4 Calculus

4.1 Differentiation

- If the limit of the denominator of a rational function is zero, you cannot substitute to find the limit: either 'juggle' or use l'Hopital's rule, e.g:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{1}\right) = 1$$

- The definition of the derivative:

$$f'(x) = \lim_{\delta x\to 0}\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right)$$

- Special derivatives:

  $\frac{d}{dx}(\sec x) = \sec x \tan x$

  $\frac{d}{dx}(\csc x) = -\csc x \cot x$

  $\frac{d}{dx}(\cot x) = -\csc^2 x$

  $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

  $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

  $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

  $\frac{d}{dx}(a^x) = a^x \ln a$

  $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

- $f(x)$ is an **increasing function** on (a, b) if $\frac{dy}{dx} \geq 0$ on that interval, or a **strictly increasing function** if $\frac{dy}{dx} > 0$.
- $f(x)$ is **concave upwards** on (a, b) if $\frac{d^2y}{dx^2} > 0$.
- If the derivative at a point is zero, the function is stationary.
- If the derivative at a point is ∞, there is a vertical line.
- For a point of inflexion, $\frac{d^2y}{dx^2} = 0$ AND the sign of $\frac{d^2y}{dx^2}$ changes, i.e concavity changes.

- Sketching the graph of $f'(x)$ given $f(x)$:
  - Stationary point $\to$ x-intercept.
  - $f(x)$ increasing $\to$ $f'(x)$ above x-axis.
  - Point of inflexion $\to$ turning point.
- The gradient at any point on the curve: $m = \frac{dy}{dx}|_{x=x_0}$.
- The equation of a tangent to the curve at (x_0, y_0): $y - y_0 = m(x - x_0)$.
- If two variabels are related, their rates of change are also related:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

- In kinematics especially:

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$$

4.2 Integration

$$\int (px+q)^n dx = \frac{(px+q)^{n+1}}{p(n+1)} + C$$

$$\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \frac{1}{x \ln x} dx = \int \frac{x^{-1}}{\ln x} + C = \ln|\ln|x|| + C$$

$$\int \tan x \; dx = \ln|\sec x| + C$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \; dx = -\ln|\csc x + \cot x| + C$$

$$\int \frac{1}{(x+k)^2 + a^2} \; dx = \frac{1}{a} \arctan(\frac{x+k}{a}) + C$$

$$\int \frac{1}{\sqrt{a^2 - (x+k)^2}} \; dx = \arcsin(\frac{x+k}{a}) + C$$

- To integrate $\sin^2 x$ or $\cos^2 x$, we expand $\cos(2x)$ and rearrange.
- To integrate $\sin^3 x$, split into $\int \sin x(\sin^2 x)dx$, then use $\sin^2 x + \cos^2 x = 1$.
- If the integral is of the form:

$$\int \frac{px+q}{\sqrt{Ax^2+Bx+C}} dx \quad \text{or} \quad \int \frac{px+q}{Ax^2+Bx+C} dx$$

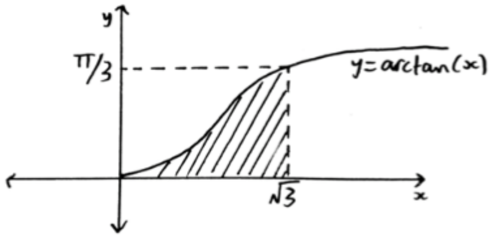
  use sorcery to change it into $\int \frac{f'(x)}{f(x)} dx$ or $\int f'(x)(f(x))^n dx$.

- Integration by substitution:
  1. Replace dx by $\frac{dx}{dt} \cdot dt$.
  2. Substitute by replacing all x with $g(t)$.
     Then: $\int f(x)\ dx = \int f(g(t)) \frac{dx}{dt} \cdot dt$
- Integration by parts:

  $\int u dv = uv - \int v du$
- To choose which one to differentiate, use LIATE: **L**ogs, **I**nverse trig, **A**lgebraic, **T**rig, **E**xponentials.

4.3 Definite integrals

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- The definite integral $\int_a^b f(x)dx$ can only be found if $f(x)$ is defined for all $x \in (a, b)$.
- The area between a curve and the y-axis: $\int_a^b f(y)dy$
- If a function is difficult to integrate, try integrating its inverse w.r.t y then subtract from a rectangle. e.g:



$$\int_0^{\sqrt{3}} \arctan x\ dx = \frac{\pi\sqrt{3}}{3} - \int_0^{\frac{\pi}{3}} \tan y\ dy$$

- The area between the curve and the axis is always $\int_a^b |f(x)|dx$.
- The area between two curves is always $\int_a^b y_1 - y_2 dx$.
- The volume of revolution:

$$V = \pi \int_a^b y^2 dx$$

- The volume of revolution of the area enclosed by two curves:

$$V = \pi \int_a^b (y_1)^2 dx - \pi \int_a^b (y_2)^2 dx$$