Theoretical Quantum Optics

Problem Sheet

Lecturer: Prof. Dr. Igor Lesanovsky **Semester:** Winter 23/24

Sheet: 13 **Hand-out:** 18.1.24 **Hand-in:** 25.1.24

Problem 33. Mach-Zehnder interferometer

In the Schwinger representation, a beam splitter is described by the unitary transformation

$$U_{\mathrm{BS}}(\alpha) = \exp\left[i\alpha(a^\dagger b + ab^\dagger)\right] = \exp(2i\alpha J_x), \tag{33.1}$$

where $\alpha \in [0, 2\pi]$ is the mixing angle. A phase shift in one of the arms of the interferometer is described by $U_{\mathrm{ps}} = \exp\left(i\varphi a^\dagger a\right)$.

a. Express $U_{\mathrm{ps}}(\varphi)$ in terms of the total photon number $\hat{N} = a^\dagger a + b^\dagger b$, and $J_z = \frac{a^\dagger a - b^\dagger b}{2}$ (see Problem 11 of Sheet 11).

b. Show that $\hat{N}$ commutes with J_x, J_z.

c. Show that the unitary transformation of the Mach-Zehnder interferometer,

$$U_{\mathrm{MZ}}(\alpha, \varphi) = U_{\mathrm{BS}}(-\alpha) U_{\mathrm{ps}}(\varphi) U_{\mathrm{BS}}(\alpha),$$

can be written as $U_{\mathrm{MZ}}(\alpha, \varphi) = \exp(i\varphi \hat{J}_\alpha)$, where $\hat{J}_\alpha$ is a linear combination of $\hat{N}$, J_y and J_z that you should determine.

d. The quantum Fisher information (QFI) I_φ for a parameter φ imprinted on a pure state $|\psi\rangle$ via the unitary $U_{\mathrm{MZ}}(\alpha, \varphi)$ is given by

$$I_\varphi = 4\,\mathrm{var}(\hat{J}_\alpha), \tag{33.2}$$

where the variance is calculated in the state $|\psi\rangle$.
Calculate I_φ at $\alpha = \frac{\pi}{4}$ for the following states:

a) $|\psi\rangle = |j, -j\rangle_z$

b) $|\psi\rangle = \frac{1}{\sqrt{2}}(|j, -j\rangle_y + |j, j\rangle_y)$,

where $|j, m\rangle_i$ are the usual angular momentum eigenstates with $\hat{J}_i|j, m\rangle_i = m|j, m\rangle_i$ and $\hat{J}^2|j, m\rangle_i = j(j+1)|j, m\rangle_i$.
Hint: Utilize the results from part **b** to argue that eigenstates of $\hat{J}^2$ are also eigenstates of $\hat{N}$, and to further simplify eq. (33.2).

Problem 34. $G^{(2)}$ for the 2-mode coherent state

Let the electromagnetic field consist of two modes, only with wavevectors $\mathbf{k}, \mathbf{k}'$, and fixed common polarization that will be suppressed in the following. Further, let the quantum state of the field be a tensor product of two coherent states

$$|\psi\rangle = |\alpha\rangle_{\mathbf{k}} |\alpha\rangle_{\mathbf{k}'} \tag{34.1}$$

a. Calculate the second order correlation function

$$G^{(2)}(\mathbf{r_1}, \mathbf{r_2}, t, t) = \langle\psi| \hat{E}^{(-)}(\mathbf{r_1}, t)\hat{E}^{(-)}(\mathbf{r_2}, t)\hat{E}^{(+)}(\mathbf{r_2}, t)\hat{E}^{(+)}(\mathbf{r_1}, t) |\psi\rangle , \tag{34.2}$$

where $\hat{E}^{(+)}(\mathbf{r}, t) = \sum_{\mathbf{j}} \mathcal{E}_j \hat{a}_{\mathbf{j}} e^{-i\omega_j t + i\mathbf{j}\cdot\mathbf{r}}$ is the positive-frequency part of the electric field expanded in a sum of plane waves with wave vector $\mathbf{j}$,and annihilation operator $\hat{a}_{\mathbf{j}}$, $j = |\mathbf{j}|$, and $|\mathbf{k}| = k = |\mathbf{k}'|$ (this implies $\mathcal{E}_k = \mathcal{E}_{k'}$ and $\omega_k = \omega_{k'}$).

b. Under what condition does one observe an interference signal as a function of $\mathbf{r_1} - \mathbf{r_2}$ and what is its contrast?

Hint: Rewrite $G^{(2)}$ as a function of $\mathbf{r}_\pm = \mathbf{r_1} \pm \mathbf{r_2}$ and look at under what condition the dependence on $\mathbf{r}_+$ vanishes.

The contrast of a function $f(x)$ is defined as $\frac{\max_x f - \min_x f}{\max_x f + \min_x f}$.