errori (23-24-27-30/9)

lunedì 23 settembre 2024 18:39

problema $\xrightarrow[1]{\text{modellizz.}}$ modello matematico continuo $\xrightarrow[1]{\text{discretizzazione}}$ modello matematico discreto $\longrightarrow$ # dati, # risultati finiti

errore dovuto alle approssimazioni nei passaggi di modellizzazione e discretizzazione

modello matematico discreto $\xrightarrow{2}$ algoritmo $\xrightarrow{3}$ risultati

2 $\longrightarrow$ errore sui dati

Dobbiamo considerare:

- COMPLESSITÀ
  - n input $\rightarrow C(n) = \#$ operaz.
  - GAUSS $\rightsquigarrow C(n) \cong n^3/3$
  - CRAMER $\rightsquigarrow C(n) \cong n!$
- ERRORI
  - analitici ①
  - inerenti ②
  - algoritmici ③

$x \in \mathbb{R} \rightsquigarrow \tilde{x}$ (errato)

$\delta := \tilde{x} - x$ errore assoluto (oppure $|\tilde{x} - x|$)

$\varepsilon := \dfrac{\tilde{x} - x}{x}$ errore relativo

$$\tilde{x} = x + \delta = x(1+\varepsilon)$$

ERRORI INERENTI: quando i dati sono sbagliati

Def: Siano

d_i = errore relativo di un input, su dato x_i

r_j = errore relativo di un output su risultato y_j

CONDIZIONAMENTO $= \dfrac{r_j}{d_i}$

esempio:

$$\begin{cases} x + y = 2 \longrightarrow \begin{matrix} x=1 \\ y=1 \end{matrix} \\ 1001x + 1000y = 2001 \end{cases}$$

$$\begin{cases} (1 + \frac{1}{100})\tilde{x} + \tilde{y} = 2 \\ 1001\tilde{x} + 1000\tilde{y} = 2001 \end{cases} \quad \begin{cases} \tilde{x} = -1/9 \longrightarrow r_x = \frac{-1/9 - 1}{1} = -\frac{10}{9} \\ \tilde{y} = 1901/900 \rightarrow r_y = \frac{1901/900 - 1}{1} = \frac{1001}{900} \end{cases}$$

errori relativi

$$|\text{cond}_x| = \left| \frac{-10/9}{10^{-2}} \right| > 100 \quad \text{(problema mal condizionato)}$$

esempio 2:

$x \mapsto f(x)$

$\tilde{x} \mapsto f(\tilde{x})$

$d = \dfrac{\tilde{x} - x}{x} \qquad r = \dfrac{f(\tilde{x}) - f(x)}{f(x)}$

$$\text{cond} = \frac{f(\tilde{x}) - f(x)}{f(x)} \cdot \frac{x}{\tilde{x} - x} \Rightarrow \underbrace{\lim_{\tilde{x} \to x} \text{cond}}_{= C_f} = \boxed{\frac{x f'(x)}{f(x)}} \longrightarrow \text{coefficiente di amplificazione} = \frac{\text{errore output}}{\text{errore input}}$$

$\hookrightarrow f'(x)$

esercizio:

calcolare C_f per $f(x) = x^2 - 7x \rightarrow f'(x) = 2x - 7 \rightarrow C_f = \dfrac{x(2x-7)}{\underbrace{x^2 - 7x}_{x(x-7)}} = \dfrac{2x-7}{x-7}$

Molto grande (forse) se $2x - 7 \to \infty$ $(x \to \infty)$ oppure $x - 7 \to 0$ $(x \to 7)$

$$\lim_{x\to\infty} C_f = \lim \frac{2x-7}{x-7} = \lim_{x\to\infty} \frac{x(2-7/x)}{x(1-7/x)} = \frac{2}{1}$$

$$\lim_{x \to 7} C_f = \lim \frac{2x-7}{x-7} = \pm\infty$$

esempio 3: $f(x) = 1 - \cos x \rightarrow f'(x) = +\sin x \rightarrow C_f = \dfrac{x \sin x}{1 - \cos x}$

$$x \to 0 = \lim_{x\to 0} \frac{x \sin x}{1 - \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} = \lim_{x\to 0} \frac{x \sin x (1 + \cos x)}{\underbrace{1 - \cos^2 x}_{\sin^2 x}} = 2 \lim_{x \to 0} \frac{x}{\sin x} = 2 \cdot 1 = 2$$

$1 + 1 = 2$

esempio 4: $f(x) = e^x \rightarrow f'(x) = e^x \rightarrow C_f = \dfrac{x \cdot e^x}{e^x} = x$

esempio 5: $f(x) = \ln x \rightarrow f'(x) = 1/x \rightarrow C_f = \dfrac{x \cdot \frac{1}{x}}{\ln x} = \dfrac{1}{\ln x} \qquad \lim_{x \to 1^\pm} C_f = \pm\infty$

esempio 6: $f(x)=\sqrt{x} \to f'(x)=\frac{1}{2\sqrt{x}} \to C_f=\frac{x\cdot\frac{1}{2\sqrt{x}}}{\sqrt{x}}=x\cdot\frac{1}{2\sqrt{x}}\cdot\frac{1}{\sqrt{x}}=\frac{x}{2\underbrace{\sqrt{x}\sqrt{x}}_{x}}=\frac{x}{2x}=\frac{1}{2}$

esempio 7: $f(x)=\sin x \rightsquigarrow f'(x)=\cos x \rightsquigarrow C_f=\frac{x\cos x}{\underbrace{\sin x}_{\to\, \simeq 0 \text{ per } x\simeq 0 \text{ o } x\simeq\pi}}$

$\lim_{x\to 0} C_f=\lim_{x\to 0}\frac{\overset{\to 0}{x}\cdot \overset{\to 1}{\cos x}}{\underset{\to 0}{\sin x}}=1$ (ben condizionato)

$\lim_{x\to \pi}\frac{\overset{\to 0}{x}\cdot \overset{\to -1}{\cos x}}{\underset{\to 0}{\sin x}}=\infty$ (mal condinzionato)

esempio 8: $f(x)=\cos x \to f'(x)=-\sin x \to C_f=\frac{x(-\sin x)}{\cos x}=-\frac{x\sin x}{\underbrace{\cos x}_{\to\,\simeq 0 \text{ per } x\simeq\frac{\pi}{2} \text{ o } x\simeq\frac{3}{2}\pi}}$

$\lim_{x\to \frac{\pi}{2}}\frac{\overset{\to \pi/2}{x}\cdot \overset{\to 1}{\sin x}}{\underset{\to 0}{\cos x}}=\infty$ (mal condizionato)

ERRORI ALGORITMICI

rappresentazione dei numeri:

- **VIRGOLA FISSA (FIXED POINT)**

$\underbrace{\square\square}_{m+1}\underbrace{\square\square}_{s}$ $m=$ parte intera, $s=$ parte decimale $\to x=\sum_{i=-s}^{m} d_i \cdot B^i$, $d_i \in\{0,\dots,B-1\}$

$x=45{,}61\ (\underset{0,\dots,9}{B=10})=4\cdot 10^1+5\cdot 10^0+6\cdot 10^{-1}+1\cdot 10^{-2}$

$x=101{,}011\ (\underset{1,2}{B=2})=1\cdot 2^2+0\cdot 2^1+1\cdot 2^0+0\cdot 2^{-1}+1\cdot 2^{-2}+1\cdot 2^{-3}$

$x=5C{,}2F\ (\underset{0,\dots,9,A,\dots,F}{B=16})=5\cdot 16^1+12\cdot 16^0+2\cdot 16^{-1}+15\cdot 16^{-2}$

$x \geq B^{m+1} \to$ OVERFLOW
$x < B^{-s-1} \to$ UNDERFLOW

- **VIRGOLA MOBILE (FLOATING POINT)**

$B=10,\ x=45{,}61=\underbrace{0{,}4561}_{\text{mantissa}}\cdot 10^{\overset{\to\text{caratteristica}}{2}}$

$\boxed{\pm}\underbrace{\square\square\square}_{\text{mantissa}}\underbrace{\square\square}_{\text{caratt.}}$

$B=2,\ x=101{,}011=0{,}101011\cdot 10^3$

$B=16,\ x=5C{,}2F=\underbrace{0{,}\overset{d_1\,d_2\,d_3\,d_4}{5C2F}}_{\text{mantissa}}\cdot 16^2$

$x=\pm B^p\sum_{i=1}^{t} d_i\cdot B^{-i}$

NUMERI DI MACCHINA $=\mathcal{F}(B,t,m,M)$ $\quad -m\leq p\leq M$ $\quad$ IEEE: singola $\to \mathcal{F}(2,24,128,127)$
doppia $\to \mathcal{F}(2,52,1024,1023)$

$x\in\mathbb{R}\longmapsto \tilde{x}\in\mathcal{F}(B,t,m,M)$ < troncamento $\longrightarrow \tilde{x}=0{,}12345$; arrotondamento $\longrightarrow \tilde{x}=0{,}12346$

$x=0{,}12345|6\cdot 10^0$ $\to 5$

troncamento: $|\varepsilon|=\left|\frac{\tilde{x}-x}{x}\right|\leq B^{-t}$

arrotondamento: $\varepsilon\leq\frac{1}{2}B^{-t}=u \to$ eps

PRECISIONE DI MACCHINA: $u=2{,}2\cdot 10^{-16}$

$d_{t+1}\geq\frac{1}{2}$: $\tilde{x}=\pm B^p\left(\sum_{i=1}^{t} d_i B^{i}+B^{-t}\right)$

$d_{t+1}<\frac{1}{2}B$: $=\pm B^p\sum_{i=1}^{+\infty} d_i B^{i}\to\tilde{x}=\pm B^p\sum_{i=1}^{t}$

esempio: $B=10,\ t=2,\ x=0{,}995\cdot 10^0\to\tilde{x}=0{,}99+0{,}01=1{,}00=0{,}10\cdot 10^1$

PRECISIONE DI MACCHINA

$x\in\mathbb{R}\mapsto\hat{x}\in\mathcal{F}(B,t,m,M),\ \hat{x}=\pm B^p\cdot\sum_{i=1}^{t} d_i\, B^{-i},\ d_i\in\{0,\dots,B-1\}$

$|\varepsilon|=\frac{|\hat{x}-x|}{|x|}\leq u=\frac{1}{2}B^{-t+1}$ $\quad$ esempio: $\mathcal{F}(2,52,1024,1023)$ $\to u=2^{-52}\simeq 2{,}2\cdot 10^{-16}$

esempio:

$x=1+\hat{\varepsilon},\ B=10,\ \hat{\varepsilon}=10^{-5}=1{,}00001=0{,}10000|1\cdot 10^1 \to \hat{x}=1\to|\varepsilon|=\frac{1-1-\hat{\varepsilon}}{1+\hat{\varepsilon}}=\frac{\hat{\varepsilon}}{1+\hat{\varepsilon}}<u=\frac{1}{2}\cdot 10^{-4}$ $\quad t=5$

$\hat{\varepsilon}=u=0{,}5\cdot 10^{-4}=0{,}00005$

$x=1+\hat{\varepsilon}=1{,}00005=0{,}10000|5\cdot 10^1\to\hat{x}=0{,}10001\cdot 10^1>1$ $\quad t=5$

ARITMETICA DI MACCHINA

- $a,b \mapsto a+b$

$\hat{a},\hat{b} \longmapsto (\hat{a}+\hat{b})(1+\varepsilon),\ |\varepsilon| \leq u$

$$\boxed{\varepsilon_{a+b} \doteq \varepsilon_a \frac{a}{a+b} + \varepsilon_b \frac{b}{a+b} + \varepsilon}$$

$\hat{a} = a(1+\varepsilon_a) \quad \hat{b} = b(1+\varepsilon_b)$

$$\varepsilon_{a+b} = \frac{(\hat{a}+\hat{b})(1+\varepsilon)-(a+b)}{a+b} = \frac{(a(1+\varepsilon_a)+b(1+\varepsilon_b))(1+\varepsilon)-a-b}{a+b} = \frac{(a+a\varepsilon_a+b+b\varepsilon_b)(1+\varepsilon)-a-b}{a+b} =$$

$$= \frac{\cancel{a}+a\varepsilon_a+\cancel{b}+b\varepsilon_b+a\varepsilon+\cancel{a\varepsilon_a\varepsilon}+b\varepsilon+\cancel{b\varepsilon_b\varepsilon}\cancel{-a}\cancel{-b}}{a+b} = \varepsilon_a \frac{a}{a+b} + \varepsilon_b \frac{b}{a+b} + \varepsilon$$

(analisi al primo ordine)

coefficienti di amplificazione

esempio: $B=10,\ t=6,\ a=0{,}123456,\ b=-0{,}123454$

$a+b = 2\cdot 10^{-6},\ \frac{a}{a+b} \approx \frac{10^{-1}}{2\cdot 10^{-6}} \approx 50000 \rightarrow 0{,}\underset{1}{2}\underset{???}{00000}\cdot 10^{-5}$ CANCELLAZIONE

- $a,b \mapsto a\cdot b$

$\hat{a} = a(1+\varepsilon_a),\ \hat{b} = b(1+\varepsilon_b) \rightarrow \hat{a}\hat{b}(1+\varepsilon)$

$$\boxed{\varepsilon_{a\cdot b} \doteq \varepsilon_a + \varepsilon_b + \varepsilon}$$

$$\varepsilon_{ab} = \frac{\hat{a}\hat{b}(1+\varepsilon)-ab}{ab} = \frac{a(1+\varepsilon_a)b(1+\varepsilon_b)(1+\varepsilon)-ab}{ab} = \frac{\cancel{ab}[(1+\varepsilon_a)(1+\varepsilon_b)(1+\varepsilon)-1]}{\cancel{ab}} = (1+\varepsilon_a+\varepsilon_b+\cancel{\varepsilon_a\varepsilon_b})(1+\varepsilon)-1=$$

$$= \cancel{1}+\varepsilon_a+\varepsilon_b+\varepsilon\cancel{-1}$$

- $a,b \mapsto a/b$

$\hat{a} = a+(1+\varepsilon_a),\ \hat{b} = b(1+\varepsilon_b) \mapsto \hat{a}/\hat{b}\,(1+\varepsilon)$

$$\boxed{\varepsilon_{a/b} \doteq \varepsilon_a - \varepsilon_b + \varepsilon}$$

$$\varepsilon_{a/b} = \frac{\hat{a}/\hat{b}\,(1+\varepsilon)-a/b}{a/b} = \frac{\frac{a(1+\varepsilon_a)}{b(1+\varepsilon_b)}(1+\varepsilon)-\frac{a}{b}}{a/b} = \frac{\frac{1+\varepsilon_a}{1+\varepsilon_b}(1+\varepsilon)-1}{1} = \frac{1+\varepsilon_a+\varepsilon-(1+\varepsilon_b)}{1+\varepsilon_b} = \frac{\varepsilon_a+\varepsilon-\varepsilon_b}{1+\varepsilon_b} = \frac{1}{1+\varepsilon_b}\cdot\frac{1-\varepsilon_b}{1-\varepsilon_b} =$$

$$= \frac{1-\varepsilon_b}{1-\cancel{\varepsilon_b^2}} = 1-\varepsilon_b = (\varepsilon_a+\varepsilon-\varepsilon_b)(1-\varepsilon_b) = \varepsilon_a+\varepsilon-\varepsilon_b$$

esempi su errori algoritmici

esempio 1: $f(x) = x^2-7x = x(x-7)$

algoritmo 1: $q := x^2 = x\cdot x$ (ε_q), $p := 7\cdot x$ (ε_p), $y_1 = q\cdot p$ (ε_{y_1})

algoritmo 2: $d := x\cdot 7$ (ε_d), $y_2 = x\cdot d$ (ε_{y_2})

$C_f = \frac{2x-7}{x-7} \qquad \varepsilon_{in} \approx C_f\cdot\varepsilon_x$

algoritmo 1) $\varepsilon_{alg_1}^{(tot)} \doteq \varepsilon_q^{(tot)}\frac{q}{q-p} \underset{\varepsilon_p''}{=} \varepsilon_q^{(tot)}\frac{p}{q-p} + \varepsilon_{y_1} = \varepsilon_q \underset{\frac{x}{x-7}}{\frac{x^2}{x^2-7x}} - \varepsilon_p \underset{\frac{7}{x-7}}{\frac{7x}{x^2-7x}} + \varepsilon_{y_1}$] ERRORE ALGORITMICO

algoritmo 2) $\varepsilon_{alg_2} = \varepsilon_x + \varepsilon_d^{(tot)} + \varepsilon_{y_2} = \varepsilon_d + \varepsilon_{y_2}$

$x \approx 7 \longrightarrow \varepsilon_{alg_1}$ ALTO ma $\varepsilon_{alg_1} \approx \varepsilon_{in} \rightarrow alg_1$ STABILE

$\varepsilon_{in} \approx C_f\ \varepsilon_x$

$C_f = \frac{x f'(x)}{f(x)} \longrightarrow \frac{\overset{\approx 7}{\boxed{2x-7}}}{x-7}$

Def: algoritmo stabile se $\varepsilon_{alg} \approx \varepsilon_{in}$ ($\circ\ \varepsilon_{alg} \ll \varepsilon_{in}$)

esempio 2: $f(x) = 1-\cos$

$C_f = \frac{x\sin x}{1-\cos x} \quad \lim_{x\to 0} C_f = 2 \rightarrow \varepsilon_{in} \approx 2\varepsilon_x$

alg 1) $c = \cos x\ (\varepsilon_c)$

$y_1 = 1-c\ (\varepsilon_{y_1})$

$\varepsilon_{alg_1} = \varepsilon_c \frac{-c}{1-c} + \varepsilon_{y_1}$

$\rightarrow -\frac{\cos x}{1-\cos x} \xrightarrow{x\to 0} -\frac{1}{0^+} = -\infty$ INSTABILE!

$f(x) = (1-\cos x)\cdot\frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{1+\cos x} = \frac{\sin^2 x}{1+\cos x}$

alg2) $c := \cos x$, $s := \sin x$
$n := s \cdot s$, $d := 1 + c$ $\quad y_2 = n/d$

$\varepsilon_n^{(tot)} = \varepsilon_s + \varepsilon_s + \varepsilon_n = 2\varepsilon_s + \varepsilon_n$

$\varepsilon_d^{(tot)} = \varepsilon_c \cdot \frac{c}{1+c} + \varepsilon_d$

$\varepsilon_{alg2} = \varepsilon_n^{(tot)} - \varepsilon_d^{(tot)} + \varepsilon_{y2}$ ALG2 STABILE per $x \simeq 0$

$\varepsilon_{alg2} = 2\varepsilon_s + \varepsilon_n - (\varepsilon_c \cdot \frac{c}{1+c} + \varepsilon_d) + \varepsilon_{y2} = 2\varepsilon_s + \varepsilon_n - \varepsilon_c \frac{c}{1+c} - \varepsilon_d + \varepsilon_d$

$\lim_{x \to 0} \frac{\cos x}{1+\cos x} = \frac{1}{2}$