1.(1) $\int_0^{+\infty} xe^{-x^2}dx = \cancel{-2e} \cancel{-2e^{-x^2}} -\frac{e^{-x^2}}{2}\Big|_0^{+\infty} = \lim_{x\to+\infty}(-\frac{1}{2e^{x^2}}-(-\frac{1}{2}))=\frac{1}{2}$

∴原式收敛.

(4) $\int_1^{+\infty}\frac{dx}{x^2(1+x)} = \int_1^{+\infty}(\frac{1}{1+x}-\frac{1}{x}+\cancel{\frac{1}{1+x}}\frac{1}{x^2})dx$

$= (\ln(1+x)-\ln x \cancel{a} -\frac{1}{x})\Big|_1^{+\infty}$

$= \lim_{x\to+\infty}[(\underbrace{\cancel{\ln(1+x)-\ln x}}_{\ln(1+\frac{1}{x})}-\frac{1}{x})-(\ln 2-0-1)]$

$= 1-\ln 2.$

原式收敛.

(7) 原式 $= \int_0^{+\infty} e^x\sin x\,dx + \int_{-\infty}^0 e^x\sin x\,dx$

$= \lim_{a\to+\infty}\int_0^a e^x\sin x\,dx + \lim_{a\to-\infty}\int_a^0 e^x\sin x\,dx$

$= \lim_{a\to+\infty}[\frac{1}{2}e^a(\sin a-\cos a)-\frac{1}{2}\times(-1)] + \lim_{a\to-\infty}[\frac{1}{2}(-1)-\cancel{e}\frac{1}{2}e^a(\sin a-\cos a)]$

由于三角函数的周期性，显然，两个极限不存在.

故原式发散.

5. 证明：若$A\neq 0$，不妨设A ~~为~~大于0，可能为某一实数或无穷大.

则一定存在$b>a$，~~使得~~ $c>0$，使当$x>b$时 $f(x)>c$.

$$\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$$
$$> \int_a^b f(x)dx + c\int_b^{+\infty} dx$$
$$= \int_a^b f(x)dx + c(x-b)\ (x\to+\infty)$$

显然，等号右侧发散，但这与条件不符

同理可证 A小于0不合理，

故 $A=0$.

2. 证明. $|f(x)g(x)| < \dfrac{f^2(x)+g^2(x)}{2}$

$\because \int_a^{+\infty} f^2(x)dx$ 与 $\int_a^{+\infty} g^2(x)$ 均收敛，$\therefore$ 不等式右侧收敛，

由比较原则 $\int_a^{+\infty} |f(x)g(x)|$ 收敛

$\therefore \int_a^{+\infty} f(x)g(x)dx$ 绝对收敛.

$$\int_a^{+\infty}[f(x)+g(x)]^2 = \int_a^{+\infty}[f^2(x)+g^2(x)+2f(x)g(x)]dx$$
$$= \int_a^{+\infty} f^2(x)dx + \int_a^{+\infty} g^2(x)dx + 2\int_a^{+\infty} f(x)g(x)dx$$

等式右侧收敛，故

$\int_a^{+\infty}[f(x)+g(x)]^2$ 收敛.

4. (1) $\lim\limits_{x\to+\infty} \frac{x^{\frac{4}{3}}}{\sqrt[3]{x^4+1}} = \lim\limits_{x\to+\infty} \frac{1}{\sqrt[3]{1+\frac{1}{x^4}}} = 1$

由比较原则推论 $\int_0^{+\infty} \frac{dx}{\sqrt[3]{x^4+1}}$ 收敛.

(4) $\lim\limits_{x\to+\infty} x^2 \cdot \frac{x\arctan x}{1+x^3} dx = \frac{\pi}{2}$

由比较原则推论, $\int_1^{+\infty} \frac{x\arctan x}{1+x^3}$ 收敛.

5. (1) $\int_1^{+\infty} \frac{\sin\sqrt{x}}{x} dx = 2\int_1^{+\infty} \frac{\sin\sqrt{x}}{\sqrt{x}} d\sqrt{x} \xrightarrow{令t=\sqrt{x}} 2\int_1^{+\infty} \frac{\sin t}{t} dt$

$\int_1^{+\infty} \frac{\sin x}{x} dx$ 条件收敛 故原式条件收敛.

(2) $|\operatorname{sgn}(\sin x)| \leq 1$ $\left|\frac{\operatorname{sgn}(\sin x)}{1+x^2}\right| \leq \frac{1}{1+x^2}$

$\int_0^{+\infty} \frac{1}{1+x^2} dx$ 收敛. 由比较原则

$\int_0^{+\infty} \frac{\operatorname{sgn}(\sin x)}{1+x^2} dx$ 绝对收敛

9. 证明，不妨假设 $\lim\limits_{x\to+\infty} f(x)\neq 0$.

$\therefore \exists \varepsilon_0>0$，对 $\forall A>a$，$\exists x_0>A$ 且 $|f(x_0)|>2\varepsilon_0$

$\because$ f 在 $[a,+\infty)$ 上一致连续，

$\therefore$ 对于 ε_0，$\exists \delta>0$ 使 $\forall x\in U_+^0(x_0;\delta)$，都有 $|f(x)-f(x_0)|<\varepsilon_0$

$\therefore |f(x)| \geq |f(x_0)|-|f(x_0)-f(x)| > \varepsilon_0$

$$\therefore \left|\int_{x_0}^{x_0+\delta} f(x)dx\right| > \varepsilon_0\int_{x_0}^{x_0+\delta} dx = \varepsilon_0\delta.$$

即~~对于~~ $\exists \varepsilon=\varepsilon_0\delta$，$\forall A>a$，$\exists x_0, x_0+\delta>G$，有 $\left|\int_{x_0}^{x_0+\delta} f(x)dx\right|>\varepsilon$.

这与无穷积分收敛的柯西准则矛盾.

故 $\lim\limits_{x\to+\infty} f(x)=0$

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2.(2) $\cancel{\int_0^1 \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{x+1}{x-1}}\Big|_0^1$

$$\int_0^1 \frac{dx}{1-x^2} = \frac{1}{2}\int_0^1\left(\frac{1}{x+1}-\frac{1}{x-1}\right)dx = \frac{1}{2}\left[\int_0^1\frac{1}{x+1}dx+\int_0^1\frac{1}{1-x}dx\right]$$

$$=\frac{1}{2}\left[\ln 2+\lim_{a\to1^-}\int_0^a\frac{1}{1-x}dx\right]=\frac{1}{2}\left[\ln 2-\lim_{a\to1^-}\ln(1-a)\right]$$

显然, $\lim_{a\to1^-}\ln(1-a)$ 不存在 故原式发散.

(3) $$\int_0^2\frac{dx}{\sqrt{|x-1|}}=\int_0^1\frac{dx}{\sqrt{1-x}}+\int_1^2\frac{dx}{\sqrt{x-1}}$$

$$=\lim_{a\to1^-}\int_0^a(1-x)^{-\frac{1}{2}}dx \quad \lim_{b\to1^+}\int_b^2(x-1)^{-\frac{1}{2}}dx$$

$$=\lim_{a\to1^-}\left(-2(1-x)^{\frac{1}{2}}\right)\Big|_0^a+\lim_{b\to1^+}\left(2(x-1)^{\frac{1}{2}}\right)\Big|_b^1$$

$$=2+2=4$$

故原式收敛.

(8) $$\int_0^1\frac{dx}{x(\ln x)^p}=\int_0^{\frac{1}{e}}\frac{1}{(\ln x)^p}d(\ln x)+\int_{\frac{1}{e}}^1\frac{1}{(\ln x)^p}d(\ln x)$$

$$=\lim_{a\to0^+}\frac{(\ln x)^{1-p}}{1-p}\Big|_a^{\frac{1}{e}}+\lim_{b\to1^-}\frac{(\ln x)^{1-p}}{1-p}\Big|_{\frac{1}{e}}^b$$

$$=\lim_{a\to0^+}\left[\frac{(-1)^{1-p}}{1-p}-\frac{(\ln a)^{1-p}}{1-p}\right]+\lim_{b\to1^-}\left[\frac{(\ln b)^{1-p}}{1-p}-\frac{(-1)^{1-p}}{1-p}\right]$$

$$=\lim_{b\to1^-}\frac{(\ln b)^{1-p}}{1-p}-\lim_{a\to0^+}\frac{(\ln a)^{1-p}}{1-p}=f(b)-g(a)$$

$f(b)\begin{cases}p<1 & \text{收敛}\\ p\geq1 & \text{发散}\end{cases}$ $\quad g(a)\begin{cases}p\leq1 & \text{发散}\\ p>1 & \text{收敛}\end{cases}$ 故等式右侧不收敛, 原式发散.

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(2) $\lim\limits_{x\to 0^+}\left(x^{\frac{1}{2}}\cdot\frac{\sin x}{x^{\frac{3}{2}}}\right)=1$ 由比较原则 ~~知~~ $\int_0^{\pi}\frac{\sin x}{x^{\frac{3}{2}}}dx$ 收敛.

(4) $\lim\limits_{x\to 1^-}(1-x)^{\frac{1}{2}}\left|\frac{\ln x}{1-x}\right| = \lim\limits_{x\to 1^-}\frac{-\ln x}{(1-x)^{\frac{1}{2}}} = \lim\limits_{x\to 1^-}\frac{-(x-1)}{(1-x)^{\frac{1}{2}}}=0$

$\therefore \int_0^1\frac{\ln x}{1-x}dx$ 绝对收敛

(6) $\lim\limits_{x\to 0^+}x^{m-2}\frac{1-\cos x}{x^m}=\lim\limits_{x\to 0^+}\frac{1-\cos x}{x^2}=\frac{1}{2}$

$m-2<1$ $m<3$ 原式收敛

$m-2\geq 1$ $m\geq 3$ 原式发散.

(7) $\int_0^1\frac{1}{x^\alpha}\sin\frac{1}{x}dx=\int_0^1-\frac{1}{x^{\alpha+2}}\sin\frac{1}{x}d\frac{1}{x} \xlongequal{t=\frac{1}{x}} \int_1^{+\infty}\frac{\sin t}{t^{2-\alpha}}dt$

1) $2-\alpha>1$ $\alpha<1$ 原式绝对收敛

2) $0<2-\alpha\leq 1$ $1\leq\alpha<2$ 原式条件收敛

3) $2-\alpha\leq 0$ $\alpha\geq 2$ 原式发散.

5、证明：$2\int_0^{\frac{\pi}{2}}\ln(\sin x)dx=\int_0^{\frac{\pi}{2}}\ln(\sin x)dx+\int_0^{\frac{\pi}{2}}\ln(\cos x)dx$

$$=\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{2}\sin 2x\right)dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}[\ln(\sin 2x)-\ln 2]\,d2x$$

令 $t=2x$

$$=\frac{1}{2}\int_0^{\pi}\ln(\sin t)dt-\frac{\ln 2}{2}\int_0^{\pi}dt$$

$$=\int_0^{\frac{\pi}{2}}\ln(\sin t)dt-\frac{\pi}{2}\ln 2$$

$$\therefore\ \int_0^{\frac{\pi}{2}}\ln(\sin x)dx=-\frac{\pi}{2}\ln 2.$$

4、解：$\int_0^{+\infty}\frac{\sin bx}{x^{\lambda}}dx\ \cancel{=b^{\lambda-1}\int_0^{+\infty}\frac{\sin bx}{(bx)^{\lambda}}dbx}=\int_0^{1}\frac{\sin bx}{x^{\lambda}}dx+\int_1^{+\infty}\frac{\sin bx}{x^{\lambda}}dx$

$$=I_1+I_2.$$

对于I_1：$\lim\limits_{x\to0^+}x^{\lambda-1}|f(x)|=|b|.$

$0<\lambda-1<1$ $1<\lambda<2$时 I_1绝对收敛　$\lambda-1\le0$ $\lambda\le1$ I_1为定积分

$\lambda-1\ge1$ $\lambda\ge2$ I_1发散.

对于I_2 $\lambda>1$，绝对收敛　$0<\lambda\le1$，条件收敛

$\lambda\le0$时，发散.

综上 原式在 $\lambda\le0$，$\lambda\ge2$时发散.　$0<\lambda\le1$条件收敛

$1<\lambda<2$ 绝对收敛.

5.证明：(1) $\int_0^{+\infty} \frac{f(ax)-f(bx)}{x}dx = \lim_{u\to 0^+}\int_u^1 \frac{f(ax)-f(bx)}{x}dx + \lim_{v\to+\infty}\frac{f(ax)-f(bx)}{x}dx$

$= \lim_{u\to 0^+}\left(\int_u^1 \frac{f(ax)}{x}dx - \int_u^1 \frac{f(bx)}{x}dx\right) +$

$\lim_{v\to+\infty}\left(\int_1^v \frac{f(ax)}{x}dx - \int_1^v \frac{f(bx)}{x}dx\right)$

$\xlongequal[s=bx]{令t=ax} \lim_{u\to 0^+}\left(\int_{au}^{a} \frac{f(t)}{t}dt - \int_{bu}^{b}\frac{f(s)}{s}ds\right) + \lim_{u\to 0^+}\int_a^b \frac{f(x)}{x}dx$

$\lim_{v\to+\infty}\left(\int_a^{av}\frac{f(t)}{t}dt - \int_b^{bv}\frac{f(s)}{s}ds\right) + \lim_{v\to+\infty}\int_b^a \frac{f(x)}{x}dx$

$= \lim_{u\to 0^+}\int_{au}^{bu}\frac{f(x)}{x}dx - \lim_{v\to+\infty}\int_{av}^{bv}\frac{f(x)}{x}dx$

$= \lim_{u\to 0^+} f(\xi)\int_{au}^{bu}\frac{dx}{x} - \lim_{v\to+\infty} f(\eta)\int_{av}^{bv}\frac{dx}{x}$

$= \ln\frac{b}{a}\left[\lim_{u\to 0^+} f(\xi) - \lim_{v\to+\infty} f(\eta)\right]$ $[\xi\in[au,bu), \eta\in[av,bv)]$

$= \ln\frac{b}{a}[f(0)-k]$

(2) 证明：$\int_a^{+\infty}\frac{f(x)}{x}dx$ 收敛，$\forall \varepsilon>0$ $\int_\varepsilon^{+\infty}\frac{f(x)}{x}dx$ 收敛

$$\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx=\lim_{\varepsilon\to0^+}\left[\int_\varepsilon^{+\infty}\frac{f(ax)}{x}dx-\int_\varepsilon^{+\infty}\frac{f(bx)}{x}dx\right]$$

$$=\lim_{\varepsilon\to0^+}\left[\int_{a\varepsilon}^{b\varepsilon}\frac{f(x)}{x}dx\right]$$

$$=\lim_{\varepsilon\to0^+}\left[f(\xi)\int_{a\varepsilon}^{b\varepsilon}\frac{dx}{x}\right]\quad (\xi\in(a\varepsilon,b\varepsilon))$$

$$=\lim_{\varepsilon\to0^+}f(\xi)\ln\frac{b}{a},\quad \xi\in(a\varepsilon,b\varepsilon).$$

$$=f(0)\ln\frac{b}{a}.$$

12.1 1.(1) $S_n=\frac{1}{5}\left(1-\frac{1}{6}+\frac{1}{6}-\frac{1}{11}+\cdots+\frac{1}{5n-4}-\frac{1}{5n+1}\right)$

$$=\frac{1}{5}\times\frac{5n}{5n+1}=\frac{n}{5n+1}$$

$$\lim_{n\to\infty}S_n=\frac{1}{5}$$

(2) $S_n=\sum_{k=1}^{n}(\sqrt{k+2}-2\sqrt{k+1}+\sqrt{k})=\sum_{k=1}^{n}\left(\frac{1}{\sqrt{k+2}+\sqrt{k+1}}-\frac{1}{\sqrt{k+1}+\sqrt{k}}\right)=\frac{1}{\sqrt{n+2}+\sqrt{n+1}}-(\sqrt{2}-1)$

$\lim_{n\to\infty}S_n=0-(\sqrt{2}-1)=1-\sqrt{2}$ 原式收敛于 $1-\sqrt{2}$

4. 证明：$S_n=\sum_{k=1}^{n}(a_k-a_{k+1})=a_1-a_{n+1}$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}(a_1-a_{n+1})=\lim_{n}a_1-\lim_{n\to\infty}a_{n+1}=a_1-a.$$