Theoretical Quantum Optics

Problem Sheet

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Sheet: 4	**Hand-out:** 2.11.23	**Hand-in:** 9.11.23

Problem 8. Complex Gaussian integrals

a. Show that the complex Gaussian integral gives,

$$\int \mathrm{d}^2\beta \; \mathrm{e}^{a\beta + b\beta^* - \delta|\beta|^2} = \frac{\pi}{\delta}\mathrm{e}^{ab},$$

with $a, b \in \mathbb{C}$ where $\delta > 0$ and $\mathrm{d}^2\beta = \mathrm{d}\beta\mathrm{d}\beta^*$.

b. Generalize the result to Gaussian integrals over N variables, that is, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ and show that,

$$\int \mathrm{d}^2\boldsymbol{\beta} \; \mathrm{e}^{-\boldsymbol{\beta}^\dagger \boldsymbol{\Gamma}\boldsymbol{\beta} + \mathbf{J}^\dagger\boldsymbol{\beta} + \boldsymbol{\beta}^\dagger\mathbf{J}} = \frac{(2\pi)^N}{\det(\boldsymbol{\Gamma})}\mathrm{e}^{\mathbf{J}^\dagger\boldsymbol{\Gamma}^{-1}\mathbf{J}},$$

where $\boldsymbol{\Gamma}$ is a Hermitian matrix, $\mathbf{J}$ is a complex vector, and $\mathrm{d}^2\boldsymbol{\beta} = \mathrm{d}\beta_1\mathrm{d}\beta_1^* \cdots \mathrm{d}\beta_N\mathrm{d}\beta_N^*$.

Problem 9. Eigenstates of the annihilation operator

Coherent states $|\alpha\rangle$ are defined as eigenstates of the annihilation operator, $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$. One may wonder what the eigenstates of the creation operator $a^\dagger$ are. It turns out, however, that $a^\dagger$ does not have any eigenstates! Prove this.

Problem 10. Fourier transform in the complex plane

Let $g(\xi)$ be a function, $g : \mathbb{C} \to \mathbb{C}$. Its Fourier transform over the complex plane, $\tilde{g}(\alpha)$, is then defined as,

$$\tilde{g}(\alpha) = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\xi \; \mathrm{e}^{\alpha\xi^* - \alpha^*\xi} g(\xi),$$

where the integration is over the complex plane, that is, $\mathrm{d}^2\xi = \mathrm{d}(\mathfrak{Re}(\xi))\mathrm{d}(\mathfrak{Im}(\xi))$.

a. Show that $g(\xi) = \frac{1}{2}\mathrm{e}^{-\frac{1}{2}|\xi|^2}$ has the Fourier transform $\tilde{g}(\alpha) = \mathrm{e}^{-2|\alpha|^2}$.

b. The convolution product of two functions $f(\xi)$ and $g(\xi)$ is defined as,

$$(f * g)(\xi) = \int \mathrm{d}^2\rho f(\rho)g(\xi - \rho).$$

Show that $\widetilde{(f * g)}(\alpha) = \pi\tilde{f}(\alpha)\tilde{g}(\alpha)$.
