3.(3) 由图 $D=\{(x,y)\mid 0\le y\le a-\sqrt{2ax-x^2},\ 0\le x\le a\}$ 为X型区域

$$\iint\limits_D \frac{d\sigma}{\sqrt{2a-x}}=\int_0^a dx\int_0^{a-\sqrt{2ax-x^2}}\frac{1}{\sqrt{2a-x}}dy=\int_0^a\frac{a-\sqrt{2ax-x^2}}{\sqrt{2a-x}}dx$$

$$=\int_0^a\frac{a}{\sqrt{2a-x}}dx-\int_0^a\sqrt{x}\,dx=\left[-2a(2a-x)^{\frac{1}{2}}\right]\Big|_0^a-\frac{2}{3}x^{\frac{3}{2}}\Big|_0^a$$

$$=(2\sqrt{2}-\frac{8}{3})a^{\frac{3}{2}}$$

4.解、在xy平面上削柱形投影 $D=\{(x,y)\mid 0\le y\le 3,\ 0\le x\le 1\}\cup\{(x,y)\mid 0\le y\le 4-x,\ 1\le x\le 2\}$

即D为x型区域

$z=4-x-y$

$$\therefore V=\iint\limits_D(4-x-y)d\sigma=\int_0^1dx\int_0^3(4-x-y)dy+\int_1^2dx\int_0^{4-x}(4-x-y)dy.$$

$$=\int_0^1(\frac{15}{2}-3x)dx+\int_1^2\frac{1}{2}(4-x)^2dx$$

$$=6+\frac{19}{6}=\frac{55}{6}$$

5.证明：$\left[\int_a^b f(x)dx\right]^2=\int_a^b f(x)dx\cdot\int_a^b f(y)dy=\iint\limits_D f(x)f(y)dxdy$ 令 $D=[a,b]\times[a,b]$

$$\le\iint\limits_D\frac{1}{2}\left[f^2(x)+f^2(y)\right]dxdy=\iint\limits_D f^2(x)dydx=\int_a^b dx\int_a^b f^2(x)dy$$

$=(b-a)\int_a^b f^2(x)dx$，当 $f(x)=f(y)$ 时成立，即对 $\forall x,y$ 等式成立，$f(x)$ 为常量函数

即 $\left[\int_a^b f(x)dx\right]^2\le(b-a)\int_a^b f^2(x)dx$，$f(x)$ 为常量函数时成立等号.

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1.(2) 原式 $= \int_0^1 dx \int_0^{\frac{\pi}{2}} dy \int_0^{\frac{\pi}{2}} x\cos y \cos z\, dz$

$= \int_0^1 x\,dx \int_0^{\frac{\pi}{2}} \cos y\,dy \int_0^{\frac{\pi}{2}} \cos z\,dz$

$= \frac{1}{2}$

(4) 在xy平面上，V投影 $D = \{(x,y) \mid 0 \le y \le \sqrt{x},\ 0 \le x \le \frac{\pi}{2}\}$.

~~$z = \frac{\pi}{2} - x$~~ ~~$z_1(x,y) =$~~ $0 < z < \frac{\pi}{2} - x$.

~~故~~ 原式 $= \int_0^{\frac{\pi}{2}} dx \int_0^{\sqrt{x}} dy \int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,dz$

$= \int_0^{\frac{\pi}{2}} dx \int_0^{\sqrt{x}} y(1-\sin x)\,dy$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} x(1-\sin x)\,dx = \frac{\pi^2}{16} - \frac{1}{2}$

2.(2) 在xy平面投影 $D_1 = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$

原式 $= \int_0^1 dy \int_0^1 dx \int_0^{x^2+y^2} f(x,y,z)\,dz$

在xz平面上投影 $D_2 = \{(x,z) \mid 0 \le z \le x^2,\ 0 \le x \le 1\} \cup \{(x,z) \mid x^2 \le z \le x^2+1,\ 0 \le x \le 1\}$

原式 $= \int_0^1 dx \int_0^{x^2} dz \int_0^1 f(x,y,z)\,dy + \int_0^1 dx \int_{x^2}^{x^2+1} dz \int_{\sqrt{z-x^2}}^1 f(x,y,z)\,dy$

$= \int_0^1 dz \int_0^{\sqrt{z}} dx \int_{\sqrt{z-x^2}}^1 f(x,y,z)\,dy + \int_0^1 dz \int_{\sqrt{z}} dx \int_0^1 f(x,y,z)\,dy$

$+ \int_1^2 dz \int_{\sqrt{z-1}}^1 dx \int_{\sqrt{z-x^2}}^1 f(x,y,z)\,dy$

在yz平面与xz平面上形式具有对称性

原式 $= \int_0^1 dy \int_0^{y^2} dz \int_0^1 f(x,y,z)dx + \int_0^1 dy \int_{y^2}^{y^2+1} dz \int_{\sqrt{z-y^2}}^1 f(x,y,z)dx$

$= \int_0^1 dz \int_{\sqrt{z}}^1 dy \int_0^1 f(x,y,z)dx + \int_0^1 dz \int_0^{\sqrt{z}} dy \int_{\sqrt{z-y^2}}^1 f(x,y,z)dx$

$+ \int_1^2 dz \int_{\sqrt{z-1}}^1 dy \int_{\sqrt{z-y^2}}^1 f(x,y,z)dx$

3.(2) 使用柱面坐标变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

原式 $= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r dr \int_r^{\sqrt{2-r^2}} z^2 dz = \frac{\pi}{6}\int_0^1 r\left[(2-r^2)^{\frac{3}{2}} - r^3\right]dr$

$= \frac{\pi}{6}\left[-\frac{1}{5}(2-r^2)^{\frac{5}{2}} - \frac{1}{5}r^5\right]\Big|_0^1$

$= \frac{\pi}{15}(2\sqrt{2}-1)$

P225 2.(2) $x^2+y^2 = x+y \iff (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = (\frac{1}{2})^2$.

原式 $= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\cos\theta+\sin\theta} (\cos\theta+\sin\theta) r^2 dr = \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos\theta+\sin\theta)^4 d\theta$

$= \frac{4}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^4(\theta - \frac{\pi}{4})d\theta \xlongequal{t=\theta-\frac{\pi}{4}} \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t dt$

$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^4 t dt = \frac{8}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}$

曲面

(3) 解 $\iint_D |xy| d\sigma = 4\iint_{\substack{x^2+y^2\le a^2 \\ x\ge 0, y\ge 0}} xy\,dx\,dy = 4\int_0^{\frac{\pi}{2}} d\theta \int_0^a (\sin\theta\cos\theta) r^3 dr$

$= 4\int_0^{\frac{\pi}{2}} \sin\theta \, d\sin\theta \cdot \int_0^a r^3 dr$

$= 4\left[\frac{1}{2}\sin^2\theta\right]\Big|_0^{\frac{\pi}{2}} \cdot \left(\frac{1}{4}r^4\right)\Big|_0^a = \frac{a^4}{2}$

3.(2) $\sqrt{x}+\sqrt{y}\le\sqrt{a}$ 变为 $u\le a$; $x\ge 0$ 和 $y\ge 0$ 变为 $u\ge 0$, $0\le v\le\frac{\pi}{2}$

$J(u,v) = 4u\sin^3 v\cos^3 v$

$\iint_D f(x,y)\,dx\,dy = \int_0^a du \int_0^{\frac{\pi}{2}} 4u\sin^3 v\cos^3 v\, f(u\cos^4 v, u\sin^4 v) dv$

4.(2) 解 令 $u=y$ $v=x+y$ $\therefore x=v-u$ $y=u$

$x+y\le 1 \Rightarrow v\le 1$ $y\ge 0 \Rightarrow u\ge 0$ $x\ge 0 \Rightarrow u\le v, v\ge 0$.

$J(u,v) = -1$

$\iint_D e^{\frac{y}{x+y}} dx\,dy = \int_0^1 dv \int_0^v e^{\frac{u}{v}} du = \int_0^1 \left[v e^{\frac{u}{v}}\right]\Big|_0^v dv.$

$= \int_0^1 v(e-1)dv = \frac{1}{2}(e-1)$

5.(1) V在xy平面投影区域为 $D=\{(x,y)\mid(x-\frac{1}{2})^2+(y-\frac{1}{2})^2\le(\frac{1}{\sqrt{2}})^2\}$

对其作极坐标变换 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$

$$D\to D'=\{(r,\theta)\mid 0\le r\le\cos\theta+\sin\theta,\ -\frac{\pi}{4}\le\theta\le\frac{3\pi}{4}\}$$

$$V=\iint_D \cancel{d\theta}\,[(x+y)-(x^2+y^2)]dxdy$$

$$=\cancel{\iint}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}d\theta\int_0^{\cos\theta+\sin\theta}r[r(\cos\theta+\sin\theta)-r^2]d\cancel{\theta}r$$

$$=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{12}(\cos\theta+\sin\theta)^4]d\theta$$

$$=\frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos^4(\theta-\frac{\pi}{4})d\theta\ \xrightarrow{t=\theta-\frac{\pi}{4}}\ \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4 t\,dt$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}\cos^4 t\,dt=\frac{2}{3}\ \frac{3!!}{4!!}\cdot\frac{\pi}{2}=\frac{\pi}{8}$$

6.(2) 作极坐标变换 $\begin{cases}x=a\cancel{r}\cos\theta\\ y=b\cancel{r}\sin\theta\end{cases}$ $\quad r=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$

$$J(r,\theta)=abr$$

$$S_D=\iint_D dxdy=4ab\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}rdr$$

$$=2ab\int_0^{\frac{\pi}{2}}(a^2\cos^2\theta+b^2\sin^2\theta)d\theta$$

$$=2a^3b\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta+2ab^3\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta$$

$$=\frac{\pi}{2}ab(a^2+b^2$$

4、(1) V在xy平面上的投影区域 $D=\{(x,y) \mid x^2 \le y \le x, 0 \le x \le 1\}$

$$z_1(x,y)=x^2+y^2, \quad z_2(x,y)=2(x^2+y^2)$$

$$V=\iiint_V dxdydz=\int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{\sin\theta}{\cos^2\theta}} dr \int_{r^2}^{2r^2} r\,dz$$

$$=\int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{\sin\theta}{\cos^2\theta}} r^3 dr$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{4}} \sin^4\theta \sec^8\theta \, d\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{4}} \tan^4\theta(1+\tan^2\theta)\,d\tan\theta$$

$$=\frac{3}{35}$$