2.(3) 令 $f(x)=\sum_{n=1}^{\infty} n^2 x^n = x\sum_{n=1}^{\infty} n^2 x^{n-1}$ 收敛半径 $R=\frac{1}{\rho}=\frac{1}{\lim_{n\to\infty}\sqrt[n]{n^2}}=1$

收敛域为 $(-1,1)$

$\int_0^x \left(\sum_{n=1}^{\infty} n^2 t^{n-1}\right)dt = \sum_{n=1}^{\infty} n x^n = x\sum_{n=1}^{\infty} n x^{n-1}$

$\int_0^x \left(\sum_{n=1}^{\infty} n x^{n-1}\right) = \frac{x}{1-x}$ $\therefore \sum_{n=1}^{\infty} n x^{n-1} = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$

$\therefore \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$ $\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{1+x}{(1-x)^3}$

$\therefore f(x)=\frac{x(1+x)}{(1-x)^3}$

4.(2) 证明: $y' = \sum_{n=0}^{\infty}\left(\frac{x^n}{(n!)^2}\right)' = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!(n-1)!}$

$y'' = \sum_{n=2}^{\infty}\frac{x^{n-2}}{n!(n-2)!}$

$xy''+y'-y = \sum_{n=2}^{\infty}\frac{x^{n-1}}{n!(n-2)!} + \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!(n-1)!} - \sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}$

$= \sum_{n=2}^{\infty}\frac{x^{n-1}}{n!(n-2)!} + \sum_{n=2}^{\infty}\frac{x^{n-1}}{n!(n-1)!} + \frac{x^{1-1}}{1!0!} - \sum_{n=1}^{\infty}\frac{x^n}{(n!)^2} - \frac{x^0}{(0!)^2}$

$= \sum_{n=2}^{\infty}\left(\frac{1}{n!(n-2)!} + \frac{1}{n!(n-1)!} - \frac{1}{[(n-1)!]^2}\right)x^{n-1}$

$=0$

$xy''+y'-y=0$ 得证

7.证明：$\because \sum_{n=0}^{\infty} a_n x^n$ 收敛半径为R，$\therefore \lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$

对$\varepsilon = \frac{K_m^{\frac{1}{n}}-1}{R}$ $(K>1)$ $\therefore \exists N>0$ 使 $\forall n>N$ 都有 $|\sqrt[n]{|a_n|}-\frac{1}{R}|<\varepsilon$. $|a_n|<\frac{K_m}{R^n}$

且有限个 ~~max~~ $\{\sqrt[1]{a_1}, \sqrt[2]{a_2}, \cdots, \sqrt[N]{a_N}\}$ ~~$\le \frac{K}{R}$~~ 都是实数

$\therefore \exists K_1, K_2, \cdots, K_N$ 使 $a_1 < \frac{K_1}{R}$ $\sqrt[2]{a_2} < \frac{K_2}{R^2}$, $\cdots$ $|a_N| < \frac{K_N}{R^N}$

取 $K = \max\{K_1, K_2 \cdots, K_N, K_m\}$

$\therefore$ 有 $|a_n| \le \frac{K}{R^n} \le KM^n$, $\forall n = 1, 2, \cdots\cdots$

11.(1) 设公差为d，$a_n = a_0 + nd$

$\therefore l = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|1+\frac{d}{a_0+nd}\right| = 1$

即收敛半径 $R=1$

(2) $a_n = a_0 + nd$ $\therefore \sum_{n=0}^{\infty}\frac{a_n}{2^n} = \sum_{n=0}^{\infty}\left(\frac{a_0}{2^n}+\frac{n}{2^n}\cdot d\right) = \sum_{n=0}^{\infty}\frac{a_0}{2^n} + d\sum_{n=0}^{\infty}\frac{n}{2^n}$

$$\sum_{n=0}^{\infty}\frac{a_0}{2^n} = \frac{1(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} \times a_0 \ (n\to\infty) = 2a_0$$

$$\sum_{n=0}^{\infty}\frac{n}{2^n} = \lim_{n\to\infty}\left(2-\frac{n+2}{2^n}\right) = 2.$$

$$\therefore \sum_{n=0}^{\infty}\frac{a_n}{2^n} = 2a_0 + 2d = 2a_1$$

2.(1) $$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!},\ x\in(-\infty,+\infty)$$

$$e^{x^2}=\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!},\ x\in(-\infty,+\infty).$$

(3) $$\frac{1}{\sqrt{1-x}}=\cancel{1+}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^n,\ x\in[-1,1).$$

$$\frac{1}{\sqrt{1-2x}}=\cancel{1+}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot(2x)^n=\sum_{n=0}^{\infty}\frac{(2n-1)!!\times 2^n}{(2n)!!}x^n$$

$$\frac{x}{\sqrt{1-2x}}=\cancel{1+}\sum_{n=0}^{\infty}\frac{(2n-1)!!\cdot 2^n}{(2n)!!}\times x^{n+1},\ x\in\left[-\frac{1}{2},\frac{1}{2}\right)$$

(6) $$\frac{x}{1+x-2x^2}=\frac{1}{3}\left(\frac{1}{1-x}-\frac{1}{1+2x}\right)$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \qquad \frac{1}{1+2x}=\sum_{n=0}^{\infty}(-1)^n(2x)^n$$

$$\frac{x}{1+x-2x^2}=\frac{1}{3}\sum_{n=0}^{\infty}[1-(-2)^n]x^n,\ |x|<\frac{1}{2}$$

(9) $$\frac{1}{\sqrt{1+t^2}}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}t^{2n},\ t\in[-1,1]$$

$$\ln(x+\sqrt{1+x^2})=\int_0^x\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}t^{2n}\right]dt$$
$$=x+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!(2n+1)}x^{2n+1},\ x\in[-1,1]$$

、以 $2l$ 为周期的函数的傅里叶级数

3.(2) $f(x)=\frac{1}{x}=\frac{1}{1+(x-1)}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n,\ x\in(0,2).$

补充：$\ln^2(1+x)$ 在 $x=0$ 展开

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad 令 a_n=\frac{(-1)^{n-1}}{n}$$

$$\ln^2(1+x)=\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\right)\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\right)$$

$$=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}a_k a_{n-k}\right)x^n$$

$$=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{(-1)^n}{k(n-k)}\right)x^n,\ x\in[-1,1]$$

1.(2) (i) 解 $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx=\frac{2}{3}\pi^2$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx$$

$$=\frac{1}{\pi}\times\frac{1}{n}\left[x^2\sin nx\Big|_{-\pi}^{\pi}+\frac{2}{n}(x\cancel{\sin}\cos nx)\Big|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\cos nx\,dx\right]$$

$$=(-1)^n\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\,dx=0$$

$$f(x)=\frac{1}{3}\pi^2+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2},\ x\in(-\pi,\pi).$$

(ii) $a_0=\frac{1}{\pi}\int_0^{2\pi}x^2dx=\frac{8}{3}\pi^2,$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,dx=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\,dx=-\frac{4}{n}\pi$$

$$f(x)=\frac{4}{3}\pi^2+4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2}-\frac{\pi\sin nx}{n}\right),\ x\in(0,2\pi).$$

三 解：$a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\frac{\pi}{4})dx+\int_{0}^{\pi}(\frac{\pi}{4})dx\right]=0$

$a_n=\frac{\pi}{4}\left[\int_{-\pi}^{0}(-\frac{\pi}{4})\cos nx dx+\int_{0}^{\pi}\frac{\pi}{4}\cos nx dx\right]=0$

$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}(-\frac{\pi}{4})\sin nx dx+\int_{0}^{\pi}\frac{\pi}{4}\sin nx dx\right]=\frac{1}{2n}[1-(-1)^n]$.

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin[(2n-1)x]}{(2n-1)},\quad x\in(-\pi,0)\cup(0,\pi).$$

(1) $x=\frac{\pi}{2}$ $\quad\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\frac{\pi}{2}]}{2n-1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$

(2) $\frac{\pi}{12}=\frac{1}{3}\times\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\frac{\pi}{2}]}{2n-1}=\frac{1}{3}-\frac{1}{9}+\frac{1}{15}-\frac{1}{21}+\frac{1}{27}$

$\therefore\frac{\pi}{3}=\frac{\pi}{4}+\frac{\pi}{12}=1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots$

(3) $\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\frac{\pi}{3}]}{2n-1}=\frac{\sqrt{3}}{2}(1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\cdots)$

$\therefore\frac{\sqrt{3}}{6}\pi=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\cdots$

7(1) $a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2} dx = \frac{1}{2\pi}\left(\pi x - \frac{x^2}{2}\right)\Big|_0^{2\pi} = 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2} \cos nx\, dx = 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2} \sin nx\, dx = \frac{1}{\pi}\left[\int_0^{2\pi} \frac{\pi}{2}\sin nx\, dx - \int_0^{2\pi} \frac{x}{2}\sin nx\, dx\right]$

$= \frac{1}{n}$

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n},\ x \in (0, 2\pi).$$

(3) (i) $a_0 = \frac{1}{\pi}\int_0^{2\pi} (ax^2 + bx + c)\, dx = \frac{8}{3}a\pi^2 + 2b\pi + 2c.$

$a_n = \frac{1}{\pi}\int_0^{2\pi} (ax^2 + bx + c)\cos nx\, dx = \frac{4a}{n^2}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} (ax^2 + bx + c)\sin nx\, dx = -\frac{4\pi a}{n} - \frac{2b}{n}.$

$$f(x) = \frac{4a}{3}\pi^2 + b\pi + c + \sum_{n=1}^{\infty}\left(\frac{4a}{n^2}\cos nx - \frac{4\pi a + 2b}{n}\sin nx\right).\ x \in (0, 2\pi).$$

(ii) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (ax^2 + bx + c)\, dx = \frac{2}{3}a\pi^2 + 2c.$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (ax^2 + bx + c)\cos nx\, dx = (-1)^n \frac{4a}{n^2}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (ax^2 + bx + c)\sin nx\, dx = (-1)^{n-1}\frac{2b}{n}$

$$f(x) = \frac{a}{3}\pi^2 + c + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 4a}{n^2}\cos nx + \frac{(-1)^{n-1} 2b}{n}\sin nx\right],\ x \in (-\pi, \pi).$$

10、证明，$\because \sup_n \{|n^3 a_n|, |n^3 b_n|\} \le M.$

$\therefore |a_n| \le \frac{M}{n^3} \quad |b_n| \le \frac{M}{n^3}.$

对 $\forall n \in N$, $|a_n \cos nx + b_n \sin nx| \le |a_n| + |b_n| \le \frac{2M}{n^3}$

且 $\sum \frac{2M}{n^3}$ 收敛，则

$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ 绝对一致收敛.

~~$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}$~~ $u_n(x) = a_n \cos nx + b_n \sin nx.$

$u_n'(x) = n b_n \cos nx - n a_n \sin nx$;

$|n b_n \cos nx - n a_n \sin nx| \le |n b_n| + |n a_n| \le \frac{2M}{n^2}.$

$\sum \frac{2M}{n^2}$ 收敛，$\therefore \sum_{n=1}^{\infty} (n b_n \cos nx - n a_n \sin nx)$ 一致收敛 且和函数连续

$\therefore \frac{d}{dx}\left(\sum_{n=0}^{\infty} u_n(x)\right) = \sum_{n=0}^{\infty} \frac{d}{dx} u_n(x) = \sum_{n=1}^{\infty} (n b_n \cos nx - n a_n \sin nx).$

即 $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ 和函数是具有连续导函数