

Heat transport through lattices of quantum harmonic oscillators in arbitrary dimensions

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In d -dimensional lattices of coupled quantum harmonic oscillators, we analyze the heat current caused by two thermal baths of different temperatures, which are coupled to opposite ends of the lattice, with a focus on the validity of Fourier's law of heat conduction. We provide analytical solutions of the heat current through the quantum system in the nonequilibrium steady state using the rotating-wave approximation and bath interactions described by a master equation of Lindblad form. The influence of local dephasing in the transition of ballistic to diffusive transport is investigated.

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I. INTRODUCTION

Ever since the discovery of anomalous heat transport through a chain of coupled harmonic oscillators in the seminal work of Rieder *et al.* [1], the topic of heat transport through systems of harmonic oscillators and the question of how classical diffusive heat transport emerges from a microscopic classical or quantum description remain interesting fields of research. The classical law of diffusive heat conduction as stated by Fourier [2],

$$J = -\kappa \nabla T \quad (1)$$

relates the heat current to the negative gradient of the temperature by the thermal conductivity κ , the latter being a property of the material. Fourier's law of heat conduction contains several key statements: (i) at constant system size, the magnitude of the heat current is proportional to the temperature difference ($J \propto \Delta T$), (ii) at constant temperature difference and constant thermal conductivity, the heat current scales inversely proportional to the distance between the heat baths, which is given by the system dimension separating the baths ($J \propto 1/L$). Rieder *et al.* found, for their one-dimensional classical system, that the heat current is proportional to ΔT but does not scale with the system size. Nonetheless, when imposing Fourier's law in such situations, one generally obtains a size-dependent thermal conductivity $\kappa = \kappa(L)$, which, in one-dimensional systems, usually is in the form of a power law $\kappa \propto L^\alpha$. In addition, they observed the absence of a temperature gradient *inside* the system. The same conclusions hold in higher-dimensional lattices of classical harmonic oscillators [3], but size dependence has only been addressed recently while also taking disorder and anharmonicity effects into account, e.g., in Refs. [4–7].

In the present paper, however, we focus on the *quantum mechanical* equivalent of heat transport through systems of harmonic oscillators. Although chains of harmonic oscillators have been investigated with an emphasis on their entanglement properties [8,9] or with the aim to scrutinize the domain of validity of master equations in Lindblad form to model thermal

baths [10], specific questions regarding the heat transport properties have only been addressed to a limited extent. Within the framework of modeling thermal baths by means of quantum Langevin equations, ballistic transport has been observed for chains of quantum harmonic oscillators [11]. More recent analyses include the study of disordered chains but without concrete conclusions regarding the exact scaling of the thermal conductivity with the system size [12]. Furthermore, explicit analytical results regarding the quantum mechanical steady state remain to be elucidated [13]. We approach this open problem by providing analytical solutions to the heat current and analytical forms of second moments of the nonequilibrium steady state in arbitrary dimensions for systems of coupled harmonic oscillators within the rotating-wave approximation.

The paper is structured as follows. In the next section, we introduce the employed quantum mechanical model of a system of coupled harmonic oscillators and the master equations describing the heat baths. In Sec. III, we then proceed to the analytical solution of the heat current in the steady state for a one-dimensional chain. After commenting on the nature of the state of individual oscillators in the chain in Sec. IV, we provide the analytical solution for the case that additional local dephasing influences the nonequilibrium steady state in Sec. V, before generalizing the result to lattices of arbitrary dimension in Sec. VI. The results are discussed in Sec. VII, and we summarize in Sec. VIII.

II. MODEL

The quantum system of interest is a lattice of coupled harmonic oscillators as illustrated in Fig. 1. In the rotating-wave approximation, we can describe it with a Hamiltonian of the following form:

$$H = \sum_{j=1}^N \omega a_j^\dagger a_j + \sum_{j=1}^{N-1} V_{j,j+1} (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j), \quad (2)$$

where we employ units of $\hbar = 1$ throughout the paper, ω denotes the frequency of each of the N identical oscillators, $V_{j,j+1}$ is the coupling strength between neighboring sites, and a_j and a_j^\dagger are the bosonic annihilation and creation operators, respectively. The interaction term, thus, describes

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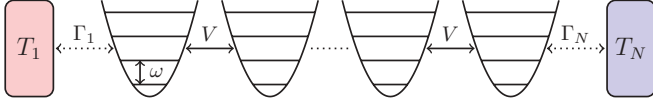


FIG. 1. (Color online) A linear array of coupled harmonic oscillators with thermal environments attached to the endpoints.

the hopping of a single excitation between neighboring sites. Within the performed rotating-wave approximation, terms proportional to $a_j a_{j+1}$ and $a_j^\dagger a_{j+1}^\dagger$ have been neglected. The Hamiltonian, thus, preserves the number of excitations and, hence, commutes with the excitation number operator. The rotating-wave approximation is valid in the weak-coupling regime of $\omega \gg V_{j,j+1}$, which is often met in quantum optical scenarios.

The dynamics of the system described by its state ρ is treated within the framework of master equations of Lindblad form, that is, the coherent transfer dynamics inside the systems on one hand and the incoherent heat exchange with the two baths on the other,

$$\frac{d\rho}{dt} = \mathcal{L}\rho = -i[H, \rho] + \mathcal{L}_1\rho + \mathcal{L}_N\rho. \quad (3)$$

Terms $\mathcal{L}_1\rho$ and $\mathcal{L}_N\rho$ describe the effective local processes of the first and last oscillators of the chain interacting with its respective heat bath. Within the Born-Markov approximation, $\mathcal{L}_j\rho$ can be given in Lindblad form, and for a thermal bath, it is given by

$$\begin{aligned} \mathcal{L}_j\rho = & \Gamma_j(n_j + 1)(a_j\rho a_j^\dagger - \frac{1}{2}\{a_j^\dagger a_j, \rho\}) \\ & + \Gamma_j n_j(a_j^\dagger\rho a_j - \frac{1}{2}\{a_j a_j^\dagger, \rho\}). \end{aligned} \quad (4)$$

The first term describes dissipation into the bath, i.e., decay of excitations into the reservoir via stimulated and spontaneous emissions, respectively, and the second term describes excitation, i.e., energy absorption from the reservoir. The quantity,

$$n_j = \frac{1}{\exp(\omega/k_B T_j) - 1} \quad (5)$$

denotes the mean excitation number of an oscillator coupled to a bath j at temperature T_j .

This master equation is of Lindblad form, and its derivation is based on both the Born-Markov and the rotating-wave approximations. The validity regime of this approximation has been analyzed in Ref. [10] by comparing the time evolution of the master equation with exact simulations generated by finite heat baths for a system of two harmonic oscillators. The analysis shows that the master equation description is valid as long as the coupling strengths between the oscillators $V_{j,j+1}$ and baths Γ_j are weak compared to ω , and it is valid even at high temperatures of the bosonic heat baths. However, due to the approximations employed in order to derive (4), the system evolving according to Eq. (3) does not thermalize in the Gibbs state with respect to the Hamiltonian (2) exactly, when both baths are at the same temperature. Instead, it evolves into a steady state that is close to the actual Gibbs state in the small parameter $V_{j,j+1}$ regime. A detailed investigation of the influence of the type of the spectral density in this context can be found in Ref. [14].

The expression for the mean heat current through the chain can be obtained for the steady state by the following argument. Starting with the time derivative of the energy expectation value, which vanishes in the steady state,

$$\frac{d}{dt}\langle H \rangle = \text{Tr}\left(H \frac{d\rho}{dt}\right) = 0, \quad (6)$$

one obtains, when inserting the master equation (3), that the coherent part vanishes exactly while the contributions of the heat baths are nonzero but cancel each other. The latter amount to the positive net heat current coming from the hotter bath and the negative net heat current exiting to the colder bath, respectively,

$$\text{Tr}(H\mathcal{L}_1\rho + H\mathcal{L}_N\rho) \equiv J_1 + J_N = 0. \quad (7)$$

The net heat current through the system is, thus, given by $J \equiv J_1 = -J_N$. A compact expression of J for the present system employs the specific form of the Hamiltonian (2) and the terms in the master equation (4) and when evaluated using commutation relation $[a_j, a_j^\dagger] = 1$ yields

$$J = \Gamma_1 \omega (n_1 - \langle a_1^\dagger a_1 \rangle) - \frac{\Gamma_1}{2} V_{1,2} (\langle a_1^\dagger a_2 \rangle + \langle a_2^\dagger a_1 \rangle). \quad (8)$$

The heat current through the chain is, therefore, given by the difference between the mean excitation number of oscillator 1 and the equilibrium state of bath 1 and by the real part of the coherences between the two oscillators at the boundary multiplied by the frequency of the oscillators and the coupling strength, respectively. An analogous expression can be derived for J_N in which the corresponding terms are expressed in terms of the last oscillator, which is connected to the colder bath.

III. ANALYTICAL SOLUTION

The coherent dynamics and the incoherent excitation exchange with the heat baths in Eq. (3) are Gaussian processes that preserve the Gaussian character of Gaussian quantum states, i.e., states that can be completely described by their first two moments. Assuming that the steady state is the unique solution to $d\rho/dt = \mathcal{L}\rho = 0$, we, therefore, know that the steady state is Gaussian. We, thus, perform an ansatz for the nonequilibrium steady state using the first and second moments only. Therefore, we first define the row vector of all bosonic operators of the system,

$$A \equiv (a_1, a_2, \dots, a_N), \quad (9)$$

and the matrix C , which captures the second moments, with matrix elements,

$$[C]_{ij} \equiv \langle a_i^\dagger a_j \rangle, \quad (10)$$

which is, thus, related to A by taking all elementwise expectation values of the matrix $A^\dagger A$.

In order to solve for the moments of the steady state, it is first necessary to transform the master equation into the equation of motion of the moments. Since the master equation is a linear differential equation and calculating the moments is also linear in the state, we can treat the coherent and incoherent parts of the master equation separately. The coherent contribution to the equation of motion for C can be obtained from the

Heisenberg equation of motion for A . From the time evolution of the annihilation operators,

$$i\dot{a}_j = [a_j, H] = \omega a_j + V_{j,j+1}a_{j+1} + V_{j-1,j}a_{j-1}, \quad (11)$$

in which we formally set $a_{-1} = 0 = a_{N+1}$, one obtains a system of coupled differential equations, the coefficients of which are contained in the matrix W . We, thus, arrive at

$$\dot{A}^\dagger = iWA^\dagger, \quad (12)$$

and further by using the product rule at the evolution equation of $A^\dagger A$. The coherent part of the differential equation for C is, thus, given by

$$\left. \frac{dC}{dt} \right|_{\text{coh.}} = i[W, C]. \quad (13)$$

The effect of the baths can be calculated in a similar way by employing the incoherent part of the master equation with the exact form of \mathcal{L}_j and the bosonic commutation relation. We first describe a slightly more general case where every oscillator is coupled to a local heat bath via \mathcal{L}_j ,

$$\frac{d\langle a_i \rangle}{dt} = \text{Tr}(a_i \mathcal{L}_i \rho) = -\frac{\Gamma_i}{2} \langle a_i \rangle, \quad (14)$$

$$\begin{aligned} \frac{d\langle a_i^\dagger a_j \rangle}{dt} &= \text{Tr}(a_i^\dagger a_j [\mathcal{L}_i \rho + \mathcal{L}_j \rho]), \quad i \neq j, \\ &= -\frac{\Gamma_i}{2} \langle a_i^\dagger a_j \rangle - \frac{\Gamma_j}{2} \langle a_i^\dagger a_j \rangle, \end{aligned} \quad (15)$$

$$\frac{d\langle a_j^\dagger a_j \rangle}{dt} = -\Gamma_j \langle a_j^\dagger a_j \rangle + \Gamma_j n_j. \quad (16)$$

The incoherent part of the evolution equation of C is, thus, given by

$$\left. \frac{dC}{dt} \right|_{\text{incoh.}} = \{L, C\} + M, \quad (17)$$

where the anticommutator is denoted by $\{L, C\} = LC + CL$, and the following diagonal matrices were introduced:

$$L = -\frac{1}{2} \text{Diag}(\Gamma_1, \Gamma_2, \dots, \Gamma_N), \quad (18)$$

$$M = \text{Diag}(\Gamma_1 n_1, \Gamma_2 n_2, \dots, \Gamma_N n_N). \quad (19)$$

The complete evolution equation of C for the master equation (3) is, thus, given by the system of linear differential equations,

$$\frac{dC}{dt} = i[W, C] + \{L, C\} + M, \quad (20)$$

where, in our scenario of two heat baths at the boundaries, $\Gamma_2 = \dots = \Gamma_{N-1} = 0$. Regarding the analytic expression of the heat current (8), the steady-state solution of C contains all the information needed and all the information necessary to describe the Gaussian steady state.

The ansatz to the solution of C in the steady state $dC_{\text{ss}}/dt = 0$ consists of two contributions: one capturing the steady state of the average temperature and one that contains the nonperturbative deviations due to the nonequilibrium. When there is no temperature bias applied, i.e., $n_1 = n_N \equiv n$, the equilibrium steady-state solution is given by a diagonal matrix $C = n\mathbb{1}$, i.e., with elements $[C]_{ii} = n$ on the diagonal and $[C]_{i,j \neq i} = 0$ for all off-diagonal elements. Each harmonic

oscillator of the chain, thus, carries the same average number of excitations as both of the heat baths. This case serves as the motivation for the contribution due to the average excitation number. In the nonequilibrium scenario, we, thus, choose the following ansatz:

$$C_{\text{ss}} = \bar{n}\mathbb{1} + \Delta n D, \quad (21)$$

where we have defined the average bath excitation number $\bar{n} \equiv (n_1 + n_N)/2$ and the amount of nonequilibrium $\Delta n \equiv (n_1 - n_N)/2$. The matrix D captures the nonequilibrium contribution to the steady state. Insertion of this ansatz into Eq. (20) yields the following equation for the nonequilibrium contribution of the steady state:

$$i[D, W] = S + \{L, D\}, \quad (22)$$

with diagonal matrix $S \equiv (2\bar{n}L + M)/\Delta n = \text{Diag}(\Gamma_1, 0, \dots, 0, \Gamma_N)$. Given the matrices W with uniform couplings $V_{j,j+1} = V$, L , and S , the structure of Eq. (22) implies the following form for the Hermitian matrix D ,

$$D = \begin{pmatrix} e_1 & x & & & \\ x^* & e_2 & x & & \\ & \ddots & \ddots & \ddots & \\ & & x^* & e_{N-1} & x \\ & & & x^* & e_N \end{pmatrix}. \quad (23)$$

Thereby, we obtain a set of algebraic equations, which, for $N > 2$ reads as follows:

$$\begin{aligned} i \frac{V}{\Gamma_1} (x - x^*) &= 1 - e_1, \\ i \frac{V}{\Gamma_1} (e_1 - e_2) &= -\frac{x}{2}, \\ (e_j - e_{j+1}) &= 0 \quad \text{for } 2 \leq j \leq N-2, \\ i \frac{V}{\Gamma_N} (e_{N-1} - e_N) &= -\frac{x}{2}, \\ i \frac{V}{\Gamma_N} (x - x^*) &= 1 + e_N. \end{aligned}$$

The only relevant parameters involved in the solution for D are the fractions V/Γ_1 and V/Γ_N . From the second and the second last equations, one can conclude that x is purely imaginary since the e_i are real numbers due to the Hermiticity of C . Furthermore, with the exception of e_1 and e_N , all the e_j 's are equal. The unique solution to the above set of equations reads

$$\begin{aligned} x &= -i \frac{4V\Gamma_1\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N)}, \\ e_1 &= \frac{4V^2(\Gamma_1 - \Gamma_N) + \Gamma_1\Gamma_N^2 + \Gamma_1^2\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N)}, \\ e_{2 \leq j \leq N-1} &= \frac{4V^2(\Gamma_1 - \Gamma_N) + \Gamma_1\Gamma_N^2 - \Gamma_1^2\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N)}, \\ e_N &= \frac{4V^2(\Gamma_1 - \Gamma_N) - \Gamma_1\Gamma_N^2 - \Gamma_1^2\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N)}. \end{aligned}$$

The average excitation numbers of the individual oscillators on the diagonal of C is, thus, given by

$$\begin{aligned}\langle a_1^\dagger a_1 \rangle &= \bar{n} + e_1 \Delta n, \\ \langle a_j^\dagger a_j \rangle &= \bar{n} + e_j \Delta n, \quad 2 \leq j \leq N-1, \\ \langle a_N^\dagger a_N \rangle &= \bar{n} + e_N \Delta n,\end{aligned}$$

that is, the mean excitation number of all oscillators in the bulk are equal, whereas, the excitation number of the boundary oscillators deviates from that of the bulk towards that of the heat bath. The coherences between neighboring oscillators in the diagonal above and below the main diagonal of C are

$$\langle a_j^\dagger a_{j+1} \rangle = x \Delta n. \quad (24)$$

We can now give the analytic expression for the quantum heat current using the matrix elements of C_{ss} ,

$$J = \frac{4\omega V^2 \Gamma_1 \Gamma_N (n_1 - n_N)}{(4V^2 + \Gamma_1 \Gamma_N)(\Gamma_1 + \Gamma_N)}. \quad (25)$$

The heat current of the chain of quantum harmonic oscillators is, thus, independent of the chain length N and, thus, constitutes a violation of Fourier's law with the thermal conductivity scaling as $\kappa \sim N$.

As an alternative expression to Eq. (8), we can give the heat flux in terms of the purely imaginary coherences,

$$J = 2\omega V i \langle a_j^\dagger a_{j+1} \rangle. \quad (26)$$

This means that, for the chain of oscillators of the same frequency, the heat current is solely given by the next-neighbor coherences. We, thus, expect any additional noise source that degrades coherent properties of the steady state to degrade the transport properties and to decrease the heat current.

This solution is specific for harmonic systems. For different kinds of systems, e.g., qubits chains, a different approach is required [15–18].

IV. LOCAL THERMALIZATION

It would be interesting to show that, in steady state, the local density matrices ρ_j at each site are given by a thermal state of the following form when written in the Fock basis:

$$\rho_j = \frac{1}{\langle n_j \rangle + 1} \sum_{m=0}^{\infty} \left(\frac{\langle n_j \rangle}{\langle n_j \rangle + 1} \right)^m |m\rangle \langle m|, \quad (27)$$

such that one can locally define a temperature for each oscillator in the chain.

Since the nonequilibrium steady state is of Gaussian form, the density operator can be reconstructed from the first and second moments. Each individual harmonic oscillator of the chain is, thus, also in a Gaussian state. A general expression for the Gaussian density operator of a single harmonic oscillator [19] is given by $\rho = D(\alpha)S(r)\rho_{th}S(r)^\dagger D(\alpha)^\dagger$, meaning that any Gaussian state of this harmonic oscillator can be constructed from a thermal state ρ_{th} by first squeezing it with $S(r, \phi) = \exp(\frac{1}{2}re^{-i2\phi}a^2 - \frac{1}{2}re^{i2\phi}a^{\dagger 2})$ and then displacing it away from the phase space origin with $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$. The mean excitation number of such a state is then given by $\langle n \rangle = |\alpha|^2 + n_{th} + (2n_{th} + 1) \sinh^2 r$, where n_{th} is the thermal contribution to the mean excitation number. Since, due to

(14) in the steady state $\langle a_j \rangle = \langle a_j^\dagger \rangle = 0$, the parameter α vanishes. Furthermore, the master equation does not involve any quadratic forms in a or a^\dagger , and it would be nice to show that the squeezing operation is not needed to describe the steady state of a single oscillator in the chain, and thus the squeezing parameter r is zero. Under these conditions, the density operator of a single oscillator at position j in the chain would be given by a thermal state of Gibbs form as written above.

For the special case of both thermal baths having the same temperature, one can show that each oscillator in the chain will have the same temperature and the same Gibbs state. The global steady state of the chains is then the N -fold direct product of all the individual (equal) Gibbs states as can be straightforwardly verified by insertion into Eq. (3). The product structure of the equilibrium steady state is a direct consequence of the baths coupling locally to the chain as modeled by Eq. (3) and their specific form (4).

This steady state, formed by the direct product of the individual Gibbs states, converges to the global thermal state in the limit $V \rightarrow 0$. In Ref. [10], an analysis of this convergence and the fidelity between the exact thermal state for Eq. (2) and the one arising from our Lindblad master equation is performed.

V. TRANSITION BETWEEN COHERENT AND INCOHERENT TRANSPORTS

The ballistic transport observed for the uniform nonequilibrium chain of quantum harmonic oscillators can be turned into diffusive transport obeying Fourier's law by adding additional noise terms that degrade next-neighbor coherences. Within our master equation approach, such local noise sources can be straightforwardly implemented by introducing additional dephasing terms. Local dephasing environments randomize coherences and, thereby, effectively degrade their magnitude. Thereby, coherently delocalized excitations are randomly localized at the sites of the harmonic oscillators, which adds a diffusive element to the transport dynamics. The Lindblad superoperator for local dephasing processes acting on each of the oscillators j is given by

$$\mathcal{L}_{\text{deph}} \rho = \sum_{j=1}^N \gamma_j \left(a_j^\dagger a_j \rho a_j^\dagger a_j - \frac{1}{2} \{ (a_j^\dagger a_j)^2, \rho \} \right). \quad (28)$$

The time evolution of the expectation value of the bosonic operators under dephasing is, therefore,

$$\begin{aligned}\frac{d\langle a_i \rangle}{dt} &= -\frac{\gamma_i}{2} \langle a_i \rangle, \\ \frac{d\langle a_i^\dagger a_j \rangle}{dt} &= -\frac{\gamma_i}{2} \langle a_i^\dagger a_j \rangle - \frac{\gamma_j}{2} \langle a_i^\dagger a_j \rangle, \quad i \neq j, \\ \frac{d\langle a_i^\dagger a_i \rangle}{dt} &= 0.\end{aligned}$$

With these equations, the corresponding evolution equation of C in the presence of the additional dephasing environments gains additional terms,

$$\begin{aligned}\frac{dC}{dt} \Big|_{\text{deph}} &= \{L_{\text{deph}}, C\} \\ &+ \text{Diag}[\gamma_1[C]_{1,1}, \gamma_2[C]_{2,2}, \dots, \gamma_N[C]_{N,N}],\end{aligned}$$

where $L_{\text{deph}} = -\frac{1}{2}\text{Diag}(\gamma_1, \gamma_2, \dots, \gamma_N)$. Let us now consider the special case in which the local dephasing operations, acting on the individual oscillators, all have equal rates, i.e., $\gamma_j = \gamma$. This allows for a more straightforward analytical solution including the dephasing effect. With the same ansatz as before, the nonequilibrium part of the steady-state solution satisfies the equation,

$$i[D, W] = S + \{L, D\} - \gamma D + \gamma \text{Diag}([D]_{1,1}, [D]_{2,2}, \dots, [D]_{N,N}). \quad (29)$$

The algebraic set of equations for the matrix elements of D takes a similar form as before but now includes the dephasing

rates,

$$\begin{aligned} i \frac{V}{\Gamma_1} (x - x^*) &= 1 - e_1, \\ i \frac{V}{\Gamma_1} (e_1 - e_2) &= 7 - \frac{x}{2} - \frac{\gamma}{\Gamma_1} x, \\ i (e_j - e_{j+1}) &= 7 \frac{\gamma}{V} x, \quad 2 \leq j \leq N-2, \\ i \frac{V}{\Gamma_N} (e_{N-1} - e_N) &= -\frac{x}{2} - \frac{\gamma}{\Gamma_N} x, \\ i \frac{V}{\Gamma_N} (x - x^*) &= 1 + e_N. \end{aligned} \quad (30)$$

The solution to the above set of equations is given by

$$\begin{aligned} x &= -i \frac{4V\Gamma_1\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N) + 2(N-1)\gamma\Gamma_1\Gamma_N}, \\ e_1 &= \frac{4V^2(\Gamma_1 - \Gamma_N) + \Gamma_1\Gamma_N^2 + \Gamma_1^2\Gamma_N + 2(N-1)\gamma\Gamma_1\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N) + 2(N-1)\gamma\Gamma_1\Gamma_N}, \\ e_{2 \leq j \leq N-1} &= \frac{4V^2(\Gamma_1 - \Gamma_N) + \Gamma_1\Gamma_N^2 - \Gamma_1^2\Gamma_N + 2(N-2j+1)\gamma\Gamma_1\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N) + 2(N-1)\gamma\Gamma_1\Gamma_N}, \\ e_N &= \frac{4V^2(\Gamma_1 - \Gamma_N) - \Gamma_1\Gamma_N^2 - \Gamma_1^2\Gamma_N - 2(N-1)\gamma\Gamma_1\Gamma_N}{(4V^2 + \Gamma_1\Gamma_N)(\Gamma_1 + \Gamma_N) + 2(N-1)\gamma\Gamma_1\Gamma_N}. \end{aligned}$$

As before, this yields the unique solution to the steady state fully contained in C_{ss} .

In order to derive the analytical expression of the heat current in the presence of the local dephasing environment, it is important to account for a possible heat current due to the dephasing environments. It is a priori not clear that local dephasing does not introduce an additional net heat current because the local dephasing processes do not leave energy eigenstates of the chain invariant. For a chain of harmonic oscillators with uniform frequencies and equal dephasing rates, the net heat current due to the dephasing environment $\text{Tr}(H\mathcal{L}_{\text{deph}}\rho) = -\frac{\gamma_1}{2}\langle a_1^\dagger a_2 \rangle - \frac{\gamma_2}{2}\langle a_2^\dagger a_1 \rangle$ vanishes for the steady state in the present scenario because coherences are purely imaginary. Thus, the equality $J = J_1 = -J_N$ holds under the considered dephasing operations.

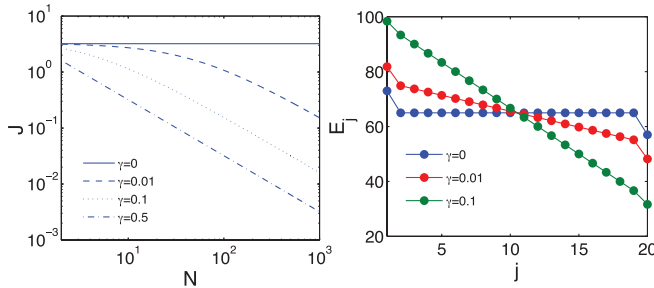


FIG. 2. (Color online) Left: Heat current vs chain length in a log-log plot for different values of the dephasing rate. Right: Mean excitation number $E_j = \langle a_j^\dagger a_j \rangle$ of all oscillators for a chain of length $N = 20$ in the steady state under dephasing. Parameter values: $\Gamma_{1,2} = 0.1$, $V = 0.1$, and $\omega = 10$.

With the nonequilibrium steady-state solution for C_{ss} , the analytical expression for the heat current in the presence of the dephasing environment is given by

$$J = \frac{4\omega V^2 \Gamma_1 \Gamma_N (n_1 - n_N)}{(4V^2 + \Gamma_1 \Gamma_N)(\Gamma_1 + \Gamma_N) + 2(N-1)\gamma \Gamma_1 \Gamma_N}. \quad (31)$$

The heat current with local dephasing noise on each oscillator, thus, acquires a dependence on the size of the system. Therefore, the heat current scales as $J \sim N^{-1}$ in the limit of large N and, thereby, recovers a size dependence as in Fourier's law. As before, next-neighbor coherences are purely imaginary but are smaller in magnitude. The expression of the heat current in terms of coherence (26) is also valid in the present case. Similar results, regarding the recovering of Fourier's law under the effect of a dephasing channel or more general sources of noise, have been recently obtained for different systems [16–18,20].

In contrast to the transport scenario without dephasing, the mean excitations of the individual oscillators in the bulk are no longer equal but decrease linearly from the hotter towards the colder heat bath. The effects of dephasing on the individual mean excitations of the oscillators are shown in Fig. 2. In the limit of large dephasing rates, the entire harmonic chain approaches the phenomenology of Fourier's law in the sense that a constant gradient emerges throughout the chain.

VI. GENERALIZATION TO d -DIMENSIONAL LATTICES

With the analytical results for heat transport through chains (dimension $d = 1$) and the strategy to solve for steady-state properties available, we generalize our results to heat transport

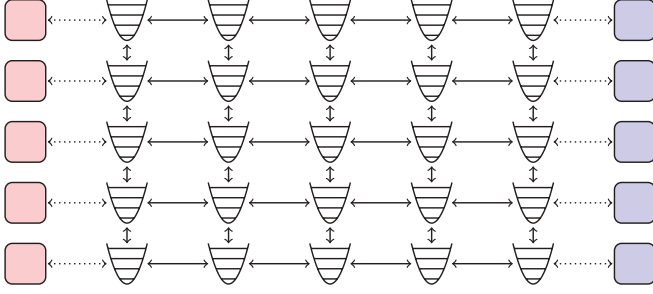


FIG. 3. (Color online) A two-dimensional harmonic lattice whose boundary oscillators of opposite edges are coupled to local heat baths of different temperatures.

scenarios through lattices of d dimensions. The Hamiltonian (2) is extended to a cubic lattice of d dimensions again with uniform oscillator frequencies for all oscillators and uniform couplings between next neighbors along all lattice edges. When advancing to higher dimensions, we consider a situation where the heat transport is only driven along *one* direction. Therefore, two opposing $(d-1)$ -dimensional hypersurfaces are coupled to heat baths of different temperatures such that there is a temperature difference along the one remaining dimension. The two heat baths are modeled such that all harmonic oscillators of the respective hypersurface are coupled to a local heat bath of identical temperatures. An example for the $d=2$ case is depicted in Fig. 3 with the two opposing edges coupled to local heat baths.

We approach the analytic solution by relating it to the one-dimensional case using a recursion argument. Clearly, due to symmetry, we do not expect a net heat current perpendicular to the applied temperature gradient. However, the coherent coupling in lattice directions transverse to the temperature gradient might constructively or destructively interfere with the heat transport in the longitudinal direction. As for the one-dimensional case, we assume the uniqueness of the nonequilibrium steady state. This again leads us to the solution with all essential information being captured in a matrix C with matrix elements $\langle a_i^\dagger a_j \rangle$, where i and j are index vectors that account for the labeling of all harmonic oscillators in d dimensions. We choose the order of indices such that the last index counts oscillators along the direction along which the temperature bias is applied. Thereby, we can relate the heat transport through a d -dimensional lattice to the heat transport through d stacked copies of $(d-1)$ -dimensional lattices that are coherently coupled. The matrix C , thus, is of block structure. For a d -dimensional hypercube of N sites along every direction, $C \equiv C_d$ can be divided into $N \times N$ blocks where each block of the diagonal captures a matrix $C_{d-1}^{(j)}$ of the j th $(d-1)$ -dimensional lattice and the off-diagonal blocks capture the coherences between these different lattices of lower dimensions. Since the stacked lower-dimensional lattices are only coupled to their direct neighbors, only coherences between two neighboring lattices exist, hence, only the blocks on the first diagonals above and below the main diagonal are nonzero. This procedure is repeated until the two-dimensional lattice is decomposed into an array of coupled chains.

Similarly, all the sets of differential equations can be related to those of the next-lower dimension with additional couplings along the transverse direction. We choose the same ansatz for the nonequilibrium steady state $C_d = \bar{n}\mathbb{1} + \Delta n D_d$, which, when inserted into the master equation, yields

$$i[D_d, W_d] = S_d + \{L_d, D_d\}. \quad (32)$$

Let us now employ the described block structure,

$$D_d = \begin{pmatrix} D_{d-1}^{(1)} & Q & & & \\ Q^* & D_{d-1}^{(2)} & Q & & \\ & \ddots & \ddots & \ddots & \\ & & Q^* & D_{d-1}^{(N-1)} & Q \\ & & & Q^* & D_{d-1}^{(N)} \end{pmatrix},$$

$$W_d = \begin{pmatrix} W_{d-1} & G & & & \\ G & W_{d-1} & G & & \\ & \ddots & \ddots & \ddots & \\ & & G & W_{d-1} & G \\ & & & G & W_{d-1} \end{pmatrix},$$

where $G = V\mathbb{1}$ is a diagonal matrix capturing the coupling of harmonic oscillators between two neighboring $(d-1)$ -dimensional sublattices. The matrix Q accounts for the coherences between these oscillators. The matrices S_d and L_d again capture the influence of the heat bath, and they are simply given by

$$L_d = \bigoplus_{j=1}^N L_{d-1} \quad \text{and} \quad S_d = \bigoplus_{j=1}^N S_{d-1}, \quad (33)$$

with L_1 and S_1 being those for the linear chain given in Sec. III.

Inserting these matrices into Eq. (32) gives rise to the set of differential equations for the lower-dimensional sublattices,

$$i[D_{d-1}^{(j)}, W_{d-1}] = S_{d-1} + \{L_{d-1}, D_{d-1}^{(j)}\} \quad \forall j, \quad (34)$$

$$\{L_{d-1}, Q_d\} = 0. \quad (35)$$

These equations imply that Q_d vanishes, and hence, there are no coherences and no heat transport between sublattices. Furthermore, since the equations are identical for all $D_{d-1}^{(j)}$, the steady state is given by the identical steady states of the lower-dimensional sublattice. The nonequilibrium steady state of the d -dimensional lattice is, thus, given by the steady state of the chains connecting heat baths at different temperatures to which we have decomposed the lattice. The heat current of the d -dimensional lattice is, thus, given by the sum of all the heat currents of the identical chains composing the lattice. Hence, the total heat current in d dimensions is given by the volume of the hypersurfaces that are in contact with the heat baths, i.e., the number of oscillators coupled to heat baths of the same temperature times the heat current of the chain as derived in Sec. III. For the special case of a d -dimensional hypercube with an edge length of N sites in each direction,

$$J_d = \text{Vol}_{d-1} J = N^{d-1} J. \quad (36)$$

This implies that the heat current per chain is invariant under a length change in the direction parallel to the heat current. The heat current of this model also is ballistic in higher dimensions.

VII. DISCUSSION

The analytic results for heat transport in harmonic lattices of arbitrary dimension show a number of differences to the existing literature. The framework of master equations in Lindblad form readily provides access not only to steady-state properties, but also to the steady state itself, and it is straightforwardly applicable to many quantum optical setups and scenarios, such as the array of coupled cavities depicted in Fig. 1. Our results differ from that of a harmonic chain where the baths are modeled by quantum Langevin equations [11,12]. On one hand, the method of treating the heat baths differs, on the other hand, the ballistic transport observed in our model is independent of the system size in the absence of dephasing. Furthermore, although the mean excitation profile of the oscillators coincides in some parameter regimes, we obtain different profiles in other cases. We attribute these differences to the performed rotating-wave approximation in the Hamiltonian of our model, but a detailed investigation of the effects of the rotating-wave approximation on the heat transport is beyond the scope of the present paper.

With respect to the open question regarding explicit results for the steady-state density matrix as stated in Ref. [13], our paper supplies analytical answers for the special case of the considered Hamiltonian and bath models, even in arbitrary dimensions. The simple relation of the heat current in lattices of higher dimensions may serve as a guiding result for numerical studies in higher dimensions.

Approaching the diffusive transport regime of Fourier's law by adding dephasing environments is consistent with other papers [16,17,21,22], which investigate the conditions under which the heat transport in quantum systems approaches a diffusive regime obeying Fourier's law. The mechanism applied in Ref. [21] in order to obtain heat diffusion in quantum chains, i.e., introducing a band of excited states rather than sharp energy levels per site, can qualitatively be recovered by our dephasing process, i.e., the effectively introduced level fluctuations and, hence, the line broadening as also observed for chains of two-level atoms [16]. The dynamic disorder introduced by the local dephasing environments allows for obtaining analytic results and a more straightforward clarification of the size dependence than approaching the diffusive regime with disorder in the Hamiltonian as pursued in Ref. [12].

In transport scenarios of molecular biology, e.g., in light-harvesting systems of photosynthesis, to what extent entanglement or (in general, nonequivalently) measures of coherence may serve as an indicator of transport properties has been investigated [23–27]. Although we have observed the emergence of entanglement in nonequilibrium systems of coupled two-level atoms [16], it does not hold for systems of harmonic oscillators.

This result is not surprising given that operators of the forms a^2 and $(a^\dagger)^2$ have been neglected in the Hamiltonian and the master equation, which are required, however, to achieve a squeezing operation that is necessary for creating an entangled Gaussian state. The conclusion to be drawn is that, although the system can exhibit large coherences and, thereby, a large heat current, i.e., efficient transport, entanglement is completely absent and cannot be generated by a large temperature difference as possible for systems of two-level atoms [16]. As spelled out by Eq. (26), the magnitude of the heat current depends on the amount of quantum coherence between the neighboring oscillators. Therefore, entanglement certainly does not qualify as a signature of quantum transport efficiency in the present model.

The validity of Fourier's law in higher dimensions strongly depends on the modeling of the system. Most of the papers in this direction highly rely on large-scale simulations [5,6,28]. Peierls showed that the phonon-phonon scattering leads to diffusive heat conduction in higher-dimensional systems [29]. In Ref. [30], it is shown that all models characterized by short range interactions and momentum conservation should exhibit the same kind of anomalous behavior in the heat conduction, i.e., a divergence of the thermal conductivity at infinite system sizes for $d < 3$. We can, thus, extend the result of Ref. [30] for the specific case of harmonic oscillator lattices to arbitrary dimensions d with an anomalous heat conduction with $\kappa \propto N$.

VIII. SUMMARY

We provide an analytical solution to the heat current and the nonequilibrium steady state of a chain of quantum harmonic oscillators, whose boundary oscillators are coupled to two heat baths of different temperatures, respectively. The heat current is independent of the chain length and, thus, is of ballistic nature exhibiting an anomalous heat current with a thermal conductivity proportional to the chain length. The diffusive heat transport regime, i.e., normal heat conduction with a constant thermal conductivity, is recovered when additional dephasing environments locally affect each of the oscillators. We observe the absence of entanglement in the chain for all parameter regimes. Finally, we provide the analytical expression for the heat current of a d -dimensional lattice of harmonic oscillators, which turns out to be the sum of the heat currents of all the chains, into which the lattice may be decomposed, each one connecting the two heat baths.

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- [1] Z. Rieder, J. L. Lebowitz, and E. Lieb, *J. Math. Phys.* **8**, 1073 (1967).
 - [2] J. Fourier, *Théorie Analytique de la Chaleur* (Didot, Paris, 1822).
 - [3] H. Nakazawa, *Prog. Theor. Phys. Suppl.* **45**, 231 (1970).

- [4] D. Roy and A. Dhar, *J. Stat. Phys.* **131**, 535 (2008).
- [5] L. W. Lee and A. Dhar, *Phys. Rev. Lett.* **95**, 094302 (2005).
- [6] K. Saito and A. Dhar, *Phys. Rev. Lett.* **104**, 040601 (2010).
- [7] D. M. Leitner and P. G. Wolynes, *Phys. Rev. E* **61**, 2902 (2000).

- [8] M. B. Plenio, J. Hartley, and J. Eisert, *New J. Phys.* **6**, 36 (2004).
- [9] F. Galve and E. Lutz, *Phys. Rev. A* **79**, 032327 (2009).
- [10] A. Rivas, A. D. Plato, S. Huelga, and M. B. Plenio, *New J. Phys.* **12**, 113032 (2010).
- [11] U. Zürcher and P. Talkner, *Phys. Rev. A* **42**, 3278 (1990).
- [12] C. Gaul and H. Büttner, *Phys. Rev. E* **76**, 011111 (2007).
- [13] A. Dhar, K. Saito, and P. Hänggi, *Phys. Rev. E* **85**, 011126 (2012).
- [14] K. Saito, S. Takesue, and S. Miyashita, *Phys. Rev. E* **61**, 2397 (2000).
- [15] J. Wu, *New J. Phys.* **12**, 083042 (2010).
- [16] D. Manzano, M. Tiersch, A. Asadian, and H. J. Briegel, *Phys. Rev. E* **86**, 061118 (2012).
- [17] M. Žnidarič, *Phys. Rev. E* **83**, 011108 (2011).
- [18] K. Saito, *Europhys. Lett.* **61**, 34 (2003).
- [19] G. Adam, *J. Mod. Opt.* **42**, 1311 (1995).
- [20] L.-A. Wu and D. Segal, *Phys. Rev. E* **77**, 060101(R) (2008).
- [21] M. Michel, G. Mahler, and J. Gemmer, *Phys. Rev. Lett.* **95**, 180602 (2005).
- [22] Y. Dubi and M. Di Ventra, *Phys. Rev. E* **79**, 042101 (2009).
- [23] M. Sarovar, A. Ishizaki, G. R. Fleming, and K. B. Whaley, *Nature Phys.* **6**, 462 (2010).
- [24] A. Ishizaki and G. R. Fleming, *New J. Phys.* **12**, 055004 (2010).
- [25] F. Fassioli and A. Olaya-Castro, *New J. Phys.* **12**, 085006 (2010).
- [26] T. Scholak, F. de Melo, T. Wellens, F. Mintert, and A. Buchleitner, *Phys. Rev. E* **83**, 021912 (2011).
- [27] M. Tiersch, S. Popescu, and H. J. Briegel, *Phil. Trans. R. Soc. A* **370**, 3771 (2012).
- [28] L. Yang, *Phys. Rev. Lett.* **88**, 094301 (2002).
- [29] R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, London, 1955).
- [30] S. Lepri, R. Livi, and A. Politi, *Phys. Rep.* **377**, 1 (2003).