1、证明. 由题意, $f(x)$与$f'(x)$导函数$f(x)$可展开为傅里叶级数.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),\ x\in(-\infty,+\infty)$$

$$f'(x)=\frac{a_0'}{2}+\sum_{n=1}^{\infty}(a_n'\cos nx+b_n'\sin nx),\ x\in(-\infty,+\infty)$$

且f是有二阶连续导函数有周期性, $\because f(-\pi)=f(\pi)$ 且 $f'(-\pi)=f'(\pi)$

易证得 $a_n'=nb_n$, $b_n'=-na_n$ $a_0'=0$ $a_n''=-n^2a_n$, $b_n''=-n^2b_n$. $a_0''=0$

由连续函数可积性, f''在$[-\pi,\pi]$可积, 由预备定理1

$$\frac{a_0''^2}{2}+\sum_{n=1}^{\infty}(a_n''^2+b_n''^2)=\sum n^4(a_n^2+b_n^2)$$ 是收敛的.

即对$\varepsilon=1$, $\exists N\in N^+$ 使$n>N$ 有 $n^4(a_n^2+b_n^2)<\varepsilon$

$\therefore 0<|a_n|<\frac{1}{n^2}$, $0<|b_n|<\frac{1}{n^2}$ $\therefore$ $|a_n|+|b_n|<\frac{2}{n^2}$

易得 $\sum\frac{2}{n^2}$收敛. $\therefore$当$n>N$时, $\frac{|a_0|}{2}+\sum_{n=1}^{\infty}(|a_n|+|b_n|)$收敛.

由定理15.1 故 $f(x)$的傅里叶级数在$(-\infty,+\infty)$上一致收敛于$f(x)$.

2、证明. 易得 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$, $x\in[-\pi,\pi]$.

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx=\cancel{\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)}=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\left[\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right]dx$$

$$=\frac{a_0}{2}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx+\underline{\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_nf(x)\cos nx+b_nf(x)\sin x)dx}$$

$$=\frac{a_0^2}{2}+$$

$\because f(x)$可积$\therefore f(x)$有界, $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$在$[-\pi,\pi]$一致收敛于$f(x)$

$\therefore \sum_{n=1}^{\infty}[a_nf(x)\cos nx+b_nf(x)\sin nx]$在$[-\pi,\pi]$一致收敛

$$\therefore \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_nf(x)\cos nx+b_nf(x)\sin nx)dx=\sum_{n=1}^{\infty}\left[\frac{a_n}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx+\frac{b_n}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx\right]$$

$$=\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$ 故有 $$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

3、(1) 由第1小题3 $f(x)=\begin{cases}-\frac{\pi}{4}, & -\pi<x<0\\ \frac{\pi}{4}, & 0\le x<\pi\end{cases}$ 的傅里叶级数为

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin[(2n-1)x]}{(2n-1)},\ x\in(-\pi,\pi)\quad a_n=0\quad b_n=\frac{1}{2n}[1-(-1)^n].$$

由帕塞瓦尔等式

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

即 $$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\pi^2}{16}dx=\sum_{n=1}^{\infty}\left(\frac{1}{2n}[1-(-1)^n]\right)^2$$

$$\Rightarrow\ \frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

5、证明、由帕塞瓦尔等式

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f'(x)]^2dx=\frac{a_0'^2}{2}+\sum_{n=1}^{\infty}(a_n'^2+b_n'^2).$$

$$a_0'=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)dx=\frac{1}{\pi}[f(\pi)-f(-\pi)]=0$$

$$a_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\cos nx\,dx=\frac{1}{\pi}f(x)\cos nx\Big|_{-\pi}^{\pi}+\frac{n}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

$$=n\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=n\cdot b_n.$$

$$b_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\sin nx\,dx=\frac{1}{\pi}f(x)\sin nx\Big|_{-\pi}^{\pi}-\frac{n}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=-n\cdot a_n.$$

显然 $\pi\left[\frac{a_0'^2}{2}+\sum_{n=1}^{\infty}(a_n'^2+b_n'^2)\right]=\pi\left[\sum_{n=1}^{\infty}(n^2a_n^2+n^2b_n^2)\right]$

$$\geq\pi\left[\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$$

即 $$\int_{-\pi}^{\pi}|f'(x)|^2dx\geq\int_{-\pi}^{\pi}|f(x)|^2dx$$

3、证明：易得 $b_n'' = -n^2 b_n$. ，$2 \geq \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$

$$\frac{1}{2}\left(2+\sum_{k=1}^{\infty}|b_k''|\right) \geq \frac{1}{2}\left[\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+|b_k''|\right)\right]$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{1}{k^2}+k^2\left(\sqrt{|b_k|}\right)^2\right] \geq \sum_{k=1}^{\infty}\frac{1}{2}\, 2\frac{1}{k}\cdot k\sqrt{|b_k|} = \sum_{k=1}^{\infty}\sqrt{|b_k|}.$$

$$\therefore \sum_{n=1}^{\infty}\sqrt{|b_n|} \leq \frac{1}{2}\left(2+\sum_{n=1}^{\infty}|b_n''|\right).$$

1.(1) f周期为π，$l=\frac{\pi}{2}$

$$a_0=\frac{2}{\pi}\int_0^{\pi}|\cos x|dx=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos x dx-\int_{\frac{\pi}{2}}^{\pi}\cos x dx\right)=\frac{4}{\pi}.$$

$$a_n=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\cos x\cos 2nx dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}[\cos(2n+1)x+\cos(2n-1)x]dx$$

$$=\frac{2}{\pi}\left[\frac{\sin(2n+1)x}{2n+1}+\frac{\sin(2n-1)x}{2n-1}\right]\Bigg|_0^{\frac{\pi}{2}}=\frac{2}{\pi}\left[\frac{(-1)^n}{2n+1}+\frac{(-1)^{n+1}}{2n-1}\right].$$

$$=(-1)^{n+1}\frac{4}{(4n^2-1)\pi},\ n=1,2,\dots$$

$$b_n=\frac{4}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\sin 2nx dx=0$$

$$|\cos x|=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos 2nx}{4n^2-1}$$

(4) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} sgn(\cos x)\,dx = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} sgn(\cos x)\cos nx\,dx = \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos nx\,dx - \int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx\right).$

$= \frac{4}{\pi}\cdot\frac{(-1)^{n+1}}{2n-1} \quad n=1,2,3\cdots$ $\qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} sgn(\cos x)\sin nx\,dx = 0$

$sgn(\cos x) = \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}.$

2、以3为周期展开

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{4}{3}$$

$$a_n = \frac{2}{3}\left(\int_0^1 x\cos\frac{2n\pi}{3}x\,dx + \int_1^2 \cos\frac{2n\pi}{3}x\,dx + \int_2^3 (3-x)\cos\frac{2n\pi}{3}x\,dx\right).$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}\left(x\sin\frac{2n\pi}{3}\Big|_0^1 + \frac{3}{2n\pi}\cos\frac{2n\pi}{3}x\Big|_0^1\right) + \right.$$

$$\frac{3}{2n\pi}\sin\frac{2n\pi}{3}x\Big|_1^2 + \frac{9}{2n\pi}\sin\frac{2n\pi}{3}x\Big|_2^3$$

$$\left. - \frac{3}{2n\pi}\left(x\sin\frac{2n\pi}{3}x\Big|_2^3 + \frac{3}{2n\pi}\cos\frac{2n\pi}{3}x\Big|_2^3\right)\right]$$

$$= \frac{2}{3}\times\frac{9}{4n^2\pi^2}\left(\cos\frac{2n\pi}{3} + \cos\frac{4n\pi}{3} - \cos 2n\pi - 1\right)$$

$$= \frac{3}{n^2\pi^2}\left((-1)^n\cos\frac{n\pi}{3} - 1\right)$$

$$f(x) = \frac{2}{3} + \frac{3}{\pi^2}\sum_{n=1}^{\infty}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\cos\frac{n\pi}{3}\right]\cos\frac{2n\pi x}{3}$$

对该级数 $\forall x \in (-\infty, +\infty)$ 收敛于 $f(x)$，上式对 $x \in (-\infty, +\infty)$ 成立.

4.解： $a_0=0 \quad a_n=0,\ n=1,2,3\cdots$

$$b_n=\frac{2}{\pi}\int_0^{\pi}\cos\frac{x}{2}\sin nx\,dx$$
$$=\frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left[\sin\left(n+\tfrac{1}{2}\right)x+\sin\left(n-\tfrac{1}{2}\right)x\right]dx$$
$$=\frac{8}{\pi}\cdot\frac{n}{4n^2-1}$$

$$\therefore f(x)=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin nx,\ x\in[0,\pi].$$

6. $a_0=2\int_0^1 f(x)\,dx=\frac{2}{3}$

$$a_n=2\int_0^1 f(x)\cos n\pi x\,dx=2\int_0^1(x-1)^2\cos n\pi x\,dx$$
$$=2\left[\frac{1}{n\pi}(x-1)^2\sin n\pi x\Big|_0^1-\frac{2}{n\pi}\int_0^1(x-1)\sin n\pi x\,dx\right]$$
$$=\frac{4}{n\pi}\cdot\frac{1}{n\pi}\left[(x-1)\cos n\pi x\Big|_0^1-\int_0^1\cos n\pi x\,dx\right]$$
$$=\frac{4}{(n\pi)^2}$$

$$f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\cos n\pi x.$$

令 $x=0$ $\quad 1=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$

$$\therefore \pi^2=6\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right).$$

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1.(2) 开集，平面上每一点都是聚点，界点为坐标轴上的点。

(5) 开集，有界集． 聚点为开集内任意点与其界点，界点为 $x=2$，$y=2$ 与 $x+y=2$ 组成三角形上的点。

3. 证明：充分性，若存在互不相同的点列 $\{P_n\} \subset E$，$P_n \neq P_0$，$\lim\limits_{n\to\infty} P_n = P_0$ 时，

则对 $\forall \varepsilon > 0$，$\exists N \in \mathbb{N}^*$，使 $n > N$ 时，有

$$P_n \in U^\circ(P_0, \varepsilon).$$

由于 ε 任意性，P_0 的任何空心邻域中都含有 E 的点，即 P_0 为 E 的聚点。

必要性，若 P_0 是 E 的聚点，则对 $\forall \varepsilon > 0$，$U^\circ(P_0, \varepsilon)$ 中必有 E 的点，

取 $\varepsilon_1 = 1$，$P_1 \subset E$ 且 $P_1 \in U^\circ(P_0; \varepsilon_1)$.

取 ~~$\varepsilon_2 = \frac{1}{2}$~~，又 $\varepsilon_2 = \min\{\frac{1}{2}, |P_1 - P_0|\}$，取一点 P_2 在 E 与 $U^\circ(P_0, \varepsilon_2)$ 中.

…… 取 $\varepsilon_n = \min\{\frac{1}{n}, |P_{n-1} - P_0|\}$，取 P_n 在 E 与 $U^\circ(P_0, \varepsilon_n)$ 中，

如此取下去，可得到互异的点列 $\{P_n\}$ $P_n \neq P_0$ $\{P_n\} \subset E$，

且 $\lim\limits_{n\to\infty} P_n = P_0$