6 Complex numbers

6.1 Forms of complex numbers

- The **Cartesian form** of a complex number: $z = x + iy$. This relates a complex number to its real and imaginary parts. $x = \text{Re}(z)$, $y = \text{Im}(z)$.
- The **Polar form**, a.k.a the **trigonometric form** or **modulus-argument form**:

$$z = r(\cos\theta + i\sin\theta) = r\text{cis}(\theta)$$

- r is the **modulus** of z: $r = |z| = \sqrt{x^2 + y^2}$.
- The **argument** of z (θ or $\arg z$) is the angle from the positive real axis to the line $\overrightarrow{OZ}$. The **principal value** of $\arg z$ is the angle in the interval $(-\pi, \pi]$.
  - The argument can be found using $\arctan(y/x)$, but you must consider the quadrant.
  - $\arg 2 = 0 \qquad \arg(-3) = \pi$
  - $\arg(3i) = \pi/2 \qquad \arg(-4i) = -\pi/2$
  - $\arg 0$ is undefined.
- Using the Maclaurin expansions of e^x, $\cos x$ and $\sin x$, we can derive Euler's beautiful formula:

$$e^{ix} = \cos x + i\sin x$$

- We can then write complex numbers in the **exponential** or **Euler** form: $z = re^{i\theta}$, for θ in radians.

Complex conjugates

- The **conjugate** of z is given by $z^* = x - iy$.
- It is interpreted on an Argand diagram as a reflection in the real axis.
- Because of this, $\arg z = -\arg z^*$ so $z^* = r\text{cis}(-\theta) = re^{-i\theta}$.
- Properties of conjugates
  - $(z^*)^* = z$
  - $(z + w)^* = z^* + w^*$
  - $(zw)^* = z^*w^* \implies (z^n)^* = (z^*)^n$
  - $z + z^* = 2\text{Re}(z)$
  - $z - z^* = 2i\text{Im}(z)$
  - $zz^* = x^2 + y^2 = |z|^2$
  - $z^* = r^2/z$

6.2 Operations on complex numbers

- When adding and subtracting complex numbers, we group real and imaginary parts.
- To multiply complex numbers in Cartesian form, we can expand the brackets.
- To multiply complex numbers in the Euler form, multiply moduli and add arguments:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

- To divide complex numbers, we subtract their arguments.
- **De Moivre's Theorem** states that, if $z = r(\cos\theta + i\sin\theta)$,

$$z^n = r^n(\cos n\theta + i\sin n\theta)\ ,\ \text{for all } n \in \mathbb{R}$$

- It follows that $|z^n| = |z|^n$.

6.3 Relation to trigonometry

$$\begin{aligned} z + z^* &= e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta \\ z - z^* &= e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta \\ \implies \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{aligned}$$

When simplifying expressions involving $e^{i\theta} \pm 1$, we can use this trick:

$$e^{i\theta} + 1 = e^{i\frac{\theta}{2}}(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}) = 2e^{i\frac{\theta}{2}}\cos\frac{\theta}{2}$$

$$e^{i\theta} - 1 = e^{i\frac{\theta}{2}}(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}) = 2ie^{i\frac{\theta}{2}}\sin\frac{\theta}{2}$$

<u>Trigonometric identities</u>

- Write $\cos 3\theta$ in terms of $\cos\theta$.

  $\cos 3\theta = \text{Re}(\cos 3\theta + i\sin 3\theta) = \text{Re}((\cos\theta + i\sin\theta)^3)$ (by De Moivre's Theorem).

  But using a binomial expansion, $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

  $\cos 3\theta = \text{Re}(\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3)$
  $\implies \cos 3\theta = \cos^3\theta + 3\cos\theta(i\sin\theta)^2 = \cos^3\theta - \cos\theta(1-\cos^2\theta)$

  $\therefore\ \cos 3\theta = 4\cos^3\theta - \cos\theta.\ \ QED.$

- Express $\sin^3\theta$ in terms of sines of multiples of θ. To begin, let $z = \text{cis}(\theta)$.

  $$\left(z - \frac{1}{z}\right)^3 = z^3 - \frac{3z^2}{z} + \frac{3z}{z^2} - \frac{1}{z^3} = \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$$

  For a complex number of unit modulus, $\left(z^n - \dfrac{1}{z^n}\right) = (z^n - (z^n)^*) = 2i\sin n\theta$

  $\implies (2i\sin\theta)^3 = 2i\sin 3\theta - 3(2i\sin\theta)$
  $\implies -8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

  $\therefore\ \sin^3\theta = \dfrac{3}{4}\sin\theta - \dfrac{1}{4}\sin 3\theta.\ \ QED.$

- For cosines, we instead use $z + \frac{1}{z}$.

6.4 Polynomials

- A quadratic will have complex roots if the discriminant $b^2 - 4ac < 0$.
- In general, the complex roots of a quadratic **with real coefficients** will always be a conjugate pair.
- A cubic will either have 3 real roots or 1 real root and 2 conjugate complex roots. If we know one of the complex roots, we know its conjugate and can multiply out. Long division will help us find the real root.

$$(x - (a + bi))(x - (a - bi)) = x^2 - 2ax + (a^2 + b^2)$$

$$(x - z)(x - z^*) = x^2 - 2Re(z) + |z|^2$$

6.5 Roots of complex numbers

- There are n values of z that solve $z^n = 1$ (because of the Fundamental Theorem of Algebra); these are known as the nth roots of unity.
- To find these, we rewrite the RHS: $1 = e^{i(0+2k\pi)}$. As a result,

$$z = e^{i\frac{2k\pi}{n}} \text{ , for } k = 1, 2, 3, ..., n.$$

- Alternatively, use $k = 0, \pm 1, \pm 2, ...$ in order to make sure that arguments will be within the principal range.
- Note that each of the roots will form on a unit circle.
- More generally, for the nth roots of a complex number c,

$$z = r^{1/n} e^{i\frac{\theta+2k\pi}{n}} \text{ , for } k = 1, 2, 3, ..., n.$$