数学物理方程习题解答

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Contents

Chapter 1

波动方程

1.1 方程的导出，定解条件

1. 细杆或弹簧受某种外界原因而产生纵向振动, 以 $u(x,t)$ 表示静止时在 x 点处的点在时刻 t 离开原来位置的偏移. 假设振动过程中所发生的张力服从胡克定律, 试证明 $u(x,t)$ 满足方程:

$$\frac{\partial}{\partial t}\left(\rho(x)\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right),$$

其中 ρ 为杆的密度, E 为杨氏模量[1].

Proof. 记杆的横截面积为 S, 取杆的左端截面的形心为原点, 杆轴为 x 轴, 任取静止时坐标为 $(x,x+\Delta x)$ 的一小段细杆 B. 在 t 时刻, B 的两段位移分别为 $u(x,t)$ 和 $u(x+\Delta x,t)$, B 的伸长为 $\Delta L=u(x+\Delta x,t)-u(x,t)$, 故 B 的应变为

$$\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}$$

令 $\Delta x\to 0$ 即得 x 点在时刻 t 的应变为 $\frac{\partial u}{\partial x}(x,t)$, 因此 B 两端的张力分别为 $ES\frac{\partial u}{\partial x}(x,t)$ 和 $ES\frac{\partial u}{\partial x}(x+\Delta x,t)$, 故 B 的运动方程为

$$ES\frac{\partial u}{\partial x}(x+\Delta x,t)-ES\frac{\partial u}{\partial x}(x,t)=S\rho(x)\Delta x\frac{\partial^2 u}{\partial t^2}(\bar{x},t).$$

消去 S 并令 $\Delta x\to 0$ 即得所证. □

2. 在杆纵向振动时, 假设 (1) 端点固定, (2) 端点自由, (3) 端点固定在弹性支承上, 试分别导出这三种情况下所对应的边界条件.

[1]杨氏模量 $E=\frac{F/S}{\Delta L/L}$ 衡量的是一个各向同性弹性体的刚度 (stiffness), 定义为在胡克定律适用的范围内, 单轴应力和单轴形变之间的比.

Proof. 内容... □

3. 试证: 圆锥形枢轴的纵振动方程为

$$E\frac{\partial}{\partial x}\left[\left(1-\frac{x}{h}\right)^2\frac{\partial u}{\partial x}\right]=\rho\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2}$$

其中 h 为圆锥的高.

Proof. 内容... □

4. 绝对柔软而均匀的弦线有一端固定, 在它本身重力的作用下, 此线处于铅锤的平衡位置, 试导出此线的微小横振动方程.

Solution. 内容... □

5. 一柔软均匀的细弦, 一端固定, 另一段是弹性支承. 设该弦在阻力与速度成正比的介质中作微小的横振动, 试写出弦的位移所满足的定解问题.

Solution. 由教材推导过程知此时满足:

$$\int_t^{t+\Delta t}\int_x^{x+\Delta x}\left(T\frac{\partial^2 u(x,t)}{\partial x^2}\ -\rho\frac{\partial^2 u(x,t)}{\partial t^2}-b\frac{\partial u(x,t)}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t=0.$$

因此定解问题为 (不妨设弹性支承在右端):

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}+\frac{b}{\rho}\frac{\partial u}{\partial t}=0,\\ u(x,0)=\varphi(x),\frac{\partial u(x,0)}{\partial t}=\psi(x)\\ u|_{x=0}=0,\left(\frac{\partial u}{\partial x}+\sigma u\right)|_{x=l}=0.\end{cases}$$ □

6. 若 $F(\xi)$, $G(\xi)$ 均为其变元的二次连续可导函数, 验证 $F(x-at)$, $G(x+at)$ 均满足弦振动方程 (1.11).

Proof. 直接验证即可. □

7. 验证

$$u(x,y,t)=\frac{1}{\sqrt{t^2-x^2-y^2}}$$

在锥 $t^2-x^2-y^2>0$ 中满足波动方程

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

Proof. 直接计算

$$\frac{\partial u}{\partial t} = -t(t^2-x^2-y^2)^{-3/2}, \quad \frac{\partial^2 u}{\partial t^2} = (t^2-x^2-y^2)^{-5/2}(2t^2+x^2+y^2),$$

$$\frac{\partial u}{\partial x} = x(t^2-x^2-y^2)^{-3/2}, \quad \frac{\partial^2 u}{\partial x^2} = (t^2-x^2-y^2)^{-5/2}(t^2-2x^2+y^2).$$

同理

$$\frac{\partial u^2}{\partial y^2} = (t^2-x^2-y^2)^{-5/2}(t^2+2y^2-x^2).$$

结合以上三式可得

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \qquad \square$$

1.2 达朗贝尔公式、波的传播

1. 设 $h>0$ 为常数, 证明方程

$$\frac{\partial}{\partial x}\left[\left(1-\frac{x}{h}\right)^2\frac{\partial u}{\partial x}\right] = \frac{1}{a^2}\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2}$$

的通解可以写成

$$u = \frac{F(x-at)+G(x+at)}{h-x},$$

其中 F, G 为任意的具有二阶连续导数的单变量函数, 并由此求它满足初始条件

$$t=0: u=\varphi(x), \frac{\partial u}{\partial t} = \psi(x)$$

的初值问题的解.

Solution. 原方程等价于

$$\frac{\partial^2}{\partial x^2}[(h-x)u] = \frac{1}{a^2}\frac{\partial^2}{\partial t^2}[(h-x)u].$$

故存在函数 F, G 使得

$$(h-x)u = F(x-at)+G(x+at),$$

也即

$$u = \frac{F(x-at)+G(x+at)}{h-x}.$$

初值问题的解为

$$u(x,t) = \frac{1}{h-x}\left[\frac{1}{2}(h-x+at)\varphi(x-at) + \frac{1}{2}(h-x-at)\varphi(x+at) + \frac{1}{2a}\int_{x-at}^{x+at}(h-\xi)\psi(\xi)\,\mathrm{d}\xi\right]. \qquad \square$$

2. 问初始条件 $\varphi(x)$ 与 $\psi(x)$ 满足怎样的条件时, 齐次波动方程初值问题的解仅由右传播波组成?

Solution. If the solution to the homogeneous IVP only consists of right propagation waves, then

$$G(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_{x_0}^{x} \psi(\alpha)\,\mathrm{d}\alpha - \frac{C}{2\alpha}$$

is a constant, which is equivalent to $\varphi'(x) + \frac{1}{a}\psi(x) = 0$. □

3. 利用传播波法, 求解波动方程的古尔萨 (Goursat) 问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u|_{x-at=0} = \varphi(x), \\ u|_{x+at=0} = \psi(x),\ \varphi(0) = \psi(0). \end{cases}$$

Solution. 设 $u(x,t)$ 具有行波解 $u(x,t) = F(x-at) + G(x+at)$, 由边界条件得

$$F(0) + G(2x) = \varphi(x),$$

$$F(2x) + G(0) = \psi(x).$$

由上式得 $F(x) = \psi(x/2) - G(0)$, $G(x) = \varphi(x/2) - F(0)$. 取 $(x,t) = (0,0)$, 得

$$u(0,0) = F(0) + G(0) = \varphi(0) = \psi(0).$$

从而

$$u(x,t) = \psi\left(\frac{x-at}{2}\right) + \varphi\left(\frac{x+at}{2}\right) - \varphi(0).$$ □

5. Solve

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x > 0, t > 0, \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = 0, \\ u_x - k u_t|_{x=0} = 0, \end{cases}$$

in which k is a constant.

Solution. Suppose the solution is

$$u(x,t) = F(x-at) + G(x+at).$$

Then according to the initial value condition we have

$$\begin{aligned} u(x,0) &= F(x) + G(x) = \varphi(x), \\ u_t(x,0) &= -aF'(x) + aG'(x) = 0. \end{aligned} \tag{1.1}$$

The solution to (**??**) is

$$\begin{cases} F(x) = \frac{1}{2}\varphi(x) + C, \\ G(x) = \frac{1}{2}\varphi(x) - C, \end{cases} \tag{1.2}$$

where C satisfies

$$F(0) + G(0) = 2C. \tag{1.3}$$

If $x - at \geqslant 0$, then

$$u(x,t) = \frac{1}{2}\big(\varphi(x+at) + \varphi(x-at)\big).$$

If $x - at \leqslant 0$, then by the boundary condition we have

$$F'(-at) = \frac{ka-1}{ka+1}G'(at). \tag{1.4}$$

Integrate on both sides to get

$$F(x) = -\frac{ka-1}{ka+1}G(-x), \tag{1.5}$$

where

$$C_1 = F(0) + \frac{ka-1}{ka+1}G(0). \tag{1.6}$$

Hence by (**??**), (**??**), (**??**) and (**??**) we have

$$\begin{aligned} u(x,t) &= F(x-at) + G(x+at) \\ &= -\frac{ka-1}{ka+1}G(at-x) + G(x+at) + C_1 \\ &= -\frac{ka-1}{2(ka+1)}\varphi(at-x) + \frac{ka-1}{ka+1}C + \frac{1}{2}\varphi(x+at) - C + C_1 \\ &= \frac{ka-1}{2(ka+1)}\varphi(at-x) + \frac{1}{2}\varphi(x+at) + \frac{ka}{ka+1}\varphi(0). \end{aligned}$$

□

6. 求解初边值问题

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad 0 < t < kx, k > 1, \\ u|_{t=0} = \varphi_0(x), \quad x \geqslant 0, \\ u_t|_{t=0} = \varphi_1(x), \quad x \geqslant 0, \\ u|_{t=kx} = \psi(x), \end{cases}$$

其中 $\varphi_0(0) = \psi(0)$.

Solution. If $x - t \geqslant 0$, by d'Alembert formula we have

$$u(x,t) = \frac{1}{2}(\varphi_0(x-t) + \varphi_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \varphi_1(\xi)\,\mathrm{d}\xi.$$

If $x - t < 0$, suppose $u(x,t) = F(x-t) + G(x+t)$, then

$$F((1-k)x) + G((1+k)x) = \psi(x). \tag{1.7}$$

Since

$$F(0) + G(2x) = \frac{1}{2}(\varphi_0(0) + \varphi_0(2x)) + \frac{1}{2}\int_0^{2x} \varphi_1(\xi)\,\mathrm{d}\xi. \tag{1.8}$$

It follows that

$$F(0) + G((1+k)x) = \frac{1}{2}(\varphi_0(0) + \varphi_0((1+k)x)) + \frac{1}{2}\int_0^{(k+1)x} \varphi_1(\xi)\,\mathrm{d}\xi. \tag{1.9}$$

By (**??**) and (**??**) we have

$$F((1-k)x) - F(0) = \psi(x) - \frac{1}{2}(\varphi_0(0) + \varphi_0((1+k)x)) - \frac{1}{2}\int_0^{(k+1)x} \varphi_1(\xi)\,\mathrm{d}\xi. \tag{1.10}$$

So

$$\begin{aligned} F(x-t) - F(0) = \psi\left(\frac{x-t}{1-k}\right) - \frac{1}{2}\left(\varphi_0(0) + \varphi_0\left(\frac{1+k}{1-k}(x-t)\right)\right) \\ - \frac{1}{2}\int_0^{\frac{k+1}{1-k}(x-t)} \varphi_1(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{1.11}$$

By (**??**) we have

$$F(0) + G(x+t) = \frac{1}{2}(\varphi_0(0) + \varphi_0(x+t)) + \frac{1}{2}\int_0^{x+t} \varphi_1(\xi)\,\mathrm{d}\xi. \tag{1.12}$$

Combining (**??**) and (**??**), we find

$$\begin{aligned} u(x,t) = F(x-t) + G(x+t) = \psi\left(\frac{x-t}{1-k}\right) + \frac{1}{2}\left[\varphi_0(x+t) - \varphi_0\left(\frac{1+k}{1-k}(x-t)\right)\right] \\ + \frac{1}{2}\int_{\frac{k+1}{1-k}(x-t)}^{x+t} \varphi_1(\xi)\,\mathrm{d}\xi. \end{aligned}$$

□

7. Solve the following initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad f(t) < x < t, \\ u|_{x=t} = \varphi(t), \\ u|_{x=f(t)} = \psi(t), \end{cases}$$

where $\varphi(0) = \psi(0) = 0$ and $x = f(t)$ is a smooth curve passing the origin point and lying between $x = t$ and $x = -t$, and $|f'(t)| \neq 1$ for all t.

Solution. Let

$$u(x,t)=F(x-t)+G(x+t).$$

Then

$$\begin{cases} u(t,t)=F(0)+G(2t)=\varphi(t), \\ u(f(t),t)=F(f(t)-t)+G(f(t)+t)=\psi(t). \end{cases} \tag{1.13}$$

By the first equality of (**??**) we have

$$G(t)=\varphi\Big(\frac{t}{2}\Big)-F(0). \tag{1.14}$$

Let $s=f(t)-t$, then by the inverse function theorem we know that t can be represented as the function of s, say, $t=g(s)$. Hence $f(t)+t=f(t)-t+2t=s+2g(s)$ and

$$F(s)=\psi(g(s))-G(s+2g(s)). \tag{1.15}$$

So by (**??**) and (**??**),

$$F(x-t)=\psi(g(x-t))-\varphi\bigg(g(x-t)+\frac{x-t}{2}\bigg)+F(0). \tag{1.16}$$

On the other hand,

$$G(x+t)=\varphi\Big(\frac{x+t}{2}\Big)-F(0). \tag{1.17}$$

By (**??**) and (**??**) we find

$$u(x,t)=\varphi\bigg(\frac{x+t}{2}\bigg)-\varphi\bigg(g(x-t)+\frac{x-t}{2}\bigg)+\psi(g(x-t)). \qquad \square$$

8. 求解波动方程的初值问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=t\sin x, \\ u|_{t=0}=0,\ \frac{\partial u}{\partial t}\Big|_{t=0}=\sin x. \end{cases}$$

Solution.

$$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t}\sin\xi\,\mathrm{d}\xi+\frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}\tau\sin\xi\,\mathrm{d}\xi\,\mathrm{d}\tau=t\sin x. \qquad \square$$

9. 求解波动方程的初值问题

$$\begin{cases} u_{tt}=a^2u_{xx}+\frac{tx}{(1+x^2)^2}, \\ u|_{t=0}=0, \\ u_t|_{t=0}=\frac{1}{1+x^2}. \end{cases}$$

Solution.
$$\begin{aligned}u(x,t)&=\frac{1}{2a}\int_{x-at}^{x+at}\frac{1}{1+\xi^2}\,\mathrm{d}\xi+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\frac{\tau\xi}{(1+\xi^2)^2}\,\mathrm{d}\xi\,\mathrm{d}\tau\\&=-\frac{1}{4a^3}\left[\frac{1}{2}\ln\frac{1+(x-at)^2}{1+(x+at)^2}-2at\arctan x\right.\\&\quad\left.+(x+at-2a^2)\arctan(x+at)-(x-at-2a^2)\arctan(x-at)\right].\end{aligned}$$
□

1.3 初边值问题的分离变量法

1. 用分离变量法求下列问题的解:

(1) $\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2},\\u(x,0)=\sin\frac{3\pi x}{l},\quad u_t(x,0)=x(l-x)\quad(0<x<l),\\u(0,t)=u(l,t)=0.\end{cases}$

(2) $\begin{cases}\frac{\partial^2u}{\partial t^2}-a^2\frac{\partial^2u}{\partial x^2}=0,\\u(x,0)=\frac{h}{l}x,\quad u_t(x,0)=0,\\u(0,t)=0,\quad u_x(l,t)=0.\end{cases}$

Solution. (2) 边界条件是线性的, 故设 $u(x,t)=X(x)T(t)$, 由边界条件知 $X(x)$ 满足定解问题
$$\begin{cases}X''(x)+\lambda X(x)=0,\\X(0)=0,\quad X'(l)=0.\end{cases}$$
根据 λ 的符号分类讨论得
$$X_k(x)=C_k\sin\frac{(2k+1)\pi}{2l}x,\quad k=0,1,2,\dots$$
其中相应的特征值 $\lambda_k=\left(\frac{2k+1}{2l}\pi\right)^2$. 由 $T''(t)+\lambda_ka^2T(t)=0$ 解得
$$T_k(t)=A_k\cos\frac{(2k+1)a\pi}{2l}t+B_k\sin\frac{(2k+1)a\pi}{2l}t.$$
于是
$$u(x,t)=\sum_{k=0}^{\infty}\left(A_k\cos\frac{(2k+1)\pi a}{2l}t+B_k\sin\frac{(2k+1)\pi a}{2l}t\right)\sin\frac{(2k+1)\pi}{2l}x.$$
再根据初始条件得
$$\sum_{k=0}^{\infty}A_k\sin\frac{(2k+1)\pi}{2l}x=\frac{h}{l}x,$$

$$\sum_{k=0}^{\infty} B_k \frac{(2k+1)\pi a}{2l} \sin \frac{(2k+1)\pi}{2l} x = 0.$$

利用三角函数序列 $\left(\sin \frac{(2k+1)\pi}{2l} x\right)_{k \geqslant 0}$ 在区间 $[0, l]$ 上的正交性, 即

$$\int_0^l \sin \frac{(2m+1)\pi}{2l} x \cdot \sin \frac{(2n+1)\pi}{2l} x \, \mathrm{d}x = \frac{l}{2} \delta_{mn},$$

得

$$A_k = \frac{2}{l} \int_0^l \frac{h}{l} x \sin \frac{(2k+1)\pi}{2l} x \, \mathrm{d}x = (-1)^k \frac{8h}{(2k+1)^2 \pi^2}, \quad B_k = 0.$$

因此

$$u(x,t) = \sum_{k=0}^{\infty} (-1)^k \frac{8h}{(2k+1)^2 \pi^2} \cos \frac{(2k+1)\pi a t}{2l} \sin \frac{(2k+1)\pi x}{2l}. \qquad \square$$

注. $\int_{-l}^{l} \sin \alpha x \sin \beta x \, \mathrm{d}x = 0$ 及 $\int_0^l \sin \alpha x \sin \beta x \, \mathrm{d}x = 0$ 的充要条件为 $(\alpha \pm \beta) l = k\pi \neq 0$. 而且正弦函数于余弦函数总是正交的, 即 $\int_{-l}^{l} \sin \alpha x \cos \beta x \equiv 0$。

2. 设弹簧一端固定, 一端在外力作用下做周期振动, 此时定解问题归结为

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u(0,t) = 0, \quad u(l,t) = A \sin^2 \omega t, \\ u(x,0) = u_t(x,0) = 0. \end{cases}$$

求解此问题.

Solution. 边界条件非齐次, 故令 $U(x,t) = \frac{x}{l} A \sin^2 \omega t$, $V(x,t) = u(x,t) - U(x,t)$, 则 $V(x,t)$ 满足的定解问题是

$$\begin{cases} \frac{\partial^2 V}{\partial t^2} - a^2 \frac{\partial^2 V}{\partial x^2} = -\frac{2\omega^2 A}{l} x \cos 2\omega t, \\ V(0,t) = 0, \quad V(l,t) = 0, \\ V(x,0) = 0, \quad \frac{\partial V(x,0)}{\partial t} = 0. \end{cases}$$

运用齐次化原理知该问题的解为

$$u(x,t) = \sum_{k=1}^{\infty} \int_0^t B_k(\tau) \sin \frac{k\pi a}{l} (t - \tau) \, \mathrm{d}\tau \cdot \sin \frac{k\pi}{l} x.$$

其中

$$\begin{aligned} B_k(\tau) &= \frac{2}{k\pi a} \int_0^l f(\xi, \tau) \sin \frac{k\pi}{l} \xi \, \mathrm{d}\xi \\ &= \frac{2}{k\pi a} \int_0^l -\frac{2\omega^2 A}{l} \xi \cos 2\omega\tau \sin \frac{k\pi}{l} \xi \, \mathrm{d}\xi \\ &= \frac{4(-1)^k \omega^2 l A \cos 2\omega\tau}{k^2 \pi^2 a}. \end{aligned} \qquad \square$$

4. 用分离变量法求解初边值问题:

$$\begin{cases} u_{tt} - a^2 u_{xx} = g, \quad 0 < x < l, \quad t > 0, \\ u|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = \sin \frac{\pi x}{2l}. \end{cases}$$

其中 g 为常数.

Solution. Let $v = u + \frac{g}{2a^2}x(x-2l)$, then it is straightforward to verify that v satisfies the following equation

$$\begin{cases} v_{tt} - a^2 v_{xx} = 0, 0 < x < l, t > 0, \\ v(0,t) = \frac{\partial v}{\partial x}(l,t) = 0, \\ v(x,0) = \frac{g}{2a^2}x(x-2l), \quad \frac{\partial v}{\partial t}(x,0) = \sin \frac{\pi x}{2l}. \end{cases}$$

Let

$$v(x,t) = X(x)T(t).$$

Then

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda.$$

First of all, since $X''(x) + \lambda X(x) = 0$ with $X(0) = X'(l) = 0$, we have

- If $\lambda \leqslant 0$, there only exists trivial solution $X \equiv 0$;
- If $\lambda > 0$, we have $\lambda_k = \left(\frac{(2k+1)\pi}{2l}\right)^2$ and $X_k(x) = C_k \sin \frac{(2k+1)\pi}{2l} x$.

On the other hand, since $T(t)$ satisfies

$$T''(t) + \left(\frac{(2k+1)\pi a}{2l}\right)^2 T = 0.$$

We can solve that

$$T_k(t) = A_k \cos \frac{(2k+1)\pi a}{2l} t + B_k \sin \frac{(2k+1)\pi a}{2l} t. \tag{1.18}$$

Therefore

$$v(x,t) = \sum_{k=0}^{\infty} \left(A_k \cos \frac{(2k+1)\pi a}{2l} t + B_k \sin \frac{(2k+1)\pi a}{2l} t \right) \sin \frac{(2k+1)\pi}{2l} x. \tag{1.19}$$

Combining with the initial value condition we have

$$\begin{aligned} A_k &= \frac{2}{l} \int_0^l \frac{g}{2a^2} x(x-2l) \sin \frac{(2k+1)\pi}{2l} x \, \mathrm{d}x = -\frac{16 l^2 g}{(2k+1)^3 a^2 \pi^3}, \\ B_0 &= \frac{2l}{\pi a}, \quad B_k = 0 \quad (k \geqslant 1). \end{aligned} \tag{1.20}$$

Therefore,

$$
\begin{aligned}
u(x,t) = \sum_{k=0}^{\infty} & \left(A_k \cos \frac{(2k+1)\pi a}{2l} t + B_k \sin \frac{(2k+1)\pi a}{2l} t \right) \sin \frac{(2k+1)\pi}{2l} x \\
& - \frac{g}{2a^2} x(x-2l),
\end{aligned}
$$

where the coefficients A_k, B_k are given by (**??**). □

5. 用分离变量法求下面问题的解:

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b \sinh x, \\
u|_{x=0} = u|_{x=l} = 0, \\
u|_{t=0} = u_t|_{t=0} = 0.
\end{cases}
$$

Solution. 利用齐次化原理, 方程的解为

$$
u(x,t) = \int_0^t W(x,t;\tau)\,\mathrm{d}\tau,
$$

其中 W 为下面方程的解:

$$
\begin{cases}
\frac{\partial^2 W}{\partial t'^2} - a^2 \frac{\partial^2 W}{\partial x^2} = 0, t' > 0, \\
W|_{t'=0} = 0, \quad \frac{\partial W}{\partial t'}|_{t'=0} = b \sinh x, \\
W|_{x=0} = W|_{x=l} = 0.
\end{cases}
$$

其通解为

$$
W(x,t;\tau) = \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi a}{l} t' + B_k \sin \frac{k\pi a}{l} t' \right) \sin \frac{k\pi}{l} x,
$$

其中

$$
A_k = 0, \quad k = 1, 2, \cdots
$$

$$
B_k = \frac{2}{k\pi a} \int_0^l b \sinh x \sin \frac{k\pi}{l} x\,\mathrm{d}x = (-1)^{k+1} \frac{2bl \sinh l}{a(l^2 + k^2\pi^2)}.
$$

故原方程解为

$$
u(x,t) = \frac{2bl^2 \sinh l}{\pi a^2} \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n(l^2 + n^2\pi^2)} \left(1 - \cos \frac{n\pi a t}{l} \right) \sin \frac{n\pi x}{l}.
$$

□

6. Solve the following problem by separation of variables:

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + 2b \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (b > 0), \\
u|_{x=0} = u|_{x=l} = 0, \\
u|_{t=0} = \frac{h}{l} x, \ \frac{u}{t}\Big|_{t=0} = 0.
\end{cases}
$$

Solution. Let $u(x,t)=X(x)T(t)$, then

$$\frac{T''+2bT'}{a^2T}=\frac{X''}{X}=-\lambda.$$

First of all, $X''+\lambda X=0$.

- If $\lambda\leqslant 0$, there only exists trivial solution $X\equiv 0$;
- If $\lambda>0$, then $\lambda_k=\left(\frac{k\pi}{l}\right)^2$ and $X_k(x)=C_k\sin\frac{k\pi x}{l}$.

Thus T satisfies the equation

$$T''+2bT'+\left(\frac{k\pi a}{l}\right)^2T=0,$$

of which the characteristic equation is

$$\lambda^2+2b\lambda+\left(\frac{k\pi a}{l}\right)^2=0.$$

We suppose that b is sufficiently small such that $\Delta=4b^2-4\left(\frac{k\pi a}{l}\right)^2<0$ for all $k\geqslant 1$, then

$$\lambda=-b\pm\sqrt{\left(\frac{k\pi a}{l}\right)^2-b^2}i,$$

and

$$T_k(t)=e^{-bt}\left(A_k\sin\sqrt{\left(\frac{k\pi a}{l}\right)^2-b^2}t+B_k\cos\sqrt{\left(\frac{k\pi a}{l}\right)^2-b^2}t\right).$$

Therefore,

$$u(x,t)=e^{-bt}\sum_{k=1}^{\infty}\sin\frac{k\pi x}{l}\left(A_k\sin\sqrt{\left(\frac{k\pi a}{l}\right)^2-b^2}t+B_k\cos\sqrt{\left(\frac{k\pi a}{l}\right)^2-b^2}t\right).$$

Combining with the initial value condition we have

$$u|_{t=0}=\sum_{k=1}^{\infty}B_k\sin\frac{k\pi x}{l}=\frac{h}{l}x,$$

and

$$u_t|_{t=0}=-b\sum_{k=1}^{\infty}B_k\sin\frac{k\pi x}{l}+\sum_{k=1}^{\infty}A_k\sqrt{\left(\frac{k\pi a}{l}\right)^2-b^2}\sin\frac{k\pi x}{l}=0,$$

from which we solve that

$$B_k=\frac{(-1)^{k+1}2h}{k\pi},\qquad A_k=\frac{(-1)^{k+1}2bh}{k\pi\sqrt{\left(\frac{k\pi a}{l}\right)^2-b^2}}.$$

□

7. (补充题目) 验证 $u(x,t)=\int_0^t W(x,t;\tau)\,\mathrm{d}\tau$ 是初边值问题 (3.23)–(3.25) 的解.

Proof. 由于 $W(x,t;\tau)$ 满足条件

$$\begin{cases}\frac{\partial^2 W}{\partial t^2}-a^2\frac{\partial^2 W}{\partial x^2}=0 \quad (t>\tau),\\ t=\tau: W=0,\ \frac{\partial W}{\partial t}=f(x,\tau),\\ W(0,t;\tau)=W(l,t;\tau)=0.\end{cases}$$

故当 $t=0$ 时, $u(x,0)=0$, $u_t(x,0)=W(x,0;0)=0$. 当 $x=0$ 时,

$$u(0,t)=\int_0^t W(0,t;\tau)\,\mathrm{d}\tau=0.$$

当 $x=l$ 时,

$$u(l,t)=\int_0^t W(l,t;\tau)\,\mathrm{d}\tau=0.$$

所以 $u(x,t)$ 满足初边值条件 (3.24), (3.25). 下面验证 $u(x,t)$ 满足方程 (3.23). 由于 $u(x,t)=\int_0^t W(x,t;\tau)\,\mathrm{d}\tau$, 故

$$\frac{\partial u}{\partial t}=W(x,t;t)+\int_0^t \frac{\partial W}{\partial t}\,\mathrm{d}\tau=\int_0^t \frac{\partial W}{\partial t}(x,t;\tau)\,\mathrm{d}\tau.$$

再求导可得

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}&=\frac{\partial W}{\partial t}(x,t;t)+\int_0^t \frac{\partial^2 W}{\partial t^2}(x,t;\tau)\,\mathrm{d}\tau\\ &=f(x,t)+\int_0^t \frac{\partial^2 W}{\partial t^2}(x,t;\tau)\,\mathrm{d}\tau.\end{aligned}$$

又因为

$$\frac{\partial^2 u}{\partial x^2}=\int_0^t \frac{\partial^2 W}{\partial x^2}(x,t;\tau)\,\mathrm{d}\tau.$$

于是

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}&=f(x,t)+\int_0^t \left(\frac{\partial^2 W}{\partial t^2}(x,t;\tau)-a^2\frac{\partial^2 W}{\partial x^2}(x,t;\tau)\right)\mathrm{d}\tau\\ &=f(x,t).\end{aligned}$$

□

1.4 高维波动方程的柯西问题

1. 利用泊松公式求解波动方程的柯西问题:

(1) $\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u|_{t=0} = 0,\ u_t|_{t=0} = x^2 + yz; \end{cases}$

(2) $\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u|_{t=0} = x^3 + y^2 z,\ u_t|_{t=0} = 0. \end{cases}$

Solution. (1) By Poisson's formula the solution is

$$\begin{aligned} u(x,y,x,t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\iint_{S_{at}^M} 0 \,\mathrm{d}S\right) + \frac{1}{4\pi a^2 t}\iint_{S_{at}^M} (\xi^2 + \eta\zeta)\,\mathrm{d}S \\ &= \frac{1}{4\pi a^2 t}\int_0^{\pi}\int_0^{2\pi}\big[(x + at\sin\theta\cos\varphi)^2 \\ &\quad + (y + at\sin\theta\sin\varphi)(z + at\cos\theta)\big]a^2t^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta \\ &= (x^2 + yz)t + a^2t^3/3. \end{aligned}$$

(2) By Poisson's formula the solution is

$$\begin{aligned} u(x,y,z,t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\iint_{S_{at}^M} \xi^3 + \zeta^2\eta\,\mathrm{d}S\right) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\int_0^{\pi}\int_0^{2\pi}\big[(x + at\sin\theta\cos\phi)^3\right. \\ &\qquad \left. + (y + at\sin\theta\phi)^2(z + at\cos\theta)\big]a^2t^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\right) \\ &= x^3 + y^2z + 3a^2t^2x + a^2t^2z. \end{aligned}$$

□

2. 试用降维法导出弦振动方程的达朗贝尔公式.

Solution. 考虑一维波动方程

$$\begin{cases} u_{tt} = a^2 u_{xx}, \\ u|_{t=0} = \varphi(x), \\ u_t|_{t=0} = \psi(x). \end{cases}$$

令 $\tilde{u}(x,y,z,t) = u(x,t)$, $\tilde{\varphi}(x,y,z) = \varphi(x)$, $\tilde{\psi}(x,y,z) = \psi(x)$, 则 $\tilde{u}$ 满足三维波动方程

$$\begin{cases} \tilde{u}_{tt} = a^2\big(\tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{u}_{zz}\big), \\ \tilde{u}|_{t=0} = \tilde{\varphi}, \\ \tilde{u}_t|_{t=0} = \tilde{\psi}. \end{cases}$$

由 Poisson 公式得

$$\tilde{u}(x,y,z,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\iint_{S_{at}^M} \tilde{\varphi}(\xi,\eta,\zeta)\,\mathrm{d}S\right) + \frac{1}{4\pi a^2 t}\iint_{S_{at}^M} \tilde{\psi}(\xi,\eta,\zeta)\,\mathrm{d}S.$$

于是

$$u(x,t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\iint_{S_{at}^M}\varphi(\xi)\,\mathrm{d}S\right)+\frac{1}{4\pi a^2 t}\iint_{S_{at}^M}\psi(\xi)\,\mathrm{d}S.$$

采用球坐标 $\xi=x+r\cos\theta$, $\eta=y+r\sin\theta\cos\psi$, $\zeta=z+r\sin\theta\sin\phi$, 其中 $r=at$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$. 则

$$\begin{aligned}u(x,t)&=\frac{\partial}{\partial t}\left(\frac{1}{4\pi a}\int_0^{2\pi}\int_0^{\pi}\varphi(x+r\cos\theta)r\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\right)\\&\quad+\frac{1}{4\pi a}\int_0^{2\pi}\int_0^{\pi}\psi(x+r\cos\theta)r\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\&=\frac{\partial}{\partial t}\left(-\frac{1}{2a}\int_0^{\pi}\varphi(x+r\cos\theta)\,\mathrm{d}(x+r\cos\theta)\right)\\&\quad-\frac{1}{2a}\int_0^{\pi}\psi(x+r\cos\theta)\,\mathrm{d}(x+r\cos\theta)\\&=\frac{\partial}{\partial t}\left(-\frac{1}{2a}\int_{x+at}^{x-at}\varphi(\xi)\,\mathrm{d}\xi\right)-\frac{1}{2a}\int_{x+at}^{x-at}\psi(\xi)\,\mathrm{d}\xi\\&=\frac{1}{2}\big(\varphi(x+at)+\varphi(x-at)\big)+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi.\end{aligned}$$

□

3. 求解平面波动方程的柯西问题:

(1) $\begin{cases}u_{tt}=a^2(u_{xx}+u_{yy}),\\u|_{t=0}=x^2(x+y),\\u_t|_{t=0}=0.\end{cases}$

(2) $\begin{cases}u_{tt}-3(u_{xx}+u_{yy})=x^3+y^3,\\u|_{t=0}=0,\\u_t|_{t=0}=x^2.\end{cases}$

Solution. (1) By Poisson's formula we have

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{\varphi(x+r\cos\theta, y+r\sin\theta)}{\sqrt{(at)^2-r^2}} r\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{(x+r\cos\theta)^2(x+y+r\cos\theta+r\sin\theta)}{\sqrt{a^2t^2-r^2}} r\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\frac{\pi\left[2x^2(x+y)+r^2(3x+y)\right]}{\sqrt{a^2t^2-r^2}} r\,\mathrm{d}r \\
&= \frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{\pi/2}\frac{2x^2(x+y)+a^2t^2\sin^2\beta(3x+y)}{at\cos\beta} at\sin\beta\cdot at\cos\beta\,\mathrm{d}\beta \\
&= \frac{1}{2a}\frac{\partial}{\partial t}\left(2x^2(x+y)at+\frac{2}{3}a^2t^2(3x+y)\right) \\
&= x^2(x+y)+at^2(3x+y).
\end{aligned}
$$

(2) First of all, we consider the homogeneous equation

$$
\begin{cases}
u_{tt} = 3(u_{xx}+u_{yy}), \\
u|_{t=0} = 0, \\
u_t|_{t=0} = x^2.
\end{cases} \tag{1.21}
$$

By Poisson's formula, the solution to (**??**) is

$$
\begin{aligned}
u_1(x,y,t) &= \frac{1}{2\sqrt{3}\pi}\int_0^{\sqrt{3}t}\int_0^{2\pi}\frac{(x+r\cos\theta)^2}{\sqrt{3t^2-r^2}} r\,\mathrm{d}\theta\,\mathrm{d}r \\
&= x^2t+t^3.
\end{aligned} \tag{1.22}
$$

Then we consider the nonhomogeneous equation with homogeneous initial condition

$$
\begin{cases}
u_{tt} = 3(u_{xx}+u_{yy})+x^3+y^3, \\
u|_{t=0} = 0, \\
u_t|_{t=0} = 0.
\end{cases} \tag{1.23}
$$

By Duhamel's principle, the solution to (**??**) is

$$
u_2(x,y,t) = \int_0^t w(x,y,t;\tau)\,\mathrm{d}\tau, \tag{1.24}
$$

where w is the solution to

$$
\begin{cases}
w_{tt} = 3(w_{xx}+w_{yy}), \\
w|_{t=\tau} = 0, \\
w_t|_{t=\tau} = x^3+y^3.
\end{cases}
$$

By Poisson's formula we have

$$w(x,y,t;\tau)=(t-\tau)(x^3+y^3)+3(t-\tau)^3(x+y).$$

Hence

$$\begin{aligned}u_2(x,y,t)&=\int_0^t(t-\tau)(x^3+y^3)+3(t-\tau)^3(x+y)\,\mathrm{d}\tau\\&=\frac{t^2}{2}(x^3+y^3)+\frac{3}{4}t^4(x+y).\end{aligned}\tag{1.25}$$

By (**??**) and (**??**) we find the solution to the original equation is

$$u(x,y,t)=x^2t+t^3+\frac{t^2}{2}(x^3+y^3)+\frac{3}{4}t^4(x+y).\tag{1.26}$$

□

4. 求二维波动方程的轴对称解 (即形如 $u=u(r,t)$ 的解, 其中 $r=\sqrt{x^2+y^2}$).

Solution. 由于 $\Delta u=u_{rr}+\frac{1}{r}u_r$, 故轴对称解满足方程

$$u_{tt}=a^2\left(u_{rr}+\frac{1}{r}u_r\right).$$

令 $u(r,t)=R(r)T(t)$, 代入上述方程得

$$\frac{T''(t)}{a^2T(t)}=\frac{R''(r)+\frac{1}{r}R'(r)}{R(r)}=-\lambda^2\quad(\lambda>0).$$

故

$$T(t)=C_1\cos a\lambda t+C_2\sin a\lambda t,\quad R(r)=\mathrm{J}_0(\lambda r),$$

其中 J_0 为 0 阶 Bessel 函数, 见附录 III. □

5. 求解柯西问题:

$$\begin{cases}u_{tt}=a^2(u_{xx}+u_{yy})+c^2u,\\ u|_{t=0}=\varphi(x,y),\\ u_t|_{t=0}=\psi(x,y).\end{cases}$$

Solution. 令 $v(x,y,z,t)=\mathrm{e}^{\frac{cz}{a}}u(x,y,t)$, 则

$$\begin{cases}v_{tt}=a^2(v_{xx}+v_{yy}+v_{zz}),\\ v|_{t=0}=\mathrm{e}^{\frac{cz}{a}}\varphi(x,y),\\ v_t|_{t=0}=\mathrm{e}^{\frac{cz}{a}}\psi(x,y).\end{cases}$$

由三维波动方程柯西问题解的 Poisson 公式得

$$v(x,y,z,t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2t}\iint_{S_{at}^M}\mathrm{e}^{\frac{c\zeta}{a}}\varphi(\xi,\eta)\,\mathrm{d}S\right)+\frac{1}{4\pi a^2t}\iint_{S_{at}^M}\mathrm{e}^{\frac{c\zeta}{a}}\psi(\xi,\eta)\,\mathrm{d}S.$$

□

6. 试用齐次化原理导出平面非齐次波动方程

$$u_{tt} = a^2(u_{xx} + u_{yy}) + f(x, y, t)$$

在齐次初始条件

$$\begin{cases} u|_{t=0} = 0, \\ u_t|_{t=0} = 0 \end{cases}$$

下的求解公式.

Solution. 由齐次化原理 $u(x, y, t) = \int_0^t w(x, y, t; \tau) \, \mathrm{d}\tau$, 其中 $w(x, y, t; \tau)$ 为以下定解问题的解:

$$\begin{cases} w_{tt} = a^2(w_{xx} + w_{yy}) \quad (t > \tau), \\ w|_{t=\tau} = 0, \quad w_t|_{t=\tau} = f(x, y, \tau). \end{cases}$$

由二维波动方程柯西问题的泊松公式有:

$$\begin{aligned} w(x, y, t; \tau) &= \frac{1}{2\pi a} \iint_{\Sigma_{at'}^M} \frac{f(\xi, \eta, \tau)}{\sqrt{(at')^2 - (\xi - x)^2 - (\eta - y)^2}} \, \mathrm{d}\xi \, \mathrm{d}\eta \quad (t' = t - \tau) \\ &= \frac{1}{2\pi a} \iint_{\Sigma_{a(t-\tau)}^M} \frac{f(\xi, \eta, \tau)}{\sqrt{a^2(t-\tau)^2 - (\xi - x)^2 - (\eta - y)^2}} \, \mathrm{d}\xi \, \mathrm{d}\eta. \end{aligned}$$

故

$$\begin{aligned} u(x, y, t) &= \int_0^t w(x, y, t; \tau) \, \mathrm{d}\tau \\ &= \frac{1}{2\pi a} \int_0^t \iint_{\Sigma_{a(t-\tau)}^M} \frac{f(\xi, \eta, \tau)}{\sqrt{a^2(t-\tau)^2 - (\xi - x)^2 - (\eta - y)^2}} \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\tau. \end{aligned}$$

□

7. 用降维法求解上面的问题.

Solution. 令 $\tilde{u}(x, y, z, t) = u(x, y, t)$, $\tilde{f}(x, y, z, t) = f(x, y, t)$, 则

$$\begin{cases} \tilde{u}_{tt} = a^2\big(\tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{u}_{zz}\big) + \tilde{f}(x, y, z, t), \\ \tilde{u}|_{t=0} = 0, \\ \tilde{u}_t|_{t=0} = 0. \end{cases}$$

上述问题的解已由教材 (4.47) 式给出, 即

$$\tilde{u}(x, y, z, t) = \frac{1}{4\pi a^2} \iiint_{r \leqslant at} \frac{\tilde{f}(\xi, \eta, \zeta, t - \frac{r}{a})}{r} \, \mathrm{d}V.$$

故

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{4\pi a^2}\iiint_{r\leqslant at}\frac{f(\xi,\eta,t-\frac{r}{a})}{r}\,\mathrm{d}V\\
&= \frac{1}{4\pi a^2}\int_0^{at}\iint_{S_r^M}\frac{f(\xi,\eta,t-\frac{r}{a})}{r}\,\mathrm{d}S_r\,\mathrm{d}r\\
&= \frac{1}{2\pi a^2}\int_0^{at}\iint_{\Sigma_r^M}\frac{f(\xi,\eta,t-\frac{r}{a})}{\sqrt{r^2-(\xi-x)^2-(\eta-y)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}r.
\end{aligned}
\qquad\square
$$

8. 解非齐次方程的柯西问题:

$$
\begin{cases}
u_{tt}=u_{xx}+u_{yy}+u_{zz}+2(y-t),\\
u|_{t=0}=0,\\
u_t|_{t=0}=x^2+yz.
\end{cases}
$$

Solution. 利用叠加原理, 考虑下面两个定解问题:

$$
(\star)\begin{cases}
u_{tt}=u_{xx}+u_{yy}+u_{zz},\\
u|_{t=0}=0,\\
u_t|_{t=0}=x^2+yz.
\end{cases}
\qquad
(\star\star)\begin{cases}
u_{tt}=u_{xx}+u_{yy}+u_{zz}+2(y-t),\\
u|_{t=0}=0,\\
u_t|_{t=0}=0.
\end{cases}
$$

首先 $(\star)$ 的解为

$$
u_1(x,y,z,t)=\frac{1}{4\pi t}\iint_{S_t^M}(\xi^2+\eta\zeta)\,\mathrm{d}S=(x^2+yz)t+\frac{1}{3}t^3.
$$

然后 $(\star\star)$ 的解为

$$
\begin{aligned}
u_2(x,y,z,t) &= \frac{1}{4\pi}\iiint_{r\leqslant t}\frac{2(\eta-t+r)}{r}\,\mathrm{d}V\\
&= \frac{1}{4\pi}\int_0^t\iint_{S_r^M}\frac{2(\eta-t+r)}{r}\,\mathrm{d}S\,\mathrm{d}r\\
&= \frac{1}{4\pi}\int_0^t\int_0^{\pi}\int_0^{2\pi}\frac{2(y+r\sin\theta\sin\varphi-t+r)}{r}r^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r\\
&= -\frac{1}{3}t^3+yt^2.
\end{aligned}
$$

所以原问题的解为

$$
u(x,y,z,t)=u_1(x,y,z,t)+u_2(x,y,z,t)=(x^2+yz)t+yt^2.\qquad\square
$$

1.5 波的传播与衰减

1. 试说明: 对一维波动方程所描述的波的传播过程一般具有后效现象.

Solution. 由教材 P12 分析知区间 $[x_1, x_2]$ 的影响区域为 $x_1 - at \leqslant x \leqslant x_2 + at$, 故一旦扰动到达某一点, 其对该点的影响将持续下去, 因此一维波动方程所描述的波的传播过程具有后效现象. □

2. 试说明: 对一维波动方程, 即使初始资料具有紧支集, 当 $t \to +\infty$ 时其柯西问题的解没有衰减性.

Solution. 设初始资料 φ, ψ 具有紧支集, 则存在一个常数 $\rho > 0$, 使得 φ, ψ 在 $[-\rho, \rho]$ 外恒等于零, 而在 $[-\rho, \rho]$ 内成立 $|\varphi| \leqslant C$, $|\psi| \leqslant C$, 那么对充分大的 t, 有

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big(\varphi(x-at) + \varphi(x+at)\big) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2a}\int_{-\rho}^{\rho} \psi(\xi)\,\mathrm{d}\xi = \text{constant}. \end{aligned}$$

故一维波动方程柯西问题的解没有衰减性. □

3. 设 u 为初始资料 φ 及 ψ 具有紧支集的二维波动方程的解. 试证明: 对任意固定的 $(x_0, y_0) \in \mathbb{R}^2$, 成立

$$\lim_{t\to+\infty} u(x_0, y_0, t) = 0.$$

Proof. For any fixed point $M = (x_0, y_0) \in \mathbb{R}^2$, there exists some $\rho > 0$ such that both φ and ψ vanish outside Σ_ρ^M and are bounded in Σ_ρ^M. For sufficient large time t such that

$\Sigma_\rho^M \subset \Sigma_{at}^M$ we have

$$
\begin{aligned}
u(x_0,y_0,t) &= \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\iint_{\Sigma_{at}^M}\frac{\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{a^2t^2-(\xi-x_0)^2-(\eta-y_0)^2}}\right.\\
&\quad\left.+\iint_{\Sigma_{at}^M}\frac{\psi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{a^2t^2-(\xi-x_0)^2-(\eta-y_0)^2}}\right]\\
&= \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{\varphi(x_0+r\cos\theta,y_0+r\sin\theta)}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}\theta\,\mathrm{d}r\right.\\
&\quad\left.+\int_0^{at}\int_0^{2\pi}\frac{\psi(x_0+r\cos\theta,y_0+r\sin\theta)}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}\theta\,\mathrm{d}r\right]\\
&= \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\int_0^{\rho}\int_0^{2\pi}\frac{\varphi(x_0+r\cos\theta,y_0+r\sin\theta)}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}\theta\,\mathrm{d}r\right.\\
&\quad\left.+\int_0^{\rho}\int_0^{2\pi}\frac{\psi(x_0+r\cos\theta,y_0+r\sin\theta)}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}\theta\,\mathrm{d}r\right]\\
&= \frac{1}{2\pi a}\left[\int_0^{\rho}\int_0^{2\pi}\frac{\partial}{\partial t}\frac{\varphi(x_0+r\cos\theta,y_0+r\sin\theta)}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}\theta\,\mathrm{d}r\right.\\
&\quad\left.+\int_0^{\rho}\int_0^{2\pi}\frac{\psi(x_0+r\cos\theta,y_0+r\sin\theta)}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}\theta\,\mathrm{d}r\right].
\end{aligned}
$$

Thus when $t\to+\infty$,

$$
\begin{aligned}
|u(x_0,y_0,t)| &\leqslant \frac{1}{2\pi a}\left[2\pi C\int_0^{\rho}-a^2t(a^2t^2-r^2)^{-\frac{3}{2}}r\,\mathrm{d}r+2\pi C\int_0^{\rho}\frac{r}{\sqrt{a^2t^2-r^2}}\,\mathrm{d}r\right]\\
&= \frac{1}{2\pi a}\left[2\pi C\left(a-\frac{a^2t}{\sqrt{a^2t^2-\rho^2}}\right)+2\pi C(at-\sqrt{a^2t^2-\rho^2})\right]\to 0,
\end{aligned}
$$

i.e.,

$$\lim_{t\to+\infty}u(x_0,y_0,t)=0.$$ □

1.6 能量不等式，波动方程解的唯一性和稳定性

Note 1.6.1 (教材 (6.15) 式解释) 第二个等式等价于

$$
\begin{aligned}
&\iint_\Omega(u_xu_{xt}+u_yu_{yt})\,\mathrm{d}x\,\mathrm{d}y+\iint_\Omega(u_{xx}u_t+u_{yy}u_t)\,\mathrm{d}x\,\mathrm{d}y\\
=&\iint_\Omega\left[\frac{\partial}{\partial x}(u_xu_t)+\frac{\partial}{\partial y}(u_yu_t)\right]\mathrm{d}x\,\mathrm{d}y=\int_\Gamma u_t(\nabla u\cdot\vec{n})\,\mathrm{d}s.
\end{aligned}
$$

而由格林公式得

$$\begin{aligned}\int_\Gamma u_t(\nabla u)\cdot\vec{n}\,\mathrm{d}s &= \iint_\Omega \nabla\cdot(u_t\nabla u)\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint_\Omega \left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}\right)\left(u_tu_x\vec{i}+u_tu_y\vec{j}\right)\mathrm{d}x\,\mathrm{d}y\\ &= \iint_\Omega \left[\frac{\partial}{\partial x}(u_xu_t)+\frac{\partial}{\partial y}(u_yu_t)\right]\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Note 1.6.2 (定理 6.2) 由能量不等式得

$$\begin{aligned}E(t)+E_0(t) &\leqslant C\left(E(0)+E_0(0)+\int_0^T\iint_\Omega (f_1-f_2)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right)\\ &= C\bigg(\iint_\Omega \left[v_t^2+a^2\left(v_x^2+v_y^2\right)\right]\Big|_{t=0}\,\mathrm{d}x\,\mathrm{d}y+\iint_\Omega v^2(x,y,0)\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad +\int_0^T\iint_\Omega (f_1-f_2)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\bigg)\\ &= C\big(\|\psi_1-\psi_2\|^2+a^2\|\varphi_{1x}-\varphi_{2x}\|^2+a^2\|\varphi_{1y}-\varphi_{2y}\|^2\\ &\qquad +\|\varphi_1-\varphi_2\|^2+\|f_1-f_2\|^2\big).\end{aligned}$$

因此初边值问题的解关于初始值 (φ,ψ) 和方程右端项 f 在定理所述意义下是稳定的.

1. 对受摩擦力作用且具固定端点的有界弦振动, 满足方程

$$u_{tt}=a^2u_{xx}-cu_t,$$

其中常数 $c>0$, 证明其能量是减少的, 并由此证明方程

$$u_{tt}=a^2u_{xx}-cu_t+f$$

的初边值问题解的唯一性以及关于初始条件及自由项的稳定性.

Proof. 能量 $E(t)=\int_0^l\left(u_t^2+a^2u_x^2\right)\mathrm{d}x$, 关于 t 求导得

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= 2\int_0^l\left(u_tu_{tt}+a^2u_xu_{xt}\right)\mathrm{d}x\\ &= 2\int_0^l\left[u_t\left(u_{tt}-a^2u_{xx}\right)+a^2\frac{\partial}{\partial x}(u_tu_x)\right]\mathrm{d}x\\ &= -2\int_0^l cu_t^2\,\mathrm{d}x+2a^2u_tu_x|_0^l=-2\int_0^l cu_t^2\,\mathrm{d}x\leqslant 0.\end{aligned}$$

因此其能量是减少的.

为了证明方程 $u_{tt}=a^2u_{xx}-cu_t+f$ 的初边值问题的解的唯一性, 只需要证明下面的齐次定解问题只有零解

$$\begin{cases} u_{tt}=a^2u_{xx}-cu_t, \\ u(0,t)=u(l,t)=0, \\ u(x,0)=u_t(x,0)=0. \end{cases}$$

由能量不等式得

$$E(t)\leqslant E(0)=\int_0^l \left[u_t^2(x,0)+a^2u_x^2(x,0)\right]\mathrm{d}x=0.$$

故 $u_t=u_x=0\Rightarrow u(x,t)\equiv 0$.

下面证明解对初始条件及自由项的稳定性:

- 关于初始条件的稳定性. 记 $E_0(t)=\int_0^l u^2\,\mathrm{d}x$, 则

$$\frac{\mathrm{d}E_0(t)}{\mathrm{d}t}=2\int_0^l uu_t\,\mathrm{d}x\leqslant\int_0^l u^2\,\mathrm{d}x+\int_0^l u_t^2\,\mathrm{d}x\leqslant E_0(t)+E(t),$$

故

$$E_0(t)\leqslant e^tE_0(0)+e^t\int_0^t e^{-\tau}E(\tau)\,\mathrm{d}\tau\leqslant e^tE_0(0)+E(0)(e^t-1).$$

根据上式, 当初值的均方模很小时, 对固定的 T, $0\leqslant t\leqslant T$ 时解的均方模也很小, 因此关于初始条件是稳定的.

- 如有外力的作用, 此时定解问题为:

$$\begin{cases} u_{tt}=a^2u_{xx}-cu_t+f, \\ u(0,t)=u(l,t)=0, \\ u(x,0)=u_t(x,0)=0. \end{cases}$$

此时 $E(0)=E_0(0)=0$, 且

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t}&=2\int_0^l u_t(-cu_t+f)\,\mathrm{d}x=-2c\int_0^l u_t^2\,\mathrm{d}x+2\int_0^l u_tf\,\mathrm{d}x \\ &\leqslant E(t)+\int_0^l f^2\,\mathrm{d}x \\ &\Rightarrow E(t)\leqslant C_0\left(E(0)+\int_0^T\int_0^l f^2\,\mathrm{d}x\,\mathrm{d}t\right)=C_0\int_0^T\int_0^l f^2\,\mathrm{d}x\,\mathrm{d}t \\ &\Rightarrow E_0(t)\leqslant e^tE_0(0)+e^t\int_0^t e^{-\tau}E(\tau)\,\mathrm{d}\tau\leqslant A\int_0^T\int_0^l f^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

故关于自由项是稳定的. □

2. 证明函数 $f(x,t)$ 在 $G\colon 0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T$ 作微小改变时, 方程

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) - q(x)u + f(x,t)$$

(其中 $k(x) > 0$, $q(x) > 0$ 和 $f(x,t)$ 都是一些充分光滑的函数) 具固定端点边界条件的初边值问题的解在 G 内的改变也是很微小的.

Proof. 令

$$E(t) = \int_0^l \left(u_t^2 + k(x)u_x^2 + q(x)u^2\right) \mathrm{d}x, \quad E_0(t) = \int_0^l u^2 \, \mathrm{d}x.$$

则

$$\begin{aligned} E'(t) &= 2\int_0^l (u_t u_{tt} + k u_x u_{xt} + q u u_t) \, \mathrm{d}x \\ &= 2\int_0^l u_t\big(u_{tt} - (k u_x)_x + q u\big) \, \mathrm{d}x + 2(k u_t u_x)|_0^l \\ &= 2\int_0^l u_t f \, \mathrm{d}x \leqslant E(t) + \int_0^l f^2 \, \mathrm{d}x. \end{aligned}$$

从 0 到 t 积分得

$$E(t) \leqslant e^t E(0) + e^t \int_0^t e^{-\tau} \int_0^l f^2(x,\tau) \, \mathrm{d}x \, \mathrm{d}\tau.$$

又

$$E_0'(t) = 2\int_0^l u u_t \, \mathrm{d}x \leqslant E_0(t) + E(t).$$

故

$$E_0(t) \leqslant e^t E_0(0) + e^t \int_0^t e^{-\tau} E(\tau) \, \mathrm{d}\tau.$$

设 $u(x,t)$ 为满足齐次初边值条件的解, 显然有 $E(0) = E_0(0) = 0$.

$$E_0(0) \leqslant C \int_0^T \int_0^l f^2 \, \mathrm{d}x \, \mathrm{d}t. \qquad \square$$

3. 证明波动方程

$$u_{tt} = a^2(u_{xx} + u_{yy}) + f(x,y,t)$$

的自由项 f 在 $L^2(K)$ 意义下作微小改变时, 对应的柯西问题的解 u 在 $L^2(K)$ 意义之下改变也是微小的, 其中 K 是由 (6.30) 式所表示的锥体.

Proof. 作特征锥 $(x-x_0)^2+(y-y_0)^2\leqslant(R-at)^2$, 记 Ω_t 为 $t=\mathrm{const}$ 与锥的交截部分, 令

$$E_1(\Omega_t)=\iint_{\Omega_t}\left(u_t^2+a^2(u_x^2+u_y^2)\right)\mathrm{d}x\,\mathrm{d}y,\quad E_0(\Omega_t)=\iint_{\Omega_t}u^2\,\mathrm{d}x\,\mathrm{d}y.$$

关于 $E_1(\Omega_t)$ 求导并分布积分得

$$\begin{aligned}\frac{\mathrm{d}E_1(\Omega_t)}{\mathrm{d}t}&=2\int_0^{R-at}\int_0^{2\pi r}u_t\left(u_{tt}-a^2(u_{xx}+u_{yy})\right)\mathrm{d}s\,\mathrm{d}r\\&\quad+2\int_{\Gamma_t}\left(a^2[u_xu_t\cos(n,x)+u_yu_t\cos(n,y)]-\frac{a}{2}[u_t^2+a^2(u_x^2+u_y^2)]\right)\mathrm{d}s\\&\leqslant 2\iint_{\Omega_t}u_tf(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\leqslant E_1(\Omega_t)+\iint_{\Omega_t}f^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

记 $F(t)=\iint_{\Omega_t}f^2(x,y,t)\,\mathrm{d}x\,\mathrm{d}y$, 则

$$E_1(\Omega_t)\leqslant e^tE_1(\Omega_0)+e^t\int_0^te^{-\tau}F(\tau)\,\mathrm{d}\tau.$$

$$\begin{aligned}\frac{\mathrm{d}E_0(\Omega_t)}{\mathrm{d}t}&=-a\int_{\Gamma_t}u^2\,\mathrm{d}s+2\iint_{\Omega}uu_t\,\mathrm{d}x\,\mathrm{d}y\leqslant E_0(\Omega_t)+E_1(\Omega_t)\\&\Rightarrow E_0(\Omega_t)\leqslant e^tE_0(\Omega_0)+e^t\int_0^te^{-\tau}E_1(\Omega_\tau)\,\mathrm{d}\tau.\end{aligned}$$

考虑柯西问题

$$\begin{cases}u_{tt}=a^2(u_{xx}+u_{yy})+f(x,y,t)\\u|_{t=0}=u_t|_{t=0}=0.\end{cases}$$

此时 $E_0(\Omega_0)=E_1(\Omega_0)=0$, 故

$$E_0(\Omega_t)\leqslant e^t\int_0^t\int_0^\tau e^{-\xi}F(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

对上式从 0 到 $T=R/a$ 积分得

$$\|u\|_{L^2(K)}^2\leqslant C\|f\|_{L^2(K)}^2.$$

□

4. 固定端点有界弦的自由振动可以分解成各种不同固有频率的驻波 (谐波) 的叠加, 试计算各个驻波的动能和位能, 并证明弦振动的总能量等于各个驻波能量的叠加. 这个物理性质对应的数学事实是什么?

Proof. 由教材 P21 知此问题的解为

$$u(x,t)=\sum_{k=1}^\infty u_k(x,t)=\sum_{k=1}^\infty N_k\cos(\omega_kt+\theta_k)\sin\frac{k\pi}{l}x.$$

其中

$$N_k = \sqrt{A_k^2 + B_k^2}, \quad \omega_k = \frac{k\pi a}{l}, \quad \cos\theta_k = \frac{A_k}{\sqrt{A_k^2 + B_k^2}}, \quad \sin\theta_k = \frac{B_k}{\sqrt{A_k^2 + B_k^2}},$$

$$A_k = \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi\,\mathrm{d}\xi, \quad B_k = \frac{2}{k\pi a}\int_0^l \psi(\xi)\sin\frac{k\pi}{l}\xi\,\mathrm{d}\xi.$$

第 k 个驻波的能量为

$$\begin{aligned}E_k &= \int_0^l \left((u_k)_t^2 + a^2(u_k)_x^2\right)\mathrm{d}x \\ &= \omega_k^2 N_k^2 \int_0^l \left[\sin^2(\omega_k t + \theta_k)\sin^2\frac{k\pi}{l}x + \cos^2(\omega_k t + \theta_k)\cos^2\frac{k\pi}{l}x\right]\mathrm{d}x \\ &= \frac{1}{2}\omega_k^2 N_k^2.\end{aligned}$$

故

$$\sum_{k=1}^{\infty} E_k = \frac{(\pi a)^2}{2l}\sum_{k=1}^{\infty} k^2 N_k^2.$$

另一方面, 我们有

$$u_t = -\sum_{k=1}^{\infty}\omega_k N_k \sin(\omega_k t + \theta_k)\sin\frac{k\pi}{l}x,$$

$$u_x = \sum_{k=1}^{\infty} N_k \frac{k\pi}{l}\cos(\omega_k t + \theta_k)\cos\frac{k\pi}{l}x,$$

故

$$\begin{aligned}E =& \int_0^l \left(u_t^2 + a^2 u_x^2\right)\mathrm{d}x \\ =& \sum_{k,j=1}^{\infty}\omega_k\omega_j N_k N_j \sin(\omega_k t + \theta_k)\sin(\omega_j t + \theta_j)\int_0^l \sin\frac{k\pi}{l}x\sin\frac{j\pi}{l}x\,\mathrm{d}x \\ &+ \sum_{k,j=1}^{\infty} a^2\frac{kj\pi^2}{l^2}N_k N_j\cos(\omega_k t + \theta_k)\cos(\omega_j t + \theta_j)\int_0^l \cos\frac{k\pi}{l}x\cos\frac{j\pi}{l}x\,\mathrm{d}x \\ =& \frac{(\pi a)^2}{2l}\sum_{k=1}^{\infty} k^2 N_k^2.\end{aligned}$$

因此

$$E = \sum_{k=1}^{\infty} E_k.$$

此事实反映了特征函数系的完备性, 即成立 Parseval 等式. □

5. 考虑波动方程的第三类初边值问题

$$\begin{aligned} &u_{tt}-a^2(u_{xx}+u_{yy})=0, \quad t>0,(x,y)\in\mathit{\Omega}, \\ &u|_{t=0}=\varphi(x,y),\ u_t|_{t=0}=\psi(x,y), \\ &\left(\frac{\partial u}{\partial \boldsymbol{n}}+\sigma u\right)\bigg|_{\mathit{\Gamma}}=0, \end{aligned}$$

其中 $\sigma>0$ 是常数, $\mathit{\Gamma}$ 为 $\mathit{\Omega}$ 的边界, $\boldsymbol{n}$ 是 $\mathit{\Gamma}$ 上的单位外法向量. 对于上述定解问题的解, 定义能量积分

$$E(t)=\iint_{\mathit{\Omega}}\big(u_t^2+a^2(u_x^2+u_y^2)\big)\,\mathrm{d}x\,\mathrm{d}y+a^2\int_{\mathit{\Gamma}}\sigma u^2\,\mathrm{d}s,$$

试证明 $E(t)$ 为常数, 并由此说明上述定解问题解的唯一性.

Proof. 直接求导得

$$\begin{aligned} \frac{\partial E(t)}{\partial t} &= 2\iint_{\mathit{\Omega}}\big(u_tu_{tt}+a^2(u_xu_{xt}+u_yu_{yt})\big)\,\mathrm{d}x\,\mathrm{d}y+2a^2\int_{\mathit{\Gamma}}\sigma uu_t\,\mathrm{d}s \\ &= 2\iint_{\mathit{\Omega}} u_t(u_{tt}-a^2(u_{xx}+u_{yy}))+a^2(u_xu_t)_x+a^2(u_yu_t)_y\,\mathrm{d}x\,\mathrm{d}y \\ &\quad +2a^2\int_{\mathit{\Gamma}}\sigma uu_t\,\mathrm{d}s \\ &= 2a^2\iint_{\mathit{\Omega}}\mathrm{div}(u_t\nabla u)\,\mathrm{d}x\,\mathrm{d}y+2a^2\int_{\mathit{\Gamma}}\sigma uu_t\,\mathrm{d}s \\ &= 2a^2\int_{\mathit{\Gamma}}u_t\left(\frac{\partial u}{\partial \boldsymbol{n}}+\sigma u\right)\mathrm{d}s=0, \end{aligned}$$

故 $E(t)$ 为常数. 由此立即可得解的唯一性. □

Chapter 2

热传导方程

2.1 热传导方程及其定解问题的导出

2.2 初边值问题的分离变量法

1. 用分离变量法求下列定解问题的解:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (t > 0, 0 < x < \pi), \\ u(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \quad (t > 0), \\ u(x, 0) = f(x) \quad (0 < x < \pi). \end{cases}$$

Solution. 利用分离变量法, 设 $u(x, t) = X(x)T(t)$, 则

$$X(x)T'(t) = a^2 X''(x)T(t).$$

由此得

$$X''(x) + \lambda X(x) = 0, \tag{2.1}$$

$$T'(t) + a^2 \lambda T(t) = 0. \tag{2.2}$$

由 (??) 及 $X(x)$ 满足的边界条件 $X(0) = 0$, $X'(\pi) = 0$, 得

(i) 当 $\lambda \leqslant 0$ 时只有零解;

(ii) 当 $\lambda > 0$ 时, $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$, 代入边界条件得固有值为 $\lambda_k = \left(k + \frac{1}{2}\right)^2$, 相应的固有函数 $X_k(x) = B_k \sin\sqrt{\lambda_k}x = B_k \sin\left(k + \frac{1}{2}\right)x$.

将 $\lambda = \lambda_k$ 代入 (??) 得 $T_k(t) = C_k e^{-a^2\lambda_k t}$, 故

$$u_k(x, t) = A_k e^{-a^2\lambda_k t} \sin\left(k + \frac{1}{2}\right)x, u(x, t) = \sum_{k=1}^{\infty} u_k(x, t).$$

利用初始条件得

$$f(x)=\sum_{k=1}^{\infty}A_k\sin\left(k+\frac{1}{2}\right)x\Rightarrow A_k=\frac{2}{\pi}\int_0^{\pi}f(\xi)\sin\left(k+\frac{1}{2}\right)\xi\,\mathrm{d}\xi.$$

因此原问题的解为

$$u(x,t)=\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^{\pi}f(\xi)\sin\left(k+\frac{1}{2}\right)\xi\,\mathrm{d}\xi\cdot e^{-a^2\lambda_k t}\sin\left(k+\frac{1}{2}\right)x.$$ □

2. 用分离变量法求解热传导方程的初边值问题:

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} & (t>0,\ 0<x<1),\\ u(x,0)=\begin{cases}x, & 0<x\leqslant\frac{1}{2},\\ 1-x, & \frac{1}{2}<x<1,\end{cases}\\ u(0,t)=u(1,t)=0 & (t>0).\end{cases}$$

Solution. Suppose $u(x,t)=X(x)T(t)$, then

$$\frac{X''(x)}{X(x)}=\frac{T'(t)}{T(t)}=-\lambda.$$

First of all, the function X satisfies the following condition

$$\begin{cases}X''(x)+\lambda X(x)=0,\\ X(0)=X(1)=0.\end{cases}\tag{2.3}$$

We can solve that the eigenvalues are $\lambda_k=(k\pi)^2$ and $X_k(x)=A_k\sin k\pi x$.

Then $T(t)$ satisfies the following equation

$$T_k(t)+(k\pi)^2T_k(t)=0,$$

from which we can solve that $T_k(t)=B_ke^{-(k\pi)^2t}$. Hence we could write $u(x,t)$ as

$$u(x,t)=\sum_{k=1}^{\infty}C_ke^{-(k\pi)^2t}\sin k\pi x.\tag{2.4}$$

Using the initial value condition we find that

$$\sum_{k=1}^{\infty}C_k\sin k\pi x=f(x)=\begin{cases}x, & 0<x\leqslant\frac{1}{2},\\ 1-x, & \frac{1}{2}<x<1.\end{cases}$$

Hence

$$C_k = 2\int_0^1 f(x)\sin k\pi x\,\mathrm{d}x = \frac{4}{k^2\pi^2}\sin\frac{k\pi}{2}. \tag{2.5}$$

From (**??**) and (**??**) we get

$$u(x,t) = \sum_{k=1}^{\infty}\frac{4}{k^2\pi^2}\sin\frac{k\pi}{2}e^{-(k\pi)^2 t}\sin k\pi x. \qquad \square$$

3. 如果有一根长度为 l 的均匀细棒, 其周围以及两端 $x=0$, $x=l$ 均为绝热, 初始温度分布为 $u(x,0)=f(x)$, 问以后的温度分布如何? 且证明当 $f(x)$ 等于常数 u_0 时, 恒有 $u(x,t)=u_0$.

Solution. 因为细棒的两端均为绝热, 故根据傅里叶定律知 $u_x|_{x=0}=u_x|_{x=l}=0$, 此初边值问题为

$$\begin{cases} u_t = a^2 u_{xx}, \\ u_x|_{x=0} = u_x|_{x=l} = 0, \\ u|_{t=0} = f(x). \end{cases}$$

直接解得

$$u(x,t) = \sum_{k=0}^{\infty} D_k e^{-a^2\lambda_k t}\cos\frac{k\pi}{l}x, \quad \lambda_k = \left(\frac{k\pi}{l}\right)^2,$$

其中

$$D_0 = \frac{1}{l}\int_0^l f(\xi)\,\mathrm{d}\xi, \quad D_k = \frac{2}{l}\int_0^l f(\xi)\cos\frac{k\pi}{l}\xi\,\mathrm{d}\xi \quad (k\geqslant 1).$$

当 $f(x)\equiv u_0$ 时, $D_0=u_0$, $D_k=0$ $(k=1,2,\cdots)$, 故 $u(x,t)=u_0$. $\square$

4. 在区域 $t>0$, $0<x<l$ 中求解如下的定解问题:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} - \beta(u-u_0), \\ u(0,t) = u(l,t) = u_0, \\ u(x,0) = f(x), \end{cases}$$

其中 a, β, u_0 均为常数, $f(x)$ 为已知函数.

Solution. 作变量代换, 令 $v(x,t)=(u-u_0)e^{\beta t}$, 则 $v(x,t)$ 满足的定解问题为:

$$\begin{cases} v_t = a^2 v_{xx}, \\ v(0,t) = v(l,t) = 0, \\ v(x,0) = f(x) - u_0. \end{cases}$$

直接解得

$$v(x,t)=\sum_{k=1}^{\infty}A_ke^{-a^2\lambda_k t}\sin\frac{k\pi}{l}x,\quad \lambda_k=\left(\frac{k\pi}{l}\right)^2,$$

其中

$$A_k=\frac{2}{l}\int_0^l(f(\xi)-u_0)\sin\frac{k\pi}{l}\xi\,\mathrm{d}\xi.$$

因此

$$u(x,t)=u_0+\sum_{k=1}^{\infty}\frac{2}{l}\int_0^l(f(\xi)-u_0)\sin\frac{k\pi}{l}\xi\,\mathrm{d}\xi\cdot e^{-\left(\frac{a^2k^2\pi^2}{l^2}+\beta\right)t}\sin\frac{k\pi}{l}x.\qquad\square$$

5. 长度为 l 的均匀细杆的初始温度为 $0\,°\mathrm{C}$, 端点 $x=0$ 保持恒温 u_0, 而在 $x=l$ 和侧面上, 热量可以发散到周围的介质中去, 介质的温度为 $0\,°\mathrm{C}$, 此时杆上的温度分布函数 $u(x,t)$ 满足下述定解问题:

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}-b^2u,\\ u(0,t)=u_0,\quad \left(\frac{\partial u}{\partial x}+Hu\right)\Big|_{x=l}=0,\\ u(x,0)=0,\end{cases}$$

其中 a, b, H 均为常数, 试求出 $u(x,t)$.

Solution. Let $u(x,t)=u_1(x,t)+u_2(x)$ in which u_1 satisfies the following problem with homogeneous boundary condition

$$\begin{cases}\frac{\partial u_1}{\partial t}=a^2\frac{\partial^2 u_1}{\partial x^2}-b^2u_1,\\ u_1(0,t)=\left(\frac{\partial u_1}{\partial x}+Hu_1\right)\big|_{x=l}=0,\\ u_1(x,0)=-u_2,\end{cases}\tag{2.6}$$

and u_2 satisfies the following ordinary differential equation

$$\begin{cases}a^2\frac{\mathrm{d}^2u_2}{\mathrm{d}x^2}-b^2u_2=0,\\ u_2(0)=u_0,\\ \left(\frac{\mathrm{d}u_2}{\mathrm{d}x}+Hu_2\right)\big|_{x=l}=0.\end{cases}\tag{2.7}$$

We first solve equation (**??**). Since $a^2\frac{\mathrm{d}^2u_2}{\mathrm{d}x^2}-b^2u_2=0$, we have that

$$u_2=C_1e^{\frac{b}{a}x}+C_2e^{-\frac{b}{a}x}.\tag{2.8}$$

By the boundary condition we get

$$\begin{cases}C_1+C_2=0,\\ C_1\frac{b}{a}e^{\frac{b}{a}l}-C_2\frac{b}{a}e^{-\frac{b}{a}l}+HC_1e^{\frac{b}{a}l}+HC_2e^{-\frac{b}{a}l}=0.\end{cases}$$

Solving C_1, C_2 and substituting them into (**??**), we have

$$u_2 = u_0 \cosh \frac{b}{a} x - \frac{H \cosh \frac{b}{a} l + \frac{b}{a} \sinh \frac{b}{a} l}{H \sinh \frac{b}{a} l + \frac{b}{a} \cosh \frac{b}{a} l} u_0 \sinh \frac{b}{a} x. \tag{2.9}$$

Now we solve equation (**??**). Let $v = e^{b^2 t} u_1$, then v satisfies

$$\begin{cases} \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \\ v(0,t) = \left(\frac{\partial v}{\partial x} + Hv\right)\big|_{x=l} = 0, \\ v(x,0) = -u_2. \end{cases} \tag{2.10}$$

The procedure of solving this problem is actually the same as that in the textbook from Page 51 to 54. Denote $v(x,t) = X(x)T(t)$, then

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{a^2 T(t)} = -\lambda$$

for some constant λ. First of all, $X(x)$ satisfies

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X'(l) + HX(l) = 0. \end{cases} \tag{2.11}$$

- If $\lambda \leqslant 0$, there only exists trivial solution $X \equiv 0$;
- If $\lambda > 0$,

  $$X(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x. \tag{2.12}$$

  Combining with the boundary conditions we have

  $$X_k(x) = B_k \sin \sqrt{\lambda_k} x, \tag{2.13}$$

  where $(\lambda_k)_{k \geqslant 1}$ is the sequence of positive solutions to $\sqrt{\lambda} + H \tan \sqrt{\lambda} l = 0$.

On the other hand, $T(t)$ satisfies

$$T'(t) + a^2 \lambda_k T(t) = 0, \tag{2.14}$$

from which we get

$$T_k(t) = C_k e^{-a^2 \lambda_k t}. \tag{2.15}$$

Therefore according to (**??**) and (**??**) we can write $v(x,t)$ as

$$v(x,t) = \sum_{k=1}^{\infty} A_k e^{-a^2 \lambda_k t} \sin \sqrt{\lambda_k} x. \tag{2.16}$$

Finally we need to utilize the initial value condition to get

$$v(x,0)=\sum_{k=1}^{\infty}A_k\sin\sqrt{\lambda_k}x=-u_0. \tag{2.17}$$

Since

$$\left(\sin\sqrt{\lambda_m}x,\sin\sqrt{\lambda_n}x\right)_{L^2}=\delta_{mn}\left(\frac{l}{2}+\frac{H}{2(H^2+\lambda_m)}\right)=:\delta_{mn}\alpha_m, \tag{2.18}$$

we have

$$A_k=-\frac{1}{\alpha_k}\int_0^l u_0(x)\sin\sqrt{\lambda_k}x\,\mathrm{d}x. \tag{2.19}$$

Hence

$$v(x,t)=-\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\int_0^l u_0(x)\sin\sqrt{\lambda_k}x\,\mathrm{d}x\cdot e^{-a^2\lambda_k t}\sin\sqrt{\lambda_k}x \tag{2.20}$$

and

$$u_1(x,t)=-\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\int_0^l u_0(x)\sin\sqrt{\lambda_k}x\,\mathrm{d}x\cdot e^{-(a^2\lambda_k+b^2)t}\sin\sqrt{\lambda_k}x \tag{2.21}$$

Finally we conclude that

$$u(x,t)=u_1(x,t)+u_2(x), \tag{2.22}$$

where u_1 and u_2 are given by (**??**) and (**??**) respectively. □

2.3 柯西问题

1. 求下列函数的 Fourier 变换:
(1) $\mathrm{e}^{-\eta x^2}$ $(\eta>0)$;
(2) $\mathrm{e}^{-a|x|}$ $(a>0)$;
(3) $\frac{x}{(a^2+x^2)^k}$, $\frac{1}{(a^2+x^2)^k}$ ($a>0$, k 为自然数).

Solution. (1) 直接计算得

$$\begin{aligned}\widehat{\mathrm{e}^{-\eta x^2}}(\xi)&=\int_{-\infty}^{\infty}\mathrm{e}^{-\eta x^2}\cdot\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x=\mathrm{e}^{-\frac{\xi^2}{4\eta}}\int_{-\infty}^{\infty}\mathrm{e}^{-\eta\left(\xi+\frac{\mathrm{i}\xi}{2\eta}\right)^2}\,\mathrm{d}x\\&=\mathrm{e}^{-\frac{\xi^2}{4\eta}}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2}\frac{1}{\sqrt{\eta}}\,\mathrm{d}y=\left(\frac{\pi}{\eta}\right)^{1/2}\mathrm{e}^{-\frac{\xi^2}{4\eta}}.\end{aligned}$$

(2) 直接计算得

$$\begin{aligned}\widehat{\mathrm{e}^{-a|x|}}(\xi)&=\int_{-\infty}^{\infty}\mathrm{e}^{-a|x|}\cdot\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x\\&=2\int_0^{\infty}\mathrm{e}^{-ax}\cos\xi x\,\mathrm{d}x=\frac{2a}{a^2+\xi^2}.\end{aligned}$$

□

2. 证明: 当 $f(x)$ 在 $(-\infty,\infty)$ 上绝对可积时, $F[f]$ 为连续函数.

Proof. 记 $F[f]=\int_{-\infty}^{\infty}f(\xi)\mathrm{e}^{-\mathrm{i}\lambda\xi}\,\mathrm{d}\xi=\tilde{f}(\lambda)$, 则

$$\begin{aligned}\left|\tilde{f}(\lambda+h)-\tilde{f}(\lambda)\right| &= \left|\int_{-\infty}^{\infty}f(\xi)\left(\mathrm{e}^{-\mathrm{i}(\lambda+h)\xi}-\mathrm{e}^{-\mathrm{i}\lambda\xi}\right)\mathrm{d}\xi\right| \\ &\leqslant \int_{-\infty}^{\infty}|f(\xi)|\cdot|\mathrm{e}^{-\mathrm{i}h\xi}-1|\,\mathrm{d}\xi\to 0\quad(\text{as } h\to 0),\end{aligned}$$

故 $F[f]$ 为连续函数. □

4. 证明 (3.29) 所表示的函数满足非齐次方程 (3.15) 以及初始条件 (3.16).

Proof. It suffices to verify that $u(x,t)$ given by (3.28) is the solution to the Cauchy problem (3.23)–(3.24). Denote the heat kernel by

$$\mathit{\Phi}(x,t)=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}},$$

which satisfies $\partial_t\mathit{\Phi}=a^2\,\partial_{xx}\mathit{\Phi}$. Then the $u(x,t)$ in (3.28) can be rewritten as

$$u(x,t)=\int_0^t\int_{-\infty}^{\infty}\mathit{\Phi}(x-\xi,t-\tau)f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{2.23}$$

Making a change of variables, we have

$$u(x,t)=\int_0^t\int_{-\infty}^{\infty}\mathit{\Phi}(\xi,\tau)f(x-\xi,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{2.24}$$

Hence

$$\begin{aligned}\frac{\partial u}{\partial t} &= \int_0^t\int_{-\infty}^{\infty}\mathit{\Phi}(\xi,\tau)f_t(x-\xi,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &\quad+\int_{-\infty}^{\infty}\mathit{\Phi}(\xi,t)f(x-\xi,0)\,\mathrm{d}\xi, &(2.25)\\ \frac{\partial^2 u}{\partial x^2} &= \int_0^t\int_{-\infty}^{\infty}\mathit{\Phi}(\xi,\tau)f_{xx}(x-\xi,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. &(2.26)\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} &= \int_0^t \int_{-\infty}^{\infty} \varPhi(\xi,\tau)(\partial_t - a^2\,\partial_{xx}) f(x-\xi, t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\quad + \int_{-\infty}^{\infty} \varPhi(\xi,t) f(x-\xi,0)\,\mathrm{d}\xi \\
&= \int_0^\epsilon \int_{-\infty}^{\infty} \varPhi(\xi,\tau)(\partial_t - a^2\,\partial_{xx}) f(x-\xi, t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\quad + \int_\epsilon^t \int_{-\infty}^{\infty} \varPhi(\xi,\tau)(-\,\partial_\tau - a^2\,\partial_{\xi\xi}) f(x-\xi, t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\quad + \int_{-\infty}^{\infty} \varPhi(\xi,t) f(x-\xi,0)\,\mathrm{d}\xi \\
&=: I_\epsilon + J_\epsilon + K.
\end{aligned}$$

We suppose that f is C^1 in t and C^2 in x and has compact support. Then

$$|I_\epsilon| \leqslant \big(\|f_t\|_\infty + a^2\|f_{xx}\|_\infty\big) \int_0^\epsilon \int_{-\infty}^{\infty} \varPhi(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \leqslant C\epsilon. \tag{2.27}$$

On the other hand, integrate by parts to get

$$\begin{aligned}
J_\epsilon &= \int_\epsilon^t \int_{-\infty}^{\infty} (\partial_\tau - a^2\,\partial_{\xi\xi})\varPhi(\xi,\tau) f(x-\xi,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\quad + \int_{-\infty}^{\infty} \varPhi(\xi,\epsilon) f(x-\xi,t-\epsilon)\,\mathrm{d}\xi \\
&\quad - \int_{-\infty}^{\infty} \varPhi(\xi,t) f(x-\xi,0)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \varPhi(\xi,\epsilon) f(x-\xi,t-\epsilon)\,\mathrm{d}\xi - K.
\end{aligned} \tag{2.28}$$

Therefore

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \varPhi(\xi,\epsilon) f(x-\xi,t-\epsilon)\,\mathrm{d}\xi = f(x,t). \tag{2.29}$$

The initial value condition (3.24) is obvious. □

5. 求解热传导方程 (3.17) 的柯西问题, 已知
(1) $u|_{t=0} = \sin x$,
(2) 用延拓法求解半有界直线上的热传导方程 (3.17), 假设

$$\begin{cases} u(x,0) = \varphi(x) \quad (0 < x < \infty), \\ u(0,t) = 0. \end{cases}$$

Solution. (1) 由泊松公式知

$$\begin{aligned}u(x,t)&=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\sin\xi\cdot\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}\,\mathrm{d}\xi\\&=\frac{1}{2a\sqrt{\pi t}}\cdot 2a\sqrt{t}\int_{-\infty}^{\infty}\sin(x-2a\sqrt{t}\zeta)\cdot\mathrm{e}^{-\zeta^2}\,\mathrm{d}\zeta\\&=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\sin x\cos 2a\sqrt{t}\zeta\cdot e^{-\zeta^2}\,\mathrm{d}\zeta\\&=\frac{2\sin x}{\sqrt{\pi}}\int_{0}^{\infty}\cos 2a\sqrt{t}\zeta\cdot e^{-\zeta^2}\,\mathrm{d}\zeta\\&=\mathrm{e}^{-a^2t}\sin x.\end{aligned}$$

(2) 对 $\varphi(x)$ 作奇延拓, 即

$$\Phi(x)=\begin{cases}\varphi(x), & x\geqslant 0,\\ -\varphi(-x), & x<0.\end{cases}$$

求解如下 Cauchy 问题

$$\begin{cases}u_t=a^2u_{xx},\\ u|_{t=0}=\Phi(x).\end{cases}$$

得

$$\begin{aligned}u(x,t)&=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\Phi(\xi)\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}\,\mathrm{d}\xi\\&=\frac{1}{2a\sqrt{\pi t}}\left[\int_{0}^{\infty}\varphi(\xi)\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}\,\mathrm{d}\xi+\int_{-\infty}^{0}-\varphi(-\xi)\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}\,\mathrm{d}\xi\right]\\&=\frac{1}{a\sqrt{\pi t}}\int_{0}^{\infty}\varphi(\xi)\mathrm{e}^{-\frac{x^2+\xi^2}{4a^2t}}\sinh\frac{x\xi}{2a^2t}\,\mathrm{d}\xi.\end{aligned}$$

□

7. 证明: 如果 $u_1(x,t)$, $u_2(y,t)$ 分别是下述两个定解问题的解:

$$\begin{cases}\frac{\partial u_1}{\partial t}=a^2\frac{\partial^2u_1}{\partial x^2},\\ u_1|_{t=0}=\varphi_1(x);\end{cases}\qquad\begin{cases}\frac{\partial u_2}{\partial t}=a^2\frac{\partial^2u_2}{\partial y^2},\\ u_2|_{t=0}=\varphi_2(y).\end{cases}$$

则 $u(x,y,t)=u_1(x,t)u_2(y,t)$ 是定解问题

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right),\\ u|_{t=0}=\varphi_1(x)\varphi_2(y)\end{cases}$$

的解.

Solution. 直接验证. $\square$

8. 导出下列热传导方程柯西问题解的表达式:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ u|_{t=0} = \sum_{i=1}^n \alpha_i(x)\beta_i(y). \end{cases}$$

Solution. 由叠加原理与上题结果或直接应用 Fourier 变换可得解为

$$u(x,y,t) = \frac{1}{4a^2\pi t}\sum_{i=1}^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_i(\xi)\beta_i(\eta)\exp\left(-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}\right)\mathrm{d}\xi\,\mathrm{d}\eta.$$ $\square$

9. 验证二维热传导方程柯西问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ u|_{t=0} = \varphi(x,y) \end{cases}$$

的解的表达式为

$$u(x,y,t) = \frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\eta)\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Proof. 本习题应该添加假设: $\varphi(x,y)$ 有界, 因为

$$u(x,y,t) = \frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\eta)\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

所以

$$\begin{aligned}\frac{\partial u}{\partial t} = {} & \frac{-1}{4\pi a^2 t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\eta)\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ & + \frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\eta)\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}} \cdot \frac{(x-\xi)^2+(y-\eta)^2}{4a^2t^2}\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

又

$$\frac{\partial u}{\partial x} = \frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\eta)\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}} \cdot \frac{-(x-\xi)}{2a^2t}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

故

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4\pi a^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\eta)\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}} \left(\frac{-1}{2a^2t} + \frac{(x-\xi)^2}{4a^4t^2}\right)\mathrm{d}\xi\,\mathrm{d}\eta.$$

显然 $\frac{\partial^2 u}{\partial y^2}$ 的结果形式同 $\frac{\partial^2 u}{\partial x^2}$, 故

$$\begin{aligned}&a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\=&\frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi,\eta)e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\left(\frac{-1}{a^2t}+\frac{(x-\xi)^2+(y-\eta)^2}{4a^4t^2}\right)\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

对比可知

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

对于初值的检验可对照教材 P61 的方法, 下面不妨简单叙述一下. 要证明当 $t\to 0$, $x\to x_0$, $y\to y_0$ 时, $u(x,y,t)\to\varphi(x_0,y_0)$, 令 $\zeta=\frac{x-\xi}{2a\sqrt{t}}$, $\theta=\frac{y-\eta}{2a\sqrt{t}}$, 则

$$u(x,y,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x-2a\sqrt{t}\zeta,y-2a\sqrt{t}\theta)\mathrm{e}^{-(\zeta^2+\theta^2)}\,\mathrm{d}\zeta\,\mathrm{d}\theta.$$

而

$$\varphi(x_0,y_0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x_0,y_0)\mathrm{e}^{-(\zeta^2+\theta^2)}\,\mathrm{d}\zeta\,\mathrm{d}\theta.$$

故

$$u(x,y,t)-\varphi(x_0,y_0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\varphi(x-2a\sqrt{t}\zeta,y-2a\sqrt{t}\theta)-\varphi(x_0,y_0)\right]\mathrm{e}^{-(\zeta^2+\theta^2)}\,\mathrm{d}\zeta\,\mathrm{d}\theta.$$

将 (ζ,θ) 平面用正方形 (四个顶点为 $(\pm N,\pm N)$) 分成两个部分. 在正方形内部, 利用 $\varphi(x,y)$ 的连续性控制, 在正方形的外部, 用积分 $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\zeta^2+\theta^2)}\,\mathrm{d}\zeta\,\mathrm{d}\theta$ 可以任意小以及 $\varphi(x,y)$ 是有界的来进行控制即可证明. $\square$

2.4 极值原理, 定解问题解的唯一性和稳定性

1. 证明方程 $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+cu$ $(c\geqslant 0)$ 具狄利克雷边界条件的初边值问题解的唯一性和稳定性.

Proof. 设 $u(x,t)$ 满足的定解问题为

$$\begin{cases}u_t=a^2u_{xx}+cu,\\u(x,0)=\varphi(x),\\u(\alpha,t)=\mu_1(t),\ u(\beta,t)=\mu_2(t).\end{cases}$$

则令 $v(x,t)=u(x,t)\mathrm{e}^{-ct}$, 可得 $v(x,t)$ 满足的定解问题为

$$\begin{cases} v_t = a^2 v_{xx}, \\ v(x,0)=\varphi(x), \\ v(\alpha,t)=\mu_1(t)\mathrm{e}^{-ct},\ v(\beta,t)=\mu_2(t)\mathrm{e}^{-ct}. \end{cases}$$

由定理 4.2 知上述定解问题的解是唯一的且稳定的, 记为 $v=v_0(x,t)$, 则原定解问题的解为 $u=u_0(x,t)=\mathrm{e}^{ct}v_0(x,t)$, 显然也是唯一的且稳定的. □

2. 利用热传导方程极值原理的方法, 证明二维调和函数在有界区域上的最大值不会超过它在边界上的最大值.

Proof. 记有界闭区域为 Ω, 其边界为 Γ. 设 $u(x,y)$ 在 Ω 上的最大值为 M, 在 Γ 上的最大值为 m. 假设在区域内部存在某点 (x_0,y_0) 使得

$$u(x_0,y_0)=M>m.$$

作辅助函数

$$V(x,y)=u(x,y)+\frac{M-m}{4R^2}\left[(x-x_0)^2+(y-y_0)^2\right],$$

其中 $\Omega\subset B(0,R)$. 我们有

$$V(x_0,y_0)=u(x_0,y_0)=M.$$

而在 Γ 上

$$V(x,y)<m+\frac{M-m}{4}=\theta M \quad (0<\theta<1).$$

因此 $V(x,y)$ 必在区域 Ω 内部某点 (x_1,y_1) 取得最大值, 在这个点应有 $V_{xx}\leqslant 0, V_{yy}\leqslant 0$, 但是

$$V_{xx}+V_{yy}=u_{xx}+u_{yy}+\frac{M-m}{R^2}>0.$$

矛盾, 故成立 $M=m$. □

3. 导出初边值问题

$$\begin{cases} u_t-a^2u_{xx}=f(x,t), \\ u|_{x=0}=\mu_1(t), \quad \left(\frac{\partial u}{\partial x}+hu\right)\Big|_{x=l}=\mu_2(t) \quad (h>0), \\ u|_{t=0}=\varphi(x) \end{cases}$$

的解 $u(x,t)$ 在 $R_T=\{0\leqslant t\leqslant T, 0\leqslant x\leqslant l\}$ 中满足估计

$$u(x,t)\leqslant \mathrm{e}^{\lambda T}\max\left\{0, \max_{0\leqslant x\leqslant l}\varphi(x), \max_{0\leqslant t\leqslant T}\left(\mathrm{e}^{-\lambda t}\mu_1(t), \frac{\mathrm{e}^{-\lambda t}\mu_2(t)}{h}\right), \frac{1}{\lambda}\max_{R_T}(\mathrm{e}^{-\lambda t}f)\right\},$$

其中 $\lambda>0$ 为任意正常数.

Proof. Let $v = \mathrm{e}^{-\lambda t}u$, then v satisfies

$$\begin{cases} v_t - a^2 v_{xx} + \lambda v = \mathrm{e}^{-\lambda t} f(x,t), \\ v(x,0) = \varphi(x), \\ v(0,t) = \mathrm{e}^{-\lambda t}\mu_1(t), \ (v_x + hv)|_{x=l} = \mathrm{e}^{-\lambda t}\mu_2(t). \end{cases}$$

It suffices to consider the case when u takes its positive maximum value at some point (x_0, t_0). Suppose that (x_0, t_0) is not on the parabolic boundary, then $v_t(x_0, t_0) \geqslant 0$, $v_{xx}(x_0, t_0) \leqslant 0$ and $v(x_0, t_0) > 0$. So

$$(v_t - a^2 v_{xx} + \lambda v)\big|_{(x_0,t_0)} > 0. \tag{2.30}$$

By choosing λ large enough, we can let

$$\mathrm{e}^{-\lambda t_0} f(x_0, t_0) \to 0. \tag{2.31}$$

Thus $v_t - a^2 v_{xx} + \lambda v = \mathrm{e}^{-\lambda t} f(x,t)$ could not hold at (x_0, t_0) and so (x_0, t_0) is on the parabolic boundary.

(i) If $(x_0, t_0) \in \{x = 0\}$, then

$$v(x,t) \leqslant \max_{0 \leqslant t \leqslant T} e^{-\lambda t}\mu_1(t). \tag{2.32}$$

(ii) If $(x_0, t_0) \in \{t = 0\}$, then

$$v(x,t) \leqslant \max_{0 \leqslant x \leqslant l} \varphi(x). \tag{2.33}$$

(iii) If $(x_0, t_0) \in \{x = l\}$, then

$$v(x,t) \leqslant \max_{0 \leqslant t \leqslant T} \frac{e^{-\lambda t}\mu_2(t)}{h}. \tag{2.34}$$

On the other hand, since $v_t(x_0, t_0) - a^2 v_{xx}(x_0, t_0) \geqslant 0$, we have that

$$\lambda v(x_0, t_0) \leqslant e^{-\lambda t_0} f(x_0, t_0) \leqslant \max_{R_T}(e^{-\lambda t} f). \tag{2.35}$$

Combining (**??**), (**??**), (**??**) and (**??**) we conclude that

$$v(x,t) \leqslant \max\left(0, \max_{0 \leqslant x \leqslant l} \varphi(x), \max_{0 \leqslant t \leqslant T}\left(e^{-\lambda t}\mu_1(t), \frac{e^{-\lambda t}\mu_2(t)}{h}\right), \frac{1}{\lambda}\max_{R_T}(e^{-\lambda t} f)\right). \tag{2.36}$$

The final estimate for $u(x,t)$ follows directly from (**??**). □

2.5 解的渐近性态

1. 证明方程

$$\begin{cases} u_t - a^2 u_{xx} = 0, \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = \varphi(x) \end{cases}$$

的解当 $t \to +\infty$ 时指数地衰减于零, 其中 $\varphi \in C^2$, 且 $\varphi(0) = \varphi(l) = 0$.

Proof. 运用分离变量法求得定解问题的解为

$$u(x,t) = \sum_{k=1}^{\infty} A_k \mathrm{e}^{-\frac{k^2\pi^2a^2}{l^2}t} \sin\frac{k\pi}{l}x.$$

其中 $A_k = \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi\,\mathrm{d}\xi$, 由 φ 有界知 $\exists C_1 > 0$, 使得 $|A_k| \leqslant C_1$, 故当 $t > 1$ 时,

$$\begin{aligned} |u(x,t)| &\leqslant C_1 \sum_{k=1}^{\infty} \mathrm{e}^{-\frac{k^2\pi^2a^2}{l^2}t} \\ &= C_1\left(1 + \sum_{k=2}^{\infty} e^{-\frac{(k^2-1)\pi^2a^2}{l^2}t}\right)\mathrm{e}^{-\frac{\pi^2a^2}{l^2}t} \\ &\leqslant C_1\left(1 + \sum_{k=2}^{\infty} e^{-\frac{(k^2-1)\pi^2a^2}{l^2}}\right)\mathrm{e}^{-\frac{\pi^2a^2}{l^2}t} \\ &< C\mathrm{e}^{-\frac{\pi^2a^2}{l^2}t}. \end{aligned}$$

因此解当 $t \to +\infty$ 时指数地衰减于零. □

2. 证明: 当 $\varphi(x,y)$ 为 $\mathbb{R}^2$ 上的有界连续函数, 且 $\varphi \in L^1(\mathbb{R}^2)$ 时, 二维热传导方程柯西问题的解, 当 $t \to +\infty$ 时, 以 t^{-1} 衰减率趋于零.

Proof.

$$\begin{aligned} |u(x,y,t)| &= \left|\frac{1}{4\pi a^2 t}\iint_{\mathbb{R}^2} \varphi(\xi,\eta)\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\,\mathrm{d}\xi\,\mathrm{d}\eta\right| \\ &\leqslant \frac{1}{4\pi a^2 t}\iint_{\mathbb{R}^2} |\varphi(x,y)|\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leqslant Ct^{-1}, \end{aligned}$$

其中 C 是仅与 a 和 $\|\varphi\|_{L^1(\mathbb{R}^2)}$ 有关的正常数. □

Chapter 3

调和方程

3.1 建立方程, 定解条件

1. 设 $u(x_1,\cdots,x_n)=f(r)$ (其中 $r=\sqrt{x_1^2+\cdots+x_n^2}$) 是 n 维调和函数 (即满足方程 $\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}=0$), 试证明

$$f(r)=c_1+\frac{c_2}{r^{n-2}}\quad(n\geqslant 2),$$

$$f(r)=c_1+c_2\ln\frac{1}{r}\quad(n=2),$$

其中 c_1, c_2 为任意常数.

Proof. 因为 $\frac{\partial u}{\partial x_i}=\frac{\mathrm{d}f}{\mathrm{d}r}\frac{x_i}{r}$, 所以

$$\frac{\partial^2 u}{\partial x_i^2}=\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\frac{x_i^2}{r^2}+\frac{\mathrm{d}f}{\mathrm{d}r}\frac{r^2-x_i^2}{r^3},$$

故

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}=\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{\mathrm{d}f}{\mathrm{d}r}\frac{n-1}{r}=f''(r)+\frac{n-1}{r}f'(r)=0.$$

由上式得

$$f'=cr^{1-n}.$$

(i) $n=2$ 时, $f(r)=c_1+c_2\ln r=c_1+c_2\ln\frac{1}{r}$,
(ii) $n\neq 2$ 时, $f(r)=c_1+\frac{c_2}{2-n}r^{2-n}=c_1+\frac{c_2}{r^{n-2}}$,
其中 c_1, c_2 为任意常数. □

2. 证明: 拉普拉斯算子在球坐标 (r,θ,φ) 下可以写成

$$\Delta u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}.$$

Proof. 球坐标变换及其逆变换为

$$\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta; \end{cases} \quad \Rightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2}, \\ \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}, \\ \varphi = \arctan\frac{y}{x}. \end{cases}$$

通过链式法则可得

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial x}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 u}{\partial \varphi^2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{\partial u}{\partial \varphi}\frac{\partial^2 \varphi}{\partial x^2}.$$

由逆变换公式求得 (所有求导项并未在下面完全列出, 因为很多项的形式是一样的)

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial^2 r}{\partial x^2} = \frac{r^2-x^2}{r^3}.$$

$$\frac{\partial \theta}{\partial x} = \frac{zx}{r^2\sqrt{x^2+y^2}}, \quad \frac{\partial \theta}{\partial z} = \frac{-\sqrt{x^2+y^2}}{r^2}.$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{zr^2y^2 - 2zx^2(x^2+y^2)}{r^4(x^2+y^2)^{3/2}}, \quad \frac{\partial^2 \theta}{\partial z^2} = \frac{2z\sqrt{x^2+y^2}}{r^4}$$

$$\frac{\partial \varphi}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial \varphi}{\partial y} = \frac{x}{x^2+y^2}, \quad \frac{\partial \varphi}{\partial z} = 0.$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}.$$

故

$$\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right] + \frac{\partial u}{\partial r}\left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right) \\
&+ \frac{\partial^2 u}{\partial \theta^2}\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2\right] + \frac{\partial u}{\partial \theta}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}\right) \\
&+ \frac{\partial^2 u}{\partial \varphi^2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + \frac{\partial u}{\partial \varphi}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}\right) \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{2}{r} + \frac{\partial^2 u}{\partial \theta^2}\frac{1}{r^2} + \frac{\partial u}{\partial \theta}\frac{z}{r^2\sqrt{x^2+y^2}} + \frac{\partial^2 u}{\partial \varphi^2}\frac{1}{x^2+y^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}.
\end{aligned}$$

证毕. □

3. 证明: 拉普拉斯算子在柱坐标 (r,θ,z) 下可以写成

$$\Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}.$$

Proof. 柱坐标变换为

$$\begin{cases}x=r\cos\theta,\\ y=r\sin\theta,\\ z=z.\end{cases}$$

故

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta,\quad \frac{\partial u}{\partial\theta}=-\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta.$$

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2}=&\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\cos\theta+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\sin\theta\right]\cos\theta\\&+\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\cos\theta+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\sin\theta\right]\sin\theta\\=&\frac{\partial^2 u}{\partial x^2}\cos^2\theta+2\frac{\partial^2 u}{\partial x\,\partial y}\sin\theta\cos\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\theta.\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial\theta^2}=&-\frac{\partial u}{\partial x}r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta-\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)(-r\sin\theta)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)r\cos\theta\right]r\sin\theta\\&+\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)(-r\sin\theta)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)r\cos\theta\right]r\cos\theta\\=&-\frac{\partial u}{\partial x}r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta+\frac{\partial^2 u}{\partial\theta^2}r^2\sin^2\theta-2\frac{\partial^2 u}{\partial x\,\partial y}r^2\sin\theta\cos\theta+\frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta.\end{aligned}$$

故

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}&=\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}\\&=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Delta u.\end{aligned}$$

□

4. 证明下列函数都是调和函数:
(1) $ax+by+c$ (a,b,c 为常数);
(2) x^2-y^2 和 $2xy$;
(3) x^3-3xy^2 和 $3x^2y-y^2$;

Proof. 直接验证即可. □

5. 证明用极坐标表示的下列函数都满足调和方程:

(1) $\ln r$ 和 θ;
(2) $r^n \cos n\theta$ 和 $r^n \sin n\theta$ (n 为常数);
(3) $r \ln r \cos\theta - r\theta \sin\theta$ 和 $r \ln r \sin\theta + r\theta\cos\theta$.

Proof. 极坐标下的 Laplace 算子为

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

代入验证 $\Delta u = 0$ 即可. □

6. 用分离变量法求解由下述调和方程的第一边值问题所描述的矩阵平板 ($0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b$) 上的稳定温度分布:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \\ u(0,y) = u(a,y) = 0, \\ u(x,0) = \sin\frac{\pi x}{a}, u(x,b) = 0. \end{cases}$$

Solution. 令 $u(x,y) = X(x)Y(y)$, 代入 $\Delta u = 0$ 得

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

由于 $u(x,0) = X(x)Y(0) = \sin\frac{\pi x}{a}$, 故 $X(x) = C\sin\frac{\pi x}{a}$ 且求导得

$$\frac{X''(x)}{X(x)} = \lambda = -\left(\frac{\pi}{a}\right)^2.$$

所以

$$Y''(y) - \left(\frac{\pi}{a}\right)^2 Y(y) = 0.$$

解得

$$Y(y) = C_1 \mathrm{e}^{\frac{\pi}{a}y} + C_2 \mathrm{e}^{-\frac{\pi}{a}y},$$

结合边界条件 $Y(b) = 0$ 得

$$Y(y) = C_3\left(\mathrm{e}^{\frac{(y-b)\pi}{a}} - \mathrm{e}^{\frac{(b-y)\pi}{a}}\right) = 2C_3 \sinh\frac{(y-b)\pi}{a}.$$

于是

$$u(x,y) = X(x)Y(y) = C_4 \sinh\frac{(y-b)\pi}{a}\sin\frac{\pi x}{a}.$$

结合 $u(x,0) = \sin\frac{\pi x}{a}$, 得

$$u(x,y) = \frac{\sinh\frac{(y-b)\pi}{a}}{\sinh\frac{-b\pi}{a}}\sin\frac{\pi x}{a} = \frac{\sinh(b-y)\pi/a}{\sinh b\pi/a}\sin\frac{\pi x}{a}.$$ □

7. 在膜型扁壳渠道闸门的设计中, 为了考察闸门在水压力作用下的受力情况, 要在矩形区域 $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ 上求解如下的非齐次调和方程的边值问题:

$$\begin{cases} \Delta u = py + q \quad (p < 0, q > 0 \text{ 常数}), \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \ u|_{x=a} = 0, \\ u|_{y=0, y=b} = 0. \end{cases}$$

试求解之.

Solution. 令 $v = u + (x^2 - a^2)(fy + g)$, 通过选取 $f = -p/2$, $g = -q/2$, 则 v 满足方程

$$\begin{cases} \Delta v = 0, \\ v_x|_{x=0} = v|_{x=a} = 0, \\ v|_{y=0} = -\frac{q}{2}(x^2 - a^2) = \alpha(x), \\ v|_{y=b} = -\frac{1}{2}(bp + q)(x^2 - a^2) = \beta(x). \end{cases}$$

再利用分离变量法求解即可. □

8. 举例说明在二维 Laplace 方程的 Dirichlet 外问题中, 如对解 $u(x, y)$ 不加在无穷远处为有界的限制, 那么定解问题的解不是唯一的.

Proof. 考虑区域 $\Omega = \{(x, y) \mid x^2 + y^2 > 1\}$ 以及相应的 Dirichlet 外问题

$$\Delta u = 0 \text{ in } \Omega, \quad u = 1 \text{ on } \partial\Omega.$$

显然 $u \equiv 1$ 和 $u = c \ln \frac{1}{r} + 1$ 都为对应的解. □

9. 设

$$J(u) = \iiint_\Omega \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iint_\Gamma \left(\frac{1}{2}\sigma u^2 - gu\right) \mathrm{d}s,$$

变分问题的提法为: 求 $u \in V$, 使

$$J(u) = \min_{v \in V} J(v),$$

其中 $V = C^2(\Omega) \cap C^1(\overline{\Omega})$. 试导出与此变分问题等价的边值问题, 并证明它们的等价性.

Proof. The equivalent BVP is

$$\begin{cases} \Delta u = 0, \\ \left.\left(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u\right)\right|_\Gamma = g. \end{cases}$$

For notational simplicity, we use $\int$ instand of $\iint$, $\iiint$ and use $\mathrm{d}x$ instand of $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

(Minimizer $\Rightarrow$ Solution) Suppose that $J(u) = \min_{v \in V} J(v)$. Define for any $v \in V$,

$$I(\epsilon) = J(u + \epsilon v) = \int_{\Omega} \frac{1}{2} |\nabla(u + \epsilon v)|^2 \,\mathrm{d}x + \int_{\Gamma} \left(\frac{1}{2} \sigma (u + \epsilon v)^2 - g(u + \epsilon v) \right) \mathrm{d}s. \quad (3.1)$$

Then $I(\epsilon)$ attains its minimum at $\epsilon = 0$ and so $I'(0) = 0$. Direct computation yields that

$$\begin{aligned} I'(0) &= \int_{\Omega} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma} (\sigma u - g) v \,\mathrm{d}s \\ &= -\int_{\Omega} \Delta u v \,\mathrm{d}x + \int_{\Gamma} \left(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u - g \right) v \,\mathrm{d}s = 0. \end{aligned} \quad (3.2)$$

Since v is arbitrary, we choose v such that $v = 0$ on Γ, then

$$\int_{\Omega} \Delta u v \,\mathrm{d}x = 0, \quad (3.3)$$

which means $\Delta u = 0$ in Ω. Substituting $\Delta u = 0$ back into (**??**), we now get

$$\int_{\Gamma} \left(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u - g \right) v \,\mathrm{d}s = 0. \quad (3.4)$$

Choosing $v = \frac{\partial u}{\partial \boldsymbol{n}} + \sigma u - g$ on Γ, we get

$$\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u = g \quad \text{on } \Gamma.$$

(Solution $\Rightarrow$ Minimizer) Suppose u is the solution of the following BVP

$$\begin{cases} \Delta u = 0, \\ \left(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u \right)\big|_{\Gamma} = g. \end{cases}$$

Since

$$\begin{aligned} J(u + v) &= \int_{\Omega} \frac{1}{2} |\nabla(u + v)|^2 \,\mathrm{d}x + \int_{\Gamma} \left(\frac{1}{2} \sigma (u + v)^2 - g(u + v) \right) \mathrm{d}s \\ &= J(u) + \int_{\Omega} \frac{1}{2} |\nabla v|^2 + \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma} \left(\frac{1}{2} \sigma v^2 + \sigma u v - g v \right) \mathrm{d}s \\ &= J(u) + \int_{\Omega} \frac{1}{2} |\nabla v|^2 \,\mathrm{d}x + \int_{\Gamma} \frac{1}{2} \sigma v^2 \,\mathrm{d}s, \end{aligned} \quad (3.5)$$

we have that $J(u + v) \geqslant J(u)$ for any $v \in V$, so $J(u) = \min_{v \in V} J(v)$. $\square$

3.2 格林公式及其应用

1. 证明 (2.7) 式对于 M_0 在 Ω 外与 Γ 上的情形成立.

Proof. 当 M_0 在 Ω 外时, $v=\frac{1}{r_{M_0M}}$ 在区域 Ω 内无奇异点, 故由格林第二公式得

$$\iiint_{\Omega}\left(u\Delta\frac{1}{r}-\frac{1}{r}\Delta u\right)\mathrm{d}V=\iint_{\Gamma}\left(u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S.$$

在 Ω 内 $\Delta u=0$, $\Delta\frac{1}{r}=0$, 故

$$-\iint_{\Gamma}\left(u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S=0.$$

当 M_0 在 Γ 上时, 将以 M_0 为球心, 以充分小正数 ε 为半径的球与 Ω 相交的部分记为 K_ε, 将 K_ε 的包含于 Ω 内的边界记为 Γ_ε, 且记 $\partial(\Omega\setminus K_\varepsilon)-\Gamma_\varepsilon=\Gamma'_\varepsilon$, 则由格林第二公式得

$$0=\iiint_{\Omega\setminus K_\varepsilon}\left(u\Delta\frac{1}{r}-\frac{1}{r}\Delta u\right)\mathrm{d}V=\iint_{\Gamma_\varepsilon\bigcup\Gamma'_\varepsilon}\left(u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S.$$

故

$$-\iint_{\Gamma'_\varepsilon}\left(u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S=\iint_{\Gamma_\varepsilon}\left(u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S.\tag{3.6}$$

而

$$\iint_{\Gamma_\varepsilon}\left(u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S=\iint_{\Gamma_\varepsilon}\left(\frac{u}{\varepsilon^2}-\frac{1}{\varepsilon}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S.$$

令 $\varepsilon\to0$, 注意到 Γ 充分光滑 (这意味着面积 $S(\Gamma_\varepsilon)\to2\pi\varepsilon^2$), 所以有

$$\lim_{\varepsilon\to0}\iint_{\Gamma_\varepsilon}\left(\frac{u}{\varepsilon^2}-\frac{1}{\varepsilon}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S=2\pi u(M_0).\tag{3.7}$$

同时 $\lim_{\varepsilon\to0}\Gamma'_\varepsilon=\Gamma$, 因此由 (**??**) 和 (**??**) 得

$$-\iint_{\Gamma}\left(u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)\mathrm{d}S=2\pi u(M_0).\qquad\square$$

2. 若函数 $u(x,y)$ 是单位圆周上的调和函数, 又它在单位圆周上的数值已知为 $u=\sin\theta$, 其中 θ 表示极角, 问函数 u 在原点之值等于多少?

Proof. 由平均值公式知原点之值为

$$u(O)=\frac{1}{2\pi}\int_{\Gamma}\sin\theta\,\mathrm{d}s=\frac{1}{2\pi}\int_0^{2\pi}\sin\theta\,\mathrm{d}\theta=0.\qquad\square$$

4. 证明: 当 $u(M)$ 在闭曲面 Γ 的外部调和, 并且在无穷远处成立

$$u(M) = O\left(\frac{1}{r_{OM}}\right), \quad \frac{\partial u}{\partial r} = O\left(\frac{1}{r_{OM}^2}\right) \quad (r_{OM} \to \infty),$$

而 M_0 是 Γ 外任意一点, 则公式 (2.6) 仍成立.

Proof. 取以 M_0 为球心, 以 R (充分大) 为半径的球 K_R 使其包含曲面 Γ, 并记该球去掉闭曲面 Γ 内部区域后得到的部分为 Ω_R. 将 K_R 的边界记为 Γ_R, 再取以 M_0 为球心, 以 ϵ 为半径的球 K_ϵ 使其完全包含在区域 Ω_R 中, 将 K_ϵ 的边界记为 Γ_ϵ. 取 $r = r_{MM_0}$, 则由格林第二公式得

$$\begin{aligned} 0 &= \iiint_{\Omega_R \setminus K_\epsilon} \left(u\Delta\frac{1}{r} - \frac{1}{r}\Delta u\right) \mathrm{d}V \\ &= \iint_{\Gamma \bigcup \Gamma_R \bigcup \Gamma_\epsilon} \left(u\frac{\partial}{\partial \vec{n}}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial \vec{n}}\right) \mathrm{d}S. \end{aligned} \tag{$\star$}$$

因为 $u(M) = O(\frac{1}{r})$, $\frac{\partial u}{\partial r} = O(\frac{1}{r^2})$ $(r \to \infty)$, 所以当 $R \to +\infty$ 时,

$$\iint_{\Gamma_R} \left(u\frac{\partial}{\partial \vec{n}}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial \vec{n}}\right) \mathrm{d}S = \iint_{\Gamma_R} \left(\frac{-u}{R^2} - \frac{1}{R}\frac{\partial u}{\partial r}\right) \mathrm{d}S \to 0.$$

又因为当 $\epsilon \to 0$ 时,

$$\iint_{\Gamma_\epsilon} \left(u\frac{\partial}{\partial \vec{n}}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial \vec{n}}\right) \mathrm{d}S = \iint_{\Gamma_\epsilon} \left(\frac{u}{\epsilon^2} + \frac{1}{\epsilon}\frac{\partial u}{\partial r}\right) \mathrm{d}S \to 4\pi u(M_0).$$

在 $(\star)$ 式中令 $R \to +\infty$, $\epsilon \to 0$, 即得

$$u(M_0) = -\frac{1}{4\pi}\iint_{\Gamma} \left(u\frac{\partial}{\partial \vec{n}}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial \vec{n}}\right) \mathrm{d}S. \qquad \square$$

5. 证明调和方程 Dirichlet 外问题解的稳定性.

Proof. 在闭曲面 Γ 上给定两个函数 f, f^*, 并且在 Γ 上满足 $|f - f^*| \leqslant \epsilon$, 设 u, u^* 是相应的狄利克雷外问题的解, 以 Γ_R 表示半径为 R 的球面, 令 $v = u - u^*$, 因为

$$\lim_{r \to \infty} v(x, y, z) = 0.$$

所以存在 R_0, 使得在 Γ_{R_0} 及其外部满足 $|v| \leqslant \epsilon$, 在 Γ 和 Γ_{R_0} 围成的有界区域中, 利用极值原理知 $|v| \leqslant \epsilon$, 故在 Γ 的外部满足 $|v| \leqslant \epsilon$, 由此证明了狄利克雷外问题的解是稳定的. $\square$

6. 对于二阶偏微分方程

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = 0,$$

其中 a_{ij}, b_i, c $(i,j=1,\ldots,n)$ 均为常数. 假设存在常数 $\lambda>0$, 使得

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

又设 $c<0$, 证明极值原理: 若 u 在 Ω 中满足方程, 在 $\Omega\cup\Gamma$ 上连续, 则 u 不能在 Ω 的内部达到正的最大值或负的最小值.

Proof. Suppose u attains its positive maximum value at some point $x_0\in\Omega$, then

$$\nabla u(x_0)=0, \qquad D^2u(x_0)\leqslant 0.$$

Thus

$$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j}(x_0)\leqslant 0, \quad \sum_{i=1}^{n} b_iu_{x_i}(x_0)=0, \quad cu(x_0)<0.$$

It follows that

$$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j}(x_0)+\sum_{i=1}^{n} b_iu_{x_i}(x_0)+cu(x_0)<0,$$

which is contradictory to the equation satisfied by u.

The proof for the case when u attains its negative minimum value in the interior of Ω is similar. □

7. 证明第 6 题中讨论的椭圆型方程的第一边值问题的唯一性与稳定性.

8. 举例说明对于方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+cu=0$ $(c>0)$, 不成立极值原理.

Solution. Consider the function

$$u(x,y)=\sin\sqrt{\frac{c}{2}}x\sin\sqrt{\frac{c}{2}}y,$$

which satisfies $\Delta u+cu=0$ in the domain $\Omega=\left[-\sqrt{\frac{2}{c}}\pi,\sqrt{\frac{2}{c}}\pi\right]^2$. However, it attains its maximum in the interior of Ω. □

3.3 格林函数

$$u(M_0)=\iint_\Gamma\left[\frac{1}{4\pi r_{M_0M}}\frac{\partial u}{\partial\vec{n}}-u\frac{\partial}{\partial\vec{n}}\frac{1}{4\pi r_{M_0M}}\right]\mathrm{d}S_M$$

$$\iint_\Gamma\left(g\frac{\partial u}{\partial\vec{n}}-u\frac{\partial g}{\partial\vec{n}}\right)\mathrm{d}S_M=0$$

相减得

$$u(M_0)=\iint_{\Gamma}\left(G\frac{\partial u}{\partial\vec{n}}-u\frac{\partial G}{\partial\vec{n}}\right)\mathrm{d}S_M,\text{其中}G(M,M_0)=\frac{1}{4\pi r_{M_0M}}-g(M,M_0)$$

1. 证明格林函数的性质 3 及性质 5.

Proof. 为了完整性，下面给出格林函数五点性质的全部证明.

性质 1 格林函数 $G(M,M_0)$ 除 $M=M_0$ 一点外处处满足方程 (1.1), 而当 $M\to M_0$ 时, $G(M,M_0)$ 趋于无穷大, 其阶数和 $\frac{1}{4\pi r_{M_0M}}$ 相同.

Proof. 除了点 $M=M_0$ 外, $\frac{1}{4\pi r_{M_0M}}$ 调和, 又因为 $g(M,M_0)$ 在 Ω 内调和, 故 $G(M,M_0)$ 除了 $M=M_0$ 外处处调和, 由极值原理知 $g(M,M_0)$ 在 Ω 上有界, 故

$$\lim_{M\to M_0}G(M,M_0)=\lim_{M\to M_0}\left(\frac{1}{4\pi r_{M_0M}}-g(M,M_0)\right)=\infty,$$

且和 $\frac{1}{4\pi r_{M_0M}}$ 同阶. □

性质 2 在边界上格林函数 $G(M,M_0)$ 恒等于零.

Proof. 由 $g(M,M_0)$ 的定义知 $G(M,M_0)|_{\Gamma}=0$. □

性质 3 在区域 Ω 上成立着不等式:

$$0<G(M,M_0)<\frac{1}{4\pi r_{M_0M}}.$$

Proof. 注意到

$$0<G(M,M_0)<\frac{1}{4\pi r_{M_0M}}\Leftrightarrow 0<g(M,M_0)<\frac{1}{4\pi r_{M_0M}}.$$

由极值原理知 $g(M,M_0)>0$ 是显然的, 下面证明 $g(M,M_0)<\frac{1}{4\pi r_{M_0M}}$:

取 δ 足够小使得在 $B(M_0,\delta)$ 上成立 $\frac{1}{4\pi r_{M_0M}}>g(M,M_0)$, 记 $D=\Omega\setminus\overline{B(M_0,\delta)}$, 则 $\frac{1}{4\pi r_{M_0M}}-g(M,M_0)$ 在 D 上调和, 且

$$\min_{\partial D}\left(\frac{1}{4\pi r_{M_0M}}-g(M,M_0)\right)=0.$$

故由极值原理知在 D 上成立 $\frac{1}{4\pi r_{M_0M}}>g(M,M_0)$, 从而在 Ω 上成立 $\frac{1}{4\pi r_{M_0M}}>g(M,M_0)$. □

性质 4 格林函数 $G(M,M_0)$ 在自变量 M 及参变量 M_0 之间具有对称性, 即设 M_1, M_2 为区域中的两点, 则

$$G(M_1,M_2)=G(M_2,M_1).$$

Proof. Let $D_\epsilon := \Omega \setminus (B(M_1,\epsilon)\bigcup B(M_2,\epsilon))$. Let

$$w(M) := G(M, M_2), \qquad v(M) := G(M, M_1).$$

We need to prove that $w(M_1) = v(M_2)$. By definition, $w(M)$ and $v(M)$ are harmonic functions in D_ϵ. So by Green's formula we have

$$\iint_{\partial D_\epsilon} \left(w\frac{\partial v}{\partial \vec{n}} - v\frac{\partial w}{\partial \vec{n}} \right) \mathrm{d}S = 0,$$

i.e.,

$$\iint_{\partial B(M_1,\epsilon)} \left(w\frac{\partial v}{\partial \vec{n}} - v\frac{\partial w}{\partial \vec{n}} \right) \mathrm{d}S = \iint_{\partial B(M_2,\epsilon)} \left(v\frac{\partial w}{\partial \vec{n}} - w\frac{\partial v}{\partial \vec{n}} \right) \mathrm{d}S. \tag{3.8}$$

We only consider the left-hand side term in (**??**). On $\partial B(M_1,\epsilon)$,

$$\begin{aligned} v(M) = G(M, M_1) = \frac{1}{4\pi r_{MM_1}} - g(M, M_1) = O\Big(\frac{1}{\epsilon}\Big), \\ \frac{\partial v}{\partial \boldsymbol{n}} = \frac{1}{4\pi\epsilon^2} + O(1), \qquad w = \frac{\partial w}{\partial \boldsymbol{n}} = O(1). \end{aligned}$$

Thus

$$\iint_{\partial B(M_1,\epsilon)} \left(w\frac{\partial v}{\partial \vec{n}} - v\frac{\partial w}{\partial \vec{n}} \right) \mathrm{d}S = \iint_{\partial B(M_1,\epsilon)} \left(w\Big(\frac{1}{4\pi\epsilon^2} + O(1)\Big) - O\Big(\frac{1}{\epsilon}\Big)O(1) \right) \to w(M_1)$$

as $\epsilon \to 0$. Similarly, the right-hand side term in (**??**) tends to $v(M_2)$ as $\epsilon \to 0$. So we conclude that $w(M_1) = v(M_2)$. □

性质 5 $\iint_\Gamma \frac{\partial G(M,M_0)}{\partial n} \mathrm{d}S_M = -1$.

Proof. 设 Γ_ϵ 是以 M_0 为球心, 以 ϵ 为半径的球面, 并且其包含在 Ω 当中, 则

$$\begin{aligned} \iint_\Gamma \frac{\partial G(M,M_0)}{\partial \vec{n}} \mathrm{d}S_M &= \iint_\Gamma \frac{\partial}{\partial \vec{n}} \left(\frac{1}{4\pi r_{M_0M}} \right) \mathrm{d}S_M - \iint_\Gamma \frac{\partial g(M,M_0)}{\partial \vec{n}} \mathrm{d}S_M \\ &= \iint_\Gamma \frac{\partial}{\partial \vec{n}} \left(\frac{1}{4\pi r_{M_0M}} \right) \mathrm{d}S_M \quad \text{(Theorem 2.1)} \\ &= \iint_{\Gamma_\epsilon} \frac{\partial}{\partial \vec{n}} \left(\frac{1}{4\pi r_{M_0M}} \right) \mathrm{d}S_M \quad \text{(Theorem 2.1)} \\ &= \iint_{\Gamma_\epsilon} \frac{-1}{4\pi\epsilon^2} \mathrm{d}S_M = -1. \end{aligned}$$

另法: 考虑定解问题

$$\begin{cases} \Delta u = 0 (in\ \Omega) \\ u|_{\Gamma=1} \end{cases}$$

由极值原理知解为 $u \equiv 1$, 代入教材 (3.4) 式即得结论. □

二维情形圆的格林函数取为 $G(M,M_0)=\frac{1}{2\pi}\left(\ln\frac{1}{r_{M_0M}}-\ln\frac{R}{\rho_0}\frac{1}{r_{M_1M}}\right)$ 也是为了使得

$$\int_{\Gamma}\frac{\partial G(M,M_0)}{\partial\vec{n}}\,\mathrm{d}s=-1$$

成立, 证明思路同上. □

2. 证明格林函数的对称性: $G(M_1,M_2)=G(M_2,M_1)$.

Proof. 见第一题. □

3. 写出球的外部区域的格林函数, 并由此导出对调和方程求解球的 Dirichlet 外问题的泊松公式.

4. 试用格林函数法导出调和方程第二边值问题解的表达式.

Solution. We need to solve

$$\begin{cases}\Delta u=0, & \text{in } \mathit{\Omega},\\ \frac{\partial u}{\partial \boldsymbol{n}}=f, & \text{on } \mathit{\Gamma}.\end{cases}$$

To this end, introduce the function $g(M,M_0)$ such that it is harmonic with respect to M in $\mathit{\Omega}$ and satisfies

$$\frac{\partial g}{\partial \boldsymbol{n}}=\frac{\partial}{\partial \boldsymbol{n}}\left(\frac{1}{4\pi r_{M_0M}}\right)\quad\text{on } \mathit{\Gamma}.$$

Hence

$$u(M_0)=\iint_{\Gamma}\left(G\frac{\partial u}{\partial \boldsymbol{n}}-u\frac{\partial G}{\partial \boldsymbol{n}}\right)\mathrm{d}S_M$$

is reduced to

$$u(M_0)=\iint_{\Gamma}fG\,\mathrm{d}S_M,$$

where $G(M,M_0)=\frac{1}{4\pi r_{M_0M}}-g(M,M_0)$ is the Green function. □

5. 求半圆区域上狄利克雷问题的格林函数.

Solution. See figure **??** for the geometric interpretation. Let D be the semicircle centered at O with radius R above the x-axis. Fix any $M_0\in D$ and let M_1 be the dual point of M_0, i.e., $|OM_0|\cdot|OM_1|=R^2$. And let M_2, M_3 be the symmetric points of M_0, M_1 with respect to the x-axis, respectively. For simplicity, denote

$$r=|MM_0|,\quad r_i=|MM_i| \text{ for } 1\leqslant i\leqslant 3.$$

Now we verify that

$$G(M,M_0):=\frac{1}{2\pi}\ln\frac{r_1r_2}{rr_3}$$

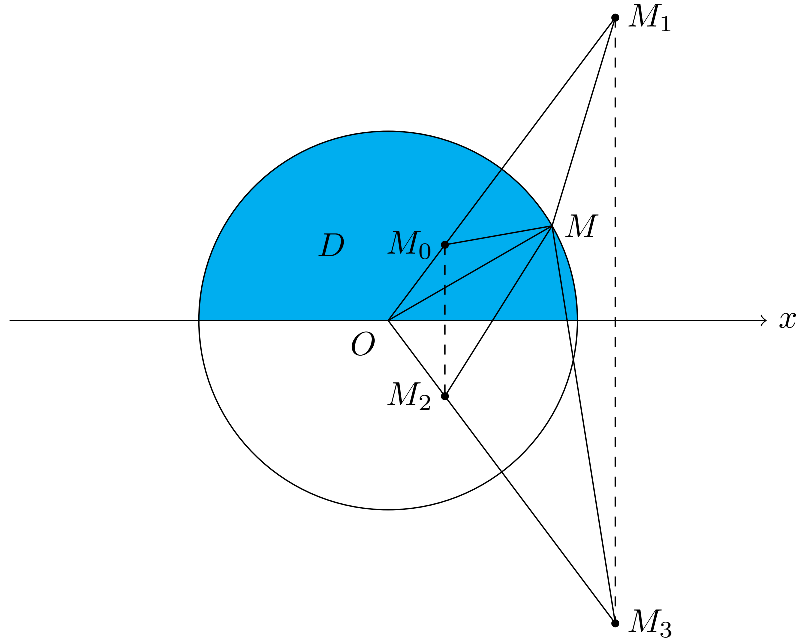

Figure 3.1: Green's function for the semicircle D

is the Green's function on D. Firstly, $G(M, M_0)$ is harmonic in D except the single point M_0. On the other hand, $G(M, M_0)$ vanishes on ∂D, which is obvious on $\partial D \cap \{y = 0\}$ since on this line segment $r = r_2$ and $r_1 = r_3$. For the curved part of ∂D, we should notice that

$$\triangle OM_0M \sim \triangle OMM_1 \tag{3.9}$$

and

$$\triangle OM_2M \sim \triangle OMM_3. \tag{3.10}$$

Therefore by (**??**) and (**??**),

$$\frac{|MM_1|}{|MM_0|} = \frac{|OM_1|}{|OM|} = \frac{|OM_3|}{|OM|} = \frac{|MM_3|}{|MM_2|}, \tag{3.11}$$

i.e., $\frac{r_1}{r} = \frac{r_3}{r_2}$, from which we see that $G(M, M_0) = 0$. $\square$

6. 利用泊松公式求边值问题

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, \quad x^2 + y^2 + z^2 < 1, \\ u(r, \theta, \varphi)|_{r=1} = 3\cos 2\theta + 1 \end{cases}$$

的解.

Solution. Let $G(x,t)$ be the generating function of the Legendre polynomials $(P_n(x))$, i.e.,

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{3.12}$$

Differentiate with respect to t in (**??**) to get

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}. \tag{3.13}$$

So

$$\begin{aligned}\frac{1-t^2}{(1-2xt+t^2)^{3/2}} &= \frac{1-2xt+t^2+2xt-2t^2}{(1-2xt+t^2)^{3/2}} \\ &= \frac{1}{\sqrt{1-2xt+t^2}} + 2t\frac{x-t}{(1-2xt+t^2)^{3/2}} \\ &= \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n.\end{aligned} \tag{3.14}$$

Since

$$\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi-\varphi_0),$$

by addition formula of Legendre polynomials we have

$$P_n(\cos\gamma) = \sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta_0)e^{im(\varphi-\varphi_0)}, \tag{3.15}$$

where P_n^m is the associated Legendre polynomials. Since $P_2(x) = \frac{1}{2}(3x^2-1)$, we have

$$4P_2(\cos\theta) = 3\cos 2\theta + 1. \tag{3.16}$$

Using (**??**), (**??**), (**??**) and Poisson's formula in $\mathbb{R}^3$, we get

$$\begin{aligned}u(\rho_0,\theta_0,\varphi_0) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{(1-\rho_0^2)(3\cos 2\theta+1)\sin\theta}{(1+\rho_0^2-2\rho_0\cos\gamma)^{3/2}}\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\gamma)\rho_0^n(3\cos 2\theta+1)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}(2n+1)\left(\sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta_0)e^{im(\varphi-\varphi_0)}\right) \\ &\qquad\cdot\rho_0^n P_2(\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\end{aligned} \tag{3.17}$$

Using

$$\int_0^{2\pi} e^{im(\varphi-\varphi_0)}\,\mathrm{d}\varphi = \begin{cases} 0, & m \neq 0, \\ 2\pi, & m = 0. \end{cases}$$

and $P_n^0 = P_n$ together with the orthogonality property of Legendre polynomials

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{mn},$$

we find that

$$\begin{aligned} u(\rho_0,\theta_0,\varphi_0) &= 2\int_0^{\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)P_n(\cos\theta_0)\rho_0^n P_2(\cos\theta)\sin\theta\,\mathrm{d}\theta \\ &= 2\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta_0)\rho_0^n\int_0^{\pi} P_n(\cos\theta)P_2(\cos\theta)\sin\theta\,\mathrm{d}\theta \\ &= 2\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta_0)\rho_0^n\int_{-1}^{1} P_n(x)P_2(x)\,\mathrm{d}x \\ &= 4P_2(\cos\theta_0)\rho_0^2 \\ &= \rho_0^2(\cos 2\theta_0 + 1). \quad \square \end{aligned} \tag{3.18}$$

7. 证明二维调和函数的可去奇点定理: 若 A 是调和函数 $u(M)$ 的孤立奇点, 在 A 点邻域中成立着

$$u(M) = o\left(\ln\frac{1}{r_{AM}}\right), \tag{$\star$}$$

则此时可以重新定义 $u(M)$ 在 $M = A$ 的值, 使它在 A 点也是调和的.

Proof. Without loss of generality, we may assume that A is the origin point and that $(\star)$ holds in $B(0,1)$. Let v be the solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0, & |x| < 1, \\ v(x) = u(x), & |x| = 1. \end{cases}$$

The existence of v is guaranteed by the Poisson integral formula. Set $M = \max_{\partial B_1}|u|$. By maximum principle, $|v| \leqslant M$ in B_1. To complete the proof, it suffices to show that $v = u$ in $B_1 \setminus \{0\}$. Let $w = v - u$ in $B_1 \setminus \{0\}$ and $M_r := \max_{\partial B_r}|w|$ for $\forall r < 1$. First, we have

$$-M_r\frac{\log|x|}{\log r} \leqslant w(x) \leqslant M_r\frac{\log|x|}{\log r} \tag{3.19}$$

for any $x \in \partial B_r \cup \partial B_1$. Note that $w(x)$ and $\log|x|$ are harmonic in $B_1 \setminus B_r$, so the maximum principle implies that

$$-M_r\frac{\log|x|}{\log r} \leqslant w(x) \leqslant M_r\frac{\log|x|}{\log r},$$

and hence

$$|w(x)| \leqslant M_r \frac{\log|x|}{\log r}, \tag{3.20}$$

for all $x \in B_1 \setminus B_r$. With

$$M_r = \max_{\partial B_r} |v - w| \leqslant \max_{\partial B_r} |v| + \max_{\partial B_r} |w| \leqslant M + \max_{\partial B_r} |u|, \tag{3.21}$$

we then have

$$|w(x)| \leqslant M \frac{\log|x|}{\log r} + \log|x| \cdot \frac{\max_{\partial B_r} |u|}{\log r}, \tag{3.22}$$

for all $x \in B_1 \setminus B_r$. For each fixed $x \in B_1 \setminus \{0\}$, we take $r < |x|$ and let $r \to 0$ in (**??**) to obtain $w(x) = 0$ since $u(x) = o(\log|x|)$ as $|x| \to 0$. □

8. 证明: 如果三维调和函数 $u(M)$ 在奇点 A 附近表示成 $\frac{N}{r_{AM}^{\alpha}}$, 其中常数 $0 < \alpha \leqslant 1$, 而 N 是不为零的光滑函数, 则当 $M \to A$ 时它趋于无穷大的阶数必与 $\frac{1}{r_{AM}}$ 同阶, 即 $\alpha = 1$.

Proof. Suppose that $\alpha < 1$, then

$$\lim_{M \to A} r_{AM} u(M) = \lim_{M \to A} N r_{AM}^{1-\alpha} = 0.$$

By removable singularity theorem we know that A is a removable singularity, which leads to a contradiction. □

9. 试求一函数 u, 使其在半径为 α 的圆的内部是调和的, 而且在圆周 C 上取下列的值:
(1) $u|_C = A\cos\varphi$;
(2) $u|_C = A + B\sin\varphi$.
其中 A, B 都是常数.

Solution. 利用泊松公式 (3.13) 式

$$u(\rho_0, \varphi_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - \rho_0^2) f(\varphi)}{R^2 + \rho_0^2 - 2R\rho_0 \cos(\varphi - \varphi_0)} \, \mathrm{d}\varphi$$

(1)

$$\begin{aligned}
u(\rho_0,\varphi_0) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-\rho_0^2)A\cos(\varphi)}{R^2+\rho_0^2-2R\rho_0\cos(\varphi-\varphi_0)}\,\mathrm{d}\varphi \\
&= \frac{A(R^2-\rho_0^2)}{2\pi}\int_0^{2\pi}\frac{\cos(\varphi)}{R^2+\rho_0^2-2R\rho_0\cos(\varphi-\varphi_0)}\,\mathrm{d}\varphi \\
&= \frac{A(R^2-\rho_0^2)}{2\pi}\int_0^{2\pi}\frac{\cos(\theta+\varphi_0)}{R^2+\rho_0^2-2R\rho_0\cos\theta}\,\mathrm{d}\theta \qquad (\text{Let } \theta=\varphi-\varphi_0) \\
&= \frac{A(R^2-\rho_0^2)}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta\cos\varphi_0-\sin\theta\sin\varphi_0}{R^2+\rho_0^2-2R\rho_0\cos\theta}\,\mathrm{d}\theta \\
&= \frac{A(R^2-\rho_0^2)}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta\cos\varphi_0}{R^2+\rho_0^2-2R\rho_0\cos\theta}\,\mathrm{d}\theta \\
&= \frac{2A(R^2-\rho_0^2)\cos\varphi_0}{2\pi}\int_0^{\pi}\frac{\cos\theta}{R^2+\rho_0^2-2R\rho_0\cos\theta}\,\mathrm{d}\theta \\
&= \frac{2A(R^2-\rho_0^2)\cos\varphi_0}{2\pi}\frac{\pi\rho_0}{R(R^2-\rho_0^2)} \\
&= \frac{A}{R}\rho_0\cos\varphi_0 = \frac{A}{\alpha}\rho_0\cos\varphi_0.
\end{aligned}$$

故

$$u(\rho,\varphi) = \frac{A}{\alpha}\rho\cos\varphi.$$

(2) 因为 $u|_C = A + B\sin\varphi = A + B\cos(\varphi - \frac{\pi}{2})$, 故由叠加原理及 (1) 中结果知

$$u(\rho,\varphi) = A + \frac{B}{\alpha}\rho\cos\left(\varphi - \frac{\pi}{2}\right) = A + \frac{B}{\alpha}\rho\sin\varphi$$

□

10. 试用静电源像法导出二维调和方程在半平面上的 Dirichlet 问题:

$$\begin{cases}\Delta u = u_{xx} + u_{yy} = 0, & y > 0, \\ u|_{y=0} = f(x).\end{cases}$$

的解.

Solution.

$$G(M,M_0) = \frac{1}{2\pi}\left[\ln\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} - \ln\frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2}}\right].$$

注意到 $\frac{\partial}{\partial \vec{n}} = -\frac{\partial}{\partial y}$, 故

$$\begin{aligned}
&u(x_0, y_0)\\
&= -\int_{\Gamma} f(x) \frac{\partial G(M, M_0)}{\partial \vec{n}} \,\mathrm{d}x\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) \frac{\partial}{\partial y}\left[\ln \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} - \ln \frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2}}\right]\Bigg|_{y=0} \mathrm{d}x\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) \left(-\frac{1}{2}\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} + \frac{1}{2}\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2}\right)\Bigg|_{y=0} \mathrm{d}x\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \frac{y_0}{(x-x_0)^2+y_0^2}\,\mathrm{d}x.
\end{aligned}$$

因此二维调和方程在半平面上的狄利克雷问题的解为:

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi) y}{(\xi - x)^2 + y^2}\,\mathrm{d}\xi.$$

□

11. 设区域 Ω 整个地包含在以原点 O 为球心、R 为半径的球 K 中, $u(r, \theta, \varphi)$ 是此区域中的调和函数, 其中 (r, θ, φ) 表示 Ω 中动点 M 的球坐标. 设 $r_1 = \frac{R^2}{r}$, 则点 $M_1 = (r_1, \theta, \varphi)$ 就是点 M 关于球 K 的反演点, 从 $M(r, \theta, \varphi)$ 到 $M_1(r_1, \theta, \varphi)$ 的变换称为逆矢径变换或反演变换. 以 Ω_1 表示 Ω 的反演区域, 证明函数

$$v(r_1, \theta, \varphi) = \frac{R}{r_1} u\left(\frac{R^2}{r_1}, \theta, \varphi\right)$$

是区域 Ω_1 中的调和函数 (无穷远处除外).

Proof. Recall that in spherical coordinate system, the Laplacian can be expressed as

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}.$$

Let

$$v(r, \theta, \varphi) := \frac{R}{r} u\left(\frac{R^2}{r}, \theta, \varphi\right),$$

and denote $x = (R^2/r, \theta, \varphi)$, then we can compute directly to get

$$\begin{aligned}
\Delta v(r, \theta, \varphi) &= \frac{R^5}{r^5} u_{rr}(x) + \frac{2R^3}{r^4} u_r(x) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{R}{r}\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{R}{r}\frac{\partial^2 u}{\partial \varphi^2}\\
&= \frac{R^5}{r^5}\left(u_{rr}(x) + \frac{2r}{R^2} u_r(x) + \frac{r^2}{R^4 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{r^2}{R^4 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\right)\\
&= \left(\frac{R}{r}\right)^5 \Delta u(x) = 0.
\end{aligned}$$

□

12. 利用开尔文变换及奇点可去性定理把 Dirichlet 外问题化为 Dirichlet 内问题.

Proof. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $\Omega \supset B(0,R)$ for some $R > 0$. Consider the following Dirichlet outer problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega^c, \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{3.23}$$

Let v be the Kelvin transform of u. Suppose $\lim_{r\to\infty} u(r,\theta,\varphi) = 0$, then

$$\lim_{r\to 0} r \cdot v(r,\theta,\varphi) = \lim_{r\to 0} Ru\left(\frac{R^2}{r},\theta,\varphi\right) = 0.$$

By removable singularity theorem we know v is harmonic in $\widetilde{\Omega^c}$, which is the reflection area of Ω^c. Hence problem (**??**) can be converted into

$$\begin{cases} \Delta v = 0, & \text{in } \widetilde{\Omega^c}, \\ v = \mathcal{K}[f], & \text{on } \partial\widetilde{\Omega^c}. \end{cases} \tag{3.24}$$

$\square$

13. 证明在空间一有界区域外定义的调和函数 u 在无穷远处趋于零, 那么它趋于零的阶数至少为 $O(\frac{1}{r})$.

Proof. Let v be the Kelvin transform of u. Since u tends to zero at infinity, we know that the origin point is a removable singularity of v, so

$$\lim_{r\to 0} v = \lim_{r\to 0} \frac{R}{r} u\left(\frac{R^2}{r},\theta,\varphi\right) < \infty,$$

which means

$$u\left(\frac{R^2}{r},\theta,\varphi\right) = O(r) \qquad \text{as } r \to 0.$$

In other words,

$$u(r,\theta,\varphi) = O(1/r) \qquad \text{as } r \to \infty.$$

$\square$

14. 证明处处满足平均值公式 (2.13) 的连续函数一定是调和函数.

Proof. See [**?**, Theorem 1.8]. Choose a test function $\varphi(x) = \varphi(|x|) \in C_0^\infty(B_1(0))$ and such that $\int_{B_1(0)} \varphi = 1$. Define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \in C_0^\infty(B_\varepsilon(0)).$$

For any $x \in \Omega$, take $\varepsilon < \operatorname{dist}(x, \partial\Omega)$. Then we have

$$\begin{aligned}
\int_\Omega u(y)\varphi_\varepsilon(y-x)\,\mathrm{d}y &= \int_{\Omega-\{x\}} u(x+y)\varphi_\varepsilon(y)\,\mathrm{d}y \\
&= \int_{|y|<\varepsilon} u(x+y)\varphi_\varepsilon(y)\,\mathrm{d}y \\
&= \int_{|y|<\varepsilon} \frac{1}{\varepsilon^n} u(x+y)\varphi\left(\frac{y}{\varepsilon}\right)\mathrm{d}y \\
&= \int_{|z|<1} u(x+\varepsilon z)\varphi(z)\,\mathrm{d}z \\
&= \int_0^1 \int_{\partial B_r} u(x+\varepsilon z)\varphi(z)\,\mathrm{d}S_z\,\mathrm{d}r \qquad (\text{let } z = rw) \\
&= \int_0^1 r^{n-1} \int_{\partial B_1} u(x+\varepsilon rw)\varphi(rw)\,\mathrm{d}S_w\,\mathrm{d}r \qquad (\varphi \text{ is radial}) \\
&= \int_0^1 r^{n-1}\varphi(r) \int_{\partial B_1} u(x+\varepsilon rw)\,\mathrm{d}S_w\,\mathrm{d}r \qquad (u \text{ satisfies MVP}) \\
&= \int_0^1 r^{n-1}\varphi(r)\omega_n u(x)\,\mathrm{d}r \\
&= u(x),
\end{aligned}$$

i.e., $u(x) = (u * \varphi)(x)$ which means that u is in fact a smooth function. Now using the MVP and divergence theorem we have

$$\begin{aligned}
\int_{B_r(x)} \Delta u &= r^{n-1}\frac{\partial}{\partial r}\int_{|w|=1} u(x+rw)\,\mathrm{d}S_w \\
&= r^{n-1}\frac{\partial}{\partial r}\big(\omega_n u(x)\big) = 0
\end{aligned}$$

for any $B_r(x) \subset \Omega$. Hence $\Delta u = 0$. □

3.4 强极值原理, 第二边值问题解的唯一性

1. 试用强极值原理来证明极值原理: 对不恒等于常数的调和函数 $u(x,y,z)$, 其在区域 Ω 的任何内点上的值不可能达到它在 Ω 上的上界或下界.

***Proof*.** 假设调和函数 $u(x,y,z)$ 不恒等于常数, 且在区域 Ω 内部某点达最小值 m, 记

$$E = \{M \in \Omega \mid u(M) = m\}.$$

则由 u 的连续性知 E 是相对闭集, 由于 u 不恒为常数, 故 $\Omega \setminus E$ 为非空开集, 取点 $M_0 \in \Omega \setminus E$ 使得 $\mathrm{dist}(M_0, E) < \mathrm{dist}(M_0, \partial\Omega)$. 取以点 M_0 为球心, 以 $\mathrm{dist}(M_0, E)$ 为半径的球 B, 取 $M_1 \in \partial B \cap E$. 对于 B 内任一点 M 均有 $u(M) > u(M_1)$, 故由强极值原理

$$\left.\frac{\partial u}{\partial \vec{\nu}}\right|_{M_1} > 0,$$

其中 $\vec{\nu}$ 与 B 在点 M_1 处的内法线方向成锐角, 但是由于 M_1 是 Ω 内部的最小值点, 故对于任意方向 $\vec{l}$ 均有

$$\left.\frac{\partial u}{\partial \vec{l}}\right|_{M_1} = 0.$$

矛盾. □

2. 利用极值原理和强极值原理证明: 当区域 Ω 的边界 Γ 满足定理 4.2 中的条件时, 调和方程第三边值问题

$$\left.\left(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u\right)\right|_{\Gamma} = f \quad (\sigma > 0)$$

的解的唯一性.

Proof. 只需要证明满足边界条件 $(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u)|_{\Gamma} = 0$ 的只有零解即可, 下面分两种情况讨论.

对于第三边值问题的内问题: 假设 u 不恒为常数, 则由极值原理知 u 在 Γ 上取得最大值和最小值, 记在 M_1 处取到最小值, 在 M_2 处取得最大值, 则

$$\frac{\partial u}{\partial \boldsymbol{n}}(M_1) + \sigma u(M_1) = 0 \Rightarrow u(M_1) = -\frac{1}{\sigma}\frac{\partial u}{\partial \boldsymbol{n}}(M_1) > 0.$$

$$\frac{\partial u}{\partial \boldsymbol{n}}(M_2) + \sigma u(M_2) = 0 \Rightarrow u(M_2) = -\frac{1}{\sigma}\frac{\partial u}{\partial \boldsymbol{n}}(M_2) < 0.$$

故 $u(M_2) < u(M_1)$, 矛盾, 故假设不成立, 所以

$$u \equiv C \Rightarrow \left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\Gamma} = 0 \Rightarrow u|_{\Gamma} = 0 \Rightarrow u \equiv 0.$$

对于第三边值问题的外问题: 记边界 Γ 的外部为 Ω', 假设存在 $M_0 \in \Omega'$, 使得 $u(M_0) > 0$, 由于 $\lim_{M \to \infty} u(M) = 0$, 故存在充分大的 R, 使得在 $\Gamma_R = \{r = R\}$ 上成立 $|u| < u(M_0)$, 则由极值原理知 u 的最大值只能在 Γ 上取, 设最大值点为 $M_1 \in \Gamma$, 则

$$\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{M_1} + \sigma u(M_1) > 0,$$

与边界条件相矛盾, 假设 $u(M_0) < 0$ 同样可以导出矛盾, 故 $u \equiv 0$. □

3. 说明在证明强极值原理过程中，不可能作出一个满足条件 (1) 和 (3) 的辅助函数 $v(x, y, z)$, 使它在整个球 $\overline{B}_R = \{x^2 + y^2 + z^2 \leqslant R^2\}$ 内满足 $\Delta v > 0$.

Proof. 若在 B_R 上有 $\Delta v > 0$, 则

$$\max_{\overline{B}_R} v = \max_{\partial B_R} v = 0.$$

又因为 $\frac{\partial v}{\partial r} < 0$, 所以

$$\min_{\overline{B}_R} v = \min_{\partial B_R} v = 0.$$

结合二者即得 $v \equiv 0$, 与 $\Delta v > 0$ 矛盾. □

4. 设 Ω 为 $\mathbb{R}^3$ 中有界区域, 边界为 Γ, u 为定解问题

$$\begin{cases} -\Delta u + cu = f, \\ \left(\dfrac{\partial u}{\partial \boldsymbol{n}} + \sigma u\right)\Big|_{\partial\Omega} = g \end{cases}$$

的解, 其中 $c, f, g, \sigma > 0$, 证明在 $\overline{\Omega}$ 上 $u > 0$.

Proof. Let x_0 be the minimum point of u in $\overline{\Omega}$.

(1) If $x_0 \in \Omega$ and $u(x_0) \leqslant 0$, then $-\Delta u(x_0) \leqslant 0$, $cu(x_0) \leqslant 0$, which is contradictory to $f > 0$;

(2) If $x_0 \in \partial\Omega$ and $u(x_0) \leqslant 0$, then $\frac{\partial u(x_0)}{\partial \boldsymbol{n}} \leqslant 0$ and $\sigma u(x_0) \leqslant 0$, which is contradictory to $g > 0$.

Therefore $u > 0$ in $\overline{\Omega}$. □

5. 举例说明: 当 $\sigma > 0$ 不成立时 (但 σ 不恒等于零), 调和方程满足边界条件 $\left(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u\right)\big|_{\partial\Omega} = g$ 的解可以不唯一.

Proof. It suffices to show that there exists non-trivial solution to the following problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} + \sigma u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = B(0, R') \setminus \overline{B(0, R)}$ and $R' > R > 0$ are constants. Let $u = 1/r$. Obviously u is harmonic in Ω and

$$\frac{\partial u}{\partial \boldsymbol{n}} = \begin{cases} -\frac{1}{R'^2}, & \text{on } \partial B(0, R'), \\ \frac{1}{R^2}, & \text{on } \partial B(0, R). \end{cases}$$

Define

$$\sigma = \begin{cases} \frac{1}{R'}, & \text{on } \partial B(0, R'), \\ -\frac{1}{R}, & \text{on } \partial B(0, R), \end{cases}$$

then $\left(\frac{\partial u}{\partial \boldsymbol{n}} + \sigma u\right)\big|_{\partial\Omega} = 0$. □

6. 对于一般的椭圆型方程

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}+cu=0,$$

其中矩阵 (a_{ij}) 正定, 即存在常数 $\alpha>0$ 使得

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j\geqslant \alpha|\xi|^2\quad \forall \xi\in\mathbb{R}^n.$$

又设 $c\leqslant 0$, 试证明它的解也成立着强极值原理. 也就是说, 如果 $u(M)$ 在球 $|x|<R$ 内满足上述方程, 在闭球 $|x|\leqslant R$ 上连续, 在球面上一点 M_0 处取到非正的最小值, 且在该点沿 $\boldsymbol{\nu}$ 方向的方向导数 $\frac{\partial u}{\partial \boldsymbol{\nu}}$ 存在, 其中 $\boldsymbol{\nu}$ 与球的内法线方向成锐角, 则在 M_0 点有 $\frac{\partial u}{\partial \boldsymbol{\nu}}>0$.

Proof. See [**?**, Hopf's Lemma in §6.4.2] and [**?**, Theorem 2.5 & Corollary 2.9]. 若 u 在球面上一点 M_0 取非正的最小值, 即 $u(M_0)\leqslant 0$, 且对球内任一点 M 有 $u(M)>u(M_0)$, 因此在 M_0 点有

$$\frac{\partial u}{\partial \boldsymbol{\nu}}\geqslant 0.$$

现在需要证明上式中的等号不能成立, 构造函数

$$v(x)=e^{-\lambda|x|^2}-e^{-\lambda R^2},$$

其中 λ 为待定的正常数, 则 v 满足如下性质:

(1) 在球面 $|x|=R$ 上 $v=0$;

(2) 通过适当选取 λ, 可以使得在区域 $D=\{R/2\leqslant|x|\leqslant R\}$ 内

$$Lv=\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 v}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial v}{\partial x_i}+cv>0.$$

事实上, 因为

$$\begin{aligned}\frac{\partial v}{\partial x_i}&=-2\lambda x_i e^{-\lambda|x|^2},\\ \frac{\partial^2 v}{\partial x_i\,\partial x_j}&=4\lambda^2 x_i x_j e^{-\lambda|x|^2},\quad i\neq j,\\ \frac{\partial^2 v}{\partial x_i^2}&=4\lambda^2 x_i^2 e^{-\lambda|x|^2}-2\lambda e^{-\lambda|x|^2},\end{aligned}$$

所以

$$
\begin{aligned}
Lv &= 4\lambda^2\left(\sum_{i,j=1}^{n} a_{ij}x_ix_j\right)e^{-\lambda|x|^2} - 2\lambda\sum_{i=1}^{n}(b_ix_i + a_{ii})e^{-\lambda|x|^2} \\
&\quad + c\left(e^{-\lambda|x|^2} - e^{-\lambda R^2}\right) \\
&= e^{-\lambda|x|^2}\left(4\lambda^2\sum_{i,j=1}^{n} a_{ij}x_ix_j - 2\lambda\sum_{i=1}^{n}(b_ix_i + a_{ii}) + c\left(1 - e^{-\lambda(R^2-|x|^2)}\right)\right).
\end{aligned}
$$

因为 $\sum_{i,j=1}^{n} a_{ij}x_ix_j \geqslant \alpha|x|^2 \geqslant \frac{\alpha R^2}{4} > 0$, 故当 λ 充分大时在区域 D 内 $Lv > 0$.

(3) v 沿球的半径方向 $\frac{\partial v}{\partial r} < 0$. 于是 $\frac{\partial v}{\partial \boldsymbol{\nu}} > 0$.

作函数

$$\tilde{u}(M) = \varepsilon v(M) + u(M_0).$$

在 M_0 点有 $\frac{\partial \tilde{u}}{\partial \boldsymbol{\nu}} = \varepsilon\frac{\partial v}{\partial \boldsymbol{\nu}} > 0$, 令函数

$$w(M) := u(M) - \tilde{u}(M) = u(M) - \varepsilon v(M) - u(M_0). \tag{3.25}$$

在区域 D 上考察 $w(M)$:

(1) $Lw = Lu - \varepsilon Lv - Lu(M_0) = -\varepsilon Lv - cu(M_0) < 0$;

(2) 在 $|x| = R/2$ 上由于 $u(M) > u(M_0)$, 取 ε 足够小可使得 $w(M) > 0$;

(3) 在 $|x| = R$ 上 $v = 0$, $u(M) > u(M_0)$, 故 $w(M) \geqslant 0$.

现在证明在整个区域 D 上 $w \geqslant 0$, 假设存在 $M_1 \in D$, 使得 $w(M_1) < 0$, 于是

$$cw(M_1) \geqslant 0, \quad \left.\frac{\partial w}{\partial x_i}\right|_{M_1} = 0, \quad \left.\left(\frac{\partial^2 w}{\partial x_i\,\partial x_j}\right)\right|_{M_1} \text{非负定}.$$

又 $a_{ij} = \sum_{r=1}^{n} g_{ri}g_{rj}$, 故

$$\sum_{i,j=1}^{n} a_{ij}\left.\frac{\partial^2 w}{\partial x_i\,\partial x_j}\right|_{M_1} = \sum_{r=1}^{n}\sum_{i,j=1}^{n}\left.\frac{\partial^2 w}{\partial x_i\,\partial x_j}\right|_{M_1} g_{ri}g_{rj} \geqslant 0,$$

因此 $Lw|_{M_1} \geqslant 0$, 与 (1) 矛盾, 因此在 D 内 $w(M) \geqslant w(M_0)$, 故

$$\left.\frac{\partial w}{\partial \boldsymbol{\nu}}\right|_{M_0} \geqslant 0.$$

从而与 (**??**) 结合得

$$\left.\frac{\partial u}{\partial \boldsymbol{\nu}}\right|_{M_0} > 0. \qquad \square$$

Chapter 4

二阶线性偏微分方程的分类与总结

4.1 二阶线性方程的分类

1. 证明: 两个自变量的二阶线性方程经过自变量的可逆变换后, 其类型不会改变, 即变换后 $\Delta = a_{12}^2 - a_{11}a_{22}$ 的符号不变.

Proof. 因为

$$\begin{cases} \bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2, \\ \bar{a}_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y, \\ \bar{a}_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2, \end{cases}$$

所以

$$\begin{aligned} \overline{\Delta} &= \bar{a}_{12}^2 - \bar{a}_{11}\bar{a}_{22} \\ &= a_{12}^2(\xi_x\eta_y + \xi_y\eta_x)^2 - 4a_{12}^2\xi_x\xi_y\eta_x\eta_y + 2a_{11}a_{22}\xi_x\xi_y\eta_x\eta_y - a_{11}a_{22}(\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2) \\ &= (a_{12}^2 - a_{11}a_{22})(\xi_x\eta_y - \xi_y\eta_x)^2 \\ &= \Delta \cdot \left[\frac{D(\xi,\eta)}{D(x,y)}\right]^2. \end{aligned}$$

故 Δ 与 $\overline{\Delta}$ 的符号相同. □

2. 判定下述方程的类型:
(1) $x^2u_{xx} - y^2u_{yy} = 0$;
(2) $u_{xx} + (x+y)^2u_{yy} = 0$;
(3) $u_{xx} + xyu_{yy} = 0$;
(4) $u_{xx} - 4u_{xy} + 2u_{xz} + 4u_{yy} + u_{zz} = 0$;
(5) $u_{xx} + (\operatorname{sgn} y)u_{yy} = 0$.

Solution.

(1) $\Delta = x^2y^2 \geqslant 0$. Parabolic on the axes and hyperbolic in other places;
(2) $\Delta = -(x+y)^2 \leqslant 0$. Parabolic on the line $x+y=0$ and elliptic elsewhere;
(3) $\Delta = -xy$. Elliptic in the 1st and 3rd quadrands, hyperbolic in the 2nd and 4th quadrands, and parabolic on the axes;
(4) The corresponding matrix is

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Since $D_1 = 1 > 0$, $D_2 = 0$ and $D_3 = -4 < 0$, the type is hyperbolic;
(5) Since

$$\Delta = -\operatorname{sgn} y \begin{cases} < 0, & \text{if } y > 0, \\ = 0, & \text{if } y = 0, \\ > 0, & \text{if } y < 0, \end{cases}$$

the type is elliptic when $y > 0$, hyperbolic when $y < 0$ and parabolic when $y = 0$.

□

3. 化下列方程为标准形式:
(1) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0$;
(2) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$;
(3) $u_{xx} + yu_{yy} = 0$;
(4) $u_{xx} - 2\cos x u_{xy} - (3 + \sin^2 x)u_{yy} - yu_y = 0$;
(5) $(1+x^2)u_{xx} + (1+y^2)u_{yy} + xu_x + yu_y = 0$.

Solution. (1) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0$. $\Delta = 4 - 5 = -1 < 0$, 故方程为椭圆型. 特征方程为 $\mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}y + 5\,\mathrm{d}x^2 = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2 \pm i \Rightarrow y = (2 \pm i)x + C$, 取 $y = (2+i)x + C$, 即 $y - 2x - ix = C$. 令

$$\begin{cases} \xi = 2x - y, \\ \eta = x. \end{cases}$$

则

$$\begin{cases} u_x = 2u_\xi + u_\eta, \\ u_y = -u_\xi, \\ u_{xx} = 2(2u_{\xi\xi} + u_{\xi\eta}) + 2u_{\xi\eta} + u_{\eta\eta} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}, \\ u_{yy} = -(-u_{\xi\xi}) = u_{\xi\xi}, \\ u_{xy} = -(2u_{\xi\xi} + u_{\xi\eta}). \end{cases}$$

代入原方程即得标准形式为

$$u_{\xi\xi} + u_{\eta\eta} + u_\eta = 0.$$

(2) $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0$. $\Delta=x^2y^2-x^2y^2=0$, 故方程为抛物型. 特征方程为 $x^2\,\mathrm{d}y^2-2xy\,\mathrm{d}x\,\mathrm{d}y+y^2\,\mathrm{d}x^2=0\Rightarrow y=Cx$. 令

$$\begin{cases}\xi=\frac{y}{x},\\ \eta=x.\end{cases}$$

则

$$\begin{cases}u_x=-\frac{y}{x^2}u_\xi+u_\eta,\\ u_y=\frac{1}{x}u_\xi,\\ u_{xx}=\frac{2y}{x^3}u_\xi-\frac{y}{x^2}\left(-\frac{y}{x^2}u_{\xi\xi}+u_{\xi\eta}\right)-\frac{y}{x^2}u_{\eta\xi}+u_{\eta\eta}=\frac{2y}{x^3}u_\xi+\frac{y^2}{x^4}u_{\xi\xi}-\frac{2y}{x^2}u_{\xi\eta}+u_{\eta\eta},\\ u_{yy}=\frac{1}{x^2}u_{\xi\xi},\\ u_{xy}=-\frac{1}{x^2}u_\xi+\frac{1}{x}\left(-\frac{y}{x^2}u_{\xi\xi}+u_{\xi\eta}\right)=-\frac{1}{x^2}u_\xi-\frac{y}{x^3}u_{\xi\xi}+\frac{1}{x}u_{\xi\eta}.\end{cases}$$

代入原方程即得标准形式为

$$x^2u_{\eta\eta}=0\Rightarrow u_{\eta\eta}=0.$$

(3) $u_{xx}+yu_{yy}=0$. $\Delta=-y$, 故 $y>0$ 时为椭圆型, $y=0$ 时为抛物型, $y<0$ 时为双曲型. 特征方程为 $\mathrm{d}y^2+y\,\mathrm{d}x^2=0$.

(i) $y>0$ 时, $\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\sqrt{y}i$, 取 $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{y}i\Rightarrow 2\sqrt{y}-ix=C$, 令

$$\begin{cases}\xi=2\sqrt{y},\\ \eta=-x.\end{cases}$$

则

$$\begin{cases}u_x=-u_\eta,\\ u_{xx}=u_{\eta\eta},\\ u_y=\frac{1}{\sqrt{y}}u_\xi,\\ u_{yy}=-\frac{1}{2}y^{-3/2}u_\xi+\frac{1}{y}u_{\xi\xi}.\end{cases}$$

代入原方程即得标准形式为

$$u_{\eta\eta}+u_{\xi\xi}-\frac{1}{\xi}u_\xi=0.$$

(ii) $y=0$ 时, $u_{xx}=0$ 即为标准形式.

(iii) $y<0$ 时, $\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\sqrt{-y}\Rightarrow 2\sqrt{-y}\pm x=C$, 令

$$\begin{cases}\xi=2\sqrt{-y}+x,\\ \eta=2\sqrt{-y}-x.\end{cases}$$

则

$$\begin{cases} u_x = u_\xi - u_\eta, \\ u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}, \\ u_y = \frac{-1}{\sqrt{-y}}(u_\xi + u_\eta), \\ u_{yy} = -\frac{1}{2}(-y)^{-3/2}(u_\xi + u_\eta) - \frac{1}{y}u_{\xi\xi} - \frac{1}{y}u_{\eta\eta} - \frac{2}{y}u_{\xi\eta}. \end{cases}$$

代入原方程即得标准形式为

$$u_{\xi\eta} - \frac{1}{2(\xi+\eta)}(u_\xi + u_\eta) = 0.$$

(4) $u_{xx} - 2\cos x u_{xy} - (3+\sin^2 x)u_{yy} - yu_y = 0$. $\Delta = \cos^2 x + 3 + \sin^2 x = 4 > 0$, 故方程为双曲型. 特征方程为 $\mathrm{d}y^2 + 2\cos x\,\mathrm{d}x\,\mathrm{d}y - (3+\sin^2 x)\,\mathrm{d}x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\cos x \pm 2 \Rightarrow y + \sin x \pm 2x = C \Rightarrow y + \sin x \pm 2x = C$. 令

$$\begin{cases} \xi = y + \sin x + 2x, \\ \eta = y + \sin x - 2x. \end{cases}$$

则

$$\begin{cases} u_x = (\cos x + 2)u_\xi + (\cos x - 2)u_\eta, \\ u_y = u_\xi + u_\eta, \\ u_{xx} = -\sin x(u_\xi + u_\eta) + (\cos x+2)^2 u_{\xi\xi} + (\cos x - 2)^2 u_{\eta\eta} + 2(\cos^2 x - 4)u_{\xi\eta} \\ u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ u_{xy} = (\cos x + 2)u_{\xi\xi} + 2\cos x u_{\xi\eta} + (\cos x - 2)u_{\eta\eta}. \end{cases}$$

代入原方程即得标准形式为

$$u_{\xi\eta} + \frac{\xi+\eta}{32}(u_\xi + u_\eta) = 0.$$

(5) $(1+x^2)u_{xx} + (1+y^2)u_{yy} + xu_x + yu_y = 0$. $\Delta = -(1+x^2)(1+y^2) < 0$, 故方程为椭圆型. 特征方程为 $(1+x^2)\,\mathrm{d}y^2 + (1+y^2)\,\mathrm{d}x^2 = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\sqrt{\frac{1+y^2}{1+x^2}}i \Rightarrow \frac{\mathrm{d}y}{\sqrt{1+y^2}} = \pm i\frac{\mathrm{d}x}{\sqrt{1+x^2}} \Rightarrow$
$\ln(y + \sqrt{1+y^2}) \pm i\ln(x + \sqrt{1+x^2}) = C$
令

$$\begin{cases} \xi = \ln(y + \sqrt{1+y^2}), \\ \eta = \ln(x + \sqrt{1+x^2}). \end{cases}$$

则

$$\begin{cases} u_x = \frac{1}{\sqrt{1+x^2}}u_\eta, \\ u_y = \frac{1}{\sqrt{1+y^2}}u_\xi, \\ u_{xx} = -x(1+x^2)^{-3/2}u_\eta + \frac{1}{1+x^2}u_{\eta\eta}, \\ u_{yy} = -y(1+y^2)^{-3/2}u_\xi + \frac{1}{1+y^2}u_{\xi\xi}. \end{cases}$$

代入原方程即得标准形式为

$$u_{\xi\xi}+u_{\eta\eta}=0. \qquad \square$$

4. 证明: 两个自变量的二阶常系数双曲型方程或椭圆型方程一定可以经过自变量及未知函数的可逆变换

$$u=\mathrm{e}^{\lambda\xi+\mu\eta}v$$

将它化成

$$v_{\xi\xi}\pm v_{\eta\eta}+cv=f$$

的形式.

Proof. 已知两个自变量的二阶常系数双曲型方程或椭圆型方程可以通过可逆变换化为标准形式:

$$u_{\xi\xi}\pm u_{\eta\eta}+au_{\xi}+bu_{\eta}+cu+f=0$$

下面以椭圆型方程为例, 因为 $u=e^{\lambda\xi+\mu\eta}v$, 所以

$$\begin{cases} u_{\xi}=e^{\lambda\xi+\mu\eta}(\lambda v+v_{\xi}) \\ u_{\eta}=e^{\lambda\xi+\mu\eta}(\mu v+v_{\eta}) \\ u_{\xi\xi}=e^{\lambda\xi+\mu\eta}(v_{\xi\xi}+2\lambda v_{\xi}+\lambda^2 v) \\ u_{\eta\eta}=e^{\lambda\xi+\mu\eta}(v_{\eta\eta}+2\mu v_{\eta}+\mu^2 v) \end{cases}$$

故

$$\begin{aligned} &u_{\xi\xi}+u_{\eta\eta}+au_{\xi}+bu_{\eta}+cu+f \\ &\quad=e^{\lambda\xi+\mu\eta}\big[v_{\xi\xi}+v_{\eta\eta}+(2\lambda+a)v_{\xi}+(2\mu+b)v_{\eta}+(\lambda^2+\mu^2+a\lambda+b\mu+c)v\big]+f \\ &\quad=0 \end{aligned}$$

令

$$\lambda=-\frac{a}{2},\mu=-\frac{b}{2},c_1=c-\frac{a^2}{4}-\frac{b^2}{4},f_1=-fe^{-(\lambda\xi+\mu\eta)}$$

即可将原方程化简为

$$v_{\xi\xi}+v_{\eta\eta}+c_1v=f_1$$

对于双曲型方程的情形可以进行类似证明. $\square$

5. 对 $\mathbb{R}^n$ 中诸点判定方程

$$\sum_{i,j=1}^{n}(\delta_{ij}-x_ix_j)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+2\sum_{i=1}^{n}x_i\frac{\partial u}{\partial x_i}+cu=f$$

的类型.

Solution. The corresponding matrix is

$$A = \begin{pmatrix} 1-x_1^2 & -x_1x_2 & \cdots & -x_1x_n \\ -x_2x_n & 1-x_2^2 & \cdots & -x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ -x_1x_n & -x_2x_n & \cdots & -x_n^2 \end{pmatrix} = E - xx^T,$$

where $x = (x_1, \ldots, x_n)^T$. By matrix determinant lemma, we have for $\lambda \neq 1$,

$$\begin{aligned} |\lambda E - A| &= |(\lambda - 1)E + xx^T| \\ &= \left(1 + x^T \frac{E}{\lambda - 1} x\right) \det\bigl((\lambda - 1)E\bigr) \\ &= (\lambda - 1)^{n-1}(\lambda - 1 + x^T x). \end{aligned}$$

Obviously the case $\lambda = 1$ can also be incorporated into the above formula. Therefore we find that the eigenvalues of A are $\lambda_1 = 1$ (with multiplicity $n-1$ or n) and $\lambda_2 = 1 - x^T x$. So

- If $x^T x = \sum x_i^2 = 1$, the equation is parabolic;
- If $x^T x = \sum x_i^2 < 1$, the equation is elliptic;
- If $x^T x = \sum x_i^2 > 1$, the equation is hyperbolic. □

4.2 二阶线性方程的特征理论

1. 求下列方程的特征方程和特征方向:

(1) $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2}$,

(2) $\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^3 \frac{\partial^2 u}{\partial x_i^2}$,

(3) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$.

2. 对波动方程 $u_{tt} - a^2(u_{xx} + u_{yy}) = 0$, 求过直线 $l : t = 0, y = 2x$ 的特征平面.

Solution. The characteristic equation is

$$\alpha_0^2 - a^2(\alpha_1^1 + \alpha_2^2) = 0.$$

Combining $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 1$, we find

$$\alpha_0 = \pm\frac{a}{\sqrt{1+a^2}}, \quad \alpha_1 = \frac{\cos\theta}{\sqrt{1+a^2}}, \quad \alpha_2 = \frac{\sin\theta}{\sqrt{1+a^2}}.$$

Since the characteristic hyperplane passes through the origin, we may write its equation as

$$at + \cos\theta\, x + \sin\theta\, y = 0. \tag{4.1}$$

On the other hand, the line $t=0, y=2x$ is on the hyperplane (**??**). So $\cos\theta + 2\sin\theta = 0$, from which we get $\cos\theta = \mp\frac{2\sqrt{5}}{5}$ and $\sin\theta = \pm\frac{\sqrt{5}}{5}$. Thus the hyperplane is

$$at \mp \frac{2\sqrt{5}}{5}x \pm \frac{\sqrt{5}}{5}y = 0. \qquad \square$$

3. 证明: 经过可逆的坐标变换 $x_i = f_i(y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$, 原方程的特征曲面变为经变换后的新方程的特征曲面, 即特征曲面关于可逆坐标变换具有不变形.

Proof. 考虑二阶线性方程

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i} + Cu = F. \tag{4.2}$$

设 $G(x_1, \ldots, x_n) = 0$ 为其特征曲面, 则

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial G}{\partial x_i}\frac{\partial G}{\partial x_j} = 0. \tag{4.3}$$

经过可逆的坐标变换 $x_i = f_i(y_1, \ldots, y_n)$, 有

$$\frac{\partial u}{\partial x_i} = \sum_{l=1}^{n}\frac{\partial u}{\partial y_l}\frac{\partial y_l}{\partial x_i},$$

$$\frac{\partial^2 u}{\partial x_i\,\partial x_j} = \sum_{k,l=1}^{n}\frac{\partial^2 u}{\partial y_l\,\partial y_k}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j} + \sum_{l=1}^{n}\frac{\partial u}{\partial y_l}\frac{\partial^2 y_l}{\partial x_i\,\partial x_j}.$$

将上面两式代入原方程 (**??**) 得

$$\sum_{i,j=1}^{n} A_{ij}\left(\sum_{k,l=1}^{n}\frac{\partial^2 u}{\partial y_l\,\partial y_k}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j} + \sum_{l=1}^{n}\frac{\partial u}{\partial y_l}\frac{\partial^2 y_l}{\partial x_i\,\partial x_j}\right) + \sum_{i=1}^{n} B_i\left(\sum_{l=1}^{n}\frac{\partial u}{\partial y_l}\frac{\partial y_l}{\partial x_i}\right) + Cu = F.$$

整理上式并简记一阶偏导数项得

$$\sum_{k,l=1}^{n}\left(\sum_{i,j=1}^{n} A_{ij}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j}\right)\frac{\partial^2 u}{\partial y_l\,\partial y_k} + \sum_{l=1}^{n}\widetilde{B}_l\frac{\partial u}{\partial y_l} + Cu = F.$$

设 $G^*(y_1, \ldots, y_n)$ 为其特征曲面, 则需满足

$$\sum_{k,l=1}^{n}\left(\sum_{i,j=1}^{n} A_{ij}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j}\right)\frac{\partial G^*}{\partial y_k}\frac{\partial G^*}{\partial y_l} = 0.$$

另一方面, 对原方程的特征曲面经过可逆变换后的特征曲面为:

$$G(x_1,\dots,x_n)=G(f_1(y_1,\dots,y_n),\dots,f_n(y_1,\dots,y_n))=:G_1(y_1,\dots,y_n),$$

由 (**??**) 得

$$\sum_{i,j=1}^n A_{ij}\left(\sum_{l=1}^n \frac{\partial G_1}{\partial y_l}\frac{\partial y_l}{\partial x_i}\right)\left(\sum_{k=1}^n \frac{\partial G_1}{\partial y_k}\frac{\partial y_k}{\partial x_j}\right)=\sum_{k,l=1}^n\left(\sum_{i,j=1}^n A_{ij}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j}\right)\frac{\partial G_1}{\partial y_k}\frac{\partial G_1}{\partial y_l}=0.$$

对比即得 $G^*=G_1$, 即特征曲面关于可逆坐标变换具有不变性. □

4. 试证二阶线性偏微分方程解的 m 阶弱间断 (即直至 $m-1$ 阶的偏导数为连续, 而 m 阶偏导数为第一类间断) 也只可能沿着特征线发生.

Proof. Similar to the proof in the textbook. For its proof, see [**?**, §2.1.2]. □

5. 试定义 n 阶线性偏微分方程的特征方程、特征方向和特征曲面.

Solution. See [**?**, §2.1.2]. For n-th order linear partial differential equation

$$\sum_{|\alpha|\leqslant n} a_\alpha(x)D^\alpha u(x)=0, \tag{4.4}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index and $D^\alpha u(x)=\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, $|\alpha|=\sum\alpha_i$. We say that the surface

$$S:\varphi(x)=0 \tag{4.5}$$

is the characteristic surface of (**??**) if

$$\sum_{|\alpha|=n} a_\alpha(x)(\nabla\varphi(x))^\alpha=\sum_{|\alpha|=n} a_\alpha(x)\prod_{i=1}^n\left(\frac{\partial\varphi}{\partial x_i}\right)^{\alpha_i}=0. \tag{4.6}$$

And a vector $v=(v_1,\dots,v_n)$ is called the characteristic vector if it satisfies the following characteristic equation

$$\sum_{|\alpha|=n} a_\alpha(x)v^\alpha=\sum_{|\alpha|=n} a_\alpha(x)v_1^{\alpha_1}\cdots v_n^{\alpha_n}=0. \tag{4.7}$$

□

4.3　三类方程的比较

1. 证明热传导方程

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

的初边值问题

$$\begin{cases} u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x) \end{cases}$$

的解关于自变量 x $(0 < x < l)$ 和 t $(t > 0)$ 可进行任意次微分.

Proof. 利用分离变量法得该初边值问题的解为

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2 a^2}{l^2}t} \sin \frac{n\pi}{l} x,$$

其中 $C_n = \frac{2}{l}\int_0^l \varphi(x) \sin \frac{n\pi}{l} x \, \mathrm{d}x$, $|C_n| \leqslant M$, 只需要证明级数逐项微分任意次后仍然是绝对且一致收敛即可, 对 t 微分 α 次, 对 x 微分 β 次需要级数

$$\sum_{n=1}^{\infty} C_n \left(-\frac{n^2\pi^2 a^2}{l^2}\right)^{\alpha} \left(\frac{n\pi}{l}\right)^{\beta} \left(\sin \frac{n\pi}{l} x\right)^{(\beta)} e^{-\frac{n^2\pi^2 a^2}{l^2}t}$$

绝对且一致收敛, 而当 $t \geqslant t_0 > 0$ 时, 上述级数以

$$\sum_{n=1}^{\infty} M \left(\frac{n\pi a}{l}\right)^{2\alpha} \left(\frac{n\pi}{l}\right)^{\beta} e^{-\frac{n^2\pi^2 a^2}{l^2}t_0}$$

为优级数, 易知此级数收敛, 故原级数绝对且一致收敛. □

4.4　先验估计

1. 设 $u(x_1, \ldots, x_n)$ 在区域 Ω 上非负, 且满足不等式

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u \geqslant 0,$$

其中 a_{ij}, b_i, c 在 $\overline{\Omega}$ 上具有一阶连续偏导数, 满足 (4.38) 式, 且 $c(x) \leqslant 0$, 证明极值原理 $\max_{\overline{\Omega}} u = \max_{\Gamma} u$ 成立.

Proof. See [**?**, Lemma 2.1 & Theorem 2.3].

Firstly, we suppose that $Lu > 0$ in Ω and u attains its nonnegative maximum at $x_0 \in \Omega$. Then $Du(x_0) = 0$ and $B = D^2u(x_0)$ is seminegative definite. By the ellipticity condition the matrix $A = (a_{ij}(x_0))$ is positive definite. Hence the matrix AB is seminegative definite with a nonpositive trace, that is, $\sum a_{ij}(x_0)D_{ij}u(x_0) \leqslant 0$. This implies $Lu(x_0) \leqslant 0$, which is a contradiction.

Then we consider the general case $Lu \geqslant 0$. For any $\varepsilon > 0$, let $w(x) = u(x) + \varepsilon e^{\lambda x_1}$ with λ to be determined. Then

$$Lw = Lu + \varepsilon e^{\lambda x_1}\big(a_{11}\lambda^2 + b_1\lambda + c\big).$$

Since b_1 and c are bounded and $a_{11}(x) \geqslant \alpha > 0$ for any $x \in \Omega$, by choosing $\lambda > 0$ large enough we get

$$a_{11}(x)\lambda^2 + b_1(x)\lambda + c(x) > 0 \quad \text{for any } x \in \Omega.$$

This implies $Lw > 0$ in Ω. By the first case we know that w attains its nonnegative maximum only at $\partial\Omega$, i.e.,

$$\sup_{\Omega} w \leqslant \sup_{\partial\Omega} w.$$

Thus

$$\sup_{\Omega} u \leqslant \sup_{\Omega} w \leqslant \sup_{\partial\Omega} u + \varepsilon \sup_{x\in\partial\Omega} e^{\alpha x_1}.$$

We finish the proof by letting $\varepsilon \to 0$. □

2. 设 $u \in C^2(\Omega) \cap C(\overline{\Omega})$ 是椭圆型方程 Dirichlet 问题

$$\begin{cases} Lu = \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u = f(x), \\ u|_{\Gamma} = \varphi(x) \end{cases}$$

的解, 系数 a_{ij}, b_i, c 满足第一题中的条件, 则成立最大模估计式 (4.4).

Proof. See [**?**, Proposition 2.15]. We will construct a function w in Ω such that

$$L(w \pm u) = Lw \pm f \leqslant 0 \quad \text{in } \Omega, \tag{4.8}$$

$$w \pm u = w \pm \varphi \geqslant 0 \quad \text{on } \partial\Omega. \tag{4.9}$$

Denote $F = \max_{\Omega}|f|$ and $\Phi = \max_{\partial\Omega}|\varphi|$. To make (**??**) and (**??**) hold, it suffices to require that

$$\begin{cases} Lw \leqslant -F, & \text{in } \Omega, \\ w \geqslant \Phi, & \text{on } \partial\Omega. \end{cases}$$

Suppose the domain Ω lies in the set $\{0 < x_1 < d\}$ for some $d > 0$. Set $w = \Phi + (e^{\lambda d} - e^{\lambda x_1})F$ with $\lambda > 0$ to be determined later. Then we have

$$\begin{aligned} Lw &= -a_{11}\lambda^2 e^{\lambda x_1} F - b_1 \lambda e^{\lambda x_1} F + c\Phi + c(e^{\lambda d} - e^{\lambda x_1})F \\ &\leqslant -(a_{11}\lambda^2 + b_1\lambda) e^{\lambda x_1} F \\ &\leqslant -(\alpha\lambda^2 + b_1\lambda) e^{\lambda x_1} F \leqslant -F \end{aligned}$$

by choosing λ large enough such that $\alpha\lambda^2 + b_1\lambda \geqslant 1$ for any $x \in \Omega$. Hence w satisfies (**??**) and (**??**). By maximum principle obtained in the last exercise, we conclude that $-w \leqslant u \leqslant w$ in Ω. In particular,

$$\sup_{\Omega} |u| \leqslant \Phi + (e^{\lambda d} - 1)F. \qquad \square$$

3. 在 $Q_T = (0, l) \times (0, T)$ 中考察下列初边值问题

$$\begin{aligned} &u_{tt} - a^2 u_{xx} + b(x,t)u_x + b_0(x,t)u_t + c(x,t)u = f(x,t), \qquad (\star) \\ &u|_{x=0} = 0, \qquad (u_x + ku)|_{x=l} = 0, \\ &u|_{t=0} = \varphi(x), \qquad u_t|_{t=0} = \psi(x), \end{aligned}$$

证明其解的唯一性及稳定性.

Proof. We suppose $k \geqslant 0$. Define the energy functional

$$E(t) := \frac{1}{2}\int_0^l (u_t^2 + a^2 u_x^2)\,\mathrm{d}x + \frac{1}{2} k a^2 u^2(l,t).$$

Multiplying both sides of the equation $(\star)$ by u_t and integrating from 0 to l with respect to x, we get

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= -\int_0^l (b u_x u_t + b_0 u_t^2 + c u u_t)\,\mathrm{d}x + \int_0^l f u_t\,\mathrm{d}x \\ &\leqslant C \int_0^l (u_t^2 + u_x^2 + f^2)\,\mathrm{d}x \\ &\leqslant C\left(E(t) + \int_0^l f^2\,\mathrm{d}x\right), \end{aligned}$$

from which it follows that

$$E(t) \leqslant E(0)e^{Ct} + C\int_0^t e^{C(t-s)} \int_0^l f^2\,\mathrm{d}x\,\mathrm{d}s.$$

$\square$

4. 建立下列初边值问题的能量估计式:

$$u_t - \Delta u + \sum_{i=1}^{n} b_i(x,t) u_{x_i} + c(x,t)u = f(x,t),$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = 0, \qquad u|_{t=0} = \varphi(x).$$

Proof. Choose any $T > 0$ and we establish the energy estimates in $[0,T]$. Let $E(t) = \int_\Omega u^2(x,t)\,\mathrm{d}x$. Then

$$\begin{aligned} E'(t) &= 2\int_\Omega u u_t \,\mathrm{d}x \\ &= 2\int_\Omega u\Delta u\,\mathrm{d}x - 2\sum_{i=1}^{n}\int_\Omega b_i u u_{x_i}\,\mathrm{d}x - 2\int_\Omega cu^2\,\mathrm{d}x + 2\int_\Omega uf\,\mathrm{d}x. \end{aligned}$$

By Green's formula we have

$$\int_\Omega u\Delta u\,\mathrm{d}x = -\int_\Omega |\nabla u|^2\,\mathrm{d}x.$$

Let M be an upper bound of $|b(x,t)|$ and $|c(x,t)|$ in $\overline{\Omega}\times[0,T]$. Then we have

$$\begin{aligned} E'(t) &\leqslant -2\int_\Omega |\nabla u|^2\,\mathrm{d}x + 2M\sum_{i=1}^{n}\int_\Omega |uu_{x_i}|\,\mathrm{d}x + 2M\int_\Omega u^2\,\mathrm{d}x + 2\int_\Omega |uf|\,\mathrm{d}x \\ &\leqslant -2\int_\Omega |\nabla u|^2\,\mathrm{d}x + M\left(\epsilon\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{n}{\epsilon}\int_\Omega u^2\,\mathrm{d}x\right) \\ &\qquad + (2M+1)\int_\Omega u^2\,\mathrm{d}x + \int_\Omega f^2\,\mathrm{d}x. \end{aligned}$$

Take $\epsilon = \frac{2}{M}$ and let $C = nM^2/2 + 2M + 1$ to have

$$E'(t) \leqslant CE(t) + \int_\Omega f^2\,\mathrm{d}x.$$

By Gronwall's inequality, it follows that

$$E(t) \leqslant e^{Ct}E(0) + \int_0^t e^{C(t-s)}\int_\Omega f^2\,\mathrm{d}x\,\mathrm{d}s$$

for all $0 \leqslant t \leqslant T$. □

5. 考察初边值问题

$$
\begin{aligned}
&\Delta u + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u = f, \\
&\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = 0.
\end{aligned}
$$

试证当 $c(x)$ 充分负时, 其解在能量模意义下的稳定性.

Proof. The proof is actually the same as that of Theorem 4.6 in the textbook. $\square$

Chapter 5

一阶偏微分方程组

5.1 引言

1. 把波动方程

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

带初始条件

$$\begin{cases} u|_{t=0} = \varphi(x,y,z), \\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x,y,z) \end{cases}$$

的柯西问题化为一个一阶方程组的柯西问题, 并证明其解的等价性.

Proof. 令 $p = \frac{\partial u}{\partial t}$, $q_1 = \frac{\partial u}{\partial x}$, $q_2 = \frac{\partial u}{\partial y}$, $q_3 = \frac{\partial u}{\partial z}$, 则

$$\begin{cases} \frac{\partial p}{\partial t} = a^2\left(\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z}\right) & (1) \\ \frac{\partial p}{\partial x} = \frac{\partial q_1}{\partial t}, \frac{\partial p}{\partial y} = \frac{\partial q_2}{\partial t}, \frac{\partial p}{\partial z} = \frac{\partial q_3}{\partial t} & (2) \\ t = 0 : p = \psi, q_1 = \frac{\partial \varphi}{\partial x}, q_2 = \frac{\partial \varphi}{\partial y}, q_3 = \frac{\partial \varphi}{\partial z} & (3) \end{cases}$$

原方程的解显然满足新方程, 而如果新方程的解为 (p, q_1, q_2, q_3), 则

$$u(x,y,z,t) = \varphi(x,y,z) + \int_{(0,0,0,0)}^{(x,y,x,t)} p\,\mathrm{d}t + q_1\,\mathrm{d}x + q_2\,\mathrm{d}y + q_3\,\mathrm{d}z$$

是原方程的解, 其中条件 (2) 确保了积分与路径无关, 故积分定义是合理的. □

2. 把方程

$$u_{tt} = u_x^2 + u_y^2$$

带初始条件

$$\begin{cases} u|_{t=0} = 0, \\ u_t|_{t=0} = e^x \sin y \end{cases}$$

的柯西问题化为一个一阶偏微分方程组的柯西问题.

Solution. Let $p = u_t$, then

$$\begin{cases} p = u_t, \\ p_t = u_x^2 + u_y^2. \end{cases}$$

And the initial value condition is

$$\begin{cases} u|_{t=0} = 0, \\ p|_{t=0} = e^x \sin y. \end{cases}$$ □

3. 证明柯瓦列夫斯卡娅型方程 (1.9) 满足初始条件

$$t = 0 : u = \varphi_0(x), \ldots, \frac{\partial^{m-1} u}{\partial t^{m-1}} = \varphi_{m-1}(x) \tag{$\star$}$$

的柯西问题可以化为一阶方程组的柯西问题, 并证明其解的等价性.

Proof. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ be the general multi-index and let

$$\begin{aligned} \alpha^0 &:= (\alpha_0 + 1, \alpha_1, \ldots, \alpha_n), \\ \alpha^i &:= (\alpha_0, \alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n). \end{aligned}$$

Decompose α as follows:

$$\alpha = \beta + \gamma,$$

where $\beta = (\alpha_0, 0, \ldots, 0)$ and $\gamma = (0, \alpha_1, \ldots, \alpha_n)$. We denote the special multi-index $\alpha^* = (m-1, 0, \ldots, 0)$ and introduce new functions by $u_\alpha = D^\alpha u$. Then the Kowalevskaya type equation can be transformed into the following first-order system

$$\begin{cases} \frac{\partial u_\alpha}{\partial t} = u_{\alpha^0}, \quad |\alpha| \leqslant m-1, \alpha_0 \leqslant m-2, \\ \frac{\partial u_\alpha}{\partial x_i} = u_{\alpha^i}, \quad |\alpha| \leqslant m-1, \\ \frac{\partial u_{\alpha^*}}{\partial t} = F(t, x, u, u_\alpha, |\alpha| \leqslant m, \alpha_0 \leqslant m-1), \end{cases} \tag{5.1}$$

together with the following initial value conditions

$$\begin{cases} u|_{t=0} = \varphi_0(x), \\ u_\alpha|_{t=0} = D^\gamma \varphi_{\alpha_0}(x), \quad 1 \leqslant |\alpha| \leqslant m, \alpha_0 \leqslant m-1. \end{cases} \tag{5.2}$$

Now we prove that equivalence of solutions. First of all, if u is a solution to (1.9) in the textbook with initial value condition ($\star$), it is straightforward to verify that (u_α) is the solution to (**??**) and (**??**).

Conversely, let (u_α) be the solution to (**??**) and (**??**). Then for all $|\alpha| \leqslant m-1$ with $\alpha_0 \leqslant m-2$ we have

$$\begin{cases} \frac{\partial u_\alpha}{\partial t} = u_{\alpha^0}, \\ \frac{\partial u_{\alpha^0}}{\partial x_i} = \frac{\partial u_{\alpha^i}}{\partial t}. \end{cases} \tag{5.3}$$

It follows that

$$\frac{\partial}{\partial t}\left(u_{\alpha^i} - \frac{\partial u_\alpha}{\partial x_i}\right) = 0. \tag{5.4}$$

So $u_{\alpha^i} - \frac{\partial u_\alpha}{\partial x_i}$ is independent of t. From (**??**) we know that it is equal to zero at $t=0$, thus

$$u_{\alpha^i} \equiv \frac{\partial u_\alpha}{\partial x_i} \quad \text{for all } t \geqslant 0. \tag{5.5}$$

Plugging (**??**) into (**??**) we find that u is a solution to (1.9) in the textbook with the given initial value condition. □

5.2 两个自变量的一阶线性偏微分方程组的特征理论

1. 求一阶方程
(1) $\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + b(x,t)u + c(x,t) = 0$,
(2) $\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + b(x,t,u) = 0$
的特征线和解沿特征线应成立的关系式.

Solution. (1) 特征线满足的方程为 $\frac{\mathrm{d}x}{\mathrm{d}t} = a(x,t)$, 解得特征线为 $x = \int_0^t a(x,\tau)\,\mathrm{d}\tau$. 在特征线上, $u(x,t) = u\left(\int_0^t a(x,\tau)\,\mathrm{d}\tau, t\right)$, 故

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}a(x,t) + \frac{\partial u}{\partial t}.$$

故 u 在特征线上满足关系式

$$\frac{\mathrm{d}u}{\mathrm{d}t} + b(x,t)u + c(x,t) = 0.$$

(2) 同理 $\frac{\mathrm{d}u}{\mathrm{d}t} + b(x,t,u) = 0$. □

2. 求下列一阶方程带初始条件 $u|_{t=0} = \varphi(x)$ 的柯西问题的解:
(1) $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$;
(2) $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = u$.

Solution. (1) 特征线为 $x=t+C$, 在特征线上, $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=0$, 故 u 在特征线上为常数. 故

$$u(x_0,t_0)=u(x_0-t_0,0)=\varphi(x_0-t_0).$$

因此 $u(x,t)=\varphi(x-t)$.

(2) 特征线为 $x=t+C$, 在特征线上, $\frac{\mathrm{d}u}{\mathrm{d}t}=u$, 故 $u=C\mathrm{e}^t$, 令 $t=0$, 得 $u|_{t=0}=C$. 故

$$u(x_0,t_0)=u(x_0-t_0,0)\mathrm{e}^{t_0}=\varphi(x_0-t_0)\mathrm{e}^{t_0}.$$

因此

$$u(x,t)=\varphi(x-t)\mathrm{e}^t.$$

□

3. 判断方程组

$$\begin{aligned}\frac{\partial u_1}{\partial t}&=a(x,t)\frac{\partial u_1}{\partial x}-b(x,t)\frac{\partial u_2}{\partial x}+f_1,\\ \frac{\partial u_2}{\partial t}&=b(x,t)\frac{\partial u_1}{\partial x}+a(x,t)\frac{\partial u_2}{\partial x}+f_2\end{aligned}$$

属于何种类型.

Solution.

$$A=\begin{pmatrix}a&-b\\b&a\end{pmatrix}\Rightarrow(\lambda-a)^2+b^2=0.$$

故当 $b=0$ 时为双曲型, 当 $b\neq0$ 时为椭圆型. □

4. 将下列各方程组化为对角型方程组:

(1) $\begin{cases}\frac{\partial u}{\partial t}+(1+\sin x)\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial x}+x=0,\\ \frac{\partial v}{\partial t}+u=0;\end{cases}$

(2) $\begin{cases}\frac{\partial u}{\partial t}=x\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x},\\ \frac{\partial v}{\partial t}=a^2\frac{\partial u}{\partial x}+x\frac{\partial v}{\partial x}\quad(a>0);\end{cases}$

(3) $\begin{cases}\frac{\partial u_1}{\partial t}+6\frac{\partial u_1}{\partial x}+5\frac{\partial u_2}{\partial x}=0,\\ \frac{\partial u_2}{\partial t}+5\frac{\partial u_1}{\partial x}+6\frac{\partial u_2}{\partial x}=2u_1,\\ 3\frac{\partial u_3}{\partial t}+6\frac{\partial u_3}{\partial x}-3\frac{\partial u_1}{\partial x}=2u_2+3u_3-3u_1.\end{cases}$

Solution. (1)

$$A=\begin{pmatrix}1+\sin x&2\\0&0\end{pmatrix}\Rightarrow\lambda=0\text{ 或 }\lambda=1+\sin x.$$

相应的特征向量为 $(-2,1+\sin x)^T,(c,0)^T$, 作变换

$$\begin{cases}u=-2v_1+cv_2\\v=(1+\sin x)v_1\end{cases}$$

则得

$$\begin{cases}(1+\sin x)\frac{\partial v_1}{\partial t}-2v_1+cv_2=0,\\ c(1+\sin x)\frac{\partial v_2}{\partial t}+c(1+\sin x)^2\frac{\partial v_2}{\partial x}+x(1+\sin x)-4v_1+2cv_2=0.\end{cases}$$

(2)

$$A=\begin{pmatrix}x & 1\\ a^2 & x\end{pmatrix}\Rightarrow(\lambda-x)^2-a^2=0\Rightarrow\lambda=x\pm a$$

对应特征向量为 $(-1,a)^T$, $(1,a)^T$, 作变换

$$\begin{cases}u=-v_1+v_2\\ v=av_1+av_2\end{cases}$$

则得

$$\begin{cases}\frac{\partial v_1}{\partial t}=(x-a)\frac{\partial v_1}{\partial x}\\ \frac{\partial v_2}{\partial t}=(x+a)\frac{\partial v_2}{\partial x}\end{cases}$$

(3)

$$A=\begin{pmatrix}6 & 5 & 0\\ 5 & 6 & 0\\ -1 & 0 & 2\end{pmatrix}\Rightarrow\begin{vmatrix}\lambda-6 & -5 & 0\\ -5 & \lambda-6 & 0\\ 1 & 0 & \lambda-2\end{vmatrix}=0.$$

解得 $(\lambda-2)(\lambda^2-12\lambda+11)=0\Rightarrow\lambda=1,2,11$, 对应特征向量为 $(1,-1,1)^T$, $(0,0,1)^T$, $(9,9,-1)^T$, 作变换

$$\begin{cases}u_1=v_1+9v_3\\ u_2=-v_1+9v_3\\ u_3=v_1+v_2-v_3\end{cases}$$

$$R=\begin{pmatrix}1 & 0 & 9\\ -1 & 0 & 9\\ 1 & 1 & -1\end{pmatrix}\Rightarrow R^{-1}=\begin{pmatrix}1/2 & -1/2 & 0\\ -4/9 & 5/9 & 1\\ 1/18 & 1/18 & 0\end{pmatrix}$$

又因为

$$B=\begin{pmatrix}0 & 0 & 0\\ -2 & 0 & 0\\ 1 & -2/3 & -1\end{pmatrix},C=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

所以

$$R^{-1}BR=\begin{pmatrix}1 & 0 & 9\\ -4/9 & -1 & -6\\ -1/9 & 0 & -1\end{pmatrix},R^{-1}C=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

即得对角型方程组

$$\begin{cases} \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x} + v_1 + 9v_3 = 0 \\ \frac{\partial v_2}{\partial t} + 2\frac{\partial v_2}{\partial x} - \frac{4}{9}v_1 - v_2 - 6v_3 = 0 \\ \frac{\partial v_3}{\partial t} + 11\frac{\partial v_3}{\partial x} - \frac{1}{9}v_1 - v_3 = 0 \end{cases}$$

□

5. 证明: 经过未知函数的任何实系数的可逆线性变换, 方程组 (2.1) 在每一点的特征线方向 (或特征曲线) 保持不变, 因此也不会改变方程组 (2.1) 所属的类型.

Proof. 原方程为

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + BU + C = 0.$$

未知函数作可逆线性变换 $U = RV$ 后, 有

$$R\frac{\partial V}{\partial t} + AR\frac{\partial V}{\partial x} + \left(\frac{\partial R}{\partial t} + A\frac{\partial R}{\partial x} + BR\right)V + C = 0.$$

两端左乘 R^{-1} 得

$$\frac{\partial V}{\partial t} + A'\frac{\partial V}{\partial x} + R^{-1}\left(\frac{\partial R}{\partial t} + A\frac{\partial R}{\partial x} + BR\right)V + R^{-1}C = 0,$$

其中 $A' = R^{-1}AR$. 由于

$$\det(A' - \lambda I) = \det(R^{-1}AR - \lambda I) = \det(A - \lambda I),$$

故方程的根保持不变, 特征方向不变, 特征线也不变. □

6. 证明: 方程组 (2.1) 在每一点的特征线方向 (或特征曲线) 经过自变量的任何可逆变换后就变成变换后方程组在对应点的特征线方向 (或特征曲线), 即特征线方向 (或特征曲线) 对可逆坐标变换具有不变性.

Proof. Suppose the original characteristic curve satisfies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda(x,t),$$

where $\lambda(x,t)$ is the solution to

$$\det(A - \lambda I) = 0. \tag{5.6}$$

Now make a change of variables

$$y = \xi(x,t), \qquad s = \eta(x,t).$$

Then for the new variables (y,s), the characteristic curve satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}s}=\frac{\xi_x\,\mathrm{d}x+\xi_t\,\mathrm{d}t}{\eta_x\,\mathrm{d}x+\eta_t\,\mathrm{d}t}=\frac{\xi_x\lambda+\xi_t}{\eta_x\lambda+\eta_t}. \tag{5.7}$$

Under the transform of variables, the original system

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}+BU+C=0$$

can be turned into

$$\left(A\frac{\partial\eta}{\partial x}+\frac{\partial\eta}{\partial t}I\right)\frac{\partial U}{\partial s}+\left(A\frac{\partial\xi}{\partial x}+\frac{\partial\xi}{\partial t}I\right)\frac{\partial U}{\partial y}+BU+C=0.$$

Suppose its characteristic curve is

$$\frac{\mathrm{d}y}{\mathrm{d}s}=\tilde{\lambda}.$$

Then $\tilde{\lambda}$ must satisfy

$$\det\left(A\frac{\partial\xi}{\partial x}+\frac{\partial\xi}{\partial t}I-\left(A\frac{\partial\eta}{\partial x}+\frac{\partial\eta}{\partial t}I\right)\tilde{\lambda}\right)=0,$$

i.e.,

$$\det\left(A-\frac{\tilde{\lambda}\eta_t-\xi_t}{\xi_x-\eta_x\tilde{\lambda}}I\right)=0. \tag{5.8}$$

By (**??**) and (**??**) we have that

$$\frac{\tilde{\lambda}\eta_t-\xi_t}{\xi_x-\eta_x\tilde{\lambda}}=\lambda,$$

from which we get

$$\tilde{\lambda}=\frac{\xi_x\lambda+\xi_t}{\eta_x\lambda+\eta_t}. \tag{5.9}$$

Combining with (**??**), the proof is finished. □

5.3 两个自变量的线性双曲型方程组的柯西问题

1. 用逐次逼近法求方程组

$$\begin{cases}u_t+u_x=v,\\ v_t-v_x=u\end{cases}$$

带下列初始条件的柯西问题的解:

(1) $u|_{t=0}=1$, $v|_{t=0}=0$;

(2) $u|_{t=0} = \sin x$, $v|_{t=0} = \cos x$.

Solution. $\lambda_1(x,t) = 1$ and $\lambda_2(x,t) = -1$, so the first characteristic curve (straight line in fact) l_1 passes through (x,t) and $(x-t,0)$ while the second l_2 passes through (x,t) and $(x+t,0)$.

(1) The equations are equivalent to

$$u(x,t) = 1 + \int_{l_1} v\,\mathrm{d}\tau,$$
$$v(x,t) = \int_{l_2} u\,\mathrm{d}\tau.$$

Let $u^{(0)} \equiv 1$ and $v^{(0)} \equiv 0$, then

$$\begin{aligned}
u^{(1)} &= 1, & v^{(1)} &= t,\\
u^{(2)} &= 1 + \frac{t^2}{2}, & v^{(2)} &= t,\\
u^{(3)} &= 1 + \frac{t^2}{2}, & v^{(3)} &= t + \frac{t^3}{6},\\
u^{(4)} &= 1 + \frac{t^2}{2} + \frac{t^4}{24}, & v^{(4)} &= t + \frac{t^3}{6}.
\end{aligned}$$

By induction we have that

$$u(x,t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{2k!} = \cosh t, \qquad v(x,t) = \sum_{k=1}^{\infty} \frac{t^{2k-1}}{(2k-1)!} = \sinh t.$$

(2) The equations are equivalent to

$$u(x,t) = \sin(x-t) + \int_{l_1} v(x_1(\tau;x,t),\tau)\,\mathrm{d}\tau,$$
$$v(x,t) = \cos(x+t) + \int_{l_2} u(x_2(\tau;x,t),\tau)\,\mathrm{d}\tau,$$

where $x_1(\tau;x,t) = x - t + \tau$ and $x_2(\tau;x,t) = x + t - \tau$. Let $u^{(0)} = \sin(x-t)$ and $v^{(0)} = \cos(x+t)$, then

$$\begin{aligned}
u^{(1)}(x,t) &= \sin(x-t) + \int_0^t v^{(0)}(x-t+\tau,\tau)\,\mathrm{d}\tau\\
&= \frac{1}{2}[\sin(x+t) - \sin(x-t)],\\
v^{(1)}(x,t) &= \cos(x+t) + \int_0^t u^{(0)}(x+t-\tau,\tau)\,\mathrm{d}\tau\\
&= \frac{1}{2}[\cos(x+t) + \cos(x-t)],
\end{aligned}$$

Similarly, we calculate that

$$\begin{aligned}
u^{(2)}(x,t) &= \frac{3}{4}\sin(x-t) + \frac{1}{4}\sin(x+t) + \frac{1}{2}t\cos(x-t),\\
v^{(2)}(x,t) &= \frac{3}{4}\cos(x+t) + \frac{1}{4}\cos(x-t) + \frac{1}{2}t\sin(x+t),\\
u^{(3)}(x,t) &= \frac{1}{2}[\sin(x+t) + \sin(x-t)] + \frac{1}{4}t[\cos(x-t) - \cos(x+t)],\\
v^{(3)}(x,t) &= \frac{1}{2}[\cos(x+t) + \cos(x-t)] + \frac{1}{4}t[\sin(x+t) - \sin(x-t)],\\
u^{(4)}(x,t) &= \frac{5}{16}\sin(x+t) + \frac{11}{16}\sin(x-t) + \frac{t}{2}\cos(x-t)\\
&\quad - \frac{t}{8}\cos(x+t) + \frac{t^2}{8}\sin(x-t),\\
v^{(4)}(x,t) &= \frac{11}{16}\cos(x+t) + \frac{5}{16}\cos(x-t) + \frac{t}{2}\sin(x+t)\\
&\quad - \frac{t}{8}\sin(x-t) - \frac{t^2}{8}\cos(x+t).
\end{aligned}$$

So $u = \sin x$ and $v = \cos x$.

□

2. 求解柯西问题:

$$\begin{cases} u_t - u_x = (x+t)v,\\ v_t + v_x + (x+t)u = 0,\\ t = 0 : u = 0, v = 1. \end{cases}$$

3. 证明用 (3.9) 式表示的函数序列 $\left\{\frac{\partial V_i^{(n)}}{\partial x}\right\}$ $(i = 1, \ldots, N,\ n = 0, 1, 2, \ldots)$ 在区域 $\overline{G}$ 上的一致收敛性.

Proof. Let

$$\begin{aligned}
A &:= \max\left\{|\alpha_{ij}|, \left|\frac{\partial \alpha_{ij}}{\partial x}\right|\right\},\\
M &:= \max\left\{|V_i^{(0)}|, |V_i^{(1)}|, \left|\frac{\partial V_i^{(0)}}{\partial x}\right|, \left|\frac{\partial V_i^{(1)}}{\partial x}\right|\right\}.
\end{aligned}$$

Then

$$\left|\frac{\partial V_i^{(1)}}{\partial x} - \frac{\partial V_i^{(0)}}{\partial x}\right| \leqslant 2M.$$

Suppose that

$$\left|\frac{\partial V_i^{(n+1)}}{\partial x} - \frac{\partial V_i^{(n)}}{\partial x}\right| \leqslant 2(n+1)M\frac{(ANt)^n}{n!}, \tag{5.10}$$

then

$$\left|\frac{\partial V_j^{(n+2)}}{\partial x}-\frac{\partial V_j^{(n+1)}}{\partial x}\right|\leqslant\int_0^t\sum_{j=1}^N\left|\frac{\partial\alpha_{ij}}{\partial x}\right|\cdot|V_j^{(n+1)}-V_j^{(n)}|$$

$$+\sum_{j=1}^N|\alpha_{ij}|\cdot\left|\frac{\partial V_j^{(n+1)}}{\partial x}-\frac{\partial V_j^{(n)}}{\partial x}\right|$$

$$\leqslant 2(n+2)M\frac{(ANt)^{n+1}}{(n+1)!}.$$

By the principle of induction, we obtain that (**??**) holds for all $n\geqslant 0$. Hence the proof is finished by the convergence of the sequence $2(n+1)M(ANt)^n/n!$. □

4. 设 $V_i(x,t)$ $(i=1,\ldots,N)$ 是方程组 (3.1) 带初始条件 (3.2) 的柯西问题在区域 $\overline{G}$ 上的解, 且设

$$\varphi_0=\max_{\substack{(x,t)\in\overline{G}\\1\leqslant i\leqslant N}}|\varphi_i(x)|,\qquad\alpha=\max_{\substack{(x,t)\in\overline{G}\\1\leqslant i\leqslant N}}\sum_{j=1}^N|\alpha_{ij}(x,t)|,$$

$$\beta(\tau)=\max_{\substack{t=\tau,(x,t)\in\overline{G}\\1\leqslant i\leqslant N}}|\beta_i(x,t)|,$$

则在区域 $\overline{G}$ 上成立着下面的 Haar 估计式

$$|V_i(x,t)|\leqslant\varphi_0e^{\alpha t}+\int_0^t\beta(\tau)e^{\alpha(t-\tau)}\,\mathrm{d}\tau\quad(i=1,\ldots,N),$$

并利用此估计式证明柯西问题 (3.1)–(3.2) 解的唯一性和对初始条件的连续依赖性.

Proof. Use (3.6) and Gronwall's inequality. □