Theoretical Quantum Optics

Problem Sheet

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Sheet: 8 | **Hand-out:** 30.11.23 | **Hand-in:** 7.12.23

Problem 19.

Calculate the Mandel Q parameter, defined by,

$$Q_M = \frac{\langle (a^\dagger)^2 a^2\rangle - \langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle} = \frac{\langle (a^\dagger a)^2\rangle - \langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle} - 1,$$

for the following states.

a. The squeezed vacuum state $|0,\xi\rangle = S(\xi)|0\rangle$ with,

$$S(\xi) = \mathrm{e}^{-(\xi(a^\dagger)^2 - \xi^* a^2)/2}.$$

b. The displaced Fock state $D(\alpha)|n\rangle$ with,

$$D(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^* a}.$$

Problem 20.

Consider the cat state, given by,

$$|\psi\rangle = \frac{1}{N}(|\gamma\rangle + \mathrm{e}^{\mathrm{i}\phi}|-\gamma\rangle),$$

where $|\gamma\rangle$ and $|-\gamma\rangle$ are coherent states and N is the normalization constant.

a. Show that the Q function is given by,

$$Q(\alpha) = \frac{4}{\pi N^2}\mathrm{e}^{-|\alpha|^2-|\gamma|^2}|\cosh(\alpha^*\gamma - \mathrm{i}\phi/2)|^2.$$

b. Show that for $\gamma \in \mathbb{R}$ that $Q(\alpha)$ has an infinite number of zeroes on the imaginary α axis and that this signals the state to be non-classical.
Hint: Consider the connection between the P and Q functions derived in the lecture,

$$Q(\alpha) = \frac{1}{\pi}\int_{\mathbb{C}} P(\beta)\mathrm{e}^{-|\alpha-\beta|^2}\,\mathrm{d}^2\beta.$$

c. Show that the Wigner function of the cat state is given by,

$$W(\alpha) = \frac{2}{\pi N^2}(\mathrm{e}^{-2|\alpha-\gamma|^2} + \mathrm{e}^{-2|\alpha+\gamma|^2} + 2\mathrm{e}^{-2|\alpha|^2}\cos(\phi + 4\mathrm{Im}(\gamma^*\alpha))).$$

d. Plot the Wigner function (e.g., using Matlab, Mathematica, or Python) for $\phi = 0, \pi/2, \pi$ and $\gamma = 1, 2, 5$. Explain your results.

e. Consider the Wigner function $W(\alpha)$ for $\phi = 0$. Is there a value of α, with $\alpha = \mathrm{i}\bar{\alpha}$ where $\gamma, \bar{\alpha} \in \mathbb{R}$ and $\gamma > 0$, for which $W(\alpha)$ is negative?

f. Decoherence resulting from an interaction with an environment destroys "coherences" (i.e., $|-\gamma\rangle\langle\gamma|$ and $|\gamma\rangle\langle-\gamma|$) by randomizing the phase ϕ such that the state $\rho = |\psi\rangle\langle\psi|$ evolves according to,

$$\rho(t) = \frac{1}{N(t)}(|\gamma\rangle\langle\gamma| + |-\gamma\rangle\langle-\gamma| + d(t)(|\gamma\rangle\langle-\gamma| + |-\gamma\rangle\langle\gamma|)).$$

Here, $d(t)$ is a positive function decreasing monotonously with time which characterizes the increasing decoherence starting from no decoherence at $t = 0$ (i.e., $d(0) = 1$). Determine the minimal value of $d(t)$ for $\gamma \in \mathbb{R}$ such that the state can still be considered quantum, based on the negativity of the Wigner function.
Hint: Instead of calculating the Wigner function, use a modification of the result in **c**.

Problem 21.

Consider the Schrödinger cat state (i.e., the prior cat state without the phase),

$$|\psi\rangle = \frac{1}{N}(|\gamma\rangle + |-\gamma\rangle).$$

a. Determine the normalization constant N.

b. Calculate $C_N(\beta) = \mathrm{Tr}(\rho \mathrm{e}^{\beta a^\dagger} \mathrm{e}^{-\beta^* a})$ for $\rho = |\psi\rangle\langle\psi|$.

c. Consider the two cases: (i) $\gamma, \beta \in \mathbb{R}$ and (ii) $\beta = ib$ with $\gamma, b \in \mathbb{R}$. For which values of γ, β is the condition $|C_N(\beta)| > 1$ satisfied and, therefore, detects quantumness?
