# Theoretical Quantum Optics

Problem Sheet

| **Lecturer:** Prof. Dr. Igor Lesanovsky | | **Semester:** Winter 23/24 |
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| **Sheet:** 12 | **Hand-out:** 11.1.24 | **Hand-in:** 18.1.24 |

## Problem 32. Maximum likelihood estimation of a Gaussian

Let $p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$ be a Gaussian distribution centered at $x=\mu$, with width $\sigma$.

**a.** Show that the maximum likelihood estimator of $\hat{\sigma}$, determined via $\frac{\partial}{\partial\sigma}p(\mathbf{x}|\mu,\sigma)=0$ (where $\mathbf{x}=(x_1,...,x_M)$ is a vector of $M$ sampled values $x_i$ and $p(\mathbf{x}|\mu,\sigma)=\prod_{i=1}^M p(x_i|\mu,\sigma)$), is given by

$$\hat{\sigma}^2 = \frac{1}{M}\sum_{i=1}^M (x_i-\mu)^2. \tag{32.1}$$

**b.** Show that $\hat{\sigma}^2$ is biased if $\mu$ is replaced by its maximum likelihood estimate $\bar{x}=\frac{1}{M}\sum_{i=1}^M x_i$, i.e. determine the expectation value $E(\hat{\sigma}^2)$ and show that it differs from $\sigma^2$ for finite $M$. What happens for $M\to\infty$?

**c.** Based on part b, show that

$$(\sigma^2)_0 \equiv \frac{1}{M-1}\sum_{i=1}^M (x_i-\bar{x})^2, \tag{32.2}$$

with $\bar{x}\equiv\frac{1}{M}\sum_{i=1}^M x_i$ is an unbiased estimator of $\sigma^2$.

This is why the variance of a Gaussian is defined in the statistics literature sometimes via eq. (32.2): It is the unbiased maximum likelihood estimate of the variance, rather than the empirical variance of the data.

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## Problem 33. Quantum Cramér-Rao bound

The Quantum Cramér-Rao bound (QCR bound) is a lower bound on the fluctuations of an unbiased estimator $\hat{\lambda}$ ( i.e. $\langle\hat{\lambda}\rangle=\lambda$) of a parameter $\lambda$ that parametrizes a quantum state $\rho_\lambda$. It is optimized over all possible POVMs (POVM = "positive operator-valued measure") and all data-analysis schemes of the measurement outcome. It can be saturated in the limit of infinitely many measurements.

The QCR bound is given by

$$\mathrm{var}(\hat{\lambda}) \geq \frac{1}{MF_q(\lambda)}, \tag{33.1}$$

where $M$ is the number of measurements. For a density matrix $\rho=\sum_{n=1}^s p_n|\psi_n\rangle\langle\psi_n|$, the QFI ("quantum Fisher information") $F_q(\lambda)$ is given by

$$F_q(\lambda) = \sum_{n=1}^s \frac{(\partial_\lambda p_n)^2}{p_n} + 2\sum_{m,n=1}^N \frac{(p_n-p_m)^2}{p_n+p_m}|\langle\psi_m|\partial_\lambda\psi_n\rangle|^2, \tag{33.2}$$

with $|\partial_\lambda\psi_n\rangle\equiv\partial_\lambda|\psi_n\rangle$, and the sums are over all terms with non-vanishing denominators.

**a.** What is the physical dimension of $F_q(\lambda)$?

**b.** Show that error-propagation holds, i.e. if $\mu = \mu(\lambda) \Rightarrow \sigma(\mu) = \left|\frac{\partial \mu}{\partial \lambda}\right| \sigma(\lambda)$, where $\sigma(A)$ is the minimal possible standard deviation of the estimator $\hat{A}$ of $A$.

**c.** Show that for pure states $\rho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$ the QFI is given by

$$F_q(\lambda) = 4\big(\langle\partial_\lambda\psi_\lambda|\partial_\lambda\psi_\lambda\rangle - |\langle\psi_\lambda|\partial_\lambda\psi_\lambda\rangle|^2\big). \tag{33.3}$$

*Hint:* Set $|\psi_1\rangle \equiv |\psi_\lambda\rangle$ and use the orthogonality $\langle\psi_\lambda|\psi_j\rangle = 0,\ \forall j > 1$ to rewrite terms in the second sum. You may also want to use $\mathbb{I} = \sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n|$ to rewrite some of the terms.

**d.** Use eqn. (33.3) to show that for a pure state $|\psi_\lambda\rangle$ that depends on $\lambda$ via a so called phase shift operation, $|\psi_\lambda\rangle = e^{i\lambda\hat{X}}|\psi_0\rangle$, where $\hat{X} = \hat{X}^\dagger$ is a hermitian generator,

$$F_q(\lambda) = 4\,\mathrm{var}(\hat{X}) = 4\big(\langle\psi_0|\hat{X}^2|\psi_0\rangle - \langle\psi_0|\hat{X}|\psi_0\rangle^2\big). \tag{33.4}$$

**e.** A thermal state is given by $\rho_\beta = \frac{e^{-\beta\hat{H}}}{Z}, \quad Z = \mathrm{tr}\, e^{-\beta\hat{H}}$, where $\beta = \frac{1}{k_B T}$ is the inverse temperature and $\hat{H}$ is the Hamiltonian of the system. Derive an uncertainty inequality relating the variance of $\hat{H}$ and the one for measurements of $\beta$. Do the same for measurements of the temperature $T$ instead of $\beta$