Theoretical Quantum Optics

Problem Sheet

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Sheet: 14	**Hand-out:** 25.1.24	**Hand-in:** 1.2.24

Problem 35. Von Neumann model of measurement

The Von Neumann model idealizes the measurement of a quantum observable of a microscopic system by a macroscopic measurement apparatus. The interaction Hamiltonian H_{int} between the system and apparatus is assumed to be so strong that during the measurement the Hamiltonian of the system H_{sys} and the Hamiltonian of the apparatus H_{app} can be neglected, such that,

$$H = H_{\text{sys}} + H_{\text{app}} + H_{\text{int}} \approx H_{\text{int}}.$$

The interaction Hamiltonian is assumed to be given by,

$$H_{\text{int}} = \sum_n |n\rangle\langle n| \otimes A_n,$$

where the $\{|n\rangle\}$ are the eigenstates of the observable acting on the Hilbert space of the system and the $\{A_n\}$ are the operators on the Hilbert space of the apparatus.

a. Give an interpretation of the expression for the interaction Hamiltonian H_{int}.

b. Let the system initially be in a coherent superposition,

$$|\psi(0)\rangle = \sum_n c_n |n\rangle ,$$

and the apparatus in the state $|\phi(0)\rangle$. What is the initial reduced density matrix,

$$\rho_{\text{sys}}(0) = \text{Tr}_{\text{app}}(\rho(0)),$$

where Tr_{app} is the partial trace over the Hilbert space of the apparatus and,

$$\rho(0) = |\Psi(0)\rangle\langle\Psi(0)| , \qquad |\Psi(0)\rangle = |\psi(0)\rangle \otimes |\phi(0)\rangle ,$$

is the initial density matrix of the combined system and apparatus?

c. Show that the reduced density matrix $\rho_{\text{sys}}(t)$ of the system at time t is given by,

$$\rho_{\text{sys}}(t) = \sum_{n,m} c_n c_m^* \langle\phi_m(t)|\phi_n(t)\rangle |n\rangle\langle m| ,$$

where $|\phi_n(t)\rangle = \mathrm{e}^{-\mathrm{i}A_n t/\hbar} |\phi(0)\rangle$. Interpret this result.

d. Assume that a subspace of the system spanned by the eigenstates $\{|k_1\rangle , \dots , |k_M\rangle\} \subseteq \{|n\rangle\}$ is described by a single operator A_k acting on the Hilbert space of the apparatus such that $A_k \equiv A_{k_1} = \cdots = A_{k_M}$. What is the effect of this on the reduced density matrix $\rho_{\text{sys}}(t)$ and the decoherences (i.e., the decay of the coherences (i.e., the off-diagonal matrix elements) of the reduced density matrix $\rho_{\text{sys}}(t)$)? What are the physical consequences of this result?

Problem 36. Rabi model with finite detuning

Consider a two-level system coupled to a classical electromagnetic field. In the case where the transition frequency and the field frequency are not at resonance, the Schrödinger equation yields in the rotating wave approximation (i.e., $\delta\omega \ll \omega$) the following expressions for the probability of each state,

$$\frac{\partial}{\partial t}c_1(t) = \frac{\mathrm{i}}{2}\Omega_\mathrm{R}\mathrm{e}^{\mathrm{i}\delta\omega t}c_2(t), \qquad \frac{\partial}{\partial t}c_2(t) = \frac{\mathrm{i}}{2}\Omega_\mathrm{R}\mathrm{e}^{-\mathrm{i}\delta\omega t}c_1(t),$$

where $\delta\omega = \omega - \omega_0$ with ω the field frequency and ω_0 the transition frequency of the system.

a. Show that,

$$\frac{\partial^2}{\partial t^2}c_2(t) + \mathrm{i}\delta\omega\frac{\partial}{\partial t}c_2(t) + \frac{\Omega_\mathrm{R}^2}{4}c_2(t) = 0.$$

b. By considering a trial solution of the form $c_2(t) \sim C\mathrm{e}^{-\mathrm{i}\zeta t}$, show that the general solution of the second order differential equation for $c_2(t)$ in **a** is,

$$c_2(t) = C_+\mathrm{e}^{-\mathrm{i}\zeta_+ t} + C_-\mathrm{e}^{-\mathrm{i}\zeta_- t},$$

with $2\zeta_\pm = \delta\omega \pm \Omega$ and $\Omega^2 = (\delta\omega)^2 + \Omega_\mathrm{R}^2$ and where $C_\pm$ are constants.

c. Hence, show for the initial conditions $c_1(0) = 1$ and $c_2(0) = 0$ that the probability to find the system in state 2 is given by,

$$|c_2(t)|^2 = \frac{\Omega_\mathrm{R}^2}{\Omega^2}\sin^2(\Omega t/2).$$

Problem 37. Pauli master equation

The Pauli master equation is a special case of the Lindblad master equation. It describes the evolution of a classical probability distribution $P(n,t)$ of a set of discrete states labelled by the index $n \in \mathbb{Z}$ due to transitions between the states. It has the form,

$$\frac{\mathrm{d}}{\mathrm{d}t}P(n,t) = \sum_m (W(n|m)P(m,t) - W(m|n)P(n,t)),$$

where $W(n|m)$ is the transition probability density (i.e., the conditional transition rate) to go from state m to state n during the time interval $\mathrm{d}t$. Now consider a particle moving on a chain,

$$x = n\Delta x, \qquad n \in \mathbb{Z},$$

that undergoes the transitions,

$$n \to n \pm 1 \text{ with rate } \gamma_\pm,$$

i.e., the particle only jumps from site n to site $n \pm 1$ with rate $\gamma_\pm$ independent of the site n.

a. Write down the Pauli master equation for this process.

b. Derive and solve the differential equations for the expectation value, i.e., mean $\mu_t \equiv \langle n\rangle_t$ and variance $\sigma_t^2 \equiv \langle n^2\rangle_t - \langle n\rangle_t^2$ of the position of the particle, where,

$$\langle n^k\rangle_t = \sum_n n^k P(n,t).$$

Let us now consider a continuous distribution by defining for the position x given above,

$$P(x,t) = \frac{1}{\Delta x} P(n,t),$$

in the limit $\Delta x \to 0$ where the quantities,

$$a = \Delta x(\gamma_+ - \gamma_-), \qquad D = (\Delta x)^2(\gamma_+ + \gamma_-),$$

are kept constant.

c. Expand the terms in the Pauli master equation to second order in Δx and show that this leads to the Fokker-Planck equation,

$$\frac{\partial}{\partial t} P(x,t) = -a\frac{\partial}{\partial x} P(x,t) + \frac{D}{2}\frac{\partial^2}{\partial x^2} P(x,t).$$

Give an interpretation of the terms on the right hand side of this equation.
