Theoretical Quantum Optics

Problem Sheet

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Sheet: 3	**Hand-out:** 26.10.23	**Hand-in:** 2.11.23

Problem 5. Coherent states

a. For two coherent states $|\alpha\rangle$ and $|\beta\rangle$, show that,

$$\langle\alpha|\beta\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\alpha^*\beta} \qquad \text{and} \qquad |\langle\alpha|\beta\rangle| = \mathrm{e}^{-|\alpha-\beta|^2}.$$

b. Prove the completeness relation,

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}.$$

Hint: Use the expansion of the coherent states in the Fock basis and use polar coordinates for the integral over the complex plane.

c. In the previous problem sheet, we found that for a coherent state $|\alpha\rangle$ with $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\theta}$, the uncertainties of the quadrature operators,

$$x = \frac{a+a^\dagger}{2}, \qquad p = \frac{a-a^\dagger}{2\mathrm{i}},$$

are given by,

$$\langle\alpha|(\Delta x)^2|\alpha\rangle = \langle\alpha|(\Delta p)^2|\alpha\rangle = \frac{1}{4}.$$

Hence, coherent states are indeed minimum uncertainty product states. Does this uncertainty of the quadrature operators for a coherent state change under time evolution with a harmonic oscillator as Hamiltonian?

Hint: Calculate the time evolution of the coherent state $|\alpha\rangle$ with respect to the Hamiltonian of the harmonic oscillator $H = \hbar\omega(a^\dagger a + \frac{1}{2})$.

d. Calculate the expectation value of the electric field,

$$E = \mathrm{i}\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} a \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{H.c.},$$

for a coherent state.

e. Furthermore, show that the fluctuations of the electric field are independent of the field amplitude, which is proportional to $|\alpha|$, and are given by

$$\Delta E = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}.$$

Problem 6. Single mode field Hamiltonian with displacement

Let the Hamiltonian of a single mode field be given by $H = \hbar\omega(a^\dagger a + \frac{1}{2}) + \hbar\gamma(a^\dagger + a)$.

a. Derive the time evolution of the annihilation operator a, given by,

$$a_{\mathrm{H}}(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar} a \mathrm{e}^{-\mathrm{i}Ht/\hbar},$$

where $a_{\mathrm{H}}(t)$ is the solution of the Heisenberg equation with the initial condition $a_{\mathrm{H}}(0) = a$.

b. Calculate the expectation value of the time evolved generalised position operator $Q_{\mathrm{H}}(t) = \frac{1}{2}(a_{\mathrm{H}}^\dagger(t) + a_{\mathrm{H}}(t))$ in an eigenstate of $a^\dagger a$.

Problem 7. Thermal photon fields

Consider a single mode field, described by the Hamiltonian $H = \hbar\omega(a^\dagger a + \frac{1}{2})$, that is in thermal equilibrium with the walls of a cavity at temperature T. The system is described by the density operator,

$$\rho_{\mathrm{th}} = \frac{1}{Z}\mathrm{e}^{-H/k_{\mathrm{B}}T}.$$

a. Calculate the partition function,

$$Z = \mathrm{Tr}(\mathrm{e}^{-H/k_{\mathrm{B}}T}).$$

b. Determine the probability p_n that the mode is thermally excited in the n^{th} level, that is, calculate $p_n = \langle n|\rho_{\mathrm{th}}|n\rangle$.

c. Calculate the average photon number of the thermal field, $\bar{n} = \mathrm{Tr}(\rho_{\mathrm{th}} n)$, where n is the number operator. Determine the approximate behaviour of $\bar{n}$ at high temperatures (i.e., $k_{\mathrm{B}}T \gg \hbar\omega$) and similarly at low temperatures (i.e., $k_{\mathrm{B}}T \ll \hbar\omega$).

d. Determine the average number of photons at optical frequencies ($\nu \approx 600$ THz) at room temperature ($T \approx 300$ K) and on the surface of the sun ($T \approx 6000$ K). What is $\bar{n}$ for microwave photons of wavelength $\lambda = 1$ cm at both temperatures?

e. Express the probability p_n as a function of the average photon number of the thermal field $\bar{n}$ and plot the thermal photon number distribution p_n for $\bar{n} = 0.1$ and $\bar{n} = 2$.
