diagonalizzazione

19/07/2024

Si consideri la matrice $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

a) $A - \lambda I$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{bmatrix} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)(1-\lambda) - (1)(1)\right] = (1-\lambda)\left[\cancel{1} - \lambda - \lambda - \lambda^2 - \cancel{1}\right] =$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda\right] = (1-\lambda)\left[\lambda(\lambda-2)\right]$$

b) $1-\lambda = 0 \rightarrow -\lambda = -1 \rightarrow \lambda_1 = 1$

$\lambda(\lambda-2)$

$\rightarrow \lambda_2 = 0$ $\qquad$ $\lambda - 2 = 0 \rightarrow \lambda_2 = 2$

c)

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = 2_2 + R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rango } 2$$

CASO $\lambda_1 = 1$ $\qquad A - 1I$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_3} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rango } 2$$

CASO $\lambda_2 = 0$

Siccome $\lambda_2 = 0$, $A - \lambda I$ risulta essere uguale ad A, e ha quindi lo stesso rango di A, ossia rango 2

CASO $\lambda_3 = 2$ $\qquad A - 2\lambda$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_1} \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rango } 2$$

d) $(A - \lambda I)X = 0$

CASO $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 = 0 \\ -x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \left\{ \underline{x_2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \underline{x_2} \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow$ il vettore non può essere nullo !!

CASO $\lambda = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1+x_3=0 \\ x_2=0 \\ 0=0 \end{cases} \quad \begin{cases} x_1=-x_3 \\ x_2=0 \\ 0=0 \end{cases} \Longrightarrow \left\{ \underline{x_3}\begin{pmatrix}-1\\0\\1\end{pmatrix} \,\middle|\, \underline{x_3} \in \mathbb{R} \right\} = \left\langle \begin{pmatrix}-1\\0\\1\end{pmatrix} \right\rangle$$

CASO $\lambda=2$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -x_1+x_3=0 \\ -x_2-2x_3=0 \\ 0=0 \end{cases} \quad \begin{cases} x_1=x_3 \\ x_2=-2x_3 \\ 0=0 \end{cases} \Rightarrow \left\{ \underline{x_3}\begin{pmatrix}1\\-2\\1\end{pmatrix} \,\middle|\, \underline{x_3} \in \mathbb{R} \right\} = \left\langle \begin{pmatrix}1\\-2\\1\end{pmatrix} \right\rangle$$

e)

$$V=\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \qquad D=\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$A = VDV^{-1}$

$A_{11} = \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} = (0\cdot 1-(-2)\cdot 1) = 2 \qquad A_{12} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = (1\cdot 1-(-2)\cdot 0) = 1 \to -1$

$A_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1\cdot 1-0\cdot 0) = 1$

$A_{21} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1-1) = -2 \to 2 \qquad A_{22} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0 \qquad A_{23} = \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} = 0$

$A_{31} = \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} = (-2\cdot -1-0) = 2 \qquad A_{32} = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1 \to 1 \qquad A_{33} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = (0\cdot 0-(-1)) = 1$

$$\mathrm{cof}(V) = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \longrightarrow \mathrm{cof}(V)^t = \begin{bmatrix} 2 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(V) = (-1)\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} -1-1 \end{vmatrix} = (-1)\begin{vmatrix} -2 \end{vmatrix} = 2$$

$$V^{-1} = \frac{1}{\det(V)} \cdot \mathrm{cof}(V)^t = \frac{1}{2}\cdot \mathrm{cof}(V)^t = \begin{bmatrix} 1 & 1 & 1 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$V\cdot D = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$VDV^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

f) il metodo delle potenze inverse applicato alla matrice A converge agli autovalori di A più vicini al parametro di shift p

- se p è vicino a $\lambda_1 = 1$, il metodo converge a λ_1
- se p è vicino a $\lambda_2 = 0$, il metodo converge a λ_2
- se p è vicino a $\lambda_3 = 2$, il metodo converge a λ_3

20/06/2024

Si consideri la matrice $\begin{bmatrix} 2 & \sqrt{5} \\ \sqrt{5} & -2 \end{bmatrix}$

a) $A-\lambda I$

$$\begin{bmatrix} 2 & \sqrt{5} \\ \sqrt{5} & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & \sqrt{5} \\ \sqrt{5} & -2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \left[(2-\lambda)(-2-\lambda) - (\sqrt{5})(\sqrt{5})\right] = -4+\lambda^2-5 = \lambda^2-9$$

b) $\lambda^2-9=0 \Rightarrow \lambda^2=9 \Rightarrow \lambda=\pm 3 \qquad \lambda_1 = 3 \qquad \lambda_2 = -3$

c) visto che $\det(A) = -9$, e quindi $-9 \neq 0$, il suo rango è 2

CASO $\lambda = 3$

$$A - 3I = \begin{bmatrix} 2 & \sqrt{5} \\ \sqrt{5} & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & \sqrt{5} \\ \sqrt{5} & -5 \end{bmatrix}$$

$$\det(A) = [(-1)(-5) - (\sqrt{5})(\sqrt{5})] = [5 - 5] = 0 \longrightarrow \text{rango } 1$$

CASO $\lambda = -3$

$$A - (-3)I = A + 3I = \begin{bmatrix} 2 & \sqrt{5} \\ \sqrt{5} & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & \sqrt{5} \\ \sqrt{5} & 1 \end{bmatrix}$$

$$\det(A) = [5 \cdot 1 - (\sqrt{5})(\sqrt{5})] = [5 - 5] = 0 \longrightarrow \text{rango } 1$$

d) $(A - \lambda I)x = 0$

CASO $\lambda = 3$

$$\begin{bmatrix} -1 & \sqrt{5} \\ \sqrt{5} & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + \sqrt{5}x_2 = 0 \\ \sqrt{5}x_1 - 5x_2 = 0 \end{cases} \begin{cases} x_1 = \sqrt{5}x_2 \\ \sqrt{5}(\sqrt{5}x_2) - 5x_2 = 0 \end{cases} \begin{cases} x_1 = \sqrt{5}x_2 \\ 5x_2 - 5x_2 = 0 \end{cases} \Rightarrow \left\{ \underline{x}_2 \begin{pmatrix} \sqrt{5} \\ 1 \end{pmatrix} \middle| \underline{x}_2 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} \sqrt{5} \\ 1 \end{pmatrix} \right\rangle$$

$0 = 0$

CASO $\lambda = -3$

$$\begin{bmatrix} 5 & \sqrt{5} \\ \sqrt{5} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 5x_1 + \sqrt{5}x_2 = 0 \\ \sqrt{5}x_1 + x_2 = 0 \end{cases} \begin{cases} 5x_1 + \sqrt{5}(\sqrt{5}x_1) = 0 \\ x_2 = -\sqrt{5}x_1 \end{cases} \begin{cases} 5x_1 - 5x_1 = 0 \\ x_2 = -\sqrt{5}x_1 \end{cases} \Rightarrow \left\{ \underline{x}_2 \begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix} \middle| \underline{x}_2 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix} \right\rangle$$

$0 = 0$

e)

$$V = \begin{bmatrix} \sqrt{5} & 1 \\ 1 & -\sqrt{5} \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

$A = VDV^{-1}$

$$\det(V) = [(\sqrt{5})(-\sqrt{5}) - 1] = [-5 - 1] = -6 \qquad \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

$A_{11} = -\sqrt{5}$ $\quad A_{12} = -1$

$A_{21} = -1$ $\quad A_{22} = \sqrt{5}$

$$\text{cof}(V) = \begin{pmatrix} -\sqrt{5} & -1 \\ -1 & \sqrt{5} \end{pmatrix} \Rightarrow \text{cof}(V)^t = \begin{pmatrix} -\sqrt{5} & -1 \\ -1 & \sqrt{5} \end{pmatrix}$$

$$V^{-1} = \frac{1}{\det(V)} \cdot \text{cof}(V)^t = \frac{1}{6} \begin{pmatrix} -\sqrt{5} & -1 \\ -1 & \sqrt{5} \end{pmatrix} = \begin{pmatrix} -\sqrt{5}/6 & -1/6 \\ -1/6 & \sqrt{5}/6 \end{pmatrix}$$

$$V \cdot D = \begin{pmatrix} \sqrt{5} & 1 \\ 1 & -\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 3\sqrt{5} & -3 \\ 3 & 3\sqrt{5} \end{pmatrix}$$

$$V \cdot D \cdot V^{-1} = \begin{pmatrix} 3\sqrt{5} & -3 \\ 3 & 3\sqrt{5} \end{pmatrix} \begin{pmatrix} -\sqrt{5}/6 & -1/6 \\ -1/6 & \sqrt{5}/6 \end{pmatrix} = \begin{pmatrix} -2 & -\sqrt{5} \\ -\sqrt{5} & 2 \end{pmatrix}$$

$$\cancel{3}\sqrt{5}\left(-\frac{\sqrt{5}}{\cancel{6}_2}\right) + (-\cancel{3})\left(-\frac{1}{\cancel{6}_2}\right) = \sqrt{5}\left(-\frac{\sqrt{5}}{2}\right) + \frac{1}{2} = -\frac{5}{2} + \frac{1}{2} = -\frac{4}{2} = -2$$

f)

$$u_1 = \frac{1}{u_i - p} \qquad u_1 = \frac{1}{\lambda_1 - p} = \frac{1}{3 - (-1)} = \frac{1}{4} = 0{,}25$$

$$u_2 = \frac{1}{\lambda_2 - p} = \frac{1}{\lambda_2 - p} = \frac{1}{-3 - (-1)} = -\frac{1}{2} = -0{,}5$$

$$V = \left|\frac{u_2}{u_1}\right|^k = \left|\frac{1/4}{-1/2}\right|^k = \left(\frac{1}{4} \cdot \frac{2}{1}\right)^k = \left(\frac{1}{2}\right)^k$$

13/09/2024

Calcolare la diagonalizzazione della matrice

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -2 & -5 \end{bmatrix}$$

a)

$A - \lambda I$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -2 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 5 \\ 0 & -2 & -5-\lambda \end{bmatrix}$$

$$\det(A) = -1-\lambda \begin{vmatrix} 2-\lambda & 5 \\ -2 & -5-\lambda \end{vmatrix} = -1-\lambda \left[(2-\lambda)\cdot(-5-\lambda) - (-2)(5)\right] = (-1-\lambda)\left[(2-\lambda)(-5-\lambda) + 10\right] =$$

$$= (-1-\lambda)\left[\cancel{-10} - 2\lambda + 5\lambda + \lambda^2 \cancel{+10}\right] = (-1-\lambda)\left[\lambda^2 + 3\lambda\right] = (-1-\lambda)\left[\lambda(\lambda+3)\right]$$

b)

$$(-1-\lambda)\,[\lambda(\lambda+3)]$$

$\lambda_1 = -1$ $\quad$ $\lambda_2 = 0$ $\quad$ $\lambda_3 = -3$

c)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -2 & -5 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rango } 2$$

CASO $\lambda = -1$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -2 & -5 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 5 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 5 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{R_3 = R_1} \begin{bmatrix} 0 & -2 & -4 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{3}{2}R_1} \begin{bmatrix} 0 & -2 & -4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rango } 2$$

CASO $\lambda = 0$

Siccome $\lambda_2 = 0$, $A - \lambda I$ risulta essere uguale ad A, e ha quindi lo stesso rango di A, ossia rango 2

CASO $\lambda = -3$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -2 & -5 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{R_2 = R_2 + \frac{5}{2}R_3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{R_2 = R_3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rango } 2$$

d)

$(A - \lambda I)\,x = 0$

CASO $\lambda = -1$

$$\begin{bmatrix} 0 & -2 & -4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_2 - 4x_3 = 0 \\ -x_3 = 0 \\ 0 = 0 \end{cases} \begin{cases} 2x_2 = -4x_3 \\ x_3 = 0 \\ 0 = 0 \end{cases} \begin{cases} x_2 = -2x_3 \\ x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \left\{ \underline{x_1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; \underline{x_1} \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

CASO $\lambda = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 = 0 \\ 2x_2 + 5x_3 = 0 \\ 0 = 0 \end{cases} \begin{cases} x_1 = 0 \\ 2x_2 = -5x_3 \\ 0 = 0 \end{cases} \begin{cases} x_1 = 0 \\ x_2 = -\frac{5}{2}x_3 \\ 0 = 0 \end{cases} \Rightarrow \left\{ \underline{x_3} \begin{pmatrix} 0 \\ -\frac{5}{2} \\ 1 \end{pmatrix} \;\middle|\; \underline{x_3} \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ -\frac{5}{2} \\ 1 \end{pmatrix} \right\rangle$$

CASO $\lambda=-3$

$$\begin{bmatrix}2&0&0\\0&-2&-2\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}2x_1=0\\-2x_2-2x_3=0\\0=0\end{cases}\quad\begin{cases}x_1=0\\x_2=-x_3\\0=0\end{cases}\Longrightarrow\left\{\underline{x_3}\begin{pmatrix}0\\-1\\1\end{pmatrix}\,\middle|\,\underline{x_3}\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\rangle$$

e)

$$V=\begin{bmatrix}1&0&0\\0&-5/2&-1\\0&1&1\end{bmatrix}\qquad D=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&0&0\\0&0&-3\end{bmatrix}$$

$A=VDV^{-1}$

$$\det(V)=1\begin{vmatrix}-5/2&-1\\1&1\end{vmatrix}=(-5/2)(1)-(-1)(1)=-5/2+1=-3/2$$

$$A_{11}=\begin{vmatrix}-5/2&-1\\1&1\end{vmatrix}=(-5/2)(1)-(-1)(1)=-5/2+1=-3/2\qquad A_{12}=\begin{vmatrix}0&-1\\0&1\end{vmatrix}=0\qquad A_{13}=\begin{vmatrix}0&-5/2\\0&1\end{vmatrix}=0$$

$$A_{21}=\begin{vmatrix}0&0\\1&1\end{vmatrix}=0\qquad A_{22}=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1-0=1\qquad A_{23}=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1-0=1\rightarrow-1\qquad\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$$

$$A_{31}=\begin{vmatrix}0&0\\-5/2&1\end{vmatrix}=0\qquad A_{32}=\begin{vmatrix}1&0\\0&-1\end{vmatrix}=-1-0=-1\rightarrow1\qquad A_{33}=\begin{vmatrix}1&0\\0&-5/2\end{vmatrix}=-5/2$$

$$\mathrm{cof}(V)=\begin{bmatrix}-3/2&0&0\\0&1&-1\\0&1&-5/2\end{bmatrix}\Longrightarrow\mathrm{cof}(V)^t=\begin{bmatrix}-3/2&0&0\\0&1&1\\0&-1&-5/2\end{bmatrix}$$

$$V^{-1}=\frac{1}{\det(V)}\cdot\mathrm{cof}(V)^t=\frac{1}{-3/2}\cdot\begin{bmatrix}3/2&0&0\\0&1&1\\0&-1&-5/2\end{bmatrix}=-\frac{2}{3}\begin{bmatrix}-3/2&0&0\\0&1&1\\0&-1&-5/2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&-2/3&-2/3\\0&2/3&5/3\end{bmatrix}$$

$$V\cdot D=\begin{bmatrix}1&0&0\\0&-5/2&-1\\0&1&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&0&0\\0&0&-3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&0&3\\0&0&-3\end{bmatrix}\qquad V\cdot D\cdot V^{-1}=\begin{bmatrix}-1&0&0\\0&0&3\\0&0&-3\end{bmatrix}\begin{bmatrix}1&0&0\\0&-2/3&-2/3\\0&2/3&5/3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&2&5\\0&-2&5\end{bmatrix}$$

f) il metodo delle potenze inverse applicato alla matrice A converge agli autovalori di A più vicini al parametro di shift p

- se p è vicino a $\lambda_1=-1$, il metodo converge a λ_1
- se p è vicino a $\lambda_2=0$, il metodo converge a λ_2
- se p è vicino a $\lambda_3=-3$, il metodo converge a λ_3

22/01/2024

Calcolare la diagonalizzazione di

$$A=\begin{bmatrix}-1&0&2\\0&0&0\\2&0&-1\end{bmatrix}$$

a)

$A-\lambda I$

$$\begin{bmatrix}-1&0&2\\0&0&0\\2&0&-1\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}=\begin{bmatrix}-1-\lambda&0&2\\0&-\lambda&0\\2&0&-1-\lambda\end{bmatrix}$$

$$\det(A)=-\lambda\begin{vmatrix}-1-\lambda&2\\2&-1-\lambda\end{vmatrix}=-\lambda\left[(-1-\lambda)(-1-\lambda)-(2)(2)\right]=-\lambda\left[(1+\lambda+\lambda+\lambda^2)-4\right]=-\lambda\left[1+2\lambda+\lambda^2-4\right]$$
$$=-\lambda\left[\lambda^2+2\lambda-3\right]$$

b)

$$-\lambda\left[\lambda^2+2\lambda-3\right]\longrightarrow\lambda_1=0$$

$$\longrightarrow\lambda^2+2\lambda-3=\frac{-2\pm\sqrt{4-4(-3)}}{2}=\frac{-2\pm\sqrt{16}}{2}\begin{cases}\frac{-2+4}{2}=\frac{2}{2}=1\\\frac{-2-4}{2}=\frac{-6}{2}=-3\end{cases}$$

$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = -3$

c)

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 = R_3 + 2R_1} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_2} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rango } 2$$

CASO $\lambda = 0$

Siccome $\lambda_1 = 0$, $A - \lambda I$ risulta essere uguale ad A, e ha quindi lo stesso rango di A, ossia rango 2

CASO $\lambda = 1$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_1} \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rango } 2$$

CASO $\lambda = -3$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rango } 2$$

d)

$(A - \lambda I)\, x = 0$

CASO $\lambda = 0$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 2x_3 = 0 \\ 3x_3 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} x_1 = -2x_3 \\ x_3 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} x_1 = 0 \\ x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \left\{ \underline{x_2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, \underline{x_2} \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

CASO $\lambda = 1$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_1 + 2x_3 = 0 \\ -x_2 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} x_1 = x_3 \\ x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow \left\{ \underline{x_3} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, \underline{x_3} \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

CASO $\lambda = -3$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 2x_3 = 0 \\ 3x_2 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} x_1 = -x_3 \\ x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow \left\{ \underline{x_3} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, x_3 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

e)

$$V = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$A = V D V^{-1}$

$$\det(V) = -1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -1\,(1 - (-1)) = -2$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & 0 & + \\ 0 & + & - \\ + & 0 & + \end{pmatrix}$$

$A_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \qquad A_{12} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \to -1 \qquad A_{13} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$A_{21} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 - (-1) = 2 \to -2 \qquad A_{22} = \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} = 0 \qquad A_{23} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0$

$A_{31} = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0 \qquad A_{32} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 0 - (-1) = 1 \to -1 \qquad A_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$$\mathrm{cof}(V)=\begin{bmatrix}0&-1&1\\-2&0&0\\0&-1&-1\end{bmatrix} \Rightarrow \mathrm{cof}(V)^t=\begin{bmatrix}0&-2&0\\-1&0&-1\\1&0&-1\end{bmatrix}$$

$$V^{-1}=\frac{1}{\det(V)}\cdot \mathrm{cof}(V)^t=-\frac{1}{2}\begin{bmatrix}0&-2&0\\-1&0&-1\\1&0&-1\end{bmatrix}=\begin{bmatrix}0&1&0\\1/2&0&1/2\\-1/2&0&1/2\end{bmatrix}$$

$$VD=\begin{bmatrix}0&1&-1\\1&0&0\\0&1&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&-3\end{bmatrix}=\begin{bmatrix}0&1&3\\0&0&0\\0&1&-3\end{bmatrix}$$

$$VDV^{-1}=\begin{bmatrix}0&1&3\\0&0&0\\0&1&-3\end{bmatrix}\begin{bmatrix}0&1&0\\1/2&0&1/2\\-1/2&0&1/2\end{bmatrix}=\begin{bmatrix}-1&0&2\\0&0&0\\2&0&-1\end{bmatrix}$$

f) il metodo delle potenze inverse applicato alla matrice A converge agli autovalori di A più vicini al parametro di shift p

- se p è vicino a $\lambda_1=0$, il metodo converge a λ_1
- se p è vicino a $\lambda_2=1$, il metodo converge a λ_2
- se p è vicino a $\lambda_3=-3$, il metodo converge a λ_3

13/02/2023

Calcolare la diagonalizzazione di

$$A=\begin{bmatrix}1&1&1\\0&-2&0\\0&0&0\end{bmatrix}$$

a) $A-\lambda I$

$$\begin{bmatrix}1&1&1\\0&-2&0\\0&0&0\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}=\begin{bmatrix}1-\lambda&1&1\\0&-2-\lambda&0\\0&0&-\lambda\end{bmatrix}$$

$$\det(A)=(1-\lambda)\begin{vmatrix}-2-\lambda&0\\0&-\lambda\end{vmatrix}=(-2-\lambda)(-\lambda)=(1-\lambda)(2\lambda+\lambda^2)=\lambda(2+\lambda)(1-\lambda)$$

b) $\lambda(2+\lambda)(1-\lambda)$

$\lambda_1=0$ $\quad 2+\lambda=0 \rightarrow \lambda_2=-2$ $\quad 1-\lambda=0 \rightarrow \lambda_3=1$

c) $\begin{bmatrix}1&1&1\\0&-2&0\\0&0&0\end{bmatrix}$ rango 2

CASO $\lambda=0$

Siccome $\lambda_1=0$, $A-\lambda I$ risulta essere uguale ad A, e ha quindi lo stesso rango di A, ossia rango 2

CASO $\lambda=-2$

$$\begin{bmatrix}1&1&1\\0&-2&0\\0&0&0\end{bmatrix}-\begin{bmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{bmatrix}=\begin{bmatrix}3&1&1\\0&0&0\\0&0&2\end{bmatrix}$$

$$\begin{bmatrix}3&1&1\\0&0&0\\0&0&2\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}3&1&1\\0&0&2\\0&0&0\end{bmatrix}$$ rango 2

CASO $\lambda=1$

$$\begin{bmatrix}1&1&1\\0&-2&0\\0&0&0\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&1\\0&-3&0\\0&0&-1\end{bmatrix}$$

$$\begin{bmatrix}0&1&1\\0&-3&0\\0&0&1\end{bmatrix}\xrightarrow{R_3\to R_3+R_1}\begin{bmatrix}0&1&1\\0&-3&0\\0&1&0\end{bmatrix}\xrightarrow[R_2\leftrightarrow R_3]{R_2\to R_2+3R_3}\begin{bmatrix}0&1&1\\0&-1&0\\0&0&0\end{bmatrix}\xrightarrow{R_2=R_2-R_1}\begin{bmatrix}0&1&1\\0&0&-1\\0&0&0\end{bmatrix}$$ rango 2

d) $(A-\lambda I)x=0$

CASO $\lambda=0$

$$\begin{bmatrix}1&1&1\\0&-2&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}x_1+x_2+x_3=0\\-2x_2=0\\0=0\end{cases}\quad\begin{cases}x_1=-x_3\\x_2=0\\0=0\end{cases}\Rightarrow\left\{\underline{x_3}\begin{pmatrix}-1\\0\\1\end{pmatrix}\middle|\ \underline{x_3}\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\rangle$$

CASO $\lambda=-2$

$$\begin{bmatrix}3&1&1\\0&0&2\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}3x_1+x_2+x_3=0\\2x_3=0\\0=0\end{cases}\quad\begin{cases}x_2=-3x_1\\x_3=0\\0=0\end{cases}\Rightarrow\left\{\underline{x_1}\begin{pmatrix}1\\-3\\0\end{pmatrix}\middle|\ \underline{x_1}\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\3\\0\end{pmatrix}\right\rangle$$

CASO $\lambda=1$

$$\begin{bmatrix}0&1&1\\0&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}x_2+x_3=0\\-x_3=0\\0=0\end{cases}\quad\begin{cases}x_2=0\\x_3=0\\0=0\end{cases}\Rightarrow\left\{\underline{x_1}\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|\ \underline{x_1}\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rangle$$

e)

$$V=\begin{bmatrix}-1&1&1\\0&3&0\\1&0&0\end{bmatrix}\qquad D=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=\begin{bmatrix}0&0&0\\0&-2&0\\0&0&1\end{bmatrix}$$

$A=VDV^{-1}$

$$\det(V)=1\begin{vmatrix}0&3\\1&0\end{vmatrix}=-3$$

$$A_{11}=\begin{vmatrix}3&0\\0&0\end{vmatrix}=0\qquad A_{12}=\begin{vmatrix}0&0\\1&0\end{vmatrix}=0\qquad A_{13}=\begin{vmatrix}0&3\\1&0\end{vmatrix}=-3$$

$$A_{21}=\begin{vmatrix}1&1\\0&0\end{vmatrix}=0\qquad A_{22}=\begin{vmatrix}-1&1\\1&0\end{vmatrix}=-1\qquad A_{23}=\begin{vmatrix}-1&1\\1&0\end{vmatrix}=1\rightarrow-1\quad\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$$

$$A_{31}=\begin{vmatrix}1&1\\3&0\end{vmatrix}=-3\qquad A_{32}\begin{vmatrix}-1&1\\0&0\end{vmatrix}=0\qquad A_{33}=\begin{vmatrix}-1&1\\0&3\end{vmatrix}=-3$$

$$\mathrm{cof}(V)=\begin{pmatrix}0&0&-3\\0&1&-1\\-3&0&-3\end{pmatrix}\Rightarrow\mathrm{cof}(V)^t=\begin{pmatrix}0&0&-3\\0&1&0\\-3&-1&-3\end{pmatrix}$$

$$V^{-1}=\frac{1}{\det(V)}\cdot\mathrm{cof}(V)^t=-\frac{1}{3}\begin{pmatrix}0&0&-3\\0&-1&0\\-3&-1&-3\end{pmatrix}=\begin{pmatrix}0&0&1\\0&1/3&0\\1&1/3&1\end{pmatrix}$$

$$VD=\begin{pmatrix}-1&1&1\\0&3&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&-2&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&-2&1\\0&-6&0\\0&0&0\end{pmatrix}$$

$$VDV^{-1}=\begin{pmatrix}0&-2&1\\0&-6&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1/3&0\\1&1/3&1\end{pmatrix}=\begin{pmatrix}1&1&1\\0&-2&0\\0&0&0\end{pmatrix}\quad\text{verificata}$$

f) il metodo delle potenze inverse applicato alla matrice A converge agli autovalori di A più vicini al parametro di shift p

- se p è vicino a $\lambda_1=0$, il metodo converge a λ_1
- se p è vicino a $\lambda_2=-2$, il metodo converge a λ_2
- se p è vicino a $\lambda_3=1$, il metodo converge a λ_3

22/06/2023

Calcolare la diagonalizzazione della matrice $A=\begin{bmatrix}-2 & -2 & 0\\ -2 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}$

a) $A-\lambda I$

$$\begin{bmatrix}-2 & -2 & 0\\ -2 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}=\begin{bmatrix}-2-\lambda & -2 & 0\\ -2 & -2-\lambda & 0\\ 0 & 0 & -\lambda\end{bmatrix}$$

$$\det(A)=-\lambda\begin{vmatrix}-2-\lambda & -2\\ -2 & -2-\lambda\end{vmatrix}=-\lambda\left[(-2-\lambda)(-2-\lambda)-(-2)(-2)\right]=-\lambda\left[\cancel{4}+4\lambda+\lambda^2-\cancel{4}\right]=$$

$$=-\lambda\left[\lambda^2+4\lambda\right]=-\lambda\left[\lambda(\lambda+4)\right]$$

b) $-\lambda[\lambda(\lambda+4)]$

$\lambda_1=0 \quad \lambda_2=0 \qquad \lambda_3=-4$

molteplicità 2

c)

$$\begin{bmatrix}-2 & -2 & 0\\ -2 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\xrightarrow{R_2=R_2-R_1}\begin{bmatrix}-2 & -2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad \text{rango } 1$$

CASO $\lambda=0$

Siccome $\lambda=0$, $A-\lambda I$ risulta essere uguale ad A, e ha quindi lo stesso rango di A, ossia rango 2

CASO $\lambda=-4$

$$\begin{bmatrix}-2 & -2 & 0\\ -2 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}-\begin{bmatrix}-4 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}=\begin{bmatrix}2 & -2 & 0\\ -2 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$\begin{bmatrix}2 & -2 & 0\\ -2 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}\xrightarrow{R_2=R_2+R_1}\begin{bmatrix}2 & -2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 4\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}2 & -2 & 0\\ 0 & 0 & 4\\ 0 & 0 & 0\end{bmatrix} \quad \text{rango } 2$$

d) $(A-I\lambda)x=0$

CASO $\lambda=0$

$$\begin{bmatrix}-2 & -2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{cases}-2x_1-2x_2=0\\ 0=0\\ 0=0\end{cases}\quad\begin{cases}x_1=-x_2\\ 0=0\\ 0=0\end{cases}\Rightarrow\left\{\underline{x}_2\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}\,\middle|\,\underline{x}_2\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}\right\rangle \qquad \text{molteplicità } \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

CASO $\lambda=-4$

$$\begin{bmatrix}2 & -2 & 0\\ 0 & 0 & 4\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{cases}2x_1-2x_2=0\\ 4x_3=0\\ 0=0\end{cases}\quad\begin{cases}x_1=x_2\\ x_3=0\\ 0=0\end{cases}\Rightarrow\left\{\underline{x}_2\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}\,\middle|\,\underline{x}_2\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}\right\rangle$$

e)

$$V=\begin{bmatrix}-1 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\qquad D=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$A=VDV^{-1}$

$$\det(V)=1\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}=-1-1=-2$$

$$A_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad A_{12} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0 \qquad A_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{21} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \to 1 \qquad A_{22} = \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = 0 \qquad A_{23} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \to 1$$

$$A_{31} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0 \qquad A_{23} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1-1 = -2 \to 2 \qquad A_{33} = \begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$\mathrm{cof}(V) = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \longrightarrow \mathrm{cof}(V)^t \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$V^{-1} = \frac{1}{\det(V)} \cdot \mathrm{cof}(V)^t = -\frac{1}{2} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & 0 & -1 \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

$$VD = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$VDV^{-1} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \\ -1/2 & -1/2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$