1.(3) 解：$L:\begin{cases} x=\cos\theta, \\ y=\sin\theta; \end{cases} 0\le\theta\le 2\pi.$

原式 $= \oint \cos\theta\sin\theta\cos\theta d\theta\ \cancel{\sin\theta d\theta} + \sin\theta\cos\theta d\theta = 2\oint \sin\theta d\sin\theta = (\sin^2\theta)\Big|_0^{2\pi} = 0$

(4) 解：记 $A(\pi,0)$. $L=\overset{\frown}{OA}+AO$, $\overset{\frown}{OA}: y=\sin x (0\le x\le\pi)$. $AO: y=0$, $x\in[0,\pi]$

原式 $= \int_{\overset{\frown}{OA}} \sin x dx + \sin x d\sin x + \int_{AO} 0 dx$

$= \int_0^{\pi} (\sin x + \sin x\cos x) dx = [-\cos x + \frac{1}{2}\sin^2 x]\Big|_0^{\pi}$

$= 2.$

4. 证明：设 $L:\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ $\alpha\le t\le\beta$ $\because P=P(x(t),y(t))$, $Q=Q(x(t),y(t))$

由柯西-施瓦兹不等式 $|Px'(t)+Qy'(t)|\le\sqrt{P^2+Q^2}\sqrt{x'^2(t)+y'^2(t)}$

$|\int_{AB} Pdx+Qdy| = |\int_\alpha^\beta [P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)]\cdot dt|$

$\le \int_\alpha^\beta |Px'(t)+Qy'(t)|dt \le \int_\alpha^\beta \sqrt{Q^2+P^2}\sqrt{x'^2(t)+y'^2(t)}dt$

$\le M\int_\alpha^\beta \sqrt{x'^2(t)+y'^2(t)}dt = ML.$ 即 $|\int_{AB} Pdx+Qdy|\le ML$

对于 $I_R = \int_{x^2+y^2=R^2} \frac{ydx-xdy}{(x^2+xy+y^2)^2}$, $P=\frac{y}{(x^2+xy+y^2)^2}$, $Q=\frac{-x}{(x^2+xy+y^2)^2}$

对于 $L: x^2+y^2=R^2$. 故 $P^2+Q^2 = \frac{R^2}{(R^2+xy)^4}$

$\therefore M=\max\limits_{(x,y)\in L}\sqrt{P^2+Q^2}=\frac{R}{(R^2+xy)^2}$ 只需要求 xy 的极值，可得 M.

$\therefore$ 令 $L(x,y,\lambda)=xy+\lambda(x^2+y^2-R^2)$. $\begin{cases} L_x=y+2\lambda x=0 \\ L_y=x+2\lambda y=0 \\ L_\lambda=x^2+y^2-R^2=0 \end{cases}$

可解得稳定点 $(\pm\frac{\sqrt{2}}{2}R,\pm\frac{\sqrt{2}}{2}R)$，$(\pm\frac{\sqrt{2}}{2}R,\mp\frac{\sqrt{2}}{2}R)$。

$L(\pm\frac{\sqrt{2}}{2}R,\pm\frac{\sqrt{2}}{2}R)=\frac{1}{2}R^2 \quad L(\pm\frac{\sqrt{2}}{2}R,\mp\frac{\sqrt{2}}{2}R)=-\frac{1}{2}R^2$

$\therefore M=\frac{R}{(R^2-\frac{1}{2}R^2)^2}=\frac{4}{R^3}$

$\therefore |I_R|\le|LM|=\frac{4}{R^3}\cdot 2\pi R=\frac{8\pi}{R^2}$

$\therefore \lim\limits_{R\to+\infty} I_R\le\lim\limits_{R\to+\infty}\frac{8\pi}{R^2}=0$

5.(2) 解：记 $A(1,0,0)$ $B(0,1,0)$ $C(0,0,1)$ 可记 $L=\overset{\frown}{AB}+\overset{\frown}{BC}+\overset{\frown}{CA}$

$\therefore \overset{\frown}{AB}:\begin{cases} x=\cos\theta \\ y=\sin\theta \quad 0\le\theta\le\frac{\pi}{2} \\ z=0 \end{cases}$, $\overset{\frown}{BC}:\begin{cases} x=0 \\ y=\cos\theta,\ 0\le\theta\le\frac{\pi}{2} \\ z=\sin\theta \end{cases}$, $\overset{\frown}{CA}:\begin{cases} x=\sin\theta \\ y=0,\ 0\le\theta\le\frac{\pi}{2} \\ z=\cos\theta \end{cases}$

由~~方程式~~图形的对称性 $\int_L(y^2-z^2)dx=\int_L(z^2-x^2)dy=\int_L(x^2-y^2)dz$

原式 $=3\int_L(y^2-z^2)dx=3\left[\int_{\overset{\frown}{AB}}(y^2-z^2)dx+\int_{\overset{\frown}{BC}}(y^2-z^2)dx+\int_{\overset{\frown}{CA}}(y^2-z^2)dx\right]$

$=3\left[\int_0^{\frac{\pi}{2}}\sin^2\theta(-\sin\theta)d\theta+0+\int_0^{\frac{\pi}{2}}(-\cos^2\theta)\cos\theta d\theta\right]$

$=-6\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta=-6\times(\frac{2}{3}\times1)=-4.$

P262

1.(1) 解: $\cancel{\iint_\Sigma (x+y+z)dS}$ $z=\sqrt{a^2-x^2-y^2}$ $\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{a^2-x^2-y^2}}$ $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{a^2-x^2-y^2}}$

$$\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}=\frac{a}{\sqrt{a^2-x^2-y^2}}.$$

$$\iint_\Sigma (x+y+z)dS = a\iint_{x^2+y^2\le a^2}\frac{x+y+\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}}dxdy$$

$$=a\iint_{x^2+y^2\le a^2}\frac{x}{\sqrt{a^2-x^2-y^2}}dxdy+a\iint_{x^2+y^2\le a^2}\frac{y}{\sqrt{a^2-x^2-y^2}}dxdy+\pi a^3$$

由 $x^2+y^2\le a^2$ 区域的对称性 与奇函数 $\iint_{x^2+y^2\le a^2}\frac{x}{\sqrt{a^2-x^2-y^2}}dxdy=\iint_{x^2+y^2\le a^2}\frac{y}{\sqrt{a^2-x^2-y^2}}=0$

∴原式 $=\pi a^3$

(4) $z=1-x-y$ $\frac{\partial z}{\partial x}=-1$ $\frac{\partial z}{\partial y}=-1$ $\sqrt{1+z_x^2+z_y^2}=\sqrt{3}$.

xy区域 $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1-x\}$.

$$\iint_\Sigma xyz\,dS=\sqrt{3}\iint_{\substack{x+y\le 1\\ x\ge 0, y\ge 0}} xy(1-x-y)dxdy$$

$$=\sqrt{3}\int_0^1 dx\int_0^{1-x}(xy-x^2y-xy^2)dy$$

$$=\sqrt{3}\int_0^1\frac{1}{6}x(1-x)^3dx$$

$$=\frac{\sqrt{3}}{120}.$$

4、解： $x_r=\cos\varphi\sin\theta$ ， $x_\varphi=-r\sin\varphi\sin\theta$

$y_r=\sin\varphi\sin\theta$ ， $y_\varphi=r\cos\varphi\sin\theta$ ：

$z_r=\cos\theta$ ， $z_\varphi=0$.

$E=x_r^2+y_r^2+z_r^2=1$ $G=x_\varphi^2$. $F=x_rx_\varphi+y_ry_\varphi+z_rz_\varphi=0$

$\sqrt{EG-F^2}=r\sin\theta$

$$\therefore \iint_S z^2 dS=\iint_D r^2\cos^2\theta\sqrt{EG-F^2}\,drd\varphi=\iint_D r^3\sin\theta\cos^2\theta\,drd\varphi.$$

$$=\frac{\pi}{2}a^4\sin\theta\cos^2\theta$$

P269

1、(1) 解 分别记S的各面投影为 $S_1: x=0$. $S_2: x=a$ $S_3: y=0$, $S_4: y=a$, $S_5: z=0$ $S_6: z=a$

$$\iint_S y(x-z)dydz+x^2dzdx+(y^2+xz)dxdy$$ ：

$$=\iint_{S_1} yz\,dydz+\iint_{D_2} y(a-z)dydz-\iint_{D_3} x^2dzdx+\iint_{D_4} x^2dzdx-\iint_{D_5} y^2dxdy$$

$$+\iint_{D_6}(y^2+ax)dxdy$$

$$=\iint_{D_2} ay\,dydz+\iint_{D_6} ax\,dxdy=a\int_0^a ydy\int_0^a dz+a\int_0^a xdx\int_0^a dy$$

$$=a^4$$

(4) 将上半部分再分为左右两部分 $S_1: y=\sqrt{1-x^2-z^2}, z\geq 0$ (右侧球面),

$S_2: y=-\sqrt{1-x^2-z^2}, z\geq 0$ (左侧球面).

$$\iint_S yz\,dz\,dx = \iint_{S_1} yz\,dz\,dx + \iint_{S_2} yz\,dz\,dx$$

$$= \iint_{\substack{x^2+z^2\leq 1\\ z\geq 0}} z\sqrt{1-x^2-z^2}\,dz\,dx - \iint_{\substack{x^2+z^2\leq 1\\ z\geq 0}} z(-\sqrt{1-x^2-z^2})\,dz\,dx$$

$$= 2\iint_{\substack{x^2+z^2\leq 1\\ z\geq 0}} z\sqrt{1-x^2-z^2}\,dz\,dx = 2\int_0^{\pi} d\theta \int_0^1 r^2\sin\theta\sqrt{1-r^2}\,dr$$

$$= 4\int_0^1 r^2\sqrt{1-r^2}\,dr \xlongequal{r=\sin t} 4\int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt$$

$$= 4\int_0^{\frac{\pi}{2}} \sin^2 t\,dt \quad 4\int_0^{\frac{\pi}{2}} \sin^4 dt$$

$$= \frac{\pi}{4}$$

2. 解. 流量 $Q=\iint_S k\,dy\,dz + y\,dz\,dx$

对于 $\iint_S k\,dy\,dz$, 由于K为常数, 前后侧投影等大反向, $\iint_S k\,dy\,dz=0$

$$E=\iint_S y\,dz\,dx = \iint_{D_{zx}} \sqrt{4-x^2-z^2}\,dz\,dx - \iint_{D_{zx}} (-\sqrt{4-x^2-z^2})\,dz\,dx$$

$$= 2\iint_{x^2+z^2\leq 4} \sqrt{4-x^2-z^2}\,dz\,dx$$

$$= 2\int_0^{2\pi} d\theta \int_0^2 r\sqrt{4-r^2}\,dr$$

$$= \frac{32}{3}\pi$$

3解：对于 $\iint_S f(x)dydz$，仅前后面投影 S_1, $x=a$ S_2, $x=0$ 需计算，其余四面积分为0

$$\iint_S f(x)dydz = \iint_{D_{yz}} f(a)dydz - \iint_{D_{yz}} f(0)dydz = bc[f(a)-f(0)].$$

由对称性同理

$$\iint_S g(y)dzdx = ac[g(b)-g(0)]$$

$$\iint_S h(z)dxdy = ab[h(c)-h(0)].$$

$$\therefore I = bc[f(a)-f(0)]+ac[g(b)-g(0)]+ab[h(c)-h(0)]$$

P216 1.(1) 可将D表示为X型区域

$$D=\{(x,y)\mid \tfrac{1}{2}(x+1)\le y\le 4x-3,\ 1\le x\le 2\}\cup\{(x,y)\mid \tfrac{1}{2}(x+1)\le y\le 11-3x,\ 2\le x\le 3\}$$

$$\frac{\partial Q}{\partial x}=-2x \quad \frac{\partial P}{\partial y}=2(x+y)$$

$$\therefore \oint_L (x+y)^2dx-(x^2+y^2)dy = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma = -\iint_D (4x+2y)d\sigma$$

$$= -\int_1^2 dx\int_{\frac{1}{2}(x+1)}^{4x-3}(4x+2y)dy - \int_2^3 dx\int_{\frac{1}{2}(x+1)}^{11-3x}(4x+2y)dy$$

$$-\frac{21}{4}x^2$$

$$= -\int_1^2\left(\frac{119}{4}x^2-\frac{71}{2}x+\frac{35}{4}\right)dx - \int_2^3\left(\cancel{-\frac{21}{4}x^2}-\frac{49}{2}x+\frac{483}{4}\right)dx$$

$$= -46\frac{2}{3}.$$