EXACT STEADY-STATE DENSITY OPERATOR FOR A COLLECTIVE ATOMIC SYSTEM IN AN EXTERNAL FIELD

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An exact steady-state density operator is obtained for a model describing the collective behaviour of a system of N two-level atoms driven by a classical field. This is used to obtain the exact steady-state expectation value of the atomic population difference for any N.

In a recent paper, Narducci et al. [1] have analysed the transient and steady-state properties of a model consisting of an ensemble of N identical two-level atoms driven by a resonant external field. The time evolution of such a system is described by the reduced atomic density operator ρ_A which obeys the master equation [2–4]

$$\partial\rho_A/\partial t = -i\Omega[S^+ + S^-, \rho_A] + 2\gamma[S^-\rho_A S^+ - \tfrac{1}{2}\rho_A S^+S^- - \tfrac{1}{2}S^+S^-\rho_A] , \tag{1}$$

where the parameters Ω and γ represent the Rabi frequency of the external field and the single-atom decay coefficient, respectively. The operators $S^\pm = \Sigma_j S_j^\pm$ are collective atomic dipole operators. Further defining the atomic energy operator $S_z = \Sigma_j S_{zj}$ one has the usual angular momentum commutation relations

$$[S^+, S^-] = 2S_z, \quad [S_z, S^\pm] = \pm S^\pm . \tag{2}$$

By converting the operator equation (1) into a c-number differential equation, Narducci et al. have subsequently obtained numerical solutions of the problem. In this paper we derive an exact form for the steady-state density operator ρ_A^{ss} for any N. It must be mentioned here that to our knowledge an exact expression for ρ_A^{ss} is hitherto available only for $N = 1$ and $N = 2$ [2].

We first introduce the eigenstates of the total angular momentum operator S^2 and the component S_z:

$$S^2|p\rangle = \sqrt{S(S+1)}\,|p\rangle , \quad S_z|p\rangle = (S-p)|p\rangle \quad (0 \leqslant p \leqslant 2S) . \tag{3}$$

We assume that all the atoms are in the ground state just before the field is switched on. The conservation of total angular momentum then implies that $S = N/2$ and that ρ_A operates in a $(N+1)$-dimensional manifold of collective atomic states. It is then convenient to use the representation of the "spin coherent states" [5] defined by

$$|\mu\rangle = (1 + |\mu|^2)^{-S}\, e^{\mu S^-}|0\rangle = (1 + |\mu|^2)^{-S} \sum_{p=0}^{2S} \left(\frac{2S!}{p!(2S-p)!}\right)^{1/2} \mu^p |p\rangle , \tag{4}$$

having the properties

$$\langle\lambda|\mu\rangle = (1 + \lambda^*\mu)^{2S}/(1 + |\mu|^2)^S(1 + |\lambda|^2)^S ,$$

$$\frac{2S+1}{\pi}\int \frac{d^2\mu}{(1+|\mu|^2)^2}\, |\mu\rangle\langle\mu| = 1 , \tag{5}$$

$$\langle\lambda|S^+|\mu\rangle = (2S\mu/(1 + \lambda^*\mu))\,\langle\lambda|\mu\rangle ,$$

$$\langle\lambda|S^-|\mu\rangle = (2S\lambda^*/(1 + \lambda^*\mu))\,\langle\lambda|\mu\rangle .$$

Using eq. (5) and the relation

$$\frac{2S+1}{\pi}\int \frac{d^2\lambda}{(1+|\lambda|^2)^{2S+2}}\, \lambda^{*m}\lambda^n = \delta_{mn}(2S-m)!\,m!/(2S)! \quad (0 \leqslant m, n \leqslant 2S) ,$$

it is straightforward to obtain the matrix elements

$$\langle\mu|S^-\rho_A|\mu\rangle = (1+|\mu|^2)^{-2S}[2S\mu^* - \mu^{*2}\partial/\partial\mu^*]\rho_A(\mu^*,\mu)\,,$$

$$\langle\mu|\rho_A S^-|\mu\rangle = (1+|\mu|^2)^{-2S}(\partial/\partial\mu)\rho_A(\mu^*,\mu)\,, \tag{6}$$

and their complex conjugates. Here $\rho_A(\mu^*,\mu) = \langle\mu|\rho|\mu\rangle(1+|\mu|^2)^{2S}$. It is clear that since $(S^\pm)^{2S+1} = 0$, $\rho_A(\mu^*,\mu)$ is a polynomial of degree at most $2S$ in μ^* and μ. The master equation (1) then reduces to

$$\partial\rho_A(\mu^*,\mu)/\partial\tau = (2S\mu^* - \mu^{*2}\partial/\partial\mu^* - \partial/\partial\mu) \times (2S\mu - \mu^2\partial/\partial\mu + g^*)\rho_A + \text{c.c.}\,, \tag{7}$$

where $g = (-)\mathrm{i}\Omega/\gamma$ and $\tau = \gamma t$. The steady-state solution $(\partial\rho_A/\partial t = 0)$ of eq. (7) may now be shown to be given by

$$\rho_A^{ss}(\mu^*,\mu) = N_f \sum_{m,n=0}^{2S} (g^*)^{2S-m}(g)^{2S-n} \times (2S\mu^* - \mu^{*2}\partial/\partial\mu^*)^m(2S\mu - \mu^2\partial/\partial\mu)^n \times (1+|\mu|^2)^{2S}\,, \tag{8}$$

where N_f is the normalization factor. Expression (8) may now be cast in the operator form

$$\rho_A^{ss} = N_f \sum_{m,n=0}^{2S} (g^*)^{2S-m}(g)^{2S-n}(S^-)^m(S^+)^n\,, \tag{9}$$

with

$$N_f^{-1} = \mathrm{Tr} \sum_{m,n=0}^{2S} (g^*)^{2S-m}(g)^{2S-n}(S^-)^m(S^+)^n = \sum_{p=0}^{2S}\sum_{k=0}^{p} \frac{(2S-p+k)!p!}{(p-k)!(2S-p)!}|g|^{2(2S-k)}\,. \tag{10}$$

It is of interest to obtain the limiting behaviour of ρ_A^{ss}. One can show that

$$\rho_A^{ss} \xrightarrow[|g|\to 0]{} (S^-)^{2S}(S^+)^{2S}\,, \tag{11a}$$

and

$$\rho_A^{ss} \xrightarrow[|g|\to\infty]{} I\,, \tag{11b}$$

where I is the identity operator. The exact expression (9) is rather convenient to use for obtaining the steady-state expectation values of any atomic operator. For example, the steady-state expectation value of the quantity $(S_z/N)^m$ will be given by

$$\langle S_z^m\rangle/N^m = N^{-m}\mathrm{Tr}(S_z^m\rho_A^{ss}) = \left(\frac{N_f}{N^m}\right)\sum_{p=0}^{2S}\sum_{k=0}^{p}(S-p)^m \times \frac{(2S-p+k)!p!}{(p-k)!(2S-p)!}|g|^{2(2S-k)}\,. \tag{12}$$

From expression (12) one may show that for any finite N the steady-state expectation value $\langle S_z\rangle/N$ of the atomic population difference per atom varies smoothly between $-1/2$ (as $|g|\to 0$) and zero (as $|g|\to\infty$). The detailed behaviour is shown in fig. 1 where $\langle S_z\rangle/N$ is plotted as a function of the scaled parameter $\beta = 2|g|/N$. As discussed in ref. [1] one observes a smooth variation of $\langle S_z\rangle/N$ with β for all finite values of N. For $N\to\infty$, β remaining finite, the atomic system seems to exhibit a sharp transition at $\beta = 1$ reminiscent of a typical second-order phase transition [1]. This is indicated in fig. 1 by dash–dotted curve.

Finally, it is of interest to consider the fluctuations of the atomic population difference in steady state given by

$$\langle\Delta S_z^2\rangle/N^2 = (\langle S_z^2\rangle - \langle S_z\rangle^2)/N^2\,.$$

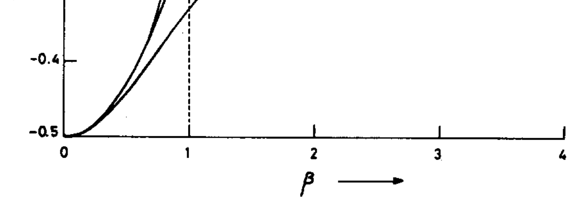

Fig. 1. Steady-state expectation value of the atomic population difference (per atom) as a function of $\beta = 2\Omega/\gamma N$. The curves marked 1 to 4 correspond to $N = 1, 3, 50, 125$, respectively. The dash–dotted curve indicates the expected behaviour as $N\to\infty$.

Using expression (12), it is seen that for any N this quantity varies between zero and $(1 + 2/N)/12$ corresponding to $\beta = 0$ and $\beta \to \infty$, respectively.

References

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