1.(1) 对原式作广义球坐标变换，$T=\begin{cases} x=ar\sin\varphi\cos\theta \\ y=br\sin\varphi\sin\theta, 0\le r\le 1, & 0\le\varphi\le\pi \\ z=cr\cos\varphi & 0\le\theta\le 2\pi \end{cases}$

$J=abcr^2\sin\varphi.$

$$\iiint\limits_V\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dxdydz=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1 abcr^2\sqrt{1-r^2}\sin\varphi\,dr$$

$$=abc\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^1 r^2\sqrt{1-r^2}\,dr$$

$$=2\pi abc(-\cos\varphi)\Big|_0^{\pi}\cdot\left[\frac{r}{8}(2r^2-1)\sqrt{1-r^2}+\frac{1}{8}\arcsin r\right]\Big|_0^1$$

$$=\frac{1}{4}\pi^2abc$$

2.解：$\begin{cases} z=\sqrt{x^2+y^2} \\ z^2=2x \end{cases}\Rightarrow x^2+y^2=2x$，即曲面在xy平面上的投影区域.

$D=\{(x,y)\mid x^2+y^2\le 2x\}$

$$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}\qquad\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}.\qquad\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{2}$$

$$\Delta S=\iint\limits_D\sqrt{2}\,dxdy=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cdot\int_0^{2\cos\theta}r\,dr=4\sqrt{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\sqrt{2}\pi$$

3.(1) 设其密度ρ，由于半椭圆关于y轴对称，$\bar{x}=0$

$$M=\iint\limits_D\rho\,dxdy=\rho\iint d\theta\int_0^1 abr\,dr=\frac{1}{2}\pi ab\rho.$$

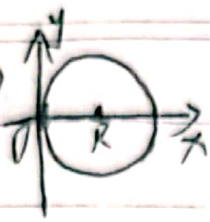
对x轴的静力矩 $M_x=\iint\limits_D\rho y\,dxdy=\rho\int_0^{\pi}d\theta\int_0^1(ab^2\sin\theta)r^2dr.$

$$=\rho ab^2\int_0^{\pi}\sin\theta\,d\theta\cdot\int_0^1 r^2dr.$$

$$=\frac{2}{3}ab^2\rho.$$

$\bar{y}=\frac{M_x}{M}=\frac{4b}{3\pi}$ $\quad\therefore$ 重心为$\left(0,\frac{4b}{3\pi}\right)$

5-(1)



如图，轴为圆切线，ρ为密度

则其转动惯量：$I_y = \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2R\cos\theta} r^3 \cos^2\theta \, dr$

$$= 8\rho R^4 \int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta$$

$$= 8\rho R^4 \frac{5!!}{6!!} \frac{\pi}{2}$$

$$= \frac{5}{4} \pi \rho R^4$$

7. 解：$x_\psi = -a\cos\varphi \sin\psi \quad x_\varphi = -(b+a\cos\psi)\sin\varphi$

$y_\psi = -a\sin\varphi \sin\psi \quad y_\varphi = (b+a\cos\psi)\cos\varphi$

$z_\psi = a\cos\psi \qquad z_\varphi = 0.$

$E = x_\psi^2 + y_\psi^2 + z_\psi^2 = a^2 \quad F = x_\psi x_\varphi + y_\psi y_\varphi + z_\psi z_\varphi = 0$

$G = x_\varphi^2 + y_\varphi^2 + z_\varphi^2 = (b+a\cos\psi)^2.$

$\sqrt{EG-F^2} = a(b+a\cos\psi)$

$\Delta S = \iint_D \sqrt{EG-F^2} \, d\varphi \, d\psi = a\int_0^{2\pi} d\varphi \int_0^{2\pi} (b+a\cos\psi) \, d\psi = 4ab\pi^2$

1.(1) $L = OA + AB + BO$，可分段计算

$OA: \begin{cases} x = x \\ y = 0, \end{cases} 0 \le x \le 1$ $\quad AB: \begin{cases} x = x \\ y = 1-x \end{cases} 0 \le x \le 1$ $\quad BO: \begin{cases} x = 0 \\ y = y \end{cases}, 0 \le y \le 1$

$$\int_L (x+y)\,ds = \int_0^1 (x+0)\sqrt{1+0^2}\,dx + \int_0^1 (x+1-x)\sqrt{1^2+(-1)^2}\,dx + \int_0^1 (0+y)\sqrt{1+1^2}\,dy$$

$$= 1+\sqrt{2}$$

(4) 单位圆周关于x轴对称，作极坐标变换 $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}, 0 \le \theta \le \pi$

$$\int_L |y|\,ds = 2\int_{\substack{x^2+y^2=1 \\ y>0}} y\,ds = 2\int_0^{\pi} \sin\theta \cdot \sqrt{(-\sin\theta)^2+(\cos\theta)^2}\,d\theta = 2(-\cos\theta)\Big|_0^{\pi}$$

$$= 4$$

(5) $\int_L (x^2+y^2+z^2)\,ds$

$$= \int_0^{2\pi} (a^2\cos^2 t + a^2\sin^2 t + b^2t^2)\sqrt{(-a\sin t)^2+(a\cos t)^2+b^2}\,dt$$

$$= \int_0^{2\pi} (a^2+b^2t^2)\sqrt{a^2+b^2}\,dt$$

$$= \sqrt{a^2+b^2}\left(a^2 t + \frac{b^2}{3}t^3\right)\Big|_0^{2\pi} = \sqrt{a^2+b^2}\left(2a^2\pi + \frac{8\pi^3}{3}b^2\right)$$

(7) $$\int_L \sqrt{2y^2+z^2}\,ds = 2a\int_{-\frac{a}{\sqrt{2}}}^{\frac{a}{\sqrt{2}}} \sqrt{1^2+1^2+\left(\frac{2x}{\sqrt{a^2-2x^2}}\right)^2}\,dx$$

$$= 4\sqrt{2}a\int_0^{\frac{a}{\sqrt{2}}} \frac{1}{a\sqrt{1-\frac{2x^2}{a^2}}}\,dx = 2\pi a^2$$

2.解: $m = \int_L \rho ds = \int_L \sqrt{\frac{2z}{a}} ds$

$= \int_0^1 t\sqrt{0+a^2+(at)^2}\, dt$

$= a\int_0^1 t\sqrt{1+t^2} dt$

$= \frac{a}{3}(2\sqrt{2}-1)$

4.(2)解: $L: \rho=\rho(\theta)$. 由直角坐标与极坐标之间变换

$L: \begin{cases} x=\rho\cos\theta=\rho(\theta)\cos\theta \\ y=\rho\sin\theta=\rho(\theta)\sin\theta \end{cases} \theta_1\leq\theta\leq\theta_2$

$\sqrt{x'^2(\theta)+y'^2(\theta)} = \sqrt{(\rho'\cos\theta-\rho\sin\theta)^2+(\rho'\sin\theta+\rho\cos\theta)^2}$

$= \sqrt{\rho'^2(\theta)+\rho^2(\theta)}$

$\int_L f(x,y)ds = \int_{\theta_1}^{\theta_2} f(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)\sqrt{\rho'^2(\theta)+\rho^2(\theta)}\, d\theta$

$\rho=ae^{k\theta}(k>0)$ 在 $r=a$ 内的部分为 $\rho=ae^{k\theta}$, $-\infty<\theta\leq 0$.

$\sqrt{\rho'^2+\rho^2} = a\sqrt{1+k^2}\, e^{k\theta}$

$\int_L x ds = \int_{-\infty}^0 ae^{k\theta}\cos\theta\cdot a\sqrt{1+k^2}\, e^{k\theta} d\theta$

$= a^2\sqrt{1+k^2}\int_{-\infty}^0 e^{2k\theta}\cos\theta d\theta$

$= a^2\sqrt{1+k^2}\left[\frac{e^{2k\theta}}{1+4k^2}(2k\cos\theta+\sin\theta)\right]\Big|_{-\infty}^0$

$= \frac{2k a^2\sqrt{1+k^2}}{1+4k^2}$

1.(1) (i) $L = OB: y = 2x^2,\ 0 \le x \le 1$

$$\int_L x\,dy - y\,dx = \int_0^1 (x \cdot 4x - 2x^2)\,dx$$
$$= \int_0^1 2x^2\,dx$$
$$= \frac{2}{3}$$

(ii) $L = OB: y = 2x,\ 0 \le x \le 1$, $\int_L x\,dy - y\,dx = \int_0^1 (2x - 2x)\,dx = 0$

(iii) $OA: y = 0,\ 0 \le x \le 1$, $AB: x = 1,\ 0 \le y \le 2$ $BO: y = 2x$, x从1到0.

$$\int_L x\,dy - y\,dx = \int_0^1 0\,dx + \int_0^2 dy + \int_1^0 (2x - 2x)\,dx$$
$$= 2$$