Theoretical Quantum Optics

Problem Sheet

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Problem 26. Young double slit interference

Consider two 2-level atoms instead of the traditional two pin-holes in Young's double slit experiment. Let the two atoms and field be initially in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|eg\rangle + e^{i\varphi}|ge\rangle\right)|0\rangle\,, \tag{26.1}$$

where $\{|e\rangle, |g\rangle\}$ are exited and ground states of energies E_e, E_g, respectively, φ a phase, and $|0\rangle$ the vacuum of the field. Due to spontaneous emission the state will decay into a state which upon detection of a photon at point P has the form

$$|\psi(\infty)\rangle = \frac{1}{\sqrt{2}}|gg\rangle\left(|1_{\mathbf{k}}\rangle + e^{i\varphi}|1_{\mathbf{k}'}\rangle\right), \tag{26.2}$$

where $|1_{\mathbf{k}}\rangle$ means one photon in mode $\mathbf{k}$, and $\mathbf{k}$, $(\mathbf{k}')$ are wave-vectors pointing in directions $\overrightarrow{A_1P}$ $(\overrightarrow{A_2P})$, respectively, where A_1, A_2 are the positions of the two atoms, and $|\mathbf{k}| = |\mathbf{k}'| = (E_e - E_g)/(\hbar c)$.

a. Calculate the correlation function

$$G^{(1)}(\mathbf{r},\mathbf{r},t,t) = \langle\psi(\infty)|\,E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)\,|\psi(\infty)\rangle\,, \tag{26.3}$$

where $E^{(+)}(\mathbf{r},t) = \sum_{\mathbf{l}} \mathcal{E}_l a_{\mathbf{l}} \exp\{-i\omega_l t + i\mathbf{l}\cdot\mathbf{r}\}$, $E^{(-)}(\mathbf{r},t) = \left(E^{(+)}(\mathbf{r},t)\right)^\dagger$, $l = |\mathbf{l}|$, and $\mathcal{E}_l = [\hbar\omega_l/(2\epsilon_0 V)]^{1/2}$.

b. What happens to the interference pattern if the phase φ is randomly and evenly distributed over the interval $[0, 2\pi]$? What is the corresponding mixed initial state?

c. Show that the phase factor $e^{i\varphi}$ between the states $|1_{\mathbf{k}}\rangle$ and $|1_{\mathbf{k}'}\rangle$ in Hilbert space has exactly the same effect as a classical phase shift of the mode $\mathbf{k}'$,

$$\exp\{-i\omega_{k'}t + i\mathbf{k}'\cdot\mathbf{r}\} \to e^{i\varphi}e^{-i\omega_{k'}t + i\mathbf{k}'\cdot\mathbf{r}}.$$

d. Repeat the calculation in **a.** for two pin-holes radiating a superposition of two coherent states $|\alpha\rangle$ in the two modes $\mathbf{k}, \mathbf{k}'$, i.e.

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_{\mathbf{k}}|0\rangle_{\mathbf{k}'} + e^{i\varphi}|0\rangle_{\mathbf{k}}|\alpha\rangle_{\mathbf{k}'}\right) \tag{26.4}$$

for the field, and compare the interference pattern to the one obtained from (26.2).

Problem 27. General properties of correlation functions

The first and second order correlation functions $g^{(1)}$ and $g^{(2)}$ are defined as

$$g^{(1)}(\mathbf{r},\tau) = \frac{\langle E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t+\tau)\rangle}{\sqrt{\langle E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)\rangle\langle E^{(-)}(\mathbf{r},t+\tau)E^{(+)}(\mathbf{r},t+\tau)\rangle}}, \tag{27.1}$$

$$g^{(2)}(\mathbf{r},\tau) = \frac{\langle E^{(-)}(\mathbf{r},t)E^{(-)}(\mathbf{r},t+\tau)E^{(+)}(\mathbf{r},t+\tau)E^{(+)}(\mathbf{r},t)\rangle}{\langle E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)\rangle\langle E^{(-)}(\mathbf{r},t+\tau)E^{(+)}(\mathbf{r},t+\tau)\rangle}, \tag{27.2}$$

with $E^{(-)}(\mathbf{r},t) = \left(E^{(+)}(\mathbf{r},t)\right)^\dagger$.

a. Show that $|g^{(1)}(\mathbf{r},\tau)| \leqslant 1$.
Hint: You can start with the property $\mathrm{Tr}(\rho f^\dagger f) \geqslant 0$, with

$$f = \alpha E^{(+)}(r,t) + \beta E^{(+)}(r,t+\tau) \quad \forall \alpha,\beta$$

b. The field operators $E^{(+)}$ and $E^{(-)}$ are given in terms of annihilation operators $a_{\mathbf{k}}$ for mode $\mathbf{k}$ and given polarization, as

$$E^{(+)}(\mathbf{r},t) = \sum_{\mathbf{k}} \mathcal{E}_k a_{\mathbf{k}} \exp\{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}\}. \tag{27.3}$$

For Gaussian states Wick's theorem states that, for any two linear combinations (with complex amplitudes) a_1 and a_2 of annihilation operators $a_{\mathbf{k}}$, one has

$$\langle a_1^\dagger a_2^\dagger a_2 a_1\rangle = \langle a_1^\dagger a_2^\dagger\rangle\langle a_1 a_2\rangle + \langle a_1^\dagger a_1\rangle\langle a_2^\dagger a_2\rangle + \langle a_1^\dagger a_2\rangle\langle a_2^\dagger a_1\rangle. \tag{27.4}$$

Show that for thermal states this implies

$$g^{(2)}(\mathbf{r},\tau) = 1 + |g^{(1)}(\mathbf{r},\tau)|^2. \tag{27.5}$$

Problem 28. Properties of the covariance matrix

Let $x = (x_1,\cdots,x_n)^t \in \mathbb{R}^n$ be a random real vector. The covariance matrix $\Gamma = (\Gamma_{ij})_{i,j=1,\cdots,n}$ is given through

$$\Gamma_{ij} = \mathrm{cov}(x_i,x_j), \tag{28.1}$$

where the covariance of any two variables x,y is defined as $\mathrm{cov}(x,y) = \mathrm{E}\left[(x-\mathrm{E}(x))(y-\mathrm{E}(y))\right]$, and $\mathrm{E}(x)$ is the expectation value (=mean value) of x.

a. Show that $\mathrm{cov}(x,y) = \mathrm{E}(xy) - \mathrm{E}(x)\mathrm{E}(y)$.

b. Show that Γ is a real, symmetric, positive semi-definite matrix.

c. A multivariate Gaussian distribution is defined as

$$p(x;\mu,\Gamma) = \sqrt{\frac{1}{(2\pi)^n \det\Gamma}} \exp\left[-\frac{1}{2}(x-\mu)^t\,\Gamma^{-1}(x-\mu)\right], \tag{28.2}$$

where $\mu = \mathrm{E}(x)$, and Γ is the covariance matrix introduced above which we assume to have full-rank.

i. Show that $p(x; \mu, \Gamma)$ is normalized correctly, i.e. $\int \mathrm{d}^n x\, p(x) = 1$.
Hint: Diagonalize Γ.

ii. We define the multivariate characteristic function

$$\chi(k) = \int \mathrm{d}^n x\, e^{ikx}\, p(x), \qquad (k \in \mathbb{R}^n), \tag{28.3}$$

i.e. $\chi(k)$ is the multidimensional Fourier transformation of the probability distribution $p(x)$, $(x \in \mathbb{R}^n)$. Show that

$$\mathrm{E}(x_j) = -i\,\frac{\partial}{\partial k_j}\chi(k)\bigg|_{k=0}, \quad \mathrm{E}(x_i x_j) = -\frac{\partial^2}{\partial k_i \partial k_j}\chi(k)\bigg|_{k=0}. \tag{28.4}$$

d. Calculate $\chi(k)$ explicitly for the multivariate Gaussian $p(x; \mu, \Gamma)$.

e. Calculate $\mathrm{E}(x_i)$ and $\mathrm{cov}(x_i, x_j)$ for the multivariate Gaussian. What is hence the meaning of μ and Γ in Eq. (28.2)?