Theoretical Quantum Optics

Problem Sheet

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Sheet: 6	**Hand-out:** 16.11.23	**Hand-in:** 23.11.23

Problem 14. Ordering of operators

A monomial of creation- and annihilation operators $a^\dagger, a$ of a single mode is said to be in "normal order" if all the $a^\dagger$ are on the left, and in "anti-normal order" if all the $a^\dagger$ are on the right. One can always use the commutation relation $[a, a^\dagger] = 1$ in order to bring a given polynomial into a form where all monomials are in normal or anti-normal order, and this is the procedure to follow when one wants to calculate expectation values of a polynomial $a^\dagger, a$ in a state represented by the $P-$function or the Q-function.

However, for a given function of $f(a^\dagger, a)$, its normally ordered form $\{f(a^\dagger, a)\}_N$ is in the literature simply defined by expanding the function into monomials without using the commutation relation, and then shifting all the $a^\dagger$s to the left in each monomial, as if they commuted (see e.g. Gardiner/Zoller, "Quantum Noise", p.102f). Sometimes, this is also denoted as $: f(a^\dagger, a) : \equiv \{f(a^\dagger, a)\}_N$. Similarly, the anti-normal order $\{f(a^\dagger, a)\}_A$ is defined by expanding the function into monomials without using the commutation relation, and then shifting all the $a^\dagger$s to the right in each monomial, as if they commuted. In addition, one defines the symmetric order, where one averages over all distinguishable orders with equal weight.
Example: $f(a^\dagger, a) = aa^\dagger a$

$$
\begin{aligned}
\Rightarrow \{f(a^\dagger, a)\}_N &= \; : f(a^\dagger, a) : \; = a^\dagger a^2 \\
\{f(a^\dagger, a)\}_A &= a^2 a^\dagger \\
\{f(a^\dagger, a)\}_S &= \frac{1}{3}(a^\dagger a^2 + aa^\dagger a + a^2 a^\dagger).
\end{aligned}
$$

a. Determine the differences between the three operator orderings and $f(a^\dagger, a)$ for:

  1. $f(a^\dagger, a) = (a^\dagger + a)^2$

  2. $f(a^\dagger, a) = (a^\dagger)^2 a^2$

b. Show that while it can be uniquely determined whether or not a polynomial is in a specific order, different equivalent forms of a function $f(a^\dagger, a)$ can lead to different (anti-)normally or symmetrically ordered forms of it.

c. How many terms are there in $\{((a^\dagger)^m a^n)\}_S$ for $m, n \in \mathbb{N}$?

Problem 15. Correspondence between operator and phase space formalism

a. Show that for the Wigner function $W_\rho(x,p) = \frac{1}{2\pi}\int \mathrm{d}\xi e^{-i\xi p}\left\langle x+\frac{\xi}{2}\right|\rho\left|x-\frac{\xi}{2}\right\rangle$ of a state ρ with ($\hbar = 1$), one has the operator correspondence

   i. "$\hat{x}\rho \leftrightarrow \left(x + \frac{i}{2}\frac{\partial}{\partial p}\right)W(x,p)$", i.e., $W_{\hat{x}\rho}(x,p) = \left(x + \frac{i}{2}\frac{\partial}{\partial p}\right)W_\rho(x,p)$,

   ii. "$\rho\hat{x} \leftrightarrow \left(x - \frac{i}{2}\frac{\partial}{\partial p}\right)W(x,p)$", i.e., $W_{\rho\hat{x}}(x,p) = \left(x - \frac{i}{2}\frac{\partial}{\partial p}\right)W_\rho(x,p)$,

   iii. "$\hat{p}\rho \leftrightarrow \left(p - \frac{i}{2}\frac{\partial}{\partial x}\right)W(x,p)$", i.e., $W_{\hat{p}\rho}(x,p) = \left(p - \frac{i}{2}\frac{\partial}{\partial x}\right)W_\rho(x,p)$,

   iv. "$\rho\hat{p} \leftrightarrow \left(p + \frac{i}{2}\frac{\partial}{\partial x}\right)W(x,p)$", i.e., $W_{\hat{p}\rho}(x,p) = \left(p + \frac{i}{2}\frac{\partial}{\partial x}\right)W_\rho(x,p)$.

   Here hats are used to distinguish between operators and arguments of the Wigner function. In the following, this will once again be omitted whenever it is not necessary.

b. Using $\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$, $\alpha = \frac{1}{\sqrt{2}}(x + ip)$ and the correspondence stated in the previous part, show that:

   i. "$a\rho \leftrightarrow \left(\alpha + \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right)W(\alpha,\alpha^*)$", i.e., $W_{a\rho}(\alpha,\alpha^*) = \left(\alpha + \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right)W_\rho(\alpha,\alpha^*)$,

   ii. "$\rho a \leftrightarrow \left(\alpha - \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right)W(\alpha,\alpha^*)$", i.e., $W_{\rho a}(\alpha,\alpha^*) = \left(\alpha - \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right)W_\rho(\alpha,\alpha^*)$,

   iii. "$a^\dagger\rho \leftrightarrow \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial \alpha}\right)W(\alpha,\alpha^*)$", i.e., $W_{a^\dagger\rho}(\alpha,\alpha^*) = \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial \alpha}\right)W_\rho(\alpha,\alpha^*)$,

   iv. "$\rho a^\dagger \leftrightarrow \left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial \alpha}\right)W(\alpha,\alpha^*)$", i.e., $W_{\rho a^\dagger}(\alpha,\alpha^*) = \left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial \alpha}\right)W_\rho(\alpha,\alpha^*)$.
