$A+B,\ \lambda A,\ A\cdot B,\ \det A=\sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det A_{ij}\quad (A\in\mathbb{R}^{n\times n})$

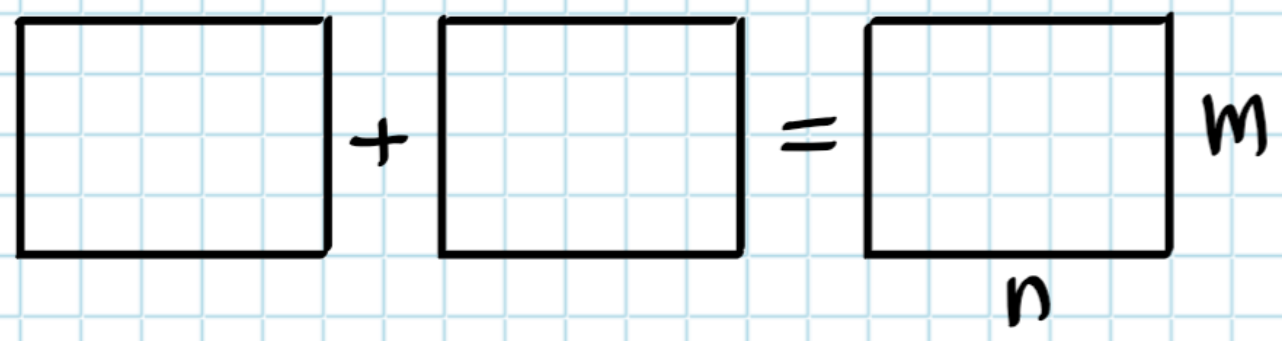
$\det(A\cdot B)=\det A\det B$
$\det(A+B)\neq\det A+\det B$
$\det(\lambda A)\neq\lambda\det A$

tesi: $\det A\neq 0 \iff \exists A^{-1}$ (invertibile)(non singolare)

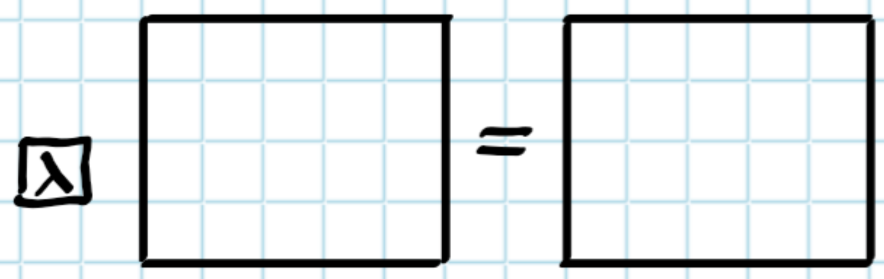
esempio: $\det(-I_n)=\det\begin{pmatrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{pmatrix}=(-1)^n$

$\exists$ metodo di calcolo di A^{-1} basato su det (ma non efficiente)

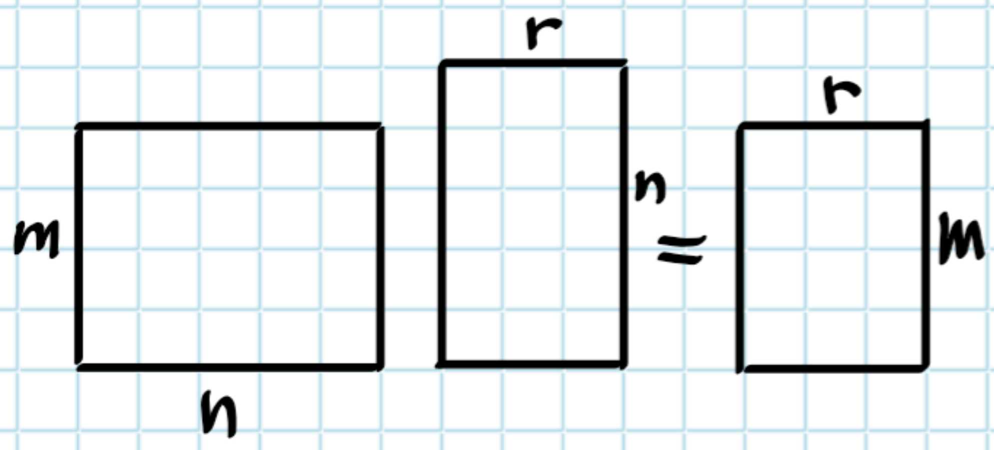
caso $n=2$: $\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}=\frac{1}{\underbrace{ad-bc}_{\det}}\cdot\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$



COSTO DI $A+B$: $m\cdot n$ addizioni



COSTO DI λA: $m\cdot n$ moltiplicazioni



COSTO DI $A\cdot B$: $m\cdot r$[n moltiplicazioni $+\,n-1$ addizioni]

ES1: $A,B\in\mathbb{R}^{n\times n}$ $\to m\cdot r$ n moltiplicazioni $\to n^3$

ES2: $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times 1}\to n^2$

C_n = costo di det $n\times n$ (con Laplace)

$C_1=0\qquad C_2=2\qquad C_n=n\cdot C_{n-1}+n>n\cdot C_{n-1}\Rightarrow C_n>n!$

→ SISTEMI LINEARI

esempio 1: $\begin{cases}x+y=2\\ 1001x+1000y=2001\end{cases}\to A=\begin{pmatrix}1 & 1\\ 1001 & 1000\end{pmatrix}\quad b=\begin{pmatrix}2\\ 2001\end{pmatrix}\quad x=\begin{pmatrix}x\\ y\end{pmatrix}$

$\to A\cdot x=b$

in generale:

$A\cdot x=\begin{pmatrix}x+y\\ 1001x+1000y\end{pmatrix}$

$\begin{cases}a_{11}x_1+\ldots+a_{1n}x_n=b_1\\ a_{21}x_1+\ldots+a_{2n}x_n=b_2\\ \vdots\\ a_{m1}x_1+\ldots+a_{mn}x_n=b_n\end{cases}\to A=\begin{pmatrix}a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{pmatrix}\in\mathbb{R}^{m\times n}\quad b=\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}\in\mathbb{R}^{n\times 1}\quad x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in\mathbb{R}^{n\times 1}$

matrice dei coefficienti (matrice incompleta) — vettore termini noti — vettore incognite

esempio 2: $\begin{cases}2x_1-x_2+x_4=5\\ x_2+x_3-2x_4=0\end{cases}\quad b=\begin{pmatrix}5\\ 0\end{pmatrix}\quad x=\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}\quad A=\begin{pmatrix}2 & -1 & 0 & 1\\ 0 & 1 & 1 & -2\end{pmatrix}\in\mathbb{R}^{2\times 4}$

$m=2\qquad n=4$

esempio 3: $\begin{cases}x_1-x_3=0\\ x_2+2x_3=0\end{cases}\quad b=\begin{pmatrix}0\\ 0\end{pmatrix}\quad x=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\quad A=\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{pmatrix}$

$m=2\qquad n=3$

$A\cdot x=b\qquad A\cdot x=0$ sistema omogeneo

caso $m=n$: $\underset{n\cdot n}{A}\cdot\underset{n\cdot 1}{x}=b\qquad \det A\neq 0\iff\exists A^{-1}\qquad \underbrace{A^{-1}\cdot A}_{I}x=A^{-1}b$

$=x$ (unica soluzione)

ALGORITMO: SOSTITUZIONE ALL'INDIETRO

$A \in \mathbb{R}^{n \cdot n}$ triangolo superiore $\quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$a_{n-1,n-1}$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ a_{22} + \dots + a_{2n}x_2 = b_2 \\ \ddots \\ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \\ a_{nn}x_n = b_n \end{cases}$$

$x_n = \frac{b_n}{a_{nn}}$

$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$

for $i = n-1, n-2, \dots, 1 : x_i := \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$

$\det A \neq 0 \iff a_{ii} \neq 0 \quad \forall i = 1 \dots n$

$\det A = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$

COSTO: $1\ (x_n) + \forall i : 1\ (/a_{ii}) + n-i\ (a_{ij}x_j)$ $\quad$ COSTO TOT $= 1 + \sum_{i=1}^{n-1} (n-i+1)$

$\underset{i=n-1}{1} + \underset{i=n-2}{2} + 3 + \dots + \underset{i=1}{n} = \frac{n(n+1)}{2} \approx \frac{n^2}{2}$

MATRICI ELEMENTARI

- $I_n = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$
- $E_{ij} \longleftrightarrow$ scambio righe i e j in I_n: $\overset{n=3}{E_{13}} = \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}$
- $E_i(\lambda) \longleftrightarrow$ moltiplico per λ riga i-esima in I_n: $E_2(3) = \begin{pmatrix} 1&0&0 \\ 0&3&0 \\ 0&0&1 \end{pmatrix}$
- $E_{ij}(\lambda) \longleftrightarrow$ sommo alla riga i-esima λ volte la riga j-esima in I_n: $E_{2\underset{j}{1}}(4) = \begin{pmatrix} 1&0&0 \\ 4&1&0 \\ 0&0&1 \end{pmatrix}$

RICHIAMO:

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ a_{m1}x_1 + \dots \end{cases} \longleftrightarrow Ax = b$$

$A' = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}_{n \times n} \rightsquigarrow$ SOSTITUZIONE ALL'INDIETRO $\left(\frac{n^2}{2}\right)$ $\quad$ A arbitraria $\rightsquigarrow A'$

MATRICI ELEMENTARI

$I_n \xrightarrow{R_i \leftrightarrow R_j} E_{ij}$, esempio: $E_{13} = \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix} \to \det E_{13} = +1 \cdot \det \begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix} = -1$

$I_n \xrightarrow{R_i \leftarrow \lambda R_i} E_i(\lambda)$ $(\lambda \neq 0)$, esempio: $E_2(3) = \begin{pmatrix} 1&0&0 \\ 0&3&0 \\ 0&0&1 \end{pmatrix} \to \det E_2(3) = 1 \cdot 3 \cdot 1 = 3$

$I_n \xrightarrow{R_i \leftarrow R_i + \lambda R_j} E_{ij}(\lambda)$, esempio: $E_{21}(4) = \begin{pmatrix} 1&0&0 \\ 4&1&0 \\ 0&0&1 \end{pmatrix} \to \det E_{21}(\lambda) = 1 \cdot 1 \cdot 1 = 1$

$\forall n, i, j, \lambda$: $\det_{ij} = -1$, $\det E_i(\lambda) = \lambda$, $\det E_{ij}(\lambda) = 1$

⇒ MATRICI ELEMENTARI INVERTIBILI

$A \mapsto E_{ij} \cdot A$
$R_i \leftrightarrow R_j$

$A = \begin{pmatrix} 1&0&-2&4 \\ 3&-1&0&2 \\ -1&0&1&0 \end{pmatrix}_{3 \times 4} \xrightarrow{R_1 \leftrightarrow R_2} E_{12} \cdot A = \begin{pmatrix} 3&-1&0&2 \\ 1&0&-2&4 \\ -1&0&1&0 \end{pmatrix}_{3 \times 4}$

$E_{12} = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$

$$A \mapsto E_2(3)\cdot A = \begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&-2&4\\3&-1&0&2\\-1&0&1&0\end{pmatrix} = \begin{pmatrix}1&0&-2&4\\9&-3&0&6\\-1&0&1&0\end{pmatrix}$$

$R_2 \leftarrow 3\cdot R_2$

$$A \mapsto E_{31}(2)\cdot A = \begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}\begin{pmatrix}1&0&-2&4\\3&-1&0&2\\-1&0&1&0\end{pmatrix} = \begin{pmatrix}1&0&-2&4\\3&-1&0&2\\1&0&-3&8\end{pmatrix}$$

$R_3 \leftarrow R_3 + 2\cdot R_1$

$\Rightarrow$ INVERSE: $\quad -E_{ij}^{-1}\cdot{}_{ij} = I \rightarrow E_{ij}^{-1} = E_{ij} \quad \bullet E_i(\lambda)^{-1}\cdot E_i(\lambda) = I \rightarrow E_i(1/\lambda) \quad -E_{ij}(\lambda)^{-1}\cdot E_{ij}(\lambda) = I \rightarrow E_{ij}(-\lambda)$

MATRICI RIDOTTE (O A SCALINI)

$$\text{PIVOT}\quad\begin{pmatrix} * & * & * & * & * & 0 & * & \dots \\ 0 & 0 & * & 0 & * & 0 & * & \dots \\ 0 & 0 & 0 & * & * & 0 & 0 & * & \dots \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & \dots \end{pmatrix} \leftarrow \text{PIVOT } i+1\text{-esima riga a destra del pivot della } i\text{-esima}$$

$$\begin{pmatrix}0&0&2&1\\1&0&3&-1\\0&0&0&1\\0&1&-1&0\end{pmatrix}\ \text{NON RIDOTTA}$$

esempio 1:

$$\begin{pmatrix}0&0&2&1\\1&0&3&-1\\0&0&0&1\\0&1&-1&0\end{pmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix}1&0&3&-1\\0&0&2&1\\0&0&0&1\\0&1&-1&0\end{pmatrix} \xrightarrow{R_2\leftrightarrow R_4} \begin{pmatrix}1&0&3&-1\\0&1&-1&0\\0&0&0&1\\0&0&2&1\end{pmatrix} \xrightarrow{R_3\leftrightarrow R_4} \begin{pmatrix}1&0&3&-1\\0&1&-1&0\\0&0&2&1\\0&0&0&1\end{pmatrix}$$

esempio 2:

$$A = \begin{pmatrix}1&0&-2&4\\9&-3&0&6\\-1&6&1&0\end{pmatrix} \xrightarrow{R_2\leftarrow R_2+(-9)R_1} \begin{pmatrix}1&0&-2&4\\0&-3&18&-30\\-1&6&1&0\end{pmatrix} \xrightarrow{R_3\leftarrow R_3+1\cdot R_1} \begin{pmatrix}1&0&-2&4\\0&-3&18&-30\\0&6&-1&4\end{pmatrix} \xrightarrow{R_3\leftarrow R_3+(2)R_2} \begin{pmatrix}1&0&-2&4\\0&-3&18&-30\\0&0&35&-56\end{pmatrix}$$

in generale: per azzerare a_{ij} avendo già il pivot a_{11}, si sceglie $\lambda = -a_{i1}/a_{11}$

$E_{32}(2)\cdot E_{31}(1)\cdot E_{21}(-9)\cdot A = A'$ RIDOTTA (1 pivot $\forall$ riga, ma 4ª colonna senza pivot)

esempio 3:

$$A = \begin{pmatrix}0&0&1&1&1\\2&2&1&-5&5\\3&3&-1&0&0\\1&1&0&-1&1\end{pmatrix} \xrightarrow{R_4\leftrightarrow R_1} \begin{pmatrix}1&1&0&-1&1\\2&2&1&-5&5\\3&3&-1&0&0\\0&0&1&1&1\end{pmatrix} \xrightarrow{R_2\leftarrow R_2+(-2)R_1} \begin{pmatrix}1&1&0&-1&1\\0&0&1&-3&3\\3&3&-1&0&0\\0&0&1&1&1\end{pmatrix} \xrightarrow{R_3+(-3)R_1} \begin{pmatrix}1&1&0&-1&1\\0&0&1&-3&3\\0&0&-1&3&-3\\0&0&1&1&1\end{pmatrix} \xrightarrow{R_3+1\cdot R_2} \begin{pmatrix}1&1&0&-1&1\\0&0&1&-3&3\\0&0&0&0&0\\0&0&1&1&1\end{pmatrix}$$

$$\xrightarrow{R_4-R_2} \begin{pmatrix}1&1&0&-1&1\\0&0&1&-3&3\\0&0&0&0&0\\0&0&0&4&-2\end{pmatrix} \xrightarrow{R_3\leftrightarrow R_4} \begin{pmatrix}1&1&0&-1&1\\0&0&1&-3&3\\0&0&0&4&2\\0&0&0&0&0\end{pmatrix} = A' = E_{34}\cdot E_{42}(-1)\cdot E_{32}(1)\cdot E_{31}(-3)\cdot E_{21}(-2)\cdot E_{14}\cdot A$$

RIDOTTA (4ª riga senza pivot, 2ª e 5ª colonna senza pivot)

CASO SISTEMI LINEARI:

$A\cdot X = b$

trasformazioni

$$A \xrightarrow[R_i\leftarrow R_i+\lambda R_j]{R_i\leftrightarrow R_j,\ R_i\leftarrow\lambda R_i} A' \text{ ridotta} \qquad X \text{ rimane invariata se } b \rightsquigarrow b'$$

"matrice completa": $(A\,|\,b)_{m\times(n+1)} \longrightarrow (A'\,|\,b')$

$A X = b \qquad\qquad A' X = b'$

esempio:

$$A = \begin{pmatrix}2&-4&1\\6&-14&8\\-2&0&6\end{pmatrix}\ b = \begin{pmatrix}1\\-1\\1\end{pmatrix} \rightsquigarrow \left(\begin{array}{ccc|c}2&-4&1&1\\6&-14&8&-1\\-2&0&6&1\end{array}\right) \xrightarrow{R_2\leftarrow R_2+(-3)R_1} \left(\begin{array}{ccc|c}2&-4&1&1\\0&-2&5&-4\\-2&0&6&1\end{array}\right) \xrightarrow{R_3\leftarrow R_3+R_1} \left(\begin{array}{ccc|c}2&-4&1&1\\0&-2&5&-4\\0&-4&7&2\end{array}\right) \xrightarrow{R_3\leftarrow R_3+(-2)R_2} \left(\begin{array}{ccc|c}2&-4&1&1\\0&-2&5&-4\\0&0&-3&10\end{array}\right) = (A'|b')$$

$$\begin{cases} 2x_1 - 4x_2 + x_3 = 1 \\ -2x_2 + 5x_3 = -4 \\ -3x_3 = 10 \end{cases} \xrightarrow{\text{SOSTITUZIONE INDUTTIVA}} \begin{cases} 2x_1 = 4x_2 - x_3 + 1 = 4\left(-\frac{19}{3}\right) + \frac{10}{3} + 1 \rightarrow 2x_1 = \frac{-76+10+3}{3} = \frac{-63}{3} = -21 \rightarrow -\frac{21}{2} \\ -2x_2 = -4 - 5\left(-\frac{10}{3}\right) = \frac{-12+50}{3} = \frac{38}{3} \rightarrow x_2 = -\frac{19}{3} \\ x_3 = -\frac{10}{3} \end{cases}$$

$$X = \begin{pmatrix}-21/2\\-19/3\\-10/3\end{pmatrix}$$

eliminazione di Gauss (riduzione)

RIDUZIONE DI GAUSS

(ELIMINAZIONE)

$$\underbrace{Ax=b}_{\substack{\text{matrice}\\\text{completa}\\(A\,|\,b)}} \xrightarrow[R_i \leftarrow R_i+\lambda R_j]{R_i \leftrightarrow R_j} A'x=b'$$

$R_i \leftarrow \lambda R_i$

$(A'\,|\,b')$ → a scalini (ridotta)

esempio:

$$\begin{cases} x_3+x_4=1 \\ 2x_1+2x_2+x_3-5x_4=5 \\ 3x_1+3x_2-x_3=0 \\ x_1+x_2-x_4=1 \end{cases}$$

$$\left(\begin{array}{cccc|c} 0&0&1&1&1\\ 2&2&1&-5&5\\ 3&3&-1&0&0\\ 1&1&0&-1&1 \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1&1&0&-1&1\\ 2&2&1&-5&5\\ 3&3&-1&0&0\\ 0&0&1&1&1 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2+2R_1} \left(\begin{array}{cccc|c} 1&1&0&-1&1\\ 0&0&1&-3&3\\ 3&3&-1&0&0\\ 0&0&1&1&1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3-3R_1} \left(\begin{array}{cccc|c} \boxed{1}&1&0&-1&1\\ 0&0&\boxed{1}&-3&3\\ 0&0&-1&3&-3\\ 0&0&1&1&1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3+R_2} \left(\begin{array}{cccc|c} 1&1&0&-1&1\\ 0&0&1&-3&3\\ 0&0&0&0&0\\ 0&0&1&1&1 \end{array}\right) \xleftrightarrow{R_4 \leftarrow R_4-R_2} \left(\begin{array}{cccc|c} \boxed{1}&1&0&-1&1\\ 0&0&\boxed{1}&-3&3\\ 0&0&0&0&0\\ 0&0&0&\boxed{4}&-2 \end{array}\right) \xleftrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccc|c} \boxed{1}&1&0&-1&1\\ 0&0&\boxed{1}&-3&3\\ 0&0&0&\boxed{4}&-2\\ 0&0&0&0&0 \end{array}\right)$$

$\underbrace{}_{A}$ $\underset{B}{\downarrow}$

$$\equiv \begin{cases} x_1+x_2-x_4=1 \longrightarrow x_1=1-x_2+x_4=\frac{1}{2}-x_2 \\ x_3-3x_4=3 \longrightarrow x_3=3+3x_4=3-\frac{3}{2}=\frac{6-3}{2}=\frac{3}{2} \\ 4x_4=-2 \longrightarrow x_4=-\frac{1}{2} \\ 0=0 \end{cases} = \begin{pmatrix} \frac{1}{2}-x_2 \\ x_2 \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} = (\infty^1)$$

esempio:

$$\begin{cases} x_1+2x_2=1 \\ 2x_1+4x_2=3 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1&2&1\\ 2&4&3 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2-2R_1} \left(\begin{array}{cc|c} 1&2&1\\ 0&0&1 \end{array}\right) \equiv \begin{cases} x_1+2x_2=1 \\ 0=1 \end{cases} \quad \emptyset$$

Teorema: un sistema ammette soluzioni (è compatibile) se tutti i pivot sono in A'

Teorema: un sistema compatibile ha
- una sola soluzione se p (# pivot) $= n$ (# colonne di A)
- ∞^{n-p}

Osservazione: sistema omogeneo ($\leftrightarrow Ax=0$) sempre compatibile; inoltre $b'=0$

esempio:

$$\begin{cases} x_1+x_2-3x_3=0 \\ x_1+3x_4=0 \end{cases} \longrightarrow \begin{pmatrix} \boxed{1}&1&-3&0\\ 1&0&0&3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2-R_1} \begin{array}{c} \begin{pmatrix} \boxed{1}&1&-3&0\\ 0&\boxed{-1}&3&3 \end{pmatrix} \\ \quad\quad\quad\quad x_3 \;\; x_4 \end{array} \equiv \begin{cases} x_1+x_2-3x_3=0 \rightarrow x_1=-x_2+3x_3=\cancel{-3x_3}-3x_4+\cancel{3x_3}=-3x_4 \\ -x_2+3x_3+3x_4=0 \rightarrow x_2=3x_3+3x_4 \end{cases}$$

$$x=\begin{pmatrix} -3x_4 \\ 3x_3+3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3\begin{pmatrix} 0\\3\\1\\0 \end{pmatrix} + x_4\begin{pmatrix} -3\\3\\0\\1 \end{pmatrix}$$

esempio:

$$\begin{cases} 3x_1-x_2-x_3=4 \\ 3x_1+6x_2-2x_3=-6 \\ 2x_1+x_2-3x_3=-6 \\ x_1-x_2-x_3=0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3&-1&-1&4\\ 3&6&-2&-6\\ 2&1&-3&-6\\ 1&-1&-1&0 \end{array}\right) \xrightarrow[\substack{R_2 \leftarrow R_2-3R_1\\ R_3 \leftarrow R_3-2R_1}]{R_1 \leftrightarrow R_4} \left(\begin{array}{ccc|c} 1&-1&-1&0\\ 0&9&1&-6\\ 0&3&-1&-6\\ 3&-1&-1&4 \end{array}\right) \xrightarrow{R_4 \leftarrow R_4-3R_1} \left(\begin{array}{ccc|c} \boxed{1}&-1&-1&0\\ 0&9&1&-6\\ 0&3&-1&-6\\ 0&2&2&4 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_4]{R_4 \leftarrow \frac{1}{2}R_4} \left(\begin{array}{ccc|c} 1&-1&-1&0\\ 0&1&1&2\\ 0&3&-1&-6\\ 0&9&1&-6 \end{array}\right) \xrightarrow[R_4 \leftarrow R_4-9R_2]{R_3 \leftarrow R_3-3R_2} \left(\begin{array}{ccc|c} 1&-1&-1&0\\ 0&1&1&2\\ 0&0&-4&-12\\ 0&0&-8&-24 \end{array}\right) \xrightarrow{R_4 \leftarrow R_4-2R_3} \left(\begin{array}{ccc|c} 1&-1&-1&0\\ 0&1&1&2\\ 0&0&-4&-12\\ 0&0&0&0 \end{array}\right)$$

$$\begin{cases} x_1-x_2-x_3=0 \rightarrow x_1=x_2+x_3=-1+3=2 \\ x_2+x_3=2 \rightarrow x_2=2-x_3=2-3=-1 \\ -4x_3=-12 \rightarrow x_3=3 \\ 0=0 \end{cases} \qquad x=\begin{pmatrix} 2\\-1\\3 \end{pmatrix}$$

esempio:

$$\begin{cases} 3x_1+3x_2-5x_3=-12 \\ x_1+2x_2-2x_3=-6 \\ 2x_1+x_2-3x_3=-6 \\ x_1-x_2-x_3=0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 2 & -2 & -6 \\ 2 & 1 & -3 & -6 \\ 3 & 3 & -5 & -12 \end{array}\right) \xrightarrow[R_4\leftarrow R_4-3R_1]{R_2\leftarrow R_2-R_1,\ R_3\leftarrow R_3-2R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3 & -1 & -6 \\ 0 & 3 & -1 & -6 \\ 0 & 6 & -2 & -12 \end{array}\right) \xrightarrow[R_4\leftarrow R_4-2R_2]{R_3\leftarrow R_3-R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3 & -1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1-x_2-x_3=0 \to x_1+2-\frac{1}{3}x_3-x_3=0 \to x_1=-2+\frac{4}{3}x_3 \\ 3x_2-x_3=-6 \to x_2=-2+\frac{1}{3}x_3 \end{cases}$$

esempio:

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 3 & 6 & 10 & 15 \end{pmatrix} \; x=0 \xrightarrow{\infty^2} \begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 3 & 3 & 6 & 10 & 15 \end{pmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 4 \\ 3 & 3 & 6 & 10 & 15 \end{pmatrix} \to x_1=0$$

$$\xrightarrow[R_3\leftarrow R_3-3R_1]{R_2\leftarrow R_2-2R_1} \begin{pmatrix} \boxed{1} & 1 & 2 & 3 & 4 \\ 0 & \boxed{-1} & -2 & -3 & -4 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{pmatrix} \equiv \begin{cases} x_1+x_2+2x_3+3x_4+4x_5=0 \\ -x_2-2x_3-3x_4-4x_5=0 \\ x_4+3x_5=0 \to x_4=-3x_5 \end{cases}$$

$x_3 \quad x_5$

$$x = x_3\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} 0 \\ 5 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

SISTEMI LINEARI IMPLEMENTAZIONE

$A\in\mathbb{R}^{n\times n}$, $\det A\neq 0 \to \exists!\, X$

$$\sum_{k=1}^{n-1} \overbrace{(n-k)(n-k+2)}^{g(k)} = \int_0^n g(k)\,dk \quad \begin{matrix} h=n-k \\ dh=-dk \end{matrix} =$$

$$= \int_n^0 h(h+2)\,dh = \int_0^n (h^2+2h)\,dh =$$

$$= \frac{n^3}{3}+n^2 = \frac{n^3}{3}$$

$Ax=b$ | Matlab: $x=A\backslash b$

$\downarrow R_i\leftrightarrow R_j$, $R_i\leftarrow R_i+\lambda R_j$

$A'x=b'$

SOST IND

$\left(\text{costo} = \frac{n^2}{2}\right)$

$A=A^{(1)}\mapsto A^{(2)}\mapsto \dots \mapsto A^{(n)}=A'$

$$A^{(k)} = \begin{pmatrix} a_{11}^{(k)} & \dots & a_{1n}^{(k)} \\ & \boxed{a_{kk}^{(k)}} & \\ 0 & a_{k+1,k}^{(k)} & \\ & a_{nk}^{(k)} & \end{pmatrix}$$

$A = \begin{pmatrix} \diagdown & \\ 0 & \diagdown \end{pmatrix} \to$ costo $O(n)$

$b=b^{(1)}\mapsto b^{(2)}\mapsto\dots\mapsto b^{(n)}=b'$

COSTO TOT GAUSS $= \sum_{k=1}^{n-1}(n-k+2)(n-k)$

$(n-k+2)(n-k)\ \forall k$

per $k=1,\dots,n-1$ $(A^{(k)}\mapsto A^{(k+1)})$

- pivot $:= a_{kk}^{(k)}$ (se $\neq 0$, altrimenti $R_k\leftrightarrow R_i$)
- per $i=k+1,\dots,n$: ($R_i\leftarrow R_i+\lambda R_k$, $\lambda=:-m_{ik}$)
- $m_{ik}=a_{ik}^{(k)}/\text{pivot}$ $\qquad b_i^{(k+1)}:=b_i^{(k)}-m_{ik}b_k^{(k)}$ + 2 op
- per $j=k+1,\dots,n$: $a_{ij}=a_{ij}^{(k)}-m_{ik}\cdot a_{kj}^{(k)}$ $\qquad$ $n-k$ op

$\Big\}\ n-k+2\ \forall i\ \forall k$

esempio:

$A=\begin{pmatrix} \boxed{\varepsilon} & 1 \\ 1 & 0 \end{pmatrix}$, $b=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad x=\begin{pmatrix} 1 \\ 1-\varepsilon \end{pmatrix}$ $\quad \varepsilon>0$, piccolo

$m_{21}=1/\varepsilon \quad R_2\leftarrow R_2-\frac{1}{\varepsilon}R_1 \quad A^{(2)}=\begin{pmatrix} \varepsilon & 1 \\ 0 & -1/\varepsilon \end{pmatrix} \quad b^{(2)}=\begin{pmatrix} 1 \\ 1-1/\varepsilon \end{pmatrix} \xrightarrow{\text{SOST IND}} x_2=\frac{1-1/\varepsilon}{-1/\varepsilon}$, $(\tilde{x}_2=1)$

$\varepsilon x_1+x_2=1 \to \tilde{x}_1=\frac{1-\tilde{x}_2}{\varepsilon}=0$

$\tilde{x}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

⚠ $A^{(k)}$ ha elementi >> elementi di A (→ Gauss instabile)

$$\frac{|a_{ik}^{(k)}|}{|a_{kk}^{(k)}|}=|m_{ik}|\gg 1$$

strategia di pivoting parziale:
$\forall k$ scambio sempre $R_k\longleftrightarrow R_{i_0}$
i_0 t.c. $|a_{i_0k}^{(k)}|=\max_{i=k\dots n}|a_{ik}^{(k)}| \Rightarrow |m_{ik}|\leq 1\ \forall i$

$Ax^{(1)}=b_1,\ Ax^{(2)}=b_2,\dots,Ax^{(r)}=b_r$

$$(A\,|\,b_1\,|\,b_2\,|\dots|\,b_r) \xrightarrow{\text{GAUSS}} (A'\,|\,b_1'\,|\dots|\,b_r')$$

$A'x^{(1)}=b_1',\dots,A'x^{(r)}=b_r' \to r$ SOST IND.

CALCOLO A^{-1}

$(A\cdot A^{-1}=I)$

$(x^{(1)}|\dots|x^{(n)})$ $\quad Ax^{(1)}=\underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_1}$ $\quad Ax^{(2)}=\underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{e_2}$ $\dots$ $Ax^{(n)}=e_n$ → vettori canonici

$$(A\,|\,\underbrace{e_1\,|\,e_2\,|\dots|\,e_n}_{I})$$

METODO DI GAUSS-JORDAN (per A^{-1})

Def: A totalmente ridotta se
- è ridotta
- i pivot sono $=1$
- sopra i pivot ho zeri

$A \in \mathbb{R}^{n\times n}$, $\det A \neq 0 \rightarrow$ tot rid $\equiv \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$

$E_i(\lambda) = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \leftarrow i \quad R_i \leftarrow \lambda R_i$

esempio:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \begin{pmatrix} 2 & 1 \\ 0 & 9/2 \end{pmatrix} \xrightarrow[R_2 \leftarrow \frac{2}{9}R_2]{R_1 \leftarrow \frac{1}{2}R_1} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ (tot. ridotta)}$$

$E_{21}\left(\frac{1}{2}\right) A = \begin{pmatrix} 2 & 1 \\ 0 & 9/2 \end{pmatrix}$

$E_2\left(\frac{2}{9}\right) E_1\left(\frac{1}{2}\right)\begin{pmatrix} 2 & 1 \\ 0 & 9/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$

$E_{12}\left(\frac{1}{2}\right)\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = I$

$\overbrace{E_{12}\left(\frac{1}{2}\right) \cdot E_2\left(\frac{2}{9}\right) \cdot E_1\left(\frac{1}{2}\right) \cdot E_{21}\left(\frac{1}{2}\right)}^{A^{-1}} A = I \qquad A^{-1} \cdot A = I$

$A^{-1} = E_{12}\left(\frac{1}{2}\right) E_2\left(\frac{2}{9}\right) E_1\left(\frac{1}{2}\right) E_{21}\left(\frac{1}{2}\right) \cdot I$

Gauss-Jordan: applico op. elementari ad I

$(A \mid e_1 \mid e_2 \mid \dots \mid e_n) \longrightarrow (I \mid A^{-1})$

---

metodo di Gauss-Jordan ($A \mapsto A^{-1}$)

esempio:

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \begin{pmatrix} 2 & 1 \\ 0 & 9/2 \end{pmatrix} \xrightarrow[R_2 \leftarrow \frac{2}{9}R_2]{R_1 \leftarrow \frac{1}{2}R_1} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\overbrace{E_{12}\left(-\frac{1}{2}\right) E_2\left(\frac{2}{9}\right) E_1\left(\frac{1}{2}\right) E_{21}\left(\frac{1}{2}\right)}^{A^{-1}} A = I$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\quad // \quad} \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \xrightarrow{\quad // \quad} \begin{pmatrix} 1/2 & 0 \\ 1/9 & 2/9 \end{pmatrix} \xrightarrow{\quad // \quad} \begin{pmatrix} 4/9 & -1/9 \\ 1/9 & 2/9 \end{pmatrix} = A^{-1}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 9/2 & 1/2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1/9 & 2/9 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 4/9 & -1/9 \\ 0 & 1 & 1/9 & 2/9 \end{array}\right)$$

$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} = \frac{1}{8+1}\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

esempio:

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \leftarrow \frac{1}{2}R_3]{R_2 \leftarrow \frac{1}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -4/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 + \frac{4}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -3/2 \\ 0 & 1 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2/3 & -1/6 \\ 0 & 1 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

A^{-1}

$A \in \mathbb{R}^{n\times n} \rightarrow$ costo $= n^3$

sistema lineare (Gauss): $n^3/3$

DETERMINANTI

Gauss: $\dots E_{21}(\lambda) E_{12} \dots A = A' = A^{(n)} = \begin{pmatrix} \diagdown & \\ 0 & \diagdown \end{pmatrix}$ $\quad A^{(1)} \rightarrow A^{(2)} \rightarrow \dots \rightarrow A^{(n)}$

$m_{ik} = a_{ik}^{(k)} / a_{kk}^{(k)}$

Regola di Binet: $\det(E \cdot A) = \det(E) \cdot \det(A)$

i) $R_i \leftrightarrow R_j \mid E_{ij} = \begin{pmatrix} 1 & & \boxed{1} \\ & \ddots & \\ \boxed{1} & & 1 \end{pmatrix} \begin{matrix} \leftarrow i \\ \\ \leftarrow j \end{matrix} \Rightarrow \det E_{ij} = -1$ $\qquad$ i) $R_i \leftrightarrow R_j \Rightarrow \det A \mapsto -\det A$

ii) $R_i \leftarrow \lambda R_i \mid E_i(\lambda) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \lambda & \\ & & & 1 \end{pmatrix} \leftarrow i \Rightarrow \det E(\lambda) = \lambda$ $\qquad$ ii) $R_i \leftarrow \lambda R_i \Rightarrow \det A \mapsto \lambda \det A$

iii) $R_i \leftarrow R_i + \lambda R_j \mid E_{ij}(\lambda) = \begin{pmatrix} 1 & & \\ \boxed{\lambda} & \ddots & \\ & & 1 \end{pmatrix} \}i \Rightarrow \det E_{ij}(\lambda) = 1$ $\qquad$ iii) $R_i \leftarrow R_i + \lambda R_j \Rightarrow \det A$ non cambia

$$A \xrightarrow[i)\,iii)]{\text{GAUSS}} A' = \begin{pmatrix} a'_{11} & \cdots & \vdots \\ & \ddots & \vdots \\ 0 & & a'_{nn} \end{pmatrix} \Rightarrow \det A' = \pm \det A$$

(-1) #scambi

$a'_{11}\, a'_{22} \dots a'_{nn}$

FATTORIZZAZIONE "LU"

→ L: lower, U: upper

$$m_{ik} = a_{ik}^{(k)} / a_{kk}^{(k)}$$

se non si fanno scambi $L = \begin{pmatrix} 1 & & 0 \\ m_{ik} & \ddots & \\ & & 1 \end{pmatrix} \leftarrow i$, $U = A^{(n)} \Rightarrow A = L \cdot U$ (k = colonna)

se faccio scambi $P \cdot A = L \cdot U$

P → prodotto delle E_{ij}

esempio:

$$A = \begin{pmatrix} 0&1&2&4 \\ -1&1&2&0 \\ 3&1&0&4 \\ 0&1&1&1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} \boxed{-1}&1&2&0 \\ 0&1&2&4 \\ 3&1&0&4 \\ 0&1&1&1 \end{pmatrix} \xrightarrow{R_3+3R_1} \begin{pmatrix} \boxed{-1}&1&2&0 \\ 0&\boxed{1}&2&4 \\ 0&4&6&4 \\ 0&1&1&1 \end{pmatrix} \xrightarrow{R_3-4R_2} \begin{pmatrix} \boxed{-1}&1&2&0 \\ 0&\boxed{1}&2&4 \\ 0&0&-2&-12 \\ 0&1&1&1 \end{pmatrix} \xrightarrow{R_4-R_2} \begin{pmatrix} \boxed{-1}&1&2&0 \\ 0&\boxed{1}&2&4 \\ 0&0&\boxed{-2}&-12 \\ 0&0&-1&-3 \end{pmatrix} \xrightarrow{R_4-\frac{1}{2}R_3} \begin{pmatrix} \boxed{-1}&1&2&0 \\ 0&\boxed{1}&2&4 \\ 0&0&\boxed{-2}&-12 \\ 0&0&0&\boxed{3} \end{pmatrix}$$

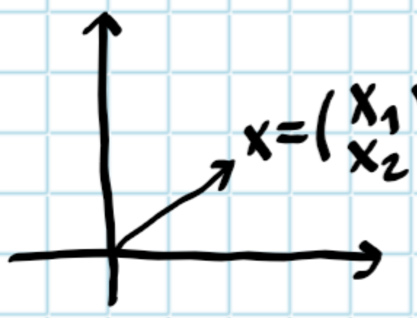
$= A^{(4)} \to \det A^{(4)} = -1 \cdot 1 \cdot (-2) \cdot 3 = 6 \to \det A = -6$

CONDIZIONAMENTO DEL PROBLEMA

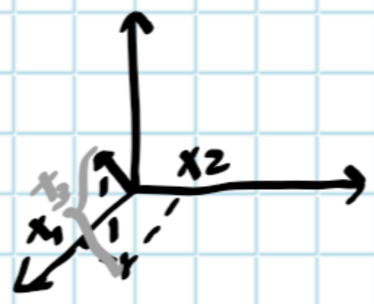
$Ax = b$ (errore inerente)

x → output, A, b → dati

→ NORME VETTORIALI



$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $|x| = \sqrt{x_1^2 + x_2^2}$



$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{x^t \cdot x}$$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x^t \cdot x = (x_1 \dots x_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

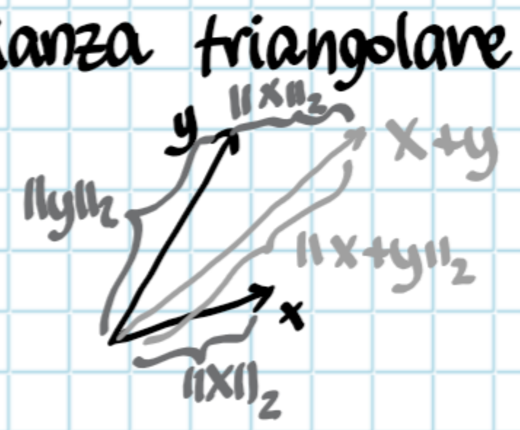
$\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$

PROPRIETÀ:

i) $\|x\|_2 \geq 0 \ \forall x \in \mathbb{R}^n$; inoltre $\|x\|_2 = 0 \Leftrightarrow x = 0$

ii) omogeneità: $\|\alpha \cdot x\|_2 = |\alpha| \cdot \|x\|_2$

iii) disuguaglianza triangolare: $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$



DEF. norma vettoriale $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ tale che

i) $\|x\| \geq 0 \ \forall x \in \mathbb{R}$ e $\|x\| = 0 \leftrightarrow x = 0$

ii) $\|\alpha x\| = |\alpha| \cdot \|x\| \ \forall \alpha \in \mathbb{R} \ \forall x \in \mathbb{R}^n$

iii) $\|x+y\| \leq \|x\| + \|y\| \ \forall x, y \in \mathbb{R}^n$

esempi: $\|x\|_\infty = \max_{i=1\dots n} |x_i|$ $\quad \|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$ $\quad \|x\|_2 = \sqrt{x^t x}$

SFERA UNITARIA $\{x \in \mathbb{R}^n \ \|x\| = 1\}$

teorema: $\forall \|\cdot\|', \|\cdot\|'' \ \exists \alpha, \beta > 0 : \alpha \|x\|'' \leq \|x\|' \leq \beta \|x\|'' \ \forall x \in \mathbb{R}^n$ (equivalenza metrica delle norme)

$\|\cdot\|'$	$\|\cdot\|''$	α	β
1	∞	1	n
1	2	1	$\sqrt{n}$
2	∞	1	$\sqrt{n}$

$\to \|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$

NORME DELLE MATRICI

esempio: $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$ norma di Frobenius

i) $\|A\| \geq 0$, $\|A\| = 0 \leftrightarrow A = 0$

ii) $\|\alpha A\| = |\alpha| \|A\|$

iii) $\|A+B\| < \|A\| + \|B\|$

iv) $\|A \cdot B\| < \|A\| \cdot \|B\|$

$A \in \mathbb{R}^{m \times n}$

$v \in \mathbb{R}^n \longrightarrow A v \in \mathbb{R}^m$

(es: $A = I \in \mathbb{R}^{n \times n}$ $\quad v \mapsto I \ v = v$)

$A = 0 \quad v \mapsto 0 \quad v = 0$

$A = 2I \quad v \mapsto 2(I v) = 2v$

PROBLEMA DATO $v \in S \mapsto \|A \cdot v\|$

Def: norma matriciale indotta $\rightarrow \|A\| = \max\limits_{v \in S} \|A v\|$ $\begin{cases} \|I\| = 1 \\ \|0\| = 0 \\ \|2I\| = 2 \end{cases}$

Teor: $\|A\|$: indotta verifica $i) - iv)$ e inoltre

$v)$ $\|Av\| \le \|A\| \cdot \|v\| \quad \forall A \in \mathbb{R}^{n \times n} \ \forall v \in \mathbb{R}^n$

Dim $v)$ $\forall v \in \mathbb{R}^n \rightarrow$ "normalizzazione" : $x = \frac{v}{\|v\|}$ (versore di v) $\Rightarrow \|x\| = \|\alpha v\| = |\alpha| \cdot \|v\| = 1 \Rightarrow x \in S$

$\alpha = \frac{1}{\|v\|} \Rightarrow \|Ax\| \le \|A\|$

$|Av| = \|A(\beta x)\| = \|\beta(Ax)\| = |\beta| \cdot \|Ax\| = \|v\| \|Ax\| \le \|v\| \|A\|$ $\square$

esempio: $\|I\|$ ($\|\cdot\|$ indotta)

$= \max\limits_{v \in S} \|I v\| = \max\limits_{v \in S} |v| = 1$

$\|I\|_F = \left\| \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\| = \sqrt{n} \quad \Rightarrow \|\cdot\|_F$ non indotta

ERRORE INERENTE IN SISTEMI LINEARI

$Ax = b$ (dati, output) $\quad A \in \mathbb{R}^{n \times n}$, $\det A \neq 0 \qquad A\hat{x} = \hat{b} \qquad \varepsilon_b \leftrightarrow \varepsilon_x$

CONDIZIONAMENTO DEL PROBLEMA

$\frac{\text{err output}}{\text{err input}} \rightarrow \frac{\varepsilon_x}{\varepsilon_b}$

errore in input $\leftrightarrow b \mapsto \hat{b} = b + \delta b \rightarrow \varepsilon_b := \frac{\|\delta b\|}{\|b\|}$

in output: $x \mapsto \hat{x}$ (risolve $A\hat{x} = \hat{b}$) $\rightarrow \varepsilon_x = \frac{\|\hat{x} - x\|}{\|x\|}$

Teor: $\varepsilon_x \le \|A\| \cdot \|A^{-1}\| \varepsilon_b \qquad$ condizionamento problema $= \frac{\varepsilon_x}{\varepsilon_b} \le \|A\| \cdot \|A^{-1}\| = \mu(A)'$ condizionamento della matrice

Dim: $A\hat{x} = \hat{b} + \delta b = Ax + \delta b \rightarrow \underbrace{A\hat{x} - Ax}_{A(\hat{x}-x)} = \delta b \Rightarrow A^{-1}A(\hat{x} - x) = A^{-1}\delta b \rightarrow \hat{x} - x = A^{-1}\delta b \Rightarrow \|\hat{x} - x\| = \|A^{-1}\delta b\| \le \|A^{-1}\| \cdot \|\delta b\|$

$b = Ax \Rightarrow \|b\| = \|Ax\| \le \|A\| \cdot \|x\| \Rightarrow \frac{1}{\|b\|} \ge \frac{1}{\|A\|\|x\|} \rightarrow \frac{1}{\|A\|\|x\|} \le \frac{1}{\|b\|} \Rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$

$\varepsilon_x = \frac{\|\hat{x} - x\|}{\|x\|} \le \frac{\|A^{-1}\|\|\delta b\|}{\|x\|} = \|A^{-1}\| \|\delta b\| \frac{1}{\|x\|} \le \|A^{-1}\| \|\delta b\| \cdot \frac{\|A\|}{\|b\|} = \|A\| \cdot \|A^{-1}\| \cdot \frac{\|\delta b\|}{\|b\|} = \mu(A)\, \varepsilon_b$

esempio: $\begin{cases} x_1 + x_2 = 2 \\ 1001 x_1 + 1000 x_2 = 2001 \end{cases} \rightarrow$ soluz. $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 2001 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 1001 & 1000 \end{pmatrix} \qquad \|A\|_\infty = \max(2, 2001) = 2001 \simeq 2 \cdot 10^3$

$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 1000 & -1 \\ -1001 & 1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 1000 & -1 \\ -1001 & 1 \end{pmatrix} = \begin{pmatrix} -1000 & 1 \\ 1001 & -1 \end{pmatrix} \rightarrow \|A^{-1}\|_\infty = \max(1001, 1002) = 1002 \simeq 10^3$

$\rightarrow \mu(A) \simeq 2 \cdot 10^6$

prop. $\forall A \cdot \mu(A) \ge 1$

dim. $\underset{1 \text{ (per norme indotte)}}{\|I\|} = \|A \cdot A^{-1}\| \le \|A\| \|A^{-1}\| = \mu(A)$