Many Body Quantum Optics (VFMBQO)

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Contents

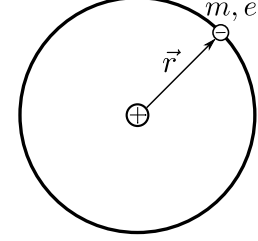# 1 Introduction and review of basic concepts in Quantum optics

1.1 Atom-field interactions

Our starting point will be to consider an atom in the absence of external field. We can view this atom as just an electron (with charge e) bound to the nucleus via the Coulomb interaction $V(r)$. The dynamics of this system is described by a Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = H_0\psi(\vec{r},t), \qquad (1.1)$$

where $\psi(\vec{r},t)$ is the wave function the contains all information about the system. The Hamiltonian on the right hand side of the equation here reads simply

$$H_0 = \frac{\vec{P}^2}{2m} + V(r) , \qquad (1.2)$$

where m is the mass of the electron, $\vec{P} = -i\hbar\vec{\nabla}$ the momentum operator, and $V(|\vec{r}| = r)$ the Coulomb potential.

In the presence of external fields, the Hamiltonian that describes this problem is the so-called *minimal-coupling Hamiltonian*

$$H(\vec{r},t) = \frac{1}{2m}\left[\vec{P} - e\vec{A}(\vec{r},t)\right]^2 + e\Phi(\vec{r},t) + V(r), \qquad (1.3)$$

which accounts for the action of the vector and scalar potentials $\vec{A}$ and Φ, respectively, which describe the external fields. In the following, we will use the concept of gauge invariance and the dipole approximation to obtain a simplified version of this Hamiltonian, which will be the starting point of many of the calculations throughout the course.

Gauge invariance:

We know from Classical Field Theory that the electric and magnetic fields can be written in terms of the scalar and vector potentials as

- $\vec{E}(\vec{r},t) = -\vec{\nabla}\Phi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$

- $\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$.

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We also know that these fields (and in general the physics of the problem) must remain the same under a **gauge transformation**:

- $\Phi'\left(\vec{r}, t\right) = \Phi\left(\vec{r}, t\right) - \frac{\hbar}{e}\frac{\partial \chi(\vec{r},t)}{\partial t}$

- $\vec{A}'\left(\vec{r}, t\right) = \vec{A}\left(\vec{r}, t\right) + \frac{\hbar}{e}\vec{\nabla}\chi\left(\vec{r}, t\right),$

this is the so-called gauge invariance. In quantum mechanics, this gauge invariance is established instead by saying that the Schrödinger equation remains invariant under a transformation of the wave function of the form

$$\psi'\left(\vec{r}, t\right) = U\psi\left(\vec{r}, t\right) = e^{i\chi(\vec{r},t)}\psi\left(\vec{r}, t\right) . \tag{1.4}$$

Let us see what this means for the Hamiltonian in the original Schrödinger equation:

$$i\hbar\frac{\partial\psi\left(\vec{r}, t\right)}{\partial t} = H\left(\vec{r}, t\right)\psi\left(\vec{r}, t\right). \tag{1.5}$$

Our aim is to write a Schrödinger equation for ψ'. We do this as

$$i\hbar\frac{\partial\,\overbrace{U\psi}^{\psi'}}{\partial t} = i\hbar\left[\frac{\partial U}{\partial t}\psi + U\frac{\partial\psi}{\partial t}\right] = i\hbar\,\underbrace{\frac{\partial U}{\partial t}}_{(1.4)}\psi + U\,\underbrace{i\hbar\frac{\partial\psi}{\partial t}}_{(1.5)}$$

$$= -\hbar\frac{\partial\chi}{\partial t}\,\underbrace{U\psi}_{\psi'} + UH\,\underbrace{\psi}_{U^\dagger\psi'} \quad \left(UU^\dagger = U^\dagger U = \mathbb{1}\right)$$

$$= \underbrace{\left[UHU^\dagger - \hbar\frac{\partial\chi}{\partial t}\right]}_{H'(\vec{r},t)}\psi'$$

Hence, the Hamiltonian after the transformation reads

$$H' = UHU^\dagger - \hbar\frac{\partial\chi}{\partial t} = e^{i\chi}\frac{1}{2m}\left[i\hbar\vec{\nabla} + e\vec{A}\right]^2 e^{-i\chi} + e\Phi + V(r) - \hbar\frac{\partial\chi}{\partial t}$$

$$= e\Phi' + V(r) + \frac{e^{i\chi}}{2m}\left[-\hbar^2\vec{\nabla}^2 + e^2\vec{A}^2 + i\hbar e\vec{\nabla}\cdot\vec{A} + i\hbar e\vec{A}\cdot\vec{\nabla}\right]e^{-i\chi}$$

$$= e\Phi' + V(r) + \frac{1}{2m}\left[-\hbar^2\vec{\nabla}^2 + \hbar^2\left(\vec{\nabla}\chi\right)^2 + i\hbar^2\vec{\nabla}^2\chi + 2i\hbar^2\vec{\nabla}\chi\cdot\vec{\nabla}\right.$$

$$\left. + e^2\vec{A}^2 + i\hbar e\vec{\nabla}\cdot\vec{A} + \hbar e\vec{\nabla}\chi\cdot\vec{A} + i\hbar e\vec{A}\cdot\vec{\nabla} + \hbar e\vec{A}\cdot\vec{\nabla}\chi + i\hbar e\vec{A}\cdot\vec{\nabla}\right]$$

$$\overset{(*)}{=} \frac{1}{2m}\left[i\hbar\vec{\nabla} + e\vec{A}'\left(\vec{r}, t\right)\right]^2 + e\Phi' + V(r)$$

Our aim is now to choose a gauge and a gauge function $\chi(\vec{r}, t)$ that simplifies (1.3). We now make a choice of gauge, the *Coulomb or radiation gauge*, such that

$$\Phi = 0 \text{ and } \vec{\nabla} \cdot \vec{A} = 0.$$

This simplifies the Hamiltonian to

$$H'(\vec{r}, t) = \frac{1}{2m} \left[\vec{P} - e \left(\vec{A} + \frac{\hbar}{e} \vec{\nabla}\chi \right) \right]^2 - \hbar \frac{\partial \chi}{\partial t} + V(r) \tag{1.6}$$

where the vector potential satisfies the *wave equation*

$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0.$$

A general solution for this equation is given by

$$\vec{A}(\vec{r}, t) = \vec{A_0} e^{i(\vec{k} \cdot \vec{r} - \omega t)} + \text{c.c.},$$

where $\vec{A_0}$ is the amplitude of the vector potential, and $\vec{k}$ is the wave vector of the radiation with $|\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$.

Dipole approximation:

In the expression of $\vec{A}$, we will be interested in the variation of the vector potential in the position of the electron $\vec{r}$ around the position of the atom $\vec{r_0}$ (which here I can assume to be, without loss of generality, the origin). Hence, here the position $\vec{r}$ in which I am interested is approximately the position of the electron with respect to the nucleus. The typical distance between nucleus and electron is a few Ångströms. On the other hand, for typical optical transitions $\lambda \approx$ a few hundred nanometers. Under these circumstances we find that $\vec{k} \cdot \vec{r} \ll 1$, which allows us to take a Taylor expansion such that

$$\vec{A}(\vec{r}, t) = \vec{A_0} e^{-i\omega t} \left(1 + i\vec{k} \cdot \vec{r} + \cdots \right) + \text{h.c.}$$
$$\approx \vec{A_0} e^{-i\omega t} + \vec{A_0}^\dagger e^{i\omega t} \equiv \vec{A_0}(t) \tag{1.7}$$

where we have kept only the first term of the expansion. This is the *dipole approximation*, and it is equivalent to consider that the external field is constant over the extension of the atom.

Taking now (1.7) into the Hamiltonian (1.6) we have:

$$H'\left(\vec{r}, t\right) = \frac{1}{2m}\left[\vec{P} - e\left(\vec{A}_0 + \frac{\hbar}{e}\vec{\nabla}\chi\right)\right]^2 - \hbar\frac{\partial \chi}{\partial t} + V(r) \,,$$

where now $\vec{A}_0$ is only a function of time. We now proceed to choose a gauge function χ. By choosing

$$\chi\left(\vec{r}, t\right) = -\frac{e}{\hbar}\vec{r}\cdot\vec{A}_0(t)$$

the Hamiltonian can be notably simplified, as here

- $\vec{\nabla}\chi = -\frac{e}{\hbar}\vec{A}_0(t)$,

- $\frac{\partial \chi}{\partial t} = -\frac{e}{\hbar}\vec{r}\cdot\frac{\partial \vec{A}_0(t)}{\partial t} \equiv \frac{e}{\hbar}\vec{r}\cdot\vec{E}_0(t)$,

such that

$$H'\left(\vec{r}, t\right) = \frac{\vec{P}^2}{2m} - e\vec{r}\cdot\vec{E}_0(t) + V(r),$$

and, finally,

$$\boxed{H' = H_0 - \vec{d}\cdot\vec{E}_0(t)} \tag{1.8}$$

where we have used the dipole moment operator:

$$\vec{d} = e\vec{r} \tag{1.9}$$

> In summary, here we have chosen a gauge and a gauge transformation to obtain (within the dipole approximation) the simplest possible expression for the Hamiltonian that describes the interaction between a bound electron in an atom and an external field.

1.2 Interaction of a two-level atom with a single-mode classical field

In this section, we will start from the Hamiltonian (1.8) and describe the dynamics of a single atom with a laser field.

The atom is modelled as a two-level system, i.e., the electron can be either in the ground $|g\rangle$ or the excited $|e\rangle$ state. Within this *two-level approximation* the two levels are separated energetically by:
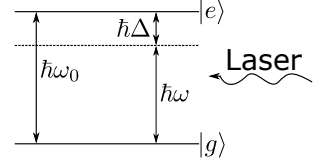
$$\hbar\omega_0 = E_e - E_g, \tag{1.10}$$

where E_e and E_g are the energies of the excited and ground state, respectively.

The external field is here considered to be a monochromatic laser field with frequency ω very close to ω_0. The difference between the two frequencies is called *detuning*:

$$\Delta = \omega_0 - \omega \tag{1.11}$$

This field can be expressed as

$$\vec{E}_0(t) = E_0 \cos\left(\vec{k} \cdot \vec{r_0} - \omega t\right)\hat{\epsilon} \tag{1.12}$$

where E_0 is the amplitude of the field, $\vec{r_0}$ is the position of the atom, and $\hat{\epsilon}$ is the unit polarization vector, which is perpendicular to the wave vector $\vec{k}$.
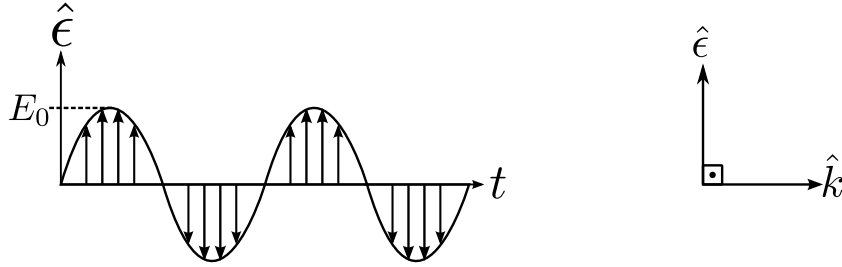
Let us now come back to the Hamiltonian we obtained before and see how it looks in this particular case. Let us start with H_0. The eigenstates of H_0 are precisely $|g\rangle$ and $|e\rangle$, i.e.,

$$H_0 |g\rangle = E_g |g\rangle$$
$$H_0 |e\rangle = E_e |e\rangle .$$

Since there are no other levels considered in the atom, $\{|g\rangle, |e\rangle\}$ form an *orthonormal basis*, which then satisfies

- $|g\rangle\langle g| + |e\rangle\langle e| = \mathbb{1}$

- $\langle g|g \rangle = \langle e|e \rangle = 1$

- $\langle g|e \rangle = \langle e|g \rangle = 0$

Using these properties, we can write H_0 as

$$
\begin{aligned}
H_0 &= \left(|g\rangle\langle g| + |e\rangle\langle e| \right) H_0 \left(|g\rangle\langle g| + |e\rangle\langle e| \right) \\
&= E_g\, |g\rangle\langle g| + E_e\, |e\rangle\langle e| .
\end{aligned}
\tag{1.13}
$$

The part of the Hamiltonian that describes the interaction between the atoms and the field can be written as

$$
\begin{aligned}
-\vec{d} \cdot \vec{E}_0 \overset{(1.12)}{=} &-E_0 \cos\left(\vec{k} \cdot \vec{r}_0 - \omega t \right) \hat{\epsilon} \cdot \vec{d} \\
= &-E_0 \cos\left(\vec{k} \cdot \vec{r}_0 - \omega t \right) \hat{\epsilon} \cdot \left[\left(|g\rangle\langle g| + |e\rangle\langle e| \right) \vec{d} \left(|g\rangle\langle g| + |e\rangle\langle e| \right) \right] \\
= &-E_0 \cos\left(\vec{k} \cdot \vec{r}_0 - \omega t \right) \hat{\epsilon} \cdot \Big[\langle g|\vec{d}|e\rangle\, |g\rangle\langle e| + \langle e|\vec{d}|g\rangle\, |e\rangle\langle g| \\
&+ \underbrace{\cancel{\langle g|\vec{d}|g\rangle\, |g\rangle\langle g|} + \cancel{\langle e|\vec{d}|e\rangle\, |e\rangle\langle e|}}_{\text{selection rules, parity *}} \Big]
\end{aligned}
$$

such that

$$
-\vec{d} \cdot \vec{E}_0 = \hbar \cos\left(\vec{k} \cdot \vec{r}_0 - \omega t \right) \left[\Omega_0\, |e\rangle\langle g| + \Omega_0^*\, |g\rangle\langle e| \right]
$$

where we have defined the *Rabi frequency* as

$$
\boxed{\Omega_0 = -\frac{E_0}{\hbar} \hat{\epsilon} \cdot \langle e|\, \vec{d}\, |g\rangle}
\tag{1.14}
$$

Let us now write everything together:

$$
H' = E_g\, |g\rangle\langle g| + E_e\, |e\rangle\langle e| + \hbar \Omega_0 \cos\left(\vec{k} \cdot \vec{r}_0 - \omega t \right) |e\rangle\langle g| + \hbar \Omega_0^* \cos\left(\vec{k} \cdot \vec{r}_0 - \omega t \right) |g\rangle\langle e| .
$$

Without loss of generality, we can set $E_g = 0$, such that $E_e = \hbar\omega_0$ [according to Eq. (1.10)], and then

$$
H' = \hbar\omega_0\, |e\rangle\langle e| + \hbar \cos\left(\vec{k} \cdot \vec{r}_0 - \omega t \right) \left[\Omega_0\, |e\rangle\langle g| + \Omega_0^*\, |g\rangle\langle e| \right].
$$

This Hamiltonian is *time-dependent*, which makes the corresponding Schrödinger equa-

tion a bit harder to solve (although possible!). In order to eliminate this time-dependence we perform yet another unitary transformation, which reads

$$U = |g\rangle\langle g| + e^{i\omega t} |e\rangle\langle e|. \tag{1.15}$$

This transformation is just a change into a *rotating frame*, i.e. we go into a frame of reference that oscillates with the frequency of the laser. The change is done the same way as before, i.e., we look for the Hamiltonian of the Schrödinger equation for the new wave function $|\psi''\rangle = U |\psi'\rangle$, i.e.,

$$H' |\psi'\rangle = i\hbar\frac{\partial |\psi'\rangle}{\partial t} \rightarrow H'' |\psi''\rangle = i\hbar\frac{\partial |\psi''\rangle}{\partial t}$$

such that

$$
\begin{aligned}
i\hbar\frac{\partial |\psi''\rangle}{\partial t} &= i\hbar\frac{\partial U |\psi'\rangle}{\partial t} = i\hbar\frac{\partial U}{\partial t} |\psi'\rangle + i\hbar U\frac{\partial |\psi'\rangle}{\partial t} \\
&= i\hbar\frac{\partial U}{\partial t}U^\dagger |\psi''\rangle + UH'U^\dagger |\psi''\rangle \\
&= \underbrace{\left(UH'U^\dagger + i\hbar\frac{\partial U}{\partial t}U^\dagger\right)}_{H''} |\psi''\rangle.
\end{aligned}
$$

For the unitary operation (1.15) we obtain (*)

$$
\begin{aligned}
H'' &= \hbar\omega_0 |e\rangle\langle e| + \hbar\cos\left(\vec{k}\cdot\vec{r_0} - \omega t\right)\left[\Omega_0 e^{+i\omega t} |e\rangle\langle g| + \Omega_0^* e^{-i\omega t} |g\rangle\langle e|\right] - \hbar\omega |e\rangle\langle e| \\
&\overset{(1.11)}{=} \hbar\Delta |e\rangle\langle e| + \hbar\frac{\Omega_0}{2}\left[e^{i\vec{k}\cdot\vec{r_0}} + \boxed{e^{-i\vec{k}\cdot\vec{r_0}}e^{2i\omega t}}\right] |e\rangle\langle g| \\
&\quad + \hbar\frac{\Omega_0^*}{2}\left[\boxed{e^{i\vec{k}\cdot\vec{r_0}}e^{-2i\omega t}} + e^{-i\vec{k}\cdot\vec{r_0}}\right] |g\rangle\langle e|
\end{aligned}
$$

This Hamiltonian still depends on time. But we now perform the so-called *Rotating Wave Approximation*. This approximation consists on neglecting the terms that rotate with frequency 2ω, i.e. the terms $\boxed{}$ above. This is well justified since ω is a very high frequency (typically in the THz order) compared with Ω_0 (kHz $-$ MHz). We are not interested in changes in the dynamics on such small timescales (order $1/\omega$). After

this, the final Hamiltonian reads

$$H'' = \hbar\Delta \, |e\rangle\langle e| + \frac{\hbar\Omega_0}{2} e^{i\vec{k}\cdot\vec{r_0}} \, |e\rangle\langle g| + \frac{\hbar\Omega_0^*}{2} e^{-i\vec{k}\cdot\vec{r_0}} \, |g\rangle\langle e|$$

Now that we have the Hamiltonian we can solve the Schrödinger equation and hence obtain the dynamics of any observable in the system. To do so, we write the wave function as

$$|\psi''(t)\rangle = c_g(t) \, |g\rangle + c_e(t) \, |e\rangle$$

where c_g and c_e are the probability amplitudes of the atom to be in the ground and excited state, respectively. Hence, the Schrödinger equation yields

- $i\hbar\frac{\partial|\psi''\rangle}{\partial t} = i\hbar\frac{\partial c_g(t)}{\partial t} \, |g\rangle + i\hbar\frac{\partial c_e(t)}{\partial t} \, |e\rangle$

- $H'' \, |\psi''\rangle = c_g(t)\frac{\hbar\Omega_0}{2} e^{i\vec{k}\cdot\vec{r_0}} \, |e\rangle + \hbar\Delta c_e(t) \, |e\rangle + c_e(t)\frac{\hbar\Omega_0^*}{2} e^{-i\vec{k}\cdot\vec{r_0}} \, |g\rangle$

Multiplying each equation on the left by $\langle g|$ and $\langle e|$, we obtain two coupled equations

$$\rightarrow i\hbar\frac{\partial c_g}{\partial t} = \frac{\hbar\Omega_0^*}{2} e^{-i\vec{k}\cdot\vec{r_0}} c_e \tag{a}$$

$$\rightarrow i\hbar\frac{\partial c_e}{\partial t} = \frac{\hbar\Omega_0}{2} e^{i\vec{k}\cdot\vec{r_0}} c_g + \hbar\Delta c_e. \tag{b}$$

We can uncouple the equations by obtaining c_e and $\frac{\partial c_e}{\partial t}$ from (a), subbing in (b), so that

$$\rightarrow \frac{\partial^2 c_g}{\partial t^2} + i\Delta\frac{\partial c_g}{\partial t} + \frac{|\Omega_0|^2}{4}c_g = 0$$

$$\rightarrow \frac{\partial^2 c_e}{\partial t^2} + i\Delta\frac{\partial c_e}{\partial t} + \frac{|\Omega_0|^2}{4}c_e = 0.$$

Note that here, for simplicity, we have assumed that $\Omega_0 = \Omega_0^*$. To solve the dynamics we only need now the initial conditions. For example, we can take that $c_e(0) = 0$, $c_g(0) = 1$, i.e. the atom is at $t = 0$ in the ground state (one can take others, just think that $|c_g(t)|^2 + |c_e(t)|^2 = 1 \quad \forall t$). With these initial conditions the solution is

$$c_e(t) = -i\frac{\Omega_0}{\Omega}\sin\left(\frac{\Omega t}{2}\right)e^{-i\frac{\Delta}{2}t}$$

$$c_g(t) = \left[\cos\left(\frac{\Omega t}{2}\right) + \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right]e^{-i\frac{\Delta}{2}t},$$
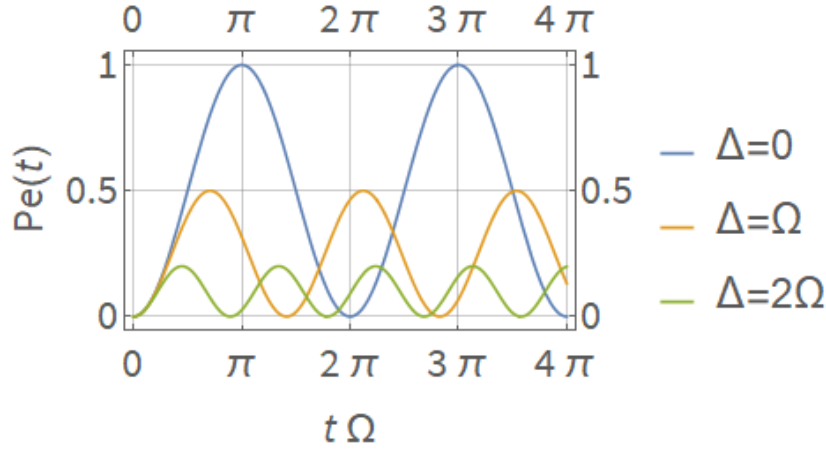
where we have introduced $\Omega = \sqrt{\Omega_0^2 + \Delta^2}$.

Rabi oscillations

Now we are in position calculate any observable we want. For example, the population of the ground and excited states is given, by $|c_g(t)|^2$ and $|c_e(t)|^2$, respectively, where

$$|c_g(t)|^2 + |c_e(t)|^2 = 1 \quad \forall t.$$

Let us then look into the probability of the atom being in the excited state. $P_e(t) \equiv |c_e(t)|^2$ for various situations:

On resonance ($\Delta = 0$):

The population simply oscillates between 0 and 1 with frequency Ω_0:

- When $\Omega_0 t = \pi$ the atom is on the excited state (π- pulse).

- When $\Omega_0 t = \pi/2$, the state of the atom becomes:

$$\left| \Psi'' \left(\frac{\pi}{2\Omega_0} \right) \right\rangle = \frac{1}{\sqrt{2}} \left[|g\rangle - i\, |e\rangle \right] \,.$$

Not on resonance $\Delta \neq 0$:

The frequency of the oscillation is the modified $\Omega = \sqrt{\Omega_0^2 + \Delta^2}$, i.e. it is a faster oscillation, but the excited state population reaches a maximum

$$P_{e,max} = \frac{\Omega_0^2}{\Omega_0^2 + \Delta^2} \,.$$

Light shifts

Let us now consider how the energies of the atom change when the light field is present. Originally, when there is no light field, the eigenenergies are simply $E_g = 0$ and $E_e = \hbar\Delta$ (in the rotating frame). When the light field is present, we need to diagonalize the Hamiltonian

$$H'' = \hbar\Delta \, |e\rangle\langle e| + \frac{\hbar\Omega_0}{2} e^{i\vec{k}\cdot\vec{r_0}} \, |e\rangle\langle g| + \frac{\hbar\Omega_0^\dagger}{2} e^{-i\vec{k}\cdot\vec{r_0}} \, |g\rangle\langle e|$$
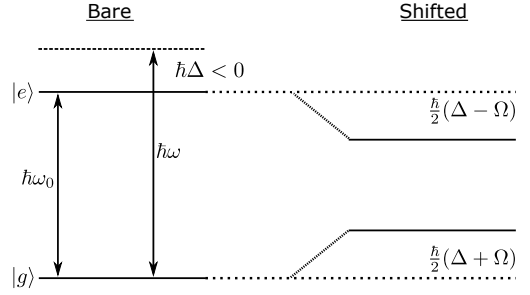
in order to obtain the eigenenergies of the system, which read

$$E_\pm = \frac{\hbar\Delta}{2} \pm \frac{\hbar\Omega}{2},$$

i.e., the energies are shifted due to the presence of the field as
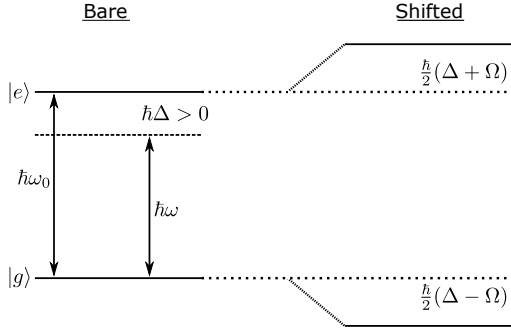
Figure 1: Red-detuned laser

Figure 2: Blue-detuned laser

An interesting limit to consider is when $|\Delta| \gg \Omega$. Here, we can use second-order perturbation theory to obtain the eigenenergies and states. We find that the eigenstates are simply the ground and excited <u>dressed</u> with properties of the other one, and the energies read

$$E_g = -\frac{\hbar\Omega_0^2}{4\Delta} \quad \text{and} \quad E_e = \hbar\Delta + \frac{\hbar\Omega_0^2}{4\Delta}.$$

This is the mechanism used to trap atoms in a so-called dipole trap. There, the field is not homogenous, but a standing wave (in practice created by two counterpropagating laser beams), such that $\Omega_0^2 \to \Omega_0^2 \sin^2(|k|R)$. The light shift for a red-detuned laser becomes:

ground state atom is trapped
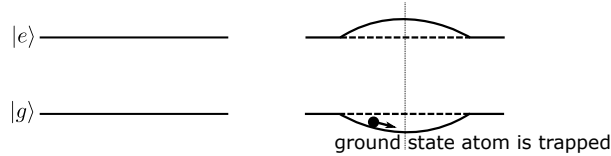
1.3 Interaction of a two-level atom with a quantum field: radiative decay

In this section, we will again start from the Hamiltonian (1.8) to describe the dynamics of a single atom (again a two-level system), this time interacting with a quantum field. In particular, we will aim to find the equation that describes the dynamics of the internal degrees of freedom of the atom when it is in contact with the full electromagnetic field. At the end of the section, we will find the *Master equation* for an open quantum system, which describes the decay of the electron from $|e\rangle$ to $|g\rangle$ with a rate γ.

The Hamiltonian here is again similar to (1.8)

$$H' = H_0 + H_F - \vec{d} \cdot \vec{E},$$

where $H_0 = \hbar\omega_0 |e\rangle\langle e|$. We have also now the field Hamiltonian H_F, which accounts for the energy of the photons in the electromagnetic field, and reads

$$H_F = \sum_{\vec{k}} \hbar\nu_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}.$$

Here, $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ are bosonic operators that represent the creation and annihilation, respectively, of a photon with momentum $\vec{k}$ and energy $\hbar\nu_{\vec{k}}$. They satisfy the bosonic commutation relation $\left[a_{\vec{k}}, a_{\vec{k'}}^\dagger\right] = \delta_{\vec{k},\vec{k'}}$. Finally, we have the term that describes the interaction between the atom and the field. Here, the electric field can be written as an operator that –evaluated at the position of the atom, which we consider here to be the origin– reads

$$\vec{E} = \sum_{\vec{k}} \hat{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{k}} \left(a_{\vec{k}} + a_{\vec{k}}^\dagger\right)$$

where

$$\mathcal{E}_{\vec{k}} = \sqrt{\frac{\hbar\nu_{\vec{k}}}{2\epsilon_0 V}}$$

with ϵ_0 being the free space permittivity, and V the quantization volume. $\hat{\epsilon}_{\vec{k}}$ is the unit vector in the direction of the polarization, such that

$$\vec{k} \cdot \hat{\epsilon}_{\vec{k}} = 0 \ .$$

We have considered for simplicity polarization vector to be real (linear polarization basis).

With these definitions, we end up with a Hamiltonian that reads

$$H' = \hbar\omega_0\sigma^\dagger\sigma + \sum_{\vec{k}} \hbar\nu_k a_{\vec{k}}^\dagger a_{\vec{k}} + \hbar\underbrace{\sum_{\vec{k}} g_{\vec{k}}\left(\sigma^\dagger + \sigma\right)\left(a_{\vec{k}} + a_{\vec{k}}^\dagger\right)}_{H_{int}}$$

with

$$g_{\vec{k}} = -\frac{\langle g|\vec{d}|e\rangle \cdot \hat{\epsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}}{\hbar}$$

and where we have defined the ladder operators

$$\sigma = |g\rangle\langle e| \qquad \sigma^\dagger = |e\rangle\langle g| \ .$$

It is convenient to work in the interaction picture, which means to perform again a unitary transformation, here given by

$$U = e^{i(H_0+H_F)t/\hbar}$$

which leads to a modified Hamiltonian: (*)

$$H'' = e^{i(H_0+H_F)t/\hbar}H_{int}e^{-i(H_0+H_F)t/\hbar}$$
$$= \sum_{\vec{k}} \hbar g_{\vec{k}}e^{i\omega_0\sigma^\dagger\sigma t}\left(\sigma^\dagger + \sigma\right)e^{-i\omega_0\sigma^\dagger\sigma t}e^{i\nu_k a_{\vec{k}}^\dagger a_{\vec{k}}t}\left(a_{\vec{k}} + a_{\vec{k}}^\dagger\right)e^{-i\nu_k a_{\vec{k}}^\dagger a_{\vec{k}}t}$$

To go any further, we need the Baker–Campbell–Hausdorff formula

$$e^{\alpha A}Be^{-\alpha A} = B + \alpha\left[A,B\right] + \frac{\alpha^2}{2!}\left[A,\left[A,B\right]\right] + \dots$$

such that (*)

- $e^{i\omega_0 t\sigma^\dagger\sigma}\sigma^\dagger e^{-i\omega_0 t\sigma^\dagger\sigma} = \sigma^\dagger + i\omega_0 t\overbrace{\left[\sigma^\dagger\sigma,\sigma^\dagger\right]}^{\sigma^\dagger} - \frac{\omega_0^2 t^2}{2}\overbrace{\left[\sigma^\dagger\sigma,\left[\sigma^\dagger\sigma,\sigma^\dagger\right]\right]}^{\sigma^\dagger} + \dots$

$$= \sigma^\dagger\left(1 + i\omega_0 t - \frac{\omega_0^2 t^2}{2} + \dots\right) = \sigma^\dagger e^{i\omega_0 t}$$

- $e^{i\omega_0 t\sigma^\dagger\sigma}\sigma e^{-i\omega_0 t\sigma^\dagger\sigma} = \sigma e^{-i\omega_0 t}$

- $e^{i\nu_k ta_{\vec{k}}^\dagger a_{\vec{k}}}a_{\vec{k}}e^{-i\nu_k ta_{\vec{k}}^\dagger a_{\vec{k}}} = a_{\vec{k}}e^{-i\nu_k t}$

- $e^{i\nu_k ta_{\vec{k}}^\dagger a_{\vec{k}}}a_{\vec{k}}^\dagger e^{-i\nu_k ta_{\vec{k}}^\dagger a_{\vec{k}}} = a_{\vec{k}}^\dagger e^{i\nu_k t}$

and then

$$H'' = \sum_{\vec{k}} \hbar g_{\vec{k}} \left[e^{i(\omega_0 - \nu_k)t} \sigma^\dagger a_{\vec{k}} + e^{-i(\omega_0 - \nu_k)t} \sigma a_{\vec{k}}^\dagger + \boxed{e^{i(\omega_0 + \nu_k)t} \sigma^\dagger a_{\vec{k}}^\dagger} + \boxed{e^{-i(\omega_0 + \nu_k)t} \sigma a_{\vec{k}}} \right]$$

Again, these $\boxed{}$ terms rotate very fast, and under the rotating wave approximation they disappear. The final Hamiltonian reads

$$H'' = \sum_{\vec{k}} \hbar g_{\vec{k}} \left[e^{i(\omega_0 - \nu_k)t} \sigma^\dagger a_{\vec{k}} + e^{-i(\omega_0 - \nu_k)t} \sigma a_{\vec{k}}^\dagger \right] \tag{1.16}$$

which will be used in the following to describe the dynamics of the system.

1.3.1 A quick introduction to open quantum systems

The *density operator*, is defined as

$$\rho = \sum_{\Psi} P_\Psi \, |\Psi\rangle\langle\Psi|$$

and it is introduced to describe a physical situation where we know that the system is in the state $|\Psi\rangle$ with probability P_Ψ. If all P_Ψ are zero but one, the system is in a *pure state*. Otherwise, the state is *mixed*. The density matrix possesses the following properties:

- $\mathrm{tr}(\rho) = 1$ (conservation of probability)

- $\rho^\dagger = \rho$ (Hermitian)

- Positive semi-definite

For pure states, moreover, $\mathrm{tr}(\rho^2) = 1$, while for mixed states $\mathrm{tr}(\rho^2) < 1$. Starting from the Schrödinger equation

$$\frac{\partial |\Psi\rangle}{\partial t} = -\frac{i}{\hbar} H |\Psi\rangle$$

one can obtain the equation obeyed by ρ as

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \sum_{\Psi} P_{\Psi} \left[\frac{\partial \left| \Psi \right\rangle}{\partial t} \left\langle \Psi \right| + \left| \Psi \right\rangle \frac{\partial \left\langle \Psi \right|}{\partial t} \right] \\
&= -\frac{i}{\hbar} \sum_{\Psi} P_{\Psi} \left[H \left| \Psi \right\rangle\!\left\langle \Psi \right| - \left| \Psi \right\rangle\!\left\langle \Psi \right| H \right] \\
&= -\frac{i}{\hbar} \left[H, \rho \right],
\end{aligned}
$$

usually referred to as the *Liouville or Von Neumann equation.*

Let us now go back to our problem. Our starting point is the equation of motion for the atom (A) and the electromagnetic field (E) which is determined by the von Neumann equation for the density matrix ρ_{AE} under the action of the time-dependent Hamiltonian H'', such that
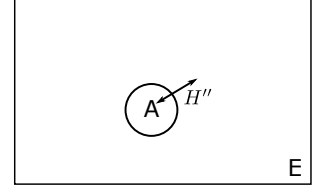
$$
\dot{\rho}_{AE} = -\frac{i}{\hbar} \left[H''(t), \rho_{AE} \right]. \tag{1.17}
$$

Actually, we want to find the dynamics of the atomic degrees of freedom only. These are encoded in the reduced density matrix

$$
\rho = \mathrm{tr}_E(\rho_{AE}) \tag{1.18}
$$

where tr_E represents the trace over all degrees of freedom of the electromagnetic field. In order to obtain such equation, we follow a number of steps:

1. **Formal integration.** In this step we simply integrate the von Neumann equation from 0 to t, such that

$$
\rho_{AE}(t) = \rho_{AE}(0) - \frac{i}{\hbar} \int_0^t \left[H''(t'), \rho_{AE}(t') \right] dt'.
$$

   Substituting this solution back into the von Neumann eq. (1.17) gives

$$
\dot{\rho}_{AE} = -\frac{i}{\hbar} \left[H''(t), \rho_{AE}(0) \right] - \frac{1}{\hbar^2} \int_0^t \left[H''(t), \left[H''(t'), \rho_{AE}(t') \right] \right] dt'. \tag{1.19}
$$

2. **Weak coupling approximation (I).** We assume here that the coupling between the atom and the field, determined by H'' is weak, and hence we are looking for a

solution of the dynamics that satisfies

$$\rho_{AE}(t) = \rho(t) \otimes \rho_E(t) + \rho_c(t)$$

with

$$\text{tr}_E(\rho_c(t)) = 0 \ ,$$

such that (1.18) is satisfied.

3. **Trace over E.** We now take the trace over the degrees of freedom of the electro-magnetic field in equation (1.19). We then obtain

$$\dot{\rho} = -\frac{i}{\hbar} \text{tr}_E \left[H''(t), \rho(0) \otimes \rho_E(0) \right] - \frac{1}{\hbar^2} \text{tr}_E \int_0^t \left[H''(t), [H''(t'), \rho(t') \otimes \rho_E(t')] \right] dt',$$

$$(1.20)$$

where we have used already that $\text{tr}_E(\rho_c(t)) = 0$.

4. **Equilibrium state.** Here we will assume that the reservoir (the electromagnetic field) starts the evolution (at time $t = 0$) in its equilibrium state (e.g. thermal equilibrium at temperature T). Since we have assumed before that the coupling is weak, we may assume that the system and the environment are non-correlated during all the time evolution, and hence $\rho_E(t') = \rho_E(0)$. However, note that this is just an approximation, as in general correlations will build up during the time evolution. This assumption is only valid if the time scales of correlation and relaxation of the environment are much smaller than the typical system time scale.

5. **Weak coupling approximation (II).** Formally integrating (1.20) between t and t' we obtain that

$$\rho(t') - \rho(t) = \mathcal{O}(H'').$$

If we were to substitute this result back inside the integral in (1.20), we would realize that the result would be

$$\dot{\rho} = -\frac{i}{\hbar} \text{tr}_E \left[H''(t), \rho(0) \otimes \rho_E(0) \right] \tag{1.21}$$

$$- \frac{1}{\hbar^2} \text{tr}_E \int_0^t \left[H''(t), [H''(t'), \rho(t) \otimes \rho_E(0)] \right] dt' + \mathcal{O}\left(H''^3\right). \tag{1.22}$$

However, we have assumed before that the coupling is weak. We choose here to keep terms in this equation only up to order $\mathcal{O}\left(H''^2\right)$. Note that effectively we

have substituted then $\rho(t') \to \rho(t)$.

6. **Markov approximation.** In this last step, we assume that the kernel in the integration (given by $H''(t)H''(t')$) decays very fast. This is ensured by a reservoir (in our case the electromagnetic field) which has a very large number of degrees of freedom, and hence leads to a kernel which is almost a delta function $\delta(t' - t)$. This also allows us to extend the upper limit of the integration from t to infinity with no real change in the outcome.

After all of these steps, we have the equation

$$\dot{\rho} = -\frac{i}{\hbar} \operatorname{tr}_E \left[H''(t), \rho(0) \otimes \rho_E(0) \right]$$
$$- \frac{1}{\hbar^2} \operatorname{tr}_E \int_0^\infty \left[H''(t), \left[H''(t'), \rho(t) \otimes \rho_E(0) \right] \right] dt',$$

where we are now in position to substitute our Hamiltonian

$$H''(t) = \sum_{\vec{k}} \hbar g_{\vec{k}} \left[e^{i(\omega_0 - \nu_k)t} \sigma^\dagger a_{\vec{k}} + e^{-i(\omega_0 - \nu_k)t} \sigma a_{\vec{k}}^\dagger \right] .$$

The result is (*)

$$\dot{\rho} = -i \sum_{\vec{k}} g_{\vec{k}} \left\langle a_{\vec{k}} \right\rangle [\sigma^\dagger, \rho(0)] e^{i(\omega_0 - \nu_k)t}$$
$$- \int_0^\infty dt' \sum_{\vec{k}\vec{k}'} g_{\vec{k}} g_{\vec{k}'} \cdot \left\{ [\sigma\sigma\rho - 2\sigma\rho\sigma + \rho\sigma\sigma] \, e^{-i(\omega_0 - \nu_k)t - i(\omega_0 - \nu_{k'})t'} \left\langle a_{\vec{k}}^\dagger a_{\vec{k}'}^\dagger \right\rangle \right\}$$
$$+ \left[\sigma\sigma^\dagger \rho - \sigma^\dagger \rho \sigma \right] e^{-i(\omega_0 - \nu_k)t + i(\omega_0 - \nu_{k'})t'} \left\langle a_{\vec{k}}^\dagger a_{\vec{k}'} \right\rangle$$
$$+ \left[\sigma^\dagger \sigma \rho - \sigma \rho \sigma^\dagger \right] e^{i(\omega_0 - \nu_k)t - i(\omega_0 - \nu_{k'})t'} \left\langle a_{\vec{k}} a_{\vec{k}'}^\dagger \right\rangle + \text{H.c.} \ ,$$

where we have used the definition of an expectation value of an operator O composed of field operators

$$\langle O \rangle = \operatorname{tr}_E(\rho_E O) = \operatorname{tr}_E(O \rho_E).$$

We can further simplify the resulting equation by choosing a particular equilibrium state of the field. As we said before, we choose this to be the thermal equilibrium state

$$\rho_E = \prod_{\vec{k}} \left[1 - e^{-\frac{\hbar \nu_k}{k_B T}} \right] e^{-\frac{\hbar \nu_k}{k_B T} a_{\vec{k}}^\dagger a_{\vec{k}}}, \tag{1.23}$$

such that

- $\left\langle a_{\vec{k}} \right\rangle = \left\langle a_{\vec{k}}^{\dagger} \right\rangle = 0$

- $\left\langle a_{\vec{k}} a_{\vec{k}'} \right\rangle = \left\langle a_{\vec{k}}^{\dagger} a_{\vec{k}'}^{\dagger} \right\rangle = 0$

- $\left\langle a_{\vec{k}}^{\dagger} a_{\vec{k}'} \right\rangle = \bar{n}_{\vec{k}} \delta_{\vec{k},\vec{k}'}$

- $\left\langle a_{\vec{k}} a_{\vec{k}'}^{\dagger} \right\rangle = (\bar{n}_{\vec{k}} + 1) \delta_{\vec{k},\vec{k}'}$,

where the thermal average boson number is given by

$$\bar{n}_{\vec{k}} = \frac{1}{e^{\frac{\hbar \nu_k}{k_B T}} - 1}. \tag{1.24}$$

In particular, if we choose the state at zero temperature we obtain $\bar{n}_{\vec{k}} = 0$, and thus

- $\left\langle a_{\vec{k}}^{\dagger} a_{\vec{k}'} \right\rangle = 0$

- $\left\langle a_{\vec{k}} a_{\vec{k}'}^{\dagger} \right\rangle = \delta_{\vec{k},\vec{k}'}$

such that the equation of motion reads

$$\dot{\rho} = -\int_0^{\infty} dt' \sum_{\vec{k}} g_{\vec{k}}^2 \left\{ e^{i(\omega_0 - \nu_k)(t-t')} \left[\sigma^{\dagger} \sigma \rho - \sigma \rho \sigma^{\dagger} \right] + e^{-i(\omega_0 - \nu_k)(t-t')} \left[\rho \sigma^{\dagger} \sigma - \sigma \rho \sigma^{\dagger} \right] \right\}$$

The next step is to substitute the sum over $\vec{k}$ by an integral, as we can assume a continuum of photon energies. In particular:

$$\sum_{\vec{k}} \rightarrow \overbrace{2}^{\text{polarizations}} \underbrace{\frac{V}{(2\pi)^3}}_{\text{Volume of each mode}} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2 \ .$$

Using that

$$g_{\vec{k}}^2 = \left| d_{ge} \cdot \hat{\epsilon}_{\vec{k}} \right|^2 \frac{\nu_k}{2\hbar\epsilon_0 V} \quad \text{with} \quad \nu_k = kc$$

and defining the azimuthal angle θ to be the one formed by the two vectors: $\vec{d}_{ge} \cdot \hat{\epsilon}_{\vec{k}} =$

$\left|\vec{d}_{ge}\right|\cos\theta$, we obtain

$$\int_0^\infty dt' \sum_{\vec{k}} g_{\vec{k}}^2 e^{i(\omega_0-\nu_k)(t-t')} = \frac{\left|\vec{d}_{ge}\right|^2}{(2\pi)^3\hbar\epsilon_0} \overbrace{\int_0^{2\pi} d\varphi}^{=2\pi} \overbrace{\int_0^\pi d\theta \sin\theta \cos^2\theta}^{=2/3} \frac{1}{c^3} \int_0^\infty d\nu_k \nu_k^3 \overbrace{\int_0^\infty dt' e^{i(\omega_0-\nu_k)(t-t')}}^{=\delta(\nu_k-\omega_0)\cdot\pi}$$

$$= \frac{\left|\vec{d}_{ge}\right|^2 \omega_0^3}{6\pi\hbar\epsilon_0 c^3} \equiv \frac{\gamma}{2}.$$

The *markovian Lindblad master equation* (or Gorini-Kossakowski-Sudarshan-Lindblad master equation) is finally given by

$$\dot{\rho} = \gamma\left[\sigma\rho\sigma^\dagger - \frac{1}{2}\left\{\sigma^\dagger\sigma, \rho\right\}\right]$$

Let us now interpret this equation by, for example, obtaining $P_e(t) \equiv \langle e|\rho|e\rangle$ (the occupation of the excited state), as we did in the case of the interaction with a laser field. Here, our initial state is the atom in the excited state, i.e. $P_e(0) = 1$, and the evolution yields (*)

$$\left.\begin{array}{rl}\dot{P}_e &= -\gamma P_e \\ \dot{P}_g &= \gamma P_e \\ P_e + P_g &= 1\end{array}\right\}$$

$$P_e(t) = P_e(0)e^{-\gamma t}$$
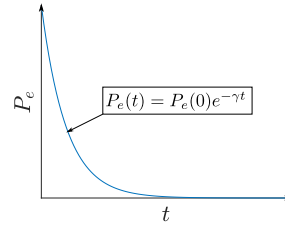
which is why γ is called the *decay rate* of the excited state. An atom in $|e\rangle$ in contact with the radiation field will eventually decay to the ground state.

As a final note, one can actually describe the two-level system interacting with both fields via the master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H_L, \rho] + \gamma\left[\sigma\rho\sigma^\dagger - \frac{1}{2}\left\{\sigma^\dagger\sigma, \rho\right\}\right]$$
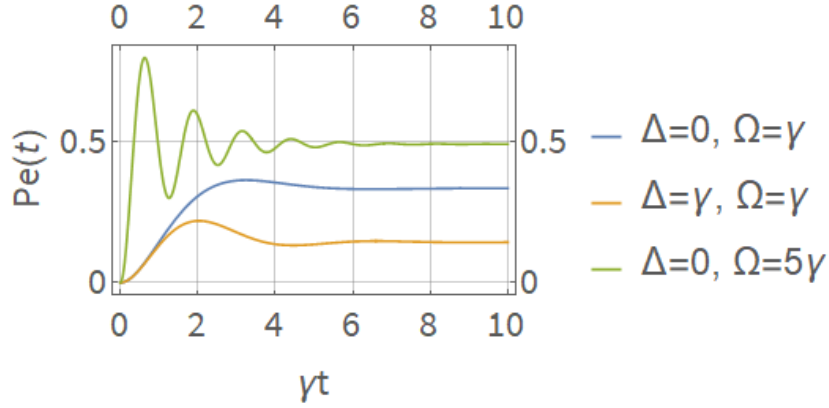
with

$$H_L = \hbar\Delta \overbrace{\sigma^\dagger\sigma}^{|e\rangle\langle e|} + \frac{\hbar\Omega_0}{2}\left(\overbrace{\sigma}^{|g\rangle\langle e|} + \overbrace{\sigma^\dagger}^{|e\rangle\langle g|}\right).$$

If we now solve again the equation of motion, for the probability $P_e(t)$, we can see that it presents damped oscillations towards a stationary state given by

$$P_e(t \to \infty) = \frac{\Omega^2}{\gamma^2 + 4\Delta^2 + 2\Omega^2}$$

as one can see in the figure below.

Literature for Chapter 1:

- Scully, M.O. and Zubairy, M.S., Quantum Optics, Cambridge University Press, 1997

- Metcalf, H.J. and van der Straten, P., Laser cooling and trapping, Springer-Verlag (New York), 1999

- Gerry, C.C. and Knight, P.L., Introductory Quantum Optics, Cambridge University Press, 2005

- Daniel Manzano, "A short introduction to the Lindblad master equation", AIP Advances 10, 025106 (2020)

2 Cavity QED
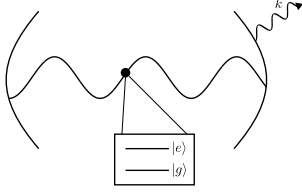
What we have studied until now is an atom coupled to the quantized electromagnetic field, with infinitely many modes. Now we will consider what happens when the atom interacts with a single-mode field instead, like in the case of microwave or optical cavities. In this chapter, we will study first how a single atom interacts with the cavity (that contains a single-mode quantum field), then add coupling of the cavity to the electromagnetic field (damped cavity). We will then study two models that describe the interaction of many atoms with a cavity: the Tavis Cummings and Dicke models.

2.1 The Jaynes-Cummings model

We will be considering the interaction of a two-level atom with a quantized field, so we have again the same Hamiltonian as in Section 1.3, i.e.

$$H' = H_0 + H_F - \vec{d} \cdot \vec{E} \ ,$$

where again our two-level atom is described by

$$H_0 = \hbar \omega_0 \sigma^\dagger \sigma$$

and the field Hamiltonian reads now

$$H_F = \hbar \nu a^\dagger a,$$

where a and $a^\dagger$ are the bosonic creation and annihilation operator of a photon with frequency ν, and $\left[a, a^\dagger \right] = 1$. To fully determine the interaction part of the Hamiltonian, we will need to write explicitly the electric field, which here reads

$$\vec{E} = \hat{\epsilon} \sqrt{\frac{\hbar \nu}{2 \epsilon_0 V}} U(\vec{R}) \left(a + a^\dagger \right) .$$

Here, we have considered that the mode of the cavity changes spatially with the position inside the cavity $\vec{R}$, with the variation encoded in $U(\vec{R})$. For example, this variation could be, in the case of a standing wave in the z direction, given by $U(\vec{R}) = \sin(kZ)$, where $k = \nu/c$ is the corresponding wave vector. The exact variation with the position

in the cavity will become more important later on, when we have more atoms in the cavity, such that each one "feels" a different field.

Considering all of the above, the total Hamiltonian reads

$$H' = \hbar\omega_0\sigma^\dagger\sigma + \hbar\nu a^\dagger a + \hbar g(\vec{R})\left(\sigma^\dagger + \sigma\right)\left(a^\dagger + a\right)$$

with

$$g(\vec{R}) = -\langle g|\vec{d}|e\rangle \cdot \hat{\epsilon}\sqrt{\frac{\nu}{2\epsilon_0\hbar V}}U(\vec{R})$$

being the coupling function between the atom and the cavity. We can skip the next steps, as they are the same as we have already done before:

A) Go into a rotating frame with the frequency of the cavity and the atom, i.e., perform a unitary transformation $U = e^{i(H_0 + H_F)t/\hbar}$ to the Hamiltonian, such that it reads

$$H'' = \hbar g\left[e^{i\Delta t}\sigma^\dagger a + e^{-i\Delta t}\sigma a^\dagger + e^{i(\omega_0+\nu)t}\sigma^\dagger a^\dagger + e^{-(\omega_0+\nu)t}\sigma a\right],$$

with the detuning between the atom and the cavity being $\Delta = \omega_0 - \nu$.

B) Perform the rotating wave approximation by neglecting the action of the fast rotating terms, such that we end up with

$$\boxed{H_{JC} = \hbar g\left[e^{i\Delta t}\sigma^\dagger a + e^{-i\Delta t}\sigma a^\dagger\right]}$$

which is the so-called *Jaynes-Cummings* Hamiltonian (in the interaction picture).

2.1.1 Time-evolution

Our next step will be to solve the dynamics of this system. In particular, here we will need to solve the Schrödinger equation

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = H_{JC}|\Psi\rangle$$

with Hamiltonian H_{JC} and wave function

$$|\Psi(t)\rangle = \sum_n \left[c_{en}(t)|en\rangle + c_{gn}(t)|gn\rangle\right]$$

where our basis now is formed by the states

- $|en\rangle \equiv |e\rangle \otimes |n\rangle \Rightarrow$ atom in $|e\rangle$ and n photons in the cavity

- $|gn\rangle \equiv |g\rangle \otimes |n\rangle \Rightarrow$ atom in $|g\rangle$ and n photons in the cavity.

Here, the states $|n\rangle$ are the eigenvectors of the bare field Hamiltonian H_F, and the action of the ladder operators on them reads (as, for example, the harmonic oscillator!)

- $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$

- $a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle.$

Let us now then substitute both the Hamiltonian and the wave function in the Schrödinger equation, such that

$$i\hbar \sum_n \left[\frac{\partial c_{en}}{\partial t}\,|en\rangle + \frac{\partial c_{gn}}{\partial t}\,|gn\rangle \right] = \hbar g \sum_n \left[c_{en} \left(e^{i\Delta t}\,\overbrace{\sigma^\dagger a\,|en\rangle}^{\sigma^\dagger|e\rangle=0} + e^{-i\Delta t}\sigma a^\dagger\,|en\rangle \right) \right.$$

$$\left. + c_{gn} \left(e^{i\Delta t}\sigma^\dagger a\,|gn\rangle + e^{-i\Delta t}\,\overbrace{\sigma a^\dagger\,|gn\rangle}^{\sigma|g\rangle=0} \right) \right]$$

$$= \hbar g \sum_n \left[c_{en}e^{-i\Delta t}\sqrt{n+1}\,|gn+1\rangle + c_{gn}e^{i\Delta t}\sqrt{n}\,|en-1\rangle \right].$$

Now, if we look back to our Hamiltonian, we can see that the only process that it drives is the exchange of an excitation in the atom with a photon in the cavity. This can be seen mathematically by observing that, by multiplying from the left by $\langle en|$ and $\langle gn+1|$, which gives two coupled equations

- $\langle en| \Rightarrow i\hbar\frac{\partial c_{en}}{\partial t} = \hbar g e^{i\Delta t}\sqrt{n+1}c_{gn+1}$ (*)

- $\langle gn+1| \Rightarrow i\hbar\frac{\partial c_{gn+1}}{\partial t} = \hbar g e^{-i\Delta t}\sqrt{n+1}c_{en}$ (*)

where we have used $\langle g|e\rangle = 0$ and we have also shifted $n \to n+1$ whenever necessary.

We have seen very similar equations before, such as in the case of an atom coupled to a classical field, Section 1.2. Now that we have the equations, we need again initial conditions to solve them. For example, when the atom is considered in the excited state initially, i.e. $c_{gn+1}(0) = 0$ and $c_{en}(0) = 1$ we get

- $c_{en}(t) = c_n(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right) \right] e^{i\Delta t/2}$

- $c_{gn+1}(t) = -c_n(0)\frac{2ig\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{-i\Delta t/2}$

where $c_n(0)$ is the probability amplitude for the field alone and

$$\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}$$

plays a role similar to Ω (Rabi frequency) in Section 1.2.

Now we are in position to extract information on the state of the atom as a function of time (which will depend on the initial cavity occupation, i.e. how many photons n are there initially) and also on the photon occupation of the cavity as a function of time. Let us start by calculating the photon occupation in the cavity. One can obtain it by calculating:

$$P_t(n) = |c_{en}(t)|^2 + |c_{gn}(t)|^2$$
$$= |c_n(0)|^2\left[\cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Omega_n t}{2}\right)^2\sin^2\left(\frac{\Omega_n t}{2}\right)\right] + |c_{n-1}(0)|^2\frac{4g^2 n}{\Omega_{n-1}^2}\sin^2\left(\frac{\Omega_{n-1}t}{2}\right).$$

Here, $|c_n(0)|^2$ is the probability of there being n photons in the cavity initially. A good approximation for this is given by the coherent state

$$|c_n(0)|^2 = \frac{\langle n\rangle^n\, e^{-\langle n\rangle}}{n!},$$

which is a Poisson distribution with mean value $\langle n\rangle$ that can be represented graphically as
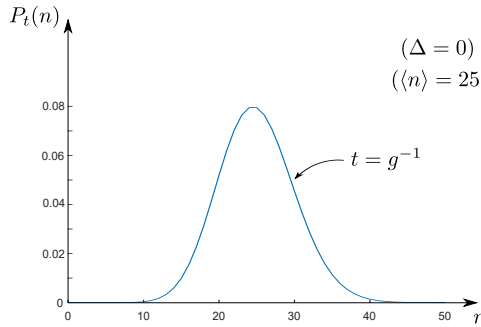
Question(*):

Does the distribution vary with time? If so, how? Does it vary with $\langle n\rangle$?

The dynamics of the internal states can also be easily obtained as

$$P_e(t) = \sum_{n=0}^{\infty} |c_{en}(t)|^2 = \sum_{n=0}^{\infty} |c_n(0)|^2 \left[\cos^2 \left(\frac{\Omega_n t}{2} \right) + \left(\frac{\Delta}{\Omega_n} \right)^2 \sin^2 \left(\frac{\Omega_n t}{2} \right) \right] .$$

For $P_g(t)$ it is possible to find a similar equation. This equation still needs the input of the initial occupation of the cavity. Let us first have a look at what happens when $|c_n(0)|^2 = \delta_{n0}$, that is, there are initially no photons in the cavity. In this case, the system can only be in states where $n = 0$ or $n = 1$, and hence the probability of being in the excited state undergoes Rabi-like oscillations:
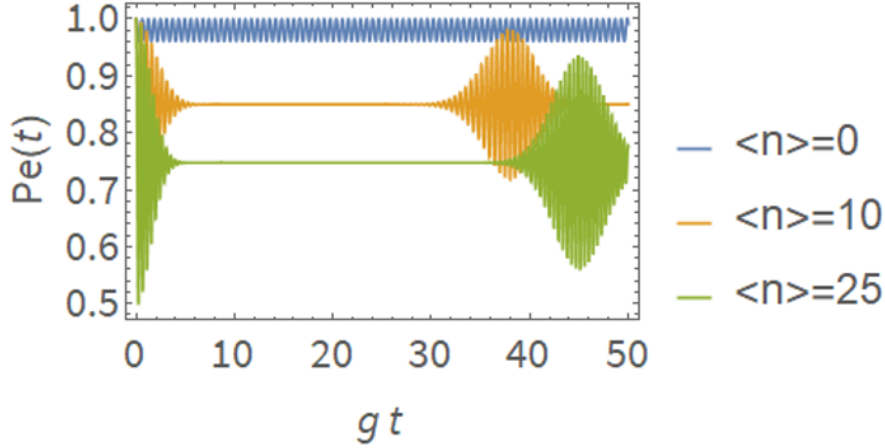
$$P_e(t) = \cos^2 \left(\frac{\Omega_0 t}{2} \right) + \left(\frac{\Delta}{\Omega_0} \right)^2 \sin^2 \left(\frac{\Omega_0 t}{2} \right)$$

with $\Omega_0 = \sqrt{\Delta^2 + 4g^2}$. However, as $\langle n \rangle$ is increased, $P_e(t)$

a) Decays to $P_e(t) \approx \frac{1}{2} \left(1 + \frac{\Delta^2}{\Delta^2 + 4g^2 \langle n \rangle} \right)$ after a time t_c

b) Experience **revivals** after a time t_r.
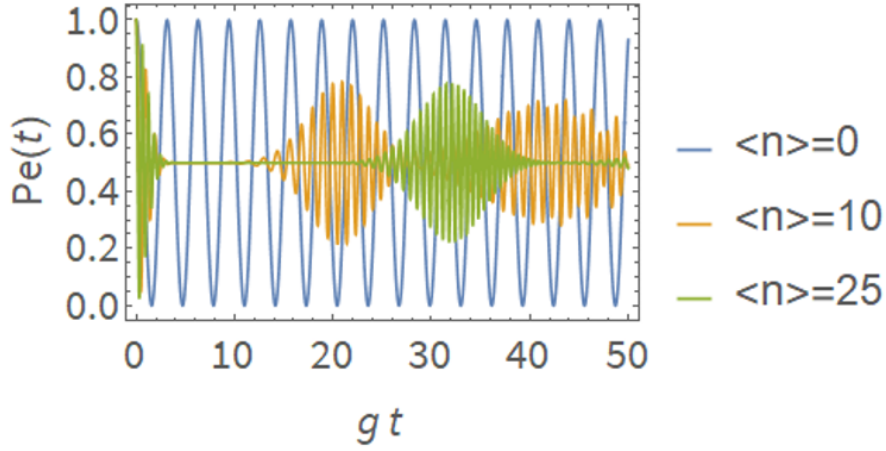
Figure 3: On resonance.

Figure 4: Out of resonance.

The reason for this behaviour can be understood by seeing $P_e(t)$ as a sum of many terms with slightly different Rabi frequencies Ω_n. Like this, one can estimate the decay time t_c as the time when oscillations with different value of n become uncorrelated. In the limit of $\langle n \rangle \gg 1$, $\Omega_n \approx \Omega_{\langle n \rangle}$, and since the photon distribution is a Poissonian, $\Delta n = \sqrt{\langle n \rangle}$, such that

$$\left(\Omega_{\langle n \rangle + \sqrt{\langle n \rangle}} - \Omega_{\langle n \rangle - \sqrt{\langle n \rangle}} \right) t_c \approx 1$$

which gives

$$t_c \approx \frac{1}{2g} \sqrt{1 + \frac{\Delta^2}{4g^2 \langle n \rangle}}$$

so for $\Delta = 0$, t_c is independent of $\langle n \rangle$, but for $\Delta \neq 0$ the larger $\langle n \rangle$ the faster the decay. Similarly, the revival time can be calculated as the time at which the neighbouring Ω_n give rise to a phase difference multiple of 2π, i.e.,

$$m \in \mathbb{Z} \qquad t_r \approx \frac{2\pi m}{\Omega_{\langle n \rangle} - \Omega_{\langle n \rangle - 1}} \approx \frac{2\pi m \sqrt{\langle n \rangle}}{g} \sqrt{1 + \frac{\Delta^2}{4g^2 \langle n \rangle}} \ ,$$

which shows revivals happening at regular intervals.

2.1.2 Dressed states

So far we have been analysing the dynamics of the system. Instead, as we did in Section 1.2 for an atom coupled to a (classical) laser field, we might learn something from looking at the eigenvalues and eigenstates of the Hamiltonian, the *dressed* state picture.

To do so, it is convenient to undo the unitary transformation and start form the

Hamiltonian

$$H' = \underbrace{\hbar\omega_0\sigma^\dagger\sigma + \hbar\nu a^\dagger a}_{H_0'} + \underbrace{\hbar g\left(\sigma^\dagger a + \sigma a^\dagger\right)}_{V'}$$

where the RWA has already been done. Without interactions, the eigenstates of H_0' are simply combinations of $\{|g\rangle, |e\rangle\}$ with $\{|n\rangle\}_{n=0}^\infty$, such that the lowest energy states are the ones depicted below.
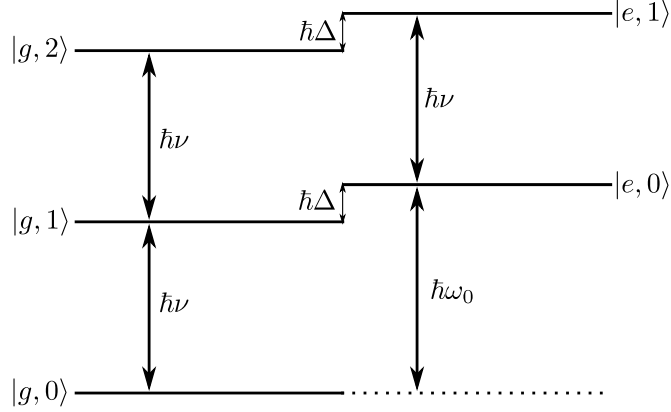
As we said before, only the states $|g, n+1\rangle$ and $|e, n\rangle$ are coupled by the interaction V'. Hence, for fixed number of photons n, the dynamics is determined, in the $\{|g, n+1\rangle, |e, n\rangle\}$ basis, by

$$H'_{(n)} = \hbar \begin{pmatrix} n\nu + \omega_0 & \sqrt{n+1}g \\ \sqrt{n+1}g & (n+1)\nu \end{pmatrix},$$

which can be then diagonalized, to obtain that the new eigenenergies of the system are

$$E_\pm = \frac{1}{2}\hbar\omega_0 + \left(n + \frac{1}{2}\right)\hbar\nu \pm \frac{\hbar}{2}\overbrace{\sqrt{\Delta^2 + 4g^2(n+1)}}^{\Omega_n}$$
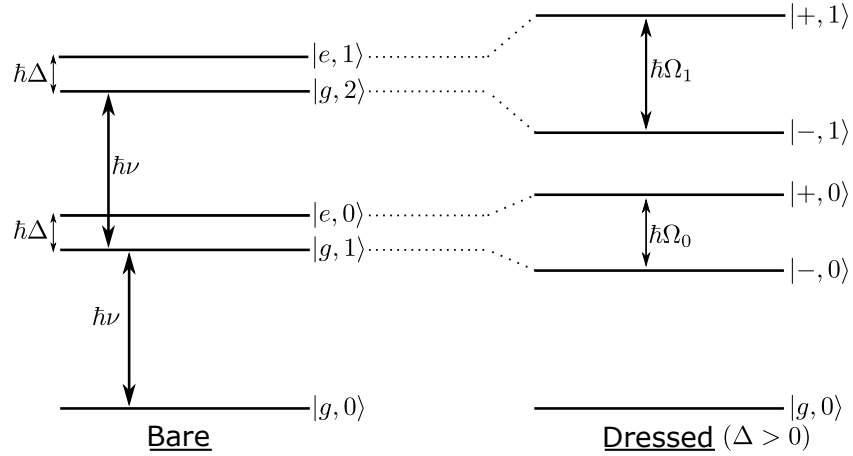
with corresponding states

- $|+, n\rangle = \sin(\theta_n)|g, n+1\rangle + \cos(\theta_n)|e, n\rangle$

- $|-, n\rangle = \cos(\theta_n)|g, n+1\rangle - \sin(\theta_n)|e, n\rangle$

where the angle θ_n is

$$\tan(2\theta_n) = -\frac{2g\sqrt{n+1}}{\Delta} \qquad (0 \le \theta_n < \pi/2).$$

Let us now compare these energies with the ones of the bare states

Each pair of dressed states form a *Jaynes-Cummings doublet*. In the limit of $\Delta = 0$, the bare states are degenerate while the dressed are forming these doublets, which can be observed experimentally, for example, in the following papers:

- PRL 93, 233603 (2004)

- PRL 94, 033002 (2005)

whose experimental data you can see below.

30
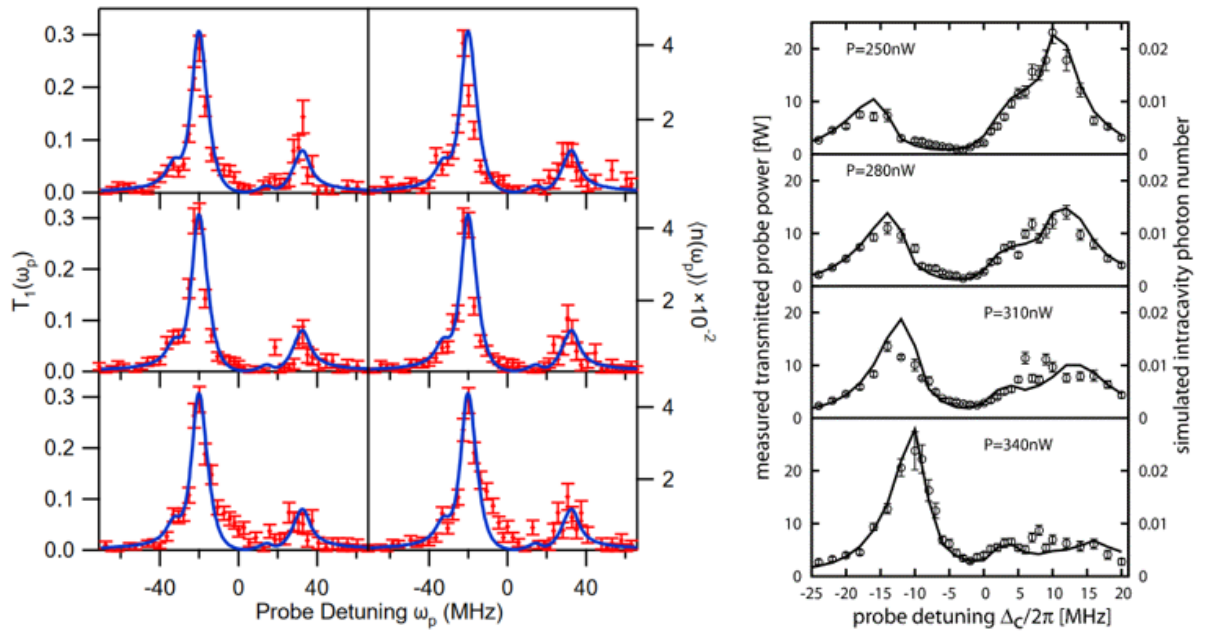
An interesting application of this doubling is the so-called *photon blockade.* The idea is that one makes it impossible for two photons to be simultaneously in the cavity.
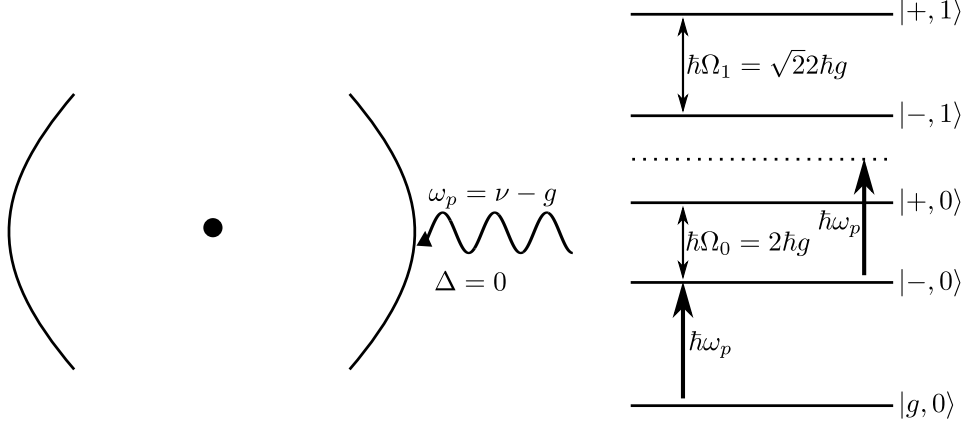
Initially, the state of the system is $|g, 0\rangle$, i.e., no photons in the cavity and the atom in the ground state. A *probe* laser field excites the atom, but is detuned, such that it hits exactly the state $|-, 0\rangle$ (that actually has 1 photon in the cavity). The laser is then off-resonant with respect to any other transition, and hence only 1 photon can be absorbed at a time. This can also be observed experimentally (see Nature 436, 87 (2005)).

2.2 Damped cavity

Up to now we have considered that everything is *coherent*, that is, no losses have been accounted for and one can describe the system with a Schrödinger equation. However, in reality there are indeed losses due to the contact of the system (both the atom and the cavity) with the radiation field.
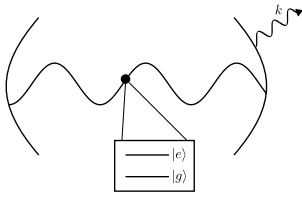
The dissipation process we will consider in this section is the case where the cavity is damped, that is, the cavity can lose photons into the environment (again the electromagnetic field, as we did in section 1.3). To account for these loses, consider that the Hamiltonian which describes only the cavity and the radiation field reads

$$H'_{CR} = \underbrace{\hbar\nu a^\dagger a}_{\text{cavity}} + \underbrace{\hbar \sum_{\vec{k}} \nu_k b^\dagger_{\vec{k}} b_{\vec{k}}}_{\text{reservoir (e.m. field)}} + \hbar \sum_{\vec{k}} \kappa_{\vec{k}} \left(b^\dagger_{\vec{k}} a + a^\dagger b_{\vec{k}} \right)$$

where the RWA has been done already, and where $\kappa_{\vec{k}}$ represents the coupling strength

between the cavity and the field. We now follow the same steps as in section 1.3 in order to obtain a master equation for the cavity degrees of freedom only. The final master equation reads here

$$\dot{\rho}_C = \kappa \left[a\rho_C a^\dagger - \frac{1}{2} \left\{ a^\dagger a, \rho_C \right\} \right]$$

where ρ_C contains the cavity degrees of freedom and κ is the cavity damping rate.

Now we can obtain an equation that describes the system of the atom plus the damped cavity as:

$$\dot{\rho} = -\frac{i}{\hbar} \left[\overbrace{\hbar\omega_0 \sigma^\dagger \sigma + \hbar\nu a^\dagger a + \hbar g \left(a^\dagger \sigma + \sigma^\dagger a \right)}^{H}, \rho \right] + \kappa \left[a\rho a^\dagger - \frac{1}{2} \left\{ a^\dagger a, \rho \right\} \right]$$

where ρ now contains <u>both</u> cavity and atom degrees of freedom. For convenience, we will solve this equation using the *Heisenberg picture*, i.e. by looking for an equation of motion for the operators instead of the density matrix. Such equation for an operator O in the Heisenberg picture is obtained by first considering the definition of the expectation value

$$\langle O \rangle = \text{tr}(O\rho)$$

and the the cyclic property of the trace

$$\text{tr}(ABC) = \text{tr}(CBA).$$

We are interested in the time evolution of this operator, which one can write as

$$\left\langle \dot{O} \right\rangle = \text{tr}\left(\dot{O}\rho \right) = \text{tr}(O\dot{\rho}) = \text{tr}\left(-\frac{i}{\hbar} O\left[H, \rho \right] + \kappa \left(Oa\rho a^\dagger - \frac{1}{2} O\left\{ a^\dagger a, \rho \right\} \right) \right)$$

$$= \text{tr}\left(-\frac{i}{\hbar} \left(OH\rho - O\rho H \right) + \kappa Oa\rho a^\dagger - \frac{1}{2} Oa^\dagger a\rho - \frac{1}{2} O\rho a^\dagger a \right)$$

$$\overset{\substack{\text{Push } \rho \\ \text{to the right}}}{=} \text{tr}\left(-\frac{i}{\hbar} \left(OH - HO \right)\rho + \kappa a^\dagger Oa\rho - \frac{1}{2} a^\dagger a\rho - \frac{1}{2} a^\dagger aO\rho \right)$$

$$= \text{tr}\left(\underbrace{\left(-\frac{i}{\hbar} \left[O, H \right] + \kappa \left(a^\dagger Oa - \frac{1}{2} \left\{ a^\dagger a, O \right\} \right) \right)}_{\dot{O}} \rho \right).$$

In particular, we are interested in the time evolution of operators such as the photon

and atom number operators $a^\dagger a$ and $\sigma^\dagger \sigma$, respectively. One can obtain these equation of motion as (*)

$$(a^{\dot{\dagger}}a) = -\frac{i}{\hbar}\left[a^\dagger a, \hbar\omega_0\sigma^\dagger\sigma + \hbar\nu a^\dagger a + \hbar g\left(a^\dagger\sigma + \sigma^\dagger a\right)\right] + \kappa\left[a^\dagger a^\dagger aa - \frac{1}{2}\{a^\dagger a, a^\dagger a\}\right]$$

$$\overset{aa^\dagger = a^\dagger a+1}{=} ig\left(\sigma^\dagger a - a^\dagger\sigma\right) - \kappa a^\dagger a$$

$$(\sigma^{\dot{\dagger}}\sigma) = -\frac{i}{\hbar}\left[\sigma^\dagger\sigma, \hbar\omega_0\sigma^\dagger\sigma + \hbar\nu a^\dagger a + \hbar g\left(a^\dagger\sigma + \sigma^\dagger a\right)\right] + \kappa\left[a^\dagger\sigma^\dagger\sigma a - \frac{1}{2}\{a^\dagger a, \sigma^\dagger\sigma\}\right]$$

$$\overset{\sigma\sigma = \sigma^\dagger\sigma^\dagger = 0}{=} -ig\left(\sigma^\dagger a - a^\dagger\sigma\right).$$

A new problem arises now. In order to solve the equations above we will need also the equation for $\sigma^\dagger a - a^\dagger\sigma$, which in turn gives rise to an open hierarchy of equations. The solution is to consider the expectation value of the operators, assuming that initially one is in the $|e, 0\rangle$ state. As we have discussed, this state is only coupled to $|g, 1\rangle$ (and to $|g, 0\rangle$ via dissipation of a cavity photon into the radiation field). This means that the expectation value of all operators involving a^2 or $a^{\dagger 2}$ or above are zero. This closes the set of relevant operators to $\langle a^\dagger a\rangle$, $\langle \sigma^\dagger\sigma\rangle$, $\langle \sigma^\dagger a - a^\dagger\sigma\rangle$ and $\langle a^\dagger a\sigma^\dagger\sigma\rangle$, which lead to a closed set of differential equations that can then be solved. For $\langle \sigma^\dagger\sigma\rangle$, which is nothing but the probability of being in the excited state, we get

$$P_e(t) = \langle \sigma^\dagger\sigma\rangle \overset{(\Delta=0)}{=} \frac{2e^{-\frac{\kappa t}{2}}}{\Omega_\kappa^2}\left[-4g^2 + e^{\frac{\Omega_\kappa t}{2}}\left[\frac{\kappa^2}{4} - 2g^2 + \frac{\kappa}{4}\Omega_\kappa\right]e^{-\frac{\Omega_\kappa t}{2}}\left[\frac{\kappa^2}{4} - 2g^2 - \frac{\kappa}{4}\Omega_\kappa\right]\right],$$

where we have defined $\Omega_\kappa = \sqrt{\kappa^2 - 16g^2}$. We can now plot the function for any combination between g and κ.
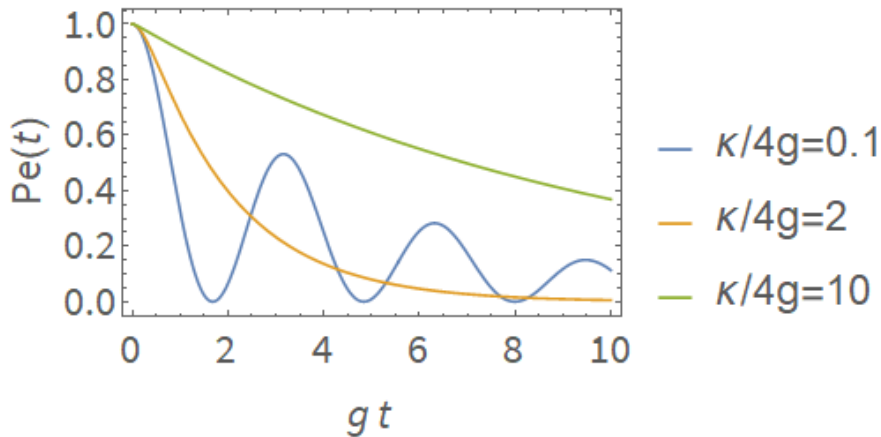
Here, one can distinguish two particularly relevant limiting cases:

- $\kappa \ll 4g$ (weak damping) $P_e(t) \approx \underbrace{\dfrac{e^{-\kappa t/2}}{2} \left[1 + \cos(2gt)\right]}_{\text{damped oscillations}}$

- $\kappa \gg 4g$ (overdamped cavity) $P_e(t) \approx \underbrace{e^{-\frac{4g^2}{\kappa}t}}_{\text{Decay with constant } \frac{4g^2}{\kappa}}$

So, overall, we have learnt that the coupling of the cavity to the environment acts as all effective coupling of the atom to the environment $\Rightarrow$ the atom decays with rate $\frac{4g^2}{\kappa}$.

2.3 Many atoms: Dicke and Tavis-Cummings models in and out of equilibrium
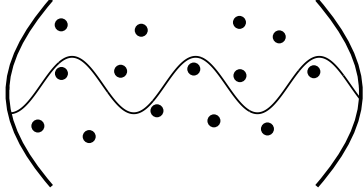
Now we will describe the coupling of an ensemble of N atoms with a single cavity mode. The Hamiltonian that describes this system is, in the most general form, very similar to the Jaynes-Cummings Hamiltonian, but simply adding the contribution of each of the atoms as

$$H = \hbar \nu a^\dagger a + \hbar \omega_0 \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i + \frac{\hbar g}{\sqrt{N}} \sum_{i=1}^{N} \left(\sigma_i^\dagger + \sigma_i\right)\left(a^\dagger + a\right) \ .$$

Note that the $\sqrt{N}$ in the denominator in front of the sum is introduced such that the sum behaves well in the thermodynamic limit, that is, in the limit $N \to \infty$. Moreover, here we have introduced the ladder operator for the i-th atom as $\sigma_i = |g_i\rangle\langle e_i|$. This model has been widely studied in the literature due to the different phases and phase transitions it can display, both in equilibrium and out of equilibrium, i.e., both in the coherent, closed version and the one where dissipation mechanisms are included. This model has also been widely studied experimentally.

2.3.1 Equilibrium Dicke transition

We will start now by considering the equilibrium case, i.e., we leave the dissipation for the moment out of the problem. We will see that this Hamiltonian displays a phase transition, i.e., that the properties (a so-called order parameter) of the thermal equilibrium state of the Hamiltonian change dramatically when varying a parameter (here it will

be the coupling g). This change becomes sudden, or sharp, only in the thermodynamic limit $N \to \infty$.

The most common approach to see this is using *mean-field theory*. For us, this will mean to break the correlations between photons and atoms, assuming the cavity photons to be described as a classical field in the thermodynamic limit. In practice, we introduce the operator

$$\alpha = \frac{a}{\sqrt{N}},$$

which satisfies

$$\left[\alpha, \alpha^\dagger\right] = \frac{1}{N}.$$

In the thermodynamic limit this commutator vanishes, i.e., α can in this limit be considered as a number. The Hamiltonian thus becomes

$$H_{MF} = \hbar\nu N|\alpha|^2 + \sum_i \left[\hbar\omega_0\sigma_i^\dagger\sigma_i + \hbar g\left(\sigma_i^\dagger + \sigma_i\right)(\alpha + \alpha^*)\right]$$

$$= \sum_i \underbrace{\left[\hbar\nu\alpha^2 + \hbar\omega_0\sigma_i^\dagger\sigma_i + \hbar g\left(\sigma_i^\dagger + \sigma_i\right)(\alpha + \alpha^*)\right]}_{\text{Each atom experiences the same Hamiltonian: } h(\alpha)}$$

We will find the thermal equilibrium state of the system varying α to find the minimum *free energy*, which is defined as

$$F(\alpha) = -\frac{1}{\beta} \ln\left[Z(\alpha)\right]$$

with $\beta = \frac{1}{k_B T}$ ($T \equiv$ temperature, $k_B \equiv$ Boltzmann constant), and $Z(\alpha)$ being the partition function, which can be calculated as

$$Z(\alpha) = \text{tr}\left(e^{-\beta H_{MF}}\right) = \text{tr}\left(e^{-\beta h(\alpha)}\right)^N$$

$$= \left[e^{-\beta E_+} + e^{-\beta E_-}\right]^N$$

where E_+ and E_- are the eigenvalues of $h(\alpha)$. These can be found as:

$$h(\alpha) = \hbar \begin{pmatrix} \nu|\alpha|^2 + \omega_0 & g\alpha \\ g\alpha^* & \nu|\alpha|^2 \end{pmatrix} \Rightarrow \frac{E_\pm}{\hbar} = \nu|\alpha|^2 + \frac{\omega_0}{2} \pm \frac{\sqrt{\omega_0^2 + 4g^2|\alpha|^2}}{2}$$

such that

$$Z(\alpha) = \left[2e^{-\beta\hbar\left(\nu|\alpha|^2 + \frac{\omega_0}{2}\right)} \cosh\left(\frac{\beta\hbar}{2}\sqrt{\omega_0^2 + 4g^2|\alpha|^2}\right)\right]^N.$$

Now we can calculate the free energy as:

$$F(\alpha) = N\left[\hbar\left(\nu|\alpha|^2 + \frac{\omega_0}{2}\right) - \frac{1}{\beta}\ln\left(2\cosh\left(\frac{\beta\hbar}{2}\sqrt{\omega_0^2 + 4g^2|\alpha|^2}\right)\right)\right].$$

We calculate now the minimum of this free energy:

$$\frac{\partial F(\alpha)}{\partial\alpha} = N\left[2\hbar\nu|\alpha| - 2\hbar g^2|\alpha|\frac{\tanh\left(\frac{\hbar\beta}{2}\sqrt{4g^2|\alpha|^2 + \omega_0^2}\right)}{\sqrt{4g^2|\alpha|^2 + \omega_0^2}}\right]$$

- Solution 1: $\alpha = 0$

- Solution 2: α finite and solution of the transdencental equation:

$$\sqrt{4g^2|\alpha|^2 + \omega_0^2} = \frac{g^2}{\nu}\tanh\left(\frac{\hbar\beta}{2}\sqrt{4g^2|\alpha|^2 + \omega_0^2}\right)$$

We can now find the transition point, i.e., the value of g at which the system changes from Solution 1 to Solution 2. We can do this by expanding the transcendental equation around $\alpha = 0$, which gives

$$\underbrace{\omega_0\sqrt{1 + \frac{4g^2}{\omega_0^2}|\alpha|^2}}_{\text{use: } \alpha\to 0} = \frac{g^2}{\nu}\tanh\left(\frac{\hbar\beta\omega_0}{2}\sqrt{1 + \frac{4g^2}{\omega_0^2}|\alpha|^2}\right)$$

$$1 + \frac{2g^2}{\omega_0^2}|\alpha|^2 = \frac{g^2}{\omega_0\nu}\underbrace{\tanh\left(\frac{\hbar\beta\omega_0}{2}\right)}_{=1 \text{ for } \beta\to\infty (T\to 0)}$$

$$\frac{2g^2}{\omega_0^2}|\alpha|^2 = \overbrace{\frac{g^2}{\omega_0\nu} - 1}^{\substack{\text{this has to be } > 0 \\ \text{for this solution to be valid}}}.$$

Putting the right hand side to be exactly equal to zero gives us the transition point between the two phases, which is here given by
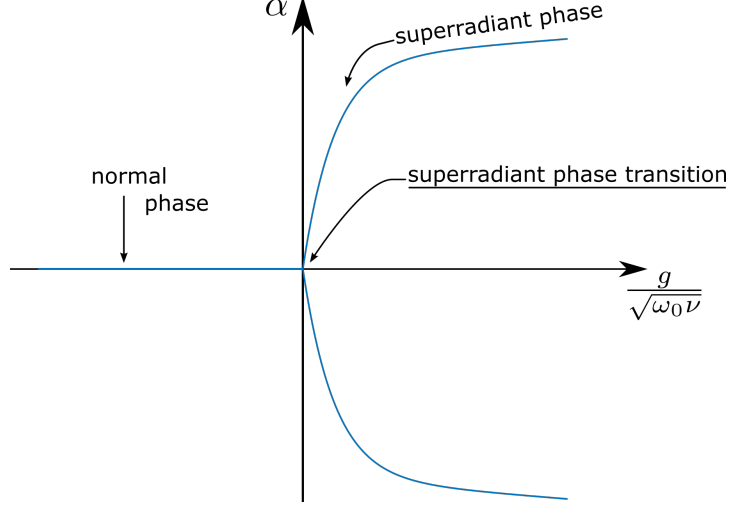
$$g_c = \sqrt{\omega_0\nu}.$$

For $g > g_c$, but close to the transition point we obtain that the order parameter grows

as

$$|\alpha| \approx \frac{\omega_0}{\sqrt{2}gg_c}\sqrt{g^2 - g_c^2}\ .$$

The phase diagram, which here is nothing but the variation of the order parameter α as a function of the coupling strength, is depicted below.

Here, we can observe that there are indeed two clearly distinct phases. In the so-called normal phase $\alpha = 0$, which means that the cavity has zero occupation. Moreover, here one can obtain as well that all the atoms are in the ground state. In the superradiant phase, on the other hand, $\alpha \neq 0$, which indicates a non-zero occupation of the cavity. This superradiant phase transition can also be related to the symmetries of the Hamiltonian. One can easily see that the Hamiltonian conserves the parity of the total number of excitations $P = (-1)^{N_{ex}}$ where

$$N_{ex} = a^\dagger a + \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i$$

(either N_{ex} is conserved or it goes to $N_{ex} \pm 2$). This can also be see as the Hamiltonian remaining unchanged as $a \to -a$ and $\sigma \to -\sigma$. While the normal phase does indeed conserve this symmetry, the superradiant one does not, and hence this superradiant transition is accompanied, or signalled, by a *spontaneous symmetry breaking* of the superradiant phase.

2.3.2 Non-equilibrium Dicke transition

We now consider again the very same situation of N atoms in a cavity, but now the cavity is *damped*. Here, we have to use what we learnt in Section 2.2 and write first the master equation that determines the dynamics of the system as

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H, \rho] + \kappa \left[a\rho a^\dagger - \frac{1}{2} \left\{ a^\dagger a, \rho \right\} \right] ,$$

with

$$H = \hbar \nu a^\dagger a + \hbar \omega_0 \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i + \frac{\hbar g}{\sqrt{N}} \sum_{i=1}^{N} \left(\sigma_i^\dagger + \sigma_i \right) \left(a^\dagger + a \right) .$$

We will now investigate this system, and obtain that it also undergoes a phase transition. However, here we are not interested in the ground or equilibrium state of the system, but rather on its *stationary state* (the state of the system and at very long times).

Before we start, it is convenient to introduce the collective atomic operators

$$J = \sum_{i=1}^{N} \sigma_i \quad ; \quad J^\dagger = \sum_{i=1}^{N} \sigma_i^\dagger \quad ; \quad J_z = \frac{1}{2} \sum_{i=1}^{N} \left(\sigma_i^\dagger \sigma_i - \sigma_i \sigma_i^\dagger \right)$$

such that the Hamiltonian reads (*)

$$H = \hbar \nu a^\dagger a + \hbar \omega_0 \left(J_z + \frac{N}{2} \right) + \frac{\hbar g}{\sqrt{N}} \left(a^\dagger + a \right) \left(J + J^\dagger \right) .$$

These collective operators satisfy the commutation relations

$$\left[J^\dagger, J \right] = 2J_z \quad ; \quad \left[J, J_z \right] = J \quad ; \quad \left[J^\dagger, J_z \right] = -J^\dagger .$$

Now, using the Heisenberg picture again, as we did in Section 2.2, we can obtain the equations of motion for the expectation values of a, J, and J_z, as: (*)

- $\frac{\partial \langle a \rangle}{\partial t} = - \left(i\nu + \frac{\kappa}{2} \right) \langle a \rangle - i \frac{g}{\sqrt{N}} \left(\langle J^\dagger \rangle + \langle J \rangle \right)$

- $\frac{\partial \langle J \rangle}{\partial t} = -i\omega_0 \langle J \rangle + i \frac{2g}{\sqrt{N}} \left\langle \left(a^\dagger + a \right) J_z \right\rangle$

- $\frac{\partial \langle J_z \rangle}{\partial t} = i \frac{g}{\sqrt{N}} \left\langle \left(a^\dagger + a \right) \left(J^\dagger - J \right) \right\rangle .$

Now we invoke again mean-field theory in order to be able to solve these equations. In particular, we break the correlations between the atoms and the cavity and factorize the expectation values as

- $\left\langle \left(a^\dagger + a \right) J_z \right\rangle = \left\langle a^\dagger + a \right\rangle \left\langle J_z \right\rangle$

- $\left\langle \left(a^\dagger + a \right) \left(J^\dagger - J \right) \right\rangle = \left\langle a^\dagger + a \right\rangle \left\langle J^\dagger - J \right\rangle$.

Finally, we look back into the Hamiltonian and realize that it actually commutes with the total angular momentum $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$, which means that it only connects states within the same *Dicke manifold*, with the same total angular momentum number j, which for us is $j = N/2$. Hence, we can impose the condition (*)

$$\langle J_z \rangle^2 + \left\langle J^\dagger \right\rangle \langle J \rangle = \frac{N^2}{4} \ .$$

With all of the above, one obtains two possible solutions of the expectation values in the stationary state, obtained by putting:

$$\frac{\partial \langle a \rangle}{\partial t} = \frac{\partial \langle J \rangle}{\partial t} = \frac{\partial \langle J_z \rangle}{\partial t} = 0.$$

The first solution is:

A) $\langle a \rangle_A = 0 \quad ; \quad \langle J \rangle_A = 0 \quad ; \quad \langle J_z \rangle_A = -\frac{N}{2}$
$\Rightarrow$ No cavity photons and all atoms in the ground state..
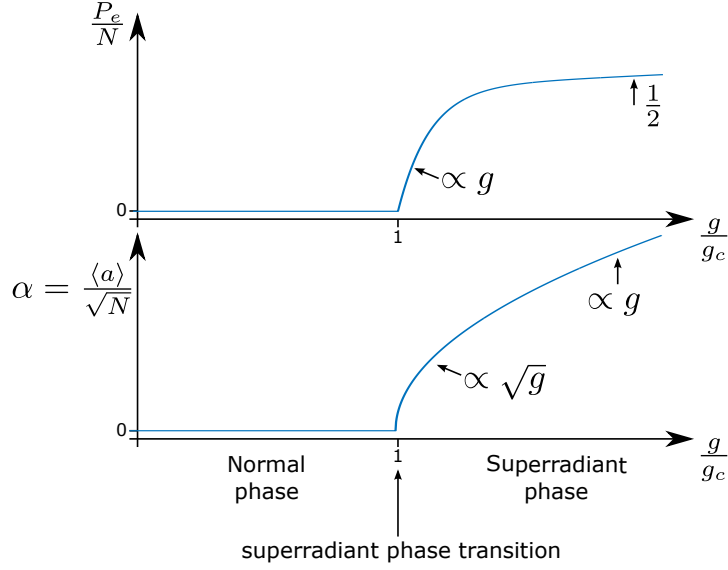
The second solution is:

B) $\langle J_z \rangle_B = -\frac{N}{2} \underbrace{\frac{\omega_0}{4\nu} \left(\nu^2 + \frac{\kappa^2}{4} \right) \frac{1}{g^2}}_{\equiv g_c^2 \text{ because } |\langle J_z \rangle| \leq \frac{N}{2}}$

so we have found the value of g at which the transition happens to be

$$\boxed{g_c = \frac{1}{2} \sqrt{\frac{\omega_0}{\nu} \left(\nu^2 + \frac{\kappa^2}{4} \right)}}$$

The values of $\langle J \rangle_B$ and $\langle a \rangle_B$ are

1) $\left| \langle J \rangle_B \right| = \frac{N}{2} \sqrt{1 - \frac{g_c^4}{g^4}}$
2) $\frac{\langle a \rangle_B}{\sqrt{N}} = \pm \frac{g}{\nu - \frac{i\kappa}{2}} \sqrt{1 - \frac{g_c^4}{g^4}}$

Note, that even though the two equilibrium and non-equilibrium phase transitions are called superradiant, they have slightly different behaviours (at least at zero temperature). In particular, they belong to different so-called universality classes.

2.3.3 Tavis-Cummings model with damping

Up to now we have used the Dicke model, which contains also the counterrotating terms $\sigma_i^\dagger a^\dagger, \sigma_i a$. In this last section, we will investigate the so-called Tavis-Cummings (TC) model

$$H_{TC} = \hbar\nu a^\dagger a + \hbar\omega_0 \sum_{i=1}^{N} \sigma_i^\dagger \sigma_i + \frac{\hbar g}{\sqrt{N}} \sum_{i=1}^{N} \left(a\sigma_i^\dagger + a^\dagger \sigma_i \right)$$

where these counterrotating terms have been neglected. Both the Dicke and Tavis-Cummings Hamiltonians can be experimentally realized.

This Hamiltonian conserves the number of excitations, and not only their parity, as it was the case in the Dicke model. Now, if we add cavity damping to this problem (κ photon loss rate), the stationary state is always the one with $\alpha = 0$, i.e. a normal phase, which in turn means that no phase transition can take place. In order to recover a transition, we have to find a mechanism that adds excitations/photons in some way.
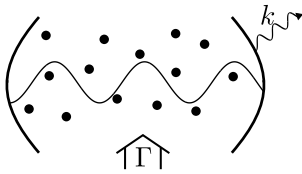
 Here, we choose to study the system in the presence of an external *incoherent pumping* of the atoms at a rate Γ. This process can be accounted for in the master equation by adding a dissipation term with jump operator $\sigma^\dagger$, such that the full

master equation reads now

$$\dot{\rho} = -i\left[\nu a^\dagger a + \omega_0 \sum_{i=1}^N \sigma_i^\dagger \sigma_i + \frac{g}{\sqrt{N}}\sum_{i=1}^N\left(a\sigma_i^\dagger + a^\dagger\sigma_i\right),\rho\right]$$

$$+\kappa\left(a\rho a^\dagger - \frac{1}{2}\left\{a^\dagger a, \rho\right\}\right) + \Gamma\underbrace{\sum_{i=1}^N\left(\sigma_i^\dagger\rho\sigma_i - \frac{1}{2}\left\{\sigma_i\sigma_i^\dagger,\rho\right\}\right)}$$

like decay
but upward!

Again, we can obtain again the equations of motion for the expectation value of the operators a, σ_i and $\sigma_z^{(i)}$, which yields (*)

- $\frac{\partial\langle a\rangle}{\partial t} = -\left(i\nu + \frac{\kappa}{2}\right)\langle a\rangle - i\frac{g}{\sqrt{N}}\sum_i\langle\sigma_i\rangle$

- $\frac{\partial\langle\sigma_i\rangle}{\partial t} = -\left(i\omega_0 + \frac{\Gamma}{2}\right)\langle\sigma_i\rangle + i\frac{g}{\sqrt{N}}\left\langle a\sigma_z^{(i)}\right\rangle$

- $\frac{\partial\left\langle\sigma_z^{(i)}\right\rangle}{\partial t} = i\frac{2g}{\sqrt{N}}\left(\left\langle\sigma_i a^\dagger\right\rangle - \left\langle\sigma_i^\dagger a\right\rangle\right) + \frac{\Gamma}{2}\left(1 - \left\langle\sigma_z^{(i)}\right\rangle\right)$

We can solve these equations in the stationary state again using mean-field theory

$$\left\langle Ja^\dagger\right\rangle = \langle J\rangle\left\langle a^\dagger\right\rangle$$
$$\langle aJ_z\rangle = \langle a\rangle\langle J_z\rangle \ .$$

Again, one finds that a solution (for small g) is the normal phase:

$$\langle a\rangle_A = 0 \quad\text{and}\quad \langle J_z\rangle_A = \frac{N}{2}$$
(inverted population!).

However, another solution yields $\langle a\rangle_B \neq 0$, and here one obtains

$$\langle J_z\rangle_B = \frac{N}{2}\frac{1}{g^2}\left[\frac{\kappa\Gamma}{4} - \omega_0\nu + \frac{i}{2}\underbrace{\left(\omega_0 k + \nu\Gamma\right)}\right].$$

imaginary part
cannot be made 0!

Since the expectation value $\langle J_z\rangle$ must always be real by definition, and the imaginary part cannot be put to zero, this means there is no stationary state for large coupling g. This in turn means that the solution for large time t is *oscillating*. Hence, this is a so-called *no-lasing* $\Rightarrow$ *lasing* transition.

Literature for Chapter 2:

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- P. Kirton, M.M. Roses, J. Keeling, E.G. Dalla Torre, Advanced Quantum Technologies, 2(1-2), 1970013 (2019)

- F. Dimer, B. Estienne, A.S. Parkins, and H.J. Carmichael, Phys. Rev. A 75, 013804 (2007)

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3 Rydberg Atoms

In a nutshell, a Rydberg atom is an atom whose valence electron in a state with very high principal quantum number $n \gg 1$. As a consequence, these atoms possess very exaggerated properties, such as very long lifetimes and, most importantly, extremely large dipole moments which lead these atoms to display interactions among them which are very strong over very long distances.
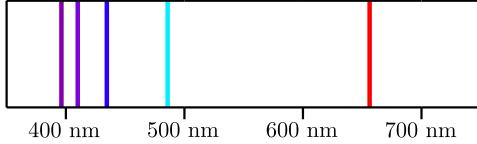
Historically, the first time this concept appeared was in 1885, with Balmer's formula for the wavelengths of the visible absorption lines of hydrogen, given by

$$\lambda = \frac{bn^2}{n^2 - 4}$$

with $b = 3645.6$ Å. Actually, this formula gives the wavelengths corresponding to electronic transitions from a state of hydrogen with $n = 2$ to higher lying levels with principal quantum number n, as it can be seen writing it as

$$\frac{1}{\lambda} = \frac{4}{b} \left(\frac{1}{4} - \frac{1}{n^2} \right) \ .$$

Similar formulas were found for other series by Lyman, Brackett, Paschen ... and then summarized by Johannes Rydberg in 1890 as

$$\frac{1}{\lambda} = \frac{R_y}{hc} \left(\frac{1}{n_2^2} - \frac{1}{n_1^2} \right)$$

where n_1 and n_2 represent the initial and final state of the electron, respectively. Here, $R_y = 13.6$ eV is the Rydberg unit of energy, which also determines the binding energy of the electron in the hydrogen atom
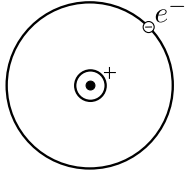
$$W = -\frac{R_y}{n^2},$$

which is the minimum energy necessary for an electron in a state with principal quantum number n to leave the atom (obtained by putting n_2 to infinity).

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3.1 Properties of Rydberg atoms

In this subsection we will get to know a few of the very exaggerated properties that make Rydberg atoms unique.

3.1.1 Binding energy

The description of alkali atoms such as Rb, Na and K with their valence electron in a Rydberg state is similar to the hydrogen, as they are both to first order formed by an electron orbiting a core of net positive $+e$ charge. The difference between the two is that in the alkali atom the positive core is formed by a nucleus with charge $+Z$ and $Z-1$ electrons. Hence, it has a structure and it can only be considered point-like, as a single proton, when the electron is very far away from the nucleus. This is actually the case when the electron is in a high angular momentum state (i.e., with $l \geq 4$). Here, the basic properties of the two systems are the same, since in these states the orbits of the valance electron are such that they do not penetrate the ionic core, and hence they experience a potential very similar to the usual Coulomb one ($\propto -1/r$). On the other hand, when l takes low values (e.g. $l = 0, 1, 2 \ldots$, corresponding to s,p or d states), the valence electron penetrates the inner core, probing its structure, and, consequently, the wave functions and energy levels of an alkali Rydberg atom differ from their hydrogen counterparts.

This effect can be accounted for in the binding energy of an alkali Rydberg atom, which is accurately described by

$$ W = -\frac{R_y}{n^{*2}} = -\frac{R_y}{(n - \delta_l)^2}. $$

Here, we have introduced δ_l, which stands for the so-called *quantum defect*, and $n^* = n - \delta_l$, called the *effective principal quantum number*. The quantum defect δ_l is particularly high for low l quantum states. The consequence of this quantum defect in the spectrum of an alkali atom is that, while in the hydrogen all states with the same principal quantum number n form degenerate manifolds, In the alkali atoms this degeneracy is broken, and the low l states are notably shifted in energy, as shown schematically in the figure.
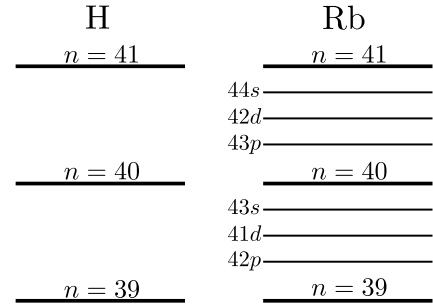
3.1.2 Dipole matrix elements

Transitions between atomic states primarily occur due to coupling with the electric dipole moment $\vec{d} = -e\vec{r}$ (remember section 1.3). In order to find the scaling of this dipole moment with the principal quantum number in the case of a Rydberg state, let us consider the situation where the electron performs a transition between two states with the same n and l differing by one, such that the transition is dipole permitted, i.e.

$$|n, l, m_l\rangle \longrightarrow |n, l \pm 1, m_l\rangle \ .$$

Note that the energy difference between these two levels is not very large. The transition dipole moment here is calculated as

$$
\begin{aligned}
\left\langle \vec{d} \right\rangle &= \langle n, l, m_l | \vec{d} | n, l \pm 1, m_l \rangle = -e \langle n, l, m_l | \vec{r} | n, l \pm 1, m_l \rangle \\
&= -e \Big[\langle n, l, m_l | r \sin(\theta)\cos(\varphi)\hat{x} + r\sin(\theta)\sin(\varphi)\hat{y} + r\cos(\theta)\hat{z} | n, l \pm 1, m_l \rangle \Big],
\end{aligned}
$$

where we have separated the components of the dipole into each individual direction, i.e. $\left\langle \vec{d} \right\rangle = \langle d_x \rangle \hat{x} + \langle d_y \rangle \hat{y} + \langle d_z \rangle \hat{z}$.

In order to calculate each of these components (here, for example, the z-component), we need to consider that the wave functions, just like in the hydrogen case, can be separated into radial and angular coordinates, i.e.

$$|n, l, m_l\rangle \propto R_{nl}(r) Y_{lm}(\theta, \varphi),$$

such that

$$\langle d_z \rangle \propto \int_\Omega d\Omega \ Y_{lm_l}(\theta, \varphi) \cos(\theta) Y^*_{l\pm 1 m_l}(\theta, \varphi) \int_0^\infty dr \ r^3 R_{nl}(r) R^*_{nl\pm 1}(r) \ .$$

Here, one can see that the first term (angular component) is independent of the principal quantum number n. On the other hand, for large enough orbital quantum numbers l, the two radial components are extremely similar, such that the second component is very close to the average distance of the electron to the nucleus, $\langle r \rangle$, which in turn is proportional to n^2. Since here $n \gg 1$, this gives rise to *very strong dipole moments*. Note, however, that these are not permanent dipoles, but induced dipoles due to the electron being most of the time <u>very far</u> ($\langle r \rangle$ can be up to a micrometer) from the nucleus.

3.1.3 Lifetime

Another attractive feature of Rydberg atoms (i.e. atoms with the valence electron in a Rydberg state) is that these states have very long radiative lifetimes. This seems perhaps counter-intuitive at first, because the electron is so highly excited and consequently close to the ionization threshold. We can estimate the scaling of the lifetime with n by considering what we know from section 1.3.

Let us consider a two-level system as the one in the figure. We calculated that the rate of decay from the initial to final state is given by
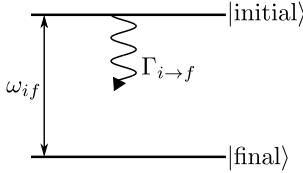
$$\Gamma_{i \to f} \propto \omega_{if}^3 \left| \left\langle \vec{d} \right\rangle_{if} \right|^2 ,$$

where $\hbar \omega_{if}$ is the energy difference between the two states and $\left\langle \vec{d} \right\rangle_{if}$ is the corresponding transition dipole moment. Let us now consider the scaling of these observables to obtain the one of the lifetime of the initial state, τ, which is inversely proportional to the decay rate.

First, for a state with large orbital quantum number (l close to n), we have calculated that $\left| \left\langle \vec{d} \right\rangle_{if} \right| \propto n^2$. Moreover, here the quantum defect is very small and hence the energy difference between states with the same n scales as $\omega_{if} \propto 1/n^3$. This overall gives that

$$\Gamma_{i \to f}^{(l \approx n)} \propto \frac{1}{n^9} n^4 = \frac{1}{n^5} .$$

On the other hand, when $l = 0, 1 \ldots$ the calculation needs to be done considering the specific form of wave functions of the alkali atom, slightly different from the hydrogen ones. This calculation results in

$$\Gamma_{i \to f}^{(l \approx 0)} \propto \frac{1}{n^3} .$$

Hence, the lifetimes of the Rydberg states $\tau \propto 1/\Gamma_{i \to f}$ grow rapidly with the principal quantum number, between the two limiting scalings n^5 and n^3. As an example, for a rubidium atom:

$$n = 60 \quad l = 1 \longrightarrow \tau \approx 7.2 \ \mu s$$
$$n = 60 \quad l = 54 \longrightarrow \tau \approx 70 \ ms$$

in both cases having values orders of magnitude larger than the lifetimes of low-lying

excited states (for example 11 ns for the first excited state of rubidium), and in some cases larger than the timescale in which experiments with cold atoms are realized, making them effectively stable over the experimental time.

3.2 Interactions between Rydberg atoms

In this section we will study how two or more Rydberg atoms interact with each other. We will then study the consequences of these interactions in the dynamics of a Rydberg gas driven by a laser field, in particular focussing on the phenomenon so-called Rydberg blockade.

3.2.1 Dipole-dipole Interactions
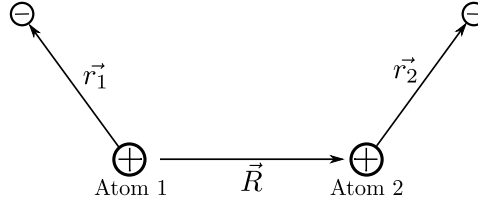
Above we have a schematic of two Rydberg atoms separated by a distance $\vec{R}$. Each atom as a nucleus and valence electron separated by the relative distances $\vec{r_1}$ and $\vec{r_2}$ (for atom 1 and 2, respectively). One can obtain the interaction between the two atoms (here represented for simplicity as dipoles) simply by considering the Coulomb interactions between the charges and using that $\left|\vec{R}\right| \gg |\vec{r_1}|, |\vec{r_2}|$:

$$V = e^2 \left[\underbrace{\frac{1}{\left|\vec{R}\right|}}_{++} + \underbrace{\frac{1}{\left|\vec{R} + \vec{r_2} - \vec{r_1}\right|}}_{--} - \underbrace{\frac{1}{\left|\vec{R} + \vec{r_2}\right|}}_{+-} - \underbrace{\frac{1}{\left|\vec{R} - \vec{r_1}\right|}}_{-+} \right]$$

$$\overset{(*)}{=} e^2 \left[\frac{1}{R} + \frac{1}{R\sqrt{1 + \frac{r_1^2 + r_2^2 - 2\vec{r_1}\cdot\vec{r_2}}{R^2} + 2\frac{\vec{r_2}\cdot\hat{R} - \vec{r_1}\cdot\hat{R}}{R}}} - \frac{1}{R\sqrt{1 + \frac{r_2^2}{R^2} + 2\frac{\vec{r_2}\cdot\hat{R}}{R}}} - \frac{1}{R\sqrt{1 + \frac{r_1^2}{R} - 2\frac{\vec{r_1}\cdot\hat{R}}{R}}} \right]$$

$$\underset{\text{up to } 1/R^3}{\approx} (R \gg r_1, r_2) \approx e^2 \frac{\left[\vec{r_1}\cdot\vec{r_2} - 3\left(\vec{r_1}\cdot\hat{R}\right)\left(\vec{r_2}\cdot\hat{R}\right) \right]}{R^3}.$$

Note that this is simply a multipole expansion. Since the atoms are neutral, the 1st and 2nd order terms are zero and the $1/R^3$ term is the dominant one.

3.2.2 Simple model for interactions
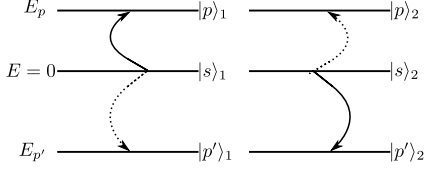
In a real atom, the spectrum can be very complicated and the amount of levels one may need to consider to obtained a detailed and realistic model for the interactions, large. However, let us consider a simplified model that gives a quite accurate depiction of the interactions.

To do so, we consider two atoms with three energy levels, $|s\rangle \equiv |ns\rangle$, $|p\rangle \equiv |np\rangle$ and $|p'\rangle \equiv |n'p\rangle$. Both p-states can be dipole-coupled to the s-state. For simplicity, let us put the energy of the s-state to zero, and $E_p > 0$ and $E_{p'} < 0$.

Now we will consider the possible states of the two atoms, and to do so we consider that the atoms are both initially in the $|s\rangle$ state. The only states directly coupled to this state are states where both atoms are in a p-state. Hence, we can consider as out basis the states

$$|S\rangle \equiv |s\rangle_1 \otimes |s\rangle_2$$
$$|P\rangle \equiv |p\rangle_1 \otimes |p\rangle_2$$
$$|P'\rangle \equiv |p'\rangle_1 \otimes |p'\rangle_2$$
$$|PP'\rangle \equiv \frac{1}{\sqrt{2}} \left[|p\rangle_1 \otimes |p'\rangle_2 + |p'\rangle_1 \otimes |p\rangle_2 \right] \ .$$

In order to further simplify our model, we will consider a regime in which

$$\delta = E_p + E_{p'} \ll 2E_p, 2E_{p'} \ .$$

As a consequence, $|P\rangle$ and $|P'\rangle$ are well out of resonance and the transition probabilities to these states is negligible. Within these approximations, the Hamiltonian describing the interaction between the states $|S\rangle$ and $|PP'\rangle$ is:

$$H_{\text{int}} = \hbar \begin{pmatrix} 0 & V \\ V & \delta \end{pmatrix}$$

where we have used as basis states $|S\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |PP'\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In order to obtain the actual interactions between two Rydberg atoms, we need to

diagonalize this Hamiltonian, which gives the eigenenergies

$$E_\pm = \frac{\delta}{2} \pm \frac{1}{2}\sqrt{\delta^2 + 4V^2} \ ,$$

where one can see that the energy of the pair of states depends upon the separation between the atoms via V. The specific form of the spatial dependence can be derived in two distinct regimes:

i) *Long range* $(V(R) \ll \delta)$

Here the energies $E_\pm = \frac{\delta}{2} \pm \frac{\delta}{2}\sqrt{1 + \frac{4V^2}{\delta^2}}$ can be expanded for large δ as

$$E_+ \approx \delta + \frac{V^2}{\delta}$$
$$E_- \approx -\frac{V^2}{\delta}$$

Moreover, here the eigenstate $|-\rangle \approx |S\rangle$, such that here two atoms in the ns-state experience an energy shift level determined by the *van-der-Waals potential*:

$$\boxed{V_{vdW} = -\frac{C_6}{R^6}} \ ,$$

with $C_6 = \alpha^2/\delta$ being the van-der-Waals coefficient and

$$\alpha \equiv \langle s|e^2 \left(\vec{r_1} \cdot \vec{r_2} - 3\left(\vec{r_1} \cdot \hat{R}\right)\left(\vec{r_2} \cdot \hat{R}\right)\right)|PP'\rangle$$

the *polarizability*. Note now, that since we know that the transition dipole moment of a Rydberg atom is proportional to n^2, the polarizability α is proportional to n^4. Moreover, we also know that the energy difference between nearby levels goes as $\delta \propto 1/n^3$. Hence,

$$\boxed{C_6 \propto n^{11}}$$

which in turn gives rise to very strong interactions, even at large distances. As an example, consider that for Rb, two atoms in the $43s$ state give rise to

$$C_6 = -4.45 \cdot 10^{-27} \text{ MHz} \cdot \text{m}^6$$

which, for two atoms at $R = 3 \ \mu$m gives an interaction of

$$V_{vdW} = 3.35 \text{ MHz} \ ,$$

orders of magnitude larger than between ground state atoms.
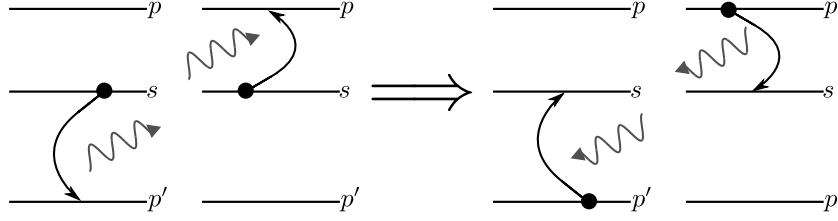
ii) *Short range* $(V(R) \gg \delta)$

In this limit, the expansion of the energies for small δ gives:

$$E_\pm \approx \pm V = \pm \frac{C_3}{R^3} \ .$$

This is the resonant *dipole-dipole interaction* regime, where $C_3 = \alpha$, such that

$$\boxed{C_3 \propto n^4} \ .$$

Here, the eigenstates of the interactions are superpositions of $|S\rangle$ and $|PP'\rangle$. In this limit, the interactions can be better understood as exchange interactions between these two states:

The transition between the two regimes $1/R^3$, $1/R^6$ occurs at the van der Waals radius

$$V(R_{vdW}) = \delta \Rightarrow R_{vdW} = \sqrt[6]{\left| \frac{C_6}{\delta} \right|} \propto n^{7/3} \qquad\qquad (*)$$

An estimate of the interactions strengths can be found below, where a comparison with the interactions between the Rydberg and ground state interactions is shown. Moreover, here one can see clearly the onset of the van der Waals radius (where the gradient of the interactions change). Finally, note that there is an angular dependence of the interaction that we have not discussed in depth. This can become important, particularly for high l states, where the interactions can become highly anisotropic, or in the presence of external fields that fix the dipole direction.
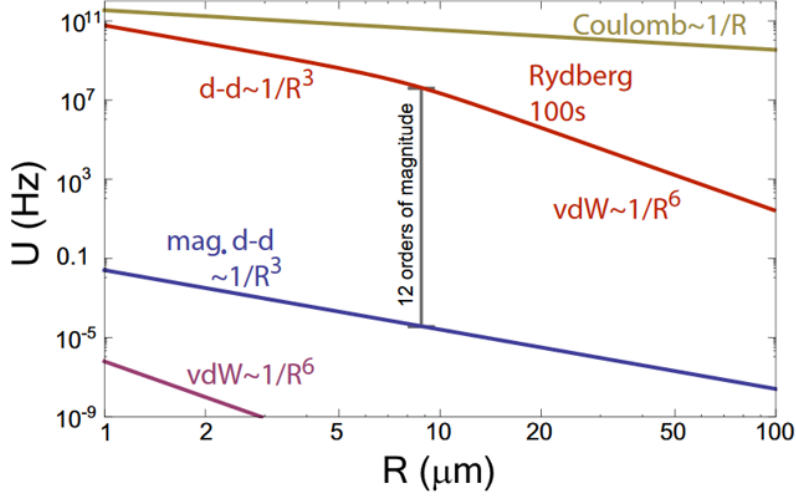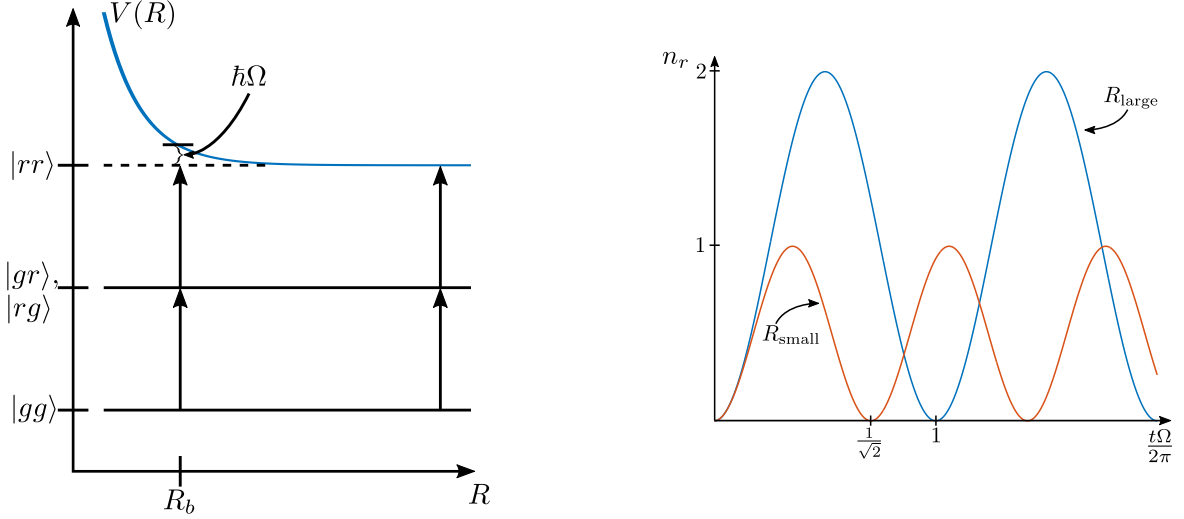
3.2.3 Blockade effect

We have now established that for high principal quantum number n the Rydberg interactions lead to a large energy shift of the state where the two atoms are in the Rydberg state. We will now describe the dynamics of a system of two two-level atoms where the lower and upper level are the ground state and a Rydberg one, respectively.

In particular, let us consider a pair of atoms resonantly driven (by a laser), from their ground state $|g\rangle$ to a Rydberg state $|r\rangle$, with a Rabi frequency Ω. If the atoms are very far form each other, both atoms can be excited to the Rydberg state and hence the dynamics will be such that the atoms go from $|gg\rangle$ to $|rr\rangle$ with frequency Ω, as one can see in the figure below. If the atoms are closer together, however, the interaction causes the $|rr\rangle$ state to be detuned out of resonance for the laser, eventually preventing the excitation of the $|rr\rangle$ state. This can be observed by simply solving the Schrödinger equation with Hamiltonian (remember section 1.2):

$$ H = \frac{\hbar\Omega}{2} \sum_{i=1,2} \left(\sigma_i^\dagger + \sigma_i \right) + V(R) n_1 n_2 $$

with $\sigma_i = |g_i\rangle\langle e_i|$ and $n_i = \sigma_i^\dagger \sigma_i$ being the operator that counts how many excitations there are in atom i.

This is the so-called *dipole blockade*, or *Rydberg blockade effect*. In the simplest model, the so-called *blockade radius* R_b, i.e., the distance at which we consider the $|rr\rangle$ state out of resonance, can be estimated by equating the van-der-Waals interaction with the power-broadened linewidth of the laser, given by its Rabi frequency, i.e. $V(R_b) = \hbar\Omega$, such that

$$R_b = \sqrt[6]{\frac{C_6}{\Omega}} \quad .$$

The consequence of the blockade effect is that within the blockade radius there can only be one Rydberg excitation taking place at a time. Coming back to the dynamics, one can easily obtain from the Hamiltonian (by changing the basis states $|gr\rangle$ or $|rg\rangle$ to their symmetric and antisymmetric superposition states) that in the case of two atoms, the state excited by the laser in the blockaded limit is not $|gr\rangle$ nor $|rg\rangle$, but rather

$$|\Psi_s\rangle = \frac{1}{\sqrt{2}} \left[|gr\rangle + |rg\rangle \right] \quad ,$$

i.e. a symmetric superposition between the two singly excited states. Moreover, the frequency at which this state is excited is increased to be $\sqrt{2}\Omega$.

Finally, note that for an ensemble of N_b atoms localized within a radius $R < R_b$ the blockade also applies. Here, as only one atom can be excited within the full blockaded sphere, the system oscillates between the ground state and the symmetric superposition state

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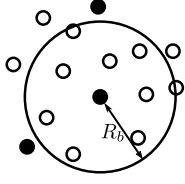
$$\left|g^{N_b-1}r\right\rangle = \frac{1}{\sqrt{N_b}} \sum_{i=1}^{N_b} \left|g_1 g_2 \ldots r_i \ldots g_{N_b}\right\rangle \ .$$

The coupling between these two states gives an enhanced collective Rabi frequency

$$\Omega_{N_b} = \sqrt{N_b}\Omega \ .$$

This effect has been observed in a large amount of experiments, and even exploited for quantum information and computation purposes.

3.3 Rydberg quantum optics

In this section we will focus on one example of application of the interactions between Rydberg atoms (i.e. the Rydberg blockade): the increase of non-linearity of the response of a medium to light.

3.3.1 Electromagnetically induced transparency
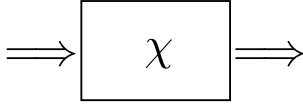
As light (an electric field $\vec{E}$) passes through a medium, it is both attenuated (absorbed) and phase-shifted. This optical response can be characterized in terms of the *susceptibility* χ, which is related to the refractive index $n = \sqrt{1+\chi}$ of a medium. In particular, the dielectric polarization of a medium can be written as $\vec{P} = \varepsilon_0 \chi \vec{E}$. The susceptibility is a complex quantity, with its real part accounting for the phase shift and the imaginary the absorption. The susceptibility can be expressed as a power expansion of the electric field $E = |\vec{E}|$ as
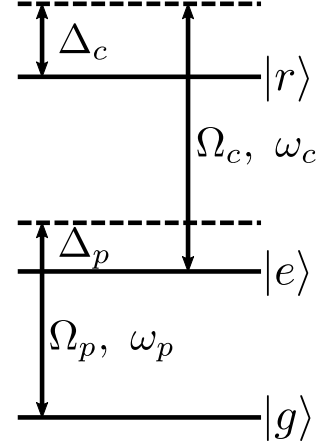
$$\chi = \underbrace{\chi^{(1)}}_{\substack{\text{linear} \\ \text{optical response}}} + \underbrace{\chi^{(2)}E + \chi^{(3)}E^2 + \ldots}_{\text{non-linear processes}} \ .$$

Usually, $\chi^{(1)}$ is the dominant contribution over the higher order effects. However, it is interesting to be able to increase the non-linear response of the medium to be able, for example, to change the velocity of propagation of light in a medium
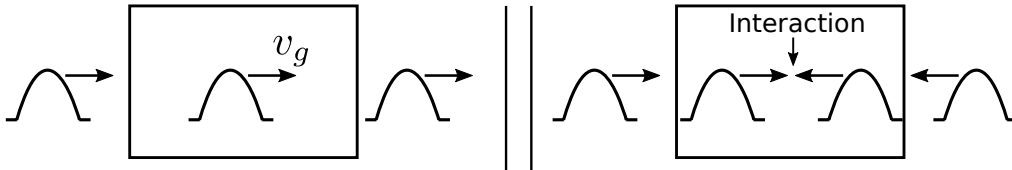
or to create interactions between two or more photons in a medium, inducing, for example, controlled phase shifts between the light pulses that may find applications in quantum information processes.

When the medium is formed by a gas made out of two-level systems, the susceptibility or response to a probe field close to resonance with the transition is dominated by a large and absorptive $\chi^{(1)}$ linear component. However, non-linear effects can be induced by adding a third level and second (control) field as shown in the figure. In particular, we consider a three-level atom with $|g\rangle$ ground, $|e\rangle$ excited and $|r\rangle$ second excited states separated by $\hbar\omega_{eg} = E_e - E_g$ and $\hbar\omega_{re} = E_r - E_e$. We consider now two lasers which drive the $|g\rangle \to |e\rangle$ (*probe* laser) and $|e\rangle \to |r\rangle$ (*control* laser) with detunings $\Delta_p = \omega_{eg} - \omega_p$ and $\Delta_c = \omega_{re} - \omega_c$, respectively (note that we will be interested in the response of the medium to the probe light only). Applying what we know from section 1.2 to the two transitions, we can obtain that the Hamiltonian that describes this physics reads:

$$H = \hbar\Delta_p \, |e\rangle\langle e| + \hbar \left(\Delta_p + \Delta_c \right) |r\rangle\langle r| + \hbar\frac{\Omega_p}{2} \Big(\underbrace{|g\rangle\langle e|}_{\sigma_{eg}} + \underbrace{|e\rangle\langle g|}_{\sigma_{eg}^\dagger} \Big) + \hbar\frac{\Omega_c}{2} \Big(\underbrace{|e\rangle\langle r|}_{\sigma_{er}} + \underbrace{|r\rangle\langle e|}_{\sigma_{er}^\dagger} \Big) \; .$$

Now we will also consider that the intermediate $|e\rangle$ state has a decay rate γ. Then, as we know, we need to resort to the *master equation* for the description of the system such that the dynamics are given by

$$\dot\rho = -\frac{i}{\hbar} \left[H, \rho \right] + \gamma \left(\sigma_{eg}\rho\sigma_{eg}^\dagger - \frac{1}{2} \left\{ \sigma_{eg}^\dagger\sigma_{eg}, \rho \right\} \right) \; .$$

Now one can solve this equation and, in particular, find out the *stationary state* of the system (when $\dot\rho = 0$). Considering that the polarization of the medium is proportional to the expectation value of the dipole moment of the $|g\rangle \to |e\rangle$ transition, one can obtain that the χ_{probe} to the probe light is proportional to

$$\Omega_p\chi_{probe} \propto \rho_{eg} = \langle e|\rho|g\rangle \; .$$

One can solve this easily from the master equation (for example with MATHEMATICA) and find then an expression for χ_{probe}. Since this expression is rather complicated, let us investigate two specific cases:

i) *Probe-only* (Ω_c, $\Delta_c = 0$)

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Without the control laser, the system reduces to a driven two-level atom. Here, it is simple to calculate that (*)

$$\chi_{\text{probe}}^{2L} \propto \frac{\left(\Delta_p + i\frac{\gamma}{2}\right)}{\Delta_p^2 + \frac{\gamma^2}{4} + \frac{\Omega_p^2}{2}},$$

a response that, as expected, is largely linear (you can check this by performing a Taylor expansion for small Ω_p and plotting the full result and the first term of the expansion, to see they are extremely similar!) The absorption (imaginary part of the susceptibility), has a Lorentzian shape and has a maximum on resonance, i.e., at $\Delta_p = 0$ (see Figure below, dashed lines).

ii) *Weak probe limit* $\Omega_p \ll \Omega_c,\ \gamma$

For the full three-level system with a very weak probe laser (for simplicity we put the control laser on resonance, $\Delta_c = 0$), a Taylor expansion for small Ω_p gives a steady-state susceptibility:

$$\chi_{\text{probe}}^{3L} \propto i\frac{1}{2}\overbrace{\frac{1}{\frac{\gamma_l}{2} + i\left(\Delta_p - \frac{\Omega_c^2}{4(\Delta_p + \Delta_c)}\right)}}^{\chi^{(1)}} + \overbrace{\mathcal{O}\left(\Omega_p^2\right)}^{\chi^{(2)}\Omega_p}$$

As one can see in the figure (solid lines), this dramatically changes the response of the medium to the light, particularly around resonance, where the absorption (χ_I) of the medium becomes zero, i.e., the medium is completely transparent to the probe light. As Ω_c is further increased the bandwidth of the Electromagnetically Induced Transparency (EIT) also increases.

Moreover, that the group velocity of the light as it passes through the medium is given by

$$v_g = \frac{c}{n(\omega_p) + \omega_p \frac{\mathrm{d}n}{\mathrm{d}\omega_p}}.$$

Hence, due to the gradient of the real part of χ changing sign, the group velocity decreases, leading to light being slowed. Finally, the last row of the figure below illustrates how the transparency is destroyed by the addition of decay from the upper level.

Note, however, that even though all of these changes are very interesting, they still only change the linear response of the light, with a very small non-linear contribution.
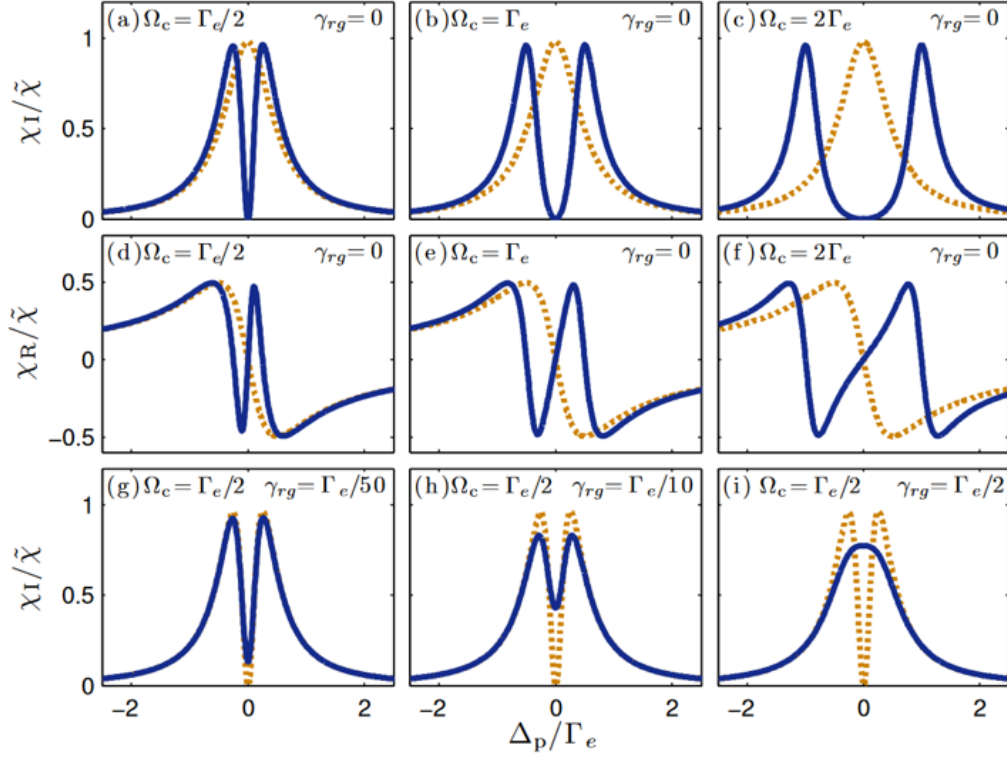
Alternative view of EIT
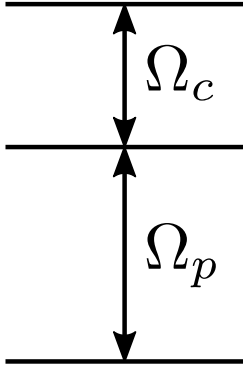
To get a better understanding on EIT, it can be useful to think in terms of dressed states. To do so, we diagonalize H when both lasers are on resonance $(\Delta_p = \Delta_c = 0)$, where the transparency appears. The eigenstates of the Hamiltonian are given by: (*)

$$|+\rangle = \frac{1}{N_+}\left(\frac{\Omega_c}{\Omega_p}|r\rangle + \frac{\sqrt{\Omega_p^2 + \Omega_c^2}}{\Omega_p}|e\rangle + |g\rangle\right)$$

$$|-\rangle = \frac{1}{N_-}\left(\frac{\Omega_c}{\Omega_p}|r\rangle - \frac{\sqrt{\Omega_p^2 + \Omega_c^2}}{\Omega_p}|e\rangle + |g\rangle\right)$$

$$|D\rangle = \frac{1}{N_0}\left(-\frac{\Omega_p}{\Omega_c}|r\rangle + |g\rangle\right)$$

Here, the only state that emits photons is $|e\rangle$. Hence, $|D\rangle$, which has no overlap with $|e\rangle$ is called a *dark* state. The states $|+\rangle$ and $|-\rangle$ do emit photons, and eventually populate the state $|D\rangle$ more and more, such that in the stationary state only $|D\rangle$ is populated. Since $|D\rangle$ is not coupled to the probe beam, this means that the medium becomes transparent. Note that here,

changing adiabatically from a regime where $\Omega_p \ll \Omega_c$ to one where $\Omega_p \gg \Omega_c$ allows to stay always in the $|D\rangle$ state, which goes from being the $|g\rangle$ state to the $|r\rangle$ state. This method of bringing the population from the ground to an excited state is called STIRAP.
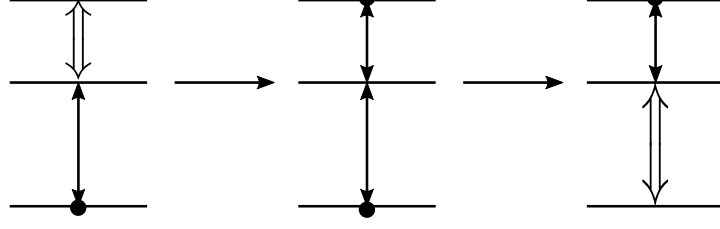
3.3.2 Rydberg EIT

Even though with the EIT one can get a non-negligible amount of non-linearity in the system, it is still very limited. In the following, we will see how to use the Rydberg interactions and, in particular, the blockade effect, to induce even higher non-linearities in the susceptibility and, in turn, use the response of the medium to inform about the interactions between Rydberg states.

To do so, let us consider first two atoms only. Here, the Hamiltonian is very similar to the one we had for one atom, with the difference that the interactions between the two atoms when they are both in the Rydberg state need to be included, and thus

$$H = \frac{\hbar}{2} \sum_{i=1}^{2} \left[\Omega_p \left(\sigma_{eg}^{(i)} + \sigma_{eg}^{\dagger\,(i)} \right) + \Omega_c \left(\sigma_{er}^{(i)} + \sigma_{er}^{\dagger\,(i)} \right) \right] + V(R) \underbrace{n_r^{(1)}}_{|r_1\rangle\langle r_1|} \cdot \underbrace{n_r^{(2)}}_{|r_2\rangle\langle r_2|} ,$$

where we have considered $\Delta_p = \Delta_c = 0$ in order to study the transparency on resonance. We may understand the effect of interactions by using again the dressed states. Remember that for N non-interacting atoms the system goes into a product state of all atoms being in the dark state, i.e.,

$$|D\rangle^N = \bigotimes_{i=1}^{N} |D_i\rangle = \bigotimes_{i=1}^{N} \frac{1}{N_D} \left(-\frac{\Omega_p}{\Omega_c} |r_i\rangle + |g_i\rangle \right) ,$$

which leads to transparency seen before. As Ω_p increases, more population is transferred to the Rydberg state and the Rydberg atoms start to interact, such that the blockade effect takes place, i.e. there can only be one Rydberg excitation in total in the full gas. This can be understood by considering that the dark state for a blockaded pair of atoms

is

$$|D\rangle^{N=2} = \frac{1}{N_D} \left(\underbrace{\frac{\Omega_p^2}{\Omega_c^2 - \Omega_p^2} |ee\rangle}_{\substack{\uparrow \\ \text{this state} \\ \underline{\text{scatters}}}} + \frac{\Omega_p \Omega_c}{\Omega_p^2 - \Omega_c^2} (|gr\rangle + |rg\rangle) + |gg\rangle \right) .$$

I.e., the state $|ee\rangle$, which scatters on resonance forms now part of the dark state and hence the medium is not transparent anymore. This effect becomes stronger as we increase the number of "blockaded" atoms per blockade sphere N_b, as one can observe solving the master equation and calculating again the susceptibility in the stationary state.
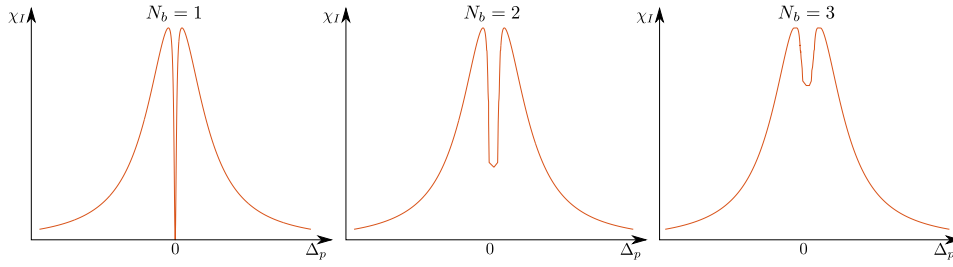
Since this loss of transparency only happens due to the interactions, it is a clear indicator of cooperativity in this system.

Finally, note that the blockade also increases the non-linearity of the medium's response: As N_b atoms are blockaded, the maximum non-linearity on resonance is enhanced by a factor of N_b, i.e.

$$\chi^{(3)} \propto N_b \quad .$$

For a typical Rydberg experiment, with $n = 60$, a blockade radius of $R_b = 5$ μm and density $\rho_0 = 10^{12}$ cm^{-3}, N_b is 1500, which indeed leads to huge non-linearity.

Literature for Chapter 3:

- T.F. Gallagher, Rydberg Atoms, Cambridge University Press, 1994

- M. Saffman, T.G. Walker, and K. Mølmer, Rev. Mod. Phys. 82, 2313 (2010)

- O. Firstenberg, C.S. Adams and S. Hofferberth J. Phys. B: At. Mol. Opt. Phys. 49 152003 (2016)

- E. Urban, T. A. Johnson, T. Henage, L. Isenhower, D. D. Yavuz, T. G. Walker and M. Saffman, Nature Phys. 5, 110 (2009)

4 Collective light-matter coupling

In this chapter we come back to two-level systems where the levels are low-lying, that is close to the ground state.

We will analyze here how a gas of N atoms of those 2-level atoms interact with the electromagnetic field, following the same steps we did in section 1.3. The result is, though, very different: the atoms <u>interact</u> with each other by <u>exchanging photons</u> and the dissipation (the way the full system of atoms emit photons back into the radiation field) becomes **collective**, with the appearance of **super -** and **subradiant** states.
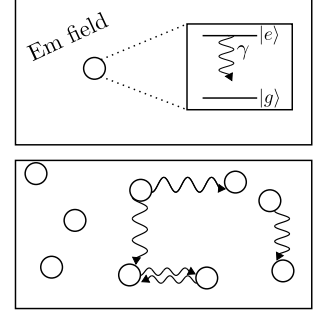
4.1 Many-body master equation

We are considering here exactly the same we did in section 1.3:
A system (which now has N atoms) interacting with a Markovian environment (again the electromagnetic field). Hence, we can use again the general expression for the master equation:

$$\dot{\rho} = -\frac{1}{\hbar^2}\operatorname{tr}_E \int_0^\infty \left[H''(t), \left[H''(t', \rho(t) \otimes \rho_E(0))\right]\right] dt' \tag{4.1}$$

Where we have assumed that Born and Markov approximations are valid, and where $\underline{H''}$ is the interaction Hamiltonian between system and environment in the **interaction picture**. Hence, in order to continue, the first step is to find this Hamiltonian.

4.1.1 Atom-field Hamiltonian

Exactly as in the case of a single atom, the Hamiltonian under the dipole approximation reads:

$$H = H_0 H_F - \underbrace{\sum_{j=1}^N \vec{d_j} \cdot \vec{E}(\vec{r_j}, t)}_{V}$$

where, while the field Hamiltonian stays the same:

$$H_F = \sum_{\vec{k}} \hbar \nu_k a_{\vec{k}}^\dagger a_{\vec{k}} \ ,$$

the atomic part is modified to consider the N atoms

$$H_0 = \hbar\Omega_0 \sum_{j=1}^{N} \sigma_j^\dagger \sigma_j \ ,$$

and the interaction part now considers the dipole moment for all atoms being the same for all atoms,

$$\vec{d}_j = \left(|g\rangle_j \langle g| + |e\rangle_j \langle e| \right) \vec{d} \left(|g\rangle_j \langle g| + |e\rangle_j \langle e| \right) = \vec{d}_{eg}\sigma_j + \vec{d}_{eg}^{\,*}\sigma_j^\dagger$$

with $\vec{d}_{eg} = \langle g| \, \vec{d} \, |e\rangle$. Moreover, now we need to consider now the **spatial dependence** of the electric field, which we did not need for the case of one atom. It reads now:

$$\vec{E}\left(\vec{r}_j, t\right) = \sum_{\vec{k}} \hat{\epsilon}_{\vec{k}} C_{\vec{k}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_j} + a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}_j} \right) \ ,$$

where $\hat{\epsilon}_{\vec{k}}$ is the polarization unit vector and $C_{\vec{k}} = \sqrt{\frac{\hbar\nu_k}{2\epsilon_0 V}}$ is the amplitude. Assuming for simplicity that both the dipole moment and the polarization are real, the interaction Hamiltonian becomes:

$$V = -\hbar \sum_{j=1}^{N} \sum_{\vec{k}} \left(\sigma_j^\dagger + \sigma_j \right) g_{\vec{k}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_j} + a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}_j} \right) \ ,$$

where $g_{\vec{k}} = \frac{\vec{d}_{eg}\cdot\hat{\epsilon}_{\vec{k}} \mathcal{E}_{\vec{l}}}{\hbar}$ is the coupling constant between the atoms and the field.

The next step is again to go into the interaction picture, such that the Hamiltonian reads: (*)

$$H''(t) = -\hbar \sum_{j=1}^{N} \sum_{\vec{k}} \left(\sigma_j^\dagger e^{i\omega_0 t} + \sigma_j e^{-i\omega_0 t} \right) g_{\vec{k}} \left(a_{\vec{k}} e^{i\left(\vec{k}\cdot\vec{r}_j - \nu_k t\right)} + a_{\vec{k}}^\dagger e^{-i\left(\vec{k}\cdot\vec{r}_j - \nu_k t\right)} \right) \ .$$

Note that here we <u>do not</u> perform the RWA just yet. It will come later, but if we do it now we lose terms we will need later!

4.1.2 Obtaining the master equation

We put H'' back into the master equation (4.1). First, we make a change of variables: $t' \to t - \tau$.

$$\dot{\rho} = -\frac{1}{\hbar^2} \int_0^\infty d\tau \, \mathrm{tr}_E \left\{ [H''(t), [H''(t-\tau), \rho(t) \otimes \rho_E(0)]] \right\} \, .$$

We perform then the trace operation, as this reduces dramatically the amount of terms we have to deal with! (only 1 H'' has 4, the total would be ...64?). We simplify the problem by assuming we are in the vacuum state (zero temperature) such that

- $\left\langle a_{\vec{k}} a_{\vec{k}'} \right\rangle = \left\langle a_{\vec{k}}^\dagger a_{\vec{k}'}^\dagger \right\rangle = \left\langle a_{\vec{k}}^\dagger a_{\vec{k}'} \right\rangle = 0$

- $\left\langle a_{\vec{k}} a_{\vec{k}'}^\dagger \right\rangle = \delta_{\vec{k}\vec{k}'}$

This gets rid of many terms (left with "only' 16!') and the $\delta_{\vec{k}\vec{k}'}$ also makes one of the sums over $\vec{k}$ disappear: (*)

$$\dot{\rho} = \int_0^\infty d\tau \sum_{j,l=1}^N \sum_{\vec{k}} \left\{ B_{j\vec{k}}(t) B_{l\vec{k}}^*(t-\tau) \left[A_l(t-\tau)\rho A_j(t) - A_j(t) A_l(t-\tau)\rho \right] + \mathrm{h.c.} \right\} \, ,$$

where

$$B_{j\vec{k}}(t) = g_{\vec{k}} e^{i\left(\vec{k}\cdot\vec{r}_j - \nu_k t\right)}$$
$$A_j(t) = \sigma_j^\dagger e^{i\omega_0 t} + \sigma_j e^{-i\omega_0 t} \, .$$

Now we simplify further, starting with the terms that contain only <u>atomic</u> operators:

$$A_j(t) A_l(t-\tau)\rho = \left[\underbrace{\sigma_j^\dagger \sigma_l^\dagger e^{i\omega_0(2t-\tau)}}_{\text{RWA}} + \sigma_j^\dagger \sigma_l e^{i\omega_0 \tau} + \sigma_j \sigma_l^\dagger e^{-i\omega_0 \tau} + \underbrace{\sigma_j \sigma_l e^{-i\omega_0(2t-\tau)}}_{\text{RWA}} \right]$$

... and <u>here</u> we make the RWA, throwing away the terms $\sigma^\dagger \sigma^\dagger$ and $\sigma\sigma$. This in turn means that we are not interested in very "fast" processes, happening on times $\Delta t < 1/\omega_0$ (really tiny!).

Now the other terms that come from the electric field:

$$B_{j\vec{k}}(t) B_{l\vec{k}}^*(t-\tau) = \left| g_{\vec{k}} \right|^2 e^{i\vec{k}\cdot\vec{r}_{jl}} e^{-i\nu_k \tau}$$

with $\vec{r}_{jl} = \vec{r}_j - \vec{r}_l$. This leaves the master equation as:

$$\dot{\rho} = \sum_{\vec{k}} \left|g_{\vec{k}}\right|^2 \int_0^\infty d\tau \left(e^{i\vec{k}\cdot\vec{r}_{jl}} \left\{ e^{-i(\nu_k-\omega_0)\tau} \left[\sigma_l\rho\sigma_j^\dagger - \sigma_j^\dagger\sigma_l\rho\right] + e^{i(\nu_k-\omega_0)\tau} \left[\sigma_l^\dagger\rho\sigma_j - \sigma_j\sigma_l^\dagger\rho\right]\right\} + \text{h.c.}\right)$$

One can see here that the easiest step to take now is to perform the integral over $\underline{\tau}$: this is done using the **Heitler** function:

$$\int_0^\infty d\tau e^{-i(\nu_k\pm\omega_0)\tau} = \pi\delta\left(\nu_k\pm\omega_0\right) - iP\left(\frac{1}{\nu_k\pm\omega_0}\right)$$

where P represents the principal Cauchy value.

The next step requires us to do the summation over all modes of the field. We have done similar things before: first convert the sum into an integral:

$$\sum_{\vec{k}} \longrightarrow \frac{V}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta \int_0^\infty dk k^2$$

and use that $\left|g_{\vec{k}}\right|^2 = \frac{\left|\vec{d}_{eg}\right|^2 \nu_k}{2\hbar\epsilon_0 V} \sin^2\theta$ to obtain

$$\dot{\rho} = \frac{\left|\vec{d}_{eg}\right|^2}{8\pi^2\hbar\epsilon_0 c^3} \sum_{j,l=1}^N \left(\int_0^\infty \nu_k^3 d\nu_k \int_0^\pi d\theta \sin^3\theta \int_0^{2\pi} d\varphi e^{i\vec{k}\cdot\vec{r}_{jl}} \times \right.$$

$$\left\{ \delta(\nu_k - \omega_0) \left[\sigma_l\rho\sigma_j^\dagger - \frac{1}{2}\left\{\sigma_j^\dagger\sigma_l,\rho\right\}\right] + \underbrace{\delta(\nu_k + \omega_0) \left[\sigma_l^\dagger\rho\sigma_j - \frac{1}{2}\left\{\sigma_j\sigma_l^\dagger,\rho\right\}\right]}_{=0 \text{ since: } \nu_k\geq 0!} + \right.$$

$$\left.\left. \frac{i}{2}P\left[\frac{\left[\sigma_j^\dagger\sigma_l,\rho\right]}{\nu_k - \omega_0} + \frac{\left[\sigma_j\sigma_l^\dagger,\rho\right]}{\nu_k + \omega_0}\right]\right\}\right) .$$

And here is where many atoms enters: the integral over the angles does not simply give

a number $(4\pi/3)$ as now we have:

$$\int_0^\pi d\theta \sin^3\theta \int_0^{2\pi} d\varphi\, e^{i\vec{k}\cdot\vec{r}_{jl}}$$

$$= 4\pi\left[\left[1-\left(\hat{d}\cdot\hat{r}_{jl}\right)^2\right]\overbrace{\frac{\sin(kr_{jl})}{(kr_{jl})}}^{\text{angular}} + \left[1-3\left(\hat{d}\cdot\hat{r}_{jl}\right)^2\right]\overbrace{\left[\frac{\cos(kr_{jl})}{(kr_{jl})^2} - \underbrace{\frac{\sin(kr_{jl})}{(kr_{jl})^3}}_{\text{distance dependence}}\right]}^{\text{dependence}}\right]$$

After some more mathematical manipulation one obtains the final form of the master equation:

$$\dot\rho = \underbrace{-i\sum_{j\neq l} V_{jl}\left[\sigma_j^\dagger\sigma_l, \rho\right]}_{\text{``Hamiltonian''}} + \underbrace{\sum_{j,l=1}^{N} \Gamma_{jl}\left[\sigma_l\rho\sigma_j^\dagger - \frac{1}{2}\left\{\sigma_j^\dagger\sigma_l, \rho\right\}\right]}_{\text{``Dissipation''}} \qquad (4.2)$$

(handwritten: $-i[H_{dd},\rho]^\dagger$ above the Hamiltonian term; $\emptyset$ next to "with")

with

(handwritten: $\vec{r}_{j\ell} = r_{jl}\cdot\hat{r}_{j\ell}$)

$$V_{j}l = -\frac{3}{4}\gamma\left[\left[1-\left(\hat{d}\cdot\hat{r}_{jl}\right)^2\right]\frac{\cos(k_{jl})}{k_{jl}} - \left[1-3\left(\hat{d}\cdot\hat{r}_{jl}\right)^2\right]\left[\frac{\sin(k_{jl})}{k_{jl}^2} + \frac{\cos(k_{jl})}{k_{jl}^3}\right]\right]$$

$$\Gamma_{j}l = +\frac{3}{2}\gamma\left[\left[1-\left(\hat{d}\cdot\hat{r}_{jl}\right)^2\right]\frac{\sin(k_{jl})}{k_{jl}} + \left[1-3\left(\hat{d}\cdot\hat{r}_{jl}\right)^2\right]\left[\frac{\cos(k_{jl})}{k_{jl}^2} - \frac{\sin(k_{jl})}{k_{jl}^3}\right]\right]$$

(handwritten: $=\cos^2\theta$)

where $\boxed{k_{jl} = \omega_0/cr_{jl} = 2\pi r_{jl}/\lambda_0}$ is the **reduced distance**, between the j-th and l-th atom (scaled by the wavelength λ of the transition $|g\rangle \to |e\rangle$).

4.2 Collective behavior

Let's try now to make sense of the master equation (4.2) from a physical point of view. The equation can be rewritten as

$$\dot\rho = -\frac{i}{\hbar}[H_{dd}, \rho] + \mathcal{D}(\rho)\,.$$

(handwritten: $\binom{1}{0}=|g\rangle$, $\binom{0}{1}=|e\rangle$)

Here, the Hamiltonian is

$$H_{dd} = \sum_{j\neq l}\hbar V_{jl}\sigma_j^\dagger\sigma_l$$

(handwritten: $\sigma = |g\rangle\langle e| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\sigma^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$)

and we refer to this as the **coherent dipole-dipole** interaction, which conserves the number of excitations. The second term reads

$$\mathcal{D}(\rho) = \sum_{j,l=1}^{N} \Gamma_{jl} \left(\sigma_l \rho \sigma_k^\dagger - \frac{1}{2} \left\{ \sigma_j^\dagger \sigma_l, \rho \right\} \right) ,$$

and is responsible for the **collective dissipation** in the system.

4.2.1 Coherent dipole-dipole interactions

Lets begin by analyzing the coherent dipole-dipole interactions and the rate or strength of the interactions, given by V_{jl}. The de-excitation of one atom creates a **virtual** photon in the electromagnetic field, which can in turn be absorbed by a neighboring atoms. This process is labelled dipole-dipole interaction, or "flip-flop" interactions.
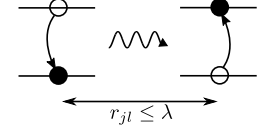
The likelihood of this process occurring is given by V_{jl}, and depends on <u>two key parameters</u>:

a) The angle formed by the atomic dipole moments (aligned) and the atomic separation vector $\vec{r}_{jl}$.

b) The (reduced!) distance between the atoms.
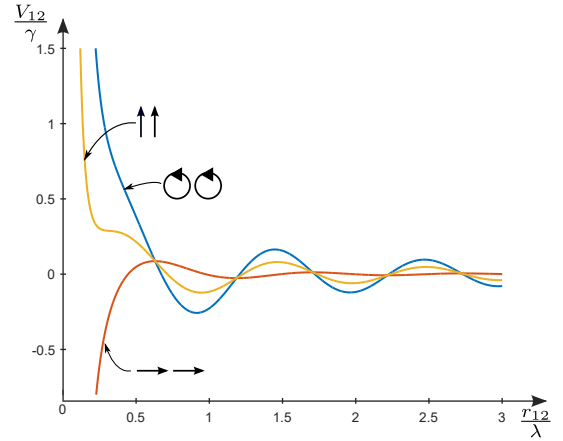
Independently of the dipole orientation, one can see separations r_{jl}/λ, decaying approximately as $1/k_{jl}^3$ and diverging for $|r_{jl}| \to 0$. Note, that while those interactions are much weaker than the ones between Rydberg atoms, they are still quite strong, although they only become noticeable for distances smaller than the wavelength λ of the atomic transition, typically a <u>few hundred nanometers</u>. As k_{jl} grows, the interaction strength oscillates and decay close to zero. This in turn means that no interactions will take place.

4.2.2 Collective dissipation

Next we consider the dissipator containing the coefficients Γ_{jl}. The form of the dissipator is similar to the one we have seen for a single atom, but the indices j and l make it difficult to identify a **rate** and **jump operator**. In order to recreate that "diagonal" case, let us diagonalize the matrix Γ_{jl}, as
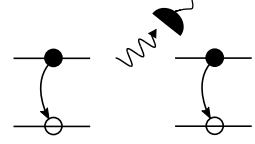
$$\Gamma_{jl} = \sum_{m=1}^{N} M_{jm} \Gamma_D^m M_{ml}^\dagger \ ,$$

where Γ_D is a diagonal matrix that will contain the **collective decay rates**. Using this expression we obtain:

$$\mathcal{D}(\rho) = \sum_{jl} \Gamma_{jl} \left(\sigma_l \rho \sigma_j^\dagger - \frac{1}{2} \left\{ \sigma_j^\dagger \sigma_l, \rho \right\} \right)$$

$$= \sum_{m=1}^{N} \Gamma_D^m \left(J_m \rho J_m^\dagger - \frac{1}{2} \left\{ J_m^\dagger J_m, \rho \right\} \right) \ ,$$

where we have defined the **collective jump operators**,

$$J_m = \sum_{l=1}^{N} \sigma_l M_{lm} \ .$$

The good thing of this form of $\mathcal{D}(\rho)$ is that it is literally a sum of $\underline{N}$ terms, each one of the shape

$$\Gamma_D^m \left(J_m \rho J_m^\dagger - \frac{1}{2} \left\{ J_m^\dagger J_m, \rho \right\} \right)$$

Where one can now really identify Γ_D^m as a rate associated with the process determined by J_m. Therese processes are the ones that are associated with the emission of a photon from the system: every photon emission is the result of one of these **collective jumps** occurring. The difference with a system where no interactions are present is that there each emission has a decay rate γ, and is associated with a photon emitted from $\underline{\text{one}}$ atom. When all atoms are interacting, as it is the case here, though, the rates are either larger or smaller than γ:

$$\Gamma_D^m < \gamma \longrightarrow \text{superradiant}$$
$$\Gamma_D^m > \gamma \longrightarrow \text{subradiant}$$

Moreover, the jump operators corresponding to these collective rates are also collective: i.e. the emission of a photon is coming from a **superposition** of atoms in the excited state.

In order to understand this a bit better, let us discuss the case of 2 atoms. Here, the dissipation matrix reads:

$$\begin{pmatrix} \gamma & \Gamma_{12} \\ \Gamma_{12} & \gamma \end{pmatrix}$$

The decay rates here are simply calculated as:

$$\Gamma_D^1 = \gamma + |\Gamma_{12}| \qquad ; \qquad \Gamma_D^2 = \gamma - |\Gamma_{12}|$$
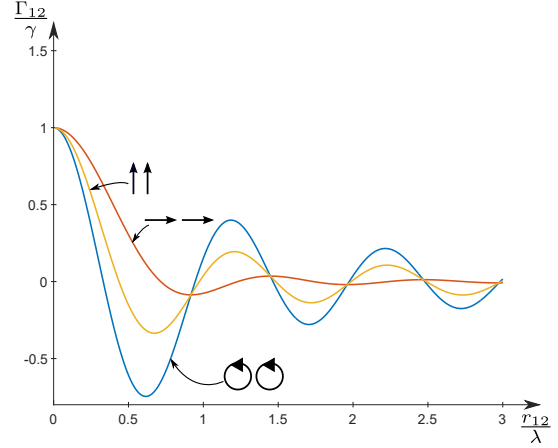
Now, as we can see in the figure, when the atoms are very far away $\Gamma_{12} \to 0$, independently of the angle between the dipoles and $\hat{r}_{12}$. This means that indeed here we recover the "simple" situation where

$$\Gamma_D^1 \propto \gamma \qquad ; \qquad J_2 \propto \sigma_2$$

which go accompanying jump operators:

$$J_1 \propto \sigma_1 \qquad ; \qquad J_2 \propto \sigma_2 \ .$$

Now, if the atoms are very close together, $r_{12} \ll \lambda$, then $\Gamma_{12} \propto \gamma$, and the collective rates and jumps become in this limit

- $\Gamma_D^1 \propto 2\gamma \longrightarrow J_1 \propto \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_2)$ **super**

- $\Gamma_D^1 \propto 0 \longrightarrow J_2 \propto \frac{1}{\sqrt{2}} (\sigma_1 - \sigma_2)$ **sub**

This means that the photons coming from this system can come at a rate $\underline{2\gamma}$, and also that one can excite in the system a state:

$$|\Psi_D\rangle = \frac{1}{\sqrt{2}} (|eg\rangle - |ge\rangle)$$

66

which will <u>not</u> emit photons into the environment as

$$J_1 \left| \Psi_D \right\rangle = 0$$
$$J_2 \left| \Psi_D \right\rangle = 0$$

This is a special subradiant state, it is completely dark. Usually, the situation is such that one has some states that are superradiant and some subradiant but not always completely dark.

Let us try to understand what happens when more than 2 atoms are in the ensemble. In particularly, let's make our life even easier and consider atoms trapped in a one-dimensional periodic chain such as
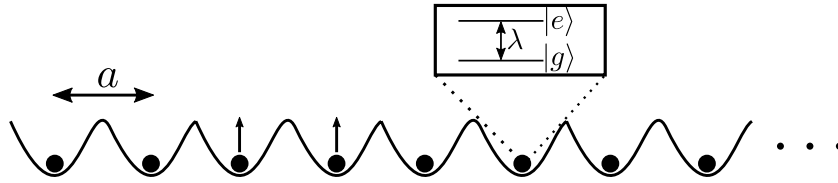
and vary the nearest-neighbor distance a. We look then at the collective decay rates (slides) for $N = 10$ atoms. One can see there that indeed when the atoms are well separated ($a \gg 1$), one recovers 10 decay rates very close to γ. However, when $a < \gamma$ many collective decay rates become much smaller than γ (subradiant states), and a few superradiant states arise.