

Introduction to dissipation and decoherence in quantum systems

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Abstract

These lecture notes address an audience of physicists or mathematicians who have been exposed to a first course in quantum mechanics. We start with a brief discussion of the general “system-bath” paradigm of quantum dissipative systems, analyze in some detail the simplest example of “pure dephasing” of a two-level system, and review the basic concept of the density matrix. We then treat the general dissipative time-evolution, introducing completely positive maps, their relation to entanglement theory, and their Kraus decomposition. Restricting ourselves to Markovian evolution, we discuss the Lindblad form of master equations. The notes conclude with an overview of topics of current interest that go beyond Lindblad Markov master equations.

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1 Introduction

The following general situation is of interest in many fields of quantum physics, ranging from quantum optics to condensed matter physics: A single quantum system interacts with a large reservoir, alternatively called “bath” or “environment”. Whenever the system is driven out of equilibrium by external perturbations, this coupling makes the system relax back to equilibrium.

The system itself might be a single atom, the spin degree of freedom of an electron or a nucleus, a single particle moving through some potential landscape (such as in a man-made interferometer in a semiconductor), or even a many-particle system. In most of these cases the system has only a few degrees of

freedom, and sometimes it even has a finite-dimensional Hilbert space (such as for the spin). In contrast, the bath needs to have infinitely many degrees of freedom in order to generate truly irreversible dynamics. It might be the electromagnetic field into which the atom can radiate its energy, or the crystal lattice that gets distorted by an electron moving along.

In a classical setting, the bath introduces dissipation (friction), as the system's energy can irreversibly be transferred to the bath. By necessity, this also means that the system will experience a fluctuating force. The balance of those two effects makes the system settle into thermal equilibrium at a temperature set by the bath. In quantum dynamics, there is yet another feature beyond these two: The system can display coherent effects, i.e. interference phenomena in space or time. These will be destroyed gradually by the coupling to the environment, an effect known as “decoherence” or “dephasing”.

In the following lecture notes, we will start with a simple example that introduces the basic physics and requires only knowledge of elementary quantum mechanics, applied to the two-level system. It will be used in subsequent sections to illustrate the concepts. We then review the description of “mixed” (incoherent) quantum states by way of the density matrix. Afterwards, we discuss the most general framework for describing dissipative quantum dynamics: completely positive maps. In the next step, we restrict ourselves to the important and simple class of Markov dynamics and introduce the Lindblad master equation, which is a workhorse of open systems dynamics in many branches of physics. Finally, we outline a few topics of modern research that go beyond this elementary tool.

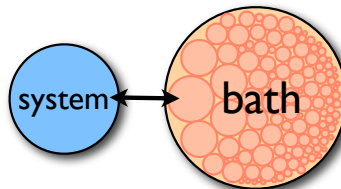


Figure 1: A quantum system interacting with the infinitely many degrees of freedom of an environment.

2 Example: Pure dephasing of a two-level system

2.1 Effect of a classical stochastic process

We consider a system described by a two-dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^2$. This is a two-level system, called “qubit” in some modern applications. Since the spin 1/2 is the most important physical realization, we denote the two basis states as “spin up” $|\uparrow\rangle$ and “spin down” $|\downarrow\rangle$. The vector describing the system's state contains the two complex probability amplitudes for these basis states:

$$|\psi(t)\rangle = \begin{pmatrix} \psi_{\uparrow}(t) \\ \psi_{\downarrow}(t) \end{pmatrix}. \quad (1)$$

In the absence of a magnetic field (i.e. in the absence of an energy difference between the two spin states), $\psi_{\uparrow/\downarrow}(t)$ would be time-independent. We now look

at the simplest possible model for dephasing of this two-level system: The energy difference between both basis states fluctuates in the course of time. For the moment, we will not even keep the bath as a large quantum system in its own right. Rather, we assume that the fluctuations it produces can be treated as a classical stochastic process. This becomes a good approximation for the limit of high temperatures. The time-dependent Hamiltonian we thus want to study is

$$H(t) = \begin{pmatrix} \epsilon(t) & 0 \\ 0 & 0 \end{pmatrix}. \quad (2)$$

Here $\epsilon(t)$ is the stochastic process, which has the meaning of a fluctuating magnetic field along the z -direction if we think of the case of a spin. The Schrödinger equation $i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$ can be solved directly. The components of the solution are

$$\psi_{\downarrow}(t) = \psi_{\downarrow}(0) \quad (3)$$

$$\psi_{\uparrow}(t) = e^{i\varphi(t)} \psi_{\uparrow}(0), \quad (4)$$

where the fluctuating phase is the integral over $\epsilon(t)$:

$$\varphi(t) = -\frac{1}{\hbar} \int_0^t \epsilon(t') dt'. \quad (5)$$

First of all, we observe that the probabilities $|\psi_{\uparrow/\downarrow}(t)|^2$ do not evolve, and only the relative phase between both states is affected by the noise. This is the reason for speaking of “pure” dephasing in this example. Next, we turn to look at observables that are sensitive to the relative phase between the two basis states, e.g. the operator σ_x that is proportional to the spin’s component in the x -direction:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (6)$$

Its quantum-mechanical expectation value, for a fixed realization of the process $\epsilon(t)$, is given by

$$\langle \psi(t) | \sigma_x | \psi(t) \rangle = \psi_{\downarrow}^*(0) \psi_{\uparrow}(0) e^{i\varphi(t)} + \text{c.c.} \quad (7)$$

[Remarks on notation: We employ standard “bra-ket” notation, $\langle \psi | A | \phi \rangle \equiv \langle \psi | A \phi \rangle \equiv \psi^\dagger A \phi$, where $\langle \psi | \psi \rangle = 1$. The complex conjugated version of the first term on the right-hand-side is abbreviated as c.c.]

In reality, it will be necessary to repeat the experiment many times in order to obtain an estimate for the expectation value, by averaging over the outcomes of σ_x measurements (which yield either $+1$ or -1 in each individual run). In general, each run will correspond to a different realization of $\epsilon(t)$ as well. Therefore, the result obtained by this averaging procedure actually also involves taking

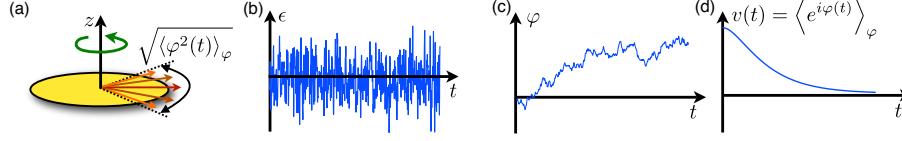


Figure 2: (a) Pure dephasing can be viewed as arising from different, fluctuating precession frequencies for an ensemble of spins; (b) Frequency fluctuations as a stochastic process; (c) Resulting phase fluctuations; (d) Visibility

the classical (stochastic) expectation value of the random phase factor $e^{i\varphi}$. We will denote this average by $\langle \cdot \rangle_\varphi$ in order to distinguish it from the quantum mechanical expectation value. Thus, the experimentalist will observe

$$\langle \langle \psi(t) | \sigma_x | \psi(t) \rangle \rangle_\varphi = \psi_\downarrow^*(0) \psi_\uparrow(0) \langle e^{i\varphi(t)} \rangle_\varphi + \text{c.c.} \quad (8)$$

Apparently, the factor

$$v(t) \equiv \langle e^{i\varphi(t)} \rangle_\varphi \quad (9)$$

describes the modification of the “interference term” $\psi_\downarrow^*(0) \psi_\uparrow(0)$ due to the noise. The same factor would occur if we were to evaluate, say, the expectation value of σ_y . We will here call this factor “visibility”, as it has to do with the contrast or visibility of interference experiments that rely on the coherence between the two basis states. In particular, the overall magnitude of the interference term is suppressed by $|v(t)| \leq 1$. This will usually decrease in the course of time, though that is not guaranteed and will depend on the properties of the process $\epsilon(t)$.

If ϵ is a Gaussian stochastic process (of zero mean), and therefore $\varphi(t)$ is a Gaussian random variable with $\langle \varphi(t) \rangle = 0$, we can express the visibility explicitly in terms of the variance of the phase:

$$v(t) = e^{-\frac{1}{2} \langle \varphi(t)^2 \rangle_\varphi}. \quad (10)$$

2.2 Several ways to obtain the same mixed state

If we are only concerned with the system and the expectation values of any of its observables (i.e. operators acting on the system alone), there are indeed many more situations that will lead to the same predictions. Let us enumerate them:

(a) *The influence of a classical stochastic process*, with the subsequent average, has been treated above.

(b) *Interaction with a bath in a random initial state*: Suppose we are considering a large Hilbert space, consisting of system and bath: $\mathcal{H}_{SB} = \mathcal{H}_S \otimes \mathcal{H}_B$. The product states of the form $|\psi_S\rangle \otimes |\chi_B\rangle \in \mathcal{H}_{SB}$ (where $|\psi_S\rangle \in \mathcal{H}_S$ and

$|\chi_B\rangle \in \mathcal{H}_B$) will be denoted as $|\psi_S\rangle |\chi_B\rangle$ for brevity, where it is understood that the first part refers to the system, and the second to the bath.

In order to describe the interaction between system and bath we might either write down the full Hamiltonian, or else (more conveniently for our purposes here) state the action of the full unitary time-evolution operator $U_{SB}(t)$ that maps an initial state in \mathcal{H}_{SB} onto a final state at time t . Given an orthonormal basis of states $|\chi_j\rangle$ in \mathcal{H}_B , we postulate that the time-evolution simply results in a phase factor that depends on the state, in the form:

$$U_{SB}(t) |\uparrow\rangle |\chi_j\rangle = e^{i\varphi_j(t)} |\uparrow\rangle |\chi_j\rangle \quad (11)$$

$$U_{SB}(t) |\downarrow\rangle |\chi_j\rangle = |\downarrow\rangle |\chi_j\rangle. \quad (12)$$

This defines U_{SB} . Now suppose furthermore that the initial state of the bath is picked out of the $|\chi_j\rangle$ at random, each of them occurring with a probability w_j (where $\sum_j w_j = 1$). This means that the full initial state is $|\psi_{SB}^{(j)}(0)\rangle = |\psi(0)\rangle |\chi_j\rangle$ with probability w_j . Such a situation actually arises if the bath is at a finite temperature, where different energy eigenstates occur with classical probabilities fixed by the Boltzmann distribution.

We now want to calculate the observed average value of σ_x at time t . To this end, we must first evaluate the quantum-mechanical expectation value of the product operator $(\sigma_x)_S \otimes (\mathbf{1})_B$, with respect to the state $|\psi_{SB}^{(j)}(t)\rangle = U_{SB}(t) |\psi_{SB}^{(j)}(0)\rangle$, and then average according to the weights w_j . We write the operator as σ_x for short, it being understood that σ_x acts only on the system and leaves the bath's state untouched. Straightforward calculation using the rules (11) and (12) shows that we obtain exactly the same result as before:

$$\sum_j w_j \langle \psi_{SB}^{(j)}(t) | \sigma_x | \psi_{SB}^{(j)}(t) \rangle = \psi_{\downarrow}^*(0) \psi_{\uparrow}(0) \langle e^{i\varphi(t)} \rangle_{\varphi} + \text{c.c.}, \quad (13)$$

if we identify $\langle e^{i\varphi(t)} \rangle_{\varphi}$ with $\sum_j w_j e^{i\varphi_j}$ [the extension to a continuous probability distribution is obvious]. Moreover, the results are the same for *any* system operator whose expectation value we care to evaluate. In this sense, the incoherent (“mixed”; see below for definition) system states produced according to (a) and (b) are indistinguishable.

(c) “*Quantum randomness*”: Instead of postulating classical randomness in the choice of the initial bath state, we could as well have postulated that it is a superposition of the basis states,

$$|\chi(0)\rangle = \sum_j \sqrt{w_j} |\chi_j\rangle, \quad (14)$$

which is normalized since $\sum_j (\sqrt{w_j})^2 = 1$. Under the action of $U_{SB}(t)$, the initial state

$$|\psi_{SB}(0)\rangle = (\psi_{\uparrow}(0) |\uparrow\rangle + \psi_{\downarrow}(0) |\downarrow\rangle) \otimes |\chi(0)\rangle \quad (15)$$

will evolve into

$$|\psi_{SB}(t)\rangle = \psi_{\uparrow}(0) |\uparrow\rangle |\chi_{\uparrow}(t)\rangle + \psi_{\downarrow}(0) |\downarrow\rangle |\chi_{\downarrow}(t)\rangle. \quad (16)$$

Here we have defined the two bath states $|\chi_{\uparrow}(t)\rangle$ and $|\chi_{\downarrow}(t)\rangle$ that have evolved out of $|\chi(0)\rangle$ under the influence of the system being in state $|\uparrow\rangle$ or $|\downarrow\rangle$, respectively:

$$|\chi_{\uparrow}(t)\rangle = \sum_j \sqrt{w_j} e^{i\varphi_j(t)} |\chi_j\rangle \quad (17)$$

$$|\chi_{\downarrow}(t)\rangle = \sum_j \sqrt{w_j} |\chi_j\rangle = |\chi(0)\rangle. \quad (18)$$

Again, when calculating the expectation value of σ_x , we obtain the same result as above. However, in addition we realize that the visibility $v(t)$ can now be expressed as the overlap of the two bath states $|\chi_{\uparrow/\downarrow}(t)\rangle$:

$$\langle e^{i\varphi(t)} \rangle_{\varphi} = \sum_j w_j e^{i\varphi_j(t)} = \langle \chi_{\downarrow}(t) | \chi_{\uparrow}(t) \rangle. \quad (19)$$

This is a result that is much more general than the present example would suggest. The two states are sometimes called “pointer states”, referring to the quantum theory of measurement. As it were, the environment can be thought of as measuring the state of the system, which means that $|\chi(0)\rangle$ evolves into either $|\chi_{\uparrow}(t)\rangle$ or $|\chi_{\downarrow}(t)\rangle$, depending on whether the system had been in the state $|\uparrow\rangle$ or $|\downarrow\rangle$. This is reminiscent of the pointer of a measuring device pointing either way depending on the signal it picks up. When the two pointer states become orthogonal, this implies that perfect knowledge about the system could be inferred (in principle) from the state of the bath. As a consequence, interference effects in the system itself are completely destroyed, analogous to the gedanken experiment of the “Heisenberg microscope” [9].

In the most general framework, one can set up a path-integral analysis, where the bath’s state would evolve according to the particular trajectory of the system. The overlap (19) between two states having evolved under the influence of two different system trajectories is then called the “Feynman-Vernon influence functional” [7]. This forms the starting point for path-integral evaluations of the dissipative dynamics of some systems, such as the damped quantum harmonic oscillator or tunneling decay under the influence of dissipation. For further details see the book of Weiss [18].

3 The density matrix

The examples of the previous subsection all yield the same results for the expectation values of any system operator. It is therefore useful to introduce a description of the system’s state that makes this equivalence explicit and which is general enough to deduce from it any arbitrary expectation value. This is

achieved by the density matrix, whose properties we review in this section. It forms the main object of study in the field of open quantum systems, where the goal will usually be to calculate the time-evolution of the density matrix of a system subject to a fluctuating environment. While originally the word “state of a quantum system” referred only to “wave functions”, i.e. vectors in a Hilbert space, it is now extended to include the “mixed states” that have to be described by a density matrix. Most readers will know this material and can skip directly to subsection 3.4, which returns to the example of pure dephasing.

3.1 Classical uncertainty

There are different ways in which the density matrix concept may arise. First, let us assume there is some classical uncertainty about the system’s state, i.e. the state $|\psi_j\rangle \in \mathcal{H}_S$ occurs with probability w_j . As explained above, when trying to predict the experimentally observed average value of an observable A , we thus should first take quantum-mechanical expectation values and then perform an additional, “classical” average:

$$\langle A \rangle \equiv \sum_j w_j \langle \psi_j | A | \psi_j \rangle \quad (20)$$

We can obtain the same result by introducing the density matrix as a weighted sum over projectors $|\psi_j\rangle \langle \psi_j|$,

$$\rho \equiv \sum_j w_j |\psi_j\rangle \langle \psi_j|, \quad (21)$$

and taking the trace over ρA :

$$\langle A \rangle = \text{tr}(\rho A). \quad (22)$$

Here $\rho \in B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the set of bounded operators on the Hilbert space \mathcal{H} .

In general, this step (from the set of $|\psi_j\rangle$ to ρ) involves a compression of information. Regardless of the number of states $|\psi_j\rangle$ we started out with in the beginning (which need not be orthogonal!), ρ itself is an operator on \mathcal{H}_S , represented by a $\dim \mathcal{H}_S \times \dim \mathcal{H}_S$ matrix in the case of a finite-dimensional Hilbert space.

3.2 General properties

The following properties follow directly from the definition (21):

$$\text{tr} \rho = 1 \quad (23)$$

$$\rho^\dagger = \rho \quad (24)$$

$$\rho \geq 0 \quad (25)$$

For a “pure” state $|\psi\rangle$, the density matrix is a projector onto that state: $\rho = |\psi\rangle\langle\psi|$. As a consequence, $\rho^2 = \rho$ and $\text{tr}\rho^2 = \text{tr}\rho = 1$. In general, the “purity” $\text{tr}\rho^2$ obeys $\text{tr}\rho^2 \leq 1$. All states that are not pure are called “mixed”, and their density matrix has a rank larger than one:

$$\text{tr}\rho^2 < 1 \Leftrightarrow \rho \text{ is mixed.} \quad (26)$$

When making a measurement that is able to distinguish the particular state $|\phi\rangle$ from other, orthogonal states, we can phrase this by saying that we measure the operator $|\phi\rangle\langle\phi|$, which will yield the value 1 only when we indeed find $|\phi\rangle$. Thus, the probability of observing $|\phi\rangle$ is the expectation value of its projector:

$$P_\phi = \text{tr}(\rho |\phi\rangle\langle\phi|) = \langle\phi|\rho|\phi\rangle. \quad (27)$$

In the special case where ρ describes the pure state $|\psi\rangle$, this correctly reduces to the well-known postulate of quantum mechanics, $P_\phi = |\langle\phi|\psi\rangle|^2$.

After diagonalizing ρ , its eigenvalues p_j can therefore be interpreted as the probabilities of finding the respective eigenvectors in a measurement that distinguishes between those eigenvectors. The properties $\rho \geq 0$ and $\text{tr}\rho = 1$ are then seen to correspond to the simple fact that these probabilities are non-negative and normalized.

3.3 Reduced density matrix

We now turn to a different setting in which the density matrix arises: Consider the world made up of a system and a bath, as explained in the introduction. For generality, we will describe the overall state of the world by a density matrix ρ^{SB} (understood in the sense explained above). Suppose we are now only interested in evaluating expectation values of system operators A_S . Adapting (22), we have

$$\langle A_S \rangle = \langle A_S \otimes \mathbf{1}_B \rangle = \text{tr}_{SB}(\rho^{SB}(A_S \otimes \mathbf{1}_B)). \quad (28)$$

Here we have indicated that the trace is taken in the full Hilbert space \mathcal{H}_{SB} . However, since the operator acts only on the system alone, we can break this trace into two steps:

$$\langle A_S \rangle = \text{tr}_S(\text{tr}_B(\rho^{SB})A_S) \equiv \text{tr}_S(\rho^S A_S). \quad (29)$$

Here we have introduced the reduced density matrix of the system, by taking the partial trace over the bath degrees of freedom:

$$\rho^S \equiv \text{tr}_B(\rho^{SB}) \quad (30)$$

We can make this explicit by choosing a product basis in \mathcal{H}_{SB} . Then the matrix elements of ρ^S are given by

$$\rho_{i'i}^S = \sum_j \rho_{(i',j)(i,j)}^{SB}, \quad (31)$$

where (i, j) refers to the basis state $|\psi_i\rangle |\chi_j\rangle$, with $|\psi_i\rangle \in \mathcal{H}_S$ and $|\chi_j\rangle \in \mathcal{H}_B$.

It is straightforward to confirm that the general properties (23), (24), and (25), continue to hold for ρ^S .

3.4 Example: Pure dephasing

For the example described in section 2, the density matrix is given by

$$\rho(t) = \begin{pmatrix} |\psi_\uparrow(0)|^2 & \psi_\uparrow(0)\psi_\downarrow^*(0)v(t) \\ \psi_\uparrow^*(0)\psi_\downarrow(0)v^*(t) & |\psi_\downarrow(0)|^2 \end{pmatrix}. \quad (32)$$

This is independent of the specific model, i.e. it holds regardless of whether we think of a classical noise process or of the interaction with a quantum-mechanical environment (where ρ would be the reduced system density matrix). Only the off-diagonal element $\rho_{\uparrow\downarrow}(t) = \langle \uparrow | \rho(t) | \downarrow \rangle = \psi_\uparrow(0)\psi_\downarrow^*(0)v(t)$ is affected by dephasing. The decay of $|v(t)|$ will reduce the purity of the state [we abbreviate $p = |\psi_\uparrow(0)|^2$]:

$$\text{tr}(\rho^2(t)) = 1 - 2p(1-p)(1-|v(t)|^2) \quad (33)$$

When starting from an equal-weight superposition, $p = 1/2$, this tends to $1/2$ if $v \rightarrow 0$. In that case, one ends up in the fully mixed state: $\rho = \mathbf{1}/2$.

Let us have a look at the simplest possible case for the decay of v , postulating that it decays exponentially in time:

$$v(t) = \left\langle e^{i\varphi(t)} \right\rangle_\varphi = e^{-\Gamma_\varphi t}, \quad (34)$$

where Γ_φ is called the “dephasing rate”. Such a decay will occur when the underlying fluctuations of ϵ are of the “white noise” type, and therefore $\varphi(t)$ undergoes Brownian motion (φ is proportional to a Wiener process). Then $\langle \varphi^2(t) \rangle_\varphi \propto t$, which leads to exponential decay according to $\langle \exp[i\varphi(t)] \rangle_\varphi = \exp[-\langle \varphi(t)^2 \rangle_\varphi / 2] = \exp[-\Gamma_\varphi t]$. For this particular case, we can write down a simple first-order differential equation for the elements of $\rho(t)$:

$$\dot{\rho}_{\uparrow\downarrow}(t) = -\Gamma_\varphi \rho_{\uparrow\downarrow}(t) \quad \dot{\rho}_{\uparrow\uparrow} = \dot{\rho}_{\downarrow\downarrow} = 0. \quad (35)$$

This is the simplest example of a “Markov master equation”. The general structure of such an equation, which we will discuss later, is of the form

$$\dot{\rho} = L\rho. \quad (36)$$

However, it is not permissible to postulate an arbitrary operator L on phenomenological grounds, because that might turn out to violate the basic properties of ρ . This can be seen clearly in the following example: Suppose we want to describe the spontaneous decay of the excited state of an atom by emission of a photon. The first reasonable (and indeed correct) ansatz that comes to mind would be to postulate an exponential decay of the probability $\rho_{\uparrow\uparrow}$ of finding the

atom in the excited state: $\dot{\rho}_{\uparrow\uparrow} = -\Gamma\rho_{\uparrow\uparrow}$, and consequently $\dot{\rho}_{\downarrow\downarrow} = -\dot{\rho}_{\uparrow\uparrow} = +\Gamma\rho_{\uparrow\uparrow}$, to conserve probability. Suppose we were to assume that the off-diagonal elements do not change ($\dot{\rho}_{\uparrow\downarrow} = \dot{\rho}_{\downarrow\uparrow} = 0$), which seems to be the simplest possible assumption. In the long-time limit $t \rightarrow \infty$, this would lead to

$$\rho(t \rightarrow \infty) = \begin{pmatrix} 0 & \rho_{\uparrow\downarrow}(0) \\ \rho_{\downarrow\uparrow}(0) & 1 \end{pmatrix} \text{ (wrong)}. \quad (37)$$

This is not a positive semidefinite matrix if $\rho_{\uparrow\downarrow}(0) \neq 0$. In other words, such an ansatz would violate a basic requirement, the positivity of probabilities. The rest of these lecture notes is concerned with describing the correct general structure of the time evolution of density matrices, which makes sure that such pitfalls are avoided.

4 Completely positive maps and Kraus operators

4.1 Complete positivity

Let us consider the linear map that takes the initial density matrix $\rho(0)$ to its value at time t ,

$$\begin{aligned} \Phi : B(\mathcal{H}) &\rightarrow B(\mathcal{H}) \\ \rho(0) &\mapsto \rho(t). \end{aligned} \quad (38)$$

Later on we will consider the map Φ for different times, but for now we suppress the corresponding subscript for brevity.

Let us list the properties which we will require of Φ , which follow from the properties of ρ that we want to be respected by the time-evolution:

$$\text{tr}\Phi(\rho) = \text{tr}\rho \quad (\text{norm - conserving}) \quad (39)$$

$$\Phi(\rho^\dagger) = \Phi(\rho)^\dagger \quad (\text{preserves hermiticity}) \quad (40)$$

$$\rho \geq 0 \Rightarrow \Phi(\rho) \geq 0 \quad (\text{"}\Phi \text{ is a positive map"} \text{)} \quad (41)$$

It would seem that these three features are all that is needed to have a permissible time-evolution. Surprisingly, that is not the case. The third requirement, of Φ being a positive map, has to be replaced by the stronger condition of Φ being “completely positive”.

Definition– We call a linear map $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ “*completely positive*” (CP) iff the following holds: Given any other Hilbert space \mathcal{H}' , we consider the product space $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ and construct a linear map $\tilde{\Phi} : B(\tilde{\mathcal{H}}) \rightarrow B(\tilde{\mathcal{H}})$ that derives from Φ by acting like Φ onto the original Hilbert space \mathcal{H} and not affecting the space \mathcal{H}' , that is: $\tilde{\Phi}(\rho \otimes \rho') = \Phi(\rho) \otimes \rho'$ for $\rho \in B(\mathcal{H})$ and $\rho' \in B(\mathcal{H}')$. Then Φ is a positive map.

4.2 Example of a positive but not completely positive map (relation to entanglement theory)

At first sight, it is surprising that positivity of Φ itself is not enough to guarantee the positivity of the enlarged map $\tilde{\Phi}$. We will now look at the simplest example of a map that is positive but not completely positive. This example is of interest in the theory of entanglement.

Claim— $\Phi : \rho \mapsto \rho^t$ is positive but not CP.

It is obvious that Φ is positive, as transposition does not change the eigenvalues of a matrix. Now let us consider the enlarged map $\tilde{\Phi}$, operating on a product basis in $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$. It induces a “partial transposition” of a density matrix $\tilde{\rho} \in B(\tilde{\mathcal{H}})$:

$$[\tilde{\Phi}(\tilde{\rho})]_{(i',j')(i,j)} = \tilde{\rho}_{(i,j)(i',j')}, \quad (42)$$

where we observe the interchange of indices i and i' referring to \mathcal{H} . In order to prove that $\tilde{\Phi}$ is not a positive map, it is enough to find one example of a state $\tilde{\rho} \geq 0$ (for one suitably chosen Hilbert space \mathcal{H}') for which the partial transposition fails to make $\tilde{\Phi}(\tilde{\rho}) \geq 0$. It is clear that any product state, of the type $\tilde{\rho} = \rho \otimes \rho'$, will not be sufficient for this purpose, since $\rho^t \geq 0$ and therefore $\tilde{\Phi}(\tilde{\rho}) \geq 0$ as well. More generally, due to the linearity of $\tilde{\Phi}$, the same holds for any so-called “*separable*” state which is a mixture of product states:

$$\tilde{\rho} = \sum_j w_j \rho^{(j)} \otimes \rho'^{(j)}. \quad (43)$$

Here $\rho^{(j)}, \rho'^{(j)}$ are valid density matrices in $B(\mathcal{H})$ and $B(\mathcal{H}')$, respectively, and $w_j \geq 0$, $\sum_j w_j = 1$. Any separable state will yield a positive semi-definite partial transpose: $\tilde{\Phi}(\tilde{\rho}) \geq 0$. This is the important PPT (“positive partial transpose”) criterion, a *necessary* condition for separability of a state $\tilde{\rho}$, discovered by Asher Peres in 1996 [15].

In order to find an example without a positive partial transpose, we thus have to consider non-separable, i.e. so-called “*entangled*” states. The simplest case is having both \mathcal{H} and \mathcal{H}' two-dimensional Hilbert spaces, and considering the pure state $\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$, with

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle). \quad (44)$$

The partial transposition acts like $\tilde{\Phi}(|\uparrow\uparrow\rangle\langle\downarrow\downarrow|) = |\downarrow\uparrow\rangle\langle\uparrow\downarrow|$ (note the interchange in the first position, referring to \mathcal{H}), and likewise on other combinations occurring in the projector $\tilde{\rho}$. As a result, we find

$$\tilde{\Phi}(\tilde{\rho}) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (45)$$

where the matrix has been written down with respect to the basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$. It has eigenvalues $+1/2$ (three-fold degenerate) and $-1/2$. As a consequence, Φ is not positive, and Φ is not a CP map.

As a side-note, we mention that PPT is even a necessary and *sufficient* criterion for separability if $\dim\mathcal{H} = \dim\mathcal{H}' = 2$ as in our example [and this even holds when the dimensions are 2 and 3, respectively]. One has to go to higher-dimensional Hilbert spaces in order to find states that are not separable (i.e. entangled) but still have a positive partial transpose.

Physically, the property of complete positivity ensures that one ends up with a permissible state even if the dissipative system had been entangled initially with another system. As a consequence, there is no analogon to the concept of complete positivity for classical dissipative systems, since classical physics does not know entangled states.

4.3 Kraus decompositions of a CP map

It turns out that all CP maps that fulfill the other properties mentioned above can be decomposed in a simple way. We state the theorem, due to Kraus [11], without proof:

Theorem— Provided we are given a map Φ that fulfills the properties (39) and (40), as well as complete positivity, there exists a set of “Kraus operators” $K_j : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ that are normalized in the sense $\sum_j K_j^\dagger K_j = \mathbf{1}$ and that can be used to represent $\Phi(\rho)$:

$$\Phi(\rho) = \sum_j K_j \rho K_j^\dagger. \quad (46)$$

The converse also holds (i.e. the three properties follow from the existence of such a decomposition). In the case of an infinite-dimensional Hilbert space, the set of values for the index j can be countably infinite.

We note that this decomposition is not at all unique. For example, multiplying K_j by a phase factor $e^{i\theta_j}$ changes nothing. More generally, a unitary matrix with matrix elements $U_{j'j}$ can be used to convert to a new set of Kraus operators that represent the same map: $K'_{j'} = \sum_j K_j U_{jj'}$ [where the cardinality of the two sets is assumed to be the same, adding Kraus operators equal to zero if need be]. It can be shown [16] that any two equivalent Kraus decompositions are connected in this way. We also note that for a finite-dimensional Hilbert space any map can be represented using at most $(\dim\mathcal{H})^2$ Kraus operators.

4.4 Examples

We now list a few examples for the applications of Kraus decompositions:

- Purely unitary time-evolution: $|\psi(t)\rangle = U(t) |\psi(0)\rangle$ leads to $\rho(t) = U(t)\rho(0)U^\dagger(t)$, and $\rho(t) = \Phi(\rho(0))$ can therefore be represented by the single Kraus operator $K = U(t)$.

- Random unitary evolution: If we pick some U_j at random with probability w_j , as in the example with pure dephasing by classical noise, then the set $K_j = \sqrt{w_j}U_j(t)$ will yield the correct $\rho(t) = \Phi(\rho(0)) = \sum_j w_j U_j(t)\rho(0)U_j^\dagger(t)$, and normalization follows from $\sum_j w_j = 1$.
- However, in the example of pure dephasing of a two-level system, we can also choose a more economical decomposition. For example, for $v(t) \in \mathbb{R}$, just two Kraus operators suffice: $K_1 = \mathbf{1}\sqrt{(1+v)/2}$ and $K_2 = \sigma_z\sqrt{(1-v)/2}$.
- The σ_z Kraus operator in the previous example can be thought of as describing a “phase flip” (changing the sign of the off-diagonal element of the density matrix). In the same sense, σ_x would describe a “bit flip” (turning $|\uparrow\rangle$ into $|\downarrow\rangle$ and vice versa), and σ_y a combination of the two.
- In quantum information processing, two-level systems are viewed as “quantum bits” (qubits). After sending such a qubit through a communication channel (where it can be subject to technical noise, or even interact with, and get entangled with, some bath), its state will have been changed by a map Φ that is characteristic for this quantum channel. Thus a quantum channel can be described by giving a set of Kraus operators.
- As indicated by the two-level example from above, for a finite-dimensional Hilbert space a finite number of Kraus operators will suffice (even if the underlying physical description of the noise involved an infinite number of possible different time evolutions U_j). This is very important in the theory of quantum error correction. It means that, contrary to first appearances, quantum computers are not as bad as classical analog computers when it comes to error correction.
- Kraus operators can be used to describe measurements: After an ideal von-Neumann measurement that distinguishes between the orthogonal states $|\phi_j\rangle$, the system ends up in one of those states with probability $p_j = \text{tr}(\rho|\phi_j\rangle\langle\phi_j|)$. However, from the point of view of someone who is not told the measurement outcomes, the state after the measurement is described by the density matrix $\rho' = \sum_j p_j |\phi_j\rangle\langle\phi_j|$, and the map from ρ to ρ' can be described by the Kraus operators $K_j = |\phi_j\rangle\langle\phi_j|$. The more general case of measurements that reveal only partial information (POVM: positive operator valued measurements) is described by arbitrary Kraus operators that are not necessarily projectors, and the probability of finding a particular value is then $\text{tr}(\rho K_j^\dagger K_j)$.
- For systems with high- (or infinite-) dimensional Hilbert spaces (like those with a continuous degree of freedom, e.g. a harmonic oscillator), the Kraus decomposition, though still possible in principle, becomes less useful in practice due to the large number of Kraus operators.

4.5 Construction of Kraus operators for system-bath interaction

We now give an explicit construction of the Kraus operators for the important example where the time-evolution of the reduced density matrix is due to the interaction with an environment. Suppose that at time $t = 0$ the environment is in the initial state $|\chi(0)\rangle$, uncorrelated with the arbitrary initial system state $\rho^S(0)$. This means for the total (system+bath) density matrix: $\rho^{SB}(0) = \rho^S(0) \otimes |\chi(0)\rangle \langle \chi(0)|$. In addition, we allow for an arbitrary unitary time-evolution U_{SB} acting on the product Hilbert space \mathcal{H}_{SB} , taking the full initial state to the final state at time t . Therefore, the system's reduced density matrix at time t is:

$$\begin{aligned} \rho(t) &= \text{tr}_B \left(U_{SB} \rho^S \otimes |\chi(0)\rangle \langle \chi(0)| U_{SB}^\dagger \right) \\ &= \sum_j \underbrace{\langle \chi_j | U_{SB} | \chi(0) \rangle}_{\equiv K_j} \rho^S \langle \chi(0) | U_{SB}^\dagger | \chi_j \rangle \\ &= \sum_j K_j \rho^S K_j^\dagger \end{aligned} \quad (47)$$

In the second line we have introduced the sum over a basis $|\chi_j\rangle$ of the bath Hilbert space \mathcal{H}_B to perform the trace over the bath. The resulting matrix elements of the full time-evolution U_{SB} with respect to the bath states define the operators K_j , as indicated. These operators act on the system Hilbert space \mathcal{H}_S . They are normalized, since

$$\sum_j K_j^\dagger K_j = \sum_j \langle \chi(0) | U_{SB}^\dagger | \chi_j \rangle \langle \chi_j | U_{SB} | \chi(0) \rangle = 1. \quad (48)$$

As a consequence, we can identify them as the Kraus operators needed to describe the map of $\rho(0) = \rho^S(0)$ to $\rho(t)$. If the bath initially were in a mixed state, we would find $\rho(t)$ to be the weighted average of expressions of this type, and consequently the full set of Kraus operators would consist of the individual sets, multiplied by the factors $\sqrt{w_j}$, where w_j are the weights. In this way, the time-evolution due to interaction with an environment (starting from an uncorrelated state) can always be expressed using Kraus operators, i.e. as a CP map. The idea of the general construction of Kraus decompositions, presented in [16], makes use of this fact.

5 Markov master equations of Lindblad form

In the previous section, we have been dealing with the completely arbitrary time-evolution of a density matrix. Let us now specialize to evolutions of the Markov type, i.e. where the density matrix follows an equation of the form $\dot{\rho} = L\rho$, with L representing a linear map. This is called Markovian since the

evolution of the state only depends on the present state itself, not on its history. Such an equation is the direct quantum analogue of the evolution equation for the probability density for the case of a classical stochastic process of Markov type.

5.1 Lindblad's theorem

It is clear that the Markov property, when combined with the restrictions discussed in the previous section, will lead to a particular form of L . This problem has first been considered in its full generality by Lindblad [13] (1976). In writing down the assumptions for his proof, which we will list below, Lindblad considers the map in the Heisenberg picture, where the time-evolution is applied to the observable A instead of the density matrix ρ :

$$\text{tr}[\rho(t)A] = \text{tr}[\Phi_t(\rho(0))A] = \text{tr}[\rho(0)\Phi_t^H(A)], \quad (49)$$

for any choice of initial density matrix $\rho(0)$.

Then a “*completely positive dynamical semigroup*” is defined as follows: Let \mathcal{A} be a W^* -algebra¹ and Φ_t be a family of completely-positive maps of \mathcal{A} into itself, depending on the real-valued time parameter $t \in \mathbb{R}_0^+$. In addition, we require the following properties:

$$\Phi_t^H(\mathbf{1}) = \mathbf{1} \quad (50)$$

$$\Phi_s^H \cdot \Phi_t^H = \Phi_{s+t}^H \quad (\forall s, t \geq 0) \quad (51)$$

$$\lim_{t \rightarrow 0} \|\Phi_t^H - \mathbf{1}\| = 0 \quad (52)$$

$$\Phi_t^H \quad \text{is normal (ultraweakly continuous)} \quad (53)$$

The first line guarantees conservation of the trace of the density matrix. The second line is the central semi-group property, which will guarantee Markov dynamics and represents a strong restriction on the allowed physical situations. It tends to work as a good approximation in situations where the correlation time of the fluctuations characterizing the environment is short in comparison with other time scales, such as those set by decay rates and oscillation periods. Under these conditions, Lindblad proved the following:

The action of Φ_t can be expressed in the form of a Markov master equation, where $\rho(t) = \Phi_t(\rho(0))$ fulfills the differential equation

$$\dot{\rho}(t) = L\rho(t), \quad (54)$$

¹A W^* -algebra (also called a von-Neumann algebra), is a $*$ -algebra of bounded operators on a Hilbert space that is closed in the weak $*$ topology and contains the identity operator. This essentially means it is an algebra of operators A , which can be added and multiplied with a scalar as usual (thus forming a vector space), have the usual rules of non-commutative multiplication between operators, the possibility of taking the adjoint (that is what the $*$ refers to) with all the features you would expect, and an operator norm that is compatible with the adjoint operation: $\|A^\dagger A\| = \|A\|^2$. See books on functional analysis, such as [17].

with the Liouvillian operator of “Lindblad form”:

$$L\rho = -\frac{i}{\hbar} [H, \rho] + \sum_j (R_j \rho R_j^\dagger - \frac{1}{2} R_j^\dagger R_j \rho - \frac{1}{2} \rho R_j^\dagger R_j). \quad (55)$$

Here the first part, involving some hermitian Hamiltonian H , describes the standard unitary time-evolution of ρ (possibly with renormalized matrix elements of H due to the presence of the environment, i.e. $H \neq H_0$, where H_0 would be the intrinsic Hamiltonian). The dissipative dynamics is generated by the second term, with the relaxation (or Lindblad) operators R_j that do not have to fulfill any special constraint (unlike the normalized Kraus operators, from which they derive). We note that, just as for the Kraus operators, the choice of R_j is not unique, and even the separation into a unitary and a dissipative part is not unique either.

5.2 Obtaining the Lindblad form from the Kraus decomposition

We now give the main ideas of the derivation, building on the Kraus decomposition, without attempting mathematical rigour.

The density matrix at a small time δt deviates from $\rho(t=0)$ only to first order in δt , due to continuity and the Markov structure. In addition, it can be written in the form of a Kraus decomposition:

$$\rho(\delta t) = \rho(0) + \mathcal{O}(\delta t) = \sum_j K_j \rho(0) K_j^\dagger, \quad (56)$$

where $\sum_j K_j^\dagger K_j = \mathbf{1}$. In order to satisfy this structure, we need one Kraus operator close to unity, which we will write as

$$K_0 = \mathbf{1} + (-\frac{i}{\hbar} H + A) \delta t + \dots \quad (57)$$

The second term has been decomposed into “real” and “imaginary” parts, with two hermitean operators $H^\dagger = H$ and $A^\dagger = A$. All the other Kraus operators K_j (with $j \geq 1$) must be of order $\mathcal{O}(\sqrt{\delta t})$, to obtain the desired expression (56). We can now obtain A from the normalization requirement:

$$\mathbf{1} = \sum_{j \geq 0} K_j^\dagger K_j = \mathbf{1} + 2\delta t A + \sum_{j \geq 1} K_j^\dagger K_j, \quad (58)$$

which yields

$$A = -\frac{1}{2\delta t} \sum_{j \geq 1} K_j^\dagger K_j, \quad (59)$$

that is an expression of $\mathcal{O}(\delta t^0)$. Inserting this result first into (57) and then into the decomposition (56), we obtain the Lindblad structure (55), by identifying the relaxation operators as

$$R_j = \lim_{\delta t \rightarrow 0} \frac{K_j}{\sqrt{\delta t}} \quad (j \geq 1). \quad (60)$$

5.3 Examples

We now list a few examples for the relaxation operators occurring for different dissipative processes. In the present lecture notes, we will not address the techniques used for obtaining microscopic derivations of these operators and the rates occurring in there, which depend on the specific type of environment and the assumed coupling. In general, however, the relaxation operators are of a simple form, describing the operator that induces the dissipative transition, multiplied with the square root of the corresponding rate.

- Pure dephasing of a two-level system, with exponential decay of the visibility at a rate Γ_φ : $R = \sqrt{\frac{\Gamma_\varphi}{2}} \sigma_z$ yields the Markov master equation (35).
- Exponential relaxation from the excited state to the ground state, at a rate Γ : $R = \sqrt{\Gamma} \sigma_-$, where $\sigma_- = |\downarrow\rangle\langle\uparrow|$. This yields the proper decay of the probabilities, $\dot{\rho}_{\uparrow\uparrow} = -\Gamma\rho_{\uparrow\uparrow}$ and $\dot{\rho}_{\downarrow\downarrow} = -\dot{\rho}_{\uparrow\uparrow} = +\Gamma\rho_{\uparrow\uparrow}$, but it also gives non-trivial information on the decay of the off-diagonal elements:

$$\dot{\rho}_{\uparrow\downarrow} = -\frac{i}{\hbar}(\epsilon_\uparrow - \epsilon_\downarrow)\rho_{\uparrow\downarrow} - \frac{\Gamma}{2}\rho_{\uparrow\downarrow}. \quad (61)$$

The first term results from the unitary evolution (i.e. from $H = \epsilon_\uparrow |\uparrow\rangle\langle\uparrow| + \epsilon_\downarrow |\downarrow\rangle\langle\downarrow|$), whereas the second term describes dephasing at a rate $\Gamma_\varphi^{\text{eff}} = \Gamma/2$ that is exactly half the total decay rate. If pure dephasing is present in addition, one just needs to introduce a second relaxation operator, as given above, and the rates would add: $\Gamma_\varphi^{\text{total}} = \Gamma_\varphi + \Gamma/2$. In the context of nuclear magnetic resonance or qubit physics, the fact that $\Gamma_\varphi^{\text{total}} \geq \Gamma/2$ is often expressed as an inequality for the corresponding time-scales (inverses of the rates): $T_2 \leq 2T_1$, where $T_2 = (\Gamma_\varphi^{\text{total}})^{-1}$ and $T_1 = \Gamma^{-1}$. We have encountered this inequality here ultimately as a consequence of the requirement that the time-evolution preserve positivity of ρ [which any decay rate smaller than $\Gamma/2$ for the off-diagonal elements would not ensure].

- If the bath is at finite temperature, the two-level system might also absorb a quantum of energy, i.e. become thermally excited. This is described by $R = \sqrt{\Gamma_{\text{up}}} \sigma_+$, where $\sigma_+ = \sigma_-^\dagger = |\uparrow\rangle\langle\downarrow|$.
- Taking into account all the three processes discussed up to now leads to the so-called “Bloch equations” first invented to describe the dissipation and decoherence of systems such as atoms or spins.
- Damping of a harmonic oscillator (due to a coupling to the bath that is linear in the coordinate of the oscillator) is described by $R = \sqrt{\Gamma}a$. Here

Γ is the damping rate (which can be observed in the linear response of the system), and a is the annihilation operator that reduces the occupation number of the oscillator by one: $a|n\rangle = \sqrt{n}|n-1\rangle$. [$a = (x + i\frac{p}{m\omega})/(2x_0)$ with $x_0 = \sqrt{\hbar/(2m\omega)}$; x and p being position and momentum operators for the oscillator] There are many important applications: For example, a single standing wave mode of the electromagnetic field inside an optical or microwave cavity is described as a harmonic oscillator. That oscillator is damped because the photons can leak out of the cavity through the semi-transparent mirrors of the cavity, and the individual decay process corresponds to the destruction of a single photon inside the cavity: $n \mapsto n-1$. Another example of current interest are nanomechanical systems (small beams on the micrometer scale) which are harmonic oscillators to a good approximation, damped because of their connection to the mechanical structure onto which they are attached and into which they can radiate phonons.

We have not described how to obtain the rates and the operators, and under which conditions one may expect the physical dynamics to be well approximated by a Markov master equation. While a thorough discussion of these points would go beyond the scope of the present notes, and it is hard to list simple, generally valid conditions, the following rule-of-thumb may be offered: most systems in which this approximation works have decay rates that are both much smaller than the typical transition frequencies of the system, and also smaller than the inverse correlation time of the fluctuating force that the environment imposes on the system. This condition is usually achieved in the limit of a weak coupling between system and bath, as the decay and dephasing rates become arbitrarily small in that limit.

6 Beyond Lindblad equations

Although in practice Lindblad Markov master equations are used in the majority of applications in the various subfields of physics, current research in quantum dissipative systems focuses on the interesting effects that arise outside of this framework. Here we just list a few of those physical situations and features, to give the reader a sense of what lies beyond Lindblad dynamics.

- In some areas it is experimentally feasible to measure the quanta which are emitted into the environment by the system, thereby learning more about the system's state. For example, the photons having been emitted from an atom or leaking out of an optical cavity can be registered by photo-detectors. The dynamics of the reduced density matrix, conditioned on the observed detector results, may then often be described by a modified master equation that depends on these results. Such an approach sometimes is referred to as “quantum jump trajectories simulations”, because it was initially developed to describe quantum jumps in individual atoms that have been observed by light scattering.

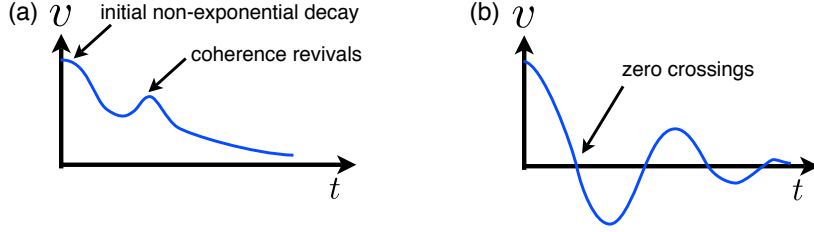


Figure 3: Possible features of the visibility in the example of pure dephasing (section 2), due to non-Markovian dynamics (a) and due to non-Gaussian noise (b).

- Non-Markovian dynamics is generated when the environment's fluctuations display long correlation times. A generic consequence of taking into account these effects are short-time deviations from exponential decay. In addition, under some circumstances the coherence of the system can even be revived at later times, after having decayed initially. Non-Markovian dynamics is generated when the environment's fluctuations display long correlation times. A generic consequence of taking into account these effects are short-time deviations from exponential decay. In addition, under some circumstances the coherence of the system can even be revived at later times, after having decayed initially. Non-Markovian dynamics is generated when the environment's fluctuations display long correlation times. A generic consequence of taking into account these effects are short-time deviations from exponential decay. In addition, under some circumstances the coherence of the system can even be revived at later times, after having decayed initially.
- A favorite microscopic model for the environment is a bath of harmonic oscillators, which is, for example, an exact representation of the electromagnetic field or the phonons in a crystal lattice. When such a bath is coupled to a single particle (bilinearly in the coordinates of particle and bath oscillators), one speaks of the “Caldeira-Leggett model” [3, 4, 18]. This allows to answer questions that go beyond Markov dynamics. For example, when such a bath gives rise to standard diffusive motion ($\langle r^2(t) \rangle \propto t$) at high temperatures, the “Quantum Brownian motion” resulting at zero temperature is sub-diffusive, with the distance to the origin obeying $\langle r^2(t) \rangle \propto \ln(t)$. One can use the same kind of model to study the suppression of quantum tunneling due to the influence of a dissipative environment. These nonperturbative studies are usually carried out in a path-integral framework.
- As the coupling to the environment is increased, there can be qualitative changes in behaviour at some critical coupling strength. The best known example occurs in a model where a particle that tunnels between two states is coupled to a bath of harmonic oscillators. This is commonly

referred to as the “spin-boson model”, where the “spin” refers to the two-level system and “boson” refers to the bath of oscillators. For a certain distribution of bath oscillator frequencies (“Ohmic bath”), increasing the coupling strength beyond some point makes the system undergo a quantum phase transition towards a symmetry-broken phase where the particle remains trapped in one of the two states for all times. Ergodicity is broken due to the strong dissipation.

- Even if the actual physical environment is not a bath of oscillators, it often may be treated as such to a very good degree of approximation as long as the coupling is weak. The fluctuating force generated by a bath of oscillators is Gaussian-distributed. However, for strong coupling, this approach will fail, and the dissipative dynamics can become qualitatively different due to the non-Gaussian fluctuations acting on the system. An example of current interest are the current fluctuations generated by the passage of discrete, single electrons through nanostructures.

7 Further reading

Mathematical treatments of quantum dissipative dynamics may be found in the books by Davies [6] (published in 1976, still without reference to Lindblad, but developing the same concepts) and Kraus [11], who emphasizes the connections with the theory of measurement. The treatise of von Neumann on the mathematical foundations of quantum mechanics [10] forms the basis.

A thorough introduction to the density matrix and its uses, at an elementary level for the physicist, can be found in the book by Blum [1], which also presents a microscopic derivation of the rates and relaxation operators appearing in master equations. Dissipative quantum systems in general (often with emphasis on quantum optics) are described in more detail in the books by Carmichael [5], Gardiner and Zoller [8], and Breuer and Petruccione [2]. The book by Weiss [18] emphasizes those aspects that go beyond the Lindblad master equations, such as the Feynman-Vernon influence functional formalism, some exact solutions, and a very detailed discussion of the spin-boson model. In that context, the classic review by Leggett and co-workers on the spin-boson model is also highly recommended [12].

Kraus operators in the context of quantum information processing are described in the book by Nielsen and Chuang [14], as well as in the quantum information lecture notes by Preskill ([16], chapter 3), who also describes the Lindblad master equation and gives a nice proof of the Kraus representation theorem.

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