CHAPTER 2

2.1 Sets

Proper subset：D⊂E#⇔#∀ {#{∈D#→#{∈E,#∧#∃ {+{∈E∧ {∉D, $\Leftrightarrow A \subseteq B$ *and* $A \neq B$

1. The cardinality of set

【Definition】Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite* set and that n is the *cardinality* of S.

Notation: $|S|$ ---- the cardinality of S

2. The power set

【Definition】Given a set S, the *power set* of S is the set of all subsets of the set S.

Notation: $P(S)$ ---- the power set of S. $P(S) = \{x | x \subseteq S\}$

- $|S|=n$ implies $|P(S)| = 2^n$
- S is finite and so is $P(S)$.
- $x \in P(S) \Rightarrow x \subseteq S$, $x \in S \Rightarrow \{x\} \in P(S)$, $S \in P(S)$

3. Cartesian Products

【Definition】*The ordered n-tuple* $(a_1,a_2,\ldots,a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, … , and a_n as its nth element.

$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \quad \Leftrightarrow \quad a_i = b_i \ (i=1,2,\ldots, n)$

In particular, 2-tuples are called *ordered pairs.*

- If $x \neq y$, then$(x, y) \neq (y, x)$. $(x, y) = (u, v) \Rightarrow x = u$ *and* $y = v$

The *Cartesian product* of A and B: $A \times B = \{(a, b) | a \in A \wedge b \in B\}$

The *Cartesian product* of $A_1, A_2, \ldots, A_n$: $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i$ for $i=1, 2, \ldots, n\}$.

Note:

- If $|A|=m$, $|B|=n$, Then $|A \times B| = |B \times A| = mn$
- $A \times B \neq B \times A$
- $A \times \varphi = \varphi \times A = \varphi$
- $(x, y) \in A \times B \Rightarrow x \in A \wedge y \in B$; $(x, y) \notin A \times B \Rightarrow x \notin A \vee y \notin B$

4. Truth Sets of Quantifiers

Given a predicate P, and a domain D. The *truth set* of P is the set of elements x in D for which $P(x)$ is true. namely, The *truth set* of $P = \{x \in D \mid P(x)\}$

2.2 Set Operations

1. Set Operations

1) Union

$A \cup B = \{x | x \in A \vee x \in B\}$

(1)$A \subseteq A \cup B, B \subseteq A \cup B$

(2)$A \subseteq C, B \subseteq C \Rightarrow A \cup B \subseteq C$

(3)$|A \cup B| \leq |A| + |B|$

(4)$A \cup B = B \Leftrightarrow A \subseteq B$

2) Intersection

$A \cap B = \{x | x \in A \wedge x \in B\}$

(1)$A \cap B \subseteq A, A \cap B \subseteq B$

(2)$C \subseteq A, C \subseteq B \Rightarrow C \subseteq A \cap B$

(3)$|A \cap B| \leq |A|, |A \cap B| \leq |B|$

(4)$A \cap B = A \Leftrightarrow A \subseteq B$

Two sets are called *disjoint* if their intersection is the empty set, namely $A \cap B = \emptyset$

❖ The cardinality of the union of two finite sets: $|A \cup B| = |A| + |B| - |A \cap B|$

3) Difference of A and B

$A - B = \{x | x \in A \wedge x \notin B\}$

4) The complement of a set

$\overline{A} \quad = \{x | x \notin A, x \in U\}$，Where U is the universal set

Note: $A - B = A \cap \overline{B}$

5) Symmetric difference $A \oplus B = (A \cup B) - (A \cap B)$

Four ways to prove set identities 相同

1. Show that A $\subseteq$ B and that A $\supseteq$ B.
2. Use logical equivalences to prove equivalent set definitions.
3. Use a membership table.
4. Use previously proven identities.

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

4. Generalized Unions and Intersections

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i, \qquad A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

5. Computer Representation of Set

Using bit strings to represent sets.

（1） Specify an arbitrary ordering of the elements of *U*, for instance $a_1, a_2, \cdots, a_n$

（2） Represent a subset *A* of *U* with the bit string of length *n*, where the *i*th bit is 1 if a_i belongs to *A* and is 0 if a_i does not belong to *A*.

Union: bitwise OR

Intersection: bitwise AND

2.3 Functions

【Definition】Let *A* and *B* be nonempty sets. A *function* (*mapping* or *transformations*) *f* from *A* to *B*: $f: A \to B$ $\forall a(a \in A \to \exists! b\ (b \in B \wedge f(a)=b))$

A is called the *domain* of *f* , *B* is called the *codomain* of *f* .

$f(a) = b$

- *b* is called the *image* of *a* under *f*
- *a* is called a *preimage* of *b*

$f(A)$: the *range* of *f* is the set of all images of elements in *A* under *f*.

If *f* is a function from *A* to *B*, we say that *f maps A* to *B*.

【Definition】Let f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and f_1f_2 are also functions from A to R defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, $(f_1f_2)(x) = f_1(x)\, f_2(x)$

【Definition】Let *f* be a function from *A* to *B* and let *S* be a subset of *A*. The image of *S* is the subset of *B* that consists of the images of the elements of *S*. We denote the image of *S* by $f(S)$, so that $f(S) = \{ f(s) \mid s \in S \}$

- $f(\varnothing) = \varnothing$
- $f(\{a\}) = \{f(a)\}$
- $f(A \cup B) = f(A) \cup f(B)$
- $f(A \cap B) \subseteq f(A) \cap f(B)$

1) One-to-One Functions

A function *f* is *one-to-one* (denoted 1-1), or *injective* $\forall a \forall b(f(a) = f(b) \to a = b)$

Note:

- This means that if $x \neq y$ then $f(x) \neq f(y)$.
- A function is said to be an ***injection*** if it is 1-1.

2) Onto Functions

A function *f* from *A* to *B* is called *onto*, or *surjective* $\forall b \in B \exists a \in A(f(a) = b)$

Note:

- This means that for every *b* in *B* there must be an *a* in *A* such that $f(a) = b$. //
- Every *b* in *B* has a preimage.
- A function is called a ***surjection*** if it is onto.

3) One-to-one Correspondence Functions

The function *f* is a *one-to-one correspondence*, or a ***bijection***, if it is both *one-to-one* and *onto*.

Note:

- Whenever there is a bijection from *A* to *B*, the two sets must have the same number of elements or the same *cardinality.*

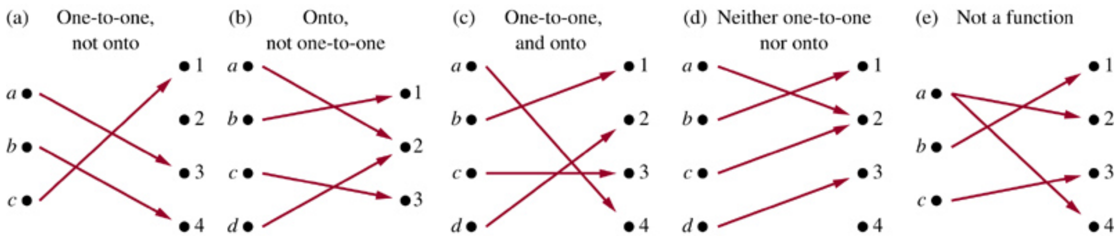
Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.



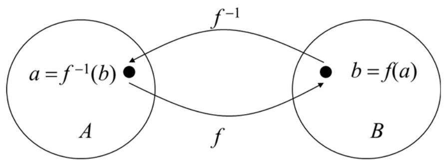
Monotonic Functions

A monotonic 单调 function f is

- ➢ either *monotonically* (*strictly*) *increasing* $\forall x \forall y (x < y \to f(x) < f(y))$
- ➢ or *monotonically* (*strictly*) *decreasing* $\forall x \forall y (x < y \to f(x) > f(y))$

Inverse Functions

Let f be a bijection from A to B. Then the *inverse function* of f, denoted as f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$



Note:

- ➢ No inverse function exists unless f is a bijection.
- ➢ Function f is invertible iff f is bijective

Compositions of Functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted by $f \circ g$, is defined by: $f \circ g(a) = f(g(a))$

Note:

- ➢ $f \circ g$ can't be defined unless the range of g is a subset of the domain of f.

The Floor Functions

【Definition 】The *floor function* $f(x)$ is the largest integer less than or equal to the real number x.

Notation: $\lfloor x \rfloor$

The Ceiling Functions

【Definition 】The *ceiling function* $f(x)$ is the smallest integer greater than or equal to the real number x.

Notation: $\lceil x \rceil$

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

2.4 Sequence and Summations

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

2.5 Cardinality of Sets

The cardinality of an infinite set

【Definition 1】The cardinality of a set A is equal to the cardinality of a set B, denoted $|A| = |B|$, iff there exists a bijection from A to B.

【Definition 2】If there is an injection from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

【Definition】A set that is either finite or has the same cardinality as the set of positive integers called *countable*.

A set that is not countable is called *uncountable*.

When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (*aleph null*).

If $|A| = |Z^+|$, the set A is *countable infinite*.

- An infinite set is countable iff it is possible to list all the elements of the set in a sequence

The properties of the countable sets:

- No infinite set has a smaller cardinality than a countable set.
- The union of two countable sets is countable.
- The union of finite number of countable sets is countable.
- The union of a countable number of countable sets is countable.

3. Cantor Diagonalization Argument

【Theorem】The set of real numbers between 0 and 1 is uncountable.

【Theorem】The set of real numbers $R = (-\infty, +\infty)$ has the same cardinality as the set $(0,1)$.

Computability

We say that a function is computable if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is uncomputable.

【Schrőder-Bernstein Theorem】If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one to –one correspondence between a A and B.

【Theorem】The cardinality of the power set of an arbitrary set has a greater cardinality than the original arbitrary set.

The Continuum Hypothesis: The continuum hypothesis (CH) asserts that there is no cardinal number a such that $\aleph_0 < a < \aleph_1$.