Theoretical Quantum Optics

Problem Sheet

Lecturer: Prof. Dr. Igor Lesanovsky | **Semester:** Winter 23/24

Sheet: 9 | **Hand-out:** 7.12.23 | **Hand-in:** 14.12.23

Problem 22. Photon number conservation by beam splitters

The action of a beam splitter on two modes with annihilation operators a and b can be described by the unitary operator:

$$U(\theta) = e^{i\theta(a^\dagger b + ab^\dagger)}. \tag{22.1}$$

Show that the beam splitter conserves the total photon number $n \equiv a^\dagger a + b^\dagger b$.

Problem 23. Beam splitter and the 2-mode Wigner function

Show that under the transformation

$$\begin{pmatrix} c \\ d \end{pmatrix} = U \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta & e^{i\varphi}\sin\theta \\ -\sin\theta e^{-i\varphi} & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{23.1}$$

the P, Q and W functions transform as follows:

$$F_{\text{out}}(\alpha,\beta) = F_{\text{in}}(\alpha',\beta'), \quad \text{with} \quad \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{23.2}$$

where F stands for any of the functions P, Q, and W.

Hint: Show that incoming coherent states $|\alpha'\beta'\rangle = D_{\text{a}}(\alpha')D_{\text{b}}(\beta')\,|00\rangle_{\text{ab}}$ are transformed into outgoing coherent states $|\alpha\beta\rangle_{\text{cd}}$ with labels linked by equation (23.2).

Problem 24. Phase shifter

A common optical element is a phase shifter. It simply means an operation that corresponds classically to a phase shift of the complex amplitude E of the electric field, $E \to Ee^{-i\varphi}$. In quantum mechanics a phase shifter is described by the unitary transformation $U(\varphi) \equiv e^{-ia^\dagger a\varphi}$.

a. Show that $U(\varphi)$ transforms the annihilation operators as

$$a \to U(\varphi) a U^\dagger(\varphi) = ae^{i\varphi}. \tag{24.1}$$

b. Show that a coherent state is transformed as

$$|\alpha\rangle \to U(\varphi)\,|\alpha\rangle = |\alpha e^{-i\varphi}\rangle\ . \tag{24.2}$$

c. Show the action of the phase shifter on a squeezed vacuum state, $|\xi,0\rangle = S(\xi)\,|0\rangle$ with

$$S(\xi) = \exp\left(-\frac{\xi}{2}(a^\dagger)^2 + \frac{\xi^*}{2}a^2\right)\ , \tag{24.3}$$

can be found as

$$|\xi,0\rangle \to U(\varphi)\,|\xi,0\rangle = \left|\xi e^{-2i\varphi},0\right\rangle. \tag{24.4}$$

Problem 25. Photon-subtracted squeezed vacuum state

The photon-subtracted squeezed vacuum state is given by

$$|\psi\rangle = \frac{1}{N_\psi} a S(\xi)\,|0\rangle = S(\xi)\,|1\rangle\,. \tag{25.1}$$

a. Show the second equality in (25.1) holds up to a phase factor. You need not determine the normalization factor N_ψ.

b. Calculate the Wigner function for $\tilde{\rho} \equiv \rho_1 = |1\rangle\langle 1|$.
Hint: Start with the Wigner function of the vacuum state $W_0(\alpha,\alpha^*) = \frac{2}{\pi}e^{-2|\alpha|^2}$ and apply the second half of exercise 15.

Using the transformation identities from Sheet **2** Excercise **4**

$$\begin{aligned} S(\xi)aS^\dagger(\xi) =& a\cosh(r) - a^\dagger e^{i\theta}\sinh(r) \equiv a(\xi) \\ S(\xi)a^\dagger S^\dagger(\xi) =& a^\dagger\cosh(r) - a e^{-i\theta}\sinh(r) \equiv a^\dagger(\xi), \end{aligned} \tag{25.2}$$

where $\xi = re^{i\theta}$, we have

$$\rho(a,a^\dagger) \equiv S(\xi)\tilde{\rho}(a,a^\dagger)S^\dagger(\xi) = \tilde{\rho}(a(\xi),a^\dagger(\xi)) \tag{25.3}$$

for any function $\tilde{\rho}(a,a^\dagger)$ that can be expanded as a power series. The linearity property of the Wigner function then implies

$$W_\rho(\alpha,\alpha^*) = W_{\tilde{\rho}}(\alpha(\xi),\alpha^*(\xi)), \tag{25.4}$$

where $\alpha(\xi) = \alpha\cosh(r) - \alpha^* e^{i\theta}\sinh(r)$ and $\alpha^*(\xi) = \alpha^*\cosh(r) - \alpha e^{-i\theta}\sinh(r)$, and W_ρ ($W_{\tilde{\rho}}$) is the Wigner function for the state ρ ($\tilde{\rho}$), respectively.

c. Determine the Wigner function for the photon-subtracted squeezed vacuum state (25.1) using your result from **b.** and equation (25.4). For what values of $|\alpha(\xi)|^2$ does the Wigner function become negative?

d. Plot the Mandel Q-parameter given in exercise **19** by using the results

$$\begin{aligned} \langle\psi|\,(a^\dagger)^2 a^2\,|\psi\rangle =& \frac{3t^2(3+2t^2)}{(1-t^2)^2} \\ \langle\psi|\,a^\dagger a\,|\psi\rangle =& \frac{1+2t^2}{1-t^2}, \end{aligned} \tag{25.5}$$

with $t = \tanh(r) \in (0,1)$. For approximately what values of r is Q_M negative ? Compare your results to those from **c**.
