Theoretical Quantum Optics

Problem Sheet

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Sheet: 11	**Hand-out:** 21.12.23	**Hand-in:** 11.1.24

Problem 29. Hanbury Brown-Twiss effect with fermions

One can sometimes read the statement that the Hanbury Brown-Twiss correlations are due to the bunching properties of bosons. In order to examine that claim, we calculate the correlation function $G^{(2)}$ for two fermions,

$$G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t, t) = \langle \psi | \hat{\psi}^\dagger(\mathbf{r}_1, t) \hat{\psi}^\dagger(\mathbf{r}_2, t) \hat{\psi}(\mathbf{r}_2, t) \hat{\psi}(\mathbf{r}_1, t) | \psi \rangle \,, \tag{29.1}$$

where now,

$$\hat{\psi}(\mathbf{r}_1, t) = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{-i\epsilon_k t + i\mathbf{k}\cdot\mathbf{r}_1},$$
$$\hat{\psi}^\dagger(\mathbf{r}_1, t) = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} e^{i\epsilon_k t - i\mathbf{k}\cdot\mathbf{r}_1},$$

and the $c_{\mathbf{k}}, c^\dagger_{\mathbf{k}}$ obey the fermionic anti-commutation relations,

$$\{c_{\mathbf{k}}, c^\dagger_{\mathbf{k}'}\} = c_{\mathbf{k}} c^\dagger_{\mathbf{k}'} + c^\dagger_{\mathbf{k}'} c_{\mathbf{k}} = \delta_{\mathbf{k}\mathbf{k}'}, \qquad \{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = \{c^\dagger_{\mathbf{k}}, c^\dagger_{\mathbf{k}'}\} = 0.$$

a. Show that for an initial state $|1_{\mathbf{k}} 1_{\mathbf{k}'}\rangle = c^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{k}'} |0\rangle$ that $G^{(2)}$ can be written as,

$$G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t, t) = |\psi^{(2)}(\mathbf{r}_1, t; \mathbf{r}_2, t)|^2, \tag{29.2}$$

with $\psi(\mathbf{r}_1, t; \mathbf{r}_2, t) = \langle 0 | \hat{\psi}(\mathbf{r}_2, t) \hat{\psi}(\mathbf{r}_1, t) | 1_{\mathbf{k}} 1_{\mathbf{k}'} \rangle$.

b. Calculate $G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t, t)$ explicitly and compare it to the bosonic case where,

$$G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t, t) = 2\mathcal{E}_k^4 (1 + \cos((\mathbf{k} - \mathbf{k}') \cdot (\mathbf{r}_1 - \mathbf{r}_2))).$$

Explain your findings.

Problem 30. One- and two-photon correlations with coherent states

Let $|\psi\rangle = |\alpha\rangle_{\mathbf{k}} |\alpha'\rangle_{\mathbf{k}'}$ be a two-mode coherent state of two electromagnetic modes $\mathbf{k}, \mathbf{k}'$ corresponding to plane waves with,

$$E^{(-)}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathcal{E}_k a^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{r})},$$

being the scalar electric field operator and we take $\alpha, \alpha' \in \mathbb{R}$ and $\mathbf{k} \neq \mathbf{k}'$.

a. Calculate the correlation function,

$$G^{(1)}(\mathbf{r}, \mathbf{r}, t, t) = \langle E^{(-)}(\mathbf{r}, t) E^{(+)}(\mathbf{r}, t) \rangle. \tag{30.1}$$

What happens for $|\mathbf{k}| = |\mathbf{k}'|$? Compare $G^{(1)}(\mathbf{r}, \mathbf{r}, t, t)$ to the two-photon correlation function $G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t, t) = \langle E^{(-)}(\mathbf{r}_1, t)E^{(-)}(\mathbf{r}_2, t)E^{(+)}(\mathbf{r}_2, t)E^{(+)}(\mathbf{r}_1, t)\rangle$ for a thermal state.
Hint: In the lecture, it was shown that,

$$G^{(2)}_{\mathrm{th}}(\mathbf{r}_1, \mathbf{r}_2, t, t) = 2\mathcal{E}_k^4(2\langle n\rangle^2 + \langle n\rangle^2(1 + \cos((\mathbf{k} - \mathbf{k}') \cdot (\mathbf{r}_1 - \mathbf{r}_2)))),$$

where $\langle n\rangle = \alpha^2$.

b. Calculate $G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t, t)$ for $|\mathbf{k}| = |\mathbf{k}'|$ and $\alpha = \alpha'$.

Problem 31. Schwinger representation

With the annihilation and creation operators a, b and $a^\dagger$, $b^\dagger$ from two mode of the electromagnetic field, one can construct pseudo-angular momentum operators,

$$J_x = \frac{1}{2}(a^\dagger b + ab^\dagger), \qquad J_y = \frac{1}{2\mathrm{i}}(a^\dagger b - ab^\dagger), \qquad J_z = \frac{1}{2}(a^\dagger a - b^\dagger b). \tag{31.1}$$

a. Show that the J_i obey the SU(2) commutation relations (i.e., $[J_j, J_k] = \mathrm{i}\epsilon_{jkl}J_l$ with the implicit summation over l and where ϵ_{jkl} is the Levi-Civita symbol).

b. Express the total angular momentum in terms of a, $a^\dagger$, b, $b^\dagger$. Conclusions?

c. Examine the commutation and anti-commutation relations of the J_i and see if the a, $a^\dagger$, b, $b^\dagger$ obey the fermionic anti-commutation relations defined above.

d. What is the maximum value of J_z in the fermionic case? Show that this result implies once more the commutation relation of the J_i for the two-mode fermions.
