matrici (4, 7, ... /10)

domenica 6 ottobre 2024 09:21

matrici

- struttura algebrica
- problema matematico : sistema lineare → teoria / metodi numerici (+ condizione problema)
- interpretazione geometrica

Una matrice è una tabella di numeri

$$\underline{\text{matrice}}_{m\times n} \longrightarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix}}_{n = \text{colonne}} \Big\} m = \text{righe}$$

$M \in \mathbb{R}^{m \cdot n}$

$M = (a_{ij})_{\substack{i=1\dots m \\ j=1\dots n}}$

esempio → con $a_{ij} = \frac{1}{i+j-1} \longrightarrow M = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots \\ \frac{1}{3} & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{pmatrix}$ matrice di Hilbert

vari tipi di matrici:

$\begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix} \in \mathbb{R}^{2\cdot 2} \longrightarrow m = n \rightarrow$ matrice quadrata

$(a_{i1} \ a_{i2} \ \dots \ a_{in}) \in \mathbb{R}^{1\cdot n} \longrightarrow m = 1 \longrightarrow R_i$ riga

$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} \in \mathbb{R}^{m\cdot 1} \longrightarrow n = 1 \longrightarrow C_j$ colonna → vettore

esempio:

$$\begin{cases} x + y = 2 \\ 1001x + 1000y = 2001 \end{cases} \longrightarrow A = \begin{pmatrix} 1 & 1 \\ 1001 & 1000 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 2001 \end{pmatrix} \quad x = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$ matrice triangolare superiore $\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}$ matrice triangolare inferiore

$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$ matrice diagonale (è presente sia triangolo superiore che inferiore)

$I_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{n\cdot n}$ matrice identica

matrice trasposta

$X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\cdot 3} \longrightarrow X^t = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 5 \end{pmatrix} \in \mathbb{R}^{2\cdot 3} \longrightarrow X^t = \begin{pmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 5 \end{pmatrix} \in \mathbb{R}^{3\cdot 2}$

$\begin{pmatrix} 7 & 5 & 4 \\ 5 & 1 & 0 \\ 4 & 0 & -2 \end{pmatrix}$ matrice simmetrica → $X = X^t$, quindi è anche quadrata $\Rightarrow X \in \mathbb{R}^{n\cdot n}$

OPERAZIONI: (1) somma

OPERAZIONI:

1. somma
2. moltiplicazione per "scalare" $\longrightarrow \equiv$ numero reale
3. prodotto (moltiplicazione)

1 $\underset{(x_{ij})}{X}, \underset{(y_{ij})}{Y} \in \mathbb{R}^{m \cdot n} \longmapsto X+Y \in \mathbb{R}^{m\cdot n}, \; X+Y = (x_{ij} + y_{ij})_{\substack{i=1\dots m \\ j=1\dots n}}$

esempio:

$$X = \begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 6 & 4 \end{pmatrix}, \; Y = \begin{pmatrix} 5 & -3 \\ 1 & 2 \\ -4 & -5 \end{pmatrix} \longrightarrow X+Y = \begin{pmatrix} 7 & 0 \\ 0 & 2 \\ 2 & -1 \end{pmatrix} \in \mathbb{R}^{3\cdot 2}$$

2 $X \in \mathbb{R}^{m\cdot n}, \; \lambda \in \mathbb{R}^{m \cdot n}, \; \lambda X = (\lambda x_{ij})_{\substack{i=1\dots m \\ j=1\dots n}}$

esempio:

$$X = \begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 6 & 4 \end{pmatrix} \longmapsto 3X \begin{pmatrix} 6 & 9 \\ -3 & 0 \\ 18 & 12 \end{pmatrix}$$

3 prodotto riga x colonna

$$(a_1 \dots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n$$

esempio:

$$(2 \; 3 \; 0) \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = (2 \cdot -2) + (3 \cdot 4) + (0 \cdot 5) = -4 + 12 + 0 = 8$$

$A = (a_{ij}) \in \mathbb{R}^{m\cdot n}, \; B = (b_{ij}) \in \mathbb{R}^{n \cdot r}$

$R_i^A = (a_{i1} \dots a_{in}), \; C_j^B = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} \Rightarrow R_i^A, C_j^B$ $\qquad A \cdot B = i\left\{ \underbrace{\begin{pmatrix} \quad & \quad \end{pmatrix}}_{j} \right. \in \mathbb{R}^{m \cdot r}$

esempio:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 7 & 5 \end{pmatrix} \in \mathbb{R}^{2\cdot 3}, \; B = \begin{pmatrix} -2 & 1 \\ 4 & 0 \\ 5 & -3 \end{pmatrix} \in \mathbb{R}^{3\cdot 2} \Rightarrow A \cdot B \in \mathbb{R}^{2 \cdot 2}$$

$$A \cdot B = \begin{pmatrix} 8 & 2 \\ 55 & -16 \end{pmatrix}$$

$$R_1^A \cdot C_1^B = (2 \; 3 \; 0) \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = -4 + 12 + 0 = 8$$

$$R_1^A \cdot C_2^B = (2 \; 3 \; 0) \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = 2 + 0 + 0 = 2$$

$$R_2^A \cdot C_1^B = (-1 \; 7 \; 5) \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = 2 + 28 + 25 = 55$$

$$R_2^A \cdot C_2^B = (-1 \; 7 \; 5) \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = -1 + 0 - 15 = -16$$

esempio:

$$B = \begin{pmatrix} -2 & 1 \\ 4 & 0 \\ 5 & -3 \end{pmatrix} \in \mathbb{R}^{3\cdot 2}, \; A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 7 & 5 \end{pmatrix} \in \mathbb{R}^{2\cdot 3} \Rightarrow B \cdot A \in \mathbb{R}^{3\cdot 3}$$

$$B \cdot A = \begin{pmatrix} 5 & 1 & 5 \\ 8 & 12 & 0 \\ 13 & -6 & -15 \end{pmatrix}$$

esempio:

$$B \cdot A = \begin{pmatrix} 8 & 12 & 0 \\ 13 & -6 & -15 \end{pmatrix}$$

esempio:

$$V = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad V^t = (1 \ -1 \ 2) \quad \begin{cases} \underset{1\times3}{V^t} \cdot \underset{3\times1}{V} = \underset{1\times1}{6} \\ \underset{3\times1}{V} \cdot \underset{1\times3}{V^t} = \underset{3\times3}{\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}} \end{cases}$$

quindi **operazioni tra matrici**

$\underset{(a_{ij})}{A} \in \mathbb{R}^{m \cdot n}$, $\underset{(b_{ij})}{B} \in \mathbb{R}^{m \cdot n} \longrightarrow \underbrace{(A+B)}_{\mathbb{R}^{m \cdot n}} = a_{ij} + b_{ij}$

$A \in \mathbb{R}^{m \cdot n}, \lambda \in \mathbb{R} \longrightarrow (\lambda A) = \lambda a_{ij}$

$A \in \mathbb{R}^{m \cdot n}, B \in \mathbb{R}^{n \cdot r} \longrightarrow \underbrace{(A \cdot B)_{ij}}_{\mathbb{R}^{n \cdot r}} = R_i^A \cdot C_j^B = \sum_{k=1}^{n} a_{ik} b_{kj} \longrightarrow$ **$A \cdot B \neq B \cdot A$**

$(A+B)(A-B) = A^2 + \underbrace{BA - AB}_{\neq 0} - B^2$

- $A, B \in \mathbb{R}^{n \cdot n}$ diagonali $\Rightarrow A \cdot B$ diagonale
  $\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix} \cdot \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$
- $A, B \in \mathbb{R}^{n \cdot n}$ triangolari superiori $\Rightarrow A \cdot B$ triangolare superiore
- $A, B \in \mathbb{R}^{n \cdot n}$ triangolari inferiori $\Rightarrow A \cdot B$ triangolare inferiore
- A, B simmetriche $\Rightarrow A \cdot B$ non simmetrica $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow A \cdot B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
- $A \cdot B = 0 \quad \left(A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \longrightarrow A \cdot B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$
- $A \cdot B = A \cdot C \not\Rightarrow B = C \quad , C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- $A^2 = 0 \not\Rightarrow A = 0 \quad \left(A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \right)$
- $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ esempio:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{matrix} \begin{pmatrix} 2 & 3 & 8 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 14 \end{pmatrix} \end{matrix} = \begin{pmatrix} 38 \\ 14 \end{pmatrix}$$

- $A(B+B') = A \cdot B + A \cdot B'$
- $(A+A')B = A \cdot B + A' \cdot B$
- $(\lambda A)B = \lambda(AB)$
- $A(\lambda B) = \lambda(AB)$
- $(AB)^t = B^t \cdot A^t$

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \text{ elemento neutro prodotto}$$

- $A \cdot I = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & & \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & & \end{pmatrix} = A$
- $I \cdot A = A$

data $A \in \mathbb{R}^{n \cdot n}$, $\exists B \in \mathbb{R}^{n \cdot n}$ t.c. $\underset{BA}{AB} = I$?
(A "invertibile", $B = A^{-1}$ inversa)

$$A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \longrightarrow A \cdot B = \begin{pmatrix} 6-5 & 3-3 \\ -10+10 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B \cdot A = \begin{pmatrix} 6-5 & -2+2 \\ 15-15 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \rightarrow B = A^{-1}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ cerco } A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow A \cdot A^{-1} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ A non invertibile

$A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, cerco $A^{-1}=\begin{pmatrix}a&b\\c&d\end{pmatrix} \longrightarrow A\cdot A^{-1}=\begin{pmatrix}c&d\\0&0\end{pmatrix}$ $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ A non invertibile

$\begin{pmatrix}-2&1\\6&-3\end{pmatrix}$ non invertibile

- A^{-1}, se esiste, è unica

$\exists B_1, B_2$ inverse $\longrightarrow AB_1 = I \xrightarrow{\text{molt per } B_2} (B_2A)B_1 = B_2 I = B_2$

$I\,B_1 = B_1$

- $I^{-1}=I$
- $(A^{-1})^{-1}=A \cdot (AB)^{-1}=B^{-1}A^{-1}$

$A\in\mathbb{R}^{n\cdot n} \longrightarrow$ determinante (det A) $\in \mathbb{R}$

- $n=1 \longrightarrow A=(a) \longrightarrow \det A = a$
- $n=2 \rightarrow A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} \xrightarrow{\text{regola di Cramer}} \det A = +a_{11}a_{22}-a_{12}a_{21}$

esempio: $\det\begin{pmatrix}2&3\\4&5\end{pmatrix}=2\cdot5-3\cdot4=10-12=-2$

- $n>2 \longrightarrow$ regola di Laplace

$\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$ $n=3$ $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix} \longrightarrow \det A = +a_{11}\cdot\det A_{11} - a_{12}\cdot\det A_{12} + a_{13}\cdot\det A_{13}$

A_{ij} = cancello riga i e colonna j

esempio:

$$\det\begin{pmatrix}1&-2&0\\1&1&2\\0&2&3\end{pmatrix} = 1\cdot\det\begin{pmatrix}1&2\\2&3\end{pmatrix}-(-2)\det\begin{pmatrix}1&2\\0&3\end{pmatrix}+0\det\begin{pmatrix}1&1\\0&2\end{pmatrix}=$$
$$=1\cdot(1\cdot3-2\cdot2)+2(1\cdot3-2\cdot0)+0(1\cdot2-1\cdot0)=$$
$$=1\cdot(3-4)+2(3)+0(2)=-1+6=5$$

in generale $\boxed{\det A = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\cdot\det A_{ij} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det A_{ij}}$

esempio:

$$\det\begin{pmatrix}1&-2&0\\1&1&2\\0&2&3\end{pmatrix}=0\cdot\det\begin{pmatrix}1&1\\0&2\end{pmatrix}-2\det\begin{pmatrix}1&-2\\0&2\end{pmatrix}+3\det\begin{pmatrix}1&-2\\1&1\end{pmatrix}=$$
$$=-2(1\cdot2-(-2)\cdot0)+3(1\cdot1-(-2)\cdot1)=$$
$$=-2(2)+3(1+2)=-4+9=5$$

- $\det(A^t)=\det A$
- $\det(A\cdot B)=\det A\cdot\det B$
- A invertibile $\Leftrightarrow \det A\neq 0$

$A\cdot A^{-1}=I \Rightarrow \det(A\cdot A^{-1})=\det(I)=\det\begin{pmatrix}1&0\\0&1\end{pmatrix}=1\cdot\det\begin{pmatrix}1&0\\0&1\end{pmatrix}=\ldots=\det\begin{pmatrix}1&0\\0&1\end{pmatrix}=1$

- $\det(A^{-1})=\frac{1}{\det A}$
- $\det\begin{pmatrix}a_{11}&&0\\&a_{22}&\\&&\ddots\\0&&a_{nn}\end{pmatrix}=a_{11}\cdot a_{22}\cdot\ldots\cdot a_{nn}$
- $\det\begin{pmatrix}a_{11}&\ldots&a_{1n}\\0&a_{22}&\vdots\\\vdots&\ddots&\\0&\ldots\,0&a_{nn}\end{pmatrix}=a_{11}\cdot a_{22}\cdot\ldots\cdot a_{nn}$