Assignment Digital Signal & Image Processing

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Linear Time-Invariant Systems

Compute the convolution product between the functions f and g where $f, g : [a, b] \rightarrow \mathrm{R}$ with:

$$f(t) = pe^{qt}sin(ln(1 + rt^2))$$

and

$$g(t) = p_T(t)$$

and p, q, r and T sampled uniformly in the interval $[0, 2]$. Here again sample a uniformly in the interval $[-2, -1]$ and b uniformly in the interval $[1, 2]$.
Verify that you obtain the same result in both the temporal and the frequency domain.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy.signal as signal
from pylab import rcParams
rcParams['figure.figsize'] = 15, 5
```

Uniform sampling

p, q, r, T, a and b uniformly sampled in their given intervals with the `numpy. random. uniform` function.

In [2]:
```
p = np.random.uniform(0,2)
q = np.random.uniform(0,2)
r = np.random.uniform(0,2)
T = np.random.uniform(0,2)
a = np.random.uniform(-2,-1)
b = np.random.uniform(1,2)
```

In [3]:
```
print('T = {}'.format(T))
print('a = {}'.format(a))
print('b = {}'.format(b))
```

```
T = 1.0130836285165628
a = -1.5209800941171325
b = 1.4838252299137198
```

Choice of sampling values

Take a point every delta = 0.0001 in a choosen interval of t, so we're choosing the points in which our function will be evaluated along the x axis.

In [4]:
```
delta = 0.0001

t = np.arange(-8, 8, delta)

# number of samples
print("We consider %d samples"%t.size)
```

```
We consider 160000 samples
```

Description of the signals

Implementation of the functions called $f(t)$ and $g(t)$, these two functions are defined on the interval $[a, b]$.

Note that T sampled uniformly in the interval $[0, 2]$, a is sampled uniformly in $[-2, -1]$ and b is sampled uniformly in the interval $[1, 2]$ so it's possible that $T > b$ and $-T < a$ **BUT** $g(t)$ is defined in the interval $[a, b]$ so to handle this case I added an `if` statement which checks if the value of t that is considered is between a and b and if it's smaller than T, if it is then we assign to $g(t)$ the value 1 else the value will be 0.

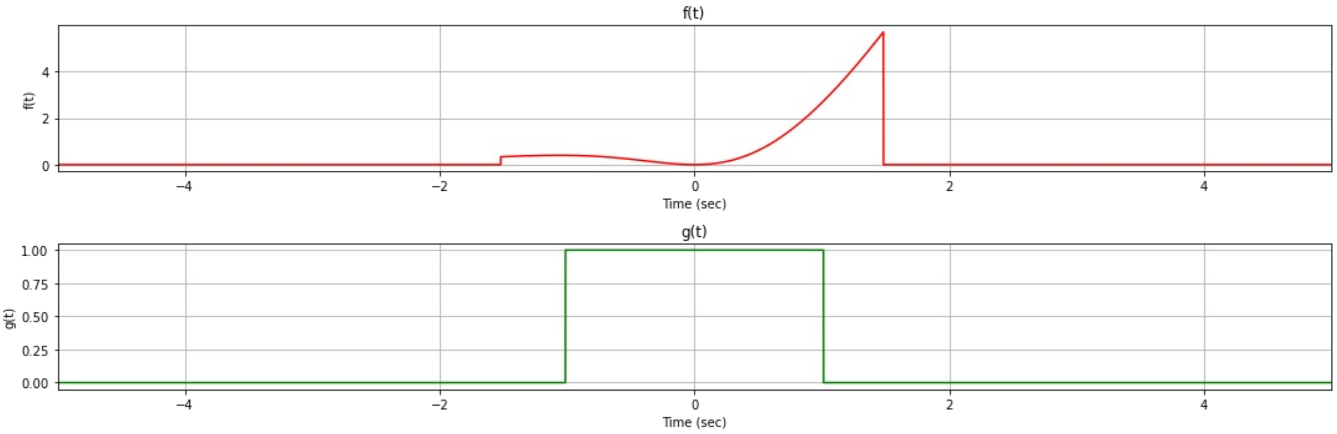
Note that even if the functions are defined over the interval $[a, b]$ I evaluate them over the entire window of time that I considered, and outside that interval I stored in the arrays a sequence of zeros, this is a sort of zero padding I implemented since without it the comparison between convolution in time and product in frequency is wrong because of the way in which the two are computed in python.

In [5]:
```
# Definition of the signal f(t):
f_t = np.zeros(len(t))
```

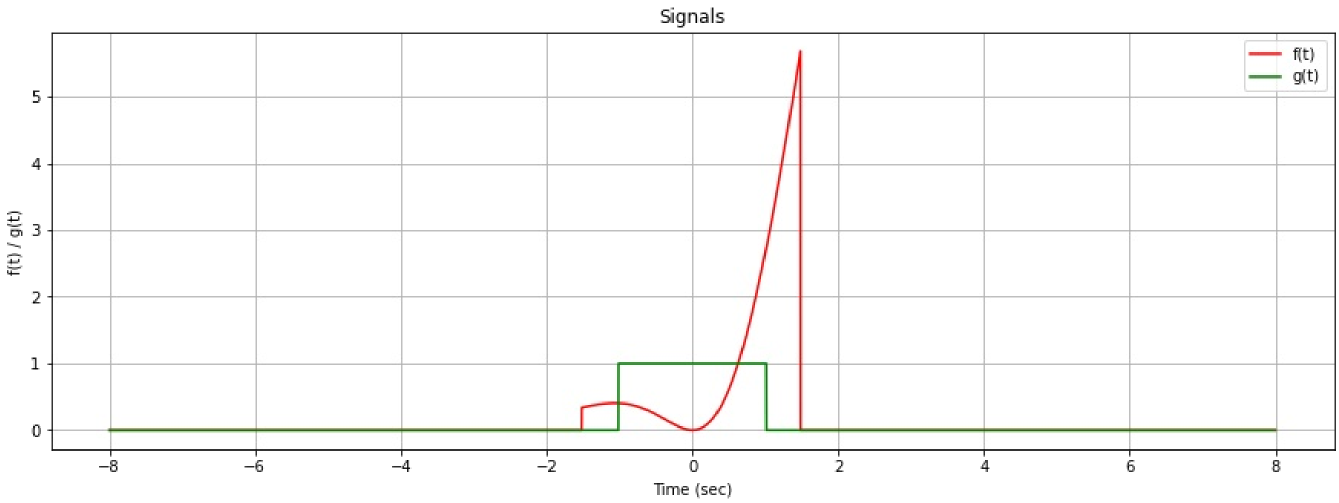
```python
def f(t, a, b):
    for i in range(len(t)):
        if t[i] >= a and t[i] <= b :
            f_t[i] = p * (np.exp(q * t[i])) * (np.sin(np.log(1 + (r * (t[i])**2))))
        else : f_t[i] = 0
    return f_t


# Definition of the signal g(t):
g_t = np.zeros(len(t))

def g(t, T, a, b):
    for i in range(len(t)):
        if t[i] >= a and t[i] <= b :
            if abs(t[i]) <= T : g_t[i] = 1
            else : g_t[i] = 0
        else : f_t[i] = 0
    return g_t
```

In [6]:
```python
# Evaluation of the functions:
f_t = f(t, a, b)
g_t = g(t, T, a, b)
```

In [7]:
```python
# Plot f(t):
plt.subplot(2, 1, 1)
plt.plot(t, f_t, color='r')
plt.title('f(t)')
plt.xlabel("Time (sec)")
plt.ylabel("f(t)")
plt.xlim(-5, +5)
plt.grid(True)
plt.tight_layout()

# Plot g(t):
plt.subplot(2, 1, 2)
plt.plot(t, g_t, color='g')
plt.title('g(t)')
plt.xlabel("Time (sec)")
plt.ylabel("g(t)")
plt.xlim(-5, +5)
plt.grid(True)
plt.tight_layout()

plt.show()
```



In [8]:
```python
# Plot the two functions:
plt.plot(t, f_t, label='f(t)', color='r')
plt.plot(t, g_t, label='g(t)', color='g')
plt.legend(loc='upper right')
plt.title("Signals")
plt.xlabel("Time (sec)")
plt.ylabel("f(t) / g(t)")
plt.grid(True)
plt.show()
```



Convolution between f(t) and g(t)

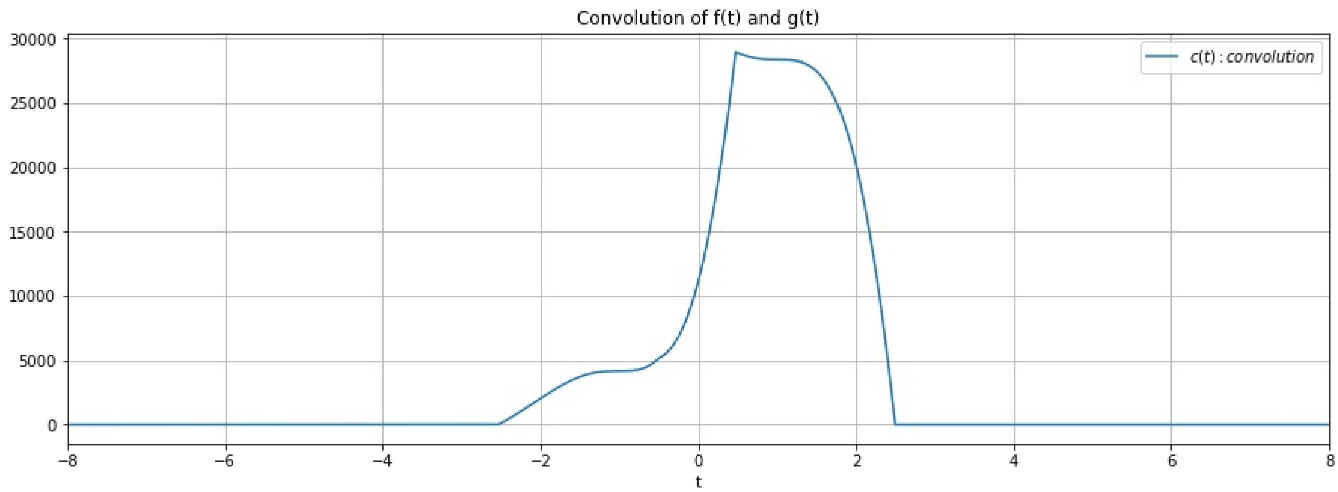
In the time domain, convolution is an operation that combines two functions, $f(t)$ and $g(t)$, to create a third function, $h(t)$. The convolution of two functions is defined as the integral of the product of one function with a time-reversed version of the other function, shifted by a variable τ. So, mathematically the convolution is:

$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$ <\center>

In simpler words, the convolution operation slides one function over the other and multiplies their values at each position, then integrate the resulting values over the entire domain.

In [9]:
```
conv = np.convolve(f_t, g_t, mode='same')
```

In [10]:
```
plt.figure(figsize=(15, 5))
plt.plot(t, conv, label="$c(t): convolution$")
plt.title("Convolution of f(t) and g(t)")
plt.xlim(-8, +8)
plt.xlabel('t')
plt.legend()
plt.grid()
plt.show()
```



Convolution Theorem

Now we have to verify that you obtain the same result in both the temporal and the frequency domain. This means that we have to verify the *Convolution Theorem* which states that:

- $x(t) * y(t) \longleftrightarrow X(f)Y(f)$
- $x(t)y(t) \longleftrightarrow X(f) * Y(f)$

So the Fourier transform of the convolution of two functions is equal to the product of their individual Fourier transforms.

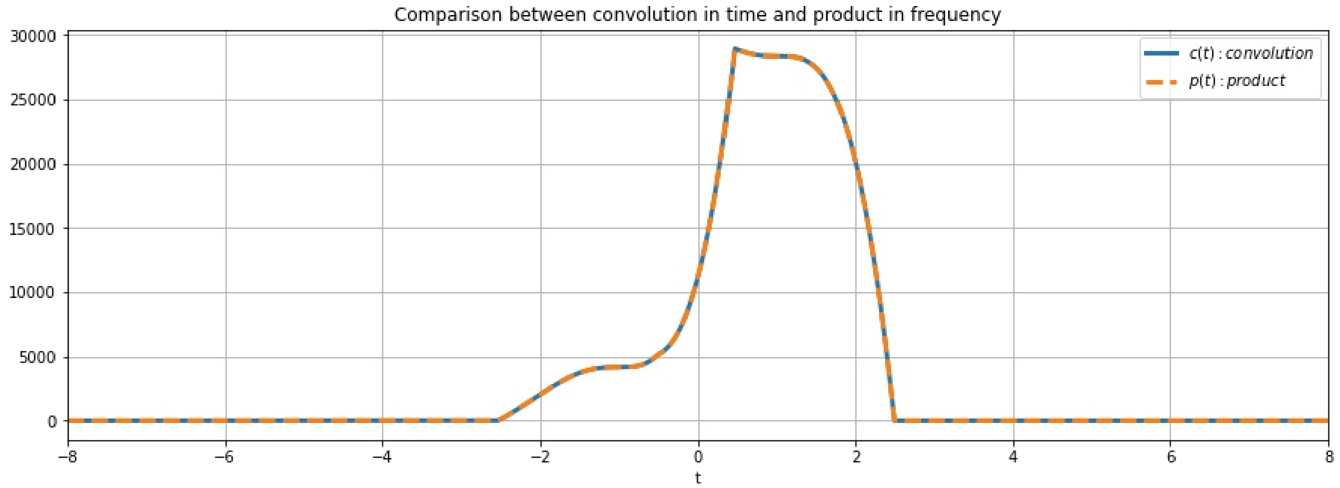
In order to verify it I followed these steps:

- I computed the Fourier Transform of $f(t)$ and $g(t)$ getting $F(w)$ and $G(w)$
- I multiplied $F(w)$ and $G(w)$
- I applied the Inverse Fourier Transform to get back into the time domain

- I shifted the result because the output of the Inverse Fourier Transform is not centered in zero, this means that the zero-frequency component (DC component) in not at the center of the time domain signal.

In [11]:
```
F = np.fft.fft(f_t)
G = np.fft.fft(g_t)

FG = np.multiply(F, G)

FG_inverse = np.fft.ifft(FG)
FG_shift = np.fft.fftshift(FG_inverse)

# Array of frequencies values:
N = len(t)
freq = np.fft.fftfreq(N, delta)
```

In [12]:
```
plt.figure(figsize=(15, 5))
plt.title("Comparison between convolution in time and product in frequency")
plt.plot(t, conv, linewidth = '3', label=r"$c(t): convolution$")
plt.plot(t, np.real(FG_shift), linewidth = '3',  linestyle = 'dashed', label=r"$p(t): product$")
plt.xlim(-8, +8)
plt.xlabel('t')
plt.legend()
plt.grid()
plt.show()
```



Final Comment

As we can see, the convolution in time between $f(t)$ and $g(t)$ and the product in frequency between $F(w)$ and $G(w)$ are equal, so we obtained the same result in both the temporal and the frequency domain, this meas that the *Convolution Theorem* is verified.

Processing math: 100%