Homework 1 – Xavier Gitiaux

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1 Exercise 1

Joint Distribution Since B and C are independent given A, $P(B,C|A) = P(B|A)P(C|A)$ so by Baye's rule,

$$Pr(A,B,C) = P(A)P(B,C|A) = P(A)P(B|A)P(C|A).$$

Therefore

$$P(A = a, B = b, C = c) = (0.2)(0.9)(0.7) = 0.126$$
$$P(A = a, B = \neg b, C = c) = (0.2)(0.1)(0.7) = 0.014$$
$$P(A = a, B = \neg b, C = \neg c) = (0.2)(0.1)(0.3) = 0.006$$
$$P(A = a, B = b, C = \neg c) = (0.2)(0.9)(0.3) = 0.054$$
$$P(A = \neg a, B = b, C = c) = (0.8)(0.4)(0.5) = 0.16$$
$$P(A = \neg a, B = \neg b, C = c) = (0.8)(0.6)(0.5) = 0.24$$
$$P(A = \neg a, B = \neg b, C = \neg c) = (0.8)(0.6)(0.5) = 0.24$$
$$P(A = \neg a, B = b, C = \neg c) = (0.8)(0.4)(0.5) = 0.16$$

Then $P(B,C) = P(B,C,A = a)P(A = a) + P(B,C,A = \neg a)P(A = \neg a) = P(B,C,A = a)(0.2) + P(B,C,A = \neg a)(0.8)$. So

$$P(B = b, C = c) = 0.1532$$

$$P(B = b, C = \neg c) = (0.054)(0.2) + (0.16)(0.8) = 0.1388$$
$$P(B = \neg b, C = \neg c) = (0.006)(0.2) + (0.24)(0.8) = 0.1932$$
$$P(B = \neg b, C = c) = (0.014)(0.2) + (0.24)(0.8) = 0.1948$$

Using Bayes' rule,

$$P(A|B = b) = \frac{P(A, B = b)}{P(B = b)} = \frac{P(B = b|A)}{P(B = b)}P(A)$$

Moreover, $P(B = b) = P(A = a)P(B = b|A = a) + P(A = \neg a)P(B|A = \neg a) = (0.2)(0.9) + (0.8)(0.4) = 0.5$ So

$$P(A = a|B = b) = \frac{0.9}{0.5}(0.2) = 0.36.$$

And

$$P(A = \neg a | B = b) = 0.64.$$

Similarly,

$$P(A|C = c) = \frac{P(A, C = c)}{P(C = c)} = \frac{P(C = c|A)}{P(C = c)} P(A) =$$

Moreover, $P(C = c) = P(A = a)P(C = c|A = a) + P(A = \neg a)P(C = c|A = \neg a) = (0.2)(0.7) + (0.5)(0.4) = 0.34$ So

$$P(A = a|B = b) = \frac{0.7}{0.34}(0.2) = 7/17$$

and

$$P(A = \neg a|B = b) = 10/17$$

Lastly,

$$P(A = a|B = b, C = c) = \frac{P(A = a, C = c, B = b)}{P(B = b, C = c)} = 0.126/0.1532$$

We have

$$P(A|B = b, C = c) = \frac{P(B = b, C = c|A)}{P(B = b, C = c)} P(A) = \frac{P(B = b|A)P(C = c|A)}{P(B = b, C = c)} P(A), \tag{1}$$

the last equation holding because B and C are independent given A.

2 Exercise 2

First,

$$P(x, y|e) = \frac{P(x, y, e)}{P(e)} = \frac{P(x|y, e)P(y, e)}{P(e)} = P(x|y, e)P(y|e). \tag{2}$$

Secondly,

$$P(y|x, e) = \frac{P(x, y, e)}{P(x, e)} = \frac{P(x|y, e)P(y, e)}{P(x, e)} = P(x|y, e)\frac{P(y|e)}{P(x, e)}. \tag{3}$$

Lastly,

$$\frac{P(a, b, c}{P(b, c)} = P(a|b, c) = P(b|a, c) = \frac{P(a, b, c}{P(a, c)}. \tag{4}$$

Therefore, if $P(a|b, c) = P(b|a, c)$, then $P(b, c) = P(a, c)$, which implies that

$$P(b|c) = \frac{P(b, c)}{P(c)} = \frac{P(a, c)}{P(c)} = P(a|c). \tag{5}$$

3 Exercise 3

3.1 Question (a)

$P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$

3.2 Question (b)

$P(cavity) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$

3.3 Question (c)

$P(toothache|cavity) = \frac{0.108+0.012}{0.2} = 0.6$

3.4 Question (d)

$$P(cavity|toothache \vee catch) = \frac{P((cavity \wedge toothache) \vee ((cavity \wedge catch))}{P(toothache \vee catch)}$$

Moreover, $P(toothache \vee catch) = P(toothache) + P(catch) - P(toothache \wedge catch) = 0.2 + (0.108 + 0.016 + 0.072 + 0.144) - 0.108 = 0.432$. And

$$P(cavity \wedge (toothache \vee catch)) = 0.108 + 0.012 + 0.072 = 0.192$$

.

Therefore,

$$P(cavity|toothache \vee catch) = \frac{0.192}{0.432} = \frac{4}{9}.$$

4 Exercise 4

4.1 Question (a)

$P(movie) = 0.5 = P(song)$.

4.2 Question (b)

$$P(Perfect|movie) = \frac{2}{8} = \frac{1}{4}$$
$$P(Perfect|song) = \frac{1}{8}$$
$$P(Storm|movie) = \frac{0}{8} = 0$$
$$P(Storm|song) = \frac{1}{8}$$

4.3 Question (c)

With Laplace smoothing,

$$P(Perfect\ Storm|movie) = \frac{1}{8+12} = \frac{1}{20} \quad (6)$$

$$P(Perfect\ Storm|song) = \frac{1}{8+12} = \frac{1}{20} \quad (7)$$

Therefore, $P(Perfect\ Storm) = \frac{1}{20}$ and

$$P(movie|Perfect\ Storm) = \frac{P(Perfect\ Storm|movie)}{P(Perfect\ Storm)} P(movie) = 0.5$$

It makes sense: since "Perfect Storm" does not occur in movie nor in song, we assign a probability 0.5 to the events $(movie|Perfect\ Storm)$ and $(song|Perfect\ Storm)$

4.4 Question (d)

Without Laplacian smoothing, since $P(Perfect\ Storm) = 0$, $P(movie|Perfect\ Storm)$ is indefinite.