Theoretical Quantum Optics

Problem Sheet

Lecturer: Prof. Dr. Igor Lesanovsky **Semester:** Winter 23/24

Sheet: 5 **Hand-out:** 9.11.23 **Hand-in:** 16.11.23

Problem 11. Symmetric order and the Wigner function

The Wigner function for a harmonic oscillator can be defined as,

$$W_{\mathrm{h.o}}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\ \mathrm{e}^{-\beta\alpha^*+\beta^*\alpha}\mathrm{Tr}\big(\mathrm{e}^{-|\beta|^2/2}\mathrm{e}^{\beta a^\dagger}\mathrm{e}^{-\beta^* a}\rho\big),$$

where $\alpha,\beta\in\mathbb{C}$ and $\mathrm{d}^2\beta = \mathrm{d}\beta\mathrm{d}\beta^*$ is the integration over the complex plane.

a. Determine the Wigner function $W_{\mathrm{c.s.}}(\alpha,\alpha^*)$ corresponding to a coherent state $|\alpha_0\rangle$ with density operator $\rho_{\mathrm{c.s.}} = |\alpha_0\rangle\langle\alpha_0|$.

b. Show that for a harmonic oscillator in a coherent state $|\alpha_0\rangle$ that,

$$\frac{\langle\alpha_0|a^\dagger a + aa^\dagger|\alpha_0\rangle}{2} = \int \mathrm{d}^2\alpha\ W_{\mathrm{c.s.}}(\alpha,\alpha^*)|\alpha|^2,$$

holds, where $W_{\mathrm{c.s.}}(\alpha,\alpha^*)$ is the Wigner function corresponding to the coherent state.

Problem 12. Shift operator and quasi-probability distributions

Show that for the shift operator,

$$D(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^* a},$$

we have,

$$\mathrm{Tr}(D(\alpha)) = \pi\delta^{(2)}(\alpha) \qquad \text{and} \qquad \mathrm{Tr}(D(\alpha)D^\dagger(\beta)) = \pi\delta^{(2)}(\alpha-\beta),$$

where $\delta^{(2)}(\alpha)$ is the Dirac delta function in the complex plane,

$$\delta^{(2)}(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi\ \mathrm{e}^{\alpha\xi^*-\alpha^*\xi},$$

that is, $\delta^{(2)}(\alpha) = \delta(a)\delta(b)$ with $\alpha = a + \mathrm{i}b$ for $a,b\in\mathbb{R}$.

Problem 13. Space inversion and time reversal for the Wigner function

Determine how the Wigner function,

$$W(q,p) = \frac{1}{2\pi\hbar}\int \mathrm{d}\xi\ \mathrm{e}^{-\mathrm{i}p\xi/\hbar}\left\langle q+\xi/2|\rho|q-\xi/2\right\rangle, \tag{13.1}$$

transforms under the following transformations.

a. Space inversion, $q\mapsto -q$.

b. Time reversal, $t\mapsto -t$.

Hint: Decompose ρ into a mixture of pure states $|\psi_i\rangle$ and find out how the wavefunction $\psi(q)$ transforms in these situations. Recall that time reversal will result in $\langle q|\rho(-t)|q'\rangle = \langle q|\rho(t)|q'\rangle^*$ due to the anti-unitarity of the time reversal operator which results from classical requirements imposed on the operator.
