1.解：(1) $y<0$ 时，因为 $0\le x\le 1$，$\operatorname{sgn}(x-y)=1$

$$F(y)=\int_0^1 1\,dx=1$$

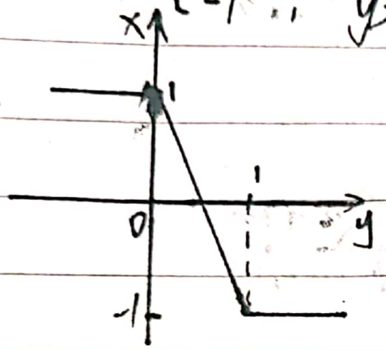
(2) $y>1$ 时 $\operatorname{sgn}(x-y)=-1$ $\quad F(y)=\int_0^1(-1)\,dx=-1$

(3) $0\le y\le 1$ 时 $F(y)=\int_0^y(-1)\,dx+\int_y^1 1\,dx=1-2y$.

$$F(y)=\begin{cases}1, & y<0\\ 1-2y, & 0\le y\le 1\\ -1, & y>1\end{cases}$$

显然，$F(y)$ 在 $(-\infty,+\infty)$ 上连续；

图像：



2.解：令 $f(x,\alpha)=\lim\limits_{\alpha\to 0}\int_0^2 x^2\cos\alpha x\,dx$ ~~$=\int_0^2\lim\limits_{\alpha\to 0}x^2\cos\alpha x=\int_0^2 x^2dx=\frac{8}{3}$~~

$f(x,\alpha)$ 在 $[0,2]\times U(0)$ 上连续，由

$$f(x,\alpha)=\int_0^2\lim_{\alpha\to 0}x^2\cos\alpha x\,dx=\int_0^2 x^2dx=\frac{1}{3}x^3\Big|_0^2=\frac{8}{3}$$

3.解：~~余斜~~ 令 $f(x,y)=e^{-xy^2}$，则 $f_x(x,y)=-y^2e^{-xy^2}$

两者在 R^2 上连续，

$$F'(x)=\int_x^{x^2}f_x(x,y)\,dy+f(x,x^2)(x^2)'-f(x,x)(x)'$$

$$=-\int_x^{x^2}y^2e^{-xy^2}dy+2xe^{-x^5}-e^{-x^3}.$$

4.(2) 对于 $I(\alpha)=\int_0^{\pi}\ln(1-\alpha\cos x)dx$.

$$I'(\alpha)=\int_0^{\pi}\frac{-\cos x}{1-\alpha\cos x}dx=\frac{1}{\alpha}\int_0^{\pi}\left(1-\frac{1}{1-\alpha\cos x}\right)dx$$

$$=\frac{\pi}{\alpha}-\frac{1}{\alpha}\int_0^{\pi}\frac{1}{1-\alpha\cos x}dx$$

$$=\frac{\pi}{\alpha}-\frac{1}{\alpha}\left[\frac{2}{\sqrt{1-\alpha^2}}\arctan\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\tan\frac{x}{2}\right)\right]\Bigg|_0^{\pi}$$

$$=\frac{\pi}{\alpha}-\frac{\pi}{\alpha\sqrt{1-\alpha^2}}$$

$$I(\alpha)=\int_0^{\alpha}I'(\alpha)d\alpha=\pi\left[\ln|\alpha|+\ln\frac{1+\sqrt{1-\alpha^2}}{|\alpha|}\right]\Bigg|_0^{\alpha}$$

$$=\pi\ln\cancel{\frac{1+\sqrt{1+\alpha^2}}{2}}\ \frac{1+\sqrt{1-\alpha^2}}{2}$$

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)dx=\int_0^{\pi}\ln\left[(1+a^2)\cdot\left(1-\frac{2a}{1+a^2}\cos x\right)\right]dx$$

$$=\pi\ln(1+a^2)+\int_0^{\pi}\ln\left[1-\frac{2a}{1+a^2}\cos x\right]dx$$

$$=\pi\ln(1+a^2)+\pi\ln\left\{\frac{1}{2}\left[1+\sqrt{1-\left(\frac{2a}{1+a^2}\right)^2}\right]\right\}$$

$$=\pi\ln\left[\frac{1}{2}(1+a^2)\left(1+\frac{|1-a|}{1+a^2}\right)\right]$$

$$=\begin{cases}0, & |a|\le 1\\ \pi\ln a^2, & |a|>1.\end{cases}$$

5、(1) 解 $\because \int_a^b x^y dy = \frac{x^b - x^a}{\ln x}$

$$I = \int_0^1 \sin(\ln\frac{1}{x}) \frac{x^b - x^a}{\ln x} dx = \int_0^1 dx \int_a^b \sin(\ln\frac{1}{x}) \cdot x^y dy.$$

$|\sin(\ln\frac{1}{x})| \le 1$, $\lim_{x \to 0^+} x^y = 0$, $\lim_{x \to 0^+} \sin(\ln\frac{1}{x}) \cdot x^y = 0$

$f(x,y) = x^y \sin(\ln\frac{1}{x})$ 可视为 $[0,1] \times [a,b]$ 上的连续函数

$$I = \int_a^b dy \int_0^1 \sin(\ln\frac{1}{x}) \cdot x^y dx$$

$$J = \int_0^1 \sin(\ln\frac{1}{x}) \cdot x^y dx = \int_0^1 \sin(\ln\frac{1}{x}) d(\frac{1}{y+1} x^{y+1}).$$

$$= [\frac{1}{y+1} x^{y+1} \sin(\ln\frac{1}{x})]\Big|_0^1 + \int_0^1 \frac{1}{y+1} x^y \cos(\ln\frac{1}{x}) dx$$

$$= \frac{1}{y+1} \int_0^1 \cos(\ln\frac{1}{x}) d(\frac{1}{y+1} x^{y+1})$$

$$= [\frac{1}{(y+1)^2} x^{y+1} \cos(\ln\frac{1}{x})]\Big|_0^1 - \int_0^1 \frac{1}{(y+1)^2} \sin(\ln\frac{1}{x}) \cdot x^y dx$$

$$= \frac{1}{(y+1)^2} - \frac{1}{(y+1)^2} J \qquad J = \frac{1}{1+(y+1)^2}$$

$$I = \int_a^b \frac{1}{1+(y+1)^2} dy = \arctan(b+1) - \arctan(a+1)$$

9.解：$F_x=\int_{\frac{x}{y}}^{xy} f(z)dz+(x-y\cdot xy)f(xy)(xy)'_x$

$-(x-y\cdot\frac{x}{y})f(\frac{x}{y})(\frac{x}{y})'_x$

$=\int_{\frac{x}{y}}^{xy} f(z)dz+y(x-xy^2)f(xy)$

$F_{xy}=xf(xy)+\frac{x}{y^2}f(\frac{x}{y})+(x-3xy^2)f(xy)+y(x-xy^2)f'(xy)\cdot x$

$=(2x-3xy^2)f(xy)+\frac{x}{y^2}f(\frac{x}{y})+(x^2y-x^2y^3)f'(xy).$

~~1.(1)~~

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1.(1)证明：对$\forall y\in(-\infty,+\infty)$，有

$$\left|\frac{y^2-x^2}{(x^2+y^2)^2}\right|<\frac{1}{x^2+y^2}\le\frac{1}{x^2}$$

广义积分 $\int_1^{+\infty}\frac{1}{x^2}dx=(-\frac{1}{x})\Big|_1^{+\infty}=1$

由M判别法知 $\int_1^{+\infty}\frac{y^2-x^2}{(x^2+y^2)^2}dx$ 在$(-\infty,+\infty)$上一致收敛.

(4)解：显然 $y=0$ 是瑕点。 $\int_0^1 \ln(xy)dy=\int_0^{\frac{1}{b}}\ln(xy)dy+\int_{\frac{1}{b}}^1\ln(xy)dy$.

1) 含参量正常积分 $\int_{\frac{1}{b}}^1\ln(xy)dy$.

$$\int_{\frac{1}{b}}^1\ln(xy)dy=\int_{\frac{1}{b}}^1\ln x dy+\int_{\frac{1}{b}}^1\ln y dy$$

$$=(1-\frac{1}{b})\ln x+(y\ln y-1)\Big|_{\frac{1}{b}}^1$$

$$=(1-\frac{1}{b})\ln x+(\frac{1}{b}\ln b+\frac{1}{b}-1)$$

2) 含参量反常积分 $(\int_0^{\frac{1}{b}}\ln(xy)dy)$.

$\because 0\le y\le\frac{1}{b}$, $\frac{1}{b}\le x\le b$, $\therefore$ $\frac{1}{b}y\le xy\le by$.

~~ln xy~~ $\ln\frac{y}{b}\le\ln(xy)\le\ln(by)\le 0$.

$|\ln(xy)|\le|\ln\frac{y}{b}|=|\ln y-\ln b|=\ln b-\ln y$

$$\int_0^{\frac{1}{b}}(\ln b-\ln y)dy=\frac{1}{b}\ln b-\int_0^{\frac{1}{b}}\ln y dy=\frac{1}{b}\ln b-\lim_{\varepsilon\to0^+}\int_\varepsilon^{\frac{1}{b}}\ln y dy$$

$$=\frac{1}{b}\ln b-\lim_{\varepsilon\to0^+}[y\ln y-y]\Big|_\varepsilon^{\frac{1}{b}}$$

$$=\frac{1}{b}\ln b-\lim_{\varepsilon\to0^+}[-\frac{1}{b}\ln b-\frac{1}{b}-\varepsilon\ln\varepsilon-\varepsilon]$$

$$=\frac{2}{b}\ln b+\frac{1}{b}$$

由M判别法 $\int_0^1\ln(xy)dy$ 在 $[\frac{1}{b},b]$ $(b>1)$ 上一致收敛.

3.证明. 令 $t=x-y$

$$F(y)=\int_0^{+\infty} e^{-(x-y)^2}dx=\int_{-y}^{+\infty} e^{-t^2}dt$$

$$=\int_{-y}^{0} e^{-t^2}dt+\int_0^{+\infty} e^{-t^2}dt$$

$$=\frac{\sqrt{\pi}}{2}+\int_{-y}^{0} e^{-t^2}dt$$

由于 e^{-t^2} 在 $(-\infty,+\infty)$ 上连续，由变下限的定积的性质，对 $\forall y\in(-\infty,+\infty)$

$\int_{-y}^{0} e^{-t^2}dt$ 在 y 处连续，从而 $F(y)=\int_0^{+\infty} e^{-(x-y)^2}dx$ 在 $(-\infty,+\infty)$ 上连续.

4.(2) 解. $I(x)=\int_0^{+\infty} e^{-t}\frac{\sin xt}{t}dt$，$f(x,t)=e^{-t}\cdot\frac{\sin xt}{t}$，

由于 $\lim\limits_{t\to+\infty} t^2\cdot|f(x,t)|=\lim\limits_{t\to+\infty}\frac{t|\sin xt|}{e^t}=0$

$\therefore$ 广义积分在 $(-\infty,+\infty)$ 收敛，又 $\lim\limits_{t\to0^+} e^{-t}\frac{\sin xt}{t}=x$

所以 $f(x,t)$ 在 $(-\infty,+\infty)\times[0,+\infty)$ 上连续，而

$$f_x(x,t)=|e^{-t}\cos xt|\le e^{-t}$$

$$\int_0^{+\infty} e^{-t}dt=-e^{-t}\Big|_0^{+\infty}=1$$

由 M 判别法 $\int_0^{+\infty} f_x(x,t)dt=\int_0^{+\infty} e^{-t}\cos xt\,dt$ 在 $(-\infty,+\infty)$ 一致收敛.

$$I'(x)=\int_0^{+\infty} f_x(x,t)\,dt=\int_0^{+\infty} e^{-t}\cos xt\,dt$$

$$=\lim_{b\to+\infty}\int_0^{b} e^{-t}\cos xt\,dt=\lim_{b\to+\infty}\left[\frac{e^{-t}}{1+x^2}(x\sin xt-\cos xt)\right]\Big|_0^{b}$$

$$=\frac{1}{1+x^2}$$

$$I(x)=\int_0^{+\infty} e^{-t}\frac{\sin xt}{t}dt=\int_0^{x} I'(x)dx=\int_0^{x}\frac{dt}{1+t^2}=\arctan x.$$

2、解：$\int_0^{\frac{\pi}{2}} \sin^{2n} u\,du = \frac{1}{2}B\left(\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$

$$= \frac{\sqrt{\pi}\cdot(2n-1)!!\sqrt{\pi}}{2\cdot n!\,2^n} = \frac{\pi(2n-1)!!}{n!\cdot 2^{n+1}}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} u\,du = \frac{1}{2}B\left(\frac{1}{2}, n+1\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{\sqrt{\pi}\,n!\,2^{n+1}}{2(2n+1)!!\sqrt{\pi}} = \frac{n!\,2^n}{(2n+1)!!}$$

3、(1) 令 $t=\ln\frac{1}{x}$，$x=e^{-t}$，$dx=-e^{-t}dt$，则

$$\int_0^1 \left(\ln\frac{1}{x}\right)^{a-1}dx = -\int_{+\infty}^0 t^{a-1}e^{-t}dt = \int_0^{+\infty} t^{a-1}e^{-t}dt = \Gamma(a)$$

(4) 令 $\frac{1}{t}=x^4+1$，$4x^3dx=-\frac{1}{t^2}dt$，则

$$\int_0^{+\infty}\frac{1}{1+x^4}dx = \int_1^0 \frac{1}{4}t\left(\frac{1}{t}-1\right)^{-\frac{3}{4}}\left(-\frac{1}{t^2}\right)dt = \frac{1}{4}\int_0^1 t^{-\frac{1}{4}}(1-t)^{-\frac{3}{4}}dt$$

$$= \frac{1}{4}B\left(\frac{3}{4},\frac{1}{4}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}\pi}{4}$$

2、证明：当 $0\le x\le 1$ 时，有

$$u(x)=\int_0^1 k(x,y)v(y)\,dy=\int_0^x y(1-x)v(y)\,dy+\int_x^1 x(1-y)v(y)\,dy$$

$$u'(x)=x(1-x)v(x)-\int_0^x y\,v(y)\,dy+\int_x^1(1-y)v(y)\,dy-x(1-x)v(x)$$

$$=-\int_0^x y\,v(y)\,dy+\int_x^1(1-x)v(y)\,dy$$

$$u''(x)=-x\,v(x)-(1-x)v(x)=-v(x)$$