Basic Probability

Random variables:

Definition:

A real variable X whose values is determined by the outcome of a random experiment is called a random variable.

i.e A random variable X be a real valued function defined on the sample space 'S' of a random experiment such that.

for each point s of the sample space $X(s)$ is a real value 'x' in $\mathbb{R}$

$$X : S \longrightarrow \mathbb{R}$$

$X(s) =$ is a real value 'x'

$$X(S) = \{ x \in \mathbb{R} \,/\, X(s) = x,\ s \in S \}.$$

$X(S) =$ Range of 'X'

Types of Random Variables:

Random variables are two types.

(1) Discrete Random Variable:

A Random variable X which can take only a finite number of discrete values in an interval of domain is called a Discrete Random Variable.

(2) Continuous Random Variable:

A Random variable X which can take values continuously.

i.e which takes all possible values in given interval is called Continuous Random variable.

Examples: (Discrete Random variables)

(1) The number of defective bulbs in a sample of electric bulbs.

(2). The no of telephone calls received by the telephone operator

(3) The sum of numbers appearing on the faces of two dice

(4) No' of children in a family.

Examples: (continuous Random variables)

(1) The height, age and weight of individuals.

(2) temperature and time are continuous Random variables.

(3) The time taken to complete Examination.

(4) The price of a product.

Examples:

Random Experiment: two coins are tossed.

Sample space: $S = \{ s_1 = (H, H),\ s_2 = (H, T)$

$s_3 = (T, H),\ s_4 = (T, T) \}$

$n(S) = 4 = 2^2$.

Random variable is defined as

$$X : S \longrightarrow \mathbb{R} \text{ as}$$

$X(s) =$ number of heads.

Then $X(s_1) = 2,\ X(s_2) = 1\ \ X(s_3) = 1\ \ X(s_4) = 0$

Range of $X = \{ X(s) : s \in S \}$

$= \{ X(s_1), X(s_2)\ X(s_3)\ X(s_4) \}$

$= \{0, 1, 2\}$

$\therefore X(S) = \{0, 1, 2\}$.

Example:

Random Experiment:

throwing a pair of fair dice

Sample space:

$S = \{ s_1(1,1)\ \ s_2(1\ 2)\ \ s_3(1\ 3)\ \cdots\ s_6(1,6)$

$s_7(2,1)\ \ s_8(2,2)\ \ \cdots\ s_{12}(2,6)$

$\vdots$

$s_{31}(6,1)\ \ s_{32}(6\ 2)\ \ \cdots\ s_{36}(6,6) \}$.

Note:

2/1 3/2 4/3 5/4 6/5 7/6

S =

(1,1)	(1 2)	(1 3)	(1 4)	(1 5)	(16)
(2 1)	(2 2)	(2 3)	(2 4)	(25)	(26)
(3 1)	(3 2)	(3 3)	(3 4)	(35)	(36)
(4 1)	(4 2)	(4 3)	(4 4)	(4 5)	(46)
(5 1)	(52)	(53)	(5 4)	(55)	(5,6)
(6,1)	(6 2)	(6 3)	(6 4)	(65)	(6,6)

8/5 9/4 10/3 11/2 12/1

Random variable is defined as.

$X : S \longrightarrow \mathbb{R},$

$X(s)$ = sum of numbers appearing on the faces of dice.

$X((a,b)) = a+b. \quad \forall\ (a,b) \in S.$

$X((2,3)) = 2+3 = 5,$

$X((5,4)) = 5+4 = 9$

$\therefore X(S) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$

$n(S) = 36, \quad n(X(S)) = 11$

$n(S)$ ↓ size of the Domain

$n(X(S))$ ↓ size of the Range.

Probability Mass function (Pmf):

If X is discrete random variable. The function given by

$$f(x) = P(X=x) = p_X(x)$$

for each 'x' within the range of X is called the Probability mass function.

$f(x)$ is the probability of occurrence of the event represented by x.

Example:

Random Experiment:

throwing a pair of dice.

Random variable.

$X((a,b)) = a+b$ = sum of numbers appearing on the faces

Probability mass function:

$$f(x) = P(X=x)$$

Ex. → $f(3) = P(X=3)$

= Probability of getting sum=3

$= \frac{2}{36} = \frac{1}{18}$.

→ $f(7) = P(X=7)$

= Probability of getting sum=7

$= \frac{6}{36} = \frac{1}{6}$.

Properties of Probability mass function:

1. $P(x) > 0 \quad \forall x \in X(S)$

2. $\sum P(x) = 1$

3. $P(x) = 0$ for all other x.
   i.e $x \notin X(S)$.

4. $0 \le P(x) \le 1$.

Example:

Random Experiment: tossed three coins

Sample space: $S = \{(HHH)\ (HHT)\ (HTH)\ (HTT)$
$(THH)\ (THT)\ (TTH)\ (TTT)\}$
$n(S) = 8 = 2^3$

Random variable: $X(s)$ = number of Heads.

$X(S) = \{0, 1, 2, 3\}$.

Probability mass function:

$p_X(x) = P(X = x) = f(x)$

= Probability of getting Exactly 'x' heads.

$P(0)$ = Probability of getting '0' Heads $= \frac{1}{8}$

$P(1)$ = Probability of getting '1' Head $= \frac{3}{8}$

$P(2) = \frac{3}{8}, \quad P(3) = \frac{1}{8}$.

Note: * $P(x) > 0 \ \forall x \in X(S)$, $\quad 0 \le P(x) \le 1$,

* $\Sigma P(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

Discrete Probability Distribution:

Suppose x is a discreat Random variable with Possible out comes (values) $x_1, x_2 \dots x_n \dots$ The Probability of each Posible outcome x_i is

$p_i = P(x = x_i) = P(x_i)$ for $i = 1, 2, 3, \dots$

satisfies two conditions.

(i) $P(x_i) \geq 0 \quad \forall i$

(ii) $\sum_i P(x_i) = 1$ (Total Probability (or) sum of the Probabilities $= 1$)

The set $\{P(x_i)\}_i$ is called the discreate Probability distribution.

<u>Note:</u> where $P(x)$ is probability mass function.

→ The Probability distribution of the discrete randomvariable X ~~it~~ is given by means of the following table.

X	x_1	x_2	x_3	....	x_n
$P(X=x)$	$p_1 = P(X=x_1)$	$p_2 = P(X=x_2)$	$p_3 = P(X=x_3)$	...	$p_n = P(X=x_n)$

Example:

Random Experiment: "tossing of Four coins at a time"

sample space :

$$S = \{ (HHHH), (THHH), (HHHT), (THHT), (HHTH), (THTH), (HHTT), (THTT), (HTHH), (TTHH), (HTHT), (TTHT), (HTTH), (TTTH), (HTTT), (TTTT) \} \quad n(S) = 16 = 2^4$$

Random variable:

$X(S)$ = number of tails.

$= \{0, 1, 2, 3, 4\}$.

Probability mass function:

$P(x) = P(X = x)$ = probability of getting 'x' tails.

$P(0)$ = probability of getting '0' tails $= \frac{1}{16}$

$P(1) = \frac{4}{16}$, $P(2) = P(X=2) = \frac{6}{16}$

$P(3) = \frac{4}{16}$, $P(4) = \frac{1}{16}$.

Discreate Probability distribution

X	$x_1 = 0$	$x_2 = 1$	$x_3 = 2$	$x_4 = 3$	$x_5 = 4$
P(X)	$p_1 = \frac{1}{16}$	$p_2 = \frac{4}{16}$	$p_3 = \frac{6}{16}$	$p_4 = \frac{4}{16}$	$p_5 = \frac{1}{16}$

(i) $P_i \geq 0 \quad \forall i$

ii. $\sum P(x_i) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{6}$

$= \frac{16}{16} = 1$

=o=.

Probability Distribution Function:

Let X be a random variable. (Discrete random variable). Then the probability distribution function associated with 'X' is defined as.

$$F_X(x) = P(X \leq x) = P\{s : X(s) \leq x\},$$
$$-\infty < x < \infty.$$

It is also called cumulative distribution function (cdf). of X.

Properties of Distribution Function:

If F is the distribution function of the random variable X and if $a<b$, then.

(I) i) $P(a < X \leq b) = F(b) - F(a)$

ii. $P(a \leq X \leq b) = [F(b) - F(a)] + P(X=a)$

iii $P(a < x < b) = F(b) - F(a) - P(X=b)$

iv $P(a \leq X < b) = F(b) - F(a) - P(X=b) + P(X=a)$

Note If $P(X=a) = P(X=b) = 0$ then

$$P(a< X\leq b) = P(a \leq X \leq b) = P(a< X< b) = P(a\leq X< b)$$

$$= F(b) - F(a)$$

(II) All distribution functions are monotonically increasing and lie between 0 and 1.

i.e (i) $0 \leq F(x) \leq 1$

(ii) $F(x) < F(y)$ when $x < y$.

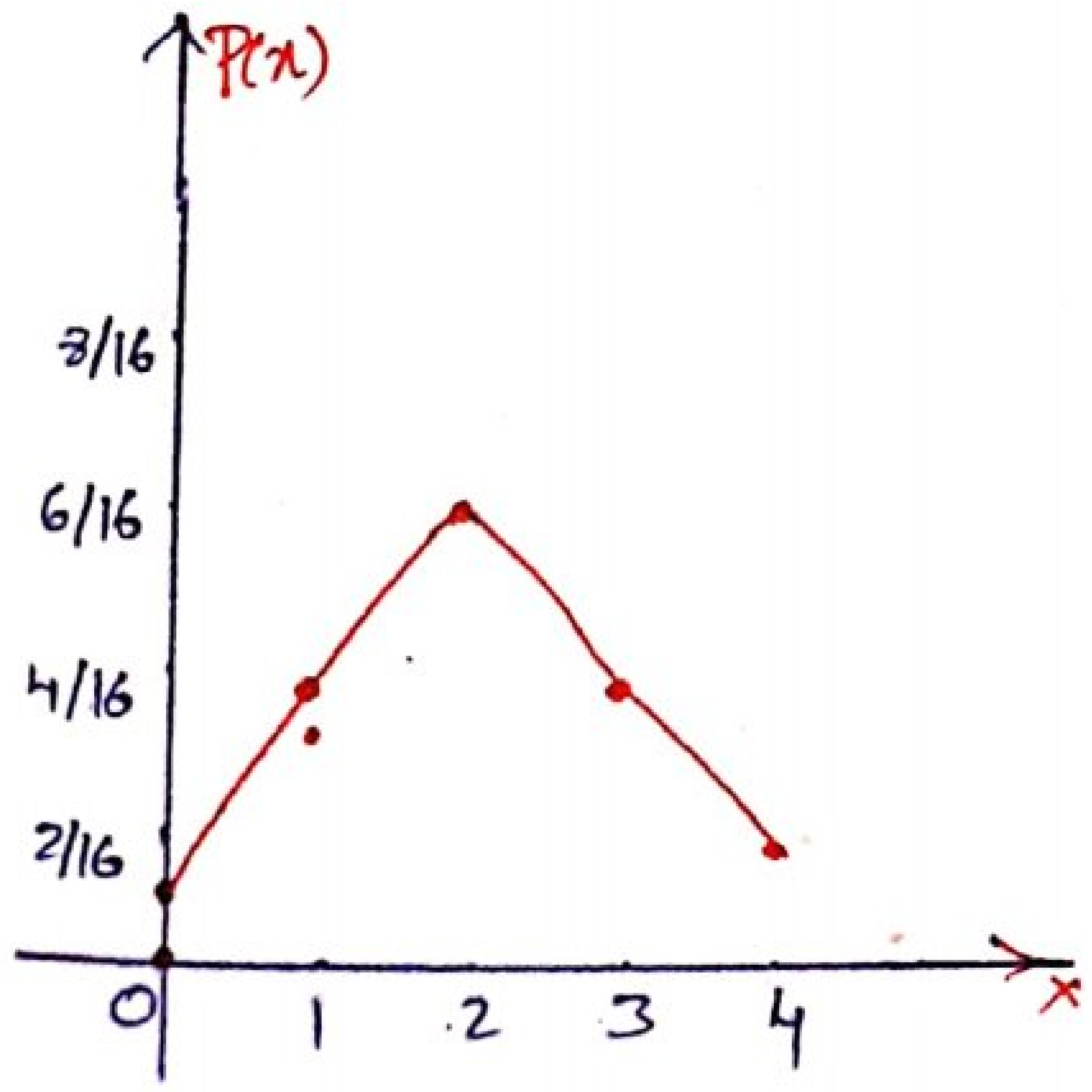
III

(i) $F(-\infty) = \lim_{x\to -\infty} F(x) = 0$

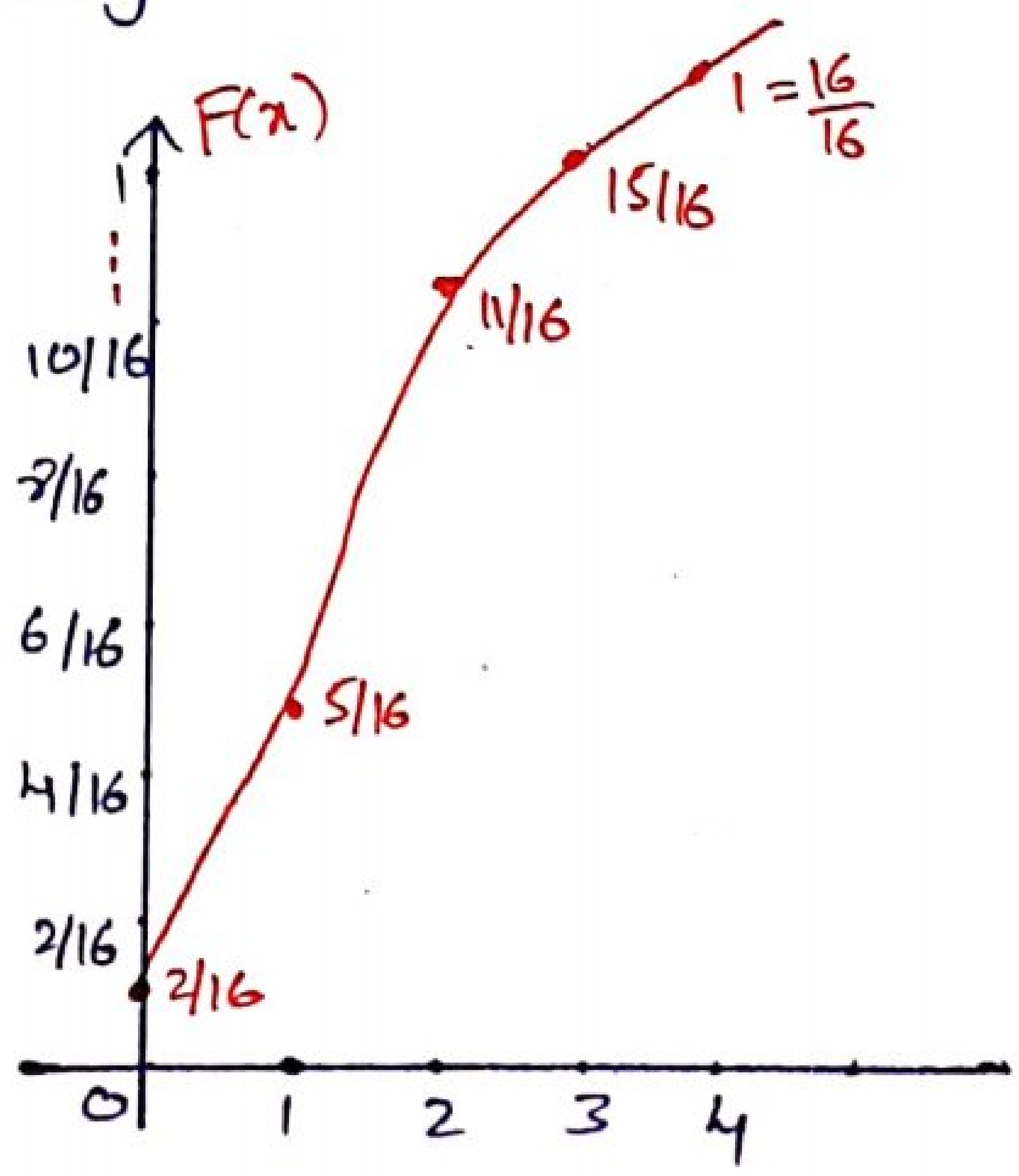
(ii) $F(\infty) = \lim_{x\to \infty} F(x) = 1$

Note:

Random Experiment:- "tossing of 4 coins"



Probability mass function



Probabity distribution function.

Note:

→ $F(x) = P(X \le x) = \sum_{x_i \le x} P(x_i)$

→ If the probability distribution of a random variable x is given by

X	x_1	x_2	x_3	...	x_n
P(x)	p_1	p_2	p_3	...	p_n

Then

$$F(x_1) = P(X = x_1) = p_1$$

$$F(x_2) = P(X \le x_2) = P(X = x_1) + P(X = x_2)$$
$$= p_1 + p_2$$

$$F(x_3) = P(X \le x_3) = P(X = x_1) + P(X = x_2) + P(X = x_3)$$
$$= p_1 + p_2 + p_3$$

...

$$F(x_n) = P(X \le x_n) = p_1 + p_2 + \dots + p_n = 1$$

EXPECTATION

The behaviour of a random variable is completely characterized by the distribution function $F(x)$ or Probability mass function (or) density function $P(x)$ or $f(x)$.

instead of a function, a more compact description can be made by a single number such as mean, median, and mode (measures of central tendency), of random variable X, also variance, standard deviation (measures of dispersion). etc.,

EXPECTATION:

Let X be a Discrete random variable which takes the values $x_1, x_2 \cdots x_n$ with respective probabilities $p_1, p_2 \cdots p_n$.

The Expectation or Mean of the Random variable X is defined as.

$$\boxed{E(X) = \sum_{i=1}^{n} p_i x_i}$$

in general

$$E(g(x)) = \sum_{i=1}^{n} p_i \cdot g(x_i).$$

$\therefore$ Mean $(\mu) = E(X) = \sum_{i=1}^{n} P(X=x_i)\, x_i$

Similarly $E(X^2) = \sum_{i=1}^{n} P(X=x_i)\, x_i^2$

$$= \sum_{i=1}^{n} p_i x_i^2$$

- - -

$$E(X^r) = \sum_{i=1}^{n} p_i x_i^r.$$

Properties:

(i) $E(X+K) = E(X)+K$ for a constant K
X is a Random variable.

ii. $E(K) = K$.

iii $E(K\cdot X) = K\cdot E(X)$.

iv, for X, Y be two Random variables, a, b are constants Then

$\rightarrow$ $E(aX+b) = a\,E(X)+b$

$\rightarrow$ $E(X+Y) = E(X)+E(Y)$

$\rightarrow$ $E(aX+bY) = a\,E(X)+b\,E(Y)$

$\rightarrow$ If X, Y are independent then

$$E(XY) = E(X)\cdot E(Y)$$

$\rightarrow$ $E(X-\mu) = 0$.

Mean (μ):

Let X be the discrete random variable with probability distribution

X	x_1	x_2	x_3	...	x_n
$P(x)$	p_1	p_2	p_3		p_n

then Mean (μ) = $\dfrac{\sum_{i=1}^{n} p_i x_i}{\sum p_i}$

$$\Rightarrow \mu = \sum_{i=1}^{n} p_i x_i \quad (\because \sum p_i = 1)$$

$$\Rightarrow \boxed{\mu = E(X)}$$

Variance (σ^2)

Variance $\sigma^2 = E[(x-\mu)^2]$

$$= E(X^2) - \mu^2$$

$$\boxed{\sigma^2 = \sum_{i=1}^{n} p_i x_i^2 - \mu^2}$$

Standard Deviation (σ):

S.D $\sigma = +\sqrt{\sigma^2}$

$$= \sqrt{\sum_{i=1}^{n} p_i x_i^2 - \mu^2}.$$

Problem:

Find (i) K (ii) $P(X \leq 2)$ (iii) $P(2 \leq X \leq 5)$

For a random variable X has the following Probability distribution.

X	1	2	3	4	5	6	7	8
P(x)	K	2K	3K	4K	5K	6K	7K	8K

Solution:

given that X be a random variable with Probability distribution

X	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	$x_4 = 4$	$x_5 = 5$
P(x)	$p_1 = K$	$p_2 = 2K$	$p_3 = 3K$	$p_4 = 4K$	$p_5 = 5K$

$x_6 = 6$	$x_7 = 7$	$x_8 = 8$
$p_6 = 6K$	$p_7 = 7K$	$p_8 = 8K$

(i) Find K

sum of the Probabilities = 1

$$\sum_{i=1}^{n} p_i = 1$$

$$\Rightarrow \sum_{i=1}^{8} p_i = 1$$

$$\Rightarrow p_1 + p_2 + \cdots + p_8 = 1$$

$$\Rightarrow K + 2K + 3K + 4K + 5K + 6K + 7K + 8K = 1$$

$$\Rightarrow 36K = 1 \Rightarrow \boxed{K = \frac{1}{36}}$$

(ii) Find $P(X \le 2)$

$$P(X \le 2) = P(x=1) + P(x=2)$$
$$= K + 2K$$
$$= 3K = 3\left(\frac{1}{36}\right) \quad \left(\because K = \frac{1}{36}\right)$$
$$= \frac{1}{12}$$

(iii) Find $P(2 \le x \le 5)$

$$P(2 \le x \le 5) = P(x=2) + P(x=3) + P(x=4) + P(x=5)$$
$$= 2K + 3K + 4K + 5K$$
$$= 14K = 14\left(\frac{1}{36}\right)$$
$$= \frac{7}{18}.$$

Problem:

A random variable x has the following Probability distribution.

X	-3	-2	-1	0	1	2	3
P(X)	K	0.1	K	0.2	2K	0.4	2K

Find (i) K (ii) Mean (iii) Variance.

(iv) standard deviation

Solution: given that.

X	$x_1=-3$	$x_2=-2$	$x_3=-1$	$x_4=0$	$x_5=1$
P(x)	$p_1=K$	$p_2=0.1$	$p_3=K$	$p_4=0.2$	$p_5=2K$

$x_6=2$	$x_7=3$
$p_6=0.4$	$p_7=2K$.

(i) Find (K)

Sum of the Probabilities = 1

$$\sum_{i=1}^{7} p_i = 1$$

$\Rightarrow p_1 + p_2 + p_3 + \cdots + p_7 = 1$

$\Rightarrow K + 0.1 + K + 0.2 + 2K + 0.4 + 2K = 1$

$\Rightarrow 6K + 0.7 = 1$

$\Rightarrow 6K = 1 - 0.7 = 0.3$

$\Rightarrow K = \frac{0.3}{6} = \frac{0.1}{2} = \frac{1}{20}$

$\therefore$ ~~$K = \frac{1}{60}$~~ $\boxed{K = \frac{1}{20}}$

(ii) Mean (μ)

$$\text{Mean } \mu = E(X) = \sum_{i=1}^{n} x_i p_i$$

$$\Rightarrow \mu = E(X) = \sum_{i=1}^{7} x_i p_i$$

$= x_1 p_1 + x_2 p_2 + \cdots + x_7 p_7$

$= -3(K) + (-2)(0.1) + (-1)K + 0 + 1(2K) + 2(0.4) + 3(2K)$

$= -3K - 0.2 - K + 2K + 0.8 + 6K$

$= 4K + 0.6$

$= 4\left(\frac{1}{20}\right) + 0.6 = \frac{1}{5} + 0.6 = 0.8$

$\therefore \boxed{\mu = E(X) = 0.8}$

(iii) Variance σ^2

$$\text{Variance } \sigma^2 = E(x^2) - \mu^2$$

$$= \sum_{i=1}^{7} x_i^2 p_i - \mu^2$$

$$= (x_1^2 p_1 + x_2^2 p_2 + \cdots + x_7^2 p_7) - \mu^2$$

$$= 9K + 4(0.1) + 1(K) + 0 + 1(2K) + 4(0.4) + 9(2K) - (0.8)^2$$

$$= 30K + 2 - 0.64$$

$$= 30\left(\frac{1}{20}\right) + 2 - 0.64$$

$$= 2.86.$$

$$\therefore \boxed{\sigma^2 = 2.86}$$

(iv) standard deviation (σ)

$$\sigma = +\sqrt{\sigma^2} = \sqrt{2.86}$$

$$\boxed{\sigma = 1.69}$$

Problem:

Let X denote the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the

(i) Discrete Probability distribution.

(ii) Expectation (iii) Variance.

Solution:

Random Experiment: throwing two dice at a time.

Sample space: $S = \{ (1,1)\ (1,2) \cdots (6,6) \}$.

Random variable: $X : S \to R$

$$X(a,b) = \text{Min}\{a,b\},$$

$$X(S) = \{1, 2, 3, 4, 5, 6\}.$$

$P(X=x)$ = Probability of getting Minimum number 'x'.

$$P(X=1) = \frac{11}{36}$$

$\because$ Favourable cases
(1,1) (1,2) (1 3) (1 4) (1 5) (1,6)
(2 1) (3 1) (4 1) (5,1) (6,1) — 11

||'ly.
$P(X=2) = \frac{9}{36}$, $P(X=3) = \frac{7}{36}$

$P(X=4) = \frac{5}{36}$ $\quad P(X=5) = \frac{3}{36}$

$P(X=6) = \frac{1}{36}$.

The discrete Probability distribution:

x	$x_1=1$	$x_2=2$	$x_3=3$	$x_4=4$	$x_5=5$	$x_6=6$
$P(X=x)$	$P_1=\frac{11}{36}$	$P_2=\frac{9}{36}$	$P_3=\frac{7}{36}$	$P_4=\frac{5}{36}$	$P_5=\frac{3}{36}$	$P_6=\frac{1}{36}$

Expectation: $(E(X))$

$$E(X) = \sum_{i=1}^{6} x_i P_i = x_1P_1 + x_2P_2 + \cdots + x_6P_6$$

$$= 1\left(\frac{11}{36}\right) + 2\left(\frac{9}{36}\right) + 3\left(\frac{7}{36}\right) + 4\left(\frac{5}{36}\right) + 5\left(\frac{3}{36}\right) + 6\cdot\left(\frac{1}{36}\right)$$

$$= \frac{11}{36} + \frac{18}{36} + \frac{21}{36} + \frac{20}{36} + \frac{15}{36} + \frac{6}{36}$$

$$= \frac{91}{36} = 2.53. \quad (\mu = E(X))$$

Variance (σ^2)

$$\sigma^2 = E(X^2) - \mu^2$$

$$= \sum_{i=1}^{6} x_i^2 P_i - \mu^2$$

$$= x_1^2 P_1 + x_2^2 P_2 + \cdots + x_6^2 P_6 - \mu^2$$

$$= 1\left(\frac{11}{36}\right) + 4\left(\frac{9}{36}\right) + 9\left(\frac{7}{36}\right) + 16\left(\frac{5}{36}\right) + 25\left(\frac{3}{36}\right) + 36\left(\frac{1}{36}\right) - (2.53)^2$$

$$= \frac{1}{36}\left[1 + 36 + 63 + 80 + 75 + 36\right] - (2.53)^2$$

$$= \frac{291}{36} - (2.53)^2$$

$$= \underline{1.68}$$

Problem:

Find the Mean and variance of the uniform probability distribution given by $f(x)=\frac{1}{n}$ for $x=1,2,3,\dots n$.

Solution:

The Probability distribution is given by

X	1	2	3	...	n
$P(X)=f(x)$	$1/n$	$1/n$	$1/n$		$1/n$

$$\text{Mean } (\mu) = E(X) = \sum_{i=1}^{n} x_i p_i$$

$$= 1\cdot\frac{1}{n} + 2\cdot\frac{1}{n} + \dots + n\cdot\frac{1}{n}$$

$$= \frac{1}{n}(1+2+3+\dots+n)$$

$$= \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2}$$

$$\text{Variance}\cdot \sigma^2 = E(X) - \mu^2$$

$$= \sum_{i=1}^{n} x_i^2 p_i - \mu^2$$

$$= 1\cdot\frac{1}{n} + 2^2\frac{1}{n} + 3^2\frac{1}{n} + \dots + n^2\frac{1}{n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}\left[1+2^2+3^2+\dots+n^2\right] - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{4}(n+1)^2$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{(n+1)}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

Problem:

A player tosses 3 fair coins. He wins Rs. 500 if 3 heads appear. Rs. 300 if 2 heads appear. Rs. 100 if 1 head occurs. On the other hand he loses Rs: 1500 if 3 tails occur. Find the expected gain of the player.

Solution:

Random Experiment: tosses 3-coins at a time.

Sample space : S = { (H H H) (T H H)
(H H T) (T H T)
(H T H) (T T H)
(H T T) (T T T) },

Random variable :

$X(s)$ = gain of the player.

$X(S) = \{-1500, 500, 300, 100\}$.

→ $P(X=x)$ = Probability of getting gain x

→ $P(X=-1500)$ = Probability of getting -1500
(i.e Probability of getting 3-tails)

$= \frac{1}{8}$

→ $P(X=500)$ = Probability of getting gain 500
(i.e Probability of getting 3-heads)

$= \frac{1}{8}$.

→ $P(X = 300)$ = Probability of getting gain 300

(i.e probability of getting 2-heads)

$= \frac{3}{8}$.

→ $P(X = 100)$ = Probability of getting gain 100

(i.e probability of getting 1 head)

$= \frac{3}{8}$.

Now the Discrete Probability distribution

X	$x_1 = -1500$	$x_2 = 500$	$x_3 = 300$	$x_4 = 100$
P(x)	$p_1 = \frac{1}{8}$	$p_2 = \frac{1}{8}$	$p_3 = \frac{3}{8}$	$p_4 = \frac{3}{8}$.

Expected gain of the player $= E(x)$

$$\Rightarrow E(x) = \sum_{i=1}^{4} p_i x_i$$

$$\Rightarrow E(x) = (-1500)\frac{1}{8} + 500\left(\frac{1}{8}\right) + 300\left(\frac{3}{8}\right) + 100\left(\frac{3}{8}\right)$$

$$= \frac{1}{8}\left[-1500 + 500 + 900 + 300\right]$$

$$= \frac{200}{8} = 25.$$

WORK OUT PROBLEMS

1. Calculate mean and variance of X, if the Probability distribution of X is given by

X	-1	0	1	2	3
$P(x)$	0.3	0.1	0.1	0.3	0.2

2. Let X be the discrete random variable with Probability distribution

X	0	1	2	3	4	5	6
$P(X)$	K	3K	5K	7K	9K	11K	13K

Find (i) K (ii) $P(X<4)$, $P(X \geq 5)$

(iii) $P(3 < X \leq 6)$ (iv) Mean (v) variance.

(vi) What will be the minimum value of K so that $P(X \leq 2) > 0.3$

3. A random variable X is defined as the sum of the numbers on the faces when two dice are thrown. Write the Probability distribution and then find Mean, variance

4. Find the distribution function which corresponding to the Probability distribution defined by $f(x) = \frac{x}{15}$, for $x = 1, 2, 3, 4, 5$.

5. A Player wins if he gets 5 on a Single throw of a die, he loses if he gets 2 or 4. If he wins, he get Rs: 50, if he loses he gets Rs: 10, otherwise he has to Pay Rs: 15. Find the value of the game. to the player. Is it favorable?

=.

CONTINUOUS PROBABILITY DISTRIBUTION:

A random variable x which can take values continuously i.e which takes all possible values in a given interval is called continuous Random variable. The Probability distribution defined on continuous Random variable is called Continuous Probability Distribution.

Probability Density Function:

Consider the small interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$ of length dx around the point x.

Let $f(x)$ be any continuous function of x so that $f(x)\,dx$ represents the probability that the variable X falls in the interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$.

i.e $P\left(x-\frac{dx}{2} \leq X \leq x+\frac{dx}{2}\right) = f(x)\,dx.$

Then $f(x)$ is called the Probability density function. (or) simply Density function. of the continuous Random variable X.

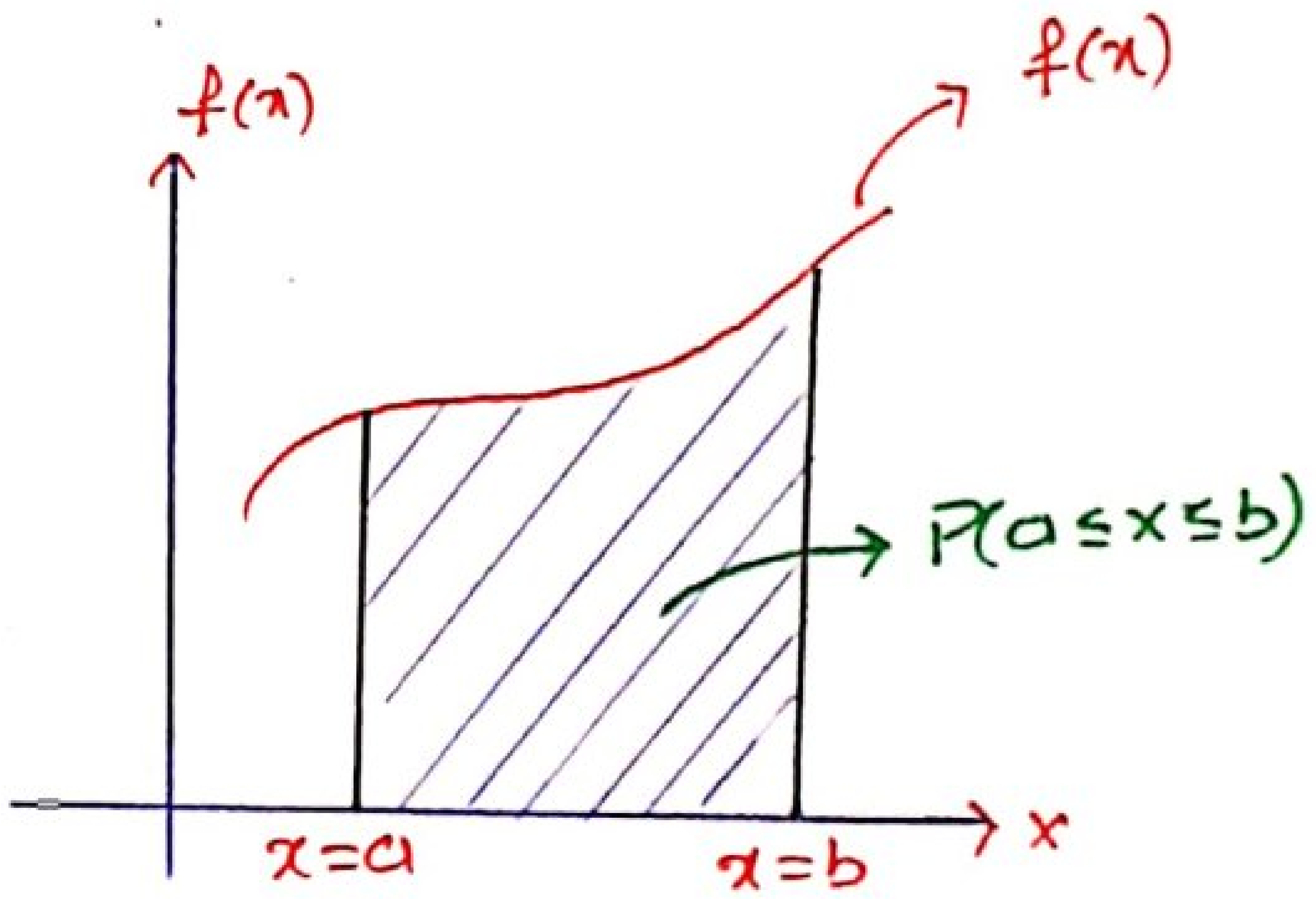
and the continuous curve $y=f(x)$ is known as Probability density curve (or) Probability curve.

As the Probability for a variate value to lie in the Interval dx is $f(x)dx$.

so the Probability for a variate value to fall in the finite interval (a,b) is $\int_a^b f(x)dx$.

i.e $$P(a \leq x \leq b) = \int_a^b f(x)\,dx.$$

Which represents the area between the curve $y=f(x)$, x-Axis and the ordinates $x=a$ and $x=b$.



Properties of Probability density Function $f(x)$:

→ $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

→ $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ ($\because$ Total Probability is 1)

→ The Probability $P(E)$ is given by

$$P(E) = \int_E f(x)\,dx.$$

$\rightarrow P(a < x \leq b) = P(a \leq x < b) = P(a < x < b)$

$= P(a \leq x \leq b)$

<u>Note</u>: Probability of the variable at a Particular Point is always zero in case of continuous Random variable.

Probability Distrubution Function $F(x)$

(cumulative distribution function)

The Probability distrubution Function $F(x)$ is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$$

Properties:

$\rightarrow$ $0 \leq F(x) \leq 1$ for $-\infty < x < \infty$

$\rightarrow$ $F(-\infty) = 0, \quad F(\infty) = 1$

$\rightarrow$ $P(a \leq x \leq b) = \int_{a}^{b} f(x)\, dx = F(b) - F(a)$

$\rightarrow$ $F'(x) = f(x) \Rightarrow \frac{d}{dx}[F(x)] = f(x).$

<u>Note</u>: $f(x) \rightarrow$ Probability density Function

$F(x) \rightarrow$ Probability Distribution Function.

Measures of central tendency for continuous Probability Distribution:

Let x be the continuous Random variable with Probability density function $f(x)$ then

Mean (μ)

Mean of the Distribution

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Note: If x define from a to b then

$$\mu = E(x) = \int_{a}^{b} x \cdot f(x)\, dx.$$

Median (M).

'M' is said to be Median of the Distribution if

$$\int_{-\infty}^{M} f(x)\, dx = \int_{M}^{\infty} f(x)\, dx = \frac{1}{2}$$

Mode: Mode is the value of x for which $f(x)$ is Maximum. Mode is given by

i.e. $f'(x) = 0$ and $f''(x) < 0$, for $-\infty < x < \infty$

Then 'x' is called Mode.

Variance (σ^2)

The variance of the distribution

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

(or)

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

Standard deviation (σ)

$$\sigma = +\sqrt{\sigma^2}$$

Mean deviation:

Mean deviation about Mean μ is

$$= \int_{-\infty}^{\infty} |x-\mu| \cdot f(x)\,dx.$$

Problem:

If X is continous Random variable with Probability density function $f(x) = \begin{cases} 2e^{-2x} & \text{for } x>0 \\ 0 & \text{for } x\le 0 \end{cases}$

then Find (i) $P(1 \le X \le 3)$ (ii) $P(X > 0.5)$

solution:

Given Probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x>0 \\ 0 & \text{for } x\le 0 \end{cases}$$

(i) $$P(1\le X\le 3) = \int_1^3 f(x)\,dx$$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3 = -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}.$$

ii. $$P(X>0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx = 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-2(0.5)}\right]$$

$$= 0 + e^{-1} = e^{-1}.$$

Problem:

If the Probability density of a random variable is given by $f(x) = \begin{cases} K(1-x^2) & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

Find the value of K, and probabilities

(i) between 0.1 and 0.2

(ii) Greater than 0.5

Solution:

Given that $f(x) = \begin{cases} K(1-x^2) & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

We know that total probability is unity

$$\Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{0}^{1} K(1-x^2)\,dx + 0 = 1$$

$$\Rightarrow K\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow K\left\{\left[1-\frac{1}{3}\right] - [0-0]\right\} = 1$$

$$\Rightarrow K\cdot\left(\frac{2}{3}\right) = 1$$

$$\Rightarrow K = \frac{3}{2}.$$

$$\therefore f(x) = \begin{cases} \frac{3}{2}(1-x^2) & \text{for } 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$$

Probability between 0.1 and 0.2

$$\Rightarrow P(0.1 \le X \le 0.2) = \int_{0.1}^{0.2} f(x)\,dx$$

$$= \int_{0.1}^{0.2} \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= \frac{3}{2}\left\{\left[0.2 - \frac{(0.2)^3}{3}\right] - \left[0.1 - \frac{(0.1)^3}{3}\right]\right\}$$

$$= \frac{3}{2}\left\{0.2 - \frac{0.008}{3} - 0.1 + \frac{0.001}{3}\right\}$$

$$= 0.1495.$$

Probability greater than 0.5

$$\Rightarrow P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \frac{3}{2}\int_{0.5}^{1} (1-x^2)\,dx + 0$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1-\frac{1}{3}\right) - \left(0.5 - \frac{(0.5)^3}{3}\right)\right]$$

$$= 0.3125$$

Problem:

The Probability density function of a continuous Random variable X is $f(x) = ce^{-|x|}$, $-\infty < x < \infty$

show that $c = 1/2$ and also Find Mean and variance of the distribution.

Solution:

The given Probability density function

$$f(x) = c\,e^{-|x|} \quad -\infty < x < \infty$$

$$\Rightarrow \quad f(x) = \begin{cases} ce^{x} & \text{if } -\infty < x \le 0 \\ ce^{-x} & \text{if } 0 < x \le \infty \end{cases}$$

$$\left[\because \; |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \le 0 \end{cases}\right.$$

We know that, total Probability is unity

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} c\cdot e^{x}\,dx + \int_{0}^{\infty} ce^{-x}\,dx = 1$$

$$\Rightarrow c\cdot\left[e^{x}\right]_{-\infty}^{0} + c\cdot\left[\frac{e^{-x}}{-1}\right]_{0}^{\infty} = 1$$

$$\Rightarrow c\left[e^{0} - e^{-\infty}\right] - c\left[e^{-\infty} - e^{0}\right] = 1$$

$$\Rightarrow c[1-0] - c[0-1] = 1$$

$$\Rightarrow c + c = 1 \Rightarrow 2c = 1 \Rightarrow \boxed{c = \tfrac{1}{2}}$$

Mean (μ)

$$\text{Mean } \mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{2} e^{-|x|} \cdot dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x \cdot e^{-|x|}\, dx.$$

$$= \frac{1}{2}(0) = 0$$

$\int_{-a}^{a} f(x)\, dx = 0$ if $f(-x) = -f(x)$

$(-x) e^{-|-x|} = -x e^{-|x|}$

(odd Function)

$\therefore$ Mean $(\mu) = 0$

Variance (σ^2)

$$\sigma^2 = E(x^2) - \mu^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx \quad (\because \mu = 0)$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} e^{-|x|}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|}\, dx$$

$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$

if $f(-x) = f(x)$

even Function

$$= \frac{1}{2} \cdot 2 \int_{0}^{\infty} x^2 e^{-|x|} \cdot dx$$

($x^2 e^{-|x|}$ is Even Function)

$$= \int_{0}^{\infty} x^2 e^{-x}\, dx \quad (\because |x| = x \text{ if } x > 0)$$

$$= \left[x^2 \cdot \frac{e^{-x}}{-1} - 2x \cdot \frac{e^{-x}}{1} + 2 \cdot \frac{e^{-x}}{-1} \right]_0^\infty$$

$$= [0 - 0 + 0] - [0 - 0 - 2]$$

$$= 2$$

$$\therefore \boxed{\sigma^2 = 2}$$

Problem:

The Probability density function of a continuous Random Variable x is given by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & \text{if } -3 \leq x < -1 \\ \frac{1}{16}(6-2x^2) & \text{if } -1 \leq x < 1 \\ \frac{1}{16}(3-x)^2 & \text{if } 1 \leq x \leq 3 \end{cases}$$

show that the area under the curve is unity, also find the Mean of the distribution.

Solution:

Area under the density curve is unity

i.e total Probability = 1

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$LHS = \int_{-\infty}^{\infty} f(x)\, dx$$

$$= \int_{-\infty}^{-3} f(x)\, dx + \int_{-3}^{-1} f(x)\, dx + \int_{-1}^{1} f(x)\, dx + \int_{1}^{3} f(x)\, dx + \int_{3}^{\infty} f(x)\, dx$$

$$= 0 + \int_{-3}^{-1} \frac{1}{16}(3+x)^2 dx + \int_{-1}^{1} \frac{1}{16}(6-2x^2) dx$$

$$+ \int_{1}^{3} \frac{1}{16}(3-x)^2 dx + 0$$

$$= \frac{1}{16}\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \frac{1}{16}\left[6x - \frac{2}{3}x^3\right]_{-1}^{1}$$

$$+ \frac{1}{16}\left[\frac{[3-x]^3}{-3}\right]_{1}^{3}$$

$$= \frac{1}{16}\left[\frac{8}{3} - 0\right] + \frac{1}{16}\left[\left(6 - \frac{2}{3}\right) - \left(-6 + \frac{2}{3}\right)\right]$$

$$+ \frac{1}{16}\left[0 + \frac{8}{3}\right]$$

$$= \frac{1}{6} + \frac{1}{16}\left[12 - \frac{4}{3}\right] + \frac{1}{6}$$

$$= \frac{1}{6} + \frac{1}{16}\left(\frac{32}{3}\right) + \frac{1}{6}$$

$$= \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = \frac{6}{6} = 1 = RHS.$$

$$\text{Mean}(\mu) = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(3+x)^2\,dx + \frac{1}{16}\int_{-1}^{1} x(6-2x^2)\,dx + \frac{1}{16}\int_{1}^{3} x(3-x)^2\,dx$$

$$= \frac{1}{16}\int_{-3}^{-1} (9x+6x^2+x^3)\,dx + 0 + \frac{1}{16}\int_{1}^{3} (9x-6x^2+x^3)\,dx$$

$\left(\because x(6-2x^2) \text{ is odd function} \Rightarrow \int_{-a}^{a} f(x)\,dx = 0 \text{ if } f(-x) = -f(x) \right)$

$$= \frac{1}{16}\left[\frac{9x^2}{2} + \frac{6x^3}{3} + \frac{x^4}{4}\right]_{-3}^{-1} + \frac{1}{16}\left[\frac{9x^2}{2} - \frac{6x^3}{3} + \frac{x^4}{4}\right]_{1}^{3}$$

$$= \frac{1}{16}\left\{\left(\frac{9}{2} - \frac{6}{3} + \frac{1}{4}\right) - \left(\frac{81}{2} - 54 + \frac{81}{4}\right)\right\} + \frac{1}{16}\left\{\left(\frac{81}{2} - 54 + \frac{81}{4}\right) - \left(\frac{9}{2} - 2 + \frac{1}{4}\right)\right\}$$

$$= \underline{\underline{0}}$$

$$\text{Mean}(\mu) = \underline{\underline{0}}$$

Problem

Find the K such that $f(x) = \begin{cases} Kx^2 & \text{if } 0<x<3 \\ 0 & \text{otherwise} \end{cases}$

is a Probability function. also Find

(i) $P(1<X\le 2)$ (ii) Mean (iii) Variance.

(v) Probability distribution function $F(x)$

Given density function. $f(x) = \begin{cases} Kx^2 & \text{if } 0<x<3 \\ 0 & \text{otherwise} \end{cases}$

We know that, total Probability is unity

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^3 Kx^2\,dx = 1$$

$$\Rightarrow K\cdot\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$\Rightarrow K\left[\frac{27}{3} - 0\right] = 1$$

$$\Rightarrow K[9] = 1 \Rightarrow \boxed{K = 1/9}$$

$$\therefore f(x) = \begin{cases} \frac{1}{9}x^2 & \text{if } 0<x<3 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow \quad P(1 < X \leq 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{1}{9} x^2 dx$

$= \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{9}\left[\frac{8}{3} - \frac{1}{3}\right]$

$= \frac{7}{27}$

$\rightarrow \quad \text{Mean } (\mu) = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \,.$

$= \frac{1}{9}\int_0^3 x^3 dx = \frac{1}{9} \cdot \left[\frac{x^4}{4}\right]_0^3$

$= \frac{1}{9}\left[\frac{81}{4} - 0\right] = \frac{9}{4}$

$\longrightarrow \quad \text{Variance } (\sigma^2) = E(X^2) - \mu^2$

$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= \frac{1}{9}\int_0^3 x^4\,dx - \left(\frac{9}{4}\right)^2$

$= \frac{1}{9}\left(\frac{x^5}{5}\right)_0^3 - \left(\frac{9}{4}\right)^2$

$= \frac{1}{9}\left[\frac{3^5}{5} - 0\right] - \left(\frac{9}{4}\right)^2$

$= \frac{27}{5} - \frac{81}{16} = \frac{27}{80} =$

$\sigma^2 = 0.3375$

Distribution Function $F(x)$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt.$$

If $x \le 0$ then

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = 0$$

If $x > 0$ then.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt.$$

$$= 0 + \int_{0}^{x} \frac{1}{9} t^2 \cdot dt.$$

$$= \frac{1}{9}\left[\frac{t^3}{3}\right]_0^x$$

$$= \frac{1}{27}\left[x^3\right]$$

$$= \frac{x^3}{27}.$$

$$\therefore F(x) = \begin{cases} \frac{1}{27}x^3 & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem:

A continuous Random variable x has the distribution function $F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ k(x-1)^4 & \text{if } 1 < x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$

then Find (i) $f(x)$ (ii) K (iii) Mean.

Solution:

→ The relation between Distribution Function $F(x)$ and density function $f(x)$ is

$$\frac{d}{dx}[F(x)] = f(x)$$

$$\therefore \quad f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 4K(x-1)^3 & \text{if } 1 < x \le 3 \\ 0 & \text{if } x > 3 \end{cases}$$

→ total Probability is unit

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{1}^{3} 4K(x-1)^3\,dx = 1$$

$$\Rightarrow 4K\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$\Rightarrow 4K\left[\frac{2^4}{4} - 0\right] = 1$$

$$\Rightarrow 16K = 1 \quad \Rightarrow K = \frac{1}{16}.$$

$\therefore$ The density function

$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ \frac{1}{4}(x-1)^3 & \text{if } 1 < x \le 3 \\ 0 & \text{if } x > 3 \end{cases}$$

$\rightarrow$ Mean $(\mu) = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{1}^{3} \frac{1}{4} \cdot x (x-1)^3\, dx.$$

Let $x - 1 = t \Leftrightarrow x = 1+t$

$\Rightarrow dx = dt$

If $x = 1, \longrightarrow t = 0$

$x = 3 \longrightarrow t = 2$

$$= \frac{1}{4} \int_{t=0}^{2} (1+t)\, t^3\, dt$$

$$= \frac{1}{4} \int_{t=0}^{2} (t^3 + t^4)\, dt$$

$$= \frac{1}{4} \left[\frac{t^4}{4} + \frac{t^5}{5} \right]_0^2$$

$$= \frac{1}{4} \left[\left(\frac{2^4}{4} + \frac{2^5}{5} \right) - (0+0) \right]$$

$$= \frac{2^4}{4} \left[\frac{1}{4} + \frac{2}{5} \right]$$

$$= 4 \left[\frac{13}{20} \right] = \frac{13}{5} = 2.6$$

$\therefore$ Mean $\mu = 2.6$

Problem:-

The troble shooting capability of an IC chip in a circuit is a random variable X whose distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x \leq 3 \\ 1 - \frac{9}{x^2} & \text{for } x > 3. \end{cases}$$

Find the Probability that IC chip will work Properly

(i) Less than 8 years (ii) Beyond 8 years.

(iii) Between 5 to 7 (iv) Anywhere from 2 to 5 years.

Solution we know that $F(x) = P(X \leq x)$

(i) $P(X < 8) = F(8) = 1 - \frac{9}{8^2} = 1 - \frac{9}{64}$

$= \frac{55}{64} = 0.8594$

(ii) $P(X > 8) = 1 - P(X \leq 8)$

$= 1 - F(8) = 1 - 0.8594$

$= 0.1406$

(iii) $P(5 < X < 7) = F(7) - F(5)$

$= 1 - \frac{9}{7^2} - 1 + \frac{9}{5^2}$

$= 9\left(\frac{1}{25} - \frac{1}{49}\right) = 0.1763$

(iv) $P(2 \leq X \leq 5) = F(5) - F(2) = 1 - \frac{9}{25} - 0$

$= 0.64$

Problem:

The Probability density function of a random variable x is $f(x) = \begin{cases} \frac{1}{2}\sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{elsewhere} \end{cases}$

Find the Mean, Median, Mode of the distribution. also find $P(0 < x < \pi/2)$

Solution:

→ Mean of the distribution $\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$\Rightarrow \mu = \int_0^{\pi} \frac{1}{2} x \cdot \sin x \, dx$$

$$= \frac{1}{2}\left[x \cdot (-\cos x) - 1 \cdot (-\sin x) \right]_0^{\pi}$$

$$= \frac{1}{2}\left[(-\pi(-1) + 0) - (0+0) \right]$$

$$\boxed{\mu = \frac{\pi}{2}}$$

→ Median. (M)

If M is Median of the distribution

then $$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{0}^{M} f(x)\,dx = \int_{M}^{\pi} f(x)\,dx = \frac{1}{2}$$

Consider $$\int_0^M f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^M \frac{1}{2} \sin x \, dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}\left[-\cos x\right]_0^M = \frac{1}{2}$$

$$\Rightarrow -\left[\cos M - \cos 0\right] = 1$$

$$\Rightarrow -\cos M + 1 = 1$$

$$\Rightarrow \cos M = 0 \Rightarrow M = \pi/2$$

$\therefore \boxed{M = \frac{\pi}{2}}$ is the Median.

→ Mode of the distribution:

Mode is the value of x for which $f(x)$ is maximum.

$f'(x) = 0$ (Necessary condition for maximum)

$$\Rightarrow \frac{1}{2}\cos x = 0$$

$$\Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2},$$

$$f''(x) = -\frac{1}{2}\sin x.$$

$$f''\left(\frac{\pi}{2}\right) = -\frac{1}{2}\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2} < 0$$

(sufficient condition)

at $x = \frac{\pi}{2}$, $f(x)$ has maximum.

$\therefore \boxed{x = \frac{\pi}{2}}$ is Mode

<u>Note</u> in this Example

Mean = Median = Mode

$$\rightarrow \quad P(0 < X < \tfrac{\pi}{2}) = \int_0^{\pi/2} f(x)\,dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\sin x\,dx$$

$$= \frac{1}{2}\left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{-1}{2}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{1}{2}\left[0-1\right] = \frac{1}{2}$$

Problem:

A continuous random variable has the probability density function

$$f(x) = \begin{cases} kxe^{-\lambda x} & \text{for } x \geq 0,\ \lambda > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find (i) k (ii) Mean (iii) variance.

Solution:

We know that total probability $= 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_0^{\infty} kxe^{-\lambda x}\,dx = 1$$

$$\Rightarrow k\cdot\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - (1)\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} = 1$$

$$\Rightarrow K\left[(0-0)-\left(0-\frac{1}{\lambda^2}\right)\right]=1$$

$$(\because e^{-\infty}=0,\ \lambda>0)$$

$$\Rightarrow \frac{K}{\lambda^2}=1 \Rightarrow \boxed{K=\lambda^2}$$

$$\therefore f(x)=\begin{cases}\lambda^2 x e^{-\lambda x} & \text{for } x\geq 0,\ \lambda>0\\ 0 & \text{otherwise}\end{cases}$$

Mean (μ)

Mean of the distribution $\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$\Rightarrow \mu = \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\infty} x f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} \lambda^2 . x^2 . e^{-\lambda x}\,dx$$

$$= \lambda^2\left[x^2 . \frac{e^{-\lambda x}}{-\lambda} - 2x\frac{e^{-\lambda x}}{\lambda^2} + \frac{2e^{-\lambda x}}{-\lambda^3}\right]_0^{\infty}$$

$$= \lambda^2\left[(0-0-0)-\left(0-0-\frac{2}{\lambda^3}\right)\right]$$

$$= \frac{+2}{\lambda}$$

$$\therefore \text{Mean } \mu = \frac{2}{\lambda}$$

Scanned with CamScanner

Variance (σ^2)

variance of the distribution

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx - \frac{4}{\lambda^2}$$

$$= 0 + \int_{0}^{\infty} \lambda^2 . x . e^{-\lambda x}\,dx - \frac{4}{\lambda^2}$$

$$= \lambda^2 \cancel{\Big[} \cancel{x^4 . \frac{e^{-\lambda x}}{-\lambda}} - 4x^3 \cancel{\frac{e^{-\lambda x}}{\lambda^2}} + 12x^2 \cancel{\frac{e^{-\lambda x}}{-\lambda^3}} -$$

$$= \lambda^2 \left[x^3 . \frac{e^{-\lambda x}}{-\lambda} - 3x^2 \frac{e^{-\lambda x}}{\lambda^2} + 6x \frac{e^{-\lambda x}}{-\lambda^3} - \frac{6e^{-\lambda x}}{\lambda^4} \right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[(0-0+0-0) - \left(0-0-0-\frac{6}{\lambda^4}\right) \right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}$$

$$\therefore \sigma^2 = \frac{2}{\lambda^2}$$

Problem:

If the probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 2kx e^{-x^2} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

then find (i) K (ii) Distribution function $F(x)$

Solutions:

We have total probability is unity

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{\infty} 2kx\, e^{-x^2} dx = 1$$

$x^2 = t \Rightarrow 2x\,dx = dt$

If $x = 0 \rightarrow t = 0$

$x = \infty \rightarrow t = \infty$

$$\Rightarrow k \int_{0}^{\infty} e^{-t} dt = 1$$

$$\Rightarrow k \left[\frac{e^{-t}}{-1} \right]_0^{\infty} = 1$$

$$\Rightarrow k \left[0 + 1 \right] = 1 \Rightarrow \boxed{k = 1}$$

$$\therefore f(x) = \begin{cases} 2x e^{-x^2} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

(ii) Probability distribution function $F(x)$

$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} f(t)\,dt$$

If $x \le 0$ then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$$

If $x > 0$ then

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

$$= 0 + \int_{0}^{x} 2t e^{-t^2}\,dt$$

Let $t^2 = u \Rightarrow 2t\,dt = du$

If $t = 0 \Rightarrow u = 0$

$t = x \Rightarrow u = x^2$

$$= \int_{0}^{x^2} e^{-u}\,du$$

$$= \left[\frac{e^{-u}}{-1}\right]_0^{x^2} = -\left[e^{-x^2} - 1\right]$$

$$= 1 - e^{-x^2}$$

$$\therefore F(x) = P(X \le x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-x^2} & \text{if } x > 0. \end{cases}$$

WORK OUT PROBLEMS

(1) If the Probability density function of a continuous Random variable X is given by

$$f(x)=\begin{cases} Kx^2e^{-x} & \text{for } x\geq 0 \\ 0 & \text{otherwise} \end{cases}$$ Find

(i) K (ii) Mean (iii) variance

(2) If the Probability density of a random variable is given by $f(x)=\begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1\leq x<2 \\ 0 & \text{elsewhere} \end{cases}$

then Find (i) $P(0.2\leq x\leq 0.8)$

(ii) $P(0.6\leq X\leq 1.2)$

(iii) Mean.

(3) If the Probability density of X is

$$f(x)=\begin{cases} 12x^3-21x^2+10x & \text{for } 0\leq x\leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find $P(X\leq \frac{1}{2})$ and $P(X>\frac{1}{2})$

(ii) Find number K such that.

$P(X\leq K)=\frac{1}{2}$.

(4) Let X be the continuous Random variable with Probability density function

$$f(x) = \begin{cases} 2/x^3 & \text{if } 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find Probability distrubution Function.

also find $F(2), F(3)$

(5) Find the standard deviation of the Probability density function

$$f(x) = \begin{cases} x^3 & 0 \le x \le 1 \\ (2-x)^3 & 1 \le x \le 2 \end{cases}.$$

Results

(1) If X is discrete random variable, a, b are constants then prove that $E(aX+b) = aE(X)+b$.

Proof: Let X be the discrete Random variable with probability distribution

X	x_1	x_2	...	x_n
$P(x)$	p_1	p_2	...	p_n

Now $$E(aX+b) = \sum_{i=1}^{n} (ax+b) p_i \quad \left(\because E(X) = \sum_{i=1}^{n} x_i p_i\right)$$

$$= \sum_{i=1}^{n} [ax p_i + b p_i]$$

$$= \sum_{i=1}^{n} a x_i p_i + \sum_{i=1}^{n} b p_i$$

$$= a \sum_{i=1}^{n} x_i p_i + b \sum_{i=1}^{n} p_i$$

$$= a\, E(X) + b(1) \quad (\because \Sigma p_i = 1)$$

$$= aE(X) + b$$

(2) If X is continuous random variable with probability density function $f(x)$ Prove that $E(aX+b) = a \cdot E(X)+b$. for any constants a, b.

Proof: Let X be the continuous Random variable with probability density function $f(x)$.

Scanned with CamScanner

Consider

$$E(ax+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx \qquad \left(E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \right)$$

$$= \int_{-\infty}^{\infty} [ax f(x) + b f(x)]\,dx$$

$$= a \cdot \int_{-\infty}^{\infty} x f(x)\,dx + b \int_{-\infty}^{\infty} f(x)\,dx.$$

$$= a\,E(X) + b \cdot (1) \qquad \left(\int_{-\infty}^{\infty} f(x)\,dx = 1 \right)$$

$$= a E(X) + b.$$

(3) If X is the continuous Random variable and K is a constant. then prove that

(i) variance $(X+K)$ = variance(X)

i.e $\operatorname{var}(X+K) = \operatorname{var}(X)$

ii. $\operatorname{var}(KX) = K^2 \operatorname{var}(X)$

Proof: Let X be the continuous Random variable density function $f(x)$ then

Mean $\mu_x = \int_{-\infty}^{\infty} x f(x)\,dx$

$$\operatorname{var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \int_{-\infty}^{\infty} x f(x)\,dx.$$

(i) $$\mathrm{Var}(X+K) = \int_{-\infty}^{\infty} (x+K)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} (x+K)\, f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} (x^2 + 2Kx + K^2)\, f(x)\,dx - \left[\int_{-\infty}^{\infty} x\, f(x)\,dx + \int_{-\infty}^{\infty} K\cdot f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + 2K\int_{-\infty}^{\infty} x f(x)\,dx + K^2\int_{-\infty}^{\infty} f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx + K\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= E(X^2) + 2K\,E(X) + K^2 - [E(X) + K]^2$$

$$= E(X^2) + 2K\,E(X) + K^2 - [E(X)]^2 - 2K\,E(X) - K^2$$

$$= E(X^2) - [E(X)]^2$$

$$= \mathrm{var}(X)$$

$$\therefore \mathrm{var}(X+K) = V(X)$$

ii. Consider

$$V(KX) = \int_{-\infty}^{\infty} (Kx)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} Kx\, f(x)\,dx\right]^2$$

$$= K^2\int_{-\infty}^{\infty} x^2 f(x)\,dx - K^2\left[\int_{-\infty}^{\infty} x\, f(x)\,dx\right]^2$$

$$= K^2\left[\int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x\, f(x)\,dx\right]^2\right]$$

$$= K^2\left[E(X^2) - [E(X)]^2\right] = K^2\,\mathrm{var}(X)$$

Moments

The "moments" of a random variable (or of its distribution) are expected values of powers or related functions of random variable.

There are two types of moments

(1) Moment about the origin

(2) Moment about the Mean.

Moment about the origin: (μ_n')

The n^{th} ($n \in N$) Moment of the Random Variable X about the origin is Expected value of X^n, it is denoted by μ_n'

$$\mu_n' = E(X^n)$$

$$\Rightarrow \mu_n' = E(X^n) = \begin{cases} \sum_{i=1}^{N} x_i^n p_i & \text{if X is Discrete} \\ \int_{-\infty}^{\infty} x^n . f(x)dx & \text{if X is continuous} \end{cases}$$

If $n=1$ then

$$\mu_1' = E(X) = \mu$$

(First moment about origin is Mean)

Moment about the Mean (μ_r)

(Central Moment).

The n^{th} Moment of the Random variable x about the Mean (μ) is Expected value of $(x-\mu)^n$. it is denoted by μ_n.

$$\mu_n = E[(x-\mu)^n]$$

$$\Rightarrow \mu_n = E[(x-\mu)^n] = \begin{cases} \sum_{i=1}^{N} (x_i-\mu)^n p_i, & x \text{ is discrete} \\ \int_{-\infty}^{\infty} (x_i-\mu)^n f(x)dx & \text{if } x \text{ is continuous} \end{cases}$$

If $n=2$ then

$$\mu_2 = E[(x-\mu)^2] = \sigma^2$$

(2nd Moment about the Mean is variance)

Note

	about origin $\mu_n' = E(x^n)$	about Mean. $\mu_n = E[(x-\mu)^n]$
n=0	$\mu_0' = E(x^0) = 1$	$\mu_0 = E[(x-\mu)^0] = 1$
n=1	$\mu_1' = E(x) = \mu$	$\mu_1 = E[(x-\mu)] = 0$
n=2	$\mu_2' = E(x^2)$	$\mu_2 = E[(x-\mu)^2] = \sigma^2$
n=3	$\mu_3' = E(x^3)$	$\mu_3 = E[(x-\mu)^3]$
n=4	$\mu_4' = E(x^4)$	$\mu_4 = E[(x-\mu)^4]$

Note (2)

→ The co-efficient of skewness using moment

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left[E[(x-\mu)^3]\right]^2}{\left[E[(x-\mu)^2]\right]^3}$$

→ The co-efficient of kurtosis using moment.

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

Absolute Moment:

the n^{th} order Absolute moment is defined as

$$\beta_n = E\left[|x|^n\right]$$

$$= \begin{cases} \sum_{i=1}^{N} |x_i|^n \cdot p_i & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} |x|^n f(x)\,dx & \text{if } x \text{ is continuous.} \end{cases}$$

Problem:

Let x be the discrete Random Variable with the Probability distribution

X	1	2	3
P(x)	1/2	1/3	1/6

then Find 2^{nd} & 3^{rd} Moments.

Given that, x is discrete Random variable

X	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$
P(x)	$p_1 = \frac{1}{2}$	$p_2 = \frac{1}{3}$	$p_3 = \frac{1}{6}$

From Definition of Moment.

$$\mu_n' = E(x^n) = \sum_{i=1}^{N} x_i^n p_i.$$

$$\mu_2' = E(x^2) = \sum_{i=1}^{3} x_i^2 p_i.$$

$$= x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3$$

$$= 1\left(\tfrac{1}{2}\right) + 4\left(\tfrac{1}{3}\right) + 9\left(\tfrac{1}{6}\right)$$

$$= \tfrac{1}{6}[3 + 8 + 9] = \tfrac{20}{6} = \tfrac{10}{3}$$

$\therefore$ The 2^{nd} Moment $\mu_2' = \frac{10}{3}$

$$\mu_3' = E(x^3) = \sum_{i=1}^{3} x_i^3 p_i$$

$$= x_1^3 p_1 + x_2^3 p_2 + x_3^3 p_3$$

$$= 1\left(\tfrac{1}{2}\right) + 8\left(\tfrac{1}{3}\right) + 27\left(\tfrac{1}{6}\right)$$

$$= \tfrac{1}{6}(3 + 16 + 27) = \tfrac{46}{6} = \tfrac{23}{3}$$

The 3^{rd} Moment $\mu_3' = 23/3$.

Problem

Let X be a discrete Random variable with Probability distribution

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Find the 2nd - central Moment.

X be the distrete Random variable with Probability distribution

X	$x_1 = 0$	$x_2 = 1$	$x_3 = 2$	$x_4 = 3$
P(X)	$p_1 = \frac{1}{8}$	$p_2 = \frac{3}{8}$	$p_4 = \frac{3}{8}$	$p_5 = \frac{1}{8}$.

From Definition.

n^{th} order central Moment (Moment about Mean)

$$\mu_n = E((X-\mu)^n)$$

(where $\mu = E(X) = \mu_1'$)

$$= \sum_{i=1}^{N} (x_i - \mu)^n \cdot p_i$$

2nd - central Moment

$$\mu_2 = E[(X-\mu)^2]$$ (Nothing but variance)

$$= \sum_{i=1}^{N} (x_i - \mu)^2 p_i$$

$$\mu = \mu_1' = E(X) = \sum_{i=1}^{N} x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4$$

$$= 0 + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{1}{8}\left[3+6+3\right] = \frac{12}{8} = \frac{3}{2}$$

$$\mu = \frac{3}{2}.$$

Now 2^{nd} order central moment

$$\mu_2 = E((x-\mu)^2)$$

$$= \sum_{i=1}^{N} (x_i - \mu)^2 \cdot p_i$$

$$= (x_1-\mu)^2 p_1 + (x_2-\mu)^2 p_2 + (x_3-\mu)^2 p_3 + (x_4-\mu)^2 p_4$$

$$= \left(0-\frac{3}{2}\right)^2 \frac{1}{8} + \left(1-\frac{3}{2}\right)^2 \frac{3}{8} + \left(2-\frac{3}{2}\right)^2 \frac{3}{8} + \left(3-\frac{3}{2}\right)^2 \frac{1}{8}$$

$$= \frac{9}{4}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{3}{8} + \frac{1}{4}\cdot\frac{3}{8} + \frac{9}{4}\cdot\frac{1}{8}$$

$$= \frac{1}{32}\left[9+3+3+9\right] = \frac{24}{32} = \frac{3}{4}.$$

$$\mu_2 = \frac{3}{4}$$ (variance $\sigma^2 = \mu_2 = 3/4$)

Problem:

If $f(x) = \begin{cases} \frac{1}{2}(x+1) & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ represents

density function of the continuous Random Variable X, then Find First, and Second Moments.

From Definition:

n^{th} - Moment of X is

$$\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx$$

$n=1$, (First Moment)

$$\mu_1' = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-1}^{1} x \cdot \frac{1}{2}(x+1)\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (x^2+x)\, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\left(\frac{1}{3}+\frac{1}{2}\right) - \left(\frac{-1}{3}+\frac{1}{2}\right) \right]$$

$$= \frac{1}{2} \left[\frac{2}{3} \right] = \frac{1}{3}.$$

$$\mu_1' = \frac{1}{3}$$

$n=2$ (2^{nd} Moment)

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx.$$

$$= \int_{-1}^{1} x^2 \frac{1}{2}(x+1)\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (x^3 + x^2)\, dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\left(\frac{1}{4} + \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right) \right]$$

$$= \frac{1}{2} \left[\frac{2}{3} \right]$$

$$\mu_2' = \frac{1}{3}$$

Chebyshev's inequality:

Let X be a random variable having finite Mean μ. and finite variance σ^2.

Let $k \in \mathbb{R}^+$ (i.e $k > 0$) then the following inequality called Chebyshev's inequality.

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Model Paper Questions.

i.	Find (i) K (ii) Mean (iii) variance in which the Probability density function of a random variable 'X' is given by $$f(x)=\begin{cases}K(1-x^2), 0<x<1\\0, otherwise\end{cases}$$
ii.	Find the expected value of 'X' and standard deviation for the following discrete distribution.

X	1	2	3	4	5	6
$P\ X$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

OR

i.	Find the expected gain of the player if a player tosses 3 fair coins & he wins Rs. 500 if 3 heads appear, Rs 300 if 2 heads appear, Rs 100 if 1 head appears. On the other hand he loses Rs 1500 if 3 tails occur.
ii.	Calculate the probability that an IC chip will work properly (i) less than 8 years (ii) Beyond 8 years (iii) Between 5 to 7 years (iv) Anywhere from 2 to 5 years if the trouble shooting capability of an IC chip in a circuit is a random variable X where distribution function is given by $$f(x)=\begin{cases}0,\ for\ x\le 3\\1-\frac{9}{x^2}\ for\ x>3\end{cases}$$

i. Find (i) the value of K (ii) P(x<6), P(x≥6) (iii) P(0<x<5) if the random variable X has the following probability function.

X	0	1	2	3	4	5	6	7
P (X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

ii. Find (i) $E\ X$, E(X^2) and V(X) of the density function of a random variable `X'

is $f\ x \ = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & otherwise \end{cases}$

OR

i. Find (i) K (ii) mean (iii) variance of a continuous random variable which has the

probability density function, $f\ x \ = \begin{cases} Kxe^{-\lambda x}, & x \geq 0, \lambda > 0 \\ 0, & otherwise \end{cases}$

ii. Prepare the distribution function, for the random variable `X' which denotes the sum of the two numbers that appear when a pair of dice is tossed. Also find mean, variance of the distribution.