Condizionamento delle matrici

20/06/2024

Si considerino

$$A=\begin{bmatrix}1&0&0\\0&-1&1\\0&50&-49\end{bmatrix}\qquad x=\begin{bmatrix}1\\-1\\-1\end{bmatrix}\qquad \delta b=\begin{bmatrix}-10^{-2}\\10^{-2}\\-10^{-2}\end{bmatrix}$$

a)

$$\det(A)=1\begin{vmatrix}-1&-1\\50&-49\end{vmatrix}=49-50=-1\qquad \begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$$

$$A_{11}=\begin{vmatrix}-1&-1\\50&-49\end{vmatrix}=49-50=-1\qquad A_{12}=\begin{vmatrix}0&-1\\0&-49\end{vmatrix}=0\qquad A_{13}=\begin{vmatrix}0&-1\\0&50\end{vmatrix}=0$$

$$A_{21}=\begin{vmatrix}0&0\\50&-49\end{vmatrix}=0\qquad A_{22}=\begin{vmatrix}1&0\\0&-49\end{vmatrix}=-49\qquad A_{23}=\begin{vmatrix}1&0\\0&50\end{vmatrix}=50\to-50$$

$$A_{31}=\begin{vmatrix}0&0\\-1&-1\end{vmatrix}=0\qquad A_{32}=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1\to-1\qquad A_{33}=\begin{vmatrix}1&0\\0&-1\end{vmatrix}=-1$$

$$\mathrm{cof}(U)=\begin{bmatrix}-1&0&0\\0&-49&-50\\0&-1&-1\end{bmatrix}\Rightarrow \mathrm{cof}(U)^t=\begin{bmatrix}-1&0&0\\0&-49&-1\\0&-50&-1\end{bmatrix}$$

$$\frac{1}{\det(U)}\cdot\mathrm{cof}(U)^t=\frac{1}{-1}\begin{pmatrix}-1&0&0\\0&-49&-1\\0&-50&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&49&1\\0&50&1\end{pmatrix}$$

b)

$$b=A\cdot x$$

$$\begin{bmatrix}1&0&0\\0&-1&1\\0&50&-49\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

c)

$$\mu_1(A)=\|A\|_1\cdot\|A^{-1}\|_1\qquad \mu_\infty(A)=\|A\|_\infty\cdot\|A^{-1}\|_\infty$$

Calcolo di $\|A\|_1$: la $\|A\|_1$ è la somma massima degli elementi in valore assoluto di ogni colonna

$$\|A\|_1=\max(|1|+|0|+|0|,\ |0|+|-1|+|50|,\ |0|+|1|+|-49|)=\max(1,51,50)=51$$

$$\|A^{-1}\|_1=\max(|1|+|0|+|0|,\ |0|+|49|+|50|,\ |0|+|1|+|1|)=\max(1,99,2)=99$$

$$\mu_1=51\cdot 99=5049$$

Calcolo $\|A\|_\infty$: somma massima degli elementi in valore assoluto di ogni riga

$$\|A\|_\infty=\max(|1|+|0|+|0|,\ |0|+|-1|+|1|,\ |0|+|50|+|-49|)=\max(1,2,99)=99$$

$$\|A^{-1}\|_\infty=\max(|1|+|0|+|0|,\ |0|+|49|+|1|,\ |0|+|50|+|1|)=\max(1,50,51)=51$$

$$\mu_\infty=99\cdot 51=5049$$

d)

$$\|x\|_1=|1|+|-1|+|-1|=3\qquad \|x\|_2=\sqrt{1^2+(-1)^2+(-1)^2}=\sqrt{3}$$

$$\|b\|_1=|1|+|-1|=2\qquad \|x\|_2=\sqrt{1^2+(-1)^2}=\sqrt{2}$$

$$\|\delta b\|_1=|-10^{-2}|+|10^{-2}|+|-10^{-2}|=\frac{1}{100}+\frac{1}{100}+\frac{1}{100}=\frac{3}{100}$$

$$\|\delta b\|_2=\sqrt{(-10^{-2})^2+(10^{-2})^2+(-10^{-2})^2}=\sqrt{10^{-4}+10^{-4}+10^{-4}}=\sqrt{\frac{1}{10^4}+\frac{1}{10^4}+\frac{1}{10^4}}=\sqrt{\frac{3}{10^4}}$$

e)

$$\varepsilon_b=\frac{\|\delta b\|_1}{\|b\|_1}=\frac{3}{100}\cdot\frac{1}{2}=\frac{3}{200}=0{,}015$$

$$\varepsilon_x=\mu_1(A)\cdot\varepsilon_b=5049\cdot 0{,}015=75{,}735$$

Quindi la maggiorazione dell'errore è approssimativamente 75,73

se $\|\tilde{x}-x\|_\infty \rightarrow \varepsilon_b = \frac{\|\delta b\|_\infty}{\|b\|_\infty}$ $\quad \varepsilon_x = \mu_\infty(A)\cdot\varepsilon_b$ $\quad \begin{matrix}\|x\|_2 \\ \|x\|_1\end{matrix} \Rightarrow \begin{matrix}\|x\|_2 \\ \|x\|_\infty\end{matrix}$

13/09/2024

Si considerino

$$A=\begin{bmatrix}1 & 0 & -31\\ 0 & -1 & 0\\ -1 & 0 & 30\end{bmatrix} \qquad x=\begin{bmatrix}-1\\1\\-1\end{bmatrix} \qquad \delta b=\begin{bmatrix}10^{-2}\\-10^{-2}\\-10^{-2}\end{bmatrix}$$

a)

$$\det(A) = -1\begin{vmatrix}1 & -31\\ -1 & 30\end{vmatrix} = (-1)[30-31] = (-1)(-1) = 1$$

$A_{11}=\begin{vmatrix}-1 & 0\\ 0 & 30\end{vmatrix}=-30$ $\quad A_{12}=\begin{vmatrix}0 & 0\\ -1 & 30\end{vmatrix}=0$ $\quad A_{13}=\begin{vmatrix}0 & -1\\ -1 & 0\end{vmatrix}=-1$

$A_{21}=\begin{vmatrix}0 & -31\\ 0 & 30\end{vmatrix}=0$ $\quad A_{22}=\begin{vmatrix}1 & -31\\ -1 & 30\end{vmatrix}=-1$ $\quad A_{23}=\begin{vmatrix}1 & 0\\ -1 & 0\end{vmatrix}=0$

$A_{31}=\begin{vmatrix}0 & -31\\ -1 & 0\end{vmatrix}=-31$ $\quad A_{32}=\begin{vmatrix}1 & -31\\ 0 & 0\end{vmatrix}=0$ $\quad A_{33}=\begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix}=-1$

$$\mathrm{cof}(V)=\begin{pmatrix}-30 & 0 & -1\\ 0 & 61 & 0\\ -31 & 0 & -1\end{pmatrix} \Rightarrow \mathrm{cof}(V)^t=\begin{pmatrix}-30 & 0 & -31\\ 0 & -1 & 0\\ -1 & 0 & -1\end{pmatrix}$$

$$A^{-1}=\frac{1}{1}\begin{pmatrix}-30 & 0 & -31\\ 0 & -1 & 0\\ -1 & 0 & -1\end{pmatrix}=\begin{bmatrix}-30 & 0 & -31\\ 0 & -1 & 0\\ -1 & 0 & -1\end{bmatrix}$$ verificata

b)

$b = A\cdot x$

$$\begin{bmatrix}1 & 0 & -31\\ 0 & -1 & 0\\ -1 & 0 & 30\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\-1\end{bmatrix}=\begin{bmatrix}30\\-1\\-29\end{bmatrix}$$

c)

$\mu_1(A)=\|A\|_1\cdot\|A^{-1}\|_1$

$\|A\|_1=\max(|1|+|0|+|-1|,|-1|,|-31|+|30|)=\max(2,1,61)=61$

$\|A^{-1}\|_1=\max(31,1,32)=32$

$\mu_1(A)=61\cdot 32=1952$

$\mu_\infty(A)=\|A\|_\infty\cdot\|A^{-1}\|_\infty$

$\|A\|_\infty=\max(1+31,1,1+30)=\max(32,1,31)=32$

$\|A^{-1}\|_\infty=\max(30+31,1,1+1)=\max(61,1,2)=61$

$\mu_\infty(A)=32\cdot 61=1952$

d)

$\|x\|_1=|-1|+|1|+|-1|=3$ $\qquad \|x\|_2=\sqrt{(-1)^2+1^2+(-1)^2}=\sqrt{3}$

$\|b\|_1=|30|+|-1|+|-29|=60$ $\qquad \|x\|_2=\sqrt{(30)^2+(-1)^2+(-29)^2}=\sqrt{900+1+841}=$

$=\sqrt{1742}=41{,}74$

$\|\delta b\|_1=|1^{-2}|+|-10^{-2}|+|-10^{-2}|=\frac{1}{100}+\frac{1}{100}+\frac{1}{100}=\frac{3}{10^2}$

$\|\delta b\|_2=\sqrt{(10^{-2})^2+(-10^{-2})^2+(-10^{-2})^2}=\sqrt{10^{-4}+10^{-4}+10^{-4}}=\sqrt{\frac{3}{10^4}}$

e)

$\varepsilon_b=\frac{\|\delta b\|_1}{\|b\|_1}=\frac{3}{100}\cdot\frac{1}{60}=\frac{3}{6000}=\frac{1}{2000}=0{,}0005$

$\varepsilon_x=\mu_1(A)\cdot\varepsilon_b=1952\cdot 0{,}0005=0{,}976$

23/01/2023

Si considerino

$$A=\begin{bmatrix}-1 & 1000\\ 1 & -1001\end{bmatrix} \qquad x=\begin{bmatrix}-1\\ 1\end{bmatrix} \qquad \delta b=\begin{bmatrix}0,1\\ 0,1\end{bmatrix}$$

a)

$$\det(A)=[(-1)(-1001)-(1)(1000)]=1001-1000=1$$

$A_{11}=-1001 \qquad A_{12}=1\rightarrow -1$

$A_{21}=1000\rightarrow -1000 \qquad A_{22}=-1$

$\begin{pmatrix}+ & -\\ - & +\end{pmatrix}$

$$\mathrm{cof}(A)=\begin{bmatrix}-1001 & -1\\ -1000 & -1\end{bmatrix} \Rightarrow \mathrm{cof}(A)^t=\begin{bmatrix}-1001 & -1000\\ -1 & -1\end{bmatrix}$$

$$A^{-1}=\frac{1}{\det(A)}\cdot \mathrm{cof}(A)^t=\frac{1}{1}\begin{bmatrix}-1001 & -1000\\ -1 & -1\end{bmatrix}=\begin{bmatrix}-1001 & -1000\\ -1 & -1\end{bmatrix}$$ verificata

b)

$b=A\cdot x$

$$b=\begin{bmatrix}-1 & 1000\\ 1 & -1001\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}1001\\ -1001\end{bmatrix}$$

c)

$\mu_1(A)=\|A\|_1\cdot\|A^{-1}\|_1$

$\|A\|_1=\max(1+1,\ 1000+1001)=\max(2,\ 2001)=2001$

$\|A^{-1}\|_1=\max(1001+1,\ 1000+1)=\max(1002,\ 1001)=1002$

$\mu_1(A)=2001\cdot 1002=2005002$

$\mu_\infty(A)=\|A\|_\infty\cdot\|A^{-1}\|_\infty$

$\|A\|_\infty=\max(1+1000,\ 1+1001)=\max(1001,\ 1002)=1002$

$\|A^{-1}\|_\infty=\max(1001+1000,\ 1+1)=\max(2001,\ 2)=2001$

$\mu_\infty(A)=1002\cdot 2001=2005002$

d)

$\|x\|_1=1+1=2 \qquad \|x\|_2=\sqrt{2}$

$\|b\|_1=1001+1001=2002 \qquad \|b\|_2=\sqrt{(1001)^2+(1001)^2}=\sqrt{1002001+1002001}$

$=\sqrt{2004002}\simeq 1415,6$

$\|\delta b\|_1=0,1+0,1=0,2 \qquad \|\delta b\|_2=\sqrt{(0,1)^2+(0,1)^2}=\sqrt{0,01+0,01}=\sqrt{0,02}$

e)

$$\varepsilon_b=\frac{\|\delta b\|_1}{\|b\|_1}=\frac{0,2}{2002}=\frac{2}{10}\cdot\frac{1}{2002}=\frac{2}{20020}\simeq 0,0001$$

$$\varepsilon_x=\mu_1(A)\cdot\varepsilon_b=2005002\cdot 0,0001\simeq 200,5002$$

21/07/2023

Si considerino

$$A=\begin{bmatrix}-49 & 50\\ 1 & -1\end{bmatrix} \qquad x=\begin{bmatrix}-1\\ -1\end{bmatrix} \qquad \delta b=\begin{bmatrix}10^{-2}\\ -10^{-2}\end{bmatrix}$$

a)

$$\det(A)=[(-49)(-1)-(50)(1)]=[49-50]=-1$$

$A_{11}=-1 \qquad A_{12}=1\rightarrow -1$

$A_{21}=50\rightarrow -50 \qquad A_{22}=-49$

$\begin{pmatrix}+ & -\\ - & +\end{pmatrix}$

$$\mathrm{cof}(A)=\begin{bmatrix}-1 & -1\\ -50 & -49\end{bmatrix} \Rightarrow \mathrm{cof}(A)^t=\begin{bmatrix}-1 & -50\\ -1 & -49\end{bmatrix}$$

$$A^{-1}=-\frac{1}{1}\begin{bmatrix}-1 & -50\\ -1 & -49\end{bmatrix}=\begin{bmatrix}1 & 50\\ 1 & 49\end{bmatrix}$$

verificato

b) $b = A \cdot x$

$$b = \begin{bmatrix} -49 & 50 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

c)

$\mu_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$

$\|A\|_1 = \max(49+1, 50+1) = \max(50, 51) = 51$

$\|A^{-1}\|_1 = \max(1+1, 50+49) = \max(2, 99) = 99$

$\mu_1(A) = 51 \cdot 99 = 5049$

$\mu_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$\|A\|_\infty = \max(49+50, 1+1) = \max(99, 2) = 99$

$\|A^{-1}\|_\infty = \max(1+50, 1+49) = \max(51, 50) = 51$

$\mu_\infty(A) = 99 \cdot 51 = 5049$

d)

$\|x\|_2 = \sqrt{2}$ $\qquad \|x\|_\infty = 1$

$\|b\|_2 = \sqrt{1} = 1$ $\qquad \|x\|_\infty = 1$

$\|\delta b\|_2 = \sqrt{(0{,}01)^2 + (-0{,}01)^2} = \sqrt{0{,}001 + 0{,}001} = \sqrt{0{,}002}$ $\qquad \|\delta b\|_\infty = 10^{-2}$

e)

$$\varepsilon_b = \frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{10^{-2}}{1} = \frac{1}{100} = 0{,}01$$

$\varepsilon_x = \mu_\infty(A) \cdot \varepsilon_b = 5049 \cdot 0{,}01 = 50{,}49$

15/09/2022

Si considerino

$$A = \begin{bmatrix} -1 & 80 \\ 1 & -81 \end{bmatrix} \qquad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \delta b = \begin{bmatrix} 10^{-3} \\ 10^{-3} \end{bmatrix}$$

a)

$\det(A) = [(-1)(-81) - (80(1)] = [81 - 80] = 1$

$A_{11} = -81$ $\qquad A_{12} = 1 \to -1$

$A_{21} = 80 \to -80$ $\qquad A_{22} = -1$

$$\mathrm{cof}(A) = \begin{bmatrix} -81 & -1 \\ -80 & -1 \end{bmatrix} \Longrightarrow \mathrm{cof}(A) = \begin{bmatrix} -81 & -80 \\ -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} -81 & -80 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -81 & -80 \\ -1 & -1 \end{bmatrix}$$

b)

$b = A \cdot x$

$$b = \begin{bmatrix} -1 & 80 \\ 1 & -81 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 81 \\ -82 \end{bmatrix}$$

c)

$\mu_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$

$\|A\|_1 = \max(1+1, 80+81) = \max(2, 161) = 161$

$\|A^{-1}\|_1 = \max(81+1, 80+1) = \max(82, 81) = 82$

$\mu_1(A) = 161 \cdot 82 = 13202$

$\mu_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$\|A\|_\infty = \max(1+80, 1+81) = \max(81, 82) = 82$

$\|A^{-1}\|_\infty = \max(81+80, 1+1) = \max(161, 2) = 161$

$\mu_\infty(A) = 82 \cdot 161 = 13202$

d)

$\|x\|_1 = 2 \qquad \|x\|_\infty = 1$

$\|b\|_1 = 81+82 = 163 \qquad \|b\|_\infty = 82$

$\|\delta b\|_1 = 0{,}0001 + 0{,}0001 = 0{,}0002 \qquad \|\delta b\|_\infty = 0{,}0001$

e)

$$\varepsilon_b = \frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{0{,}0001}{82} = \frac{1}{1000} \cdot \frac{1}{82} = \frac{1}{82000} = 0{,}00001$$

$$\varepsilon_x = \mu_\infty(A) \cdot \varepsilon_b = 13202 \cdot 0{,}00001 = 0{,}13202$$

21/01/2022

Si considerino

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -100 & 0 & 101 \end{bmatrix} \qquad x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \delta b = \begin{bmatrix} 10^{-2} \\ -10^{-2} \\ 10^{-2} \end{bmatrix}$$

a)

$$\det(A) = 1 \begin{vmatrix} 1 & -1 \\ -100 & 101 \end{vmatrix} = [101 - 100] = 1$$

$A_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 101 \end{vmatrix} = 101 \qquad A_{12} = \begin{vmatrix} 0 & 0 \\ -100 & 101 \end{vmatrix} = 0 \qquad A_{13} = \begin{vmatrix} 0 & 1 \\ -100 & 0 \end{vmatrix} = 100$

$A_{21} = \begin{vmatrix} 0 & -1 \\ 0 & 101 \end{vmatrix} = 0 \qquad A_{22} = \begin{vmatrix} 1 & -1 \\ -100 & 101 \end{vmatrix} = 1 \qquad A_{23} = \begin{vmatrix} 1 & 0 \\ -100 & 0 \end{vmatrix} = 0$

$A_{31} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1 \qquad A_{32} = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0 \qquad A_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$$\mathrm{cof}(A) = \begin{bmatrix} 101 & 0 & 100 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Longrightarrow \mathrm{cof}(A)^t = \begin{bmatrix} 101 & 0 & 1 \\ 0 & 1 & 0 \\ 100 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 101 & 0 & 1 \\ 0 & 1 & 0 \\ 100 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 101 & 0 & 1 \\ 0 & 1 & 0 \\ 100 & 0 & 1 \end{bmatrix}$$ verificato

b)

$b = A \cdot x$

$$b = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -100 & 0 & 101 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 201 \end{bmatrix}$$

c)

$\mu_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$

$\|A\|_1 = \max(1+100, 1, 1+101) = \max(101, 1, 102) = 102$

$\|A^{-1}\|_1 = \max(101+100, 1, 1+1) = \max(201, 1, 2) = 201$

$\mu_1(A) = 102 \cdot 201 = 20502$

$\mu_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$\|A\|_\infty = \max(1+1, 1, 100+101) = \max(2, 1, 201) = 201$

$\|A^{-1}\| = \max(101+1, 1, 100+1) = \max(102, 1, 101) = 102$

$\mu_\infty(A) = 201 \cdot 102 = 20502$

d)

$\|x\|_2 = \sqrt{(-1)^2 + (-1)^2 + (1)^2} = \sqrt{3} \qquad \|x\|_\infty = 1$

$\|b\|_2 = \sqrt{(-2)^2 + (-1)^2 + (201)^2} = \sqrt{4+1+40401} = \sqrt{40406} = 201{,}01 \qquad \|b\|_\infty = 201$

$\|\delta b\|_2 = \sqrt{(0{,}01)^2 + (-0{,}01)^2 + (0{,}01)^2} \qquad \|b\delta\|_\infty = 0{,}01$

e)

$$\varepsilon_b = \frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{1}{100} \cdot \frac{1}{201} = \frac{1}{20100} = 0{,}00005$$

$$\varepsilon_x = \mu_\infty(A) \cdot \varepsilon_b = 20502 \cdot 0{,}00005 = 10251$$