AVL

:) $O(\lg n)$ in worst case

:(complicate implementation

extra space h_L, h_R

non-adaptive.

Splay tree.

:($O(n)$ in worst case

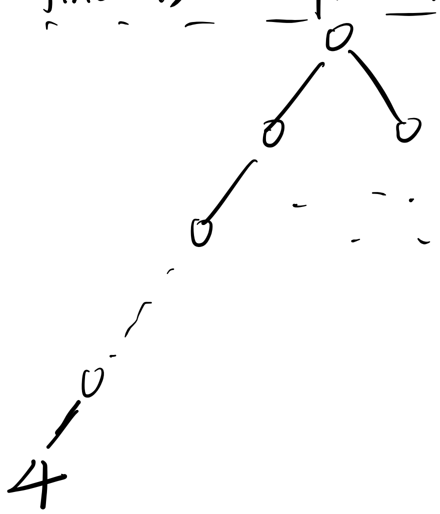
:) $O(\lg n)$ amortized cost

:) easy implementation

no extra space

adaptive

find(4) find(4) find(4) ...

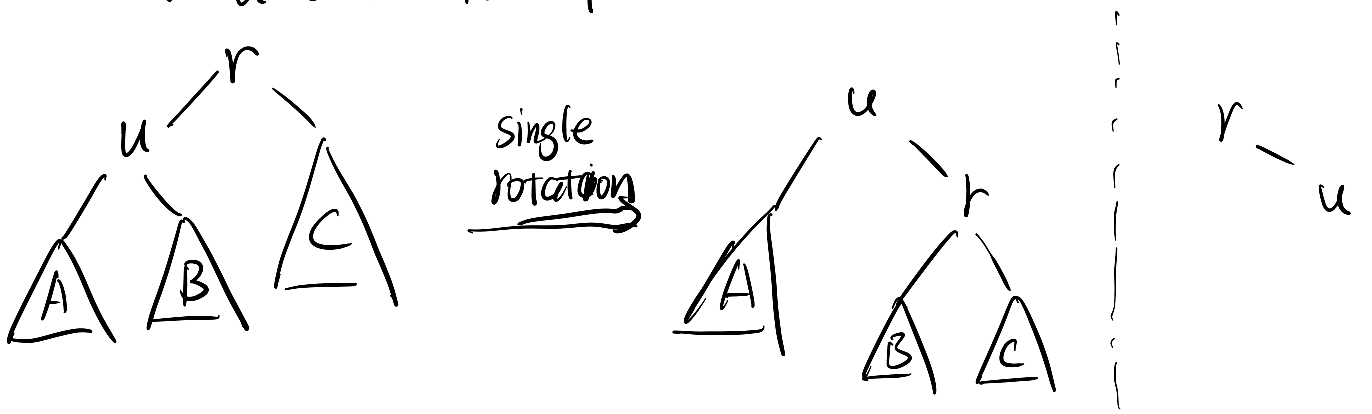


splay(u): move u to the root by rotations.

If u is the root

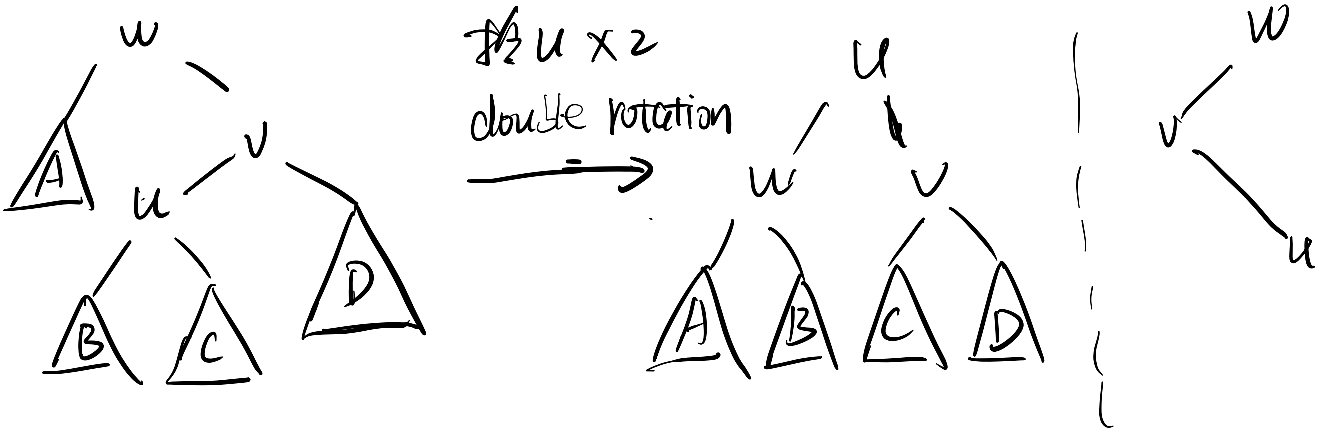
done

else if u is a child of the root

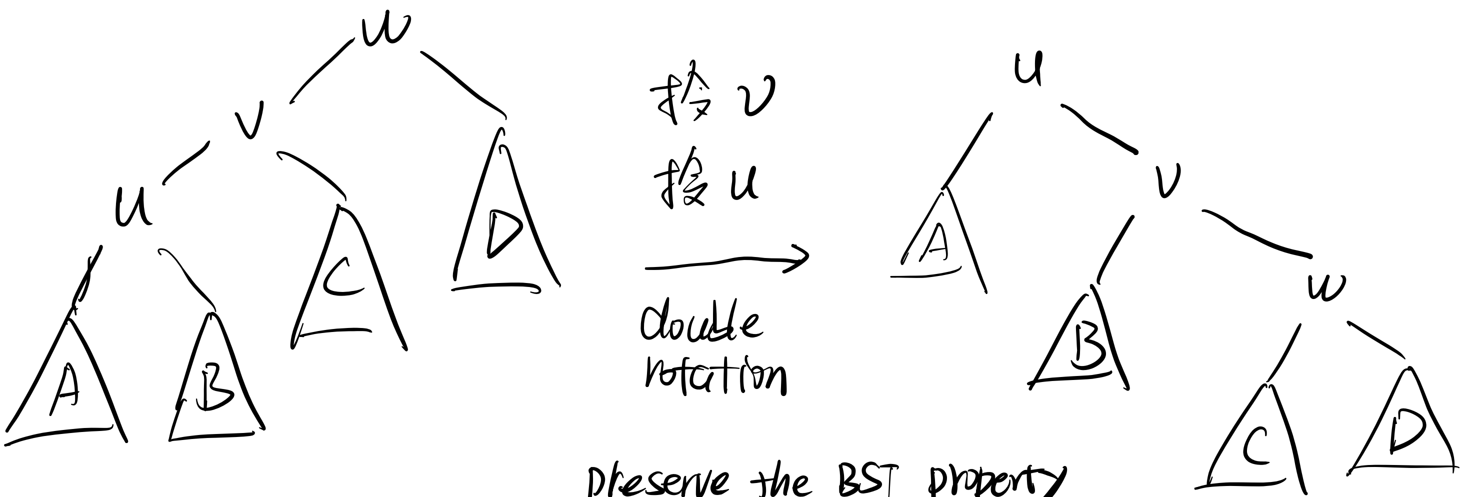


else // u has a grandparent.

case 1. zig-zag



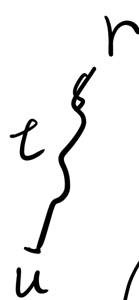
case2 zig-zig



preserve the BST property

Find key:

1. find as in BST
2. splay the node to the root



~~O(t)~~ c'· # rotation

\+ c · # rotations

rotations ≥ t/2

c · # rotations

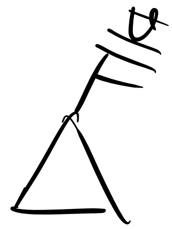
Insert:

1. insert as in BST
2. splay the newly inserted node to the root c · # rotations

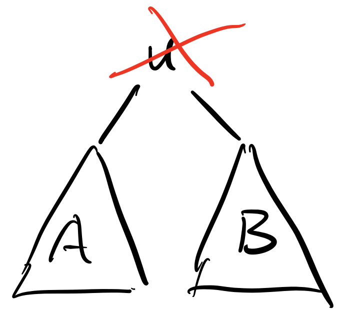
Delete: c · # rotations

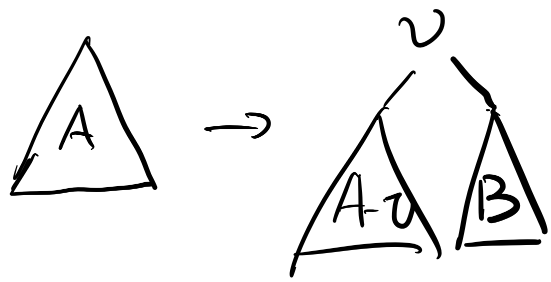
1. find the node u to be deleted and splay it to the root.

2. if u has most one child
   delete u directly



3. else // u has two children
4. delete u
5. splay the largest element v of A to the root of A
6. attach B to v





Observation

actual cost of each operation (find, Ins, Del) is $\underbrace{c}_{\text{some constant}} \cdot \#$ rotations

$\Phi(n) \geq \Phi(0)$

$\Phi(i) - \Phi(i-1) \leq c\lg n - c\cdot \#\text{rotations}$

$\hat{c}_i = c_i + \Phi(i) - \Phi(i-1) = \cancel{c\cdot\#\text{rotations}} + c\lg n - \cancel{c\cdot\#\text{rotation}}$

$= c\lg n = O(\lg n)$

Given a tree T,

- for each node $u \in T$

size(u) = # nodes in T_u

rank $\quad r(u) = \lg(\text{size}(u))$

- $\Phi(T) = c \cdot \sum_{u \in T} r(u)$

Claim: let T be a splay tree. Let $u \in T$ be a node. let T' be the tree after splay(u).

$$\Phi(T') - \Phi(T) \le 3c \cdot (\underbrace{r'(u)}_{\text{rank of } u \text{ in } T'} - r(u)) - 2c \cdot (\#\text{rotations} - 1)$$

Find key:

1. find as in BST $\qquad \Delta_1 = 0$
2. splay(u) $\qquad \Delta_2 = 3c \cdot \underbrace{(r'(u) - r(u))}_{\le r'(u) \le \lg n} - 2c \cdot (\#\text{rotations} - 1)$

$$\text{amortized cost} = \underbrace{c \cdot \#\text{rotations}}_{\text{actual cost}} + \Delta_2 \le 3c \cdot \lg n - c \cdot \#\text{rotations} + 2c$$

$$\le 3c \cdot \lg n + 2c$$

$$= O(\lg n)$$

Insertion

1. insert as in BST $\qquad \Delta_1 \le h \le 2\#\text{rotations}$
2. splay(u) $\qquad \Delta_2 \le 3c \cdot (r'(u) - r(u)) - 2c \cdot (\#\text{rotations} - 1)$

$$\text{amortized cost} = c\cdot\#\text{rotations} + A_1 + A_2$$

$$\le 3c\cdot(r'(u) - r(u)) - (c-2)\cdot\#\text{rotations} + 2c$$

assume $c \ge 2$

$$\le 3c\cdot(\underbrace{r'(u) - r(u)}_{\lg n}) + 2c \;=\; O(\lg n)$$

Del:

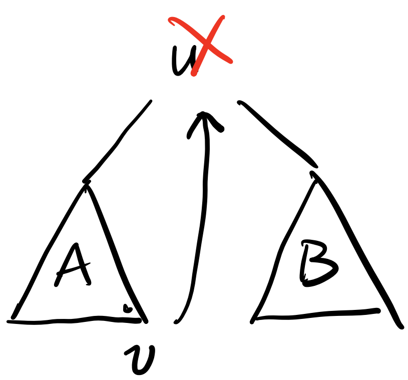
1. find u and splay(u)
2. delete u
3. find v and splay(v) (to the root of A)
4. attach B to v

$\underline{A_1} \le 3c\cdot(r'(u) - r(u))$

$\underline{A_2}$ $\quad -2c\cdot(\#\text{rotations} - 1)$ in step 1

$\underline{A_3} \le 3c\cdot(r'(v) - r(v))$

A_4 $\quad -2c(\#\text{rotation} - 1)$ in step 3



$r(v) \le r(u)$

$\Rightarrow A_2 + A_4 \le 0$

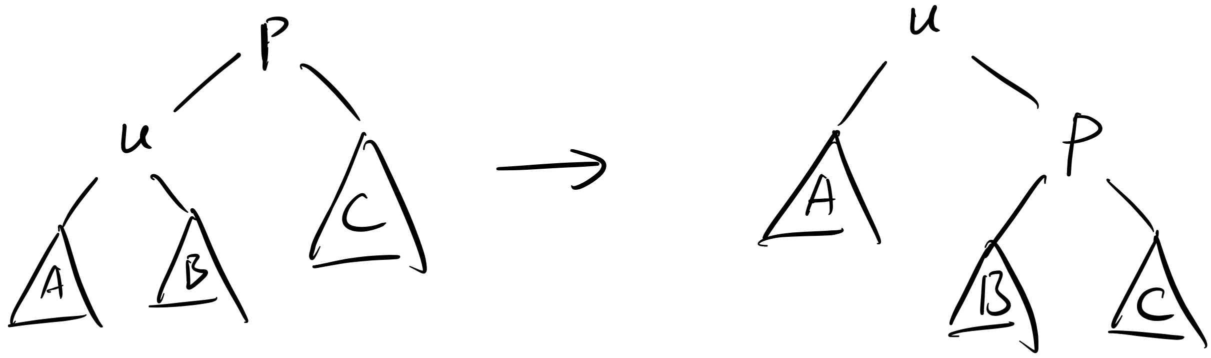
$$\text{amortized cost} = \cancel{c\cdot\#\text{rotations}} + 3c(\overbrace{r'(u) - r(u)}^{\le \lg n}) + 3c(\overbrace{r'(v) - r(v)}^{\le \lg n})$$

$$+0 \; -\cancel{2}c\cdot\#\text{rotations} + 4c$$

$$\le 6c\cdot\lg n + 4c \qquad O(\lg n)$$

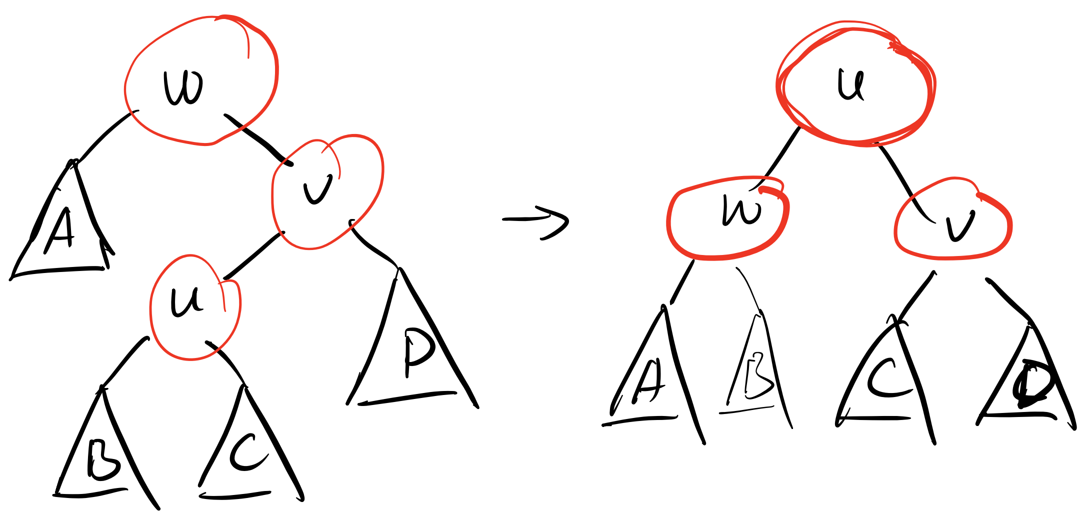
$$\Phi(T') - \Phi(T) \le 3\,(r'(u) - r(u)) - 2\,(\#\text{rotations} - 1)$$

case 1. Single rotation



$$\frac{\Phi(T') - \Phi(T)}{c} = r'(u) - r(u) + r'(p) - r(p)$$
$$\le r'(u) - r(u)$$
$$\le 3(r'(u) - r(u))$$

case 2 Zig-zag



$$\frac{\Phi(T') - \Phi(T)}{c} = \cancel{r'(u)} - r(u) + r'(v) - r(v) + r'(w) - \cancel{r(w)}$$

$r(v) > r(u)$

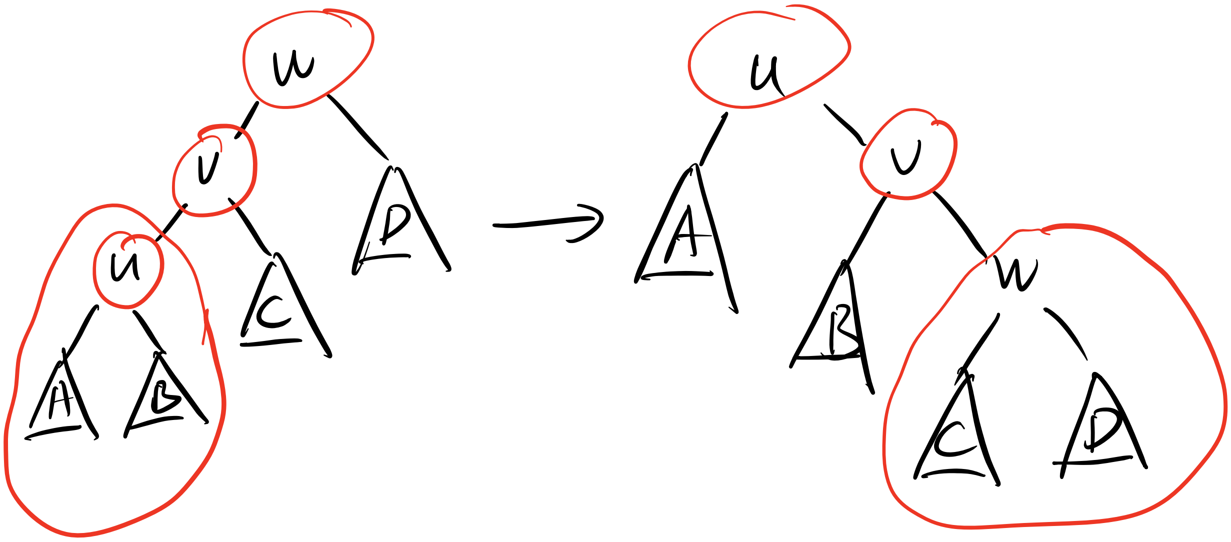
$$\le -r(u) + r'(v) - r(u) + r'(w)$$
$$= -2r(u) + r'(v) + r'(w)$$

$$\le -2r(u) + 2r'(u) - 2 \le 3(r'(u) - r(u)) - 2$$

$$size'(w) + size'(v) + 1 = size'(u)$$

$$r'(w) + r'(v) \le 2r'(u) - 2$$

case 3. zig - zig



$$\frac{\Phi(T') - \Phi(T)}{c} = \cancel{r'(u)} - r(u) + r'(v) - r(v) + r'(w) - \cancel{r(w)}$$

$$\because\ r'(v) \le r'(u), \quad r(v) \ge r(u)$$

$$\le -r(u) + r'(u) - r(u) + r'(w)$$

$$= -2r(u) + r'(u) + r'(w)$$

$$= -3r(u) + r'(u) + r'(w) + r(u) \le 3(r'(u) - r(u)) - 2$$

$$size(u) + size'(w) = A + B + C + D + 2 \le size'(u)$$

$$r(u) + r'(w) \le 2r'(u) - 2$$

splay(u):

$$\frac{\Phi(T') - \Phi(T)}{c} = 3 \cdot (r'(u) - r(u)) - 2 \cdot (\underbrace{\#\text{double rotations}}_{\ge \#\text{rotations} - 1})$$

$$\leq 3\cdot(r'(u) - r(u)) - 2\cdot(\#\text{rotations} - 1)$$