A Non-Regular Language that Satisfies the Conditions of the Pumping Theorem

Consider the language $F = \{a^i b^j c^k : i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$. We first show that F satisfies the conditions of the pumping theorem. Let the pumping length $p = 2$. Let w be any string in F whose length is at least p. There are two cases.

(i) w is in the form of $a^i b^j c^k$ with $i \neq 2$. We write w as $w = xyz$ where $x = e$, y is the first symbol of w, and z is the rest part.

(ii) w is in the form of $a^2 b^j c^k$. We write w as $w = xyz$ where $x = e$, $y = aa$, and $z = b^j c^k$.

It is easy to verify that the three conditions in the pumping theorem are satisfied in all the cases.

Next we show that F is not regular. Let $A = \{ab^i c^i : i \geq 0\}$. Let $B = \{a^i b^j c^k : i \neq 1\}$. It is easy to see that $F = A \cup B$ and $A \cap B = \emptyset$. Since $A \cap B = \emptyset$, we can write A as

$$A = (A \cup B) \cap \overline{B} = F \cap \overline{B}.$$

One can easily verify that $\overline{B}$ is regular (since B is regular). If F is regular, by the closure property of regular languages, we have that $A = F \cap \overline{B}$ is regular. But one can easily prove that A is not regular. Therefore, F cannot be regular.

References

[1] Sipser M.. Introduction to the Theory of Computation. CENGAGE Learning (2013)