Theoretical Quantum Optics

Problem Sheet

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Sheet: 2	**Hand-out:** 19.10.23	**Hand-in:** 26.10.23

Problem 3. Coherent states and the shifted harmonic oscillator

Consider the coherent state $|\alpha\rangle$ of the harmonic oscillator with Hamiltonian $H = \hbar\omega(a^\dagger a + 1/2)$ where ω is the frequency of the oscillator system given by,

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

a. Show that the coherent state $|\alpha\rangle$ can be written as,

$$|\alpha\rangle = D(\alpha)|0\rangle ,$$

where $|0\rangle$ is the vacuum state with the shift operator,

$$D(\alpha) = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha a^\dagger} \mathrm{e}^{-\alpha^* a}. \tag{3.1}$$

b. Show that the two forms of the shift operator,

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a} \qquad \text{and} \qquad D(\alpha) = \mathrm{e}^{|\alpha|^2/2} \mathrm{e}^{-\alpha^* a} \mathrm{e}^{\alpha a^\dagger}, \tag{3.2}$$

are equivalent by using the Baker-Campbell-Hausdorff (BCH) formula,

$$\mathrm{e}^{A+B} = \mathrm{e}^{-[A,B]/2} \mathrm{e}^{A} \mathrm{e}^{B},$$

valid for $[[A,B],A] = [[A,B],B] = 0$.

c. Show that the shift operator $D(\alpha)$ is a unitary operator. What is $D^{-1}(\alpha)$?

d. Show that the shift operator $D(\alpha)$ is indeed a shift operator in the sense that,

$$D^{-1}(\alpha) a D(\alpha) = a + \alpha \qquad \text{and} \qquad D^{-1}(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^*.$$

Hint: Use Eq. (3.2) for $D^{-1}(\alpha)$ and Eq. (3.1) for $D(\alpha)$ and the general BCH formula,

$$\mathrm{e}^{-\alpha A} B \mathrm{e}^{\alpha A} = B - \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \cdots , \tag{3.3}$$

valid for all operators A and B.

Problem 4. Squeezed states

A coherent state $|\alpha\rangle$ of a harmonic oscillator can be defined by $a|\alpha\rangle = \alpha|\alpha\rangle$ with the annihilation operator a. The connection to the dimensionless position x and impulse p operators is,

$$x = \frac{a + a^\dagger}{2}, \qquad p = \frac{a - a^\dagger}{2\mathrm{i}}.$$

a. Show that the coherent states $|\alpha\rangle$ are states of minimal uncertainty, with the same uncertainty in the dimensionless position x and impulse p, such that,

$$\langle\alpha|(\Delta x)^2|\alpha\rangle = \langle\alpha|(\Delta p)^2|\alpha\rangle = \frac{1}{4},$$

with $\langle(\Delta A)^2\rangle = \langle A^2\rangle = \langle A\rangle^2$ for all operators A.

b. We defined the squeeze operator $S(\xi)$ for all $\xi \in \mathbb{C}$ by,

$$S(\xi) = \mathrm{e}^{-A(\xi)}, \qquad A(\xi) = \frac{\xi(a^\dagger)^2 - \xi^* a^2}{2}.$$

i. Show that $S^\dagger(\xi) = S(-\xi)$.

ii. Show that $S(\xi)S^\dagger(\xi) = \mathbb{1}$ where $\mathbb{1}$ is the identity operator.

c. Set $\xi = r\mathrm{e}^{\mathrm{i}\theta}$ and show, with the help of the BCH formula in Eq. (3.3), that,

$$S^\dagger(\xi)aS(\xi) = a\cosh(r) - a^\dagger e^{\mathrm{i}\theta}\sinh(r) \qquad \text{and} \qquad S^\dagger(\xi)a^\dagger S(\xi) = a^\dagger\cosh(r) - a\mathrm{e}^{-\mathrm{i}\theta}\sinh(r).$$

d. Let us introduce the new Hermitian operators x' and p', defined by,

$$x' + \mathrm{i}p' = (x + \mathrm{i}p)\mathrm{e}^{-\mathrm{i}\theta/2}. \tag{4.1}$$

i. Show that the hermiticity of the operators x' and p' and, of course, x and p, implies the transformation,

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

ii. Show that $\langle\alpha|(\Delta x')^2|\alpha\rangle = \langle\alpha|(\Delta x)^2|\alpha\rangle$.

e. We now introduce the operators $\tilde{x}$ and $\tilde{p}$ given by,

$$\tilde{x} = S^\dagger(\xi)x'S(\xi) \qquad \text{and} \qquad \tilde{p} = S^\dagger(\xi)p'S(\xi).$$

Show that,

$$\tilde{x} = x'\mathrm{e}^{-r} \qquad \text{and} \qquad \tilde{p} = p'\mathrm{e}^{r}.$$

Hint: You should express $S^\dagger(\xi)xS(\xi)$ and $S^\dagger(\xi)pS(\xi)$ in terms of x and p and then use the result from Eq. (4.1) for the transformation x' and p'.

f. We define a squeezed coherent state $|\alpha,\xi\rangle$ by $|\alpha,\xi\rangle = S(\xi)\,|\alpha\rangle$.

i. Show that for all $n \in \mathbb{N}$ that,

$$\langle\alpha,\xi|(x')^n|\alpha,\xi\rangle = \langle\alpha|(\tilde{x})^n|\alpha\rangle \qquad \text{and} \qquad \langle\alpha,\xi|(p')^n|\alpha,\xi\rangle = \langle\alpha|(\tilde{p})^n|\alpha\rangle\,.$$

ii. Deduce that,

$$\langle\alpha,\xi|(\Delta x')^2|\alpha,\xi\rangle = \frac{\mathrm{e}^{-2r}}{4} \qquad \text{and} \qquad \langle\alpha,\xi|(\Delta p')^2|\alpha,\xi\rangle = \frac{\mathrm{e}^{2r}}{4},$$

and give an interpretation of this result.