Theory of Computation, Fall 2021
Assignment 7 Solutions

Exercises

Q1. We know that Σ^* is countable. We also know any subset of a countable set is countable. Since every language is a subset of Σ^*, it must be countable.

Q2. There are an uncountable number of subsets of $\{1\}^*$. There are only a countable number of Turing machines and each Turing machine can decide at most one language. By pigeonhole principle, some subset of $\{1\}^*$ cannot be decided any Turing machine, and therefore is undecidable.

Q3. We construct a Turing machine M_A that semidecides A.

$$\begin{aligned} M_A = \quad & \text{on input "}M\text{":} \\ & \text{1. For } i = 1, 2, 3, \ldots\text{:} \\ & \text{2.} \qquad \text{run } M \text{ on each of } s_1, s_2, \ldots, s_i \text{ for } i \text{ steps.} \\ & \text{3.} \qquad \text{halt if } M \text{ halts on any of these string within } i \text{ steps.} \end{aligned}$$

Q4. (a) We construct M_A as follows.

$$\begin{aligned} M_A = \quad & \text{on input } w\text{:} \\ & \text{1. compute } f(w) \\ & \text{2. run } M_B \text{ on } f(w) \end{aligned}$$

One can see that M_A halts on w iff M_B halts on $f(w)$ iff $f(w) \in B$ iff $w \in A$. The last iff is by the definition of reduction. Therefore, M_A semidecides A.

(b) Suppose that there is a reduction f from A to B. We have that if B is recursively enumerable, so is A.

Q5. (a) We construct a reduction f from H to A. For any valid input "M""w" of the halting problem, we define M_w whose input alphabet is $\{0, 1\}$ as follows.

$$\begin{aligned} M_w = \quad & \text{on input } u\text{:} \\ & \text{1. if } u == 01 \\ & \text{2.} \qquad \text{halt} \\ & \text{3. else if } u == 10 \\ & \text{4.} \qquad \text{run } M \text{ on } w \\ & \text{5. else} \\ & \text{6.} \qquad \text{looping} \end{aligned}$$

We let $f(\text{"}M\text{""}w\text{"}) = \text{"}M_w\text{"}$. We claim that "$M_w$" $\in A$ (M_w halts on both 01 and 10) if and only if "M""w" $\in H$ (M halts on w). To see this, it suffices to note that M_w halts on 01, may or may not halt on 10 (depending on whether M halts on w), and does not halt on anything else.

(b) Define

$$\mathcal{L} = \{L : L \text{ is recursively enumerable and for any string } w,\ w \in L \text{ whenever } w^R \in L\}$$

Then A is equivalent to the following problem.

Given a Turing machine M, is $L(M) \in \mathcal{L}$?

Moreover, $\mathcal{L}$ is a proper, non-empty subset of all recursively enumerable languages. (One can easily verify that $\{01, 10\} \in \mathcal{L}$ and $\{01\} \notin \mathcal{L}$ where both $\{01, 10\}$ and $\{01\}$ are recursively enumerable languages.) Therefore, the conditions of Rice's theorem are satisfied, and A is undecidable.