Homework 3 – Xavier Gitiaux

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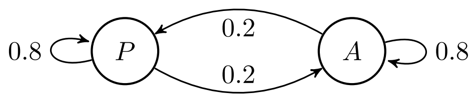
1 Exercise 1

1.1 Question a

There are two hidden states P and A:



The prior distribution is $(0.5, 0.5)$. The transition matrix is

$$Q = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix} \tag{1}$$

1.2 Question b

There are four states: P, A, C and AC. Since the lever are independent, the Markov structure is as follows:

$$P(A|A) = P[lever1 = ON|lever = 1 = ON]P[lever2 = OFF|lever2 = OFF] = 0.56 = P[P|P]$$

$$P(A|P) = P[lever1 = ON|lever = 1 = OFF]P[lever2 = OFF|lever2 = OFF] = 0.2{*}0.7 = 0.14 = P[P|A]$$

$$P[C|C] = P[lever1 = OFF|lever = 1 = OFF]P[lever2 = ON|lever2 = ON] = 0.8{*}0.7 = 0.56 = P(AC|AC)$$

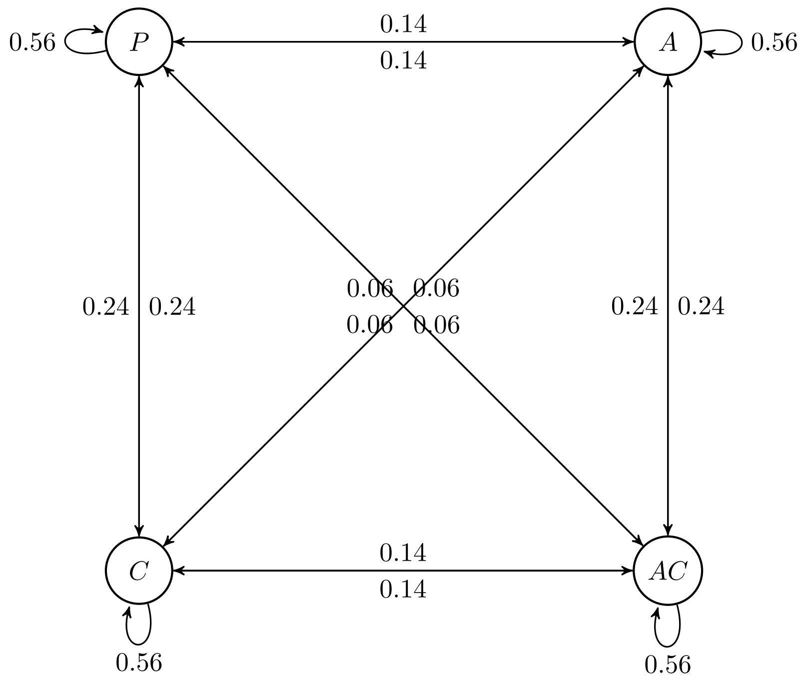
$$P(C|P) = P[lever1 = OFF|lever = 1 = OFF]P[lever2 = ON|lever2 = OFF] = 0.8{*}0.3 = 0.24 = P(P|C)$$

$$P(AC|A) = P[lever1 = ON|lever = 1 = ON]P[lever2 = OFF|lever2 = ON] = 0.8{*}0.3 = 0.24 = P(A|AC)$$

$$P(AC|P) = P[lever1 = ON|lever = 1 = OFF]P[lever2 = OFF|lever2 = ON] = 0.2{*}0.3 = 0.06 = P(P|AC)$$

$$P(C|AC) = P[lever1 = OFF|lever = 1 = ON]P[lever2 = ON|lever2 = ON] = 0.2{*}0.7 = 0.14 = P(AC|C)$$

$$P(C|A) = P[lever1 = OFF|lever = 1 = ON]P[lever2 = ON|lever2 = OFF] = 0.2{*}0.3 = 0.06 = P(A|C)$$



The prior distribution is $(0.25, 0.25, 0.25, 0.25)$. And the transition matrix is

$$Q = \begin{pmatrix} 0.56 & 0.14 & 0.24 & 0.06 \\ 0.14 & 0.56 & 0.06 & 0.24 \\ 0.24 & 0.06 & 0.56 & 0.14 \\ 0.06 & 0.24 & 0.14 & 0.56 \end{pmatrix} \tag{2}$$

1.3 Question c

$$P(P, A, AC, C) = P(P)P(A|P)P(AC|A)P(C|A) = (0.25)(0.14)(0.24)(0.06) = 0.000504 \tag{3}$$

1.4 d

At time 1:

$$\begin{aligned} P(P|D, L) &\propto P(L|P)P(P|D) \\ &\propto P(P|D)\sum_{x_2} P(L|P, x_2)P(x_2|P) \\ &\propto P(D|P)P(P)\sum_{x_2} P(L|x_2)P(x_2|P) \\ &\propto (0.8)(0.25)\left[(0.2)(0.56) + (0.4)(0.14) + (0.8)(0.24) + (0.9)(0.06)\right] \\ &\propto 0.207 \end{aligned} \tag{4}$$

$$
\begin{aligned}
P(A|D,L) &\propto P(D|A)P(A)\sum_{x_2} P(L|x_2)P(x_2|A) \\
&\propto (0.6)(0.25)\left[(0.2)(0.14)+(0.4)(0.56)+(0.8)(0.06)+(0.9)(0.24)\right] \\
&\propto 0.258
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
P(C|D,L) &\propto P(D|C)P(C)\sum_{x_2} P(L|x_2)P(x_2|C) \\
&\propto (0.2)(0.25)\left[(0.2)(0.24)+(0.4)(0.06)+(0.8)(0.56)+(0.9)(0.14)\right] \\
&\propto 0.0323
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
P(AC|D,L) &\propto P(D|AC)P(AC)\sum_{x_2} P(L|x_2)P(x_2|AC) \\
&\propto (0.1)(0.25)\left[(0.2)(0.06)+(0.4)(0.24)+(0.8)(0.14)+(0.9)(0.56)\right] \\
&\propto 0.0181
\end{aligned} \tag{7}
$$

After normalization $P(X_1|D,L) \approx (0.40, 0.50, 0.06, 0.04)$.

At time $t = 2$,

$$
\begin{aligned}
P(X_2|D,L) &\propto P(L|X_2,D)P(X_2|D) \\
&\propto P(L|P)\sum_{x_1} P(X_2|x_1)P(D|x_1))
\end{aligned} \tag{8}
$$

Therefore

$$P(X_2 = P|D,L) = 0.2 * [0.56 * 0.8 + 0.14 * 0.6 + 0.24 * 0.2 + 0.06 * 0.1] = 0.1172$$

$$P(X_2 = A|D,L) = 0.4 * [0.14 * 0.8 + 0.56 * 0.6 + 0.06 * 0.2 + 0.24 * 0.1] = 0.1936$$

$$P(X_2 = C|D,L) = 0.8 * [0.24 * 0.8 + 0.06 * 0.6 + 0.56 * 0.2 + 0.14 * 0.1] = 0.2832$$

$$P(X_2 = AC|D,L) = 0.9 * [0.06 * 0.8 + 0.24 * 0.6 + 0.14 * 0.2 + 0.56 * 0.1] = 0.2484$$

Therefore, after renormalization, $P(X_2|D,L) = (0.14, 0.23, 0.34, 0.29)$.