hw-I3-numerical optiamization of robot

Task 1

homework 1

Example

only equality constraints:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x$$
 2. $A \times = b$ 3. t. $A x = b$ 5. t. $A = b$ 5. The state of the second of

Q 矩阵严格正定情况,等式约束 KKT 条件推导 首先获得 KKT 相关条件,由于仅仅存在线性等式约束 故存在 • 原始可行性条件

$$Ax^* = b$$

• 稳定性条件

设拉格朗日乘子为v目标函数为

$$rac{1}{2}x^TQx+c^Tx+v(Ax-b)$$

对 x 求偏导, 设 KKT 点为 x^* , 那么满足 kkt 条件为

$$\left. rac{\partial}{\partial x} \left(rac{1}{2} x^T Q x + c^T x + v^T (A x - b)
ight)
ight|_{x = x^*} = 0$$

整理得

$$Qx^* + A^Tv^* = -c$$

满足上述条件的线性等式方程为

$$\left[egin{array}{cc} Q & A^T \ A & 0 \end{array}
ight] \left[egin{array}{c} x^* \ v^* \end{array}
ight] = \left[egin{array}{c} -c \ b \end{array}
ight]$$

task2 low-Dimensional QP

task2 low-Dimensional QP

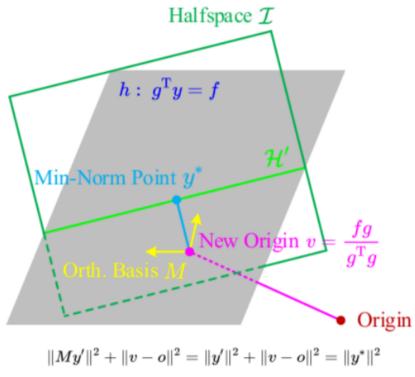
第1列

```
\begin{aligned} & \text{LowDimMinNorm}(\mathcal{H}) \\ & \text{if } \dim(\mathcal{H}) = 1 \\ & y \leftarrow \text{OneDimMinNorm}(\mathcal{H}) \\ & \text{end if} \\ & \mathcal{I} \leftarrow \{\} \\ & \text{for } h \in \mathcal{H} \text{ in a random order} \\ & \text{if } y \not\in h \\ & & \left[ \underbrace{\{M, v, \mathcal{H}'\} \leftarrow \text{HouseholderProj}(\mathcal{I}, h)}_{y' \leftarrow \text{LowDimMinNorm}(\mathcal{H}')} \right] \\ & y \leftarrow \text{My'} + v \\ & \text{end if} \\ & \mathcal{I} \leftarrow \mathcal{I} \cup \{h\} \\ & \text{end for} \\ & \text{return } y \end{aligned}
```

第2列

处理的核心逻辑是通过投影降低维数。

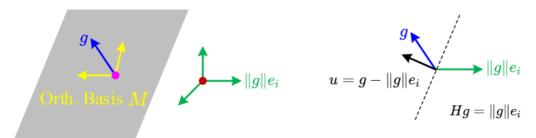
首先判断约束是否为一维,为一维直接去在满足约束下找到即可,否则需要对于违反当前 $x \notin h$ 最优解的 d 维新约束 $g^T y \leq f$,要将之前的约束进行投影,在 h 平面上: $g^T y = f$, 也就是用 h 上 的一组 d-1 维的标准正交基。



How to find the orthonormal basis on $h: g^{T}y = f$?

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All vectors in the orth. basis of h is perpendicular to g.



H is exactly the Householder reflection (orthonormal) matrix H has (d-1) orthonormal row vectors that are perpendicular to g All (d-1) column vectors of M are these (d-1) row vectors of H Choosing $-\operatorname{sgn}(g_i)\|g\|e_i$ and $i=\operatorname{argmax}_k|g_k|$ is numerically stable

$$H = I_d - rac{2uu^{
m T}}{u^{
m T}u}$$

Householder 投影具体操作:

- 首先找到新的原点 v, 投影在 h 平面上
- 寻找一组标准正交基

进行 Household 反射变换

为了数值稳定性, 镜面的法向向量 $u = g + sgn(g_i)||g||e_i$

其中 $i = argmax_k |g_k|$

则根据向量相加几何特性可知

$$g-2g^Trac{u}{\|u\|}rac{n}{\|n\|}=-\operatorname{sgn}\left(g_i
ight)\|g\|e_i$$

其中 n=u^T

$$H = I_d - rac{2uu^T}{u^Tu}$$

H 便是反射矩阵。

降维的约束 H'=Hh

源码:

C++

```
// TODO
              id=argmax(gi)
cpy<d>(plane_i, reflx);
double g_norm = std::sqrt(sqr_norm<d>(plane_i));
reflx[id]+=plane_i[id]<0.0 ? -g_norm:g_norm;//u=g+sigan(g_i)|g|</pre>
double h=-2.0/ sqr_norm<d>(reflx);
for(int j=0; j!=i; j=next[j]){
    const double *halfspace = halves + (d + 1) * j;
   double *proj_halfspace = new_halves + d * j;
   double term=h*dot<d>(halfspace, reflx);
   int temp=0;
   for(int k=0;k<d+1;k++){</pre>
       if(k==id) continue;
       proj_halfspace[temp++]=halfspace[k]+term*reflx[k];
   proj_halfspace[d-1]=dot<d>(new_origin, halfspace)+halfspace[d];
}
```

Result

```
Python optimal sol: 4.11111 9.15556 4.50022 optimal obj: 201.14 cons precision: 8.88178e-16
```

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task3 PHR ALM NMPC

```
egin{array}{ll} \min _{s_1,\ldots,s_N,u_0,\ldots,u_N} & J\left(s_1,\ldots,s_N,u_0,\ldots,u_N
ight) \ & 	ext{s.t.} & F\left(s_k,u_k
ight) = s_{k+1}, orall i \in \{0,\ldots,N\} \ & G\left(s_k,u_k
ight) \leq 0, \quad orall i \in \{0,\ldots,N\} \end{array}
```

其中, $s_k = [x_k, y_k, \phi_k, v_k]^T$ 为状态变量, $u_k = [a_k, \delta_k]^T$ 为输入变量, $F(s_k, u_k) = f(s_k, u_k) + s_k$ 为离散运动学模型:如下所示

RK45(龙格库塔)也可以用

$$\left\{egin{array}{l} \dot{x} = v\cos(\phi) \ \dot{y} = v\sin(\phi) \ \dot{\phi} = v\tan(\delta)/L \ \dot{v} = a \end{array}
ight.$$

 $G(s_k, u_k) \leq 0$ 为状态与输入约束的集成, 表示为:

$$egin{aligned} a_{\min} & \leq a_k \leq a_{\max} \ \delta_{\min} & \leq \delta_k \leq \delta_{\max} \ v_{\min} & \leq v_k \leq v_{\max} \ k \in \{0,1,\ldots,N\} \end{aligned}$$

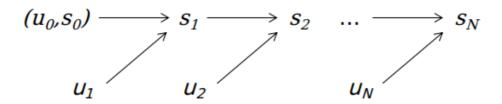
目标函数为

$$J\left(s_{1},\ldots,s_{N},u_{0},\ldots,u_{N}
ight):=\sum_{k=1}^{N}\left[\left(x_{k}-x_{k}^{ ext{ref}}
ight)^{2}+\left(y_{k}-y_{k}^{ ext{ref}}
ight)^{2}+w_{v}\left(a_{k}-a_{k}^{ ext{ref}}
ight)^{2}
ight]$$

优化量分别为横向误差与纵向误差,控制的相邻时刻变化量的,

todo: 横向误差与纵向误差的权重

为了快速求解,通过运动学模型将状态量转化为与输入相关的量,从而消除这个等式约束,如下图所示



这样 u 可以看作唯一的优化变量,

$$u_{0:N} = (u_0, \ldots, u_N)$$
 $F(s_k, u_k) = s_{k+1}, \ orall \ i \in \{0, \ldots, N\}$ $s_k(u_{0:N}), \ orall \ i \in \{1, \ldots, N\}$

于是目标函数转化为

$$egin{aligned} \min_{u_{0:N}} J\left(s_{1}\left(u_{0:N}
ight), \ldots, s_{N}\left(u_{0:N}
ight), u_{0:N}
ight) \ ext{s.t.} \ G\left(s_{k}\left(u_{0:N}
ight), u_{k}
ight) \leq 0, orall i \in \left\{0, \ldots, N
ight\} \end{aligned}$$

在不等式约束引入辅助变量 s ,使其转化为等式约束,如下所示

$$G(s_k(u_{0:N}), u_k) + [s]^2 = 0$$

可由 PHR-ALM 得到相应的增广拉格朗日函数

$$\mathcal{L}_{
ho}(u_{0:N},\mu):=J\left(s_{1}\left(u_{0:N}
ight),\ldots,s_{N}\left(u_{0:N}
ight),u_{0:N}
ight)+rac{
ho}{2}\left\Vert \max\left[G\left(s_{k}\left(u_{0:N}
ight),lpha
ight]
ight]$$

第二项对 u 的导为:

$$rac{d}{du}(rac{
ho}{2}\left\|\max\left[G\left(s_{k}\left(u_{0:N}
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外层采用 lbfgs 算法更新 ALM 的优化变量 $\;u_{0:N} \leftarrow \mathop{argmin}_{u_{0:N}} \mathcal{L}_{
ho}(u_{0:N},\mu)\;\;$ 。

对于内层的乘子更新按以下方式:

$$\left\{ egin{array}{l} \mu \leftarrow \max[\mu +
ho G\left(s_{k}\left(u_{0:N}
ight), u_{k}
ight), 0
ight] \
ho \leftarrow \min[(1 + \gamma)
ho, eta] \end{array}
ight.$$

对于存在不等式约束的 PHR-ALM 问题,需要满足稳定性条件和对偶可行性条件 迭代停止条件为:

$$egin{aligned} \left\| \max \left[G(s_k(u_{0:N}, u_k), -rac{\mu}{
ho}
ight]
ight\|_{\infty} &< \epsilon_{ ext{cons}} \;, \ \left\|
abla_x \mathcal{L}_{
ho}(u_{0:N}, \mu)
ight\|_{\infty} &< \epsilon_{ ext{prec}} \end{aligned}$$

结果

见 mpc1.mp4

