Introduction to Machine Learning

Introduction: Losses & Risk Minimization

HOW TO EVALUATE MODELS

In the training, we want to optimize $\boldsymbol{\theta}$. To score $\boldsymbol{\theta}$, we have to compare the actual output with the predicted output:

Features x	
People in Office (Feature 1) x_1	**Salary (Feature 2) x_2**
4	4300 €
12	2700 €
5	3100 €

Target y
Worked Minutes Week (Target Variable)
2220
1800
1920

$\underbrace{\text{Features } x \quad \text{Target } y}_{\mathcal{D}_{\text{train}}}$

$\overset{?}{\approx}$

Prediction $\hat{y}$
Worked Minutes Week (Target Variable)
2588
1644
1870

MOTIVATION

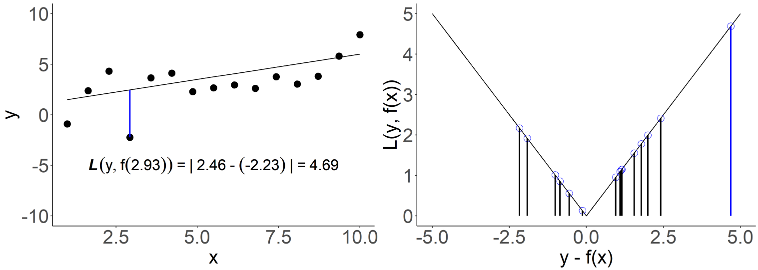
- Assume we trained a model to predict flat rent based on some features (size, location, age, ...).
- The real rent of a flat is EUR 1600, our model predicts EUR 1300.
- How do we measure the performance of our model?
- Need to define a suitable criterion, e.g.:
  - Absolute error $|1600 - 1300| = 300$
  - Squared error: $(1600 - 1300)^2 = 90000$
    (puts more emphasis on predictions that are far off the mark)
- The choice of this metric has a major influence on the final model, because it determines what constitutes a *good* model: it will determine the ranking of the different models $f \in \mathcal{H}$.
- The metric we use is called the **loss function**.

LOSS

The **loss function** $L(y, f(\mathbf{x}))$ quantifies the "quality" of the prediction $f(\mathbf{x})$ of a single observation $\mathbf{x}$:

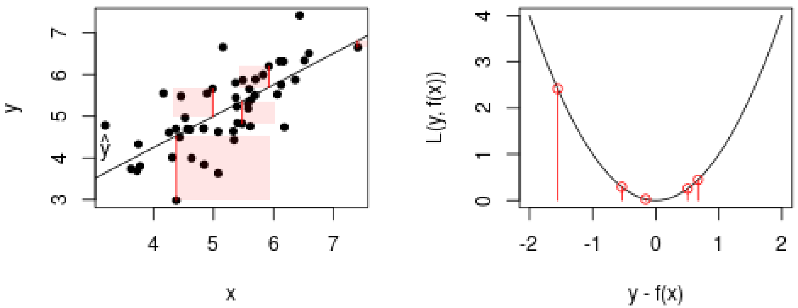
$$L : \mathcal{Y} \times \mathbb{R}^g \to \mathbb{R},$$

How "close" $f(\mathbf{x})$ is to y can be quantified e. g. by the absolute loss $L(y, f(\mathbf{x})) = |f(\mathbf{x}) - y|$.



LOSS

Often, we use the L2-loss $L(y, f(\mathbf{x})) = (y - f(\mathbf{x}))^2$:

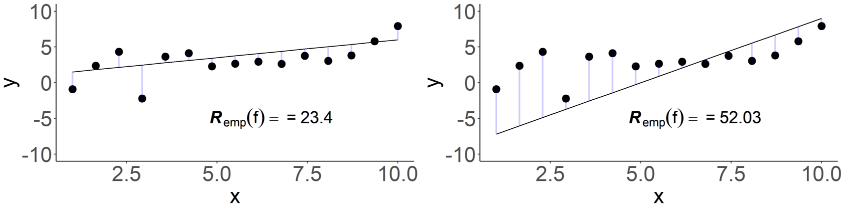


RISK

The **risk function** quantifies the "quality" of the whole model.

The ability of a model f to reproduce the association between $\mathbf{x}$ and y that is present in the data $\mathcal{D}$ can be measured by the **summed loss**, also called **"empirical risk"**:

$$\mathcal{R}_{\text{emp}}(f) = \sum_{i=1}^{n} L\left(y^{(i)}, f\left(\mathbf{x}^{(i)}\right)\right).$$



RISK

Note:
The risk is often denoted as empirical mean over $L(y, f(\mathbf{x}))$

$$\bar{\mathcal{R}}_{\text{emp}}(f) = \frac{1}{n} \sum_{i=1}^{n} L\left(y^{(i)}, f\left(\mathbf{x}^{(i)}\right)\right).$$

The factor $\frac{1}{n}$ does not make a difference in optimization, so we will consider $\mathcal{R}_{\text{emp}}(f)$ most of the time.

????

The best model is the model with the smallest risk.

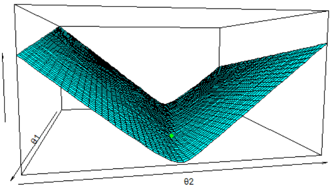
If we have a finite number of models f, we can compare the risk $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$ of all models:

Model	$\boldsymbol{\theta}_{intercept}$	$\boldsymbol{\theta}_{slope}$	$\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$
f_1	4	1	96.37
f_2	3	7	576.37
f_3	1	0.5	1.56
f_4	-9	1.8	52.03

???

But: Normally, the hyposthesis space $\mathcal{H}$ is infinitely large.

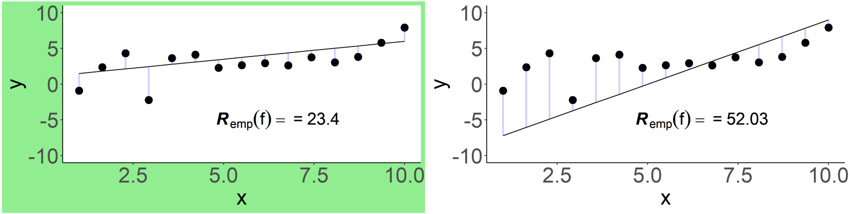
As the the mapping of the hypothesis space to its parameters is bijective, we can consider the error surface depending on the parameters:



RISK MINIMIZATION

The process of finding the best model is called **empirical risk minimization** (ERM).

$$\hat{f} = \arg\min_{f \in \mathcal{H}} \mathcal{R}_{\text{emp}}(f).$$



RISK MINIMIZATION

Since the model f is usually defined by **parameters** $\boldsymbol{\theta}$ in a parameter space Θ, this becomes:

$$
\begin{aligned}
\mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) &= \frac{1}{n} \sum_{i=1}^{n} L\left(y^{(i)}, f\left(\mathbf{x}^{(i)} \mid \boldsymbol{\theta}\right)\right) \\
\hat{\boldsymbol{\theta}} &= \underset{\boldsymbol{\theta} \in \Theta}{\arg\min}\, \mathcal{R}_{\text{emp}}(\boldsymbol{\theta})
\end{aligned}
$$

Most learners in ML try to solve the above *optimization problem*, which implies a tight connection between ML and optimization.

FURTHER REMARKS

- For regression tasks, the loss often only depends on the residual $L(y, f(\mathbf{x})) = L(y - f(\mathbf{x})) = L(\epsilon)$.
- The choice of loss implies which kinds of errors are important or not – requires *domain knowledge*!
- For learners that correspond to probabilistic models, the loss determines / is equivalent to distributional assumptions.
- Since learning can be re-phrased as minimizing the loss, the choice of loss strongly affects the computational difficulty of learning:
    - How smooth is $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$ in $\boldsymbol{\theta}$?
    - Is $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$ differentiable so that we can use gradient-based methods?
    - Does $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$ have multiple local minima or saddlepoints over Θ?