Homework Assignment 1

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Course: Time Series, Fall 2019
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Date: 13/11/2019

Contents

#+++++++NOTES+++++ 2. e distributed formula must be changed 4. Names of AR(2) Tables wrong 5. Something's off with autocorrelation in AR(2)

1. Introduction

Each stochastic process implies a theoretical structure of the autocorrelation function (ACF). A particular dataset is a *realization* of some specific stochastic process. Hence, by analyzing at a sample ACF, one can approximate the underlying stochastic process, and find the proper model to estimate it, thereby being able to make forecasts. However, in order to be capable of matching a sample ACF with a theoretical one, one must first know the properties of the theoretical stochastic process and thereby it's ACF. Hence, we need to have a catalogue of theoretical ACF:s. The purpose of this paper is to review some MA(q), AR(p) and ARMA(p,q) processes, to create such a catalogue.

Each section in the paper starts with a general discussion about the process in question, followed by a simulation excercise where we generate a sample based on the process in question. In that way, we can study the correlogram of the process. One important aspect of the discussion is whether a particular process is covariance stationary or not, since this has major implications for how to work with the process. The sample size in the simulations is set to 5000 (if nothing else is stated), but the plots only runs up to 500.

The paper is structured as follows. In section 2, the concept of covariance stationary is defined, which will be a point of reference in most of the following discussions. Section 3 and 4 discuss the MA(1) and MA(2) processes respectively, and section 5 and 6 discuss the AR(1) and AR(2). Section 7 covers the ARMA() process, and section 8 concludes.

1 2. The Concept of stationarity

A covariance stationary process is a process where the statistical properties do not change over time. This is of importance when we are trying to estimate a process. More specifically, covariance stationarity imposes three requirements: 1. $E(Y_t)$ constant over time 2. $Var(Y_t)$ constant over time 3. The autocovariance depends only on the distance between two observations, and not on where in time the observations are found

1.1 Study for MA(1)

A moving average process of order 1 -an MA(1) proccess- indicates a process where the output variable Y_t depends linearly on current white noise as well as white noise in the previous period. More specifically, we have the model:

MA(1):

$$Y_t = e_t - \theta e_{t-1}$$

where $e_t \sim NID(0, 1)$

Y_t is an output variable, θ the parameter of the process, and e_1 denotes the white noise. e_t is identically and independently distributed with mean 0 and a constant variance. Since it is independent, the autocovariance is 0 by definition, making any kind of predictions of e_t impossible. This has indications for the properties of Y_t, which are discussed further in Appendix A. Here however, it is sufficient just to present the resulting formulas for the statistical properties of Y_t:

$E(Y_t) = 0$ $\gamma_0 = V(Y_t) = (1+\theta)*\sigma^2$ $\gamma_1 = Cov(Y_t, Y_{t-1}) = \theta*\sigma^2$ $\gamma_2 = Cov(Y_t, Y_{t-2}) = 0$ $\rho_1 = \gamma_1/\gamma_0 = \theta/(\theta^2+1)$ $\rho_2 = \gamma_2/\gamma_0 = 0$

In order to have a covariance stationary MA(1) process, all three requirements for covariance stationarity must be fulfilled. Just by looking at the formula for the expected value and variance of Y_t, one understands that the first and second requirements indeed are fulfilled. Neither one of expressions are related to time. Moreover, to have a covariance stationary process, the autocovariances must be independent of where in the process the observations are found, and only depend on the distance between those observations. This is indeed the case, which is shown in Appendix A. As you see above, the first autocorrelation (using the lag 1) is a function of θ and σ^2, and the other covariances are simply 0. Hence, they do not depend on time, but only on the distance.

We can hence conclude that an MA(1) process should be stationary no matter what the parameter is, as long as it is finite. Moreover, the first autocovariance is (positively) related to the MA(1) parameter, while the following ones are not. This has of course implications for the autocorrelations, since the autocorrelation is a function of the covariance. We expect a spike in the ACF at k=1, and 0 thereafter. Just by looking at the model one realizes that this is the case. Y_t is related to the white noise only one period back. Hence, the interdependence between Y_t and $e_{t-k}, k > 1$ must be nonexistent.

The PACF shows the partial autocorrelation function, which shows the autocorrelation using the lag k while controlling for the other k-1 lags. Hence, it is the marginal correlation of Y_t and Y_{t-k}. We expect a geometrically decreasing PACF for an MA(1) process. Note here that the blue dotted lines in the correlagrams are the confidence intervals at the 95%-level. Hence, any spike reaching beyond a blue line is statistically different from 0 with 95% confidence.

To get an even deeper understanding of how the MA(1) proccess behaves, we will now turn to a simulation excercise where the parameter value will be varied. All simulation plots can be found in the Appendix B. We move through our tables from top to bottom, following the values of their parameters. We will be looking at each table at a time, with the following structure: Table X(variable = y,z,x), meaning the first model has a variable value of y, second model z, etc.

Table 1($\theta = -1, -0.45, 0$) When $\theta = -1$, Y_t depends negatively on e_{t-1}. The sample variance of the plot looks constant, which aligns with theory since the true variance is $1 * (1 + (-1)^2) = 2$. Moreover, there is no trend in our data, but rather a choppy spread centered around 0. The reason for this is the nature of e_t as an independent, identically distributed random variable. When turning to the correlograms, they look as expected. The sample ACF has a spike statistically different from 0 at k=1 (where the true $\rho = (-1/((-1)^2 + 1) = -0.5)$, and is 0 otherwise. The sample PACF looks geometrically decreasing, which is what we expect.

When θ =-0.45, the model resembles the previous one, to a large degree. The major difference is that the variance of Y_t now is smaller. One can see how the time series only moves between -3 and 3, instead of between -4 and 4 as in the previous case (note the different scales of the plots!) This makes sense, since the true variance in this case only is 1.2025. ACF looks like what we expect (true value -0.372 when k=1) and the PACF seems to be declining like we expect as well.

When $\theta = 0$, the moving average process collapses into just being white noise. Hence, we now have the model $Y_t = e_t$, with mean 0 and variance σ^2. Since the autocovariance is zero, so no matter what k is, $\rho = 0$ and hence we see no spikes whatsoever in the sample ACF. Y_t is *independently* distributed, which makes it not just covariance stationary but actually strict stationary.

Table 2($\theta = 0.45, 1, 2$) Here, θ is set to be *positive*, beginning at 0.45. Still, there is no evident trend in our data, but rather a choppy spread centered around 0. As in the previous cases, the variance of Y_t looks stable in the time series plot, being indicative of the true variance which in this case is constant at 1.2025. Note that the variance of Y_t thus does not depend on the sign of θ, since it equals the variance in the second row. Moreover, the sample value of ACF when k=1 is of the same magnitude as in row 2, but it is now positive. This also makes perfect sense, since Y_t is *positively* related to e_t in this case.

When increasing θ further to 1, the true variance increases to 2 (as in Table 1, $\theta = -1$). Moreover, the value of the ACF when k=1 is of the same magnitude as then, but now it is positive instead of negative. ACF and PACF both look correct. The plot is quite choppy, with a big ravine in the middle.

In the last row in Table 2, θ is increased to 2. This model is very similar to the previous one. The sample ACF and PACF both ressemble the previous ones. Hence, we have *doubled* θ, but we see no great changes in the simulation.

In summary, all simulations of the MA(1) process, using different parameter values, generate stationary samples of data. All realizations presented here have, by visual inpsection, constant means and variances. Moreover, the autocovariances do not depend on time but only on the time lag. Finally, in all cases except for when θ is 0, we have some interdependence, making the data covariance stationary instead of strict stationary.

1.2 Study for MA(2)

An MA(2) process includes not only white noise in the current and previous periods like the MA(1) process, but it also includes white noise *two* time periods back. More specifically, we have the model:

MA(2):

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

where

$$e_t \sim NID(0, 1)$$

As before, Y_t is an output variable, and e_t denotes the white noise. The θ_1 parameter describes how the previous period is related to Y_t, and θ_2 is the parameter describing how e two periods back is related to Y_t. Moreover, e_t is identically and independently distributed with mean 0 and a constant variance just as in the MA(1) process. Below, the statistical properties of an MA(2) process are presented. To see derivations and further discussion, please see Appendix A.

$E(Y_t) = 0$ $\gamma_0 = V(Y_t) = (1+\theta_1^2+\theta_2^2)*\sigma^2$ $\gamma_1 = Cov(Y_t, Y_{t-1}) = \theta_1*\sigma^2+\theta_1*\theta_2*\sigma^2$ $\gamma_2 = Cov(Y_t, Y_{t-2}) = \theta_2*\sigma^2$ $\rho_1 = \theta_1 * (\theta_2 + 1)/(\theta_1^2 + \theta_2^2 + 1)$ $\rho_2 = \theta_2/(\theta_2^2 + theta_1^2 + 1)$

In conclusion, an MA(2) process should be stationary no matter what the parameters are, as long as they are finite. When it comes to the ACF, we expect spikes at k=1 and at k=2, but not thereafter, since the autocovariances for k>2 are equal to 0. Also, we expect a geometrically decreasing PACF.

In the following, we will discuss simulations of the the MA(2) process using different combinations of θ_1 and θ_2. The section follows this structure: Table X(θ_2). θ_1 is set to the values, -0.8, 0 and 0.8 in that order.

Table 3($\theta_2 = 0$) In Table 3, since $\theta_2 = 0$, the MA(2) process essentially collapses into MA(1) processes or simply white noise, depending on what θ_1 is.

In the first cell $\theta_1 = -0.8$, creating an MA(1) process with the parameter value -0.8. The fact that this really is an MA(1) process is evident in the sample, since the sample ACF only has one spike, at k=1. Moreover, the PACF looks similar to the one in the MA(1) processes where θ was set to negative values in Table 1.

When $\theta_1 = 0$, we once again get a white noise process. For a further discussion on white noise, please review the section discussing the MA(1) process where $\theta = 0$ in Table 1. We can see similarities in ACF and PACF with this process, as we have no significant spikes, and a non-geometrical change in PACF.

When $\theta_1 = 0.8$, we get an MA(1) process with the parameter value 0.8. The sample ACF only has one spike, at k=1, and the PACF looks similar to the ones in the MA(1) processes where θ was set to 1 and 2 respectively. The time series looks fairly centered, however it is somewhat spread in some places.

Table 4($\theta_2 = 0.7$) In Table 4, $\theta_2 = 0.7$, creating a full MA(2) process, with terms for the white noise in the current and past white noise, as well as the white noise two periods back.

When $\theta_1 = -0.8$ and $\theta_2 = 0.7$, Y_t is positively related to e_t two periods back, and negatively related to e_t in the previous period. The sample variance look constant in the time series plot, which is according to theory since the true variance is constant at $1 * (1 + (-0.8)^2 + 0.7^2) = 1.4964$. Moreover, there is no trend in the data, but rather a choppy spread constantly around 0, just as in the MA(1) cases. The sample ACF now has two spikes, at k=1 and k=2. The first spike is negative (true value -0.638), indicating the negative relationship between Y_t and e_{t-1}. The second spike is on the other hand positive (true value 0.230), since e_{t-2} is positively related to Y_t.

When $\theta_1 = 0$, and $\theta_2 = 0.7$, the model becomes:

MA(2):

$$Y_t = 0.7 * e_{t-2} + e_t$$

where $e_t \sim NID(0, 1)$

We see only one spike in the sample ACF, at k=2, which aligns with theory. Since $\theta_1 = 0$, the true first autocorrelation becomes 0, and the second becomes -in this case- 0.329. The sample PACF is still geometrically decreasing. It does look like the sample estmation is a bit bigger, but still pretty close.

When $\theta_1 = 0.8$, and $\theta_2 = 0.7$, Y_t is positively related to e both one and two periods back. The sample variance looks constant in the time series plot, which aligns with theory; a constant true variance at 2.49. The sample ACF now has two spikes, at k=1 and k=2, and naturally, both spikes are positive. The true values of the ACF are 0.638 (k=1) and 0.329 (k=2).

Table 5($\theta_2 = 1$) In Table 5, $\theta_2 = 1$. Compared to before,

When $\theta_1 = -0.8$, and $\theta_2 = 1$, Y_t is negatively related to e one period back and positively realted to e two periods back. The variance looks constant in the time series plot, which makes sense since the true variance is constant at 2.64. The sample ACF looks as a good estimation of the true autocorrelations, as in the previous cases. The true value of the ACF when k=1 is -0.606, and when k=2 it is 0.379. The PACF is geometrically decreasing, as expected.

When $\theta_1 = 0.$ and $\theta_2 = 1$, we get the model MA(2):

$$Y_t = e_{t-2} + e_t$$

where $e_t \sim NID(0, 1)$

As was the case in Table 4 when $\theta_1 = 0$, there is only one spike in the sample ACF, at k=2. The sample PACF is geometrically decreasing, also as expected.

When $\theta_1 = 0.8$, and $\theta_2 = 1$ the process is in most aspects similar to when θ_2 is somewhat smaller.

1.3 Study for AR(1)

An autoregressive process of order 1, an AR(1), is a *recursive* process, where Y_t depends on itself one period back, as well as some white noise e_t. More specifically, we have the model:

$$Y_t = \phi_1 Y_{t-1} + e_t$$

Where we assume $\mathrm{Cov}\,(Y_t, e_t) = 0$, and where $e_t \sim iid(0, 1)$

ϕ_1 describes how Y_t depends on Y_{t-1}, and as before, e_t is identically and independently distributed with mean 0 and a constant variance. When having an AR process, it is impossible to calculate statistical properties such as expected value, variance and autocovariances without *assuming* stationarity. Thus, after the derivations have been made, one must check whether the assumptions actually are fulfilled in a particular AR(1). The full derivations are found in Appendix A, where the reasons for the neccessity of a stationarity assumption also are presented. Here, it suffices to just present the resulting formulas:

$$E(Y_t) = 0 \gamma_0 = \sigma^2/(1-\phi^2) \gamma_1 = \phi * \sigma^2/(1-\phi^2) \gamma_2 = \phi^2 * \sigma^2/(1-\phi^2) \rho_1 = \phi \rho_2 = \phi^2$$

Once again note that these formulas are derived under the *assumption* of stationarity. Consequently, they are not valid if the assumption of stationarity is violated.

Unlike as in the MA() processes, the ACF (the ρ:s) in an AR(1) will not drop to 0 when k>1. The reason is the recursiveness of the AR(1) process. Y_t depends on it's past values, which in turn depends on *it's* past values, etc. Hence, the relationship between Y_t and any Y_{t-k} can be decsribed by some specific function of ϕ_1 To illustrate, we can subtsitute Y_{t-1} in the model above:

AR(1):

$$Y_t = \phi_1(\phi_1 Y_{t-2} + e_{t-1}) + e_t Y_t = \phi_1^2(\phi_1 Y_{t-3} + e_{t-2}) + \phi_1 e_{t-1} + e_t Y_t = \phi_1^3(\phi_1 Y_{t-4} + e_{t-3}) + \phi_1 e_{t-2} + \phi_1 e_{t-1} + e_t etc.$$

It becomes evident that that Y_{t-2} is related to Y_t by ϕ_1^2, Y_{t-3} by ϕ_1^3 etc.

Any ρ_k will therefore also depend on ϕ_1, more specifically $\rho_k = \phi_1^k$. Hence, the ACF will *not* be 0 for k>1. Thus, as long as $\phi_1 < 1$, we will expect a geometric decrease in the ACF as k increases.

Also, we expect the PACF to have only one significant spike, at k=1, since this is the case for AR(1) processes.

To check for stationarity in AR(1) processes, one studies the *characteristic equation*, which is defined as $(1 - \phi_1 X) = 0$, generating the *root* $X = 1/\phi_1$. This root is to be evaluated in relation to the unit circle. If the roots in absolute value is smaller than 1, the process in question is nonstationary. Of course, this is equivalent to requiring the abosolute value of ϕ_1 to be smaller than 1. In summary, one must study a specific AR(1) process with a specific parameter value more in detail, to see whether it is stationary or not.

This will be done in the following. We will discuss in total 7 different AR(1) processes, with different of values on ϕ_1, and for each process, comments on stationarity will be made.

Table 6 In table 6 in Appendix B, we have the values $\phi_1 = -0.1$,$\phi_1 = -0.95$, and $\phi_1 = -0.75$. As we can see, we only have negative values for these cells. When $\phi_1 = -0.1$, we can immediately see how Y_t is centered around zero. Also, the variance seems to increase somewhat over time, which could indicate that this is not a covariance stationary process. However, the absolute value of ϕ is obviusly smaller than 1. Hence, we do have a covariance stationary process. This makes the formulas for the statistical prooperties above valid.

The sample ACF seems to oscillate between positive and negative values, reflecting the fact that Y_t is negatively related to Y_{t-1} in this process. It's hard to determine wether the ACF is actually declining or not, indicating a high inderdependence. The true value of the ACF when k=1 is -0.1 and when k=2 it is $(-0.1)^2 = 0.01$. Furthermore, the sample PACF only has one significant value and decreases very fast, which is to be expected for an AR(1) process.

When $\phi_1 = -0.95$ we have a rather choppy time series plot. The variance varies greatly from time period to time period, and Y_t looks centered around zero, wich would indicate nonstationarity. However ϕ_1 is still smaller than 1, prooving the process to be covariance stationarity.

The sample ACF still oscillates, and seems to be declining the way we expect it to as k increases. The true value of the ACF is -0.95 (k=1) and 0.9025 (k=2). The sample PACF looks like expected with only one significant value.

When $\phi_1 = -0.75$, we still have an uneven, choppy time series plot. It seems to be more centered around zero overall, with a reduced variance compared to the second cell - the values only strech between -4 and 4, instead of -6 and 6. We can also see that the sample ACF declines much faster than before. The sample PACF looks as expected.

Table 7 In table 6, we have the values $\phi_1 = -0$,$\phi_1 = 0.75$,$\phi_1 = -0.95$ and $\phi_1 = 1$. For this table, all values are positive. When $\phi_1 = -0$, the process just becomes white noise; we have no parameters but only the error term. For a closer discussion on white noise, please review the discussion for the MA(1) process where $\theta = 0$, in Table 1.

For $\phi_1 = 0.75$, the plot looks centered around zero, but it looks rather uneven and choppy in its variation, the points seem to be spread further apart. It is stationary since $\phi_1 < 0.75$. The sample ACF and PACF looks as expected, with a fast decline of the ACF.

The third cell $\phi_1 = 0.95$, looks even more uneven and spread out. The plot is uneven, going from 5 to -10. We seem to be looking at an increase in choppyness the more we go towards 1 in ϕ_1. It looks like the plot is still focused on zero, but there is a lot more randomness in how far away from zero the values stretch. The sample ACF decline has slowed considerably compared to the previous cell. The sample PACF looks normal.

For the fourth and final cell, $\phi_1 = 1$, creating a *random walk* process. Since $\phi_1 = 1$, the process is not stationary. In the time series plot, we can see any pretense of stationarity evaporating, as the plot has a slope. Thus, it is no longer centered around zero. It still is choppy and even more uneven than before, as the entire plot is moving downwards. The sample ACF at the top of Table 6 and this one actually look similar, if we were to disregard the oscillating ACF of $\phi_1 = -0.1$ and put all the values as positive. We can thus say that ACF starts declining slower when ϕ_1 becomes larger. PACF still remains the way we expect it to be, with only one spike

1.4 Study for AR(2)

An autoregressive process of order 2, an AR(2), is a process where Y_t depends on Y_{t-1} as well as Y_{t-2}, and some white noise e_t.The process is *recursive*; Y_t depends on it's own past values. More specifically, we have the model:

AR(2):

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

where $e_t \sim NID(0,1)$, and $Cov(Y_{t-1}, e_t) = 0$ by assumption.

As the AR(1), the AR(2) process is not allways stationairy; it depends on the parameters of the process. The statistical properties stated below are derived under the *assumption* of stationarity, just as in the AR(1) case. The full derivations are found in Appendix A, and only the resulting formulas are presented here;

Statistical properties:

$$E(Y_t) = 0 \gamma_0 = \gamma_1 = \gamma_2 = \rho_1 = \phi_1/(1-\phi_2) \rho_2 = \phi_1^2/(1-\phi_2) + \phi_2$$

As in the AR(1) case, the ACF (the ρ:s) in an AR(2) will not drop to 0 when k is greater than the order of the process (in this case when k>2). The reason is again the recursiveness of the AR process. To illustrate also in the AR(2) case, we can subtsitute Y_{t-1} and Y_{t-2} in the model above:

AR(2): $Y_t = \phi_1(\phi_1 Y_{t-2} + \phi_2 Y_{t-3} + e_{t-1}) + \phi_2(\phi 1_1 Y_{t-3} + \phi_2 Y_{t-4} + e_{t-2}) + e_t$

No need to mention that this substitution can go on forever, so that Y_t depends on all it's past values. It is harder to see the exact relationship here between Y_t and any Y_{t-k}, making it harder to establish ρ_k, but what we at least know is that any ρ_k will depend on ϕ_1 and ϕ_2 in some way.

To check for stationarity in AR(2) processes, one studies the characteristic equation, which is defined as $(1 - \phi_1 X - \phi_2 X^2) = 0$. Solving for X will in this case generate two X-values , two *roots* of the unit circle. If *either one* of these roots in absolute value is smaller than 1, the process in question is nonstationary. As in the AR(1) cases, one must study a specific AR(2) process, to see whether it is stationary or not.

This will be done in the following. We will discuss in total 9 different AR(2) processes, with different combinations of values on ϕ_1 and ϕ_2, and for each process, comments on stationarity will be made.

In Table 8 in Appendix B, ϕ_2 is held constant at 0.1, while ϕ_1 varies. In the following, each combination will be discussed.

When $\phi_1 = -0.9$, the Y_t is negatively related to Y_{t-1}, and positively related to Y_{t-2}. The time series plot is centered quite densely around the mean, which by inspection seems to be 0. Hence, the first requirement for covariance stationarity seems to be fullfilled; a constant mean. However, the variance is obviously not constant; it increases over time and in some kind of cyclic pattern. Hence, this AR(2) process does not seem to fulfill the second requirement, namely a constant variance. This can be formally checked by analyzing the characteristic equation, which in this case is:

$$(1 - (-0.9)X - 0.1X^2) = 0(1 + 0.9X - 0.1X^2) = 0$$

solving for X gives the roots $X_1 = 9.1$ and $X_2 = -0.11$, and since the latter is smaller than the unit circle, the process is indeed nonstationary. This in turn invalidates the statistical properties stated above.

Turning to the sample ACF and PACF; we see a large interdependence, and basically no decrease of the spikes in the ACF as k increases. At the first lag, the sample autocorrelation is negative, since Y_t is negatively related to itself one period back. Using the same logic the autocorrelation at the second lag is positive. Moreover, we see that the ACF do not drop to 0 when k becomes bigger than 2 as in the MA() processes. This is exactly what we expected. Also, no decrese as k increses is seen in the sample ACF, indicating a large interdependence. According to the formulas for ρ_1 and ρ_2 above, the value of the ACF when k=1 is $(-0.9)/(1 - 0.1) = -1$ and when k=1 it is $(-0.9)^2/(1 - 0.1) + 0.1 = 1$. However, those formulas were derived under the assumption of stationarity, which is not the case in this particular process.

The sample PACF shows two significant spikes at k=1 and k=2, and thereafter the spikes are undistinguishable from 0.

When $\phi_1 = 0$ instead, we get the model:

AR(2):

$$Y_t = 0.1 * Y_{t-2} + e_t$$

So that Y_t only depends on Y two time periods back, as well as the white noise. When studying the time series plot and the correlograms, it actually reminds a little bit of a white noise process. Perhaps, the reason is that ϕ_2 is pretty small, so that the relationship between Y_t and Y_{t-2} does not become that pronounced. (For a discussion about white noise, please review the discussion of the MA(1) process, where $\theta = 0$, in Table 1.) However, a vague resemblance is not an exact match.

To evaluate stationarity we turn to the characteristic equation:

$$(1 - 0.1X^2) = 0$$

Solving for X we get the roots Xś3.16, placing us outside the unit circle. Thus, this process is stationary, and as a result, the stated formulas above are valid. Calculating the first autocorrelation, we get $0/(1 + 0.9) = 0$, and the second one is $0/(1+0.9)+0.1 = 0.1$. Looking at the sample ACF, it actually looks like a good estimation of the true value, but since it is so small it is naturally harder to make precise estimations. Therefore, it falls within the 95% confidence interval.

When $\phi_1 = 0.7$ (and $\phi_2 = 0.1$ as before), the time series plot becoem less centered around 0. Instead, it looks like it is drifting a bit, although still around 0. Also, it is hard to say anything about whether the variance is constant or not. Consequently, it is hard to discern evidence on stationarity from the plot. Instead, we turn to the characteristic equation:

$$(1 - 0.7X - 0.1X^2) = 0$$

Which generates the roots $X = 1.22$ and $X = -8.22$, placing us outside the unit circle. Hence, this particular AR(2) process is stationary. The true mean is 0, and the variance is constant. Of course, the autocovariances also consequently only depends on k, and not on time, since this is the third requirement for stationarity.

The true first autocorrelation is 0.778, an the second is 0.644. THIS DOES NOT MATCH THE CORRELOGRAM!!

In Table 9 in Appenx B, $\phi_2 = 0.2$, while ϕ_1 is allowed to vary. In the following, each combination will be discussed.

When ϕ_1 simultaneously was set to -0.9, the simulation program broke down using the sample size 5000. Therefore, the plot shows a sample of 500 instead. In this AR process, Y_t depends negatively on Y one period back, and positively on Y two periods back. Perhaps, it can be thought of as a more "extreme" version of the AR(2) $\phi_2 = 0.1$, $\phi_1 = -0.9$ in Table 8. The difference between the two processes is a doubling of ϕ_2. The time series plot shows little variation in Y (note however the scale!) until very late in the process, where the variance litteraly explodes. At the original sample size, the variance probably became so very large so that the program could not handle it anymore. So the variance seems to diverge into infinity, and consequently, this process is obviously not stationary. However to formally analyze this, we take a look at the characteristic equation:

$$(1 - (-0.9)X - 0.2X^2) = 0(1 + 0.9X - 0.2X^2) = 0$$

Which generates the roots: $X_1 = 5.42$ and $X_2 = -0.922$. Since X_2 is within the unit circle, the process is nonstationary. Moreover, the sample ACF is geometrically decreasing. At k=1, the value of the ACF is negative, and at k=2 it is positive, reflecting the signs of ϕ_1 and ϕ_2 respectively. Thereafter, the sample ACF is geometrically decreasing, reflecting the recursiveness of the process. The sample PACF shows a large spike at k=1, and thereafter no significant spikes at all.

When ϕ_1 is set to 0, we get the model: AR(2):

$$Y_t = 0.2 * Y_{t-2} + e_t$$

so that Y_t depends on itself only two periods back. In many aspects, it is very similar to the AR(2) process $\phi_1 = 0, \phi_2 = 0.1$ in Table 8. However here, we do see some significant spikes in correlogram. That is logic, since some more interdependence should be expected when ϕ_2 is doubled. To check for stationarity we set up the characteristic equation:

$$(1 - 0.2X^2) = 0$$

Which generates the roots Xś2.23. Since they are outside the unit circle, this process is stationary too.

When ϕ_1 is set to 0.7, the time series plot shows clear signs of a cyclic variation, centered around a value smaller than 0.

The caracteristic equation:

$$(1 - 0.7X - 0.2X^2) = 0$$

generates the roots $X_1 = 4.59$ and $X_2 = -1.09$. Since they are both outside the unit circle, the process is stationary. The true first autocorrelation is $(0.7/(1-0.2) = 0.875$ and the second is $0.7^2/(1-0.2)+0.2 = 0.813$. They seem to be properly estimated when looking at the sample ACF graph. The spikes in the sample ACF continue to decrease towards 0 as k increases. Moreover, there are 2 significant spikes in the sample

PACF. This process has many similarities with the AR(2) $\phi_1 = 0.7, \phi_2 = 0.1$ in Table 8. Here however, ACF decreases at a greater speed. Also, the value of the sample PACF is a lot larger, while it is smaller when k=2.

In Table 10 in Appendix B, $\phi_2 = 0.8$, while ϕ_1 is allowed to vary. In the following, each combination of parameters will be discussed.

When $\phi_1 = -0.9$ the simulation program broke down at the original sample size. Therefore, we chose a sample size of 500 instead. The process very much looks like a further more "extreme" case of the AR(2) $\phi_2 = 0.1$, $\phi_1 = -0.9$ in Table 8, as well as the AR(2) $\phi_2 = 0.2$, $\phi_1 = -0.9$ in Table 9. The variance increases towards infinity, and the process is obviously not stationary. Hence, as ϕ_1 is held constant at -0.9, the processes become more and more "extreme" as ϕ_2 increases.

When ϕ_1 is set to 0, we once again (as in AR(2) $\phi_2 = 0.1\phi_1 = 0$ in Table 8, or $\phi_2 = 0.2\phi_1 = 0$ in Table 9) have an AR(2) model with only two terms: AR(2):

$$Y_t = 0.8 * Y_{t-2} + e_t$$

However this process does not at all resemble white noise. We have a lot of significant spikes in the sample ACF, and one significant spike in the sample PACF. The interpendence is naturally larger here, since ϕ_2 has been quadrupled compared to Table 9.

To check for stationarity we set up the characteristic equation:

$$(1 - 0.8X^2) = 0$$

Which generates the roots Xś1.118. Since they are outside the unit circle, this process is stationary. One interesting note is that the bigger the ϕ_2 has become (holding ϕ_1 constant at 0), the closer have the roots come to unity. It is very plausible that if ϕ_2 was increased further, the process would become nonstationary.

When $\phi_1 = 0.7$ in Table 10, the simulation program broke down at the original sample size. Therefore, the sample size was set at n=500 instead. In the time series plot, there seems to be little variation (not however the scale!) until the time periods just before 500, where it dramatically drops. Actually, this looks like an ARMA process.

We look for stationarity by analyzing the characteristic equation:

$$(1 - 0.7X - 0.8X^2)$$

which gives the roots: $X_1 = -1.64$ and $X_2 = 0.76$. Since the latter is within the unit circle, the process is -perhaps not suprisingly after inspecting the time series plot- nonstationary. The sample ACF is decreasing as k increases, and the sample PACF is showing one spike at k=1 but nothing thereafter.

2 Conclusion

2.1 Appendix A

2.2 Derivation for MA(1)

2.2.1 Model

$$Y_t = e_t - \theta e_{t-1}$$

$$e_t \sim IID\left(0, \sigma^2\right)$$

2.2.2 Mean

$$\begin{aligned}
&E\left(Y_t\right) = E\left(e_t - \quad \theta e_{t-1}\right) \\
&E\left(Y_t\right) = E\left(e_t\right) - E\left(\theta e_{t-1}\right) \\
&E\left(Y_t\right) = E\left(e_t\right) - \theta E\left(e_{t-1}\right) \\
&E\left(Y_t\right) = 0 \quad - \theta \times 0 \\
&E\left(Y_t\right) = 0
\end{aligned}$$

2.2.3 Variance

$$\begin{aligned}
&\operatorname{Var}\left(Y_t\right) = V\left(Y_t\right) \\
&\operatorname{Var}\left(Y_t\right) = V\left(e_t + -\theta e_{t-1}\right) \\
&\operatorname{Var}\left(Y_t\right) = V\left(e_t\right) + V\left(-\theta e_{t-1}\right) + 2\operatorname{Cov}\left(e_t, -\theta e_{t-1}\right) \\
&\operatorname{Var}\left(Y_t\right) = V\left(e_t\right) + (-\theta)^2 V\left(\quad e_{t-1}\right) + (-\theta) 2\operatorname{Cov}\left(e_t, \quad e_{t-1}\right) \\
&\operatorname{Var}\left(Y_t\right) = V\left(e_t\right) + \quad \theta^2 \quad V\left(\quad e_{t-1}\right) + (-\theta) 2 \times 0 \\
&\operatorname{Var}\left(Y_t\right) = V\left(e_t\right) + \quad \theta^2 \quad V\left(\quad e_{t-1}\right) \\
&\operatorname{Var}\left(Y_t\right) = V\left(e_t\right) + \theta^2 V\left(e_t\right) \\
&\operatorname{Var}\left(Y_t\right) = \sigma^2 + \theta^2 \sigma^2 \\
&\operatorname{Var}\left(Y_t\right) = \sigma^2\left(1 + \theta^2\right)
\end{aligned}$$

2.2.4 First autocovariance

$$\begin{aligned}
\operatorname{cov}\left(Y_t, Y_{t-1}\right) &= \operatorname{Cov}\left(\theta_1 e_{t-1} + e_t, \quad \theta_1 e_{t-2} + e_{t-1}\right) \\
&= \operatorname{Cov}\left(\theta_1 e_{t-1} \quad , \quad \theta_1 e_{t-2} \quad \right) \\
&+ \operatorname{Cov}\left(\theta_1 e_{t-1} \quad , \quad e_{t-1}\right) \\
&+ \operatorname{Cov}\left(\quad e_t, \quad \theta_1 e_{t-2} \quad \right) \\
&+ \operatorname{Cov}\left(\quad e_t, \quad e_{t-1}\right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{Cov}\left(Y_t, Y_{t-1}\right) &= \theta_1^2 \operatorname{Cov}\left(e_{t-1}, e_{t-2}\right) \\
&+ \theta_1 \operatorname{Cov}\left(e_{t-1}, e_{t-1}\right) \\
&+ \theta_1 \operatorname{Cov}\left(e_t, e_{t-2}\right) \\
&+ \quad \operatorname{Cov}\left(e_t, e_{t-1}\right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{Cov}\left(Y_t, Y_{t-1}\right) &= \theta_1^2 \times 0 \\
&+ \theta_1 \sigma^2 \\
&+ \theta_1 \times 0 \\
&+ \quad 0 \\
&= \theta_1 \sigma^2
\end{aligned}$$

2.2.5 Second autocovariance

$$
\begin{aligned}
\operatorname{cov}(Y_t, Y_{t-2}) &= \operatorname{Cov}(\theta_1 e_{t-1} + e_t, \quad \theta_1 e_{t-3} + e_{t-2}) \\
&= \operatorname{Cov}(\theta_1 e_{t-1} \quad , \quad \theta_1 e_{t-3} \quad) \\
&+ \operatorname{Cov}(\theta_1 e_{t-1} \quad , \quad e_{t-2}) \\
&+ \operatorname{Cov}(\quad e_t, \quad \theta_1 e_{t-3} \quad) \\
&+ \operatorname{Cov}(\quad e_t, \quad e_{t-2})
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Cov}(Y_t, Y_{t-2}) = &\ \theta_1^2 \operatorname{Cov}(e_{t-1}, e_{t-3}) \\
&+ \theta_1 \operatorname{Cov}(e_{t-1}, e_{t-2}) \\
&+ \theta_1 \operatorname{Cov}(e_t, e_{t-3}) \\
&+ \quad \operatorname{Cov}(e_t, e_{t-2})
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Cov}(Y_t, Y_{t-1}) = &\ \theta_1^2 \times 0 \\
&+ \theta_1 \times 0 \\
&+ \theta_1 \times 0 \\
&+ \quad 0 \\
&= \theta_1 \sigma^2
\end{aligned}
$$

2.2.6 First autocorrelation

$$
\begin{aligned}
\gamma_0 &= \sigma^2 \left(1 + \theta^2\right) \\
\gamma_1 &= -\theta \sigma^2 \\
\rho_1 &= \frac{-\theta \sigma^2}{\sigma^2 (1 + \theta^2)} \\
\rho_1 &= \frac{-\theta}{(1 + \theta^2)}
\end{aligned}
$$

2.2.7 Second autocorrelation

$$
\begin{aligned}
\rho_2 &= \frac{\gamma_2}{\gamma_0} \\
\rho_2 &= \frac{0}{\sigma^2 (1 + \theta^2)} \\
\rho_2 &= 0
\end{aligned}
$$

2.2.8 General expression for the autocorrelation

$$
\gamma_k = 0 \text{ for all } k \geq 2
$$
$$
\rho_k = \tfrac{\gamma_k}{\gamma_0} = 0 \text{ for all } k \geq 2
$$

2.3 Derivation for MA(2)

2.3.1 Model

$$
\begin{gathered}
Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} \\
e_t \sim \text{IID}\left(0, \sigma^2\right)
\end{gathered}
$$

2.3.2 Mean

$$
\begin{aligned}
&E(Y_t) = E(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\
&E(Y_t) = E(e_t) - E(\theta_1 e_{t-1}) - E(\theta_2 e_{t-2}) \\
&E(Y_t) = E(e_t) - \theta_1 E(e_{t-1}) - \theta_2 E(e_{t-2}) \\
&E(Y_t) = 0 - \theta_1 \times 0 - \theta_2 \times 0 \\
&E(Y_t) = 0
\end{aligned}
$$

2.3.3 Variance

$$
\begin{aligned}
&\gamma_0 = V(Y_t) \\
&\gamma_0 = V(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\
&\gamma_0 = V(e_t) + V(-\theta_1 e_{t-1}) + V(-\theta_2 e_{t-2}) \\
&\gamma_0 = \sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2 \\
&\gamma_0 = \sigma^2 \left(1 + \theta_1^2 + \theta_2^2\right)
\end{aligned}
$$

2.3.4 First autocovariance

$$
\begin{aligned}
\gamma_1 = &\quad \operatorname{Cov}\ [Y_t, Y_{t-1}] \\
\gamma_1 = &\quad \operatorname{Cov}\ [(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}), (e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3})]
\end{aligned}
$$

So all the covariances except $\operatorname{Cov}(-\theta_1 e_{t-1}), e_{t-1}$ and $\operatorname{Cov}(-\theta_2 e_{t-2}, -\theta_1 e_{t-2})$ will be zero.

We have that

$$
\begin{gathered}
\operatorname{Cov}(-\theta_1 e_{t-1}, e_{t-1}) = -\theta_1 \sigma^2 \\
\operatorname{Cov}(-\theta_2 e_{t-2}, -\theta_1 e_{t-2}) = \theta_1 \theta_2 \sigma^2
\end{gathered}
$$

So

$$
\begin{aligned}
&\gamma_1 = 0 + 0 + 0 - \theta_1 \sigma^2 + 0 + 0 + \theta_1 \theta_2 \sigma^2 + 0 \\
&\gamma_1 = \sigma^2 (\theta_1 \theta_2 - \theta_1) \\
&\gamma_1 = \sigma^2 \theta_1 (\theta_2 - 1)
\end{aligned}
$$

2.3.5 Second autocovariance

$$\gamma_2 = \quad \text{Cov} \quad [Y_t \quad , Y_{t-2}]$$
$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$
$$Y_{t-2} = e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}$$

Thus,

$$\gamma_2 = \text{Cov}[(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}), (e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4})]$$
$$\gamma_2 = 0 + 0 + 0 + 0 + 0 + 0 + \text{Cov}\left(-\theta_2 e_{t-2}, e_{t-2}\right) + 0 \ldots$$
$$\gamma_2 = 0 + 0 + 0 + 0 + 0 + 0 + \text{Cov}\left(-\theta_2 e_{t-2}, e_{t-2}\right) + 0 \ldots$$
$$\gamma_2 = -\theta_2 Var\left(e_{t-2}, e_{t-2}\right)$$
$$\gamma_2 = -\theta_2 \sigma^2$$

2.3.6 First autocorrelation

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

$$\gamma_1 = \sigma^2 \theta_1 \left(\theta_2 - 1\right)$$

$$\gamma_0 = \sigma^2 \left(1 + \theta_1^2 + \theta_2^2\right)$$

thus, the first autocorrelation is

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1 \left(\theta_2 - 1\right)}{\left(1 + \theta_1^2 + \theta_2^2\right)}$$

2.3.7 Second autocorrelation

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

$$\gamma_2 = -\theta_2 \sigma^2$$

$$\gamma_0 = \sigma^2 \left(1 + \theta_1^2 + \theta_2^2\right)$$

thus, the second autocorrelation is

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{-\theta_2}{\left(1 + \theta_1^2 + \theta_2^2\right)}$$

2.3.8 General expression for the autocorrelation

$$\gamma_k = 0 \text{ for all } k \geq 3$$
$$\rho_k = \tfrac{\gamma_k}{\gamma_0} = 0 \text{ for all } k \geq 3$$

2.4 Derivation for AR(1)

2.4.1 Model

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + e_t$$

Where we assume

$$\text{Cov}\left(Y_t, e_t\right) = 0$$

and where $e_t \sim iid(0, 1)$

2.4.2 Mean

$$\begin{aligned}
E\left(Y_t\right) &= E\left(\phi_0 + \phi_1 Y_{t-1} + e_t\right) \\
E\left(Y_t\right) &= \phi_0 + \phi_1 E\left(Y_{t-1}\right) + E\left(e_t\right)
\end{aligned}$$

Note that under covariance stationarity, we have the following:

$$E\left(Y_{t-1}\right) = E\left(Y_t\right)$$

thus,

$$\begin{aligned}
E\left(Y_t\right) &= \phi_0 + \phi_1 E\left(Y_t\right) + 0 \\
E\left(Y_t\right) - \phi_1 E\left(Y_t\right) &= \phi_0 \\
E\left(Y_t\right)\left(1 - \phi_1\right) &= \phi_0 \\
E\left(Y_t\right) &= \frac{\phi_0}{\left(1 - \phi_1\right)}
\end{aligned}$$

2.4.3 Variance

$$\begin{aligned}
&\gamma_0 = Var\left(\phi_1 Y_{t-1} + e_t\right) \\
&\gamma_0 = Var\left(\phi_1 Y_{t-1}\right) + Var\left(e_t\right) + 2\,\text{Cov}\left(\phi_1 Y_{t-1}, e_t\right) \\
&\gamma_0 = \phi_1^2 Var\left(Y_{t-1}\right) + Var\left(e_t\right) + 2\phi_1\,\text{Cov}\left(Y_{t-1}, e_t\right) \\
&\gamma_0 = \phi_1^2 Var\left(Y_t\right) + \sigma^2 + 2\phi_1 \times 0 \\
&\gamma_0 = \phi_1^2 \gamma_0 + \sigma^2 \\
&\gamma_0 = \sigma^2/(1 - \phi_1^2)
\end{aligned}$$

2.4.4 First autocovariance

$$\begin{aligned}
&\gamma_1 = Cov\left(Y_t, Y_{t-1}\right) \\
&\gamma_1 = Cov\left(\phi Y_{t-1} + e_t, Y_{t-1}\right) \\
&\gamma_1 = Cov\left(\phi Y_{t-1}, Y_{t-1}\right) + Cov(e_t, Y_{t-1}) \\
&\gamma_1 = Cov\left(\phi Y_{t-1}, Y_{t-1}\right) + 0 \\
&\gamma_1 = \phi Cov(Y_{t-1}, Y_{t-1})
\end{aligned}$$

Since we assume stationarity, the autocovariance only depends on the lag and not on where in the process we are. Hence, the covariance does not change when we lead or lag the variables, as long as we make sure to keep the distance between them. We lead the variables in the expression above, and get;

$$\gamma_1 = \phi Cov(Y_t, Y_{t-0})$$

and since

$$Cov(Y_t, Y_{t-0}) = Var(Y_{t-0}) = \gamma_0$$

we have

$$\gamma_1 = \phi\gamma_0$$

By inserting the expression for γ_0, we get;

$$\gamma_1 = \phi * \sigma^2/(1 - \phi^2)$$

2.4.5 Second autocovariance

$$\begin{aligned}
\gamma_2 &= Cov\,(Y_t, Y_{t-2}) \\
\gamma_2 &= Cov\,(\phi Y_{t-1} + e_t, Y_{t-2}) \\
\gamma_2 &= Cov\,(\phi Y_{t-1}, Y_{t-2}) = 0 \\
\gamma_2 &= \phi Cov\,(Y_{t-1}, Y_{t-2})
\end{aligned}$$

Assuming stationarity, we can lead the variables and get;

$$\gamma_2 = \phi Cov\,(Y_t, Y_{t-1})$$

Since

$$Cov(Y_t, Y_{t-1}) = Var(Y_{t-1}) = \gamma_1$$

We have;

$$\gamma_2 = \phi\gamma_1$$

By inserting the expression for γ_1, we get;

$$\gamma_2 = \Phi^2 * \sigma^2/(1 - \Phi^2)$$

2.4.6 First autocorrelation

$$\rho_k = \phi^k$$

thus we have

$$\rho_1 = \phi^1 = \phi$$

2.4.7 Second autocorrelation

$$\rho_k = \phi^k$$

thus we have

$$\rho_2 = \phi^2$$

2.5 Derivation for AR(2)

2.5.1 Model

We have the general AR(2) model:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

Where we assume

$$\text{Cov}\left(Y_{t-1}, e_t\right) = 0$$

and

$$\text{Cov}\left(Y_{t-2}, e_t\right) = 0$$

and where $e_t \sim iid(0,1)$

2.5.2 Mean

$$\begin{aligned}
&E\left(Y_t\right) = E\left(\phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t\right)\\
&E\left(Y_t\right) = \phi_0 + \phi_1 E\left(Y_{t-1}\right) + \phi_2 E(Y_{t-2}) + E\left(e_t\right)\\
&E\left(Y_t\right) = \phi_0 + \phi_1 E\left(Y_{t-1}\right) + \phi_2 E(Y_{t-2})
\end{aligned}$$

then we assume stationarity, so that the expected value is constant over time;

$$\begin{aligned}
&E\left(Y_t\right) - \phi_1 E\left(Y_t\right) + \phi_2 E(Y_t) = \phi_0\\
&E\left(Y_t\right)\left(1 - \phi_1 - \phi_2\right) = \phi_0
\end{aligned}$$

thus

$$E\left(Y_t\right) = \phi_0/(1 - \phi_1 - \phi_2)$$

2.5.3 Variance

$$\begin{aligned}
&\gamma_0 = Cov\left(Y_t, Y_t\right)\\
&\gamma_0 = Cov\left(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_t\right)\\
&\gamma_0 = Cov\left(\phi_1 Y_{t-1}, Y_t\right) + Cov\left(\phi_2 Y_{t-2}, Y_t\right) + Cov\left(Y_t, e_t\right)\\
&\gamma_0 = Cov\left(\phi_1 Y_{t-1}, Y_t\right) + Cov\left(\phi_2 Y_{t-2}, Y_t\right) + \sigma^2\\
&\gamma_0 = \phi_1 Cov\left(Y_{t-1}, Y_t\right) + \phi_2 Cov\left(Y_{t-2}, Y_t\right) + \sigma^2
\end{aligned}$$

Since we assume stationairity, we can lead the variables;

$$\gamma_0 = \phi_1 Cov\left(Y_t, Y_{t+1}\right) + \phi_2 Cov\left(Y_t, Y_{t+2}\right) + \sigma^2 \gamma_0 = \phi_1 \gamma_{-1} + \phi_2 \gamma_{-2} + \sigma^2$$

Which of course, under stationarity, also can be expressed as;

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2$$

2.5.4 The First autocovariance

$$\gamma_1 = Cov\left(Y_t, Y_{t-1}\right)$$
$$\gamma_1 = Cov\left(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_{t-1}\right)$$
$$\gamma_1 = Cov\left(\phi_1 Y_{t-1}, Y_{t-1}\right) + Cov\left(\phi_2 Y_{t-2}, Y_{t-1}\right) + Cov\left(e_t, Y_{t-1}\right)$$
$$\gamma_1 = Cov\left(\phi_1 Y_{t-1}, Y_{t-1}\right) + Cov\left(\phi_2 Y_{t-2}, Y_{t-1}\right) + 0$$
$$\gamma_1 = \phi_1 Cov\left(Y_{t-1}, Y_{t-1}\right) + \phi_2 Cov\left(Y_{t-2}, Y_{t-1}\right)$$

Assuming stationarity, we can lead the variables as many times as we want, as long as we keep the distance between them constant:

$$\gamma_1 = \phi_1 Cov\left(Y_t, Y_t\right) + \phi_2 Cov\left(Y_t, Y_{t+1}\right)$$

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_{-1}$$

Which then of course also can be expressed as:

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$$

2.5.5 The second autocovariance

$$\gamma_2 = Cov\left(Y_t, Y_{t-2}\right)$$
$$\gamma_2 = Cov\left(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_{t-2}\right)$$
$$\gamma_2 = Cov\left(\phi_1 Y_{t-1}, Y_{t-2}\right) + Cov\left(\phi_2 Y_{t-2}, Y_{t-2}\right) + Cov\left(e_t, Y_{t-2}\right)$$
$$\gamma_2 = Cov\left(\phi_1 Y_{t-1}, Y_{t-2}\right) + Cov\left(\phi_2 Y_{t-2}, Y_{t-2}\right) + 0$$
$$\gamma_2 = \phi_1 (Cov\left(Y_{t-1}, Y_{t-2}\right) + \phi_2 Cov\left(Y_{t-2}, Y_{t-2}\right)$$

Assuming stationarity, we can lead the variables as many times we wish, as long as we keep the distance between them;

$$\gamma_2 = \phi_1 (Cov\left(Y_t, Y_{t-1}\right) + \phi_2 Cov\left(Y_t, Y_t\right) \gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0$$

2.5.6 The Equation system for the variance and autocovariances

The expressions for γ_0,γ_1 and γ_2 provide an equation system with three equations and three unknowns:

$$(i) \quad \gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2 (ii) \qquad \gamma_1 = \phi_1 \gamma_0 / (1 - \phi_2)(iii) \qquad \gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0$$

This equation system has to be solved in order to get the variance and the autocovariances as functions of ϕ and σ^2. (i)

$$=> \gamma_2 = (\gamma_0 - \phi_1 \gamma_1 - \sigma^2)/\phi_2$$

Insert (ii) in (iii) =>

$$\gamma_2 = (\phi_1^2 \gamma_0)/(1 - \theta_2) + \phi_2 \gamma_0$$

Equating the both expressions:

$$\gamma_2 = \gamma_2(\gamma_0 - \phi_1 \gamma_1 - \sigma^2)/\phi_2 = (\phi_1^2 \gamma_0)/(1 - \theta_2) + \phi_2 \gamma_0$$

Solving for γ_0 gives;

$$\gamma_0 = \sigma^2 (1 - \phi_2)/(1 - \phi_2 - \phi_1^2 - \phi_1^2 \phi_2 - \phi_2^2 - \phi_2^3)$$

Inserting this in (ii) gives an expression for γ_1:

$$\gamma_1 = \phi_1 \sigma^2 / [(1 - \phi_2)(1 - \phi_2 - \phi_1^2 - \phi_1^2 \phi_2 - \phi_2^2 - \phi_2^3)]$$

And inserting the expressions for γ_0 and γ_1 into (iii) gives the expression for γ_2:

$$\gamma_2 = [\sigma^2(\phi_1^2 + \phi_2(1 - \phi_2)^2)]/[(1 - \phi_2)(1 - \phi_2 - \phi_1^2 - \phi_1^2 \phi_2 - \phi_2^2 - \phi_2^3)]$$

2.5.7 First autocorrelation

$$\rho_k = \phi_1(\gamma_{k-1}/\gamma_0) + \phi_2(\gamma_{k-2}/\gamma_0)$$

which is

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$$

Correlations are symmetrical, so

$$\rho_k = \phi_1\rho_1 + \phi_2\rho_2$$

$$\rho_2 = \phi_1\rho_0 + \phi_0\rho_1$$

meaning that

$$\rho_1 = (\phi_1/(1-\phi_2))\rho_0$$

$$\rho_0 = \gamma_0/\gamma_0 = 1$$

so the first autocorrelation is

$$\rho_1 = (\phi_1/(1-\phi_2))$$

2.5.8 Second autocorrelation

$$\rho_k = \phi_1(\gamma_{k-1}/\gamma_0) + \phi_2(\gamma_{k-2}/\gamma_0)$$

which is

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$$

Correlations are symmetrical, so

$$\rho_k = \phi_1\rho_1 + \phi_2\rho_2$$

thus

$$\rho_1 = \phi_1\rho_1 + \phi_2\rho_0$$

meaning that

$$\rho_1 = \phi_1\rho_1 + \phi_2$$

since

$$\rho_0 = \gamma_0/\gamma_0 = 1$$

so the second autocorrelation is

$$\rho_2 = \phi_1(\phi_1/(1-\phi_2)) + \phi_2$$

which is the same as

$$\rho_2 = \phi_1^2/(1-\phi_2) + \phi_2$$

we convert them to one numenator and get

$$\rho_2 = \phi_1^2 + \phi_2(1-\phi_2)/(1-\phi_2)$$

MA(1), $\theta = 1$, table 1, appendix B

2.6 Derivation for ARMA(1,1)

We merge AR with MA to get ARMA

AR-part: $\phi(B)Y_t$

MA-part: $\phi(B)Y_t$

ARMA-model: $\phi(B)Y_t = \phi(B)Y_t$

For ARMA

2.6.1 Mean

2.6.2 Variance

2.6.3 First autocovariance

2.6.4 Second autocovariance

2.6.5 First correlation

2.6.6 Second correlation

2.7 Appendix B

2.7.1 Models - MA(1)

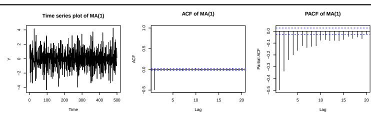
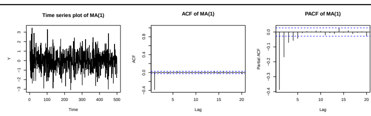
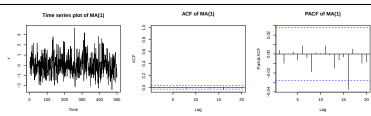
θ_2 θ_1	$\theta_2 = 0$
$\theta_1 = -1.0$	
$\theta_1 = -0.45$	
$\theta_1 = 0$	

Table 1: Table of MA(1) - (1 of 2)

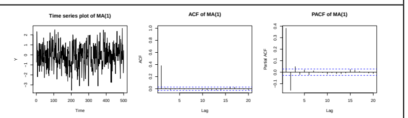
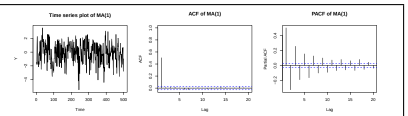
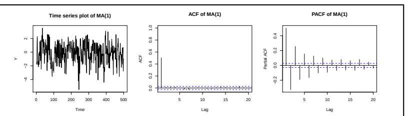
$\theta_1 \backslash \theta_2$	$\theta_2 = 0$
$\theta_1 = 0.45$	
$\theta_1 = 1$	
$\theta_1 = 2$	

Table 2: Table of MA(1) - (2 of 2)

2.7.2 Models - MA(2)

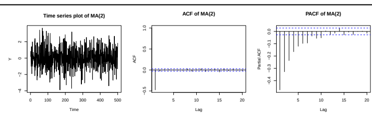
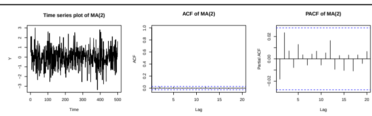
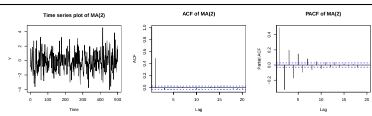
θ_1 \ θ_2	$\theta_2 = 0$
$\theta_1 = -0.8$	
$\theta_1 = 0$	
$\theta_1 = 0.8$	

Table 3: Table of MA(2) - (1 of 3)

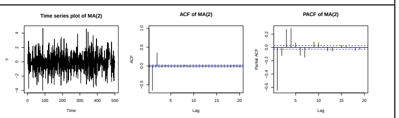
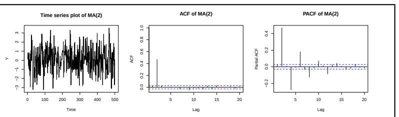
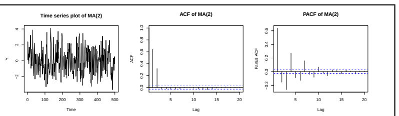
θ_1 \ θ_2	$\theta_2 = 0.7$
$\theta_1 = -0.8$	
$\theta_1 = 0$	
$\theta_1 = 0.8$	

Table 4: Table of MA(2) - (2 of 3)

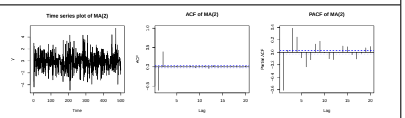
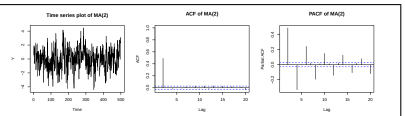
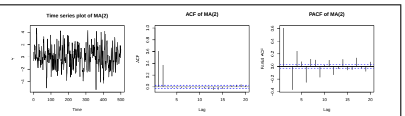
θ_1 \ θ_2	$\theta_2 = 1$
$\theta_1 = -0.8$	
$\theta_1 = 0$	
$\theta_1 = 0.8$	

Table 5: Table of MA(2) - (3 of 3)

2.7.3 Models - AR(1)

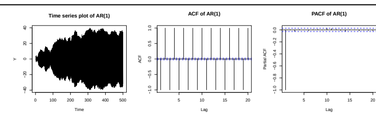
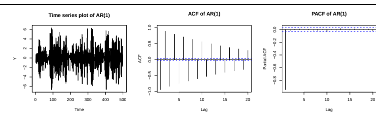
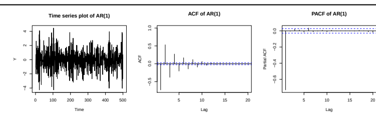
ϕ_1 \ ϕ_2	$\phi_2 = 0$
$\phi_1 = -0.1$	
$\phi_1 = -0.95$	
$\phi_1 = -0.75$	

Table 6: Table of AR(1) - (1 of 2)

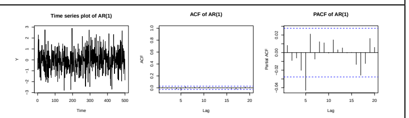
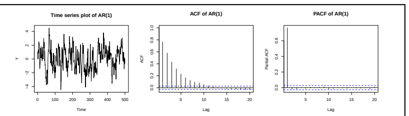
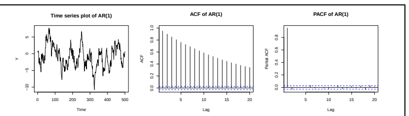
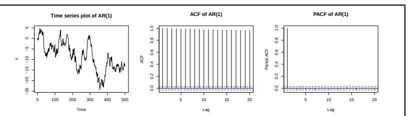
ϕ_1 \ ϕ_2	$\phi_2 = 0$
$\phi_1 = 0$	
$\phi_1 = 0.75$	
$\phi_1 = 0.95$	
$\phi_1 = 1$	

Table 7: Table of AR(1) - (2 of 2)

2.7.4 Models - AR(2)

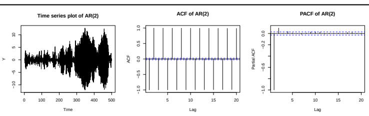
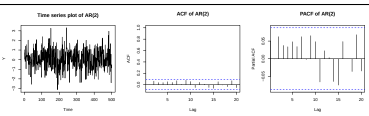
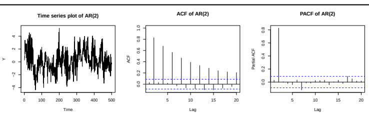
ϕ_1 \ ϕ_2	$\phi_2 = 0.1$
$\phi_1 = -0.9$	
$\phi_1 = 0$	
$\phi_1 = 0.7$	

Table 8: Table of AR(1) - (1 of 3)

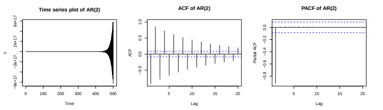
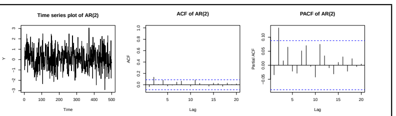
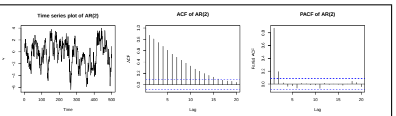
ϕ_1 \ ϕ_2	$\phi_2 = 0.2$
$\phi_1 = -0.9$	
$\phi_1 = 0$	
$\phi_1 = 0.7$	

Table 9: Table of AR(1) - (2 of 3)

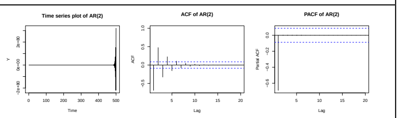
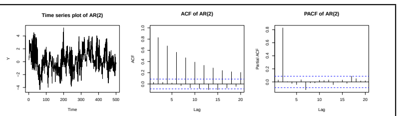
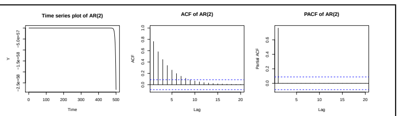
ϕ_1 \ ϕ_2	$\phi_2 = 0.8$
$\phi_1 = -0.9$	
$\phi_1 = 0$	
$\phi_1 = 0.7$	

Table 10: Table of AR(1) - (3 of 3)

2.7.5 Models - ARMA(1,1)

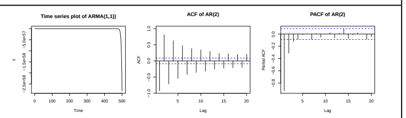
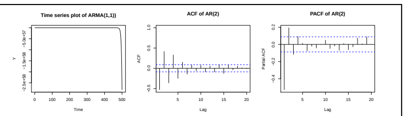
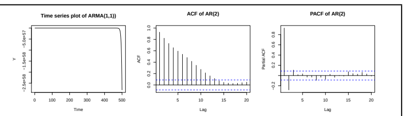
θ / ϕ	$\theta = -0.4$
$\phi_1 = -0.9$	
$\phi_1 = 0.8$	
$\phi_1 = 0.9$	

Table 11: Table of ARMA(1,1) - (1 of 2)

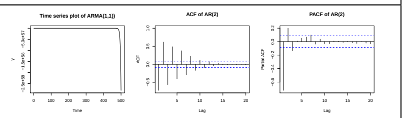
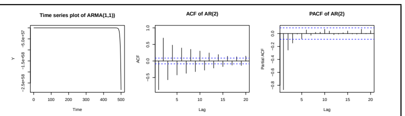
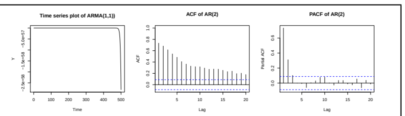
ϕ \ θ	0.4
-0.9	
0.8	
0.9	

Table 12: Table of ARMA(1,1) - (2 of 2)