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Theoretische Quantenoptik **WiSe 2023/24**

1. Übungsblatt

Aufgabe 1: Single-mode case

The components E_x and B_y of the quantized electric field and magnetic induction of a single standing mode in a resonator in vacuum are given by

$$E_x(z,t) = E_0(ae^{-i\omega t} + a^\dagger e^{i\omega t})\sin(kz) \tag{1}$$

$$B_y(z,t) = B_0\frac{1}{i}(ae^{-i\omega t} - a^\dagger e^{i\omega t})\cos(kz)\,, \tag{2}$$

where $E_0 = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}$, and $B_0 = E_0/c$, are the field amplitudes for a single photon, $k = \pi n/L$ for $n \in \mathbb{N}$ is the wave number of the mode, L the length of the resonator, V its volume, and c the speed of light. In vacuum, $\mathbf{B} = \mu_0\mathbf{H}$, where $\mathbf{H}$ is the magnetic field. The creation and annihilation operators a and $a^\dagger$ satisfy the usual commutation relations known from the quantized harmonic oscillator: $[a, a^\dagger] = 1$.

a) Calculate the commutator $[E_x(z,t), B_y(z',t')]$.

b) Deduce the Heisenberg uncertainty relation between $E_x(z,t)$, and $B_y(z',t')$.

c) What are the combinations of times or positions at which E_x and B_y can both have sharp values?

d) Calculate the fields at those times or positions and find an explanation for the results in (c).

Aufgabe 2: Multi-mode case for equal times

In the multi-mode case, the full electric and magnetic fields are decomposed into plane waves,

$$\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} \epsilon_{\mathbf{k}}^{(\lambda)} \mathcal{E}_{\mathbf{k}} a_{\mathbf{k},\lambda}\, e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.} \tag{3}$$

and

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\mu_0}\sum_{\mathbf{k},\lambda} \frac{\mathbf{k}\times\epsilon_{\mathbf{k}}^{(\lambda)}}{\omega_k} \mathcal{E}_{\mathbf{k}} a_{\mathbf{k},\lambda}\, e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}, \tag{4}$$

where $\mathcal{E}_{\mathbf{k}} = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}$ has the dimension of an electric field and $\epsilon_{\mathbf{k}}^{(\lambda)}$ is a unit polarization vector. Here, $\lambda \in \{1,2\}$ labels the two polarization directions perpendicular to the wavevector $\mathbf{k}$, whose components satisfy $k_i = 2\pi n_i/L$ with $n_i \in \mathbb{Z}$, and $\omega_k = c|\mathbf{k}|$. The commutation relations between the operators $a_{\mathbf{k},\lambda}$ and $a_{\mathbf{k},\lambda}^\dagger$ are

$$[a_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}] = [a_{\mathbf{k},\lambda}^\dagger, a_{\mathbf{k}',\lambda'}^\dagger] = 0, \tag{5}$$

$$[a_{\mathbf{k},\lambda}, a^{\dagger}_{\mathbf{k}',\lambda'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}\,, \tag{6}$$

indicating that each mode, labeled by $\mathbf{k}$ and λ, can be seen as an independent harmonic oscillator.

a) Verify that the polarization vectors satisfy

$$\epsilon^{(1)}_{\mathbf{k}i}\epsilon^{(1)}_{\mathbf{k}j} + \epsilon^{(2)}_{\mathbf{k}i}\epsilon^{(2)}_{\mathbf{k}j} = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{7}$$

where i, j represent the Cartesian components and $k^2 = |\mathbf{k}|^2$.
Hint: You may want to write the Cartesian components of the wavevector in spherical coordinates,

$$\mathbf{k} = k(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))\,, \tag{8}$$

and do so as well for the polarization vectors.

b) Show that

$$[E_x(\mathbf{r}, t), H_y(\mathbf{r}', t)] = \frac{\hbar c^2}{2V}\sum_{\mathbf{k}} k_z \left[e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right] \tag{9}$$

Hint: Write down the field components in the commutator explicitly and use (7).

c) Replace

$$\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3}\int d^3k \tag{10}$$

in equation (9) and consider the limit $V \to \infty$, to show that in the continuous case

$$[E_x(\mathbf{r}, t), H_y(\mathbf{r}', t)] = -i\hbar c^2 \frac{\partial}{\partial z}\delta^{(3)}(\mathbf{r} - \mathbf{r}')\,, \tag{11}$$

where

$$\delta^{(3)}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3k \tag{12}$$

is the three-dimensional Dirac-delta function.

d) Generalise (11) to arbitrary $j, k \in \{x, y, z\}$:

$$[E_j(\mathbf{r}, t), H_k(\mathbf{r}', t)] =? \tag{13}$$

e) Compare your results to the one mode case.