Theoretical Quantum Optics

Problem Sheet

Lecturer: Prof. Dr. Igor Lesanovsky **Semester:** Winter 23/24

Sheet: 7 **Hand-out:** 23.11.23 **Hand-in:** 30.11.23

Problem 16. Quasi-probability distributions

We define the operators $\Delta^{(\Omega)}$ and $\overline{\Delta}^{(\Omega)}$ by,

$$\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger) = \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{e}^{\Omega(\beta,\beta^*)}\mathrm{e}^{-\beta(\alpha^*-a^\dagger)+\beta^*(\alpha-a)}\,\mathrm{d}^2\beta,$$
$$\overline{\Delta}^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger) = \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{e}^{-\Omega(\beta,\beta^*)}\mathrm{e}^{\beta(\alpha^*-a^\dagger)-\beta^*(\alpha-a)}\,\mathrm{d}^2\beta,$$

where $\Omega(\beta,\beta^*)$ is a function which characterizes the different distributions used in quantum optics. The Wigner function characterized by $\Omega(\beta,\beta^*) = 0$, while for the P function, defined by,

$$\rho = \int_{\mathbb{C}} P(\alpha,\alpha^*)\,|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha,$$

one uses $\Omega(\beta,\beta^*) = -|\beta|^2/2$. Finally, $\Omega(\beta,\beta^*) = |\beta|^2/2$ represents the Husimi Q function.

a. Show that,

$$\mathrm{Tr}(\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)\overline{\Delta}^{(\Omega)}(\alpha' - a, \alpha'^* - a^\dagger)) = \frac{1}{\pi}\delta^{(2)}(\alpha - \alpha').$$

b. Show that the Wigner function $W(\alpha,\alpha^*)$ can be expressed in terms of P as,

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\int_{\mathbb{C}} P(\beta,\beta^*)\mathrm{e}^{-2|\alpha-\beta|^2}\,\mathrm{d}^2\beta.$$

Hint: Start with,

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{Tr}(\rho\mathrm{e}^{\lambda a^\dagger - \lambda^* a})\mathrm{e}^{\alpha\lambda^* - \alpha^*\lambda}\,\mathrm{d}^2\lambda,$$

and express the density matrix ρ in the form given above for the P function.

Problem 17. Quantum state representations

a. Calculate the Husimi Q function $Q = (1/\pi)\,\langle\alpha|\rho|\alpha\rangle$ for a thermal state, given by the density operator,

$$\rho = \sum_{n=0}^{\infty} \frac{\overline{n}^n}{(\overline{n}+1)^{n+1}}\,|n\rangle\langle n|\,,$$

where $|n\rangle$ are Fock states.

b. The P, Q, and Wigner functions can be expressed using the characteristic functions,

$$\{W|P|Q\}(\alpha,\alpha^*) = \frac{1}{\pi}\int_{\mathbb{C}} C_{\{S|N|A\}}(\beta,\beta^*)\mathrm{e}^{\alpha\beta^*-\alpha^*\beta}\,\mathrm{d}^2\beta,$$

where C_S, C_N, and C_A are the symmetric, normal, and anti-normal ordered characteristic functions defined by,

$$\begin{aligned}
C_S(\beta, \beta^*) &:= \mathrm{Tr}(\mathrm{e}^{\beta a^\dagger - \beta^* a}\rho) = \mathrm{Tr}(D(\beta)\rho), \\
C_N(\beta, \beta^*) &:= \mathrm{Tr}(\mathrm{e}^{\beta a^\dagger}\mathrm{e}^{-\beta^* a}\rho) = \mathrm{Tr}(D(\beta)\rho)\mathrm{e}^{|\beta|^2/2}, \\
C_A(\beta, \beta^*) &:= \mathrm{Tr}(\mathrm{e}^{-\beta^* a}\mathrm{e}^{\beta a^\dagger}\rho) = \mathrm{Tr}(D(\beta)\rho)\mathrm{e}^{-|\beta|^2/2},
\end{aligned}$$

Calculate C_S and C_N for the thermal state in **a**.

Hint: Instead of calculating the characteristic function directly, one can instead invert the above expressions for the P, Q, and Wigner functions in terms of characteristic functions by applying the complex Fourier transform detailed in a prior problem sheet to both sides, together with the definition of the Dirac delta function in the complex plane.

c. Calculate P and W functions for the thermal state given in **a** by inserting your results from **b** into the expressions for the P, Q, and Wigner functions in terms of characteristic functions above. Compare the results for P, Q, and W and give an interpretation.

Problem 18. Wavefunction for a coherent state

The defining equation for a coherent state $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$ transforms in the position representation into a differential equation for its wavefunction,

$$\frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x\right)\langle x|\alpha\rangle = \alpha\,\langle x|\alpha\rangle\,,$$

where $\hbar = l = 1$ with l the oscillator length.

a. Show that this differential equation is solved by a shifted Gaussian. Find its width, central position, and normalize it (i.e., $\int_{\mathbb{R}} |\langle x|\alpha\rangle|^2\,\mathrm{d}x = 1$).

b. Rewrite the Gaussian in terms of the mean values of x and p, defined by $\alpha = (\overline{x} + \mathrm{i}\overline{p})/\sqrt{2}$, and give an interpretation of the result.
