a)\ y = \dfrac{-2}{x^2} \quad (b)\dfrac{1}{3x^3}$

The Chain Rule

If $y = f(u)$ where $u = g(x)$ and $g(x)$ are differentiable functions, then the composite function defined by $y = f\left[g(x)\right]$ which has a derivative given by $\dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx}$

Examples

1\. Differentiate $(3x+2)^4$

Solution:

Let $y = (3x+2)^4$ and $u = 3x+2$;then $y = u^4$; $\dfrac{du}{dx} = 3$; $\dfrac{dy}{du} = 4u^3$;But by chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx}$

$= 4u^3 \times 3 = 12u^3$;But $u = 3x+2$; $\therefore \dfrac{dy}{dx} = 12(3x+2)^3$

2\. Differentiate $(x^2+3x)^7$

Solution:

Let $y=\left(x^2+3x\right)^7$; Let $u=x^2+3x$; $\therefore y=u^7$;

$$\frac{du}{dx}=2x+3,\quad \frac{dy}{du}=7u^6 \quad \therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=7u^6\left(2x+3\right)$$

$$=7\left(2x+3\right)\left(x^2+3x\right)^6$$

$$\therefore \frac{dy}{dx}=7\left(2x+3\right)\left(x^2+3x\right)^6$$

3.Diffrentiate $\frac{1}{1+\sqrt{x}}$

Solution:

Let $y=\left(1+\sqrt{x}\right)^{-1}$ and $u=1+\sqrt{x}$ or $1+x^{\frac{1}{2}}$

$$\therefore y=u^{-1};\quad \frac{dy}{du}=-1u^{-1-1}=-u^{-2};\quad \frac{du}{dx}=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-\frac{1}{u^2}\times\frac{1}{2\sqrt{x}}=-\frac{1}{2u^2\sqrt{x}}=-\frac{1}{2\left(1+\sqrt{x}\right)^2\sqrt{x}}$$

4.Diffrentiate $\sqrt{1+x^2}$

Solution:

Let $y=\left(1+x^2\right)^{\frac{1}{2}}$ and $u=1+x^2$;

$$\therefore y=u^{\frac{1}{2}};\quad \frac{dy}{du}=\frac{1}{2}u^{\frac{1}{2}-1}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}$$

$$\frac{du}{dx}=2x;\quad \frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2\sqrt{u}}\times 2x=\frac{x}{\sqrt{1+x^2}}$$

5. $y=\frac{1}{1+\sqrt{x}}=\left(1+\sqrt{x}\right)^{-1}$

Solution:

$$u=1+\sqrt{x}=1+x^{\frac{1}{2}};\quad \frac{du}{dx}=\frac{1}{2}x^{\frac{1}{2}}$$

$$y=u^{-1};\quad \frac{dy}{du}=-1u^{-2}=\frac{-1}{u^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=-\frac{1}{u^2}\times\frac{1}{2}x^{-\frac{1}{2}}=-\frac{1}{\left(1+\sqrt{x}\right)^2}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=-\frac{1}{2\left(1+\sqrt{x}\right)^2\sqrt{x}}$$

Exercise

Differentiate

1. $\left(3x^2+5\right)^3$ 2. $\left(3x^2-5x\right)^{-\frac{2}{3}}$ 3. $\left(6x^3-4x\right)^{-2}$ 4. $\dfrac{1}{\left(x^2-7x\right)^3}$ 5. $\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$

Mixed Examples

Differentiate the expression $y=\left(x^2-3\right)\left(x+1\right)^2$ and simplify the result.

Solution: Let $u=\left(x^2-3\right)$ and let $v=\left(x+1\right)^2$; $\dfrac{du}{dx}=2x;$ $\dfrac{dv}{dx}=2\left(x+1\right)1=2\left(x+1\right)$

$$\begin{aligned}\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}&=\left(x^2-3\right)\left(2\right)\left(x+1\right)+\left(x+1\right)^2 2x\\&=2\left(x+1\right)\left(x^2-3\right)+2x\left(x+1\right)^2\\&=2\left(x+1\right)\left[\left(x^2-3\right)+x\left(x+1\right)\right]\\&=2\left(x+1\right)\left[x^2-3+x^2+x\right]\end{aligned}$$

$$2\left(x+1\right)\left(2x+3\right)\left(x-1\right)=2\left(x+1\right)\left[2x^2+x-3\right]$$

2.Differentiate $\left(x^2+1\right)^3\left(x^3+1\right)^2$

Let $u=\left(x^2+1\right)^3$ and $v=\left(x^3+1\right)^2$

$$\frac{d}{dx}\left(uv\right)=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{dy}{dx}=2\left(3x^2\right)\left(x+1\right)=6x^2\left(x^3+1\right)$$

$$\frac{du}{dx}=3\left(2x\right)\left(x^2+1\right)^2$$

$$\begin{aligned}&=\left(x^2+1\right)^3 6x^2\left(x^3+1\right)+\left(x^3+1\right)^2 3\left(2x\right)\left(x^2+1\right)^2\\&=6x^2\left(x^2+1\right)^3\left(x^3+1\right)+6x\left(x^3+1\right)^2\left(x^2+1\right)^2\\&=6x\left(x^3+1\right)\left(x^2+1\right)^2\left[x\left(x^2+1\right)+\left(x^3+1\right)\right]\\&=6x\left(x^3+1\right)\left(x^2+1\right)^2\left[2x^3+x+1\right]\end{aligned}$$

3. Differentiate $\left(x-3\right)^2\left(x+2\right)^{-2}$

Solution:

$$y=(x-3)^2(x+2)^{-2}$$
$$u=(x-3)^2; \quad v=(x+2)^{-2}$$

$$\frac{du}{dx}=2(x-3); \quad \frac{dv}{dx}=-2(x+2)^{-3}$$

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}=(x-3)^2(-2)(x+2)^{-3}+(x+2)^{-2}(2)(x-3)$$

$$=-2(x-3)^2(x+2)^{-3}+2(x-3)(x+2)^{-2}$$
$$=\frac{-2(x-3)^2}{(x+2)^3}+\frac{2(x-3)}{(x+2)^2} \quad =\frac{-2(x-3)^2+2(x-3)(x+2)}{(x+2)^3}$$

$$\frac{2(x-3)\left[-(x-3)+x+2\right]}{(x+2)^3}=\frac{2(x-3)(5)}{(x+2)^3}=\frac{10(x-3)}{(x+2)^3}$$

4. Differentiate $y=\dfrac{x+1}{x^2-2}$

Solution:

Let $u=x+1; \quad v=x^2-2$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx}=1; \quad \frac{dv}{dx}=2x \quad \frac{dy}{dx}=\frac{(x^2-2)1-(x+1)(2x)}{(x^2-2)^2}$$
$$=\frac{(x^2-2)-(x+1)2x}{(x^2-2)^2}=\frac{x^2-2-2x^2-2x}{(x^2-2)^2}=\frac{-x^2-2x-2}{(x^2-2)^2}$$

5. Differentiate $\dfrac{x}{\sqrt{(1+x^2)}}$

Solution: Let $u=x; \quad v=(1+x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{(1+x^2)^{\frac{1}{2}}-x\left(\frac{1}{2}\right)(2x)(1+x^2)^{-\frac{1}{2}}}{\left(\sqrt{(1+x^2)}\right)^2}$$

$$\frac{(1+x^2)^{\frac{1}{2}}-x^2(1+x^2)^{-\frac{1}{2}}}{1+x^2}=\frac{(1+x^2)^{\frac{1}{2}}-\frac{x^2}{(1+x^2)^{\frac{1}{2}}}}{1+x^2}$$

$$\frac{(1+x^2)-x^2}{(1+x^2)(1+x^2)^{\frac{1}{2}}}=\frac{1}{(1+x^2)(1+x^2)^{\frac{1}{2}}}=\frac{1}{(1+x^2)^{\frac{3}{2}}}$$

6. $y=\sqrt{\dfrac{x^2-4}{x^2+4}}$

Solution:

$$y=\frac{\left(x^2-4\right)^{\frac{1}{2}}}{\left(x^2+4\right)^{\frac{1}{2}}};u=\left(x^2-4\right)^{\frac{1}{2}};v=\left(x^2+4\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{8x}{\left(x^2-4\right)^{\frac{1}{2}}\left(x^2+4\right)^{\frac{3}{2}}}$$

7. $y=\sqrt{\left(\dfrac{1+x}{2+x}\right)}$

Solution:

Let $u=\dfrac{1+x}{2+x}$; $\quad \dfrac{du}{dx}=\dfrac{(2+x)-(1+x)}{(2+x)^2}=\dfrac{1}{(2+x)^2}$; $y=u^{\frac{1}{2}}$; $\dfrac{dy}{du}=\dfrac{1}{2}u^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{u}}$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2\sqrt{(1+x)(2+x)^{-1}}}\times\frac{1}{(2+x)^2}=\frac{1}{2(1+x)^{\frac{1}{2}}(2+x)^{-\frac{1}{2}}(2+x)^2}=\frac{1}{2(x+1)^{\frac{1}{2}}(2+x)^{\frac{3}{2}}}$$

$$=\frac{1}{2\sqrt{(x+1)(2+x)^3}}$$

Exercise

1. $\dfrac{1-x^2}{1+x^2}$ 3. $\dfrac{x^2}{\sqrt{1+x^2}}$ 3. $\sqrt{\dfrac{(x+1)^3}{x+2}}$ 4. $\dfrac{\sqrt{x}}{\sqrt{1+x}}$ 5. $\dfrac{2x^2-x^3}{\sqrt{x^2-1}}$ 6. $y=\dfrac{(3x-x^4)}{(x^2+1)}$

End of proofreading, 6/2/12, 10.35 p.m.

Implicit functions

Explicit functions if x e.g $y = x^2 - 5x + 4$

Here, y is given as an expression in x .If however y is given implicity by an equal such as $x = y^4 - y + 1$

He cannot express y in terms of x.

Consider

$$x = y^2, \Rightarrow y = x^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}$$

$$= \frac{1}{2y}$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

But $\frac{dx}{dy} = 2y$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

When it is impracticable to express either variable explicitly in terms of the other, we can still differentiate both sides with respect to *x*.

A term like y^n can be differentiated by first differentiating wrt *y* then, as the chain rule demands, multiplying by $\frac{dy}{dx}$ thus

$$\frac{d}{dx}\left(y^n\right) = \frac{d}{dy}\left(y^n\right)\frac{dy}{dx} = ny^{n-1}\frac{dy}{dx}$$

Similarly if we have a term of a form $x^m y^m$, then we use the product rule and obtain

$$\frac{d}{dx}\left(x^m y^n\right) = x^m \frac{d}{dx}\left(y^n\right) + y^n \frac{d}{dx}\left(x^m\right)$$
$$= nx^m y^{n-1}\frac{dy}{dx} + mx^{m-1}y^n$$

Example

1.Find the gradient of the curve

$x^2 + 2xy - 2y^2 + x = 2$ at a point (-4,1)

To find the gradient, differentiate wrt *x*

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(2xy\right) - \frac{d}{dx}\left(2y^2\right) + \frac{d}{dx}\left(x\right) = \frac{d}{dx}\left(2\right)$$
$$2x + \left(2x\frac{dy}{dx} + 2y\right) - 4y\frac{dy}{dx} + 1 = 0$$
$$\frac{dy}{dx}\left(2x - 4y\right) = -1 - 2x - 2y$$
$$\frac{dy}{dx} = \frac{-1-2x-2y}{2x-4y}$$
$$x = -4, y = 1$$
$$\frac{dy}{dy} = \frac{-1+8-2}{-12} = \frac{-5}{12}$$

2.Find $\frac{dy}{dx}$ if $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$

$$2x + 2y\frac{dy}{dx} - 6\left[y + x\frac{dy}{dx}\right] + 3 - 2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\left[2y - 6x - 2\right] = 6y - 3 - 2x$$
$$\frac{dy}{dx} = \frac{6y - 2x - 3}{2y + 6x - 2}$$

Find the slope of the tangent to the curve at a point (1, 2)

$$x^2 + xy + y^2 = 7$$
$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(x + 2y) = -2x - y$$
$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$
$$x = 1, y = 2$$
$$\frac{dy}{dx} = \frac{-2 - 2}{1 + 4} = \frac{-4}{5}$$

Find $\frac{dy}{dx}$ if $(x+y)^3 + (x-y)^3 = x^4 + y^4$

$$3\left(\frac{dy}{dx}\right)(x+y)^2 + 3\left(-\frac{dy}{dx}\right) = 4x^3 + 4y^3\frac{dy}{dx}$$
$$2x^3 + 6xy^2 = x^4 + y^4$$
$$6x^2 + 6\left[y^2 + 2yx\frac{dy}{dx}\right] = 4x^3 + 4y^3\frac{dy}{dx}$$
$$6x^2 + 6y^2 + 12xy\frac{dy}{dx} = 4x^3 + 4y^3\frac{dy}{dx}$$
$$\frac{dy}{dx}\left(12xy - 4y^3\right) = 4x^3 - 6x^2 - 6y^2$$
$$\frac{dy}{dx} = \frac{4x^3 - 6x^2 - 6y^2}{12xy - 4y^3}$$

4.If $4x^2 - 2y^2 = 9$ find $\frac{d^2y}{dx^2}$

$$8x - {}^{4ydy}\!\!\diagup_{dx} = 0$$

$$\frac{dy}{dx} = \frac{8x}{4y} = \frac{2x}{y}$$

$$8 - 4\left(\frac{dy}{dx}\right)^2 - 4y\frac{d^2y}{dx^2} = 0$$

$$8 - 4\left(\frac{dy}{dx}\right)^2 = 4y\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{8 - 4\left(\frac{dy}{dx}\right)^2}{4y}$$

$$= \frac{8 - 4\left({}^{2x}\!\diagup_{y}\right)^2}{4y}$$

$$= \frac{8 - {}^{16x^2}\!\diagup_{y^2}}{4y}$$

$$= \frac{8y^2 - 16x^2}{4y^3} = \frac{2y^2 - 4x^2}{y^3}$$

$$\frac{2y - {}^{4x}\!\diagup_{y}\, x}{y^2}$$

$$= \frac{2y^2 - 4x^2}{y^3}$$

4.Find $\frac{dy}{dx}$ where $x^2 + 2xy + y^2 = 3$

$$2x + 2\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x + 2y) = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x + 2y}$$

$$= \frac{-2(x + y)}{2(x + y)}$$

$$\frac{dy}{dx} = -1$$

5. $\frac{dy}{dx}$ where $x^2 - 3xy + y^2 - 2y + 4x = 0$

$$2x-3\left[y+x\frac{dy}{dx}\right]+2y\frac{dy}{dx}-2\frac{dy}{dx}+4=0$$
$$2x-3y-3x\frac{dy}{dx}+2y\frac{dy}{dx}-2\frac{dy}{dx}+4=0$$
$$\frac{dy}{dx}\left[2y-2-3x\right]=-4+3y-2x$$
$$\frac{dy}{dx}=\frac{3y-2x-4}{2y-3x-2}$$

6. $\frac{dy}{dx}\left(3x^2-4xy=7\right)$

$$6x-4\left(y+x\frac{dy}{dx}\right)=0$$
$$6x-4y-x\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{4y-6x}{-x}=6-\frac{4y}{x}$$

$x^2-3xy-y^2=3$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point (1,1)

$$2x+3\left(y+x\frac{dy}{dx}\right)-2y\frac{dy}{dx}=0$$
$$2x+3y+3\left(x\frac{dy}{dx}\right)-2\left(y\frac{dy}{dx}\right)=0$$
$$2+3\frac{dy}{dx}+3\left[\frac{dy}{dx}+x\frac{d^2y}{dx^2}\right]-2\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]=0$$
$$2+3\frac{dy}{dx}+3\frac{dy}{dx}+3x\frac{d^2y}{dx^2}-2\left(\frac{dy}{dx}\right)^2-2y\frac{d^2y}{dx^2}=0$$
$$2+6\frac{dy}{dx}+\frac{d^2y}{dx^2}\left[3x-2y\right]-2\left(\frac{dy}{dx}\right)^2=0$$
$$\frac{d^2y}{dx^2}=\frac{2\left(\frac{dy}{dx}\right)^2-6\frac{dy}{dx}-2}{3x-2y}$$

$\frac{dy}{dx}$ if $x^2+y^2-6xy+3x-2y+5=0$

$$2x+2y\frac{dy}{dx}-6y-6x\frac{dy}{dx}+3-2\frac{dy}{dx}=0$$
$$\frac{dy}{dx}\left[2y-6x-2\right]=2x-6y+3$$
$$\frac{dy}{dx}=\frac{2x-6y+3}{2y-6x-2}$$

Given that $3x^2+5xy+4y^2-4y=0$,obtain the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (0,1)

$$6x+5y+5x\frac{dy}{dx}+8y\frac{dy}{dx}-4\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left[5x+8y-4\right]=-6x-5y$$
$$\frac{dy}{dx}=\frac{-6x-5y}{5x+8y-4}$$
$$\frac{dy}{dx}\Big/0,1=\frac{-3}{4}$$
$$6+5\frac{dy}{dx}+5\left(\frac{dy}{dx}+x\frac{d^2y}{dx^2}\right)+8\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)$$
$$6+5\frac{dy}{dx}+5\frac{dy}{dx}+5x\frac{d^2y}{dx^2}+8y\frac{d^2y}{dx^2}+8\left(\frac{dy}{dx}\right)^2-4\frac{d^2y}{dx^2}=0$$
$$\frac{d^2y}{dx^2}\left[5x-4+8y\right]+\frac{dy}{dx}\left(10+8\frac{dy}{dx}\right)=0$$
$$y=\left|\begin{array}{ll}\sqrt{4-x} & x<4\\ x-4 & x\geq 4\end{array}\right.$$
$$\frac{d^2y}{dx^2}=\frac{-\frac{dy}{dx}\left[10+8\frac{dy}{dx}\right]}{5x-4+8y}\quad(0,1)$$
$$=\frac{\frac{5}{4}\left[10-\frac{5\times 8}{4}\right]}{5x-4+8y}$$
$$=\frac{0}{4}$$
$$=0$$

Find $\frac{dy}{dx}$ if $x^3-3xy^2+y^3=1$

LOGARITHMIC FUNCTIONS

Recall:

1. $\log_b^1=0$ 2. $\log_b^b=1$. $\log_b ac=\log_b a+\log_b c$ 4. $\log_b\frac{a}{c}=\log_b a-\log_b c$ 5. $\log_b a^r=r\log_b a$

e.g. $\log_b a^{\frac{1}{2}}=\frac{1}{2}\log_b a$ 6. $\log_b\frac{1}{c}=\log_b c^{-1}=-1\log_b c$

7. If $\log_c a = y \Rightarrow c^y = a$

Taking logs on both sides

$\log c^y = \log a; \quad y\log c = \log a; \quad y = \dfrac{\log a}{\log c}; \quad y = \dfrac{\log_b a}{\log_b c}$ Change to any base.

The number e (natural number)

$e \approx 2.718281828$

$\log_e b$ (log to base e of b) is called natural logarithm of b, abbreviated as lnb

a) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e;$ and $\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$

Derivative of $\log_b x$ (b is any base)

$$\frac{d}{dx}(\log_b x) = \lim_{h\to 0}\frac{\log_b x+h-\log_b x}{h} = \lim_{h\to 0}\frac{1}{h}\left(\log_b \frac{x+h}{x}\right) = \lim_{h\to 0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right)$$

Let $\dfrac{h}{x} = t; \quad h = tx$

$$\frac{d}{dx}(\log_b x) = \lim_{t\to 0}\frac{1}{tx}\log_b(1+t)$$

$$= \frac{1}{x}\lim_{h\to 0}\frac{1}{t}\log(1+t) = \frac{1}{x}\lim_{t\to 0}\log_b(1+t)^{\frac{1}{t}} = \frac{1}{x}\log_b\lim_{t\to 0}(1+t)^{\frac{1}{t}} = \frac{1}{x}\log_b e$$

$\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x}\log_b e$;

If $b=e$, $\quad \dfrac{d}{dx}(\log_e x] = \dfrac{1}{x}\log_e e = \dfrac{1}{x}$

$\log_e x$ is the natural logarithm of x

$\log_e x = \ln x; \quad y = \ln x \quad \dfrac{dy}{dx} = \dfrac{1}{x}$

Example

Use $\dfrac{d}{dx}(\ln x)$ to find $\dfrac{d}{dx}(\log_b x)$

Solution:

$y = \log_b x; \quad b^y = x \quad \ln b^y = \ln x; \quad y\ln b = \ln x$

$\dfrac{dy}{dx}\ln b = \dfrac{1}{x}; \quad \dfrac{dy}{dx} = \dfrac{1}{x\ln b}$

Or $\quad y = \dfrac{\ln x}{\ln b}; \quad \dfrac{dy}{dx} = \dfrac{\ln bx\frac{1}{x} - \ln x(0)}{(\ln b)^2} = \dfrac{1}{x\ln b}$

Derivative of a^x

Let $y = a^x$; $\ln y = x\ln a$ differentiate both sides with x

$\dfrac{1}{y}\dfrac{dy}{dx} = \ln a; \quad \dfrac{dy}{dx} = y\ln a; \quad \dfrac{dy}{dx} = a^x\ln a \quad \therefore \dfrac{d}{dx}\left(a^x\right) = a^x\ln a$

Examples:

Differentiate the following functions with respect to x

1. $\quad y = 3^x$;

Solution: $\ln y = \ln 3^x$; $\ln y = x\ln 3$; $\frac{1}{y}\frac{dy}{dx} = \ln 3$; $\frac{dy}{dx} = y\ln 3$; $= 3^x \ln 3$

2. $y = 3^{\sin x}$

Solution:

$$\ln y = \ln 3^{\sin x};\quad \ln y = \sin x \ln 3;\quad \frac{1}{y}\frac{dy}{dx} = \cos x \ln 3;\quad \frac{dy}{dx} = y\cos x \ln 3; = 3^{\sin x}\cos x \ln 3$$

3. $y = x^x$

Solution:

$$\ln y = x\ln x;\quad \frac{1}{y}\frac{dy}{dx} = \ln x + x\frac{1}{x};\quad \frac{dy}{dx} = (\ln x + 1)y \quad = x^x(\ln x + 1)$$

4. $y = x^y$

Solution:

$$\ln y = \ln x^y;\quad \ln y = y\ln x;\quad \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\ln x + \frac{y}{x};\quad \frac{dy}{dx}(\frac{1}{y} - \ln x] = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} \div \left(\frac{1}{y} - \ln x\right);\quad = \frac{y}{x} \div \frac{1 - y\ln x}{y} \quad = \frac{y}{x} \times \frac{y}{1 - y\ln y} = \frac{y^2}{x(1 - y\ln y)} = \frac{\left(x^y\right)^2}{x(1 - y\ln y)}$$

5. $y = \left(x^2 + 1\right)^{10} + x10^{x^2+1}$

Solution:

$$\frac{dy}{dx} = 10(2x)\left(x^2 + 1\right)^9 + 2x10^{x^2+1}\ln 10 = 20x\left(x^2 + 1\right)^9 + 2x10^{x^2+1}\ln 10$$

6. $y = \sin x^{\tan x}$ $\sin x > 0$;

Solution:

$$\ln y = \tan x \ln \sin x$$

$$\frac{1}{y}\frac{dy}{dx} = \tan x \frac{\cos x}{\sin x} + \sec^2 \times \ln \sin x = \frac{\sin x}{\cos x}\,\frac{\cos x}{\sin x} + \sec^2 x \ln \sin x$$

$$\frac{dy}{dx} = \left(1 + \sec^2 x \ln \sin x\right)y = \sin x^{\tan x}\left(1 + \sec^2 x \ln \sin x\right)$$

$$(y = \ln \sin x;\quad \mathit{Let}\, u = \sin x;\quad \frac{du}{dx} = \cos x;\quad y = \ln u$$

$$\frac{dy}{du} = \frac{1}{u};\quad \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \cos x = \frac{\cos x}{\sin x})$$

7. $y = \ln\left(x^2 + 6\right)$

Solution:

Let $u = x^2 + 6$; $\frac{du}{dx} = 2x$; $y = \ln u$; $\frac{dy}{du} = \frac{1}{u}$ $\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 2x$; $= \frac{2x}{x^2 + 6}$

Or directly $\frac{2x}{x^2 + 6}$

8. $y = \ln\left(3x^3 + 2x\right)$

Solution: $\frac{dy}{dx} = \frac{9x^2 + 2}{3x^3 + 2x}$

9. $y=\ln\sqrt{x+1} \qquad =\ln\left(x+1\right)^{\frac{1}{2}}$

Solution: $\dfrac{dy}{dx}=\dfrac{\frac{1}{2}\left(x+1\right)^{-\frac{1}{2}}}{\left(x+1\right)^{\frac{1}{2}}} \quad =\dfrac{1}{2}\times\dfrac{1}{\left(x+1\right)^{\frac{1}{2}}\left(x+1\right)^{\frac{1}{2}}}=\dfrac{1}{2\left(x+1\right)}$

0r $y=\ln u;\ u=\left(x+1\right)^{\frac{1}{2}};\ \dfrac{du}{dx}=\dfrac{1}{2}\left(x+1\right)^{-\frac{1}{2}};\ \dfrac{dy}{du}=\dfrac{1}{u}\ \therefore\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=\dfrac{1}{\left(x+1\right)^{\frac{1}{2}}}\times\dfrac{1}{2\left(x+1\right)^{\frac{1}{2}}}=\dfrac{1}{2\left(x+1\right)}$

10. $y=\ln ax$

$u=ax \qquad \dfrac{dy}{du}=\dfrac{1}{u}$

$\dfrac{du}{dx}=a \qquad \dfrac{dy}{dx}=\dfrac{1}{ax}a=\dfrac{1}{x}$

11. $y=\ln\left[\dfrac{x+\sqrt{\left(x+5\right)}}{\left(x-1\right)^{3}}\right]$

Solution:

$=\ln x+\ln\left(x+5\right)^{\frac{1}{2}}-\ln\left(x-1\right)^{3} \qquad =\ln x+\dfrac{1}{2}\ln\left(x+5\right)-3\ln\left(x-1\right)$

$\dfrac{dy}{dx}=\dfrac{1}{x}+\dfrac{1}{2}\left(\dfrac{1}{x+5}\right)-\dfrac{3}{x-1}$

$\quad =\dfrac{1}{x}+\dfrac{1}{2x+10}-\dfrac{3}{x-1}$

12. $y=\ln\left(\dfrac{x^{2}\sin x}{\sqrt{1-x}}\right)$

$=\ln x^{2}+\ln\sin x-\ln\left(1-x\right)^{\frac{1}{2}}=\dfrac{2x}{x^{2}}+\dfrac{\cos x}{\sin x}-\dfrac{1}{2}\ln\left(1-x\right)=\dfrac{2x}{x^{2}}+\cot\left(x\right)+\dfrac{1}{2}\times\dfrac{1}{x-1}=\dfrac{2}{x}+\cot x+\dfrac{1}{2\left(1-x\right)}$

13. $y=\ln\left(\dfrac{\sqrt{\cos x}}{x^{2}\sin x}\right)=\ln\left(\sqrt{\cos x}\right)^{\frac{1}{2}}-[\ln x^{2}+\ln\sin x]$

$=\dfrac{1}{2}\ln\cos x-2\ln x-\ln\sin x=-\dfrac{1}{2}\dfrac{\sin x}{\cos x}-\dfrac{2}{x}-\dfrac{\cos x}{\sin x}=\dfrac{\tan x}{2}-\dfrac{2}{x}-\cot x$

Exercise

Differentiate the following with respect to x

1. $y=7^{x}$; 2. $y=3^{2-x^{2}}$ 3. $y=x^{e}+e^{x}$ 4. $y=2^{\sec x}$ 5. $y=\left(\cos x\right)^{\sqrt{x}}$

6. Find $\dfrac{d^{2}y}{dx^{2}}$ if $y=\sin t$ and $x=\cos t+\ln\left(\tan\left(\dfrac{1}{2}t\right)\right)$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Recall

1. $\lim\limits_{h\to 0}\sin h=0 \qquad \lim\limits_{h\to 0}\cosh=1;$

$$\lim_{h\to 0}\frac{\sinh}{h} = \lim_{h\to 0}\frac{\cosh}{1} = 1 \quad \text{(by L' Hospital Rule)}; \quad \therefore \lim_{h\to 0}\frac{\sinh}{h} = 1$$

Also, $\lim_{h\to 0}\frac{1-\cosh}{h} = \lim_{h\to 0}\frac{\sinh}{1} = \lim_{h\to 0}\sinh = 0$

Recall also the factor formulae

$$cosP + cosQ = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$
$$cosP - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$
$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$
$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

Derivatives of $\sin x$ and $\cos x$

Derivative of $\sin x$ *from 1st Principles*

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f(x) = y = \sin x$$
$$f(x+h) = \sin(x+h)$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h};$$
$$\lim_{h\to 0}\frac{\sin x\cos h + \sinh\cos x - \sin x}{h}$$
$$\lim_{h\to 0}\left(\frac{\sin x\cosh}{h} + \frac{\sinh\cos x}{h} - \frac{\sin x}{h}\right)$$
$$\sin x\lim_{h\to 0}\frac{\cosh}{h} + \cos x\lim_{h\to 0}\frac{\sinh}{h} - \sin x$$
$$= \cos x$$

(Or by factor method

$$\sin A - \sin B = 2\cos\frac{(A+B)}{2}\sin\frac{(A-B)}{2}$$

$\sin(x+h)-\sin x$; Let $A = (x+h)$; $B = x$

$$\sin(x+h)-\sin x = 2\cos\frac{(x+h)+x}{2}\sin\frac{(x+h-x)}{2}$$
$$\sin(x+h)-\sin x = 2\cos\frac{2x+h}{2}\sin\frac{h}{2}$$
$$\lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{1}{2}h} = \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{1}{2}h}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\times 1 = \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right) = \cos x$$

Derivative of $\cos x$ *from 1st Principles*

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$\cos A - \cos B = -2\sin\frac{(A+B)}{2}\sin\frac{A-B}{2}$$

$$\cos(x+h) - \cos x = -2\sin\frac{x+h+x}{2}\sin\frac{x+h-x}{2}$$

$$= -2\sin\frac{2x+h}{2}\sin\frac{h}{2}$$

$$\lim_{h\to 0}\frac{-2\sin\frac{(2x+h)}{2}\sin\frac{h}{2}}{h} = \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} = -\lim_{h\to 0}\frac{\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}} = -\sin x$$

Therefore $\frac{d}{dx}(\sin x) = \cos x$; $\quad \frac{d}{dx}(\cos x) = -\sin x$

Recall

1. $\cos^2 x + \sin^2 = 1 \Rightarrow \frac{\cos^2 x}{\sin^2 x} + 1 = \frac{1}{\sin^2 x} \Rightarrow \cot^2 x + 1 = \cos ec^2 x$

   Similarly, $1 + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow 1 + \tan^2 x = \sec^2 x$

Derivative of $\tan x$

$$y = \tan x; \quad y = \frac{\sin x}{\cos x}; \quad \frac{dy}{dx} = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x; \qquad \therefore \frac{d}{dx}(\tan x) = \sec^2 x$$

Derivative of $\cot x$

$$y = \cot x = \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx}=\frac{-\sin x\sin x-\cos x\cos x}{\sin^2 x}=\frac{-\left[\sin^2 x+\cos^2 x\right]}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\cos ec^2 x;\ \therefore \frac{d}{dx}(\cot x)=-\cos ec^2 x$$

Examples

Differentiate

1. $y=x^2\tan x$

Solution:

$$\frac{dy}{dx}=x^2\sec^2 x+2x\tan x$$

2. $y=\dfrac{\sin x}{1+cosx}$

Solution:

$$\frac{dy}{dx}=\frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}=\frac{1+\cos x}{(1+\cos x)^2}=\frac{1}{1+\cos x}$$

3.Let $y=\sec x$ Find $\dfrac{d^2y}{dx^2}$ at $x=\dfrac{\pi}{4}$

Solution:

$$\frac{dy}{dx}=\sec x\tan x$$

$$\frac{d^2y}{dx^2}=\sec x\sec^2 x+\tan x(\sec x\tan x)$$

$$=\sec^3 x+\tan^2 x\sec x$$

At $x=\dfrac{\pi}{4}$, $\dfrac{d^2y}{dx^2}=\sec^3\left(\dfrac{\pi}{4}\right)+\tan^2\left(\dfrac{\pi}{4}\right)\sec\dfrac{\pi}{4}=\left(\sqrt{2}\right)^3+1\left(\sqrt{2}\right)$ $\dfrac{d^2y}{dx^2}=2\sqrt{2}+\sqrt{2}=3\sqrt{2}$

4. $y=\left(x^2+1\right)\sec x;$

. *Solu*tion:

$$\frac{dy}{dx}=2x\sec x+\left(x^2+1\right)\sec x\tan x$$

5. $y=\sin 2x$

Solution: Let $u=2x;$ $\dfrac{du}{dx}=2,$ $y=\sin u;$ $\dfrac{dy}{du}=\cos u;$ $\therefore \dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=\cos u\times 2=2\cos 2x$

6. $y=\sin^2 x=(\sin x)^2$

Solution:

$$\frac{dy}{dx}=2(\cos x)\sin x=2\cos x\sin x$$

7. $\sec^3 x=(\sec x)^3$

Solution:

$$\frac{dy}{dx} = 3(\sec x \tan x)\sec x = 3\sec^2 x \tan x$$

8. Given that $y\sin x = 3x^2$, determine $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$

Solution:

$$y\cos x + \sin x\frac{dy}{dx} = 6x$$

$$\sin x\frac{dy}{dx} = \frac{6x}{y\cos x}$$

$$\frac{dy}{dx} = \frac{6x}{y\sin x\cos x}$$

$$\frac{dy}{dx}\Big/\frac{\pi}{2} = 3\pi$$

Exercise

1. Differentiate (a) *tanx* (b)*cotx* (c)*secx* (d) *cosecx* from first principles(from the definition of derivatives).

1.Show that (a) $\frac{d}{dx}(\sec x) = \sec x\tan x$ (b) $\frac{d}{dx}(\cos ex) = -\cos ecx\cot x$

In each of the following questions, find $\frac{dy}{dx}$

3. $\tan^2 5x = (\tan 5x)^2$ 4. $y = \sqrt{x}\tan^3 x^{\frac{1}{2}} = x^{\frac{1}{2}}\left(\tan x^{\frac{1}{2}}\right)^3$ 5. $y = \cos\left(\sin\sqrt{x}\right)$ 6. $y = 2\cos ec^2 4x^3 = 2\left(\cos ec 4x^3\right)^2$

7. If $y = \left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}$ Show that $\frac{dy}{dx} = \frac{1}{1-\sin x}$

Mixed Exercise

1. Find the limit of $\lim_{x\to 2}\frac{x^2 - x - 2}{x-2}$

2. Show that the function $f(x) = \frac{x}{x^2-3}$ is continous at x=3.

3. Let $g(x) = \frac{f(x)}{x}$. Given that *f(4)=3* and $f'(4) = -5$,find $g'(4)$.

4. Discuss continuity of $f(x)$ if $f(x) = \begin{cases} \frac{x^2-1}{x-1} & if\ x \neq 1 \\ 2, & if\ x = 1 \end{cases}$

5. Find the derivative of f(x) from first principles (a) $f(x) = \sqrt{x-2}$ (b) $f(x) = 2 + 8x - 5x^2$

6. Find $\frac{dy}{dx}$ of (a) $y = \left(\frac{x-1}{x+2}\right)^{\frac{3}{2}}$ (b) $y = \sqrt{\frac{x^2+1}{x^2-5}}$

INVERSE TRIGONOMETRICAL FUNCTIONS

The functions $\sin^{-1}x$, $\tan^{-1}x$, $\cos^{-1}x$ etc are called inverse functions.

Examples

Find the derivative of $\sin^{-1}x$

Solution:

Let $y=\sin^{-1}x \quad \Rightarrow \sin y = x$; Differentiating w.r.t. x, we get $\cos y\dfrac{dy}{dx}=1$; $\quad \dfrac{dy}{dx}=\dfrac{1}{\cos y}=\dfrac{1}{\sqrt{1-x^2}}$

Or $\sin^2 y = x^2$; $\quad (1-\cos^2 y)=x^2;\quad -\cos^2 y = x^2-1;\quad \cos^2 y = 1-x^2; \cos y=\sqrt{1-x^2}$

2. $y=\tan^{-1}(x^2+1)$;

$\tan y = x^2+1;\quad \sec^2 y\dfrac{dy}{dx}=2x;\quad \dfrac{dy}{dx}=\dfrac{2x}{\sec^2 y}=2x\cos^2 y = (\dfrac{2x}{\sqrt{x^4+2x^2+2}})^2=\dfrac{2x}{x^4+2x^2+2}$

3. Differentiate with respect to x

(a) $y=\cos^{-1}x$ (b) $y=\sin^{-1}(2x+1)$ (c) $y=\cos^{-1}(x^2+1)$ (d) $y=\sin^{-1}(\dfrac{x-1}{x+1})$ (e) $y=4x^2\ln(\tan^{-1}x)$

(f) $y=\tan^{-1}(\dfrac{2x}{1-x^2})$ (g) $y=\tan^{-1}(\dfrac{1+x}{1-x})$ (h) $y=\cos^{-1}(\tan x)$ (i) $y=\cos^{-1}(x^2+1)$

Exponential differentiation

Recall:

1. $a^b\cdot a^c = a^{b+c}$ $\quad$ 2. $\dfrac{a^b}{a^c}=a^b\cdot a^{-c}=a^{b-c}$ $\quad$ 3. $\left(a^b\right)^c = a^{bc}=\left(a^c\right)^b$

4. Equations involving $\ln x$ and e^x

If $y=\ln x \; = \log_e x$, $y=\log_e x \;\Rightarrow\; e^y = x$

5. $e^{\ln x}=x$

Proof:

Let $y=e^{\ln x}$; $\quad \ln y = \ln e^{\ln x}$; $\ln y = \ln x \ln e$; $\quad \ln y = \ln x;\quad y = x$

or $\quad \ln y - \ln x = 0;\quad \ln\left(\dfrac{y}{x}\right)=0;\quad e^0=\dfrac{y}{x};\quad y=x$ but $y=e^{\ln x}$; $\quad \therefore e^{\ln x}=x;\quad \therefore e^{\ln x^2}=x^2$

6. Solve for y.

(a) $e^{3y}=2+\cos x$

Solution:

$$\ln e^{3y}=\ln(2+\cos x)$$
$$3y\ln e = \ln(2+\cos x)$$
$$3y=\ln(2+\cos x)$$
$$y=\frac{1}{3}\ln(2+\cos x)$$

(b) $\ln(y-1)-\ln y = 3x$

Solution:

$\log_e\dfrac{(y-1)}{y}=3x;\quad \dfrac{y-1}{y}=e^{3x};\quad y-1=ye^{3x};\quad y-ye^{3x}=1;\quad y\left(1-e^{3x}\right)=1;\quad y=\dfrac{1}{1-e^{3x}}$

Derivative of $y=e^x$

Let $y = e^x$; $\ln y = \ln e^x$; $\ln y = x \ln e$; $\ln y = x$

Differentiate both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = 1; \quad \frac{dy}{dx} = y \text{ ; but } y = e^x \qquad \frac{dy}{dx} = e^x; \quad \therefore \frac{d}{dx}\left(e^x\right) = e^x$$

Examples

Differentiate the following with respect to x.

1. $y = e^{ax}$

Solution:

Let $u = ax$; $\frac{du}{dx} = a$; $\quad y = e^u$; $\frac{dy}{du} = e^u$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = ae^u = ae^{ax}$$

$$\therefore \frac{d}{dx}\left(e^{ax}\right) = e^{ax}\frac{d}{dx}(ax) = ae^{ax}$$

2. $y = e^{\sin x}$

Solution:

let $u = \sin x$; $\frac{du}{dx} = \cos x$; $\quad y = e^u$; $\frac{dy}{du} = e^u$; $\frac{dy}{dx} = e^u \cos x$

$= e$

$$\frac{dy}{dx} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cos x = \cos x e^{\sin x} \quad \text{or}$$

3. $y = x^2 e^x$

Solution:

$$\frac{dy}{dx} = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}x^2$$

$$= x^2e^x + 2xe^x = xe^x(x+2)$$

4. $y = e^{\sqrt{x^2+1}}$

Solution:

$$y = e^{\left(x^2+1\right)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = e^{\sqrt{x^2+1}}\frac{d}{dx}\left(x^2+1\right)^{\frac{1}{2}}; = e\sqrt{x^2+1}\left(\frac{1}{2}(2x)(x+1)^{-\frac{1}{2}}\right) = \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

Or let $u = \left(x^2+1\right)^{\frac{1}{2}}$; $\frac{du}{dx} = \frac{1}{2}(2x)\left(x^2+1\right)^{-\frac{1}{2}}$; $\quad y = e^u$; $\quad \frac{dy}{du} = e^u$; $\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

5. $e^y = x^2 - 3$

Solution:

$$e^y\frac{d}{dx}(y) = \frac{d}{dx}\left(x^2-3\right); \ \frac{dy}{dx}e^y = 2x; \quad \frac{dy}{dx} = \frac{2x}{e^y} = \frac{2x}{x^2-3} \text{ OR}$$

Take natural logarithms on both sides

$\ln e^{y} = \ln\left(x^2-3\right);\quad y\ln e = \ln\left(x^2-3\right);\quad y = \ln\left(x^2-3\right);\quad \frac{dy}{dx} = \frac{2x}{x^2-3}$

6. $e^{xy} = x^2$

Solution:

$e^{xy}\frac{d}{dx}\left(xy\right) = 2x;\quad e^{xy}\left[x\frac{dy}{dx}+y\right] = 2x;\quad xe^{xy}\frac{dy}{dx}+ye^{xy} = 2x;\quad \frac{dy}{dx} = \frac{2x-ye^{xy}}{xe^{xy}}$

$= \frac{2x-yx^2}{x\left(x^2\right)} = \frac{2x-yx^2}{x^3} = \frac{2-yx}{x^2} = \frac{2}{x^2}-\frac{y}{x} = \frac{1}{x}\left[\frac{2}{x}-y\right]$

OR

$\ln e^{xy} = \ln x^2;\quad xy\ln e = 2\ln x;\quad xy = 2\ln x;\quad x\frac{dy}{dx}+y = \frac{2}{x};\quad \frac{dy}{dx} = \frac{1}{x}\left[\frac{2}{x}-y\right]$

7. $y = \cos ye^{-x}$

Solution:

Let $u = e^{-x}$; $\quad y = \cos^{-1}u;\ \frac{dy}{du} = \frac{-u}{\sqrt{1-u^2}};\ ;\ \frac{du}{dx} = -e^{-x};\ \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{-e^{-x}}{\sqrt{1-e^{-2x}}}\times -e^{-x} = \frac{e^{-2x}}{\sqrt{1-e^{-2x}}}$

Exercise: Find $\frac{dy}{dx}$

1. $y = \sqrt{e^{2x}+2x}$; 2. $y = e^{\sqrt{x}}+\sqrt{e^{x}}$; 3. $y = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$; 4. $y = e^{\left(e^{x}\right)}$;5. $y = e^{-2x}\ln x$

6. $y = \ln\sqrt{e^{2x}+e^{-2x}}$; 7. $e^{xy}-3x+3y^2 = 11$; 8. $xe^{y}+2x-\ln\left(y+1\right) = 3$; 9. $xe^{y}-ye^{x} = 2$

PARAMETRIC EQUATIONS

Consider $x = f(t)$ and $y = g(t)$,then x and y are both functions of *(t)*.These equations are called *parametric equations* for *x* and *y* and the variable *t* is called a *parameter*.

Example of parametric equation is $x = 2t, \quad y = t^2 - 1$

Derivative of parametric equations

$x = f(t)$ and $y = g(t)$; $\quad \frac{dx}{dt} = f'(t)$; $\quad \frac{dy}{dt} = g'(t)$; $\quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ where $\frac{dx}{dt} \neq 0$

Examples

1. Find the derivative $\left(\frac{dy}{dx}\right)$ of $x = 2t, y = t^2 - 1$

Solution:

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 2t\ ; \quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{2} = t\ ; \quad \frac{dy}{dx} = t\ .$$

2. $x = t^3 + t^2 \quad , y = t^2 + t$

Solution:

$$\frac{dy}{dt} = 3t^2 + 2t \quad , \frac{dy}{dt} = 2t + 1; \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (2t+1) \times \frac{1}{3t^2 + 2t} = \frac{2t+1}{3t^2 + 2t}$$

3. $x = \cos t, \quad y = \sin t$

Solution:

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = cost\ ; \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cos t}{-\sin t} = -\cot t; \quad \frac{dy}{dx} = -\cot t = -\frac{\cos t}{\sin t} = -\frac{x}{y}$$

4. Find the gradient of the curve $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$ at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Solution:

$$\frac{dx}{dt} = \frac{(1+t) - t(1)}{(1+t)^2} = \frac{1}{(1+t)^2}\ ; \quad \frac{dy}{dt} = \frac{(1+t)3t^2 - t^3}{(1+t)^2} = \frac{3t^2 + 3t^3 - t^3}{(1+t)^2} = \frac{3t^2 + 2t^3}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2 + 2t^3}{(1+t)^2} \times (1+t)^2; \quad = 3t^2 + 2t^3$$

Where $x = \frac{1}{2}$, $\frac{t}{1+t} = \frac{1}{2}; \quad 2t = 1 + t; \quad t = 1;$

When $y = \frac{1}{2}; \ \frac{t^3}{1+t} = \frac{1}{2}; \quad 2t^3 = 1 + t; \quad t = 1;$

When $t = 1, \quad \frac{dy}{dx} = 3(1^2) + 2(1^3) = 5$

5. If $x = t^3 + t^2, \ y = t^2 + t$ find $\frac{dy}{dx}$ in terms of t.

Solution:

$$\frac{dx}{dt} = 3t^2 + 2t \quad , \frac{dy}{dt} = 2t + 1; \quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t + 1 \times \frac{1}{3t^2 + 2t} \quad = \frac{2t+1}{3t^2 + 2t} = \frac{2t+1}{t(3t+2)}$$

6. If $x=\frac{2t}{t+2}, \quad y=\frac{3t}{t+3}$, find $\frac{dy}{dx}$ in terms of t.

Solution:

$$\frac{dx}{dt}=\frac{(t+2)2-2t(1)}{(t+2)^2}=\frac{2t+4-2t}{(t+2)^2}=\frac{4}{(t+2)^2}$$

$$\frac{dy}{dt}=\frac{(t+3)3-3t(1)}{(t+3)^2} \quad =\frac{3t+9-3t}{(t+3)^2}=\frac{9}{(t+3)^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{9}{(t+3)^2}\times\frac{(t+2)^2}{4}=\frac{9}{4}\frac{(t+2)^2}{(t+3)^2}$$

Exercise

1. Find $\frac{dy}{dx}$, in terms of t when a) $x=at^2, \quad y=2at$; b) $x=(t+1)^2, \quad y=(t^2-1)$;

2. $x=\cos^2 t, \quad y=\sin^2 t$; 3. $x=t, \quad y=\frac{1}{t}$; 4. $x=t^2-\frac{\pi}{2}, \quad y=\sin(t^2)$;

Parametric formula for $\frac{d^2y}{dx^2}$

Let $x=f(t), \quad y=g(t)$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx} \quad \left(\frac{dx}{dt}\neq 0\right)=y^{'}; \ \frac{d^2y}{dx^2}=\frac{dy^{'}}{dx}$$ But by chain rule

$$\frac{dy^{'}}{dx}=\frac{dy^{'}}{dt}\times\frac{dt}{dx}; \qquad \therefore \frac{d^2y}{dx^2}=\frac{dy^{'}}{dt}\times\frac{dt}{dx}$$

Similarly,

$$\frac{d^3y}{dx^3}=\frac{\frac{d}{dx}\left(y^{''}\right)}{\frac{dx}{dt}}; \ \frac{d}{dx}\left(y^{''}\right)\times\frac{dt}{dx}$$

Example

1. Find $\frac{d^2y}{dx^2}$ if $x=t-t^2, \quad y=t-t^3$

Solution:

$$\frac{dx}{dt}=1-2t, \quad \frac{dy}{dt}=1-3t^2; \quad \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{1-3t^2}{1-2t}; \quad y^{'}=\frac{1-3t^2}{1-2t}$$

$$\frac{dy^{'}}{dt}=\frac{(1-2t)(-6t)-\left(1-3t^2\right)(-2)}{(1-2t)^2} \ =\frac{-6t+12t^2+2-6t^2}{(1-2t)^2} \quad \therefore \frac{dy^{'}}{dt}=\frac{6t^2-6t+2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2}=\frac{dy^{'}}{dx}=\frac{dy^{'}}{dt}\times\frac{dt}{dx} \ =\frac{6t^2-6t+2}{(1-2t)^2}\times\frac{1}{1-2t} \ =\frac{6t^2-6t+2}{(1-2t)^3}$$

2.Find $\frac{d^2y}{dx^2}$ where $x=\cos^2 t \quad ,y=\sin^2 t$.

Solution:

$$\frac{dx}{dt}=2(-\sin t)\cos t; \quad \frac{dy}{dt}=2(\cos t)\sin t$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{2\cos t\sin t}{-2\sin t\cos t}=-1\ \therefore\ \frac{d^2y}{dx^2}=0$$

Find $\frac{d^2y}{dx^2}$ if $x=t,\quad y=\sqrt{t}$;

Solution:

$$\frac{dx}{dt}=1;\quad \frac{dy}{dt}=\frac{1}{2}t^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{1}{2}t^{-\frac{1}{2}}\times\frac{1}{1}=\frac{1}{2\sqrt{t}}\ ;\ \frac{dy'}{dt}=-\frac{1}{4}t^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{dy'}{dt}\times\frac{dt}{dx}=-\frac{1}{4}t^{-\frac{3}{2}}=-\frac{1}{4t^{\frac{3}{2}}}=-\frac{1}{4\left(\sqrt{t}\right)^3}$$

4. $y=\sin t,\quad x=\cos t+\ln\left(\tan\left(\frac{1}{2}t\right)\right)$

Solution:

$$\frac{dy}{dt}=\cos t;\ \frac{dx}{dt}=-\sin t+\frac{\frac{1}{2}\sec^2\frac{1}{2}t}{\tan\frac{1}{2}t}=\text{-sin t}+\left(\frac{1}{2\cos^2\frac{1}{2}t}\times\frac{\cos\frac{1}{2}t}{\sin\frac{1}{2}t}\right)=-\sin t+\frac{1}{2\cos\frac{1}{2}t\sin\frac{1}{2}t}=-\sin t+\frac{1}{\sin t}$$

$$\frac{dx}{dt}=\frac{-\sin^2t+1}{\sin t}=\frac{\cos^2t}{\sin t}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dx}{dt}=\cos t\times\frac{\sin t}{\cos^2t}=\tan t$$

$$\frac{d^2y}{dx^2}=\sec^2t\times\frac{\sin t}{\cos^2t}=\frac{\sin t}{\cos^4t}=\tan t\sec^3t$$

Exercise

Find $\frac{d^2y}{dx^2}$ where 1. $x=2t-5,\quad y=4t-7$; 2. $x=\cos t,\quad y=5\sin t$; 3. $x=t^2-\frac{\pi}{2},\quad y=\sin\left(t^2\right)$

HIGHER DERIVATIVES

If *f* is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')'=f''$. This new function f'' is called the second derivative of *f* because it is the derivative of the derivative of f. Thus

$$f''(x)=\frac{d}{dx}\left(f'(x)\right)=\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$$

Examples

1. If $f(x)=x^8$, then $f'(x)=8x^7$ .So $f''(x)=\frac{d}{dx}\left(f'(x)\right)=\frac{d}{dx}\left(8x^7\right)=56x^6$

Notation: If $y=f(x)$, then $y''=f''(x)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$

$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

The process can be continued. The fourth derivative f'''' is usually denoted by f^{iv} or $f^{(4)}$.In general, the n^{th} derivative of *f* is denoted by $f^{(n)}$ and is obtained from f by differentiating *n* times. If $y = f(x)$, we write

$y^n = f^n = \frac{d^ny}{dx^n}$

2. If $y = x^3 - 6x^2 - 5x + 3$; $y' = 3x^2 - 12x - 5$; $y'' = 3x^2 6x - 12$; $y''' = 6$, $y^{(iv)} = 0$

3. Find y'' if $x^4 + y^4 = 16$

Solution: Differentiating implicitly w.r.t. *x*,

$$4x^3 + 4y^3\frac{dy}{dx} = 0;\ 4y^3\frac{dy}{dx} = -4x^3;\ \frac{dy}{dx} = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3;\ \frac{dy}{dx} = -\frac{x^3}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = \frac{d}{dx}\left(-x^3y^{-3}\right) = -x^3(-3)y^{-4}\frac{dy}{dx} + y^{-3}\left(-3x^2\right) = \frac{3x^3}{y^4}\frac{dy}{dx} - \frac{3x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{3x^3}{y^4}\times\left(\frac{-x^3}{y^3}\right) - \frac{3x^2}{y^3} = \frac{-3x^6}{y^7} - \frac{3x^2}{y^3} = \frac{-3x^6 - 3x^2y^4}{y^7} = \frac{-3x^2\left(x^4 + y^4\right)}{y^7} \quad \text{but } y^4 + x^4 = 16$$

$$\frac{d^2y}{dx^2} = -\frac{48x^2}{y^7}$$

Or

$$12x^2 + 4\left(3y^2\left(\frac{dy}{dx}\right)^2 + y^3\frac{d^2y}{dx^2}\right) = 0$$

$$12x^2 + 12y^2\left(\frac{dy}{dx}\right)^2 + 4y^3\frac{d^2y}{dx^2} = 0$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right)^2 + y^3\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2}{y^3} - \frac{3y^2}{y^3}\left(\frac{dy}{dx}\right)^2 = -\frac{3x^2}{y^3} - \frac{3}{y}\left(\frac{x^6}{y^6}\right) = -\frac{3x^2}{y^3} - \frac{3x^6}{y^7} = \frac{-3x^2y^4}{y^7} - \frac{3x^6}{y^7} = \frac{-3x^2(y^4 + x^4)}{y^7} = \frac{-3x^2(16)}{y^7} = \frac{-48x^2}{y^7}$$

Examples

1. If $4x^2 - 2y^2 = 9$, find $\frac{d^2y}{dx^2}$

Solution:

$$8x - 4y\frac{dy}{dx} = 0;\quad \frac{dy}{dx} = \frac{8x}{4y} = \frac{2x}{y};$$

$$8 - 4\left\{y\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx}\right\} = 0$$

$$8 - 4y\frac{d^2y}{dx^2} - 4\left(\frac{dy}{dx}\right)^2 = 0$$

$$-4y\frac{d^2y}{dx^2}=4\left(\frac{dy}{dx}\right)^2-8$$

$$\frac{d^2y}{dx^2}=\frac{8-4\left(\frac{dy}{dx}\right)^2}{4y}$$; But $\frac{dy}{dx}=\frac{2x}{y}$,

$$\frac{d^2y}{dx^2}=\frac{8-4\left(\frac{2x}{y}\right)^2}{4y}=\frac{8-\frac{16x^2}{y^2}}{4y}=\frac{8y^2-16x^2}{4y^3}=\frac{2y^2-4x^2}{y^3}$$

Or $\frac{dy}{dx}=\frac{2x}{y}$,

$$\frac{d^2y}{dx^2}=\frac{2y-2x\cdot\frac{dy}{dx}}{y^2}=\frac{2y-2x\left(\frac{2x}{y}\right)}{y^2}=\frac{2y-\frac{4x^2}{y}}{y^2}=\frac{2y^2-4x^2}{y^3}$$

2. $x^2+3xy-y^2=3$; Find $\frac{d^2y}{dx^2}$ at (1,1).

Solution:

$$2x+3\left[x\frac{dy}{dx}+y\right]-2y\frac{dy}{dx}=0$$

$$2x+3x\frac{dy}{dx}+3y-2y\frac{dy}{dx}=0;\quad \frac{dy}{dx}=\frac{-2x-3y}{3x-2y}$$

$$2+3\left[x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right]+3\frac{dy}{dx}-2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=0$$

$$\frac{d^2y}{dx^2}=\frac{2\left(\frac{dy}{dx}\right)^2-\frac{6dy}{dx}-2}{3x-2y}$$; But $\frac{dy}{dx}=\frac{-2x-3y}{3x-2y}=\frac{dy}{dx}=\frac{-2(1)-3(1)}{3(1)-2(1)}=\frac{-5}{1}=-5$

$$\frac{d^2y}{dx^2}=\frac{2\left(-5\right)^2-6(-5)-2}{3(1)-2(1)}=50+30-2=78$$

3. $3x^2+5xy+4y^2-4y=0$. Find $\frac{d^2y}{dx^2}$ at (0,1).

Solution:

$$6x+5(x\frac{dy}{dx}+\frac{dy}{dx})+8y\frac{dy}{dx}-4\frac{dy}{dx}=0$$

$$6x+5x\frac{dy}{dx}+5\frac{dy}{dx}+8y\frac{dy}{dx}-4\frac{dy}{dx}=0$$

$$6+5(x\frac{d^2y}{dx^2}+\frac{dy}{dx})+5\frac{d^2y}{dx^2}+8[y\frac{dy}{dx}+(\frac{dy}{dx})^2]-4\frac{d^2y}{dx^2}=0$$

$$5x\frac{d^2y}{dx^2}+5\frac{d^2y}{dx^2}-4\frac{d^2y}{dx^2}=-6-5\frac{dy}{dx}-8y\frac{dy}{dx}-8(\frac{dy}{dx})^2$$

$$5x\frac{d^2y}{dx^2}+\frac{d^2y}{dx^2}=-6-5\frac{dy}{dx}-8y\frac{dy}{dx}-8(\frac{dy}{dx})^2$$

$$\frac{d^2y}{dx^2}=\frac{-6-5\frac{dy}{dx}-8y\frac{dy}{dx}-8(\frac{dy}{dx})^2}{5x+1}$$

But $5x\frac{dy}{dx}+5\frac{dy}{dx}+8y\frac{dy}{dx}-4\frac{dy}{dx}=-6x \Rightarrow \frac{dy}{dx}=\frac{-6x}{5x+5+8y-4}=\frac{dy}{dx}=\frac{-6x}{5x+8y+1}=\frac{-6(0)}{5(0)+8(1)+1}=0$

$$\frac{d^2y}{dx^2}=\frac{-6-5(0)-8(1)(0)-8(0)^2=0}{5(0)+1}$$

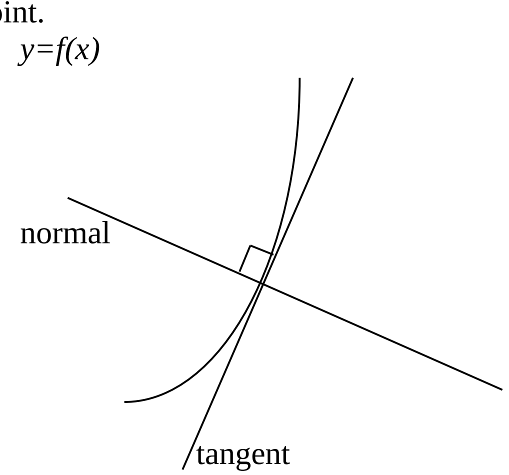
Exercise

1. Find y', y'', y''' where a) $y=\frac{x}{1-x}$ b) $y=\sqrt{x^2+1}$ c) $x^3+y^3=1$ d) $x^2+6xy+y^2=8$ e) $\sqrt{x}+\sqrt{y}=1$
2. Let $9y=x^3+3x+1$. Show that $y'''+xy''-2y'=0$
3. Let $y=\frac{1}{x}\left(x\neq 0\right)$.Show that $x^3y''+x^2y'-xy=0$

APPLICATIONS OF DIFFERENTIATION

EQUATIONS OF TANGENTS AND NORMALS

Definition: A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point.



Finding the equations of tangents and normals.

Examples

1. Find the equation of the tangent to the curve $y = x^3$ at the point (2,8).

Solution:

$y = x^3$; $\therefore$ gradient of y or $\frac{dy}{dx} = 3x^2$

When $x=2$; $\frac{dy}{dx} = 3\times 2^2 = 3\times 4 = 12$

Thus the gradient of the tangent at (2, 8) is 12.

But gradient $= \frac{\Delta y}{\Delta x} = \frac{y-8}{x-2} = 12$;

$y - 8 = 12(x-12)$

$y - 8 = 12x - 24$

$y = 12x - 24 + 8$

$y = 12x - 16$ which is the equation of the tangent.

2. Find the equation of the normal to the curve $y = (x^2 + x + 1)(x-3)$ at the point where it cuts the x-axis.

Solution:

$y = (x^2 + x + 1)(x-3)$. At the x-axis $y=0$.

When $y=0$, $(x^2 + x + 1)(x-3) = 0$; $x - 3 = 0$; $x = 3$.

Or $x^2 + x + 1 = 0$; $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2}$ (No real roots).

Hence $x = 3$ and therefore the curve cuts the x-axis at (3,0).

$\therefore$ gradient $\left(\frac{dy}{dx}\right) = (2x+1)(x-3) + 1(x^2 + x + 1) = 2x^2 - 6x + x - 3 + x^2 + x + 1$

$\frac{dy}{dx} = 3x^2 - 4x - 2$; When $x=3$, $\frac{dy}{dx} = 3(3^2) - 4(3) - 2 = 27 - 12 - 2 = 13$

The gradient of the tangent at (3,0) is 13, $\therefore$ the gradient of the normal at (3,0) is $-\frac{1}{13}$ (since for perpendicular lines with gradients m_1 and m_2, $m_1 \times m_2 = -1$)

$\frac{\Delta y}{\Delta x} = \frac{y-0}{x-3} = -\frac{1}{13}$; $13y = -x + 3$; $y = -x/13 + \frac{3}{13}$ is the equation of the normal.

3. Find the slope (gradient)of the tangent to the curve $x^2 + xy + y^2 = 7$ at the point (1,2).

Solution:

$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$; $(x + 2y)\frac{dy}{dx} = -2x - y$; $\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$.

At $x = 1, y = 2$, $\frac{dy}{dx} = \frac{-2(1) - 2}{1 + 2(2)} = \frac{-4}{5}$.

4. Find the equation of the tangent and a normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

Solution:

Equation of the tangent

$$2x+y+x\frac{dy}{dx}-2y\frac{dy}{dx}=0\ ;\ (x-2y)\frac{dy}{dx}=-2x-y\ ;\ \frac{dy}{dx}=\frac{-2x-y}{x-2y}=\frac{-2(2)-3}{2-2(3)}=\frac{7}{4}$$ at the point $x=2,\ y=3$.

Equation of the normal

Recall: For perpendicular lines with gradients m_1 and m_2, $m_1\times m_2=-1$

$\therefore$ The gradient of the normal at the point (2, 3)is $-\frac{4}{7}$.

$$\frac{\Delta y}{\Delta x}=\frac{y-3}{x-2}=-\frac{4}{7};\quad 7(y-3)=-4(x-2);\quad 7y-21=-4x+8;\ 7y=-4x+8+21;\quad y=\frac{-4x}{7}+\frac{29}{7}$$

$$=\frac{1}{7}[29-4x]$$

5. Find the normal to a curve $3xy+2y^2-x^3=0$ at the point (1,2).

   Solution:

$$3y+3x\frac{dy}{dx}+4y\frac{dy}{dx}-3x^2=0;\quad (3x+4y)\frac{dy}{dx}=3x^2-3y;\ \frac{dy}{dx}=\frac{3x^2-3y}{3x+4y}$$

At x=1, y=2, $\frac{dy}{dx}=\frac{3-6}{3+8}=\frac{-3}{11}$; $\therefore$ Gradient of the normal to the curve at (1,2) is $\frac{11}{3}$.

$$\frac{\Delta y}{\Delta x}=\frac{y-2}{x-1}=\frac{11}{3};\quad 3y-6=11x-11;\ 3y=11x-5;\quad y=\frac{11x}{3}-\frac{5}{3}$$

6. Find the equations of the tangent and the normal to the curve $x^2+2xy-y^2=4$ at the point $(2,4)$.

   Solution:

$$2x+2y+2x\frac{dy}{dx}-2y\frac{dy}{dx}=0;\quad (2x-2y)\frac{dy}{dx}=-2x-2y;\ \frac{dy}{dx}=\frac{-2x-2y}{2x-2y}=\frac{-4-8}{4-8}=\frac{-12}{-4}=3\text{at }(2,4).$$

$\therefore$ the gradient of the tangent line at $(2,4)$ is 3.

$\therefore$ gradient of the normal to the curve is $-\frac{1}{3}$.

$$\frac{\Delta y}{\Delta x}=\frac{y-4}{x-2}=-\frac{1}{3};\ 3y-12=-x+2;\ 3y=-x+14;\ y=\frac{1}{3}[14-x]$$

7. The parametric equations of a curve are $x=t^2-4$ and $y=t^3-4t$. Find the equation of the tangent to the curve at the point (-3,3).

   Solution:

$$\frac{dx}{dt}=2t\quad ,\frac{dy}{dt}=3t^2-4;\ \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{3t^2-4}{2t}.$$ But $x=t^2-4,\quad y=t^3-4t$.

When x=-3, $-3=t^2-4,\ t^2=1;\ t=\pm1$

When y=3, $3=t^3-4t;\ t^3-4t-3=0;\ when\ t=-1,\ -1+4-3=0$ and therefore *(t+1)* is a factor of t^3-4t-3.

$$\begin{array}{r|l} & t^2 - t - 3 \\ \hline (t+1) & t^3 - 4t - 3 \\ & \underline{t^3 + t^2} \\ & 0 - t^2 - 4t \\ & \underline{-t^2 - t} \\ & -3t - 3 \\ & \underline{-3t - 3} \\ & 0 \end{array}$$

$(t+1)(t^2-t-3)=0$; $t=-1$ or $t^2-t-3=0$; $t=\frac{1\pm\sqrt{1+12}}{2}=\frac{1\pm\sqrt{13}}{2}$

At t=-1, $\frac{dy}{dx}=\frac{3-4}{2(-1)}=\frac{1}{2}$;

$\frac{\Delta y}{\Delta x}=\frac{y-3}{x+3}=\frac{1}{2}$; $2(y-3)=x+3$; $2y-6=x+3$; $2y=x+9$; $y=\frac{x}{2}+\frac{9}{2}$;

Exercise

1. Find the equation of a tangent and normal to the curve $x^2+y^2-6xy+3x-2y+5=0$ at a point (3,0).

2. Find the equation of the tangent and normal to the curve $x=\frac{t}{1+t}$, $y=\frac{t^3}{1+t}$ at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

SMALL CHANGES

Recall:

$\frac{d}{dx}=f'(x)=\lim\limits_{\delta x\to o}\frac{f(x+\delta x)-f(x)}{\delta x}$ approaches the tangent line.

$\therefore$ if δx is small, then we say that $\frac{\delta y}{\partial x}\approx\frac{dy}{dx}$ $\Rightarrow \delta y\approx\frac{dy}{dx}\cdot\delta x$

This approximation can be used to estimate the value of a function close to a known value.i.e $y+\delta y$ can be approximated if y is known.

Examples

1. Use $y=\sqrt{x}$ to approximate the value of $\sqrt{1\cdot 1}$.

   Solution:

Known value $\sqrt{1}=1$.

From $\sqrt{1\cdot 1}=\sqrt{1+0\cdot 1}$, $x=1, \delta x=0\cdot 1$

$y=\sqrt{x}$; $\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

From $\delta y\approx\frac{dy}{dx}\cdot\delta x \approx \frac{1}{2\sqrt{x}}\cdot\delta x \approx \frac{1}{2\sqrt{1}}\times 0\cdot 1 \approx 0\cdot 05$ $\therefore \delta y\approx 0\cdot 05$.

$\therefore\sqrt{1\cdot 1}\approx y+\delta y \approx \sqrt{1}+0\cdot 05$; $\sqrt{1\cdot 1}\approx 1\cdot 05$

2. Approximate $\ln 1\cdot 1$

   Solution:

   Known value $=\ln 1=0$

Let $y=\ln x$; $x=1$, $\delta x=0\cdot 1$

$\frac{dy}{dx}=\frac{1}{x}$; But $\delta y \approx \frac{dy}{dx}\cdot\delta x \approx \frac{1}{x}\cdot\delta x \approx \frac{1}{1}\times 0\cdot 1 \approx 0\cdot 1$

$\therefore \ln(1\cdot 1) \approx y+\delta y \approx \ln x+\delta y \approx \ln 1+\delta y \approx 0+0\cdot 1 \approx 0\cdot 1 \quad \therefore \ln 1\cdot 1 \approx 0\cdot 1$.

3. Approximate $\sqrt{101}$.

Solution:

Known value=100

Let $y=\sqrt{x}$, $x=100$, $\delta x=1$

$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$; But $\delta y \approx \frac{dy}{dx}\cdot\delta x \approx \frac{1}{2\sqrt{100}}\times 1 \approx \frac{1}{20}$

$\therefore \sqrt{101} \approx y+\delta y \approx \sqrt{x}+\frac{1}{20} \approx \sqrt{100}+\frac{1}{20}=10+0\cdot 05; \quad \therefore \sqrt{101}=10\cdot 05$.

4. By taking $1^0=0\cdot 0175$ radians, approximate $\sin 29^0$.

Solution:

Known value $\sin 30^0=\frac{1}{2}$; Let $y=\sin x$; $x=30^0$; $\delta x=-1$

$\frac{dy}{dx}=\cos x$; $\delta y \approx \frac{dy}{dx}\cdot\delta x \approx \cos x\cdot\left(-1^0\right)$

$\approx \cos 30^0\times\left(-1^0\right)$; But $-1^0=-0\cdot 0175$ radians,

$\delta y=\frac{\sqrt{3}}{2}(-0\cdot 0175) \approx -\frac{\sqrt{3}}{2}(0\cdot 0175)$

$\therefore \sin 29^0 \approx y+\delta y \approx \sin x+\delta y \approx \sin 30+\delta y \approx \frac{1}{2}-0\cdot 015 \approx 0\cdot 4848$

5. Approximate $\sqrt[3]{65}$.

Solution:

Known value $=\sqrt[3]{64}=4$

Let $y=3\sqrt{x}$, $x=64$; $\delta x=1$

$y=x^{\frac{1}{3}}$; $\frac{dy}{dx}=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3x^{\frac{2}{3}}}$; $\partial y \approx \frac{dy}{dx}\times\delta x \approx \frac{1}{3x^{\frac{2}{3}}}\times 1 \approx \frac{1}{3\left(\sqrt[3]{64}\right)^2}\times 1=\frac{1}{16\times 3}=\frac{1}{48}$;

$\therefore \sqrt[3]{65}=y+\delta y \approx \sqrt[3]{64}+\delta y \approx 4+\frac{1}{48} \approx 4\cdot 021$

6. The side of a square is 5cm. Find the increase in the area of the square when the side expands by 0.01cm.

Solution:

Let the area of the square be $A\text{cm}^2$ when the side is x cm.

Then $A=x^2$.

Now, $\delta A \approx \frac{dA}{dx}\delta x \quad x=5; \quad \delta x=0\cdot 01$

$A=x^2$; $\frac{dA}{dx}=2x \quad \therefore \delta A \approx 2x(0\cdot 01) \approx 2\times 5(0\cdot 01) \approx 0\cdot 1$

$\therefore$ the increase in the area is $\approx 0\cdot 1$

7. Find approximation for $\sqrt{9\cdot 01}$

Solution:

Known value $= \sqrt{9} = 3$

Let $y = \sqrt{x}$, $x = 9;\ \delta x = 0\cdot 01$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}};\quad \delta y \approx \frac{dy}{dx}\cdot\delta x \approx \frac{1}{2\sqrt{x}}\times 0\cdot 01 \approx \frac{1}{2\sqrt{9}}\times 0\cdot 01 \approx \frac{1}{6}\times 0\cdot 01 = \frac{1}{600}$$

$$\therefore \sqrt{9\cdot 01} = y + \delta y \approx \sqrt{x} + \delta y \approx \sqrt{9} + \frac{1}{600} \approx 3 + \frac{1}{600} \approx 3\cdot 00167$$

8. Given that $\sin 60^0 = 0\cdot 86605, \cos 60^0 = 0\cdot 50000$,and $1^0 = 0\cdot 001745$ radians, Use $\frac{\delta y}{\partial x} \approx \frac{dy}{dx}$ to calculate the value of $\sin 60\cdot 1^0$ correct to 5.d.p.

Solution:

$y = \sin x$, $x = 60^0;\ \delta x = 0\cdot 1^0$

$$\frac{dy}{dx} = \cos x;\ \delta y \approx \frac{dy}{dx}\cdot\delta x \approx \cos x(0\cdot 1)^0 \approx \cos 60^0(0\cdot 0001745) \approx (0\cdot 5)(0\cdot 0001745) \approx 0\cdot 00008725$$

$$\therefore \sin\left(60\cdot 1^0\right) \approx y + \delta y \approx \sin x + \delta y \approx \sin 60^0 + (0\cdot 5)(0\cdot 0001745) \approx 0\cdot 86605 + (0\cdot 5)(0\cdot 0001745)$$

$$= 0\cdot 86613725$$

Assignment

ATTEMPT ALL QUESTIONS

1. (a)Use the linear approximation formula to approximate $(626)^{\frac{3}{4}}$.

(b)Find $\frac{dy}{dx}$ if (i) $y = \tan^{-1}\left(x^2 + 1\right)$ (ii) $y = \sin^{-1} x$

(c)Find $\frac{dy}{dx^2}$ and $\frac{d^2y}{dx^2}$, given $xy + x - 2y - 1 = 0$.

2.(a)If $x = \cos t$ and $y = 1 - \sin^2 t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b)Use logarithmic differentiation to evaluate $\frac{dy}{dx}$ if

(i) $y = \frac{\sin x \cos x \tan^3 x}{\sqrt{x}}$ (ii) $y = \frac{\left(x^2 + 1\right)\cot x}{3 - \cot x}$

(c)Find the equation of a tangent and normal to the curve $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$ at the point (3, 0).

3.(a)Differentiate $f(x) = \cot x$ from first principles.

(b)Find $\frac{dy}{dx}$ if (i) $y = 4^x$ (ii) $y = \ln(\cot x - \cos ecx)$ (iii) $y = x\sin^{-1}(3x) - \sqrt{1 - 9x^2}$ (iv) $y = \ln\left(\frac{1 + \sin x}{1 - \sin x}\right)^{\frac{1}{2}}$

RATE OF CHANGE

The identity $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$ is useful in solving certain rate of change problems.

$\frac{dy}{dt}$ =rate of change of y w.r.t. time t; $\frac{dx}{dt}$ =rate of change of x w.r.t time t.

Examples:

1. A spherical balloon is blown up so that its volume increases at a constant rate of $2cm^3$ per second. Find the rate of increase of a radius when the volume of the balloon is $50cm^3$

Solution:

Volume of a sphere $=\frac{4}{3}\pi r^3$; $\frac{dv}{dt}=+2$. Volume $=50cm^3$

We are required to find $\frac{dr}{dt}$; $\quad \frac{dr}{dt}=\frac{dr}{dv}\times\frac{dv}{dt}$

$v=\frac{4}{3}\pi r^3$; $\quad \frac{dv}{dr}=3r^2\times\frac{4}{3}\pi=4\pi r^2$;

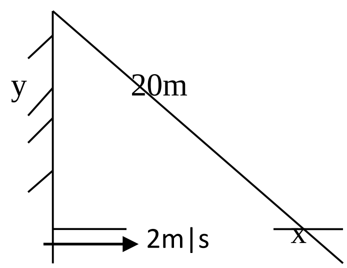
$\frac{dr}{dt}=\frac{dr}{dv}\times\frac{dv}{dt}=\frac{1}{4\pi r^2}\times 2=\frac{1}{2\pi r^2}$;

But what is r when v= $50cm^3$; v $=\frac{4}{3}\pi r^3$; $\quad 50=\frac{4}{3}\pi r^3$; $r^3=50\times\frac{3}{4}\times\frac{1}{\pi}$; $\quad r=\sqrt[3]{50\times\frac{3}{4}\times\frac{7}{22}}=2\cdot 29$

$\therefore \frac{dr}{dt}=\frac{1}{2\pi r^2}=\frac{1}{2\pi(2\cdot 29)^2}=0\cdot 03cm|s$

2. A ladder 20m long leans against a vertical wall. If the bottom of the ladder slides away from the building horizontally at a rate of 2m|s, how fast is the ladder sliding down the building when the top of the ladder is 12m above the ground?

   Solution:



$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$; $\quad$ Let y be the distance from the ground to the top of the ladder.

Given $\frac{dx}{dt}=+2$; $\quad x^2+y^2=400$; $\quad 2x+2y\frac{dy}{dx}=0$; $\quad \frac{dy}{dx}=-\frac{2x}{2y}=-x/y$.

When y=12, $x^2=400-144$; $\quad x^2=256$; $\quad x=16$.

When $x=16$, and $y=12$

$\frac{dy}{dx}=-\frac{16}{12}=-4/3$; $\therefore \frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}=-4/3\times 2;=-8/3$

Or

$\sqrt{20^2-12^2}=\sqrt{400-144}=\sqrt{256}=16$

$\frac{dy}{dt}=?$; $\frac{dx}{dt}=2$; $\quad y^2+x^2=20$; $\quad 2y\frac{dy}{dx}+2x=0$; $\quad \frac{dy}{dx}=-\frac{x}{y}=\frac{-x}{\sqrt{20x^2}}$

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt};\quad =\frac{-x}{\sqrt{20^2-x^2}}\cdot 2\ ;\quad =\frac{-2x}{\sqrt{20-x^2}}=\frac{-2\left(\overset{4}{\cancel{16}}\right)}{\underset{3}{\cancel{12}}}=\frac{-8}{3}$$

3. A container in the shape of a right circular cone of height 10cm and base radius 1cm is catching the drips from a tap leaking at a rate of $0\cdot 1\text{cm}^2 \mid s$ .Find the rate at which the surface area of water is increasing when the water is half-way up the cone.

   Solution:

$$\frac{dv}{dt}=\frac{dv}{dh}\cdot\frac{dh}{dt};\quad v=\frac{1}{3}\pi r^2 h;\quad v=\frac{1}{3}\pi(1)^2(10)=\frac{10\pi}{3}$$

$\frac{dv}{dt}=0\cdot 1\text{cm}^3 \mid s$. We need to find $\frac{dA}{dt}$ when $h=5$.

$$\frac{dA}{dt}=\frac{dA}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dv}{dr}\cdot\frac{dr}{dt}=\frac{dv}{dt}$$

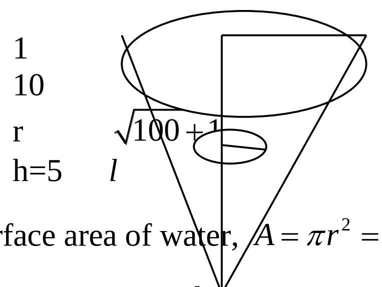
Circumference = $\frac{x}{360^0}\times 2\pi r=1$

$$\frac{x}{360^0}\times 2\times\frac{22}{7}\times 10=1;\quad 440x=360\times 7;\quad x=\frac{360\times 7}{440}=\frac{63}{11};$$

$$\frac{dA}{dt}=\frac{dA}{dh}\cdot\frac{dh}{dt}$$

$$\therefore A=\frac{63/11}{360}\times\pi h^2$$

$$\frac{dA}{dh}=\frac{63/11}{360}\times\frac{22}{7}\times 2h\ =\frac{63}{11\times 360}\times\frac{22}{7}=\frac{1}{20}\times 2h=\frac{h}{10};\qquad \therefore\frac{dA}{dt}=\frac{dA}{dh}\cdot\frac{dh}{dt}=\frac{h}{10}$$



the surface area of water, $A=\pi r^2=\frac{\pi h^2}{100}$

and we wish to find $\frac{dA}{dt}$ when $h=5$ (half-way)

$$\frac{dA}{dt}=\frac{dA}{dh}\times\frac{dh}{dt}=\frac{2\pi h}{100}\times\frac{dh}{dt}\ldots(1)$$

The volume of water, $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{h}{10}\right)^2 h$

$$V=\frac{\pi h^3}{300}$$

$$\frac{dx}{dh} = \frac{3}{300}\pi h^2$$

But $\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$

$$\frac{dv}{dt} = \frac{3}{300}\pi h^2 \times \frac{dh}{dt}$$

But we are given that $\frac{dv}{dt} = 0 \cdot 1$

$$0 \cdot 1 = \frac{3}{300}\pi h^2 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{0 \cdot 1 \times 300}{3\pi h^2} = \frac{10}{\pi h^2} \ldots (2)$$

From 1 and 2

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt} = \frac{2\pi h}{100} \times \frac{10}{\pi h^2} = \frac{1}{5h}$$

$\frac{dA}{dt} = \frac{1}{5h}$ where h=5

$$\frac{dA}{dt} = \frac{1}{25h} = 0 \cdot 04$$

Therefore when the water is half-way, the rate of change of the surface area is equal to $0 \cdot 04\text{cm}^2 / s$.

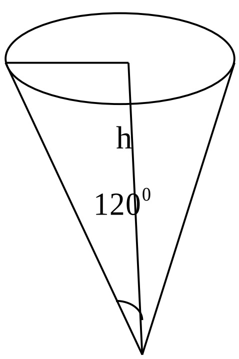
4. An inverted right circular cone of vertical angle 120^0 is collecting water from a tap at a steady rate of $18\pi\text{cm}^3 / \min$. Find

(a)The depth of the water after 12min

(b)The rate of increase of the depth at this instant

Solution:

(a) $1\min = 18\pi$; $\quad 12\min = 12 \times 18\pi = 216\pi = V \qquad \frac{dv}{dt} = 18\pi$



$$\tan 120^0 = \frac{r}{h}; \quad r = h\tan 120^0 ; \therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(h\tan 120^0\right)^2 h$$

$$= \frac{1}{3}\pi h^3 \tan^2 120^0 = \frac{1}{3} \times 3\pi h^3 ; \quad V = \pi h^3$$

$$\therefore \pi h^3 = 216\pi \text{ ; } h^3 = 216 ; \quad h = 6$$

$\frac{dh}{dt} = ?$ $\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$; $\quad$ But $\frac{dv}{dt} = 18\pi$; $v = \pi h^3$; $\frac{dv}{dh} = 3h^2\pi$;

$$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$$

$$18\pi = (3h^2\pi) \times \frac{dh}{dt} ; \; \frac{dh}{dt} = \frac{18\pi}{3h^2\pi} ; \; \frac{dh}{dt} = \frac{18}{3h^2} ; \quad \text{At } h = 6, \; \frac{dh}{dt} = \frac{18}{3 \times 36} = \frac{1}{6} .$$

KINEMATICS

VELOCITY AND ACCELERATION

The velocity *(v)* is instantaneous rate of change of position. The velocity of a moving particle can be positive or a negative, depending on whether the particle is moving in the positive or negative direction along a line of motion.

Suppose a particle moves along a horizontal straight line, with its location at time *t* given by its position function $s = f(t)$. Think of the time interval from *t* to $t + \Delta t$. The particle moves from position $f(t)$ to position $f(t + \Delta t)$ during this interval. Then *v* is given by

$$v = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{ds}{dt} = f^{'}(t)$$

Example: An object moving in a straight line has its displacement *s* meters from an origin 0 at time *t* seconds given by $s = t(t-3)^2$. Determine a)The time when the object is at the origin

b)The time when the object is at rest

c)The distance moved between *t=0* and *t=2*. Use $s = \sqrt{1 + \left(\frac{ds}{dt}\right)^2}$.

Solution:

a) The object will be at the origin when *s=0*

$$0 = t(t-3)^2; \;\; 0 = t\left(t^2 - 6t + 9\right); \quad t^3 - 6t^2 + 9t = 0; \quad t\left(t^2 - 6t + 9\right) = 0;$$

$$t = 0 \quad \text{or} \quad t^2 - 6t + 9 = 0; \quad (t-3)^2 = 0 \qquad\qquad t = 3\text{sec.}$$

b) $v = \frac{ds}{dt} = (t-3)^2 + 2(1)(t-3)^{'}t$

$$v = \frac{ds}{dt} = t^2 - 6t + 9 + 2t^2 - 6t = 3t^2 - 12t + 9; \;\; v = (t-3)(3t-3)$$

For max or min $v = 0$; $\quad (t-3)(3t-3) = 0$; *t=3* or *t=1*

The object is thus instantaneously at rest at *t=1* and *t=3* seconds.

(c)By second derivative

$$\frac{d^2v}{dt^2} = 6t - 12; \;\; \frac{d^2v}{dt^2}\Big|t = 3 \qquad = 18 - 12 = 6 > 0 \text{ minimum point.}$$

$$\frac{d^2v}{dt^2}\Big|t = 1 \qquad = 6 - 12 < 0 \qquad\qquad \text{maximum point}$$

When *t=3*, *s=0* when *t=0*, *s=0*

When *t=1*,*s=4* when *s=0*,*t=3*

When *t=1*, *s=4*(distance)

Between *t=0*, and *t=1*, the velocity is positive and the object moves from position *s=0* to $s = 1(1-3)^2 = 4$.

Between *t=1* and *t=3*, the velocity is negative and the object moves from position *s=4* to position *s=0*. Therefore the distance moved by the object between *t=0* and *t=2* will be given by 4(the positive difference between values of *s* at time *t=1*,*t=2* respectively).

when $t=1, s=4$

when $t=2, s=2$

$\therefore$ total distance is (4+2)=6metres.

Acceleration *(a)* at a time *t* is given by $a=\dfrac{dv}{dt}$.

Hence to determine the acceleration at time *t* differentiate *v* with respect to *t*.

Examples:

1. A particle is moving in a straight line and has its displacement *s* metres from the origin after *t* seconds given by $s=e^{-\sqrt{3}t}\sin t$. Determine its displacement, velocity and acceleration when $t=\dfrac{\pi}{2}$ and also the smallest positive value *t* for which the particle is at rest (i.e. *v=0).*

Solution:

$s=e^{-\sqrt{3}t}\sin t$; $\quad v=\dfrac{ds}{dt}=\cos te^{-\sqrt{3}t}+\left(-\sqrt{3}\right)e^{-\sqrt{3}t}\sin t$

$$a=\frac{dv}{dt}=-\sqrt{3}e^{-\sqrt{3}t}\cos t+(-\sin t)e^{-\sqrt{3}t}+\left(-\sqrt{3}\right)\left(-\sqrt{3}\right)e^{-\sqrt{3}t}\sin t+\cos t\left(-\sqrt{3}\right)e^{-\sqrt{3}t}$$

$$=-\sqrt{3}e^{-\sqrt{3}t}\cos t-\sin te^{-\sqrt{3}t}+3e^{-\sqrt{3}t}\sin t+\cos t\left(-\sqrt{3}\right)e^{-\sqrt{3}t}$$

$$a=-2\sqrt{3}e^{-\sqrt{3}t}\cos t+2e^{-\sqrt{3}t}\sin t$$

$\therefore$ at $t=\dfrac{\pi}{2}$, $\quad v=\left.\dfrac{ds}{dt}\right|t=\dfrac{\pi}{2}=-\sqrt{3}e^{-\sqrt{3}\frac{\pi}{2}}\sin\dfrac{\pi}{2}+e^{-\sqrt{3}\frac{\pi}{2}}\cos\dfrac{\pi}{2} \quad =-\sqrt{3}e^{-\sqrt{3}\frac{\pi}{2}}$

$a=\left.\dfrac{dv}{dt}\right|t=\dfrac{\pi}{2} \quad =2e^{-\sqrt{3}\frac{\pi}{2}}-2\sqrt{3}e^{-\sqrt{3}\frac{\pi}{2}}\cdot 0=2e^{-\sqrt{3}\frac{\pi}{2}}$

The displacement at $t=\dfrac{\pi}{2}$ is given by $\quad s=e^{-\sqrt{3}t}\sin t$; $\quad s=e^{-\sqrt{3}\frac{\pi}{2}}\sin\dfrac{\pi}{2}$; $\quad s=e^{-\sqrt{3}\frac{\pi}{2}}$ metres.

When the particle at rest *v=0*; $\quad \therefore v=-\sqrt{3}e^{-\sqrt{3}t}\sin t+e^{-\sqrt{3}t}\cos t=0$

$v=e^{-\sqrt{3}t}\left(-\sqrt{3}\sin t+\cos t\right)=0$; $\quad e^{-\sqrt{3}t}>0$ for all t

$-\sqrt{3}\sin t+\cos t=0 \quad ; \quad -\sqrt{3}\sin t=-\cos t$

$\dfrac{\sin t}{\cos t}=\dfrac{1}{\sqrt{3}}$ $\quad ;\tan t=\dfrac{1}{\sqrt{3}}$; $\quad t=\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)=30^0, 210^0$; $\quad \therefore$ the smallest is $t=30^0$.

2. A distance time graph is represented by the equation $s=2t^3-2t^2-3t$.

   Evaluate (a)The velocity at time *t*

   (b)The acceleration at time *t*

   (c)Show that the minimum distance over attained occurs went $t=\dfrac{2+\sqrt{22}}{6}$

Solution:

(a) $v=\dfrac{ds}{dt}=6t^2-4t-3$

(b) $a=\dfrac{dv}{dt}=12t-4$

(c) for minimum and maximum distance *v=0*

$v=6t^2-4t-3=0$; $\quad t=\dfrac{4\pm\sqrt{16+72}}{12} \quad =\dfrac{4\pm\sqrt{88}}{12}=\dfrac{4\pm 2\sqrt{22}}{12}$; $\quad t=\dfrac{2\pm\sqrt{22}}{6}$

$$\therefore t = \frac{2+\sqrt{22}}{r} \text{ or } t = \frac{2-\sqrt{22}}{6}$$

Using the second derivation test

$$a = \frac{dv}{dt} = 12t - 4; \quad \left.\frac{dv}{dt}\right| t = \frac{2-\sqrt{22}}{6} \quad = 12\left(\frac{2-\sqrt{22}}{6}\right) - 4 = -9\cdot 4 < 0 \text{ maximum}$$

$$\left.\frac{dv}{dt}\right| t = \frac{2+\sqrt{22}}{6} \quad = 12\left(\frac{2+\sqrt{22}}{6}\right) - 4 = 9\cdot 4 > 0 \text{ minimum}$$

Exercise

1. A particle moves along a straight line in such a way that after t seconds, its velocity is $v\text{ms}^{-1}$, where $v = t^2 - t + 2$. Find the acceleration the particle(a) after 2 seconds(b) after $\frac{1}{2}$ seconds.

2.The distance s metres that a particle has gone in t seconds is given by $s = 5t + 15t^2 - t^3$.Find the velocity and acceleration after (a)3seconds (b)6seconds .When is the acceleration zero?

3. After t seconds a particle has gone s metres where $s = t^3 - 6t^2 + 9t + 5$.

(a) After how many seconds is its velocity zero?

(b) When is its acceleration zero?

(c)Find its velocity and acceleration (i)initially (ii)after 4seconds

In summary, the motion of a particle p along a straight line is completely described by the equations $s = f(t)$, where $t > 0$ is the time and s is the distance of p from a fixed point 0 in its path.

The velocity of p at time t is $v = \frac{ds}{dt}$.

If $v=0$, p is instantaneously at rest. Acceleration of p at time t is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

CURVE SKETCHING

Asymptote: An asymptote is a straight line which the curve being studied approachs. Alternatively, an asymptote is a straight line to which the curve $y = f(x)$ approaches closer and closer as one moves along it. The asymptotes are vertical, horizontal and slant/oblique asymptotes.

1. Vertical asymptotes

These correspond to the zeroes of the denominator of a fraction.

Examples

(a) Find vertical asymptotes of $y = \frac{x^2 + 2x - 3}{x^2 - 5x - 6}$;

Solution: The denominator is $x^2 - 5x - 6$; Zero of denominator $x^2 - 5x - 6 = 0$.

$(x-6)(x+1) = 0$; $x=6$ or $x=-1$

So x cannot be 6 or -1 because in this case the denominator will be zero and dividing by zero will give an undefined value.

Therefore $x=6$ and $x=-1$ are the vertical asymptotes.

(b) Find vertical asymptotes of $y = \frac{x+2}{x^2 + 2x - 8}$

Solution: The denominator is $x^2 + 2x - 8 = 0$. Zero of denominator $x^2 + 2x - 8 = 0$

$(x+4)(x-2)=0$; $x=-4, \quad x=2$

2. **Horizontal asymptotes**

The line $y=b$ is a horizontal asymptote for $y=f(x)$ if $\lim\limits_{x\to+\infty} f(x)=b$ or $\lim\limits_{x\to-\infty} f(x)=b$

Examples

(a) Find Horizontal asymptotes of $y=\dfrac{x^2+2x-3}{x^2-5x-6}$

Solution:

$$\lim_{x\to\pm\infty}\frac{x^2+2x-3}{x^2-5x-6}=\lim_{x\to\pm\infty}\frac{1+\frac{2}{x}-\frac{3}{x^2}}{1-\frac{5}{x}-\frac{6}{x^2}}=1; \quad \therefore y=1 \text{ is the horizontal asymptote.}$$

(b) Find Horizontal asymptotes of $y=\dfrac{x+2}{x-1}$

Solution:

$$\lim_{x\to\infty}\frac{x+2}{x-1}=\lim_{x\to\infty}\frac{1+\frac{2}{x}}{1-\frac{1}{x}}=1 \quad \therefore y=1 \text{ is the horizontal asymptote.}$$

Or make x the subject of the formula; $(x-1)y=x+2$; $xy-y=x+2$; $xy-x=y+2$; $x(y-1)=y+2$

$x=\dfrac{y+2}{y-1}$ and equate $y-1=0$; $y=1$ which is the asymptote.

(c).Find vertical and horizontal asymptotes, if any, of $y=\dfrac{2x}{x+1}$ and sketch the curve.

Solution:

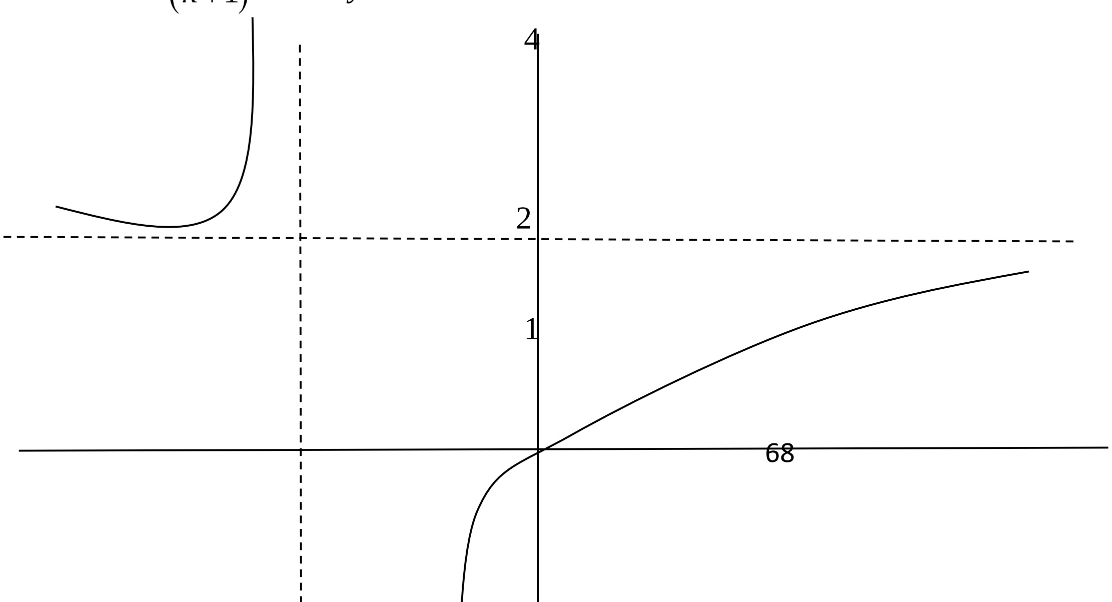
Vertical asymptote: $x+1=0$; $x=-1$ is the vertical asymptote

Horizontal asymptote: $\lim\limits_{x\to\pm\infty}\dfrac{2x}{x+1}=\lim\limits_{x\to 0}\dfrac{2}{1+\frac{1}{x}}=2$; $\therefore y=2$ is the horizontal asymptote

x-intercept, *y=0*, *x=0*; y-intercept, *x=0*, *y=0*

Turning points

$$\frac{dy}{dx}=\frac{2(x+1)-2x}{(x+1)^2}; \quad \frac{dx}{dy}=\frac{(x+1)^2}{2}=0; \ (x+1)^2=0; \ x=-1; \ y=\infty$$



-2 -1 1

3.Oblique or slant asymptote

If $y=\dfrac{h(x)}{g(x)}$ where the degree of $h(x)$ minus the degree of $g(x)$ equals to one, then $y=f(x)$ has an oblique asymptote of the form $y=mx+c$ where $y=f(x)=mx+c+\dfrac{a}{g(x)}$

Example

(a) Find all the asymptotes, if any,of $y=\dfrac{x^2-4}{x-1}$ and sketch the curve.

Solution: Vertical asymptote: $x-1=0;\ \ x=1$

Horizontal asymptote: $\lim\limits_{x\to\pm\infty}\dfrac{x^2-4}{x-1}=\lim\limits_{x\to\pm 0}\dfrac{1-\dfrac{4}{x^2}}{\dfrac{1}{x}-\dfrac{1}{x^2}}=\infty$; No limit hence no horizontal asymptote.

Oblique asymptote: We find by long division

$$\begin{array}{r|l} & \ \ \ x+1 \\ \hline x-1 & x^2-4 \\ & \underline{x^2-x} \\ & \ \ \ x-4 \\ & \ \ \ \underline{x-1} \\ & \ \ \ \ \ -3 \end{array} \qquad y=\frac{x^2-4}{x-1}=(x+1)+\frac{-3}{x-1};\ \therefore\ y=x+1$$

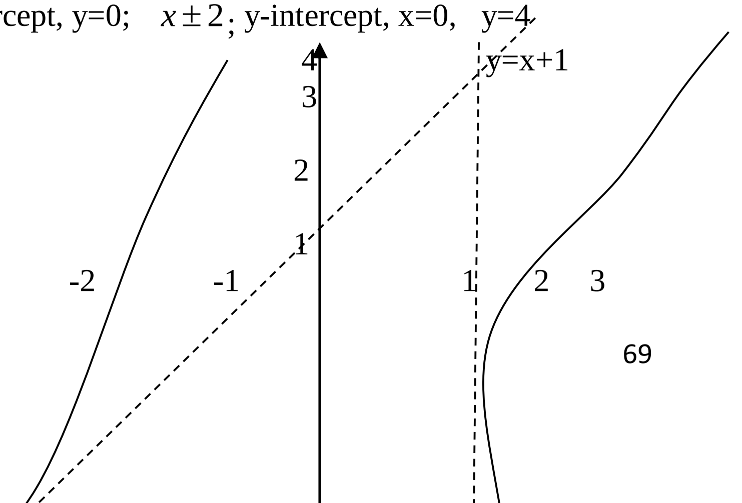
$\therefore\ y=x+1$ is the oblique asymptote

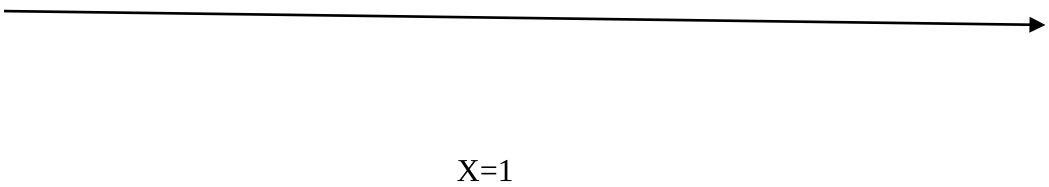
Turning points:

$y=\dfrac{x^2-4}{x-1}$; $\dfrac{dy}{dx}=\dfrac{(x-1)2x-(x^2-4)}{(x-1)^2}$ =0; $=\dfrac{2x^2-2x-x^2+4}{(x-1)^2}$ =0; $\dfrac{dy}{dx}=\dfrac{x^2-2x+4}{(x-1)^2}=0$; $y^{'}=x^2-2x+4=0$

$x=\dfrac{+2\pm\sqrt{4-16}}{2}$ no real roots so no turning points.

x-intercept, y=0; $x\pm 2$; y-intercept, x=0, y=4





(b) Determine the vertical, horizontal and oblique asymptotes if any of the function $y = f\left(x\right) = \dfrac{x^2+3x+6}{x-4}$ .Hence or otherwise sketch the graph.

Solution:

Vertical asymptote: *x-4=0* , *x=4*

Horizontal asymptote: $\lim\limits_{x\to\pm\infty} \dfrac{x^2+3x+6}{x-4} = \dfrac{1+\frac{3}{x}+\frac{6}{x^2}}{\frac{1}{x}-\frac{4}{x^2}} = \infty$ (undefined), therefore there is no horizontal asymptote.

Oblique asymptote $x-4\overline{\smash{\big)}x^2+3x+6}$ with quotient $x+7$: $\begin{array}{r} x^2+3x+6 \\ \underline{x^2-4x} \\ 7x+6 \\ \underline{7x-28} \\ 34 \end{array}$; $y=\left(x+7\right)+\dfrac{34}{x-4}$; *y=x+7* is the oblique asymptote

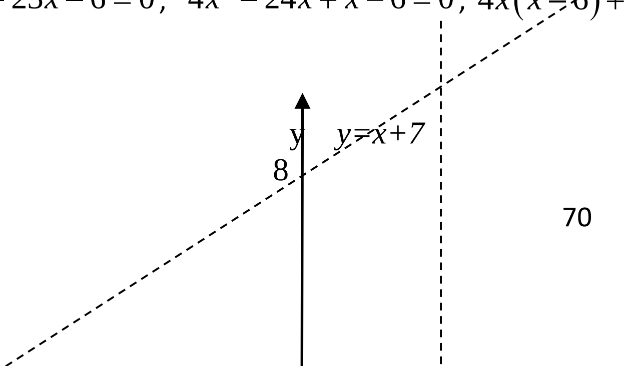
Turning points: $\dfrac{dy}{dx} = \dfrac{\left(2x+3\right)\left(x-4\right)-\left(x^2+3x+6\right)}{\left(x-4\right)^2}$ =0; $= \dfrac{x^2-8x-18}{\left(x-4\right)^2} = 0$; $x^2-8x-18=0$.

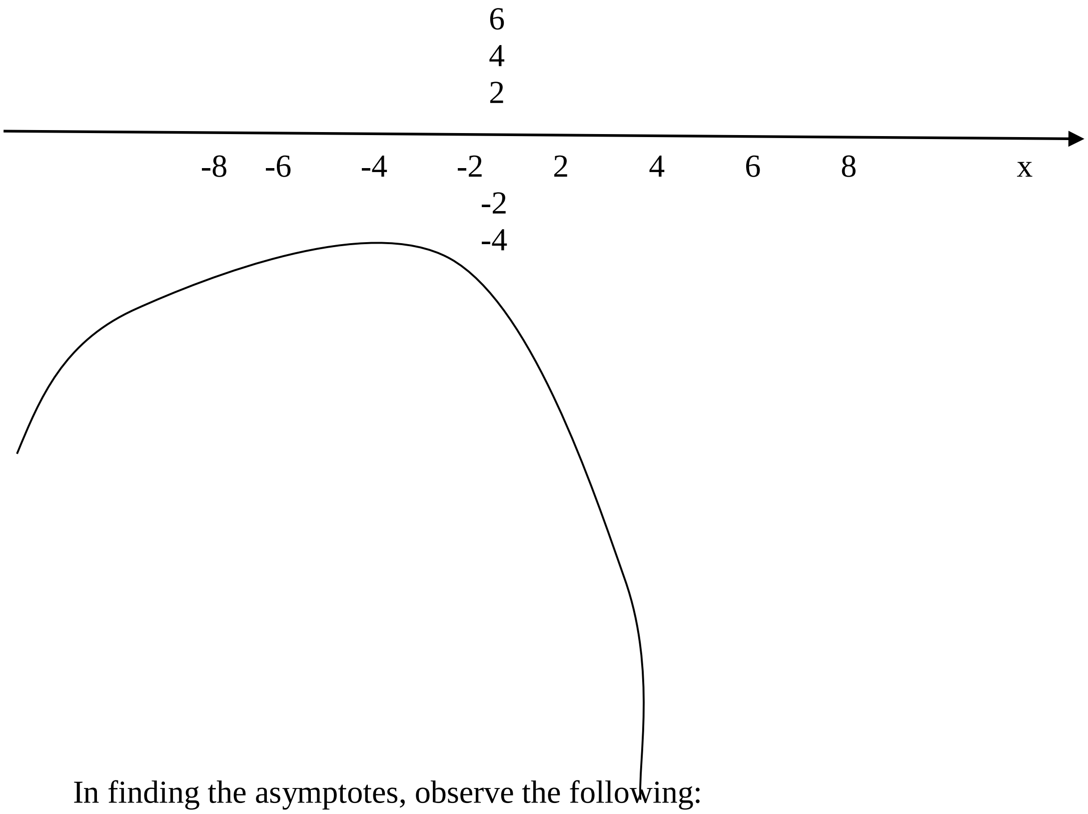
$$x = \frac{8\pm\sqrt{6++72}}{2}; \quad x = 9\cdot 8; \quad x = -1\cdot 8$$

Or

$$y = \frac{x^2+3x+6}{x-4}; \ \frac{dy}{dx} = \frac{\left(2x+3x\right)\left(x-4\right)-\left(x^2+3x+6\right)}{\left(x-4\right)^2} = 0; \ 2x^2-8x+3x^2-12x-x^2-3x-6=0$$

$$4x^2-23x-6=0; \ 4x^2-24x+x-6=0; \ 4x\left(x-6\right)+1\left(x-6\right)=0; \quad x=\frac{-1}{4}, x=6 \,|\, y=\frac{-5}{4}, y=30$$





In finding the asymptotes, observe the following:
1. To get vertical asymptotes, set the determinant equal to zero and solve for the zeros (if any).
2.Compare the degrees of the numerator and the denominator.
If the degrees are the same, then you have a horizontal asymptote.
If the degree of the denominator is greater than that of the numerator, then you have a horizontal asymptote at *y=0*
If the numerators degree is greater (by a margin 1), then you have a slant or oblique asymptote which you will find by long division.

SUMMARY OF CURVE SKETCHING
1. Determine the *y* and *x* intercepts
2. Determine the asymptotes of the graph if any.
3. Determine the turning points and distinguish them (i.e. check maximum, minimum and point of inflection).
4. Hence sketch the points.

Note:
A curve will have vertical asymptotes if, when its equation is written in the form
$$ay^n+(bx+c)y^{n-1}+\left(dx^2+ex+f\right)y^{n-2}+\ldots U_n(x)=0\ldots(1)$$
Where $U_n(x)$ is a polynomial in *x* of degree *n*.
The coefficient of the highest power of *y* is a non-constant function of *x* having one or more (real) linear factors. To each such factor, there corresponds a vertical asymptote.
A curve will have horizontal asymptotes if, when its equation is written in the form
$ax^n+(by+c)x^{n-1}+\left(dy^2+ey+f\right)x^{n-2}+=0$, the coefficient of the highest power of *x* is a non-constant function of *y* having one or more(real)linear factors. To such factors, there corresponds a horizontal asymptote.

To obtain the equations of the oblique asymptotes

1. Replace y by $mx+b$ in the equation of the curve and arrange the result in the form
$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$.**(3)**

2. Solve simultaneously the equations $a_0 = 0$ and $a_1 = 0$ for m and b.

For each pair of solutions m and b, write the equation of an asymptote $y = mx + b$.

3. If $a_1 = 0$, irrespective of the value of b, the equations $a_0 = 0$ and $a_2 = 0$ are to be used in (3).

Exercise: Find the equations of the asymptotes of $y^2(1+x) = x^2(1-x)$.