Hiring Problem (Secretary Problem)

n applications

Goal: hire the best candidate

```
for i=1 to n
    if candate i is the best so far
        hire i
```

Any deterministic alg. hires n candidates in the worst case.

$\ln n$ candidates in expectation

permute the n applications randomly

```
for i=1 to n
    if candate i is the best so far
        hire i
```

A_i = candidate i is the best among the first i candidates

$\Pr(A_i) = \frac{1}{i}$

$$\underbrace{\circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ}_{i}$$

$$X_i = \begin{cases} 1 & \text{if candidate } i \text{ is hire} \\ 0 & \text{otherwise} \end{cases}$$

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} Pr(A_i) = \sum_{i=1}^{n} \frac{1}{i} \lessapprox \ln n.$$

Goal: hire one candidate

maximize the probability that the best candidate is hired.

$\frac{1}{e}$

1. randomly permutation the n applications

2. interview k candidates

3. for $i = k+1$ to n

4. if candidate i is better than the best of the first k candidates

5. hire i

6. break

Pr[the best candidate is hired]

$= \sum^{n}$ Pr[[illegible]]

$$= \sum_{i=k+1} \Pr[\underbrace{\text{the best is at position } i}_{A_i} \text{ and } \underbrace{\text{candidate } i \text{ is hired}}_{B_i}]$$

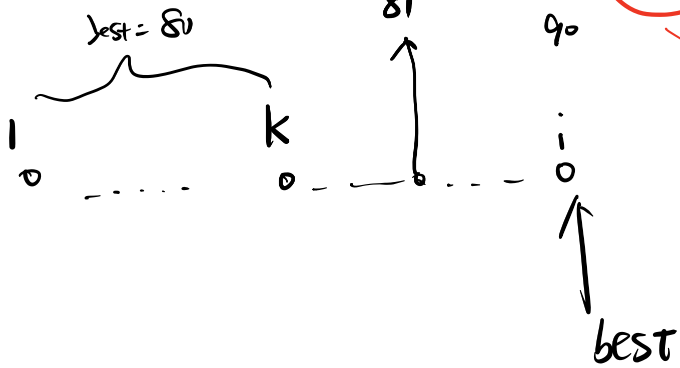
$$= \sum_{i=k+1}^{n} \Pr[A_i \cap B_i]$$

$$= \sum_{i=k+1}^{n} \Pr[A_i] \cdot \Pr[B_i \mid A_i] = \sum_{i=k+1}^{n} \frac{1}{n} \cdot \frac{k}{i-1} = \frac{k}{n} \sum_{i=k}^{n-1} \frac{1}{i} \geq \frac{k}{n} \cdot \ln \frac{n}{k}$$

$$\Pr[A_i] = \frac{1}{n}$$

$$k = \frac{n}{e}$$

$$\frac{1}{e}$$

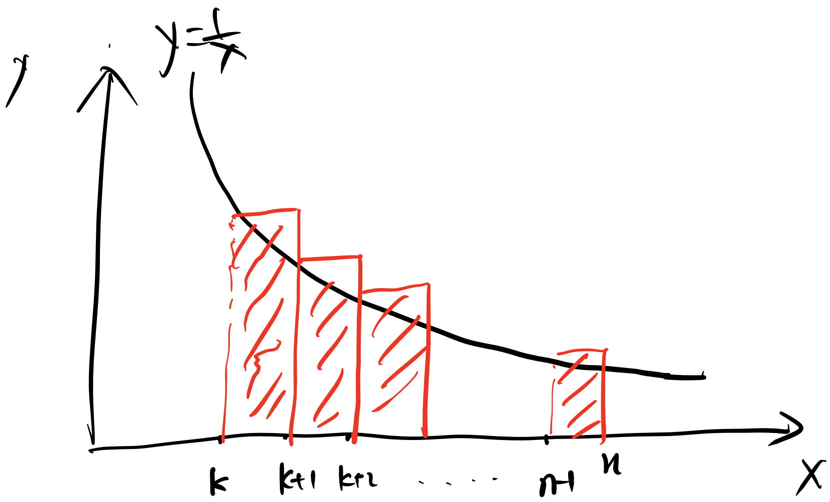


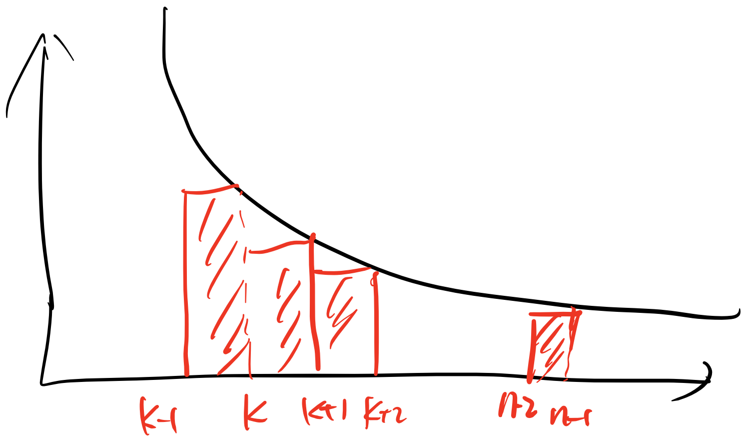
Pr[candidates k+1, …, i-1 is worse than the best of the first k candidates]

= Pr[the best of i-1 candidates is among the first k candidates]

$$= \frac{k}{i-1}$$

$$\ln \frac{n}{k} = \int_k^n \frac{1}{x} \cdot dx \leq \sum_{i=k}^{n-1} \frac{1}{i} \leq \int_{k-1}^{n-1} \frac{1}{x} \cdot dx = \ln \frac{n-1}{k-1}$$





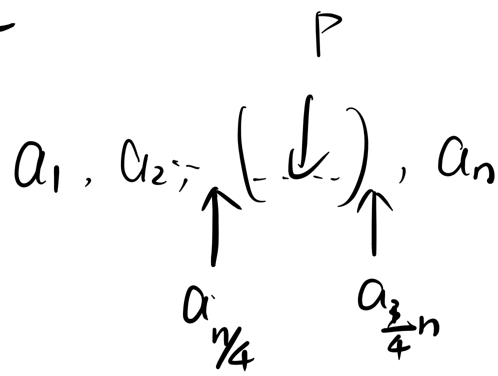

```
Quicksort(A)
    If |A| ≤ 3
        trivial
    else
        Choose a pivot p from A
        for each element a ∈ A
            put a into A⁻ if a < p
            put a into A⁺ if a > p
        Quicksort(A⁻)
        Quicksort(A⁺)
        Output A⁻, p, A⁺
```

Idea 1.

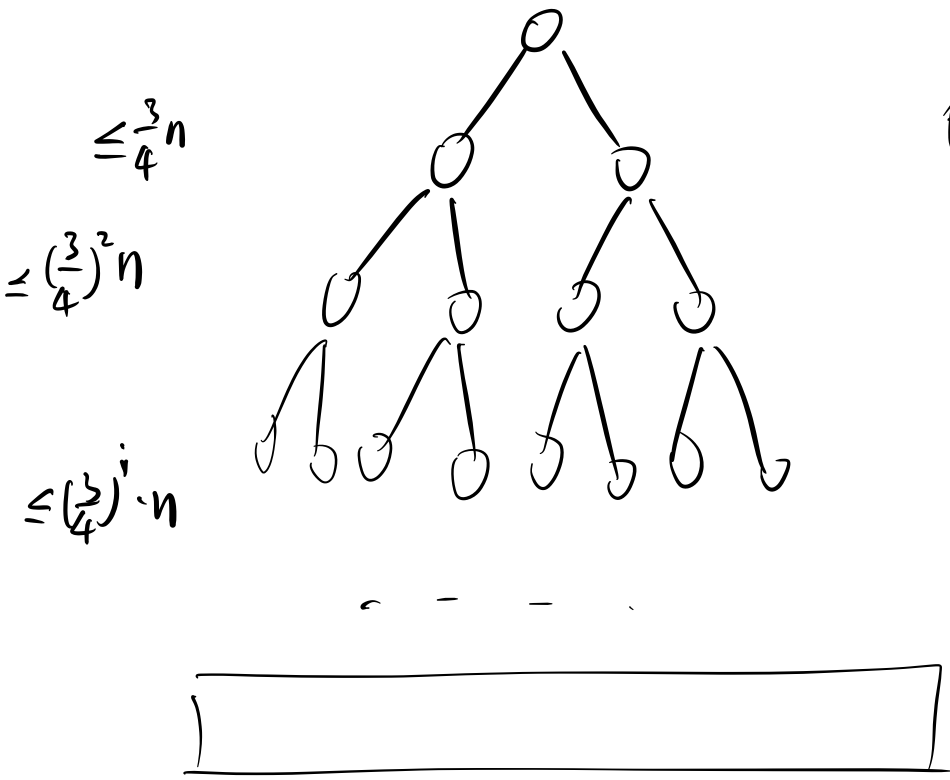
A pivot is a central splitter if $|A^+| \geq \frac{1}{4}n$ and $|A^-| \geq \frac{1}{4}n$

keep picking a random pivot p from A until get a central splitter

$\Pr[\,p$ is a central splitter$\,]$
$= 1/2$



E[time to partion A] is $O(|A|)$



$\Theta(n)$ in expectation

$(\frac{3}{4})^i \cdot n \ge 3$
$\Rightarrow i \le O(\log_{4/3} n)$

total
$= O(n \log n)$ in expectation

Algorithm Design 773

Idea 2

randomly pick a pivot
and use it anyway

$O(n \log n)$ in expectation

Quicksort(A)
```
If |A| ≤ 3
    trivial
else randomly
    choose a pivot p from A              O(1)
    for each element a ∈ A
        put a into A⁻ if a < p           O(# comparisons)
        put a into A⁺ if a > p
    Quicksort(A⁻)
    Quicksort(A⁺)
    Output A⁻, p, A⁺                     O(1)
```

Total running time $= O($total # comparison$)$

$a_1, a_2, \ldots, a_n$ in increasing order.

for $a_i, a_j \in A$

let $X_{ij} = \{ 1$ if a_i, a_j are compared define $X_{ij} = \begin{cases} \ldots \\ 0 & \text{otherwise.} \end{cases}$

$$T = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$$

$$\tau = j - i$$

$$E[T] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n} \sum_{\tau=1}^{n-i} \frac{2}{\tau+1}$$

$$\le \sum_{i=1}^{n} \sum_{t=1}^{n} \frac{2}{t} \le 2n \ln n$$

a_i $\qquad$ a_k $\qquad$ a_j

① a_i or a_j is picked as a pivot.

② no $a_{i+1}, a_{i+2}, \ldots, a_{j-1}$ is picked as a pivot before that

$\Updownarrow$

a_i or a_j is first pivot among $\{a_i, \ldots, a_j\}$

$\emptyset$ $\qquad$ $\searrow$ $\dfrac{2}{j-i+1}$