Theory of Computation, Fall 2021
Assignment 5 Solutions

Exercises

Q1. Suppose, for the sake of contradiction, that A is context-free. Let p be the pumping length given by the pumping theorem. Consider $w = a^p b^p c a^p b^p \in A$. By pumping theorem, w can be written as $w = uvxyz$ such that

(i) $uv^i xy^i z \in A$ for any $i \geq 0$,

(ii) $|v| + |y| > 0$, and

(iii) $|vxy| \leq p$.

There are three cases, and we shall see that each case leads to a contradiction. Therefore, A cannot be context-free.

- Case 1: one of v and y contains c. Then $uv^0 xy^0 z$ does not contain any c, so it cannot belongs to A. This contradicts with (i).
- Case 2: v and y are on the same side of c. Then in $uv^0 xy^0 z$, we have different number of symbols on two sides of c. As a result, $uv^0 xy^0 z \notin A$, contradicting with (i).
- Case 3: v and y are on the left and right side of c, respectively. Since $|vxy| \leq p$, it must be that $v = b^j$ for some j and $y = a^k$ for some k. $|v| + |y| > 0$ implies $j + k > 0$. We have that $uv^0 xy^0 z = a^p b^{p-j} c a^{p-k} b^p$ cannot belong to A. This contradicts with (i).

Q2. Let $P_A = (K_A, \Sigma, \Gamma_A, \Delta_A, s_A, F_A)$ be PFA accepting A. Let $M_B = (K_B, \Sigma, \Delta_B, s_B, F_B)$ be a NFA accepting B. We claim that the following PDA $P_\cap = (K_\cap, \Sigma, \Gamma_\cap, \Delta_\cap, s_\cap, F_\cap)$ accepts $A \cap B$. Therefore, $A \cap B$ is context-free.

- $K_\cap = K_A \times K_B$
- $\Gamma_\cap = \Gamma_A$
- $s_\cap = (s_A, s_B)$
- $F_\cap = F_A \times F_B$
- $\Delta_\cap = \{((p_A, p_B), a, \alpha), ((q_A, q_B), \beta) : ((p_A, a, \alpha), (q_A, \beta)) \in \Delta_A \wedge (p_B, a, q_B) \in \Delta_B\}$.

Q3. (a) Suppose that A is context-free. By the conclusion of Q2, we have that $A \cap a^* b^* c^* = \{a^n b^n c^n : n \geq 0\}$ is also context-free. But we have proved that $\{a^n b^n c^n : n \geq 0\}$ is not context-free. Therefore, A cannot be context-free.

(b) $\overline{A}$ can be written as a union of three languages.

$$\overline{A} = \{a^i b^j c^k : i \neq j\} \cup \{a^i b^j c^k : j \neq k\} \cup \{a^i b^j c^k : i \neq k\}$$

It is easy to show that each of these three languages is context-free. Since the class of context-free languages is closed under union, $\overline{A}$ is context-free.

Q4. $M_\rightarrow = (\{s, h\}, \Sigma, \delta, s, \{h\})$ where $\delta(s, a) = (h, \rightarrow)$ for any $a \in \Sigma$.

Q5. $L_\sqcup$ does not halt. It keeps moving within the first two squares of the tape.

Q6. The Turing machine is shown as follows.

$$
\begin{tikzcd}
{>}\, L \arrow[r, "a \neq \sqcup"] \arrow[d, "\sqcup"] & \sqcup R a L_{\sqcup} \arrow[l, bend right] \\
R R_{\sqcup}
\end{tikzcd}
$$