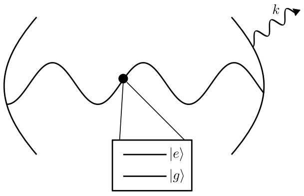
2 Cavity QED



What we have studied until now is an atom coupled to the quantized electromagnetic field, with infinitely many modes. Now we will consider what happens when the atom interacts with a single-mode field instead, like in the case of microwave or optical cavities. In this chapter, we will study first how a single atom interacts with the cavity (that contains a single-mode quantum field), then add coupling of the cavity to the electromagnetic field (damped cavity). We will then study two models that describe the interaction of many atoms with a cavity: the Tavis Cummings and Dicke models.

2.1 The Jaynes-Cummings model

We will be considering the interaction of a two-level atom with a quantized field, so we have again the same Hamiltonian as in Section 1.3, i.e.

$$H' = H_0 + H_F - \vec{d} \cdot \vec{E} \ ,$$

where again our two-level atom is described by

$$H_0 = \hbar\omega_0 \sigma^\dagger \sigma$$

and the field Hamiltonian reads now

$$H_F = \hbar\nu a^\dagger a,$$

where a and $a^\dagger$ are the bosonic creation and annihilation operator of a photon with frequency ν, and $\left[a, a^\dagger\right] = 1$. To fully determine the interaction part of the Hamiltonian, we will need to write explicitly the electric field, which here reads

$$\vec{E} = \hat{\epsilon}\sqrt{\frac{\hbar\nu}{2\epsilon_0 V}} U(\vec{R}) \left(a + a^\dagger\right) .$$

Here, we have considered that the mode of the cavity changes spatially with the position inside the cavity $\vec{R}$, with the variation encoded in $U(\vec{R})$. For example, this variation could be, in the case of a standing wave in the z direction, given by $U(\vec{R}) = \sin(kZ)$, where $k = \nu/c$ is the corresponding wave vector. The exact variation with the position in the cavity will become more important later on, when we have more atoms in the cavity, such that each one "feels" a different field.

Considering all of the above, the total Hamiltonian reads

$$H' = \hbar\omega_0 \sigma^\dagger \sigma + \hbar\nu a^\dagger a + \hbar g(\vec{R}) \left(\sigma^\dagger + \sigma\right) \left(a^\dagger + a\right)$$

with

$$g(\vec{R}) = - \langle g|\vec{d}|e\rangle \cdot \hat{\epsilon} \sqrt{\frac{\nu}{2\epsilon_0 \hbar V}} U(\vec{R})$$

being the coupling function between the atom and the cavity. We can skip the next steps, as they are the same as we have already done before:

A) Go into a rotating frame with the frequency of the cavity and the atom, i.e., perform a unitary transformation $U = e^{i(H_0+H_F)t/\hbar}$ to the Hamiltonian, such that it reads

$$H'' = \hbar g \left[e^{i\Delta t}\sigma^\dagger a + e^{-i\Delta t}\sigma a^\dagger + e^{i(\omega_0+\nu)t}\sigma^\dagger a^\dagger + e^{-(\omega_0+\nu)t}\sigma a\right],$$

with the detuning between the atom and the cavity being $\Delta = \omega_0 - \nu$.

B) Perform the rotating wave approximation by neglecting the action of the fast rotating terms, such that we end up with

$$\boxed{H_{JC} = \hbar g \left[e^{i\Delta t}\sigma^\dagger a + e^{-i\Delta t}\sigma a^\dagger\right]}$$

which is the so-called *Jaynes-Cummings* Hamiltonian (in the interaction picture).

2.1.1 Time-evolution

Our next step will be to solve the dynamics of this system. In particular, here we will need to solve the Schrödinger equation

$$i\hbar \frac{\partial |\Psi\rangle}{\partial t} = H_{JC} |\Psi\rangle$$

with Hamiltonian H_{JC} and wave function

$$|\Psi(t)\rangle = \sum_n \left[c_{en}(t) |en\rangle + c_{gn}(t) |gn\rangle\right]$$

where our basis now is formed by the states

- $|en\rangle \equiv |e\rangle \otimes |n\rangle \Rightarrow$ atom in $|e\rangle$ and n photons in the cavity
- $|gn\rangle \equiv |g\rangle \otimes |n\rangle \Rightarrow$ atom in $|g\rangle$ and n photons in the cavity.

Here, the states $|n\rangle$ are the eigenvectors of the bare field Hamiltonian H_F, and the action of the ladder operators on them reads (as, for example, the harmonic oscillator!)

- $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$
- $a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$.

Let us now then substitute both the Hamiltonian and the wave function in the Schrödinger equation, such that

$$\begin{aligned} i\hbar\sum_n\left[\frac{\partial c_{en}}{\partial t}\,|en\rangle + \frac{\partial c_{gn}}{\partial t}\,|gn\rangle\right] &= \hbar g\sum_n\Bigg[c_{en}\left(e^{i\Delta t}\overbrace{\sigma^\dagger a\,|en\rangle}^{\sigma^\dagger|e\rangle=0} + e^{-i\Delta t}\sigma a^\dagger\,|en\rangle\right) \\ &\quad + c_{gn}\left(e^{i\Delta t}\sigma^\dagger a\,|gn\rangle + e^{-i\Delta t}\overbrace{\sigma a^\dagger\,|gn\rangle}^{\sigma|g\rangle=0}\right)\Bigg] \\ &= \hbar g\sum_n\left[c_{en}e^{-i\Delta t}\sqrt{n+1}\,|gn+1\rangle + c_{gn}e^{i\Delta t}\sqrt{n}\,|en-1\rangle\right] \ . \end{aligned}$$

Now, if we look back to our Hamiltonian, we can see that the only process that it drives is the exchange of an excitation in the atom with a photon in the cavity. This can be seen mathematically by observing that, by multiplying from the left by $\langle en|$ and $\langle gn+1|$, which gives two coupled equations

- $\langle en| \Rightarrow i\hbar\frac{\partial c_{en}}{\partial t} = \hbar g e^{i\Delta t}\sqrt{n+1}c_{gn+1}$ (*)
- $\langle gn+1| \Rightarrow i\hbar\frac{\partial c_{gn+1}}{\partial t} = \hbar g e^{-i\Delta t}\sqrt{n+1}c_{en}$ (*)

where we have used $\langle g|e\rangle = 0$ and we have also shifted $n \to n+1$ whenever necessary.

We have seen very similar equations before, such as in the case of an atom coupled to a classical field, Section 1.2. Now that we have the equations, we need again initial conditions to solve them. For example, when the atom is considered in the excited state initially, i.e. $c_{gn+1}(0) = 0$ and $c_{en}(0) = 1$ we get

- $c_{en}(t) = c_n(0)\left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{i\Delta t/2}$

- $c_{gn+1}(t) = -c_n(0)\frac{2ig\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{-i\Delta t/2}$

where $c_n(0)$ is the probability amplitude for the field alone and

$$\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}$$

plays a role similar to Ω (Rabi frequency) in Section 1.2.

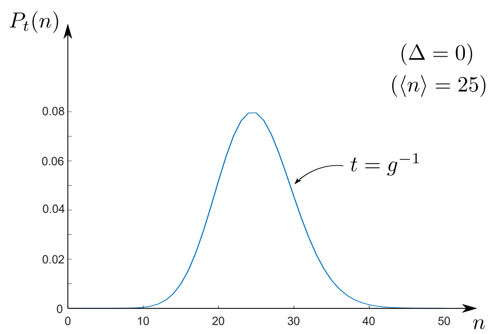
Now we are in position to extract information on the state of the atom as a function of time (which will depend on the initial cavity occupation, i.e. how many photons n are there initially) and also on the photon occupation of the cavity as a function of time. Let us start by calculating the photon occupation in the cavity. One can obtain it by calculating:

$$\begin{aligned} P_t(n) &= |c_{en}(t)|^2 + |c_{gn}(t)|^2 \\ &= |c_n(0)|^2\left[\cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Omega_n t}{2}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)\right] + |c_{n-1}(0)|^2\frac{4g^2 n}{\Omega_{n-1}^2}\sin^2\left(\frac{\Omega_{n-1} t}{2}\right) \ . \end{aligned}$$

Here, $|c_n(0)|^2$ is the probability of there being n photons in the cavity initially. A good approximation for this is given by the coherent state

$$|c_n(0)|^2 = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!},$$

which is a Poisson distribution with mean value $\langle n\rangle$ that can be represented graphically as



Question(*):

Does the distribution vary with time? If so, how? Does it vary with $\langle n\rangle$?

The dynamics of the internal states can also be easily obtained as

$$P_e(t) = \sum_{n=0}^{\infty} |c_{en}(t)|^2 = \sum_{n=0}^{\infty} |c_n(0)|^2 \left[\cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)\right] .$$

For $P_g(t)$ it is possible to find a similar equation. This equation still needs the input of the initial occupation of the cavity. Let us first have a look at what happens when $|c_n(0)|^2 = \delta_{n0}$, that is, there are initially no photons in the cavity. In this case, the system can only be in states where $n = 0$ or $n = 1$, and hence the probability of being in the excited state undergoes Rabi-like oscillations:

$$P_e(t) = \cos^2\left(\frac{\Omega_0 t}{2}\right) + \left(\frac{\Delta}{\Omega_0}\right)^2 \sin^2\left(\frac{\Omega_0 t}{2}\right)$$

with $\Omega_0 = \sqrt{\Delta^2 + 4g^2}$. However, as $\langle n \rangle$ is increased, $P_e(t)$

a) Decays to $P_e(t) \approx \frac{1}{2}\left(1 + \frac{\Delta^2}{\Delta^2 + 4g^2\langle n \rangle}\right)$ after a time t_c

b) Experience **revivals** after a time t_r.

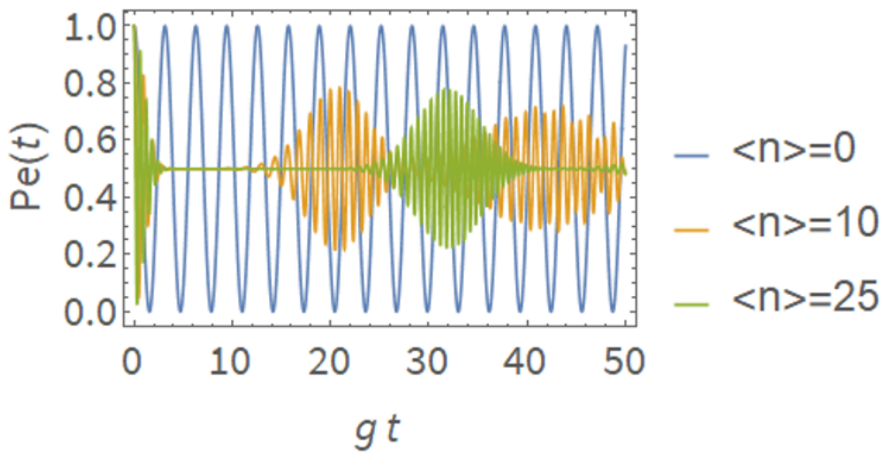


Figure 3: On resonance.

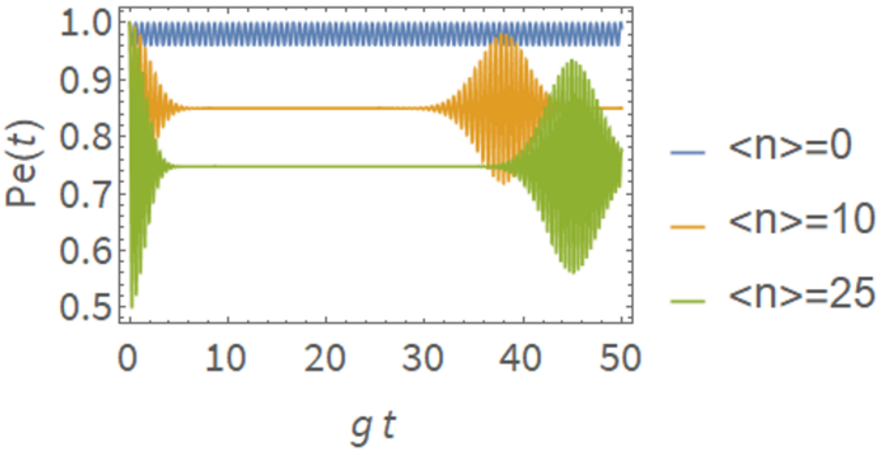


Figure 4: Out of resonance.

The reason for this behaviour can be understood by seeing $P_e(t)$ as a sum of many terms with slightly different Rabi frequencies Ω_n. Like this, one can estimate the decay time t_c as the time when oscillations with different value of n become uncorrelated. In the limit of $\langle n\rangle \gg 1$, $\Omega_n \approx \Omega_{\langle n\rangle}$, and since the photon distribution is a Poissonian, $\Delta n = \sqrt{\langle n\rangle}$, such that

$$\left(\Omega_{\langle n\rangle+\sqrt{\langle n\rangle}} - \Omega_{\langle n\rangle-\sqrt{\langle n\rangle}}\right) t_c \approx 1$$

which gives

$$t_c \approx \frac{1}{2g}\sqrt{1+\frac{\Delta^2}{4g^2\langle n\rangle}}$$

so for $\Delta = 0$, t_c is independent of $\langle n\rangle$, but for $\Delta \neq 0$ the larger $\langle n\rangle$ the faster the decay. Similarly, the revival time can be calculated as the time at which the neighbouring Ω_n give rise to a phase difference multiple of 2π, i.e.,

$$m \in \mathbb{Z} \qquad t_r \approx \frac{2\pi m}{\Omega_{\langle n\rangle} - \Omega_{\langle n\rangle-1}} \approx \frac{2\pi m\sqrt{\langle n\rangle}}{g}\sqrt{1+\frac{\Delta^2}{4g^2\langle n\rangle}}\ ,$$

which shows revivals happening at regular intervals.

2.1.2 Dressed states

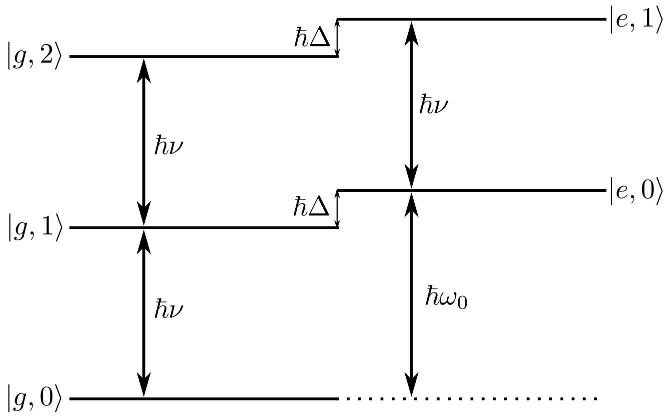
So far we have been analysing the dynamics of the system. Instead, as we did in Section 1.2 for an atom coupled to a (classical) laser field, we might learn something from looking at the eigenvalues and eigenstates of the Hamiltonian, the *dressed* state picture.

To do so, it is convenient to undo the unitary transformation and start form the

Hamiltonian

$$H' = \underbrace{\hbar\omega_0\sigma^\dagger\sigma + \hbar\nu a^\dagger a}_{H_0'} + \underbrace{\hbar g\left(\sigma^\dagger a + \sigma a^\dagger\right)}_{V'}$$

where the RWA has already been done. Without interactions, the eigenstates of H_0' are simply combinations of $\{|g\rangle, |e\rangle\}$ with $\{|n\rangle\}_{n=0}^{\infty}$, such that the lowest energy states are the ones depicted below.



As we said before, only the states $|g, n+1\rangle$ and $|e, n\rangle$ are coupled by the interaction V'. Hence, for fixed number of photons n, the dynamics is determined, in the $\{|g, n+1\rangle, |e, n\rangle\}$ basis, by

$$H'_{(n)} = \hbar \begin{pmatrix} n\nu + \omega_0 & \sqrt{n+1}g \\ \sqrt{n+1}g & (n+1)\,\nu \end{pmatrix} ,$$

which can be then diagonalized, to obtain that the new eigenenergies of the system are

$$E_\pm = \frac{1}{2}\hbar\omega_0 + \left(n + \frac{1}{2}\right)\hbar\nu \pm \frac{\hbar}{2}\overbrace{\sqrt{\Delta^2 + 4g^2\,(n+1)}}^{\Omega_n}$$

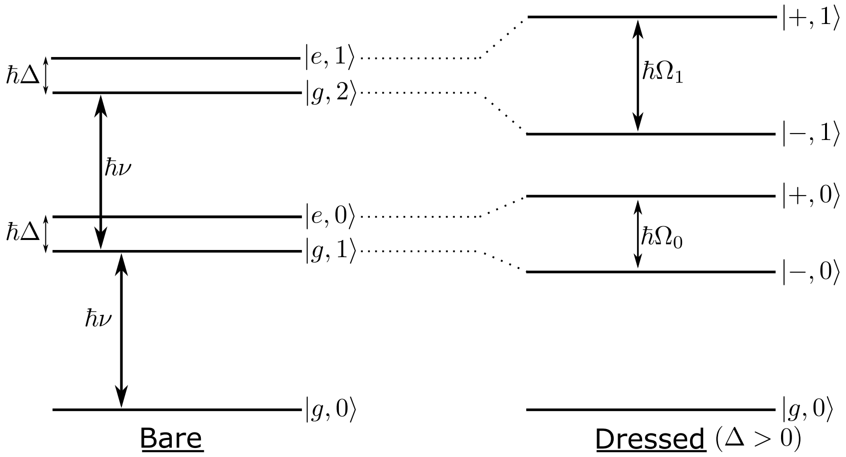
with corresponding states

- $|+, n\rangle = \sin(\theta_n)\,|g, n+1\rangle + \cos(\theta_n)\,|e, n\rangle$
- $|-, n\rangle = \cos(\theta_n)\,|g, n+1\rangle - \sin(\theta_n)\,|e, n\rangle$

where the angle θ_n is

$$\tan(2\theta_n) = -\frac{2g\sqrt{n+1}}{\Delta} \qquad (0 \leq \theta_n < \pi/2)\,.$$

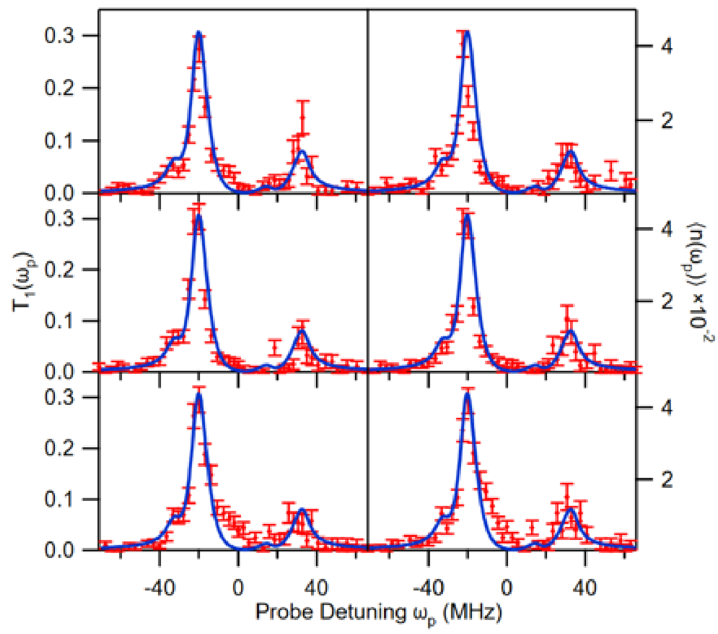
Let us now compare these energies with the ones of the bare states

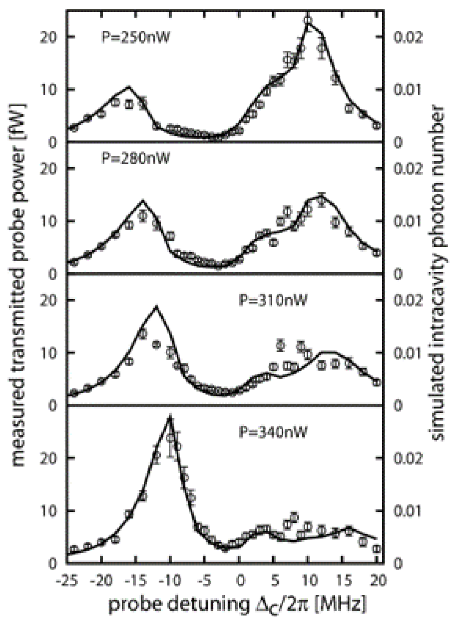


Each pair of dressed states form a *Jaynes-Cummings doublet*. In the limit of $\Delta = 0$, the bare states are degenerate while the dressed are forming these doublets, which can be observed experimentally, for example, in the following papers:

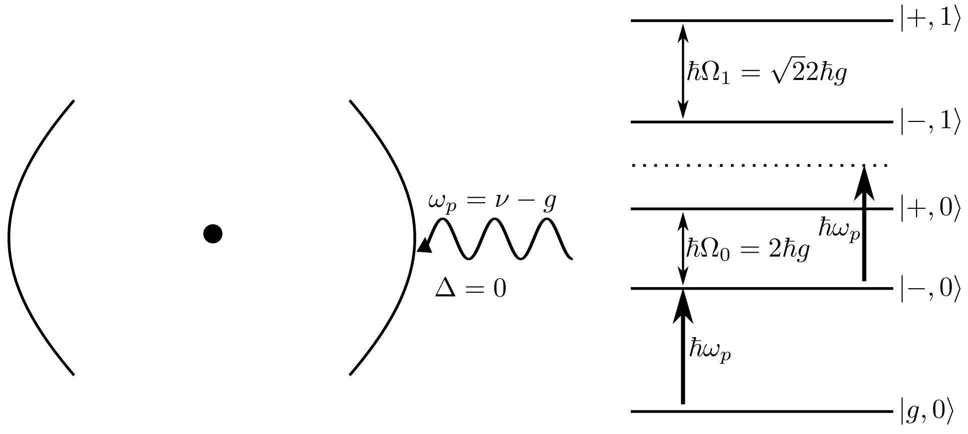
- PRL 93, 233603 (2004)
- PRL 94, 033002 (2005)

whose experimental data you can see below.





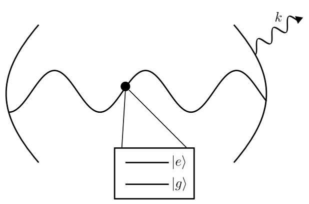
An interesting application of this doubling is the so-called *photon blockade.* The idea is that one makes it impossible for two photons to be simultaneously in the cavity.



Initially, the state of the system is $|g, 0\rangle$, i.e., no photons in the cavity and the atom in the ground state. A *probe* laser field excites the atom, but is detuned, such that it hits exactly the state $|-, 0\rangle$ (that actually has 1 photon in the cavity). The laser is then off-resonant with respect to any other transition, and hence only 1 photon can be absorbed at a time. This can also be observed experimentally (see Nature 436, 87 (2005)).

2.2 Damped cavity

Up to now we have considered that everything is *coherent*, that is, no losses have been accounted for and one can describe the system with a Schrödinger equation. However, in reality there are indeed losses due to the contact of the system (both the atom and the cavity) with the radiation field.



The dissipation process we will consider in this section is the case where the cavity is damped, that is, the cavity can lose photons into the environment (again the electromagnetic field, as we did in section 1.3). To account for these loses, consider that the Hamiltonian which describes only the cavity and the radiation field reads

$$H'_{CR} = \underbrace{\hbar\nu a^\dagger a}_{\text{cavity}} + \underbrace{\hbar\sum_{\vec{k}} \nu_k b^\dagger_{\vec{k}} b_{\vec{k}}}_{\text{reservoir (e.m. field)}} + \hbar\sum_{\vec{k}} \kappa_{\vec{k}} \left(b^\dagger_{\vec{k}} a + a^\dagger b_{\vec{k}} \right)$$

where the RWA has been done already, and where $\kappa_{\vec{k}}$ represents the coupling strength between the cavity and the field. We now follow the same steps as in section 1.3 in order to obtain a master equation for the cavity degrees of freedom only. The final master equation reads here

$$\dot{\rho}_C = \kappa \left[a \rho_C a^\dagger - \frac{1}{2} \left\{ a^\dagger a, \rho_C \right\} \right]$$

where ρ_C contains the cavity degrees of freedom and κ is the cavity damping rate.

Now we can obtain an equation that describes the system of the atom plus the damped cavity as:

$$\dot{\rho} = -\frac{i}{\hbar} \left[\overbrace{\hbar \omega_0 \sigma^\dagger \sigma + \hbar \nu a^\dagger a + \hbar g \left(a^\dagger \sigma + \sigma^\dagger a \right)}^{H}, \rho \right] + \kappa \left[a \rho a^\dagger - \frac{1}{2} \left\{ a^\dagger a, \rho \right\} \right]$$

where ρ now contains <u>both</u> cavity and atom degrees of freedom. For convenience, we will solve this equation using the *Heisenberg picture*, i.e. by looking for an equation of motion for the operators instead of the density matrix. Such equation for an operator O in the Heisenberg picture is obtained by first considering the definition of the expectation value

$$\langle O \rangle = \mathrm{tr}(O\rho)$$

and the the cyclic property of the trace

$$\mathrm{tr}(ABC) = \mathrm{tr}(CBA).$$

We are interested in the time evolution of this operator, which one can write as

$$\begin{aligned}
\left\langle \dot{O} \right\rangle &= \mathrm{tr}\left(\dot{O} \rho \right) = \mathrm{tr}(O \dot{\rho}) = \mathrm{tr}\left(-\frac{i}{\hbar} O \left[H, \rho \right] + \kappa \left(O a \rho a^\dagger - \frac{1}{2} O \left\{ a^\dagger a, \rho \right\} \right) \right) \\
&= \mathrm{tr}\left(-\frac{i}{\hbar} \left(O H \rho - O \rho H \right) + \kappa O a \rho a^\dagger - \frac{1}{2} O a^\dagger a \rho - \frac{1}{2} O \rho a^\dagger a \right) \\
&\overset{\substack{\text{Push } \rho \\ \text{to the right}}}{=} \mathrm{tr}\left(-\frac{i}{\hbar} \left(O H - H O \right) \rho + \kappa a^\dagger O a \rho - \frac{1}{2} a^\dagger a \rho - \frac{1}{2} a^\dagger a O \rho \right) \\
&= \mathrm{tr}\left(\underbrace{\left(-\frac{i}{\hbar} \left[O, H \right] + \kappa \left(a^\dagger O a - \frac{1}{2} \left\{ a^\dagger a, O \right\} \right) \right)}_{\dot{O}} \rho \right).
\end{aligned}$$

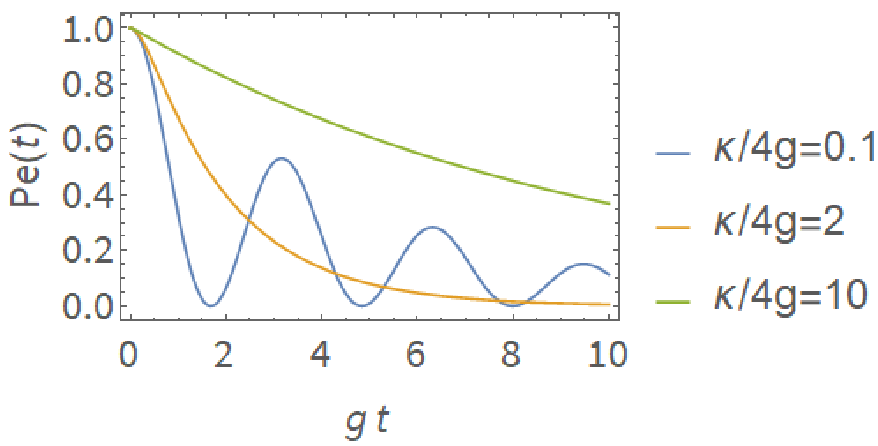
In particular, we are interested in the time evolution of operators such as the photon and atom number operators $a^\dagger a$ and $\sigma^\dagger\sigma$, respectively. One can obtain these equation of motion as (*)

$$
\begin{aligned}
(\dot{a^\dagger a}) &= -\frac{i}{\hbar}\left[a^\dagger a, \hbar\omega_0\sigma^\dagger\sigma + \hbar\nu a^\dagger a + \hbar g\left(a^\dagger\sigma + \sigma^\dagger a\right)\right] + \kappa\left[a^\dagger a^\dagger a a - \frac{1}{2}\left\{a^\dagger a, a^\dagger a\right\}\right] \\
&\overbrace{=}^{aa^\dagger = a^\dagger a + 1} ig\left(\sigma^\dagger a - a^\dagger\sigma\right) - \kappa a^\dagger a \\
(\dot{\sigma^\dagger\sigma}) &= -\frac{i}{\hbar}\left[\sigma^\dagger\sigma, \hbar\omega_0\sigma^\dagger\sigma + \hbar\nu a^\dagger a + \hbar g\left(a^\dagger\sigma + \sigma^\dagger a\right)\right] + \kappa\left[a^\dagger\sigma^\dagger\sigma a - \frac{1}{2}\left\{a^\dagger a, \sigma^\dagger\sigma\right\}\right] \\
&\overbrace{=}^{\sigma\sigma = \sigma^\dagger\sigma^\dagger = 0} -ig\left(\sigma^\dagger a - a^\dagger\sigma\right).
\end{aligned}
$$

A new problem arises now. In order to solve the equations above we will need also the equation for $\sigma^\dagger a - a^\dagger\sigma$, which in turn gives rise to an open hierarchy of equations. The solution is to consider the expectation value of the operators, assuming that initially one is in the $|e,0\rangle$ state. As we have discussed, this state is only coupled to $|g,1\rangle$ (and to $|g,0\rangle$ via dissipation of a cavity photon into the radiation field). This means that the expectation value of all operators involving a^2 or $a^{\dagger^2}$ or above are zero. This closes the set of relevant operators to $\left\langle a^\dagger a\right\rangle, \left\langle\sigma^\dagger\sigma\right\rangle, \left\langle\sigma^\dagger a - a^\dagger\sigma\right\rangle$ and $\left\langle a^\dagger a\sigma^\dagger\sigma\right\rangle$, which lead to a closed set of differential equations that can then be solved. For $\left\langle\sigma^\dagger\sigma\right\rangle$, which is nothing but the probability of being in the excited state, we get

$$
P_e(t) = \left\langle\sigma^\dagger\sigma\right\rangle \overset{(\Delta=0)}{=} \frac{2e^{-\frac{\kappa t}{2}}}{\Omega_\kappa^2}\left[-4g^2 + e^{\frac{\Omega_\kappa t}{2}}\left[\frac{\kappa^2}{4} - 2g^2 + \frac{\kappa}{4}\Omega_\kappa\right] e^{-\frac{\Omega_\kappa t}{2}}\left[\frac{\kappa^2}{4} - 2g^2 - \frac{\kappa}{4}\Omega_\kappa\right]\right],
$$

where we have defined $\Omega_\kappa = \sqrt{\kappa^2 - 16g^2}$. We can now plot the function for any combination between g and κ.



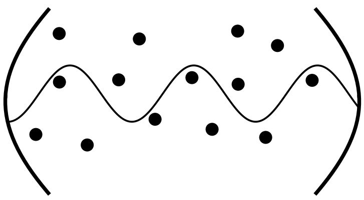
Here, one can distinguish two particularly relevant limiting cases:

- $\kappa \ll 4g$ (weak damping) $P_e(t) \approx \underbrace{\frac{e^{-\kappa t/2}}{2}\left[1+\cos(2gt)\right]}_{\text{damped oscillations}}$

- $\kappa \gg 4g$ (overdamped cavity) $P_e(t) \approx \underbrace{e^{-\frac{4g^2}{\kappa}t}}_{\text{Decay with constant } \frac{4g^2}{\kappa}}$

So, overall, we have learnt that the coupling of the cavity to the environment acts as all effective coupling of the atom to the environment $\Rightarrow$ the atom decays with rate $\frac{4g^2}{\kappa}$.

2.3 Many atoms: Dicke and Tavis-Cummings models in and out of equilibrium



Now we will describe the coupling of an ensemble of N atoms with a single cavity mode. The Hamiltonian that describes this system is, in the most general form, very similar to the Jaynes-Cummings Hamiltonian, but simply adding the contribution of each of the atoms as

$$H = \hbar\nu a^\dagger a + \hbar\omega_0 \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i + \frac{\hbar g}{\sqrt{N}} \sum_{i=1}^{N} \left(\sigma_i^\dagger + \sigma_i\right)\left(a^\dagger + a\right) \ .$$

Note that the $\sqrt{N}$ in the denominator in front of the sum is introduced such that the sum behaves well in the thermodynamic limit, that is, in the limit $N \to \infty$. Moreover, here we have introduced the ladder operator for the i-th atom as $\sigma_i = |g_i\rangle\langle e_i|$. This model has been widely studied in the literature due to the different phases and phase transitions it can display, both in equilibrium and out of equilibrium, i.e., both in the coherent, closed version and the one where dissipation mechanisms are included. This model has also been widely studied experimentally.

2.3.1 Equilibrium Dicke transition

We will start now by considering the equilibrium case, i.e., we leave the dissipation for the moment out of the problem. We will see that this Hamiltonian displays a phase transition, i.e., that the properties (a so-called order parameter) of the thermal equilibrium state of the Hamiltonian change dramatically when varying a parameter (here it will be the coupling g). This change becomes sudden, or sharp, only in the thermodynamic limit $N \to \infty$.

The most common approach to see this is using *mean-field theory*. For us, this will mean to break the correlations between photons and atoms, assuming the cavity photons to be described as a classical field in the thermodynamic limit. In practice, we introduce the operator

$$\alpha = \frac{a}{\sqrt{N}},$$

which satisfies

$$\left[\alpha, \alpha^\dagger\right] = \frac{1}{N}.$$

In the thermodynamic limit this commutator vanishes, i.e., α can in this limit be considered as a number. The Hamiltonian thus becomes

$$\begin{aligned} H_{MF} &= \hbar\nu N|\alpha|^2 + \sum_i \left[\hbar\omega_0\sigma_i^\dagger\sigma_i + \hbar g\left(\sigma_i^\dagger + \sigma_i\right)(\alpha+\alpha^*)\right] \\ &= \sum_i \underbrace{\left[\hbar\nu\alpha^2 + \hbar\omega_0\sigma_i^\dagger\sigma_i + \hbar g\left(\sigma_i^\dagger + \sigma_i\right)(\alpha+\alpha^*)\right]}_{\text{Each atom experiences the same Hamiltonian: } h(\alpha)} \end{aligned}$$

We will find the thermal equilibrium state of the system varying α to find the minimum *free energy*, which is defined as

$$F(\alpha) = -\frac{1}{\beta}\ln\left[Z(\alpha)\right]$$

with $\beta = \frac{1}{k_B T}$ ($T \equiv$ temperature, $k_B \equiv$ Boltzmann constant), and $Z(\alpha)$ being the partition function, which can be calculated as

$$\begin{aligned} Z(\alpha) &= \mathrm{tr}\left(e^{-\beta H_{MF}}\right) = \mathrm{tr}\left(e^{-\beta h(\alpha)}\right)^N \\ &= \left[e^{-\beta E_+} + e^{-\beta E_-}\right]^N \end{aligned}$$

where E_+ and E_- are the eigenvalues of $h(\alpha)$. These can be found as:

$$h(\alpha) = \hbar\begin{pmatrix} \nu|\alpha|^2 + \omega_0 & g(\alpha+\alpha^*) \\ g(\alpha+\alpha^*) & \nu|\alpha|^2 \end{pmatrix} \Rightarrow \frac{E_\pm}{\hbar} = \nu|\alpha|^2 + \frac{\omega_0}{2} \pm \frac{\sqrt{\omega_0^2 + 16g^2|\alpha|^2}}{2}$$

(where we have assumed α is real) such that

$$Z(\alpha) = \left[2e^{-\beta\hbar\left(\nu|\alpha|^2+\frac{\omega_0}{2}\right)}\cosh\left(\frac{\beta\hbar}{2}\sqrt{\omega_0^2+16g^2|\alpha|^2}\right)\right]^N \ .$$

Now we can calculate the free energy as:

$$F(\alpha) = N\left[\hbar\left(\nu|\alpha|^2+\frac{\omega_0}{2}\right) - \frac{1}{\beta}\ln\left(2\cosh\left(\frac{\beta\hbar}{2}\sqrt{\omega_0^2+16g^2|\alpha|^2}\right)\right)\right] \ .$$

We calculate now the minimum of this free energy:

$$\frac{\partial F(\alpha)}{\partial\alpha} = N\left[2\hbar\nu|\alpha| - 8\hbar g^2|\alpha|\frac{\tanh\left(\frac{\hbar\beta}{2}\sqrt{16g^2|\alpha|^2+\omega_0^2}\right)}{\sqrt{16g^2|\alpha|^2+\omega_0^2}}\right]$$

- Solution 1: $\alpha = 0$
- Solution 2: α finite and solution of the transcendental equation:

$$\sqrt{16g^2|\alpha|^2+\omega_0^2} = \frac{4g^2}{\nu}\tanh\left(\frac{\hbar\beta}{2}\sqrt{16g^2|\alpha|^2+\omega_0^2}\right)$$

We can now find the transition point, i.e., the value of g at which the system changes from Solution 1 to Solution 2. We can do this by expanding the transcendental equation around $\alpha = 0$, which gives

$$\begin{aligned}
\omega_0\underbrace{\sqrt{1+\frac{16g^2}{\omega_0^2}|\alpha|^2}}_{\text{use: }\alpha\to 0} &= \frac{4g^2}{\nu}\tanh\left(\frac{\hbar\beta\omega_0}{2}\sqrt{1+\frac{16g^2}{\omega_0^2}|\alpha|^2}\right) \\
1+\frac{8g^2}{\omega_0^2}|\alpha|^2 &= \frac{4g^2}{\omega_0\nu}\underbrace{\tanh\left(\frac{\hbar\beta\omega_0}{2}\right)}_{=1\text{ for }\beta\to\infty(T\to 0)} \\
\frac{8g^2}{\omega_0^2}|\alpha|^2 &= \overbrace{\frac{4g^2}{\omega_0\nu}}^{\substack{\text{this has to be } > 0\\ \text{for this solution to be valid}}} - 1\,.
\end{aligned}$$

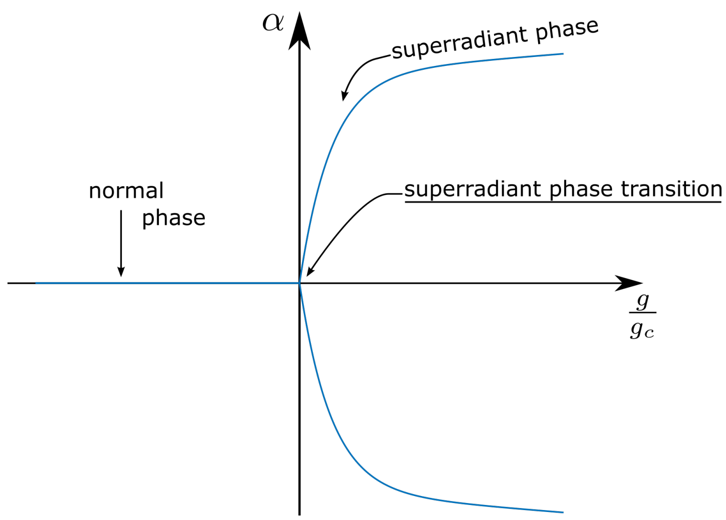
Putting the right hand side to be exactly equal to zero gives us the transition point between the two phases, which is here given by

$$g_c = \frac{1}{2}\sqrt{\omega_0 \nu}.$$

For $g > g_c$, but close to the transition point we obtain that the order parameter grows as

$$|\alpha| \approx \frac{\omega_0}{\sqrt{8} g g_c}\sqrt{g^2 - g_c^2}\ .$$

The phase diagram, which here is nothing but the variation of the order parameter α as a function of the coupling strength, is depicted below.



Here, we can observe that there are indeed two clearly distinct phases. In the so-called normal phase $\alpha = 0$, which means that the cavity has zero occupation. Moreover, here one can obtain as well that all the atoms are in the ground state. In the superradiant phase, on the other hand, $\alpha \neq 0$, which indicates a non-zero occupation of the cavity. This superradiant phase transition can also be related to the symmetries of the Hamiltonian. One can easily see that the Hamiltonian conserves the parity of the total number of excitations $P = (-1)^{N_{ex}}$ where

$$N_{ex} = a^\dagger a + \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i$$

(either N_{ex} is conserved or it goes to $N_{ex} \pm 2$). This can also be see as the Hamiltonian remaining unchanged as $a \to -a$ and $\sigma \to -\sigma$. While the normal phase does indeed conserve this symmetry, the superradiant one does not, and hence this superradiant transition is accompanied, or signalled, by a *spontaneous symmetry breaking* of the superradiant phase.

2.3.2 Non-equilibrium Dicke transition

We now consider again the very same situation of N atoms in a cavity, but now the cavity is *damped*. Here, we have to use what we learnt in Section 2.2 and write first the master equation that determines the dynamics of the system as

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H,\rho] + \kappa\left[a\rho a^\dagger - \frac{1}{2}\left\{a^\dagger a, \rho\right\}\right] ,$$

with

$$H = \hbar\nu a^\dagger a + \hbar\omega_0 \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i + \frac{\hbar g}{\sqrt{N}} \sum_{i=1}^{N} \left(\sigma_i^\dagger + \sigma_i\right)\left(a^\dagger + a\right) .$$

We will now investigate this system, and obtain that it also undergoes a phase transition. However, here we are not interested in the ground or equilibrium state of the system, but rather on its *stationary state* (the state of the system and at very long times).

Before we start, it is convenient to introduce the collective atomic operators

$$J = \sum_{i=1}^{N} \sigma_i \quad ; \quad J^\dagger = \sum_{i=1}^{N} \sigma_i^\dagger \quad ; \quad J_z = \frac{1}{2}\sum_{i=1}^{N}\left(\sigma_i^\dagger \sigma_i - \sigma_i \sigma_i^\dagger\right)$$

such that the Hamiltonian reads (*)

$$H = \hbar\nu a^\dagger a + \hbar\omega_0\left(J_z + \frac{N}{2}\right) + \frac{\hbar g}{\sqrt{N}}\left(a^\dagger + a\right)\left(J + J^\dagger\right) .$$

These collective operators satisfy the commutation relations

$$\left[J^\dagger, J\right] = 2J_z \quad ; \quad [J, J_z] = J \quad ; \quad \left[J^\dagger, J_z\right] = -J^\dagger .$$

Now, using the Heisenberg picture again, as we did in Section 2.2, we can obtain the equations of motion for the expectation values of a, J, and J_z, as: (*)

- $\frac{\partial\langle a\rangle}{\partial t} = -\left(i\nu + \frac{\kappa}{2}\right)\langle a\rangle - i\frac{g}{\sqrt{N}}\left(\langle J^\dagger\rangle + \langle J\rangle\right)$
- $\frac{\partial\langle J\rangle}{\partial t} = -i\omega_0\langle J\rangle + i\frac{2g}{\sqrt{N}}\left\langle\left(a^\dagger + a\right)J_z\right\rangle$
- $\frac{\partial\langle J_z\rangle}{\partial t} = i\frac{g}{\sqrt{N}}\left\langle\left(a^\dagger + a\right)\left(J^\dagger - J\right)\right\rangle$.

Now we invoke again mean- field theory in order to be able to solve these equations. In particular, we break the correlations between the atoms and the cavity and factorize the expectation values as

- $\left\langle \left(a^\dagger + a\right) J_z \right\rangle = \left\langle a^\dagger + a \right\rangle \langle J_z \rangle$
- $\left\langle \left(a^\dagger + a\right) \left(J^\dagger - J\right) \right\rangle = \left\langle a^\dagger + a \right\rangle \left\langle J^\dagger - J \right\rangle$.

Finally, we look back into the Hamiltonian and realize that it actually commutes with the total angular momentum $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$, which means that it only connects states within the same *Dicke manifold*, with the same total angular momentum number j, which for us is $j = N/2$. Hence, we can impose the condition (*)

$$\langle J_z \rangle^2 + \left\langle J^\dagger \right\rangle \langle J \rangle = \frac{N^2}{4} \ .$$

With all of the above, one obtains two possible solutions of the expectation values in the stationary state, obtained by putting:

$$\frac{\partial \langle a \rangle}{\partial t} = \frac{\partial \langle J \rangle}{\partial t} = \frac{\partial \langle J_z \rangle}{\partial t} = 0.$$

The first solution is:

A) $\langle a \rangle_A = 0 \quad ; \quad \langle J \rangle_A = 0 \quad ; \quad \langle J_z \rangle_A = -\frac{N}{2}$
$\Rightarrow$ No cavity photons and all atoms in the ground state..

The second solution is:

B) $\langle J_z \rangle_B = -\frac{N}{2} \underbrace{\frac{\omega_0}{4\nu} \left(\nu^2 + \frac{\kappa^2}{4}\right)}_{\equiv g_c^2 \text{ because } |\langle J_z \rangle| \leq \frac{N}{2}} \frac{1}{g^2}$

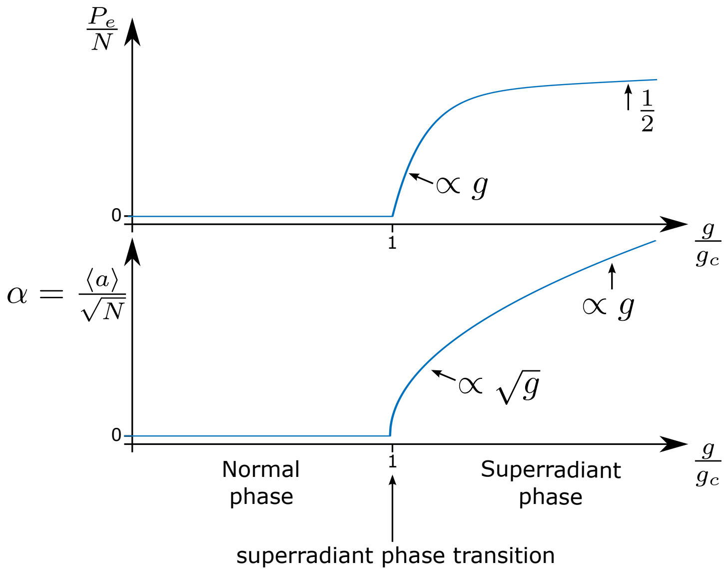
so we have found the value of g at which the transition happens to be

$$\boxed{g_c = \frac{1}{2} \sqrt{\frac{\omega_0}{\nu} \left(\nu^2 + \frac{\kappa^2}{4}\right)}}$$

The values of $\langle J \rangle_B$ and $\langle a \rangle_B$ are

1) $|\langle J \rangle_B| = \frac{N}{2} \sqrt{1 - \frac{g_c^4}{g^4}}$

2) $\frac{\langle a \rangle_B}{\sqrt{N}} = \pm \frac{g}{\nu - \frac{i\kappa}{2}} \sqrt{1 - \frac{g_c^4}{g^4}}$



Note, that even though the two equilibrium and non-equilibrium phase transitions are called superradiant, they have slightly different behaviours (at least at zero temperature). In particular, they belong to different so-called universality classes.

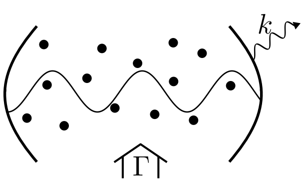
2.3.3 Tavis-Cummings model with damping

Up to now we have used the Dicke model, which contains also the counterrotating terms $\sigma_i^\dagger a^\dagger, \sigma_i a$. In this last section, we will investigate the so-called Tavis-Cummings (TC) model

$$H_{TC} = \hbar\nu a^\dagger a + \hbar\omega_0 \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i + \frac{\hbar g}{\sqrt{N}} \sum_{i=1}^{N} \left(a\sigma_i^\dagger + a^\dagger \sigma_i \right)$$

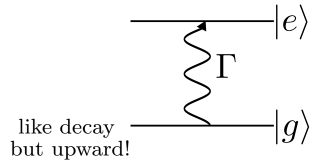
where these counterrotating terms have been neglected. Both the Dicke and Tavis-Cummings Hamiltonians can be experimentally realized.

This Hamiltonian conserves the number of excitations, and not only their parity, as it was the case in the Dicke model. Now, if we add cavity damping to this problem (κ photon loss rate), the stationary state is always the one with $\alpha = 0$, i.e. a normal phase, which in turn means that no phase transition can take place. In order to recover a transition, we have to find a mechanism that adds excitations/photons in some way.



Here, we choose to study the system in the presence of an external *incoherent pumping* of the atoms at a rate Γ. This process can be accounted for in the master equation by adding a dissipation term with jump operator $\sigma^\dagger$, such that the full master equation reads now

$$
\dot{\rho} = -i\left[\nu a^\dagger a + \omega_0 \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i + \frac{g}{\sqrt{N}} \sum_{i=1}^{N}\left(a\sigma_i^\dagger + a^\dagger \sigma_i\right), \rho\right]
+ \kappa\left(a\rho a^\dagger - \frac{1}{2}\left\{a^\dagger a, \rho\right\}\right) + \underbrace{\Gamma \sum_{i=1}^{N}\left(\sigma_i^\dagger \rho \sigma_i - \frac{1}{2}\left\{\sigma_i \sigma_i^\dagger, \rho\right\}\right)}
$$



Again, we can obtain again the equations of motion for the expectation value of the operators a, σ_i and $\sigma_z^{(i)}$, which yields (*)

- $\frac{\partial \langle a \rangle}{\partial t} = -\left(i\nu + \frac{\kappa}{2}\right)\langle a \rangle - i\frac{g}{\sqrt{N}} \sum_i \langle \sigma_i \rangle$
- $\frac{\partial \langle \sigma_i \rangle}{\partial t} = -\left(i\omega_0 + \frac{\Gamma}{2}\right)\langle \sigma_i \rangle + i\frac{g}{\sqrt{N}}\left\langle a\sigma_z^{(i)}\right\rangle$
- $\frac{\partial \left\langle \sigma_z^{(i)} \right\rangle}{\partial t} = i\frac{2g}{\sqrt{N}}\left(\left\langle \sigma_i a^\dagger \right\rangle - \left\langle \sigma_i^\dagger a \right\rangle\right) + \frac{\Gamma}{2}\left(1 - \left\langle \sigma_z^{(i)} \right\rangle\right)$

We can solve these equations in the stationary state again using mean-field theory

$$
\begin{aligned}
\left\langle J a^\dagger \right\rangle &= \langle J \rangle \left\langle a^\dagger \right\rangle \\
\langle a J_z \rangle &= \langle a \rangle \langle J_z \rangle \ .
\end{aligned}
$$

Again, one finds that a solution (for small g) is the normal phase:

$$
\langle a \rangle_A = 0 \quad \text{and} \quad \underbrace{\langle J_z \rangle_A = \frac{N}{2}}_{\text{(inverted population!).}}
$$

However, another solution yields $\langle a \rangle_B \neq 0$, and here one obtains

$$
\langle J_z \rangle_B = \frac{N}{2}\frac{1}{g^2}\left[\frac{\kappa\Gamma}{4} - \omega_0 \nu + \frac{i}{2}\underbrace{(\omega_0 k + \nu\Gamma)}_{\substack{\text{imaginary part} \\ \text{cannot be made 0!}}}\right].
$$

Since the expectation value $\langle J_z \rangle$ must always be real by definition, and the imaginary part cannot be put to zero, this means there is no stationary state for large coupling g. This in turn means that the solution for large time t is *oscillating*. Hence, this is a so-called *no-lasing* $\Rightarrow$ *lasing* transition.

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