1.(2) $\frac{\partial z}{\partial x} = e^x\cos y + \sin y(x+1)e^x = e^x[\cos y + (1+x)\sin y]$.

$\frac{\partial z}{\partial y} = e^x(-\sin y + x\cos y)$

$\frac{\partial^2 z}{\partial x^2} = e^x[\cos y + (x+2)\sin y]$ $\quad \frac{\partial^2 z}{\partial y^2} = e^x(-\cos y - x\sin y)$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x} = e^x[-\sin y + (1+x)\cos y]$$

(4) $\frac{\partial^{p+q+r} u}{\partial x^p \partial y^q \partial z^r} = (x+p)(y+q)(z+r)e^{x+y+z}$.

(7) 令 $s = x+y$, $t = xy$, $v = \frac{x}{y}$ $\quad z = f(s,t,v)$.

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = f_1' + yf_2' + \frac{1}{y}f_3'$$

$$z_{xx} = \frac{\partial}{\partial x}(f_1' + yf_2' + \frac{1}{y}f_3') = \cancel{\frac{\partial f_1'}{\partial s}\frac{\partial s}{\partial x} + y\frac{\partial f_2'}{\partial t}\frac{\partial t}{\partial x} + \frac{1}{y}\frac{\partial f_3'}{\partial v}\frac{\partial v}{\partial x}}$$

$$= \frac{\partial f_1'}{\partial x} + y\frac{\partial f_2'}{\partial x} + \frac{1}{y}\frac{\partial f_3'}{\partial x} = f_{11}'' + yf_{12}'' + \frac{1}{y}f_{13}'' + y(f_{21}'' + yf_{22}'' + \frac{1}{y}f_{23}'')$$

$$+ \frac{1}{y}(f_{31}'' + yf_{32}'' + \frac{1}{y}f_{33}'')$$

$$= f_{11}'' + y^2 f_{22}'' + \frac{1}{y^2}f_{33}'' + 2yf_{12}'' + \frac{2}{y}f_{13}'' + 2f_{23}''$$

$$z_{xy} = \frac{\partial}{\partial y}(f_1' + yf_2' + \frac{1}{y}f_3') = \frac{\partial f_1'}{\partial y} + f_2' + y\frac{\partial f_2'}{\partial y} - \frac{1}{y^2}f_3' + \frac{1}{y}\frac{\partial f_3'}{\partial y}$$

$$= f_{11}'' + xf_{12}'' - \frac{x}{y^2}f_{13}'' + f_2' + y(f_{21}'' + xf_{22}'' - \frac{x}{y^2}f_{23}'') - \frac{1}{y^2}f_3' + \frac{1}{y}(f_{31}'' + xf_{32}'' - \frac{x}{y^2}f_{33}'')$$

$$= f_{11}'' + xyf_{22}'' - \frac{x}{y^3}f_{33}'' + (x+y)f_{12}'' + (\frac{1}{y} - \frac{x}{y^2})f_{13}'' + f_2' - \frac{1}{y^2}f_3'.$$

2、$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$，$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\sin\theta$

$$\frac{\partial^2 u}{\partial r^2} = \cancel{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial x}{\partial r}} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)$$

$$= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial x\partial y}2\sin\theta\cdot\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2} r^2\cos^2\theta - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} - r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

4.证明. $\frac{\partial v}{\partial x} = -\frac{1}{r^2}g\frac{\partial r}{\partial x} + \frac{1}{r}g'\cdot\left(-\frac{1}{c}\right)\frac{\partial r}{\partial x} = -\frac{x}{r^3}g - \frac{1}{c}\frac{x}{r^2}g'$

$$\frac{\partial^2 v}{\partial x^2} = \frac{r^3 - x\cdot 3r^2\cdot\frac{x}{r}}{r^6}g - \frac{x}{r^3}g'\cdot\left(-\frac{1}{c}\right)\frac{x}{r} - \frac{1}{c}\frac{r^2 - x\cdot 2r\cdot\frac{x}{r}}{r^4}g' - \frac{1}{c}\frac{x}{r^2}g''\cdot\left(-\frac{1}{c}\right)\frac{x}{r}$$

$$= \frac{3x^2 - r^2}{r^5}g + \frac{3x^3 - r^2}{cr^4}g' + \frac{x^2}{c^2r^3}g''$$

同理 $\frac{\partial^2 v}{\partial y^2} = \frac{3y^2 - r^2}{r^5}g + \frac{3y^3 - r^2}{cr^4}g' + \frac{y^2}{c^2r^3}g''$

$$\frac{\partial^2 v}{\partial z^2} = \frac{3z^2 - r^2}{r^5}g + \frac{3z^3 - r^2}{cr^4}g' + \frac{y^2}{c^2r^3}g''$$

$$v_{xx} + v_{yy} + v_{zz} = \frac{1}{c^2 r}g'' = \frac{1}{c^2}v_{tt}$$

…只有稳定点；$f:\mathbf{R}^n\to\mathbf{R}$，且 $f(x)=\varphi(a\cdot x)$ …

解 $f_x(0,0)=\lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to0}\dfrac{x^3}{x^3}=1$

同 $f_y(0,0)=\lim\limits_{y\to0}\dfrac{f(y,0)-f(0,0)}{y-0}=\lim\limits_{y\to0}\left(\dfrac{-y^3}{y^3}\right)=-1$

$$\lim_{\rho\to0}\frac{[\Delta f-f_x(0,0)-f_y(0,0)y]}{\rho}=\lim_{(x,y)\to(0,0)}\frac{x^2y-xy^2}{(x^2+y^2)^{\frac32}}$$

令 $y=x$ $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y-xy^2}{(x^2+y^2)^{\frac32}}=0$ $y=2x$ $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y-xy^2}{(x^2+y^2)^{\frac32}}=-\dfrac{2}{5\sqrt5}$

故极限不存在，即 $f(x,y)$ 在 $(0,0)$ 不可微

2.证明：(1) $$\frac{\partial u}{\partial x_k}=\sum_{i=1}^{n-1}\begin{vmatrix}1 & 1 & \cdots & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_k & \cdots & x_n\\ \vdots & \vdots & & \vdots & & \vdots\\ 0 & 0 & \cdots & ix_k^{i-1} & \cdots & 0\\ \vdots & \vdots & & \vdots & & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_k^{n-1} & \cdots & x_n^{n-1}\end{vmatrix}$$

$$\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}=\sum_{k=1}^{n}\sum_{i=1}^{n-1}\sum_{k=1}^{n}\begin{vmatrix}1 & 1 & 1\\ x_1 & x_k & x_n\\ 0 & ix_k^{i-1} & 0\\ x_1^{n-1} & x_k^{n-1} & x_n^{n-1}\end{vmatrix}=\sum_{i=1}^{n-1}\begin{vmatrix}1 & \cdots & 1 & \cdots & 1\\ x_1 & & x_k & & x_n\\ \vdots & & \vdots & & \vdots\\ x_1^{i-1} & & x_k^{i-1} & & x_n^{i-1}\\ ix_1^{i-1} & & ix_k^{i-1} & & ix_n^{i-1}\\ x_1^{i+1} & & x_k^{i+1} & & x_n^{i+1}\\ x_1^{n-1} & & x_k^{n-1} & & x_n^{n-1}\end{vmatrix}$$

$=0$（行列式中 i 与 $i+1$ 行成 i 倍关系，所有行列式为 0）

(2) $\sum_{k=1}^{n} x_k \frac{\partial u}{\partial x_k} = \sum_{i=1}^{n-1} \sum_{k=1}^{n} \begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \vdots & & & & \vdots \\ 0 & \cdots & i x_k^i & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ x_1^{n-1} & & x_k^{n-1} & & x_n^{n-1} \end{vmatrix} = \sum_{i=1}^{n-1} i \cdot u = \frac{n(n-1)}{2} u$

P298 4.(1) 令 $x_1 = \sin x$, $x_2 = \cos x$

$$(f\circ g)'(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)\begin{pmatrix} \frac{dx_1}{dx} \\ \frac{dx_2}{dx} \end{pmatrix} = (1, -1)\begin{pmatrix} \cos x \\ -\sin x \end{pmatrix} = \cos x + \sin x$$

(2) 令 $x = x_1 - x_2$

$$(g\circ f)'(x_1, x_2) = \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_2}{\partial x} \end{pmatrix}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = \begin{pmatrix} \cos x \\ -\sin x \end{pmatrix}(1, -1)$$

$$= \begin{bmatrix} \cos(x_1 - x_2) & -\cos(x_1 - x_2) \\ -\sin(x_1 - x_2) & \sin(x_1 - x_2) \end{bmatrix}$$

(3) 令 $u = h(y) = (y_1 y_2, y_2 - y_1)^T$,

$y = h(x_1, x_2) = (x_1 x_2, x_2 - x_1)^T$,

$$(h\circ h)'(x_1, x_2) = \begin{pmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} y_2 & y_1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_2 & x_1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_2^2 - 2x_1 x_2 & 2x_1 x_2 - x_1^2 \\ -x_2 - 1 & -x_1 + 1 \end{pmatrix}$$

(4) 令 $u = s(y) = (y_1^2, 2y_2, y_2+4)^T$

$y = h(x_1, x_2) = (x_1 x_2, x_2 - x_1)^T$

$$(s \circ h)'(x_1, x_2) = \begin{pmatrix} 2y_1 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2, & x_1 \\ -1, & 1 \end{pmatrix} = \begin{bmatrix} 2x_1x_2^2 & 2x_1^2x_2 \\ -2 & 2 \\ -1 & 1 \end{bmatrix}$$

(5) $u = t(y) = (y_1 y_2 y_3, y_1 + y_2 + y_3)^T$

$y = s(x_1, x_2) = (x_1^2, \cancel{\ } 2x_2, x_2+4)^T$

$$(t \circ s)'(x_1, x_2) = \begin{pmatrix} y_2y_3 & y_1y_3 & y_1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2x_1 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4x_1x_2^2 + 16x_1x_2 & 8x_1^2 + 4x_1^2x_2 \\ 2x_1 & 3 \end{pmatrix}$$

(6) $u = s(y) = (y_1^2, 2y_2, y_2+4)^T$

$y = t(x_2, x_2, x_3) = (x_1 x_2 x_3, x_1 + x_2 + x_3)^T$

$$(s \circ t)'(x_1, x_2, x_3) = \begin{pmatrix} 2y_1 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1x_2^2x_3^2 & 2x_1^2x_2x_3^2 & 2x_1^2x_2^2x_3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

5. 解: $(x,y)^T \mapsto (x,y,u)^T \mapsto (x,y,u,v)^T \mapsto w$.

$$w'(x,y)=\left(\frac{\partial w}{\partial x},\frac{\partial w}{\partial y},\frac{\partial w}{\partial u},\frac{\partial w}{\partial v}\right)\begin{pmatrix}\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial u}\\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial u}\\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial u}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial u}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y}\\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y}\\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\end{pmatrix}$$

$$=\left(\frac{\partial h}{\partial x},0,\frac{\partial h}{\partial u},\frac{\partial h}{\partial v}\right)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial u}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{pmatrix}$$

$$=\left(\frac{\partial h}{\partial x}+\frac{\partial f}{\partial x}\frac{\partial h}{\partial u}+\frac{\partial h}{\partial v}\left(\frac{\partial g}{\partial x}+\frac{\partial f}{\partial x}\frac{\partial g}{\partial u}\right),\ \frac{\partial f}{\partial y}\frac{\partial h}{\partial u}+\frac{\partial h}{\partial v}\left(\frac{\partial g}{\partial y}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial u}\right)\right)$$

P_{133} 7.(2). $f(x,y)=\frac{x}{y}$ $\quad f(1,1)=1$ $\quad f_x=\frac{1}{y}$ $f_x(1,1)=1$. $\quad f_y=-\frac{x}{y^2}$ $f_y(1,1)=-1$.

$f_{xx}=0$, $f_{xx}(1,1)=0$ $\quad f_{yy}=\frac{2x}{y^3}$ $f_{yy}(1,1)=2$. $\quad f_{xy}(1,1)=-1$

$f_{x^3}(1,1)=0$ $\quad f_{x^2y}(1,1)=0$ $\quad f_{xy^2}(1,1)=2$ $\quad f_{y^3}(1,1)=-6$

$f_{x^4}=0$ $\quad f_{x^2y^2}=0$ $\quad f_{xy^3}=-\frac{6}{y^4}$ $\quad f_{y^4}=\frac{24x}{y^5}$

$x_0=1,\ y_0=1,\ x=1+h\ \ y=1+k,\ n=3$

$$f(x,y)=1+h-h-hk+k^2+hk^2-k^3+\left[-\frac{hk^3}{(1+\theta k)^4}+\frac{1+\theta h}{(1+\theta k)^5}k^4\right]\quad(0<\theta<1)$$

(3) x, y 位置对称 $f_{x^i y^j}(0,0) = (-1)^{i+j-1}(i+j-1)!$

$x_0=0 \quad y_0=0 \quad x=h \quad y=k$

$$\ln(1+x+y) = \sum_{r=0}^{n} \frac{1}{r!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^r f(0,0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(\theta h, \theta k)$$

$$= \sum_{r=1}^{n} (-1)^{r-1}\frac{(x+y)^r}{r} + (-1)^n \frac{(x+y)^{n+1}}{(n+1)(1+\theta h+\theta y)^{n+1}} \quad (0<\theta<1)$$

8.(1) $\begin{cases} \frac{\partial z}{\partial x} = 3ay-3x^2=0 \\ \frac{\partial z}{\partial y} = 3ax-3y^2=0 \end{cases}$ $P_1(0,0)$ $P_2(a,a)$ 是其稳定点.

$Z_{xx}=-6x \quad Z_{xy}=3a \quad Z_{yy}=-6y$

1° $Z_{xx}(0,0)=0 \quad Z_{xx}(0,0)\cdot Z_{yy}(0,0)-Z_{xy}^2(0,0)=-9a^2<0$, $P_1(0,0)$ 非极值点

2° $Z_{xx}(a,a)=-6a<0 \quad Z_{xx}(a,a)\cdot Z_{yy}(a,a)-Z_{xy}^2(a,a)=27a^2>0$,

$Z=3axy-x^3-y^3$ 在 $P_2(a,a)$ 取极大值 a^3.

(2) $\begin{cases} Z_x = e^{2x}(1+2x+4y+2y^2)=0 \\ Z_y = e^x(2+2y)=0 \end{cases}$ $P_0(\frac{1}{2},-1)$ 为稳定点.

$Z_{xx}(\frac{1}{2},-1)=e^{2x}(4+4x+8y+4y^2)=2e>0 \quad Z_{xy}=e^{2x}(4+4y) \quad Z_{yy}=2e^{2x}$

$Z_{xx}(\frac{1}{2},-1)\,Z_{yy}(\frac{1}{2},-1)-Z_{xy}^2(\frac{1}{2},-1)=4e^2>0$

Z 在 $P_0(\frac{1}{2},-1)$ 取极小值, $Z(\frac{1}{2},-1)=-\frac{1}{2}e$.

9.(3) $\begin{cases} d'_x = \cos x - \cos(x+y) = 0 \\ d'_y = \cos y - \cos(x+y) = 0 \end{cases}$ 得 $P_0(\frac{2}{3}\pi, \frac{2}{3}\pi)$ 为稳定点.

$d(P_0) = \frac{3}{2}\sqrt{3}$.

1) 在 $y=0$ 上, $0 \le x \le 2\pi$ $z = \sin x - \sin x = 0$

2) 在 $x=0$; $0 \le y \le 2\pi$ $z = 0$

3) 在 $x+y=2\pi$, $0 \le x \le 2\pi$ $z = \sin x + \sin(2\pi - x) - \sin(2\pi) = 0$

最大值 $\frac{3}{2}\sqrt{3}$, 最小值 0.

11. $d(x,y) = x^2 + y^2 + \frac{1}{5}(x+2y-16)^2$

$\begin{cases} d'_x = 2x + \frac{2}{5}(x+2y-16) = 0 \\ d'_y = 2y + \frac{4}{5}(x+2y-16) = 0 \end{cases}$ $P_0(\frac{8}{5}, \frac{16}{5})$ 是稳定点

易知 $f(P_0)$ 就是 $d(x,y)$ 的最小值点, $d(P_0) = \frac{128}{5}$,

5. 解 $\frac{\partial^2 \varphi}{\partial x^2} = \begin{vmatrix} 1 & 0 & 0 \\ d+z & e+x & f+y \\ g+y & h+z & k+x \end{vmatrix} + \begin{vmatrix} a+x & b+y & c+z \\ 0 & 1 & 0 \\ g+y & h+z & k+x \end{vmatrix} + \begin{vmatrix} a+x & b+y & c+z \\ d+z & e+x & f+y \\ 0 & 0 & 1 \end{vmatrix}$

$= (e+x)(k+x) - (f+y)(h+z) + (a+x)(k+x) - (c+z)(g+y)$
$+ (a+x)(e+x) - (b+y)(d+z)$

$\frac{\partial^2 \varphi}{\partial x^2} = k+x+e+x+a+x+k+x+a+x+e+x = 6x + 2(a+e+k)$

7、解，首先证明 若$f(x,y)$在R^2上连续，$f_x(x,y)=0$，则$f(x,y)=\varphi(y)$

对R^2上任意两点(x_1,y)，(x_2,y) 由中值定理

$$f(x_2,y)-f(x_1,y)=f_x(x_1+\theta(x_2-x_1),y)(x_2-x_1)=0.$$

$\therefore f(x_2,y)=f(x_1,y)$

由(x_1,y)，(x_2,y)对x任意性知$f(x,y)$与x无关 $f(x,y)=\varphi(y)$

$u_{xy}=0$. $u_x=\varphi(x)$

$\frac{\partial}{\partial x}(u-\int\varphi(x)dx)=0$ $u-\int\varphi(x)dx=\psi(y)$

$u=\int\varphi(x)dx+\psi(y)=\Phi(x)+\psi(y)$.

P298 8(2) $f(x)=\frac{1}{2}x^TAx$. $A=\begin{pmatrix}-2&4&6\\4&-4&-6\\6&-6&8\end{pmatrix}$

黑塞矩阵 $f''(x)=A=\begin{pmatrix}-2&4&6\\4&-4&-6\\6&-6&8\end{pmatrix}$.

驻点 $x_0=-A^{-1}b=-\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&\frac{13}{34}&-\frac{3}{34}\\0&-\frac{3}{34}&\frac{1}{17}\end{bmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$A_{11}=-2<0$ $A_{22}=\begin{vmatrix}-2&4\\4&-4\end{vmatrix}=-8<0$

黑塞矩阵不定，x_0不是极值点